Chapter 1

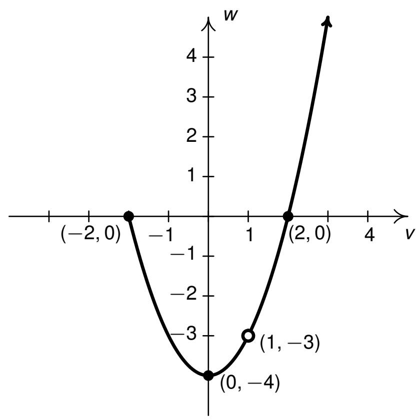
FIRST STEPS INTO CALCULUS

1.1 A (more) Formal Introduction to Limits

In this chapter, we take some more steps towards[1] Calculus. We first revisit the concept of **limit** . We've primarily used limits as a way to analyze and codify function behavior in places where we simplify could not evaluate the function.[2] We first focus on how the concept is expressed graphically.

1.1.1 Limits from Graphs

Even though we didn't introduce the limit concept or notation until Chapter **??**, we first encounter the underlying concept much earlier. Recall in Example **??** we were given the graph of a function $w = F(v)$:



The hole in the graph tells us that even though $F(1)$ is undefined, we'd **expect** $F(1)$ to be -3 based on what's happening with the graph **near** the point $(1, -3)$. Using limit notation, we'd write $\lim\limits_{v \to 1} F(v) = -3$. We take a moment below to better define what we mean when use the limit notation these types of situations.

DEFINITION 1.1. **Informal Definition of Limit:** Given a function f defined on an open interval containing $x = a$, except possibly at $x = a$, the notation $\lim\limits_{x \to a} f(x) = L$, means as input values, x, approach[a] the number a, the output values, $f(x)$, approach the number L. The notation '$\lim\limits_{x \to a} f(x) = L$' is read read 'the **limit** as x approaches a of $f(x)$ equals L.'

[a]ignoring what is happening at $x = a$

Some remarks about Definition 1.1 are in order. Note that the business about f being defined on 'an open interval containing $x = a$' is there to guarantee that we have the appropriate 'room' for inputs x to approach a from either direction.[3] (For now, we'll just assume we all understand what the word 'approach' means in this context and let a Calculus class[4] explain how this is more precisely codified mathematically.)

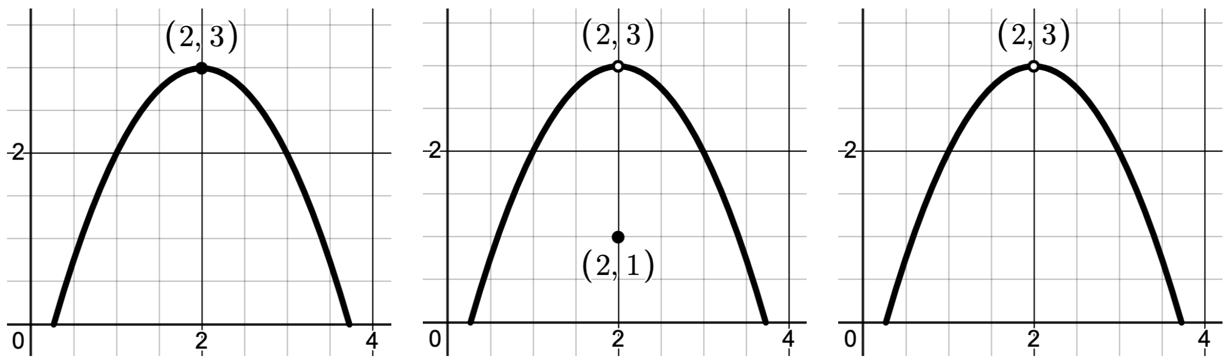
[1]into?
[2]Whether it be describing end behavior as $x \to \pm\infty$ or places where we'd be dividing by '0.'
[3]We'll get to 'one-sided' limits here shortly.
[4]Or Section **??** - if it ever gets written . . .

The phrase 'except possibly at $x = a$' which immediately follows means the limit doesn't concern itself with what is actually happening at $x = a$. The function f may or may not be defined at $x = a$. Indeed, if $\lim\limits_{x \to a} f(x) = L$, $f(a)$ could be L, $f(a)$ could be a number different than L or $f(a)$ could not exist at all.

This drives home the principle difference between the precalculus notion of '$f(a)$' and the Calculus notion of '$\lim\limits_{x \to a} f(x)$': '$\lim\limits_{x \to a} f(x)$' is what we **expect** $f(a)$ to be - which may or may not agree with what $f(a)$, if it exists.

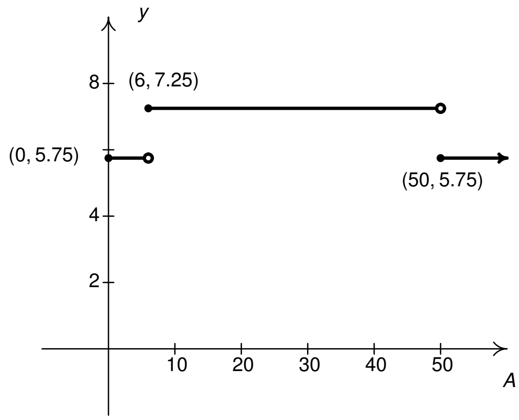
For example, using the graph from Example **??**, we write $\lim\limits_{v \to 0} F(v) = -4$ since as $v \to 0$, we see $w = F(v) \to -4$. In this particular case, $F(0) = -4$ so we get from F at $v = 0$ what we expect to get.[5]

For another example, consider the graphs of the functions f, g, and h below near $x = 2$. Through a precalculus lens, each of these functions is different at $x = 2$: $f(2) = 3$, $g(2) = 1$, and $h(2)$ doesn't exist. Through a Calculus lens, however, all three of these functions are behaving identically as x approaches 2: $\lim\limits_{x \to 2} f(x) = 3$, $\lim\limits_{x \to 2} g(x) = 3$, and $\lim\limits_{x \to 2} h(x) = 3$.



$y = f(x)$ $y = g(x)$ $y = h(x)$

Next let's head to Chapter **??** and revisit Example **??** in a piecewise-defined function is used to model matinee admission prices at a local theater:



$$y = p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50 \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$$

[5]This is another way to describe the notion of **continuity**. (See Definition 1.4.)

What can be said about $\lim_{A \to 6} p(A)$? Remember, $\lim_{A \to 6} p(A)$ is what we would **expect** $p(6)$ to be by analyzing p as $A \to 6$, ignoring what is happening at $A = 6$. If $A < 6$, $p(A)$ is always 5.75, so, based on this information, we'd **expect** $p(6)$ to be 5.75. If $A > 6$, then $p(A)$ is always 7.25, so we'd expect $p(6)$ to be 7.25. Since Definition 1.1 requires the $p(A)$ values to approach a **single** value L as $A \to 6$, we'd say in this case that $\lim_{A \to 6} p(A)$ does not exist.

Even though $\lim_{A \to 6} p(A)$ does not exist, we've used so-called 'one-sided' limit notation in Chapters **??** and **??** which we can apply here. Specifically, we write $\lim_{A \to 6^-} p(A) = 5.75$ and $\lim_{A \to 6^+} p(A) = 7.25$ to more precisely record the behavior of p as we approach $A = 6$ from either direction.[6]

DEFINITION 1.2. **One-sided Limits:**

- If f is defined on an open interval for $x < a$ except possibly at $x = a$, the notation $\lim_{x \to a^-} f(x) = L$, read 'the **limit** as x approaches a **from the left** of $f(x)$ equals L' means as input values, x, $x < a$, approach the number a (ignoring what is happening at $x = a$), the output values, $f(x)$, approach the number L.
- If f is defined on an open interval for $x > a$ except possibly at $x = a$, the notation $\lim_{x \to a^+} f(x) = L$, read 'the **limit** as x approaches a **from the right** of $f(x)$ equals L' means as input values, x, $x > a$, approach the number a (ignoring what is happening at $x = a$), the output values, $f(x)$, approach the number L.

In order for the (two-sided) limit to exist, both one-sided limits need to exist, be equal, and vice-versa. This is recorded in the following theorem.

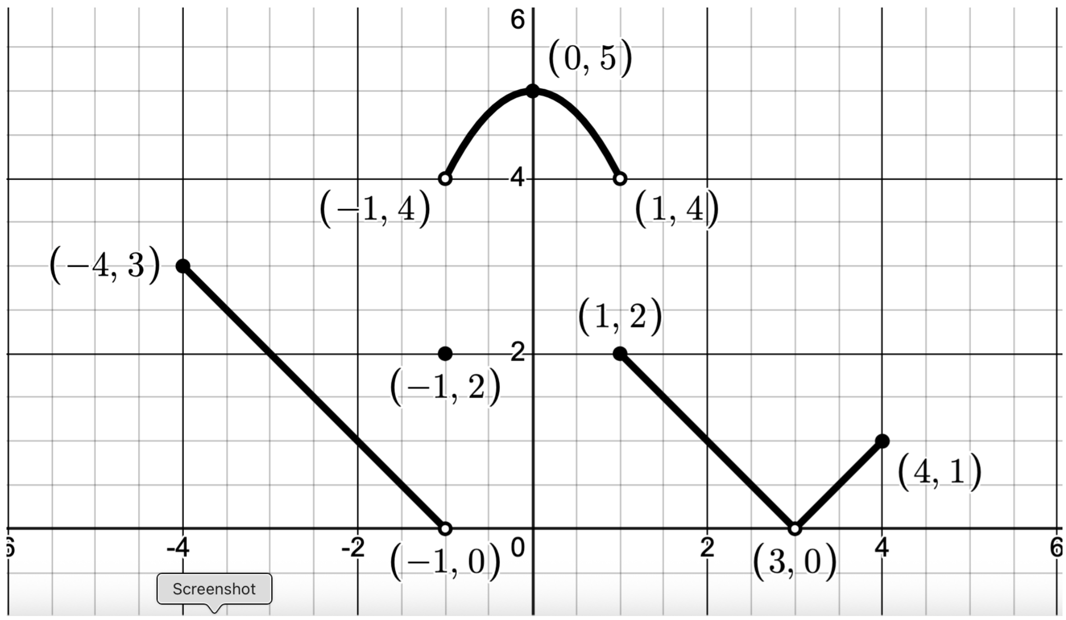
THEOREM 1.1. Given a function f defined on an open interval containing $x = a$, except possibly at $x = a$, $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

It's time for an example.

EXAMPLE 1.1.1. Use the **complete** graph[7] of $y = f(x)$ below to answer the following questions.

[6] Note that $\lim_{A \to 6^+} p(A) = 7.25 = p(6)$ in this case. So at least we 'get' what we 'expect to get' when approaching 6 from the right.

[7] Recall this means this is the entire graph of f. There's nothing hidden offscreen.



1. State the domain and range of f using interval notation.
2. Find the following values. If a value does not exist, write 'does not exist.' Explain your reasoning.

- $f(-1)$
- $\lim_{x\to -1^-} f(x)$
- $\lim_{x\to -1^+} f(x)$
- $\lim_{x\to -1} f(x)$
- $f(1)$
- $\lim_{x\to 1^-} f(x)$
- $\lim_{x\to 1^+} f(x)$
- $\lim_{x\to 1} f(x)$
- $f(0)$
- $\lim_{x\to 0^-} f(x)$
- $\lim_{x\to 0^+} f(x)$
- $\lim_{x\to 0} f(x)$
- $f(3)$
- $\lim_{x\to 3^-} f(x)$
- $\lim_{x\to 3^+} f(x)$
- $\lim_{x\to 3} f(x)$

3. Explain why $\lim_{x\to -4} f(x)$ does not exist and find $\lim_{x\to -4^+} f(x)$
4. Explain why $\lim_{x\to 4} f(x)$ does not exist and find $\lim_{x\to 4^-} f(x)$

Solution.

1. Projecting the graph of f to the $x-$axis, we find that everything from -4 to 4, inclusive, is covered except for $x = 3$. Hence the domain is $[-4, 3) \cup (3, 4]$. When projecting the graph of f to the y-axis, we see everything between 0 and 3 is covered (excluding 0 and including 3) then everything between 4 and 5 is covered (excluding 4 and including 5). Hence the range is $(0, 3] \cup (4, 5]$.

2.
   - Regarding $x = -1$: since the point $(-1, 2)$ is on the graph of f, we know $f(-1) = 2$. To determine $\lim_{x\to -1^-} f(x)$, we see that the graph to the left of $x = -1$ is headed towards the point $(-1, 0)$ as x approaches -1. Hence, $\lim_{x\to -1^-} f(x) = 0$. To determine $\lim_{x\to -1^+} f(x)$, we see that the graph to the right of $x = -1$ is headed towards the point $(-1, 4)$ as x approaches -1. Hence, $\lim_{x\to -1^+} f(x) = 4$. Since $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$ are different, $\lim_{x\to -1} f(x)$ does not exist.
   - Regarding $x = 1$: since the point $(1, 2)$ is on the graph of f, we know $f(1) = 2$. To determine $\lim_{x\to 1^-} f(x)$, we see that the graph to the left of $x = 1$ is headed towards the point $(1, 4)$ as x approaches 1. Hence, $\lim_{x\to 1^-} f(x) = 4$. To determine $\lim_{x\to 1^+} f(x)$, we see that the graph to the right of $x = 1$ is headed towards the point $(1, 2)$ as x approaches 1. Hence, $\lim_{x\to 1^+} f(x) = 2$. Since $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$ are different, $\lim_{x\to 1} f(x)$ does not exist.
   - Regarding $x = 0$: since the point $(0, 5)$ is on the graph of f, we know $f(0) = 5$. To determine $\lim_{x\to 0^-} f(x)$, we see that the graph to the left of $x = 0$ is headed towards the point $(0, 5)$ as x approaches 0. Hence, $\lim_{x\to 0^-} f(x) = 5$. To determine $\lim_{x\to 0^+} f(x)$, we see that the graph to the right of $x = 0$ is headed towards the point $(0, 5)$ as x approaches 0. Hence, $\lim_{x\to 0^+} f(x) = 5$. Since $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ are both 5, $\lim_{x\to 0} f(x) = 5$.
   - Regarding $x = 3$: since there is no point on the graph of f with an x-coordinate of 3, $f(3)$ does not exist.. To determine $\lim_{x\to 3^-} f(x)$, we see that the graph to the left of $x = 3$ is headed towards the point $(3, 0)$ as x approaches 3. Hence, $\lim_{x\to 3^-} f(x) = 0$. To determine $\lim_{x\to 3^+} f(x)$, we see that the graph to the right of $x = 3$ is headed towards the point $(3, 0)$ as x approaches 3. Hence, $\lim_{x\to 3^+} f(x) = 0$. Since $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$ are both 0, $\lim_{x\to 3} f(x) = 0$.

3. In order to find $\lim_{x\to -4} f(x)$, we need to analyze the graph of f from both the left an right of $x = -4$. Since there is no graph to the left of $x = -4$, $\lim_{x\to -4^-} f(x)$, and hence $\lim_{x\to -4} f(x)$ does not exist. However, $\lim_{x\to -4^+} f(x) = 3$, since, when coming from the right, the graph approaches the point $(-4, 3)$ as x approaches -4.

4. In order to find $\lim_{x\to 4} f(x)$, we need to analyze the graph of f from both the left an right of $x = 4$. Since there is no graph to the right of $x = 4$, $\lim_{x\to 4^+} f(x)$, and hence $\lim_{x\to 4} f(x)$ does not exist. However, $\lim_{x\to 4^-} f(x) = 1$, since, when coming from the left, the graph approaches the point $(4, 1)$ as x approaches 4. $\square$

Another use of limits we've seen is to codify unbounded behavior. Since ∞ and $-\infty$ aren't real numbers, we used limit notation to help us describe end behavior (as $x \to \pm\infty$) and unbounded function behavior ($f(x) \to \pm\infty$.) Let's take a moment to think about what it means to write $\lim_{x\to\infty} f(x) = \infty$. How does one 'approach' infinity anyhow?

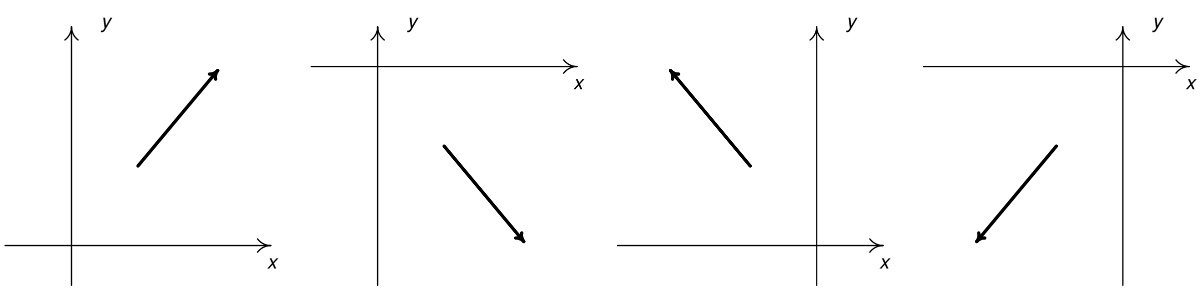
Let's consider $\lim_{x\to\infty} x^2 = \infty$. What me mean here is that as x grows larger and larger (without bound), $f(x) = x^2$ follows suit. To prove something like this, we'd need to show that for any 'arbitrarily large' real number, N, we can find some threshold M so that if the inputs, $x > M$, the outputs, $f(x) > N$. For example, if we set $N = 10000$, then to guarantee $f(x) = x^2 > 10000$, we can solve and get $x > \sqrt{10000} = 100$. So provided $x > 100$, $f(x) > 10000$. In this case, $N = 10000$ and $M = \sqrt{10000} = 100$. In general, if $x > \sqrt{N}$, $x^2 > N$, which justifies us writing $\displaystyle\lim_{x\to\infty} x^2 = \infty$.

We can adjust the inequality signs in the sort of argument[8] above to direct x or $f(x)$ to either ∞ or $-\infty$. Doing so gives us the (formal) definitions of below.

Definition 1.3.

1. Given a function f defined on an open interval (a, ∞):
   - the notation '$\displaystyle\lim_{x\to\infty} f(x) = \infty$' means that for any real number N there is a real number M so that if $x > M$, $f(x) > N$.
   - the notation '$\displaystyle\lim_{x\to\infty} f(x) = -\infty$' means that for any real number N there is a real number M so that if $x > M$, $f(x) < N$.
2. Given a function f defined on an open interval $(-\infty, a)$:
   - the notation '$\displaystyle\lim_{x\to-\infty} f(x) = \infty$' means that for any real number N there is a real number M so that if $x < M$, $f(x) > N$.
   - the notation '$\displaystyle\lim_{x\to-\infty} f(x) = -\infty$' means that for any real number N there is a real number M so that if $x < M$, $f(x) < N$.

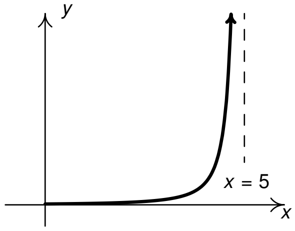
We'll explore Definition 1.3 more in the Exercises. In the meantime, the reader is encouraged to take some time and think about the inequalities in Definition 1.3 and how they force the corresponding graphical behavior showcased below:



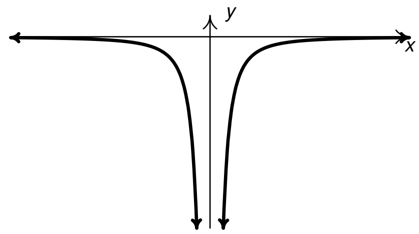
$\displaystyle\lim_{x\to\infty} f(x) = \infty$ $\quad$ $\displaystyle\lim_{x\to\infty} f(x) = -\infty$ $\quad$ $\displaystyle\lim_{x\to-\infty} f(x) = \infty$ $\quad$ $\displaystyle\lim_{x\to-\infty} f(x) = -\infty$

Combining the ideas of what it means for x or $f(x)$ to approach (finite) real numbers along with our (more precise notion) of what it means for x or $f(x)$ to approach $\pm\infty$, we can mix and match to produce expressions and graphs containing vertical and horizontal asymptotes such as the ones depicted below:

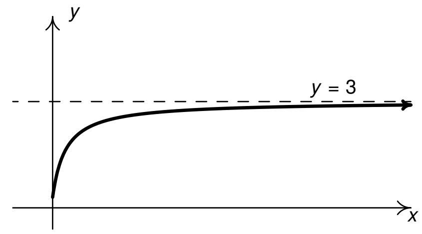
[8]If this sort of argument seems familiar, it should! Replacing '$>$' with '$=$' is how we proved the ranges of the monomial functions and Laurent monomials in Sections **??** and **??**, respectively.



$\lim\limits_{x \to 5^-} f(x) = \infty$



$\lim\limits_{x \to 0} f(x) = -\infty$



$\lim\limits_{x \to \infty} f(x) = 3$

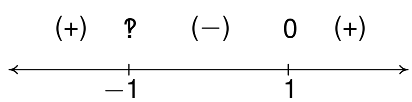
We would be remiss in our duties as (pre)Calculus instructors if we failed to point out that even though we've used notation '$= \infty$' in expressions like $\lim\limits_{x \to 5^-} f(x) = \infty$ above, since ∞ is not a real number, technically, $\lim\limits_{x \to 5^-} f(x)$ does not exist. The '$= \infty$' here just codifies better **the manner in which** the limit fails to exist.

Our last example of this section turns the tables and has you construct the graph of function given information provided by limits.

EXAMPLE 1.1.2. Sketch the graph of a function f which satisfies all of the following criteria:

- $\lim\limits_{x \to -\infty} f(x) = 0$
- $\lim\limits_{x \to -1^-} f(x) = \infty$
- $\lim\limits_{x \to -1^+} f(x) = -\infty$
- $\lim\limits_{x \to 1^-} f(x) = -\dfrac{1}{2}$
- $\lim\limits_{x \to 1^+} f(x) = 0$
- $\lim\limits_{x \to \infty} f(x) = \infty$

The sign diagram for f is:



A sign diagram for $f(x)$

Solution. Each piece of information given describes a portion of the graph of $y = f(x)$. The strategy is to sketch each individual portion and connect them together.

First off, $\lim\limits_{x \to -\infty} f(x) = 0$ tells us that $y = 0$ is a horizontal asymptote to the graph. This means as we head off to the left, the graph approaches the x-axis. Since the Sign Diagram tells is $f(x) > 0$ for $x < -1$, we know the graph must approach the x-axis from above.

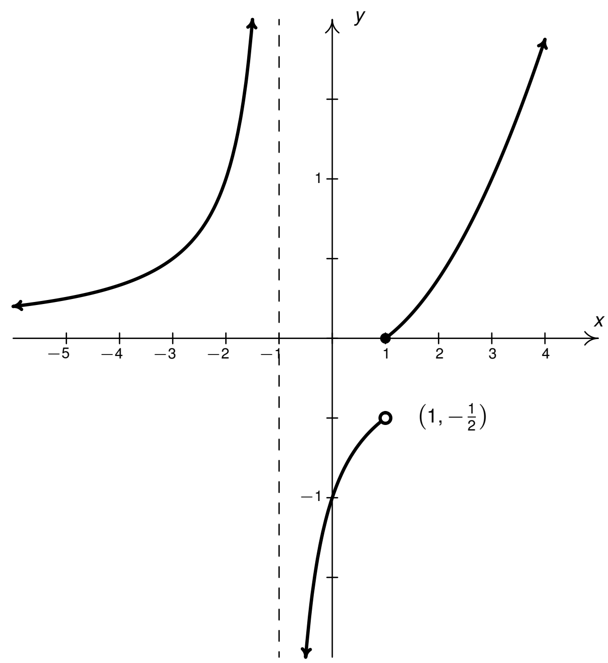
Next, we have $\lim\limits_{x \to -1^-} f(x) = \infty$ and $\lim\limits_{x \to -1^+} f(x) = -\infty$ which tells us $x = -1$ is a vertical asymptote to the graph. These behaviors agree with the Sign Diagram both in sign ('+' ∞ for $x < -1$ and '$-$' ∞ for $x > -1$) and the fact that f is undefined at $x = -1$.

Moving on we are given information about f near $x = 1$. The limit $\lim\limits_{x \to 1^-} f(x) = -\frac{1}{2}$ means as we approach $x = 1$ from the left, the y-values approach $-\frac{1}{2}$. Likewise, $\lim\limits_{x \to 1^+} f(x) = 0$ means as we approach $x = 1$ from the right, the y-values approach 0 (the x-axis).

The sign diagram tells us that, indeed, $f(1) = 0$. Hence, as $x \to 1^-$, the graph of f approaches a hole at $\left(0, -\frac{1}{2}\right)$. As $x \to 0^+$, the graph of f approaches an x-intercept, $(1, 0)$, which is included in the graph.

Finally, $\lim\limits_{x\to\infty} f(x) = \infty$ means that as we move farther to the right, the graph moves farther up which we indicate, as usual, with an arrow up to the right.

Connecting these pieces together (careful to not violate the Vertical Line Test, Theorem **??**) we get:



□

1.1.2 Limit Properties and an Introduction to Continuity

Let $f(x) = 6$. Consider $\lim\limits_{x\to5} f(x) = \lim\limits_{x\to5} 6$. Since the function values are unchanging, there is no other value other than '6' to expect from f so it stands to reason that $\lim\limits_{x\to5} f(x) = \lim\limits_{x\to5} 6 = 6$. Indeed, for any real number a, $\lim\limits_{x\to a} 6 = 6$. In general, if $f(x) = c$ is a constant function, $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} c = c$. The formal proof of this fact requires a formal definition of limit,[9] but for now, we'll just take it as true.

Next, let's consider $f(x) = x$. Consider $\lim\limits_{x\to5} f(x) = \lim\limits_{x\to5} x$. What do we expect the value of 'x' to be as $x \to 5$? Well, '5'. Indeed, it can be proved that $\lim\limits_{x\to a} x = a$ for all real numbers, a.

What about $\lim\limits_{x\to5}(x + 6)$? Since $\lim\limits_{x\to5} x = 5$ and $\lim\limits_{x\to5} 6 = 6$, it stands to reason that

$$\lim_{x\to5}(x + 6) = \lim_{x\to5} x + \lim_{x\to5} 6 = 5 + 6 = 11,$$

[9]which, incidentally, can be phrased in terms of absolute value inequalities . . . See Section **??** (once it's written...)

which is indeed the case. It turns out that in most cases, limits do respect arithmetic. This is hugely impactful and summarized below.

THEOREM 1.2. **(Some of the) Properties of Limits:**

1. **Constant Rule:** If c is a constant, then $\lim_{x\to a} c = c$ for every real number a.
2. **Identity Rule:** $\lim_{x\to a} x = a$ for every real number a.
3. **Limits Respect Function Arithmetic:** If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$, then:
   (a) **Sum and Difference Rule:** $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm K$
   (b) **Product Rule:** $\lim_{x\to a} [f(x)\, g(x)] = \left[\lim_{x\to a} f(x)\right] \left[\lim_{x\to c} g(x)\right] = L\,K$
      i. **Scalar Multiple Rule:** $\lim_{x\to a} [c\, f(x)] = c \left[\lim_{x\to a} f(x)\right] = c\,L$
      ii. **Power Rule:** $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n$, where n is any natural number.[a]
   (c) **Quotient Rule:** $\lim_{x\to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{K}$, provided $K \neq 0$.
   (d) **Radical Rules:**
      - If n is **odd**, $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$.
      - If n is **even** and $L > 0$, $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$.
      - If n is **even** and $L = 0$, $\lim_{x\to a} \sqrt[n]{f(x)} = 0$ provided $f(x) \geq 0$ for all x near a.

[a] Recall this means $n = 1, 2, 3, \ldots$

For those interested, the 'Scalar Multiple Rule' and 'Power Rule' are grouped with the Product Rule since they both follow directly from the Product Rule. For instance, using the Product Rule,

$$\lim_{x\to a} [c\, f(x)] = \lim_{x\to a} c \lim_{x\to a} f(x) = c \lim_{x\to a} f(x).$$

For powers, note that $[f(x)]^2 = f(x)\, f(x)$ so that

$$\lim_{x\to a} [f(x)]^2 = \lim_{x\to a} [f(x)\, f(x)] = \lim_{x\to a} f(x) \lim_{x\to a} f(x) = L\,L = L^2.$$

Once this is established, we can use the fact that $[f(x)]^3 = f(x)\ [f(x)]^2$ and the product rule again to get

$$\lim_{x\to a} [f(x)]^3 = \lim_{x\to a} \left[f(x)\ [f(x)]^2\right] = \lim_{x\to a} f(x) \lim_{x\to a} [f(x)]^2 = L\,L^2 = L^3.$$

Continuing in this manner gives us the Power Rule.[10]

[10] See Exercise **??** in Section **??** for more details.

A note regarding the 'Radical Rules': since $\sqrt{N}$ is not defined if $N < 0$, we have to be careful about limits involving even-indexed radicals (or exponents which indicate even-indexed radicals.) For example, consider $\lim x \to 5\sqrt{5-x}$. Since this is a 'two-sided' limit, we must consider both $x \to 5^-$ and $x \to 5^+$.

As $x \to 5^-$, the radicand, $5 - x > 0$ so $\sqrt{5-x}$ is defined as a real number. More specifically, as $x \to 5^-$, $5 - x \to 0^+$ so $\lim x \to 5^-\sqrt{5-x} = 0$. On the other hand, if $x \to 5^+$, $5 - x < 0$, and $\sqrt{5-x}$ is no longer a real number. Therefore, $\lim x \to 5^+\sqrt{5-x}$, and, hence, $\lim x \to 5\sqrt{5-x}$ does not exist.

We put the limit properties to good use in the following example.

EXAMPLE 1.1.3. Let $f(x) = \dfrac{x\sqrt{x+1}}{x^2+2x-4}$. Use Theorem 1.2 to find $\lim\limits_{x\to 3} f(x)$.

Solution.

$$
\begin{aligned}
\lim_{x\to 3}\frac{x\sqrt{x+1}}{x^2+2x-4} &= \frac{\lim_{x\to 3} x\sqrt{x+1}}{\lim_{x\to 3}\left(x^2+2x-4\right)} && \text{Quotient Rule}\\
&= \frac{\left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3}\sqrt{x+1}\right)}{\lim_{x\to 3}\left(x^2\right)+\lim_{x\to 3}(2x)-\lim_{x\to 3} 4} && \frac{\text{Product Rule}}{\text{Sum and Difference Rule}}\\
&= \frac{3\sqrt{\lim_{x\to 3}(x+1)}}{\left(\lim_{x\to 3} x\right)^2+2\lim_{x\to 3} x-4} && \frac{\text{Identity and Radical Rules}}{\text{Power, Constant Multiple, and Constant Rules}}\\
&= \frac{3\sqrt{\lim_{x\to 3} x+\lim_{x\to 3} 1}}{(3)^2+2(3)-4} && \frac{\text{Sum Rule}}{\text{Identity Rule}}\\
&= \frac{3\sqrt{3+1}}{9+6-4} = \frac{6}{11} && \text{Identity and Constant Rules, Simplify} \quad \Box
\end{aligned}
$$

It is worth noting that we could have arrived at the same (correct) answer to Example 1.1.3 by evaluating $f(3)$: $f(3) = \frac{3\sqrt{3+1}}{(3)^2+6-4} = \frac{6}{11}$. That's really the power of Theorem 1.2. Under 'nice' circumstances,[11] Theorem 1.2 allows us to compute limits using direct substitution. Functions with this property have a familiar name.

> DEFINITION 1.4. A function f is said to be **continuous** at an input $x = a$ if $\lim\limits_{x\to a} f(x) = f(a)$.
>
> Said differently, a function f is said to be continuous at a real number a if what we **get**, $f(a)$, exactly what we **expect to get**, $\lim\limits_{x\to a} f(x)$.

This is not the first time we've mentioned this property of functions. Indeed, we've discussed continuity albeit in graphical terms throughout Chapters **??** through **??**. In those chapters, we described continuous functions as those whose graphs are connected meaning they have 'no holes or breaks' in them. It is a

[11] primarily those in which we're not dealing with piecewise-defined functions or dividing by 0 . . .

great exercise to compare the description given in Definition 1.4 to the graphical description to see how those two ideas mesh.

In a standard Calculus course, you'll explore properties of continuous functions more extensively. For our purposes here, polynomial, and, more generally, rational functions are continuous **on their domains**, as well as the functions we encountered in Chapter **??**. This means in order to **evaluate limits** of these functions, we may use Definition 1.4 and simply **evaluate the function** at the corresponding value.

EXAMPLE 1.1.4. Let $f(x) = \begin{cases} 2x - 1 & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$.

1. Is f is continuous at $x = 2$? Explain.
2. Find a constant 'm' so that $g(x) = \begin{cases} mx - 1 & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$ is continuous at $x = 2$.

Solution.

1. To determine if f is continuous at $x = 2$, we need to check to see if $\lim\limits_{x \to 2} f(x) = f(2)$. We note that f is defined at $x = 2$ and that $f(2) = (2)^2 = 4$ so we set about determining $\lim\limits_{x \to 2} f(x)$.

   Since f is a piecewise-defined function which has different formulas on either side of 2, we need to check $\lim\limits_{x \to 2} f(x)$ from both directions. To find $\lim\limits_{x \to 2^-} f(x)$, we note that as $x \to 2^-$, $x < 2$ so $f(x) = 2x - 1$. Hence, $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (2x - 1) = 2(2) - 1 = 3$, the last step coming from the fact that for $x < 2$, $f(x) = 2x - 1$ is a linear function (a polynomial) and is continuous.

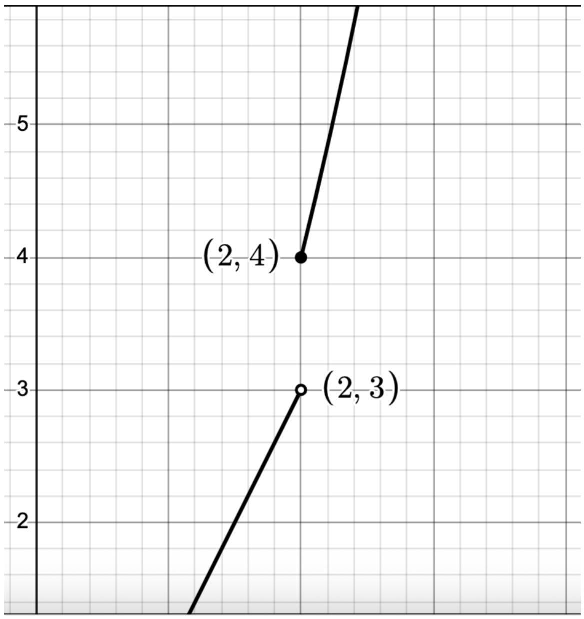
   Now on to $\lim\limits_{x \to 2^+} f(x)$. Here, $x \to 2^+$, so $x > 2$ and $f(x) = x^2$. Hence, $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} x^2 = (2)^2 = 4$, the last step courtesy of the fact that for $x > 2$, $f(x) = x^2$ is a quadratic function (a polynomial) and is continuous.

   Since $\lim\limits_{x \to 2^-} f(x) = 3$ and $\lim\limits_{x \to 2^+} f(x) = 4$, we have that $\lim\limits_{x \to 2} f(x)$ does not exist per Theorem 1.1. Hence, f is not continuous. If we graph f near $x = 2$ using desmos, we can see the vertical gap or 'jump' occurring at $x = 2$.

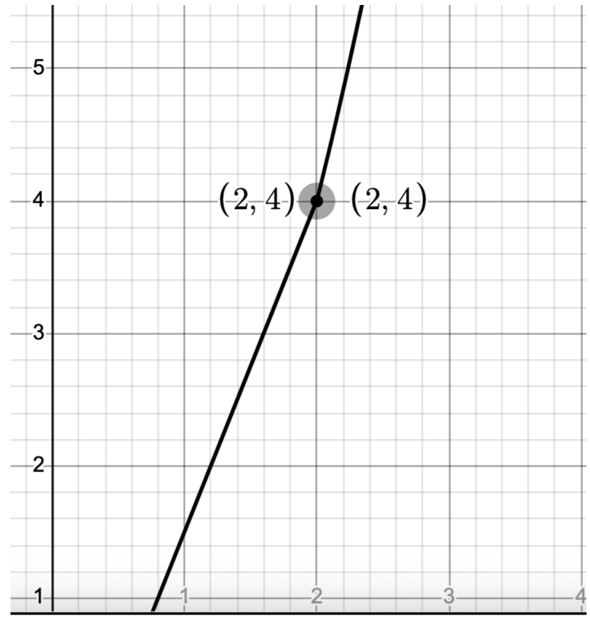
2. In this problem,[12] we're given a parameter 'm' to help us adjust the left hand side of the graph to meet the right hand side at $x = 2$. IIf $x < 2$, $g(x) = mx - 1$ so $\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^-} (mx - 1) = 2m - 1$.

   To ensure the limit exists, we need $\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^+} g(x)$. Since $g(x) = f(x)$ for $x \geq 2$, we know $\lim\limits_{x \to 2^+} g(x) = 4$. Solving $2m - 1 = 4$, we get $m = \frac{5}{2} = 2.5$. Sure enough, $\lim\limits_{x \to 2^-} (2.5x - 1) = 5 - 1 = 4$. Since $g(2) = 4$, we have $\lim\limits_{x \to 2} g(x) = g(2)$, so g is continuous at $x = 2$.

[12] For a graphically interactive take on this problem, check out this desmos worksheet.



$y = f(x)$ near $x = 2$



$y = g(x)$ near $x = 2$

□

It is worth noting that despite each 'piece' of the piecewise-defined function f in Example 1.1.4 being continuous, the pieces don't match up at $x = 2$ causing what is called a **discontinuity**. A discontinuity is a place where a function is **not** continuous. The particular variety of discontinuity appearing here is usually called a 'jump' discontinuity - a type of discontinuity belonging to a larger class of 'non-removable' or 'essential' discontinuities. We'll point out other types of discontinuities as we encounter them.[13]

We close this section with an example that ties (most of) the fundamental concepts of limits and their calculations together.

EXAMPLE 1.1.5. Let $f(x) = \frac{x^2-2x-3}{x^2-1}$. Find $\displaystyle\lim_{x\to -1} f(x)$ analytically[14] and interpret graphically.

Solution. Since f is a rational function, and rational functions are continuous on their domains, it's worth checking to see what happens when we attempt to find $f(-1)$. We get $f(-1) = \frac{(-1)^2-2(-1)-3}{(-1)^2-1} = \frac{0}{0}$, which is undefined because of the '0' in the denominator. However, the '0' in the numerator signals to us[15] there are common factors of $(x-(-1)) = (x+1)$ which will cancel and potentially help us determine the indeterminate form. To that end, we simplify: $f(x) = \frac{x^2-2x-3}{x^2-1} = \frac{(x-3)(x+1)}{(x-1)(x+1)} = \frac{(x-3)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = \frac{x-3}{x-1}, \quad x \neq -1$.

That is, for all real numbers **except** $x = -1$, $\frac{x^2-2x-3}{x^2-1} = \frac{x-3}{x-1}$. Since $\displaystyle\lim_{x\to -1} f(x)$ is concerned only with what's happening **near** $x = -1$, but not with what's happening **at** $x = -1$, it seems reasonable to suggest that $\displaystyle\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x^2-2x-3}{x^2-1} = \lim_{x\to -1} \frac{x-3}{x-1}$.

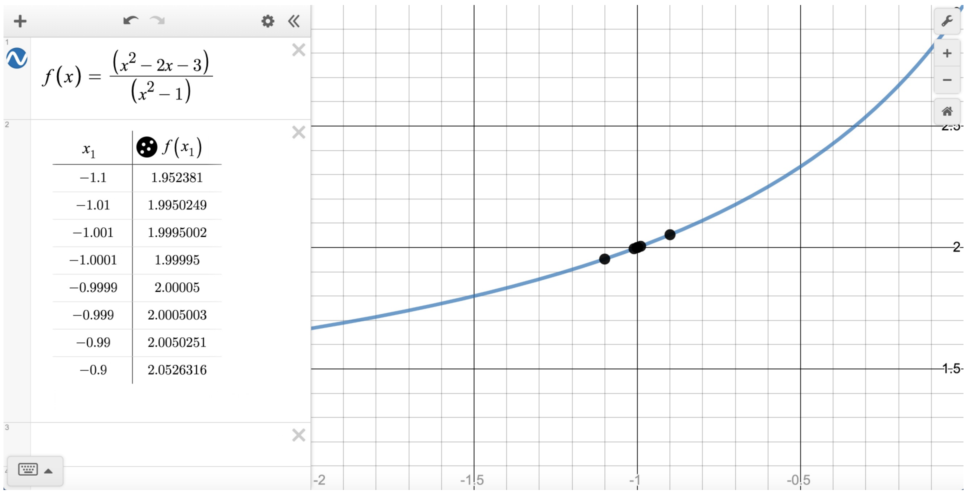
[13] Probably in the footnotes because, after all, this is (supposedly) a precalculus book, not a Calculus book . . .

[14] i.e, using properties of limits

[15] courtesy of the Factor Theorem (Theorem **??**) . . .

Note that $x = -1$ is in the domain of the function $g(x) = \frac{x-3}{x-1}$, hence g **is** continuous at $x = -1$. This means $\lim\limits_{x \to -1} g(x) = g(-1)$, that is, $\lim\limits_{x \to -1} \frac{x-3}{x-1} = \frac{-1-3}{-1-1} = 2$.

Putting all of this work together, we get $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} \frac{x-3}{x-1} = \frac{-1-3}{-1-1} = 2$. Graphically, this means there is a hole in the graph of f at the location $(-1, 2)$.[16] The table and graph below confirm our answer.



$y = f(x)$ near $x = -1$

□

The above reasoning is sound and is true in general. We'll be getting a lot of use out of the following:

THEOREM 1.3. If f and g are two functions which agree on an open interval containing $x = a$, except possibly at $x = a$, then if $\lim\limits_{x \to a} g(x)$ exists, $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$.

[16] Since $f(-1)$ is undefined but $\lim\limits_{x \to -1} f(x)$ exists means the discontinuity here is classified as 'removable.' We can 'remove' the discontinuity by 'patching' the 'hole' by defining $f(-1) = 2$. We do such repairs in Calculus.

1.1.3 Exercises

To incude:

- Limits from graphs
- Graphs from limits
- Using continuity to find limits
- Piecewise-defined functions
- Simplify to find limits
- One-sided continuity
- Explorations with formal definitions involving infinite limits - prelude to epsilonics

1.2 Introduction to Derivatives

1.2.1 Average and Instantaneous Velocity, Revisited

We begin this section by revisiting (again!) the notion of **average velocity** - a concept we first encountered in Example **??** in Section **??** and later revisited in Example **??** in Section **??**.

In this scenario, the position function function[1] $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. The average rate of change of s over an interval is the **average velocity** of the rocket over that interval. The average velocity provides two pieces of information: the average speed of the rocket along with the rocket's direction. We formalized the average velocity in Definition **??** in Section **??**:

DEFINITION. Let $s(t)$ be the position of an object at time t and t_0 a fixed time in the domain of s. The **average velocity** between time t and time t_0 for $t \neq t_0$ is given by

$$\overline{v}(t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t) - s(t_0)}{t - t_0}.$$

If we define the change in time, $\Delta t = t - t_0$, we get $t = t_0 + \Delta t$ which gives:

$$\overline{v}(\Delta t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}, \quad \Delta t \neq 0.$$

The above formula measures the average velocity between time t_0 and time $t_0 + \Delta t$ as a function of Δt.

We now revisit Example **??** in Section **??** using this new formulation.[2]

EXAMPLE 1.2.1. Let $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ give the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff.

1. Find, and simplify: $\overline{v}(\Delta t) = \dfrac{s(15 + \Delta t) - s(15)}{\Delta t}$, for $\Delta t \neq 0$.
2. Find and interpret $\overline{v}(-1)$.
3. Find and interpret $\lim\limits_{\Delta t \to 0} \overline{v}(\Delta t)$.
4. Graph $y = \overline{v}(\Delta t)$ and interpret your answer to part 3 graphically.

Solution.

1. To find $\overline{v}(\Delta t)$, we first find $s(15 + \Delta t)$:

$$\begin{aligned} s(15 + \Delta t) &= -5(15 + \Delta t)^2 + 100(15 + \Delta t) \\ &= -5(225 + 30\Delta t + (\Delta t)^2) + 1500 + 100\Delta t \\ &= -5(\Delta t)^2 - 50\Delta t + 375 \end{aligned}$$

[1] So named because $s(t)$ provides information about **where** the rocket is at time t.

[2] Along with some of the new tools we learned in Section 1.1.

Since $s(15) = -5(15)^2 + 100(15) = 375$, we get:

$$\begin{aligned}
\overline{v}(\Delta t) &= \frac{s(15+\Delta t) - s(15)}{\Delta t} \\
&= \frac{(-5(\Delta t)^2 - 50\Delta t + 375) - 375}{\Delta t} \\
&= \frac{\Delta t(-5\Delta t - 50)}{\Delta t} \\
&= \frac{\cancel{\Delta t}(-5\Delta t - 50)}{\cancel{\Delta t}} \\
&= -5\Delta t - 50 \qquad\qquad \Delta t \neq 0
\end{aligned}$$

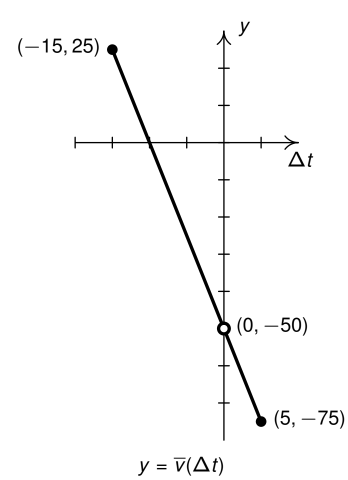
In addition to $\Delta t \neq 0$, the domain of s is restricted to $0 \leq t \leq 20$. Hence, we require $0 \leq 15+\Delta t \leq 20$ or $-15 \leq \Delta t \leq 5$. Our final answer is $\overline{v}(\Delta t) = -5\Delta t - 50$, for $\Delta t \in [-15, 0) \cup (0, 5]$.

2. We find $\overline{v}(-1) = -5(-1) - 50 = -45$. This means the average velocity over between time $t = 15 + (-1) = 14$ seconds and $t = 15$ seconds is -45 feet per second. This indicates the rocket is, on average, heading *downwards* at a rate of 45 feet per second.

3. Since $\overline{v}(\Delta t) = -5\Delta t - 50$, for all values of Δt near $\Delta t = 0$ (excluding $\Delta t = 0$), Theorem 1.3 applies. We get $\lim\limits_{\Delta t \to 0} \overline{v}(\Delta t) = \lim\limits_{\Delta t \to 0} -5\Delta t - 50 = -5(0) - 50 = -50$, where we have used the fact that the function $f(\Delta t) = -5\Delta t - 50$ is continuous to evaluate the limit.

   Recall from Example **??** that the limit value here, -50 is the so-called *instantaneous velocity* of the rocket *at* $t = 15$ seconds. That is, 15 seconds after lift-off, the rocket is heading back towards the surface of the moon at a rate of 15 feet per second.

4. Since the domain of $\overline{v}$ is $[-15, 0) \cup (0, 5]$, the graph of $y = \overline{v}(\Delta t) = -5\Delta t - 50$ is a line **segment** from $(-15, 25)$ to $(5, -75)$ with a hole at $(0, -50)$.



□

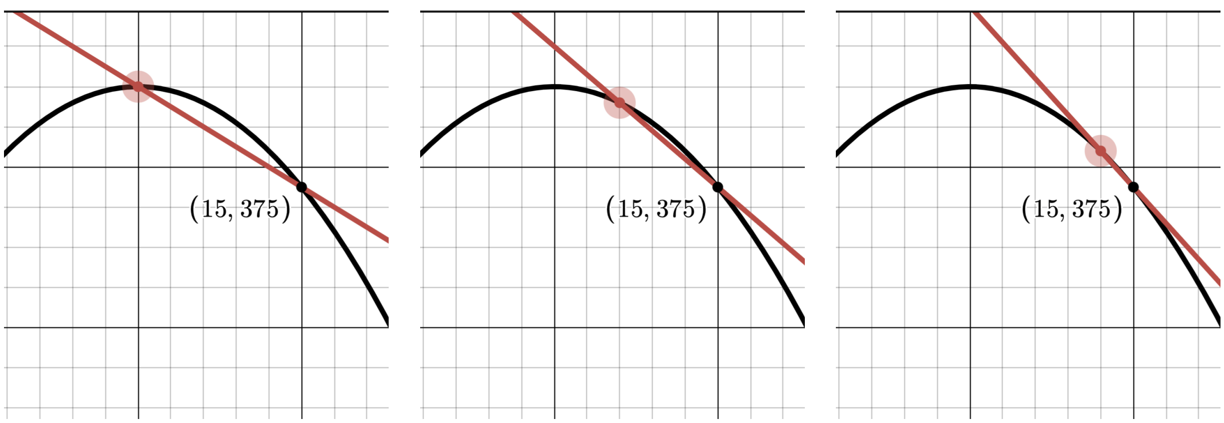
The reader is invited to compare Example **??** in Section **??** with Exercise 1.2.1 above. We obtain the exact same information because we are asking the *exact same* questions - they are just framed differently. We now take the time to formally define **instantaneous velocity**:

DEFINITION 1.5. Let $s(t)$ be the position of an object at time t and t_0 a fixed time in the domain of s. The **instantaneous velocity** at t_0 is given by:

$$v(t_0) = \lim_{\Delta t \to 0} \frac{\Delta[s(t)]}{\Delta t} = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}, \quad \text{provided this limit exists.}$$

Based on our work in Examples **??** and 1.2.1, we have $v(15) = -50$. In both of those examples, we've seen what $v(15)$ means on the graph of $\overline{v}$, but there is a more important interpretation when we analyze the graph of s. Recall that the average velocity, and, more generally, average rates of change can be visualized as slopes of **secant lines**.[3]

Below is a sequence of secant lines along with the graph of $y = s(t)$. In each case, the secant line is graphed between $(15, s(15)) = (15, 375)$ and another point on the graph.[4] As the points on the parabola approach $(15, 375)$ the secant lines approach what is known as the **tangent line**.

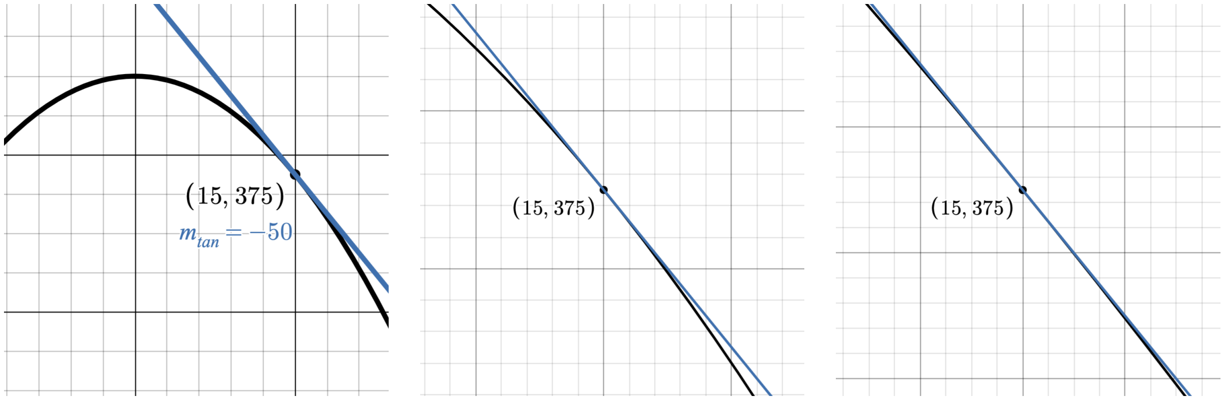


To find the equation of the tangent line in this case, which we'll call $L(t)$, we refer to the point-slope form of a line, Equation **??**:

$$L(t) = s(t_0) + m(t - t_0) = s(15) + v(15)(t - 15) = 375 - 50(t - 15) = -50t + 1125.$$

The tangent line can best be thought of as 'the best linear approximation' to the graph of $y = s(t)$ at $(15, 375)$. That is, if we zoom in near $(15, 375)$, the graph of $y = s(t)$ and this tangent line become indistinguishable. This property of $y = s(t)$ is called **local linearity** and is foundational to the analysis of functions. Below we graph $y = s(t) = -5t^2 + 100t$ along with $y = L(t) = -50t + 1125$ and observe the local linearity near $(15, 375)$.

[3] See Section **??** for a refresher, if needed.
[4] For an interactive demonstration of this process, check out this desmos worksheet.



The tangent line at $(15, 375)$). Zooming in near $(15, 375)$. Zooming in closer to $(15, 375)$.

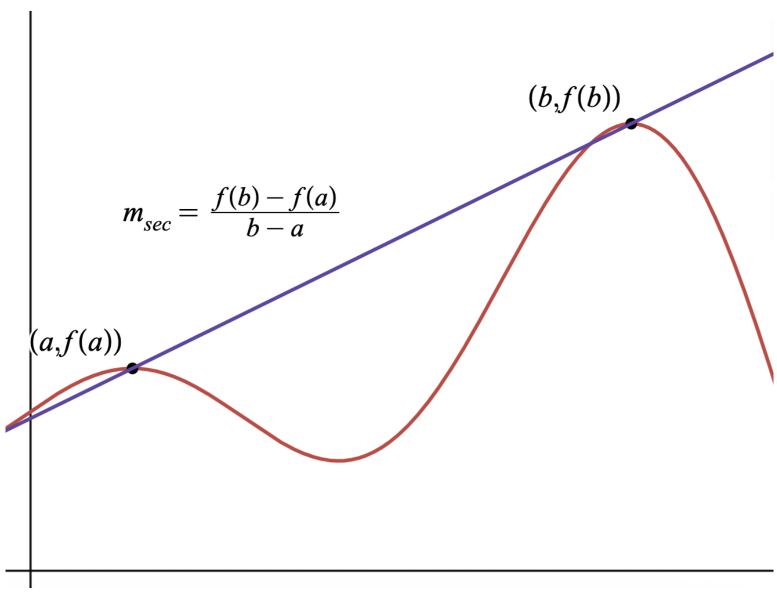
Our next step is to generalize these notions to all functions.

1.2.2 Difference Quotients and Derivatives

Recall in Section **??** the concept of the average rate of change of a function over the interval $[a, b]$ is the slope between the two points $(a, f(a))$ and $(b, f(b))$ and is given by

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, the average rate of change is the slope of the so-called **secant line** which 'cuts' through the graph of $y = f(x)$ at the points $(a, f(a))$ and $(b, f(b))$:



Consider a function f defined over an interval containing x and $x+h$ where $h \neq 0$. The average rate of change of f over the interval $[x, x+h]$ is thus given by the formula:[5]

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+h) - f(x)}{h}, \quad h \neq 0.$$

The above is an example of what is traditionally called the **difference quotient** or **Newton quotient** of f, since it is the *quotient* of two *differences*, namely $\Delta[f(x)]$ and Δx. Another formula for the difference quotient keeps with the notation Δx instead of h:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0.$$

It is important to understand that in this formulation of the difference quotient, the variables 'x' and 'Δx' are distinct - that is they do not combine as like terms.

Note that, regardless of which form the difference quotient takes, when h, Δx, or Δt is 0, the difference quotient returns the indeterminate form '$\frac{0}{0}$.' As we've seen with rational functions in Section **??**, when this happens, we can use a limit to help us **determine** the **indeterminate** form.

In Section 1.2.1, taking the limit of average velocity as $\Delta t \to 0$ produced instantaneous velocity. More generally, taking the limit of the average rate of change as the denominator approaches 0 produces the **instantaneous rate of change** of the function at that point. The instantaneous rate of change of a function is called the **derivative** of the function is defined below.

DEFINITION 1.6. Given a function f defined on an open interval containing $x = a$, the **derivative** of f at a, denoted $f'(a)$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \quad \text{provided this limit exists.}$$

The number $f'(a)$ represents the **instantaneous rate of change** of f with respect to x at the input $x = a$. If $f'(a)$ exists, we say f is **differentiable** at $x = a$.

Using the language of derivatives, Examples **??** and 1.2.1 have us computing $v(15) = s'(15)$. Moreover, since the derivative is a rate of change, it's important to note that the associated units of $f'(a)$ are $\frac{\text{units of } f(x)}{\text{units of } x}$. This tracks with the units of $v(15) = s'(15)$ being $\frac{\text{feet}}{\text{second}}$, a velocity.

As in Section 1.2.1, $f'(a)$ represents the slope of the tangent line at the point $(a, f'(a))$. We use this to formally define the **tangent line** below.

DEFINITION 1.7. If f is differentiable at $x = a$, then $f'(a) = m_{\text{tan}}$, the slope of the **tangent linetangent line** to $y = f(x)$ at $(a, f(a))$. The equation of the tangent line is therefore: $y = f'(a)(x - a) + f(a)$.

We put these definitions to good use in the following example.

[5]assuming $h > 0$; otherwise, we the interval is $[x+h, x]$. We get the same formula for the difference quotient either way.

EXAMPLE 1.2.2. Let $f(x) = -x^2 + 3x - 1$.

1. Find the equation of the tangent line to $y = f(x)$ at $x = -2$. Check your answer graphically.
2. If f represents the temperature (in degrees Celsius) x hours after Noon on a particular day, interpret $f'(-2)$ in terms of time and temperature.

Solution.

1. We first find $m_{\tan} = f'(-2)$ using Definition 1.6 with $a = -2$: $f'(-2) = \lim_{h\to 0} \dfrac{f(-2+h) - f(-2)}{h}$.

   First we find $f(-2+h)$ and are careful to apply the exponent in the expression $-(-2+h)^2$ first:

$$\begin{aligned} f(-2+h) &= -(-2+h)^2 + 3(-2+h) - 1 \\ &= -(4 - 4h + h^2) - 6 + 3h - 1 \\ &= -4 + 4h - h^2 - 6 + 3h - 1 \\ &= -h^2 + 7h - 11 \end{aligned}$$

   Next, we find $f(-2) = -(-2)^2 + 3(-2) - 1 = -11$, so the difference quotient is:

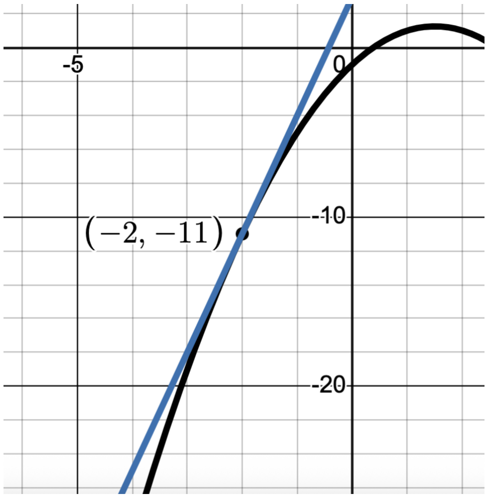
$$\begin{aligned} \frac{f(-2+h) - f(-2)}{h} &= \frac{(-h^2 + 7h - 11) - (-11)}{h} \\ &= \frac{-h^2 + 7h}{h} && \text{simplify} \\ &= \frac{h(-h+7)}{h} && \text{factor} \\ &= \frac{\cancel{h}(-h+7)}{\cancel{h}} && \text{cancel} \\ &= -h + 7 \end{aligned}$$

   Finally, we get $f'(-2) = \lim_{h\to 0} \dfrac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} (-h+7) = -(0) + 7 = 7$.

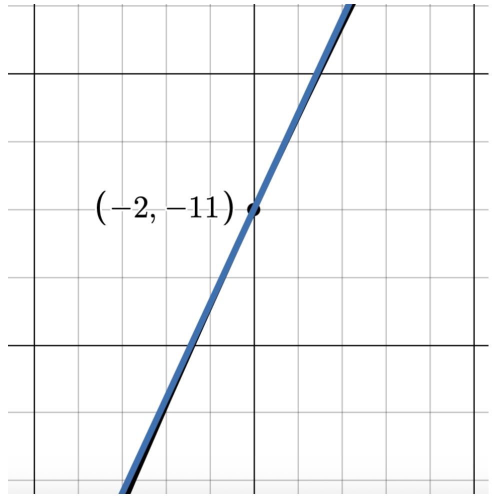
   Hence, the slope of the tangent line is $m_{\tan} = f'(-2) = 7$. Hence, the equation of the tangent line is:

$$\begin{aligned} y &= f'(-2)(x - (-2)) + f(-2) \\ &= 7(x+2) - 11 \\ &= 7x + 14 - 11 \\ y &= 7x + 3 \end{aligned}$$

   Graphing $y = 7x + 3$ and $y = f(x)$ near $(-2, -11)$ reveals the local linearity we would expect:



The tangent line at $(-2, -11)$.



Zooming in near $(-2, -11)$.

2. Since x represents the number of hours **after** Noon, $x = -2$ corresponds to 2 hours **before** Noon, or 10 AM. Since the units of $f(x)$ are degrees Celsius and the units of x are hours, the units of $f'(-2)$ are degrees Celsius per hour. Since $f'(-2)$ is positive, we know the slope is positive, so the temperature is increasing. Putting all this together, $f'(-2) = 7$ means that at 10 AM, the temperature is rising at a rate of 7 degrees Celsius per hour. □

What if we wanted to find the equation of the tangent line to the graph of the function in Example 1.2.2 at $x = 0$? $x = 1$? $x = 5$? We'd ostensibly need to run through difference quotients and limit calculations for each and every input value: $x = 0$, $x = 1$, and $x = 5$. Or we could do a single limit with a generic 'x', simplify the difference quotient and take the limit once, and substitute in particular values of x:

DEFINITION 1.8. Given a function f defined on an open interval, the **derivative** of f, denoted $f'(x)$ is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{provided the limit exists.}$$

It is worth noting that if we set $h = \Delta x$, and consider the graph $y = f(x)$, we get:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta[f(x)]}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

which is why sometimes the derivative is denoted[6] $\frac{dy}{dx}$.

[6] This is the so-called Leibniz notation . . .

EXAMPLE 1.2.3. Let $f(x) = -x^2 + 3x - 1$.

1. Find an expression for $f'(x)$.
2. Find $f'(-2)$ using your answer to part 1 and compare that to what you obtained in Example 1.2.2.
3. Find the equation of the tangent line to the graph of $y = f(x)$ at $x = 0$. Check your answer graphically.
4. Solve $f'(x) = 0$ and interpret your answer graphically.

Solution.

1. We start finding $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ by first finding $f(x+h)$:

$$\begin{aligned} f(x+h) &= -(x+h)^2 + 3(x+h) - 1 \\ &= -(x^2 + 2xh + h^2) + 3x + 3h - 1 \\ &= -x^2 - 2xh - h^2 + 3x + 3h - 1 \end{aligned}$$

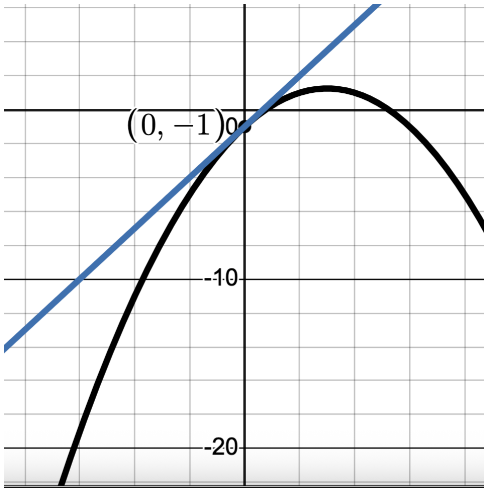
The difference quotient simplifies as follows:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(-x^2 - 2xh - h^2 + 3x + 3h - 1) - (-x^2 + 3x - 1)}{h} \\ &= \frac{-x^2 - 2xh - h^2 + 3x + 3h - 1 + x^2 - 3x + 1}{h} \\ &= \frac{-2xh - h^2 + 3h}{h} && \text{simplify} \\ &= \frac{h(-2x - h + 3)}{h} && \text{factor} \\ &= \frac{\cancel{h}(-2x - h + 3)}{\cancel{h}} && \text{cancel} \\ &= -2x - h + 3 \end{aligned}$$

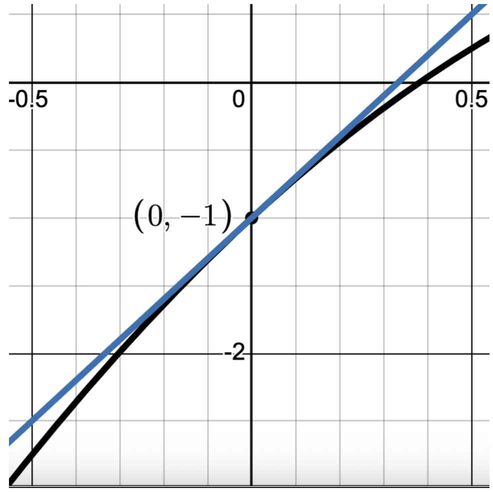
Our last step is to take the limit: $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0}(-2x - h + 3)$. Notice here that we have two variables, x and h, in the limit. Of these two variables, we are taking the limit on the h: $h \to 0$. As far as h is concerned, x may as well be just another constant like the '3'. Hence, $f'(x) = \lim\limits_{h\to 0}(-2x - h + 3) = -2x - 0 + 3 = -2x + 3$.

2. Evaluating our formula for $f'(x)$ at $x = -2$ gives $f'(-2) = -2(-2) + 3 = 7$ which matches with what we obtained in Example 1.2.2.

3. The equation of the tangent line to the graph of $y = f(x)$ at $x = 0$ is $y = f'(0)(x - 0) + f(0)$. We have $f'(0) = 2(0) + 3 = 3$ and $f(0) = -(0)^2 + 3(0) - 1 = -1$. We get $y = 3(x - 0) + (-1)$ so $y = 3x - 1$. Our graph bears this out.

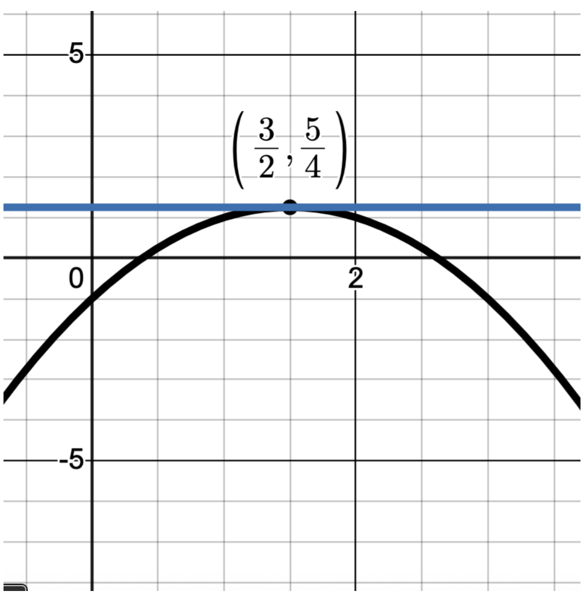


The tangent line at $(0, -1)$.

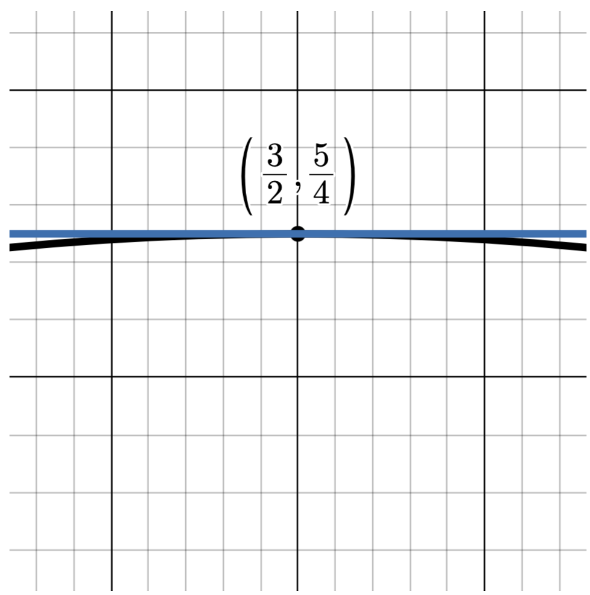


Zooming in near $(0, -1)$.

4. Solving $f'(x) = 0$ gives $-2x + 3 = 0$ so $x = \frac{3}{2}$. This means the slope of the tangent line at the point $\left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right)$ is 0, so the tangent line there is horizontal. We find $f\left(\frac{3}{2}\right) = -\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) - 1 = \ldots = \frac{5}{4}$. Hence, the tangent line at $\left(\frac{3}{2}, \frac{5}{4}\right)$ is $y = \frac{5}{4}$. Graphically, this checks out.



The tangent line at $\left(\frac{3}{2}, \frac{5}{4}\right)$.



Zooming in near $\left(\frac{3}{2}, \frac{5}{4}\right)$.

□

The astute reader will note that the graph of $f(x) = -x^2 + 3x - 1$ in Example 1.2.3 is a parabola and finding where $f'(x) = 0$ lead us right back to the vertex. Using a derivative to find the vertex may seem a bit excessive given that we've algebraically derived a handy 'vertex formula' in Section **??**. However, as the functions we aim to analyze become more and more sophisticated, the tools we use to analyze them must also become more sophisticated. The derivative is one such tool that has a near universal application.[7]

EXAMPLE 1.2.4.

1. For $f(x) = x^2 - x - 2$, find and simplify:
   (a) $f'(3)$
   (b) The equation of the tangent line to the graph $y = f(x)$ at $(3, f(3))$. Check your answer graphically.
   (c) $f'(x)$
2. For $g(x) = \dfrac{3}{2x+1}$, find and simplify:[8]
   (a) $g'(0)$
   (b) The equation of the tangent line to the graph $y = g(x)$ at $(0, g(0))$. Check your answer graphically.
   (c) $g'(x)$
3. $r(t) = \sqrt{t}$, find and simplify:[9]
   (a) $r'(9)$
   (b) The equation of the tangent line to the graph $y = r(t)$ at $(9, r(9))$. Check your answer graphically.
   (c) $r'(t)$

Solution.

1. (a) To find $f'(3) = \lim\limits_{h \to 0} \frac{f(3+h)-f(3)}{h}$ we first simplify $f(3+h)$:

$$\begin{array}{rcl} f(3+h) & = & (3+h)^2 - (3+h) - 2 \\ & = & 9 + 6h + h^2 - 3 - h - 2 \\ & = & 4 + 5h + h^2 \end{array}$$

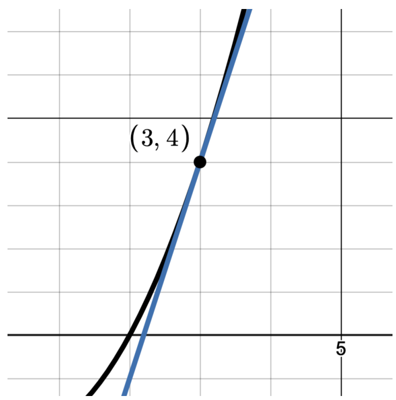
Since $f(3) = (3)^2 - 3 - 2 = 4$, we get for the difference quotient:

[7] As we'll see in Section **??**.
[8] A review of Section **??** may be in order for this problem.
[9] A review of Section **??** may be in order for this problem.

$$\begin{aligned}\frac{f(3+h)-f(3)}{h} &= \frac{(4+5h+h^2)-4}{h}\\ &= \frac{5h+h^2}{h}\\ &= \frac{h(5+h)}{h} && \text{factor}\\ &= \frac{\cancel{h}(5+h)}{\cancel{h}} && \text{cancel}\\ &= 5+h\end{aligned}$$

Hence, $f'(3) = \lim\limits_{h\to 0}(5+h) = 5+0 = 5$.

(b) The equation of the tangent line at $x = 3$ is: $y = f'(3)(x-3)+f(3) = 5(x-3)+4$, or $y = 5x-11$. We check graphically below.



$y = f(x)$ and $y = 5x - 11$ near $(3, 4)$

(c) To find $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$, we first find $f(x+h)$:

$$\begin{aligned}f(x+h) &= (x+h)^2-(x+h)-2\\ &= x^2+2xh+h^2-x-h-2.\end{aligned}$$

So the difference quotient is

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\left(x^2+2xh+h^2-x-h-2\right)-\left(x^2-x-2\right)}{h}\\ &= \frac{x^2+2xh+h^2-x-h-2-x^2+x+2}{h}\\ &= \frac{2xh+h^2-h}{h}\end{aligned}$$

$$\begin{aligned}
&= \frac{h\,(2x+h-1)}{h} && \text{factor} \\
&= \frac{\cancel{h}\,(2x+h-1)}{\cancel{h}} && \text{cancel} \\
&= 2x+h-1.
\end{aligned}$$

Hence, $f'(x) = \lim_{h\to 0}(2x+h-1) = 2x+0-1$ so $f'(x) = 2x-1$. Note that using this formula, we get $f'(3) = 2(3)-1 = 5$ which checks our answer above.

2. (a) Next we find $g'(0) = \lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = \frac{g(h)-g(0)}{h}$.

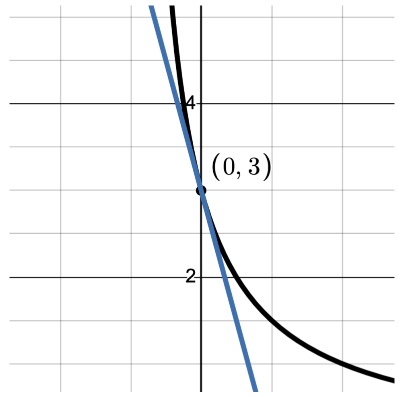
   Since $g(h) = \frac{3}{2h+1}$ and $g(0) = \frac{3}{2(0)+1} = 3$, our difference quotient contains a complex fraction. Thinking ahead, we need to (eventually) be able to cancel the factor 'h' from the denominator $\frac{g(h)-g(0)}{h}$, so we begin by simplifying the complex fraction and see where that takes us:

$$\begin{aligned}
\frac{g(0+h)-g(0)}{h} &= \frac{\frac{3}{2h+1}-3}{h} \\
&= \frac{\frac{3}{2h+1}-3}{h}\cdot\frac{(2h+1)}{(2h+1)} \\
&= \frac{3-3(2h+1)}{h(2h+1)} \\
&= \frac{3-6h-3}{h(2h+1)} \\
&= \frac{-6h}{h(2h+1)} \\
&= \frac{-6\cancel{h}}{\cancel{h}(2h+1)} && \text{cancel} \\
&= \frac{-6}{2h+1}.
\end{aligned}$$

   We are now ready to take the limit:

$$g'(0) = \lim_{h\to 0}\frac{-6}{2h+1} = \frac{-6}{2(0)+1} = -6.$$

   (b) The equation of the tangent line when $x = 0$ is: $y = g'(0)(x-0)+g(0) = (-6)(x-0)+3$ or $y = -6x+3$, which checks graphically below.



$y = g(x)$ and $y = -6x + 3$ near $(0, 3)$

(c) To find $g'(x)$, we first find $g(x + h)$:

$$\begin{aligned} g(x+h) &= \frac{3}{2(x+h)+1} \\ &= \frac{3}{2x+2h+1} \end{aligned}$$

Simplifying the difference quotient involves simplifying the resulting complex fraction, as above, keeping an eye out for an opportunity to cancel the factor 'h' from the denominator:

$$\begin{aligned} \frac{g(x+h)-g(x)}{h} &= \frac{\frac{3}{2x+2h+1} - \frac{3}{2x+1}}{h} \\ &= \frac{\frac{3}{2x+2h+1} - \frac{3}{2x+1}}{h} \cdot \frac{(2x+2h+1)(2x+1)}{(2x+2h+1)(2x+1)} \\ &= \frac{3(2x+1) - 3(2x+2h+1)}{h(2x+2h+1)(2x+1)} \\ &= \frac{6x+3-6x-6h-3}{h(2x+2h+1)(2x+1)} \\ &= \frac{-6h}{h(2x+2h+1)(2x+1)} \\ &= \frac{-6\cancel{h}}{\cancel{h}(2x+2h+1)(2x+1)} && \text{cancel} \\ &= \frac{-6}{(2x+2h+1)(2x+1)}. \end{aligned}$$

Hence,

$$g'(x) = \lim_{h \to 0} \frac{-6}{(2x+2h+1)(2x+1)} = \frac{-6}{(2x+2(0)+1)(2x+1)} = -\frac{6}{(2x+1)^2}.$$

We check $g'(0) = -\frac{6}{(2(0)+1)^2} = \ldots = -6$, as required.

3. (a) To find $r'(9) = \lim_{h\to 0} \frac{r(9+h)-r(9)}{h}$, we start with $r(9+h) = \sqrt{9+h}$ and $r(9) = \sqrt{9} = 3$. Hence our difference quotient is:

$$\frac{r(9+h)-r(9)}{h} = \frac{\sqrt{9+h}-3}{h}.$$

In order for us to determine the limit as $h \to 0$, we need to somehow cancel the factor of h from the **denominator**. To do so, we set about **rationalizing the numerator** by multiplying both numerator and denominator by the conjugate[10] of the numerator, $\sqrt{9+h}-3$:

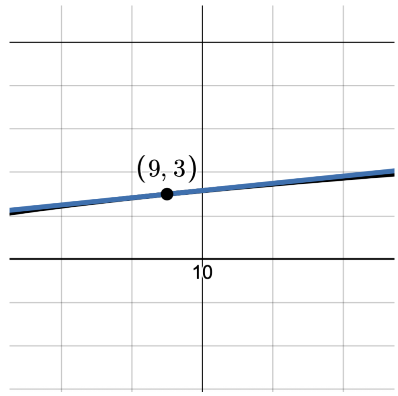
$$\begin{aligned}
\frac{r(9+h)-r(9)}{h} &= \frac{\sqrt{9+h}-3}{h} \\
&= \frac{\left(\sqrt{9+h}-3\right)}{h} \cdot \frac{\left(\sqrt{9+h}+3\right)}{\left(\sqrt{9+h}+3\right)} && \text{Multiply by the conjugate.} \\
&= \frac{\left(\sqrt{9+h}\right)^2-(3)^2}{h\left(\sqrt{9+h}+3\right)} && \text{Difference of Squares.} \\
&= \frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)} \\
&= \frac{h}{h\left(\sqrt{9+h}+3\right)} \\
&= \frac{\cancelto{1}{h}}{\cancel{h}\left(\sqrt{9+h}+3\right)} && \text{cancel} \\
&= \frac{1}{\sqrt{9+h}+3}
\end{aligned}$$

Hence,

$$r'(9) = \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{\sqrt{9+0}+3} = \frac{1}{6}.$$

(b) The equation of the tangent line to $y = r(t)$ at $(9,3)$ is therefore: $y = r'(9)(t-9)+r(9) = \frac{1}{6}(t-9)+3$ or $y = \frac{1}{6}\,t + \frac{3}{2}$. The graph below confirms this.

[10] Again, see Section **??** for a review of these sorts of machinations.



$y = r(t)$ and $y = \frac{1}{6}t + \frac{3}{2}$ near $(9, 3)$

(c) As one might expect, we use this same strategy of rationalizing numerators to simplify the difference quotient to find $r'(t)$:

$$\begin{aligned}
\frac{r(t+h) - r(t)}{h} &= \frac{\sqrt{t+h} - \sqrt{t}}{h} \\
&= \frac{\left(\sqrt{t+h} - \sqrt{t}\right)}{h} \cdot \frac{\left(\sqrt{t+h} + \sqrt{t}\right)}{\left(\sqrt{t+h} + \sqrt{t}\right)} && \text{Multiply by the conjugate.} \\
&= \frac{\left(\sqrt{t+h}\right)^2 - \left(\sqrt{t}\right)^2}{h\left(\sqrt{t+h} + \sqrt{t}\right)} && \text{Difference of Squares.} \\
&= \frac{(t+h) - t}{h\left(\sqrt{t+h} + \sqrt{t}\right)} \\
&= \frac{h}{h\left(\sqrt{t+h} + \sqrt{t}\right)} \\
&= \frac{\cancelto{1}{h}}{\cancel{h}\left(\sqrt{t+h} + \sqrt{t}\right)} \\
&= \frac{1}{\sqrt{t+h} + \sqrt{t}}
\end{aligned}$$

We get

$$r'(t) = \lim_{h \to 0} \frac{1}{\sqrt{t+h} + \sqrt{t}} = \frac{1}{\sqrt{t+0} + \sqrt{t}} = \frac{1}{2\sqrt{t}}.$$

We check $r'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}$ as above. □

1.2.3 Exercises

- difference quotients
- velocities
- other rates of change (marginals?)
- tangent lines
- derivative formulas: constant, linear, quadratic
- foreshadow inc / dec
- vertex formula (reprise)

1.3 Applications of the Derivative

In this section, we take a peek into Calculus and look at some select applications of the derivative.

1.3.1 The Shape of Graphs

We know if f is differentiable at $x = a$ then the graph of f is **locally linear** at $x = a$ and $f'(a)$ is the **slope** of the tangent line at the point $(a, f(a))$. In this section, we explore how local behavior near a point can be extrapolated to global behavior over an interval. First, we review Definition **??** from Section **??**:

DEFINITION. Let f be a function defined on an interval I. Then f is said to be:

- **increasing** on I if, whenever $a < b$, then $f(a) < f(b)$. (i.e., as inputs increase, outputs **increase**.)
  NOTE: The graph of an increasing function **rises** as one moves from left to right.
- **decreasing** on I if, whenever $a < b$, then $f(a) > f(b)$. (i.e., as inputs increase, outputs **decrease**.)
  NOTE: The graph of a decreasing function **falls** as one moves from left to right.
- **constant** on I if $f(a) = f(b)$ for all a, b in I. (i.e., outputs don't change with inputs.)
  NOTE: The graph of a function that is constant over an interval is a horizontal line.

Suppose a function satisfies $f'(x) > 0$ for all x in an open interval[1] I. Then we know that not only is the graph of f locally linear on I, but the slopes of all of the tangent lines are positive. This means that all of the tangent lines are increasing so it stands to reason that the function f is likewise increasing on I. In other words, if a function is **locally** increasing on I, then it is **globally** increasing on I as well.

We can apply the same reasoning above to situations where $f'(x) < 0$ for all x in I, which implies f is decreasing on I or $f'(x) = 0$ on I, which implies f is constant on I. In Calculus, you'll learn this fact is a consequence of the Mean Value Theorem.[2] In this text, we'll just accept the following theorem is true and hope we've done enough hand-waving to deem it reasonable.

THEOREM 1.4. Suppose f is differentiable on an open interval I:

- If $f'(x) > 0$ for all x in I, then f is increasing on I.
- If $f'(x) < 0$ for all x in I, then f is decreasing on I.
- If $f'(x) = 0$ for all x in I, then f is constant on I.

[1] We've defined derivatives as two-sided limits, so an open interval here guarantees enough 'room' on either side of any given number to take such a limit.

[2] which Carl thinks is the actual 'Fundamental Theorem of Calculus' since it relates local and global behavior . . .

Theorem 1.4 may be visualized as follows:

- $f'(x) > 0$ for all x in I:

- $f'(x) < 0$ for all x in I:

- $f'(x) = 0$ for all x in I:

We can use Theorem 1.4 to help us determine the (open) intervals over which a function f is increasing, decreasing, and constant by making a sign diagram for the derivative f'.

In order to avoid us having to go through the (somewhat lengthy) process of finding $f'(x)$ using Definition 1.8, we'll just use some properties of derivatives from Calculus behind the scenes and present you with both a function and its derivative. It's time for an example.

EXAMPLE 1.3.1. Let $f(x) = x^3 - 3x^2 - 9x + 5$. Use the fact that $f'(x) = 3x^2 - 6x - 9$ to find the open intervals over which f is increasing, decreasing, and constant. Check your answer graphically.

Solution. To make use of Theorem 1.4, we make a sign diagram for $f'(x)$. Since f' is a polynomial, f' is continuous so the per the Intermediate Value Theorem, Theorem **??**, f' will only change sign on either side of zeros. Hence, our first step is to solve $f'(x) = 0$.

We are given $f'(x) = 3x^2 - 6x - 9$. Solving $f'(x) = 3x^2 - 6x - 9 = 0$ gives $3(x^2 - 2x - 3) = 0$ or $3(x-3)(x+1) = 0$. We get two solutions: $x = -1$ and $x = 3$ which divides the x-axis into three regions: $x < -1$, $-1 < x < 3$ and $x > 3$.

Next we select a test value in each of these three regions to determine the sign of $f'(x)$. For the interval $x < -1$, we select $x = -3$: $f'(-3) = 3(-3)^2 - 6(-3) - 9 = (+)$. For $-1 < x < 3$, we select $x = 0$: $f'(0) = 3(0)^2 - 6(0) - 9 = (-)$. Finally, for $x > 3$, we select $x = 4$: $f'(4) = 3(4)^2 - 6(4) - 9 = (+)$.

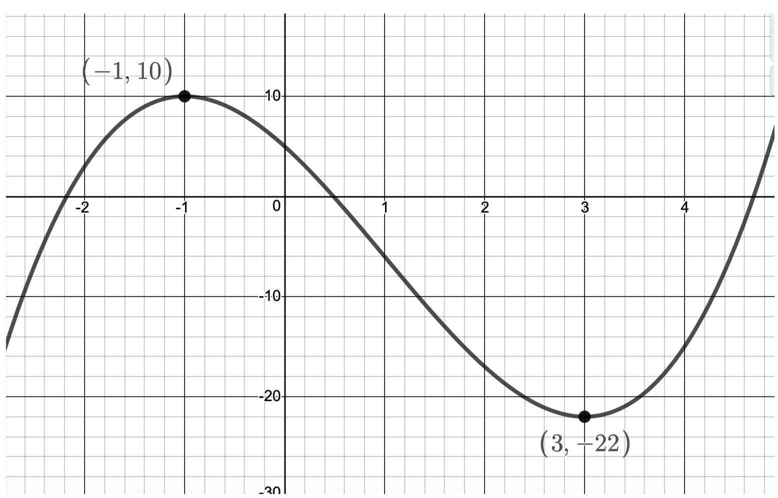
Below on the left is a sign diagram for $f'(x)$ and on the right is what this means for the graph of $y = f(x)$:

We find f is increasing on $(-\infty, -1)$ and again on $(3, \infty)$ while f is decreasing on $(-1, 3)$. At the points $x = -1$ and $x = 3$, we have $f'(x) = 0$ so the graph of f is locally flat there.

Since f changes from increasing just to the left of $x = -1$ to decreasing just to the right of $x = -1$, it stands to reason that f has a local maximum at $x = -1$. This is indeed the case and we find that the local maximum value is $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = 10$.

Similarly, since f changes from decreasing just to the left of $x = 3$ to increasing just to the right of $x = 3$, f has a local minimum at $x = 3$. The local minimum value is $f(3) = (3)^3 - 3(3)^2 - 9(3) + 5 = -22$.

A quick check using desmos confirms our results.



□

We generalize our observations about local extrema in the following result.

THEOREM 1.5. **The (First)[a] Derivative Test for Local Extema:** Suppose f is differentiable on an open interval I containing c:
Let c be a critical number for a continuous function f (i.e., $f'(c) = 0$ or $f'(c)$ does not exist.)

- If $f'(x)$ changes from (+) for $x < c$ to $(-)$ for $x > c$, f has a local maximum at $x = c$.
- If $f'(x)$ changes from $(-)$ for $x < c$ to (+) for $x > c$, f has a local minimum at $x = c$.
- If $f'(x)$ doesn't change sign going from $x < c$ to $x > c$, f does not have a local extremum at $x = c$.

[a] Why 'First'? Stay tuned ...

EXAMPLE 1.3.2. Let $f(x) = x^{4/3} - 4x^{1/3}$. Use the fact that $f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$ to help you find:

1. the open intervals over which f is increasing, decreasing, and constant.
2. the local extrema.

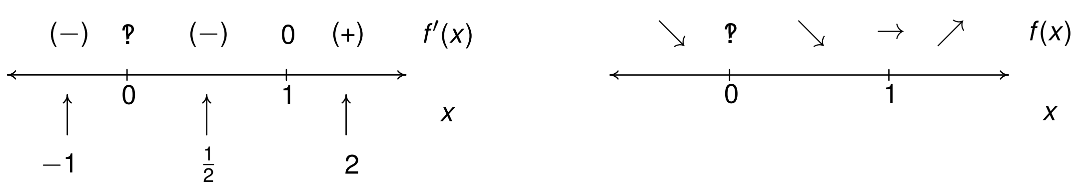
Solution.

1. In order to make a sign diagram for $f'(x)$, we rewrite $f'(x)$ as a single fraction:

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4x^{1/3}}{3} - \frac{4}{3x^{2/3}} = \frac{4x^{1/3}}{3} \cdot \frac{x^{2/3}}{x^{2/3}} - \frac{4}{3x^{2/3}} = \frac{4x}{3x^{2/3}} - \frac{4}{3x^{2/3}} = \frac{4x-4}{3x^{2/3}}.$$

Unlike the derivative in Example 1.3.1, $f'(x) = \frac{4x-4}{3x^{2/3}}$ is undefined when $3x^{2/3} = 0$, that is, when $x = 0$, so we need to record this on our sign diagram with the customary '?.'

Next, we solve $f'(x) = \frac{4x-4}{3x^{2/3}} = 0$ to get $4x - 4 = 0$ or $x = 1$. The usual machinations produces the sign diagram for $f'(x)$ below on the left. We interpret what this means for f below on the right.

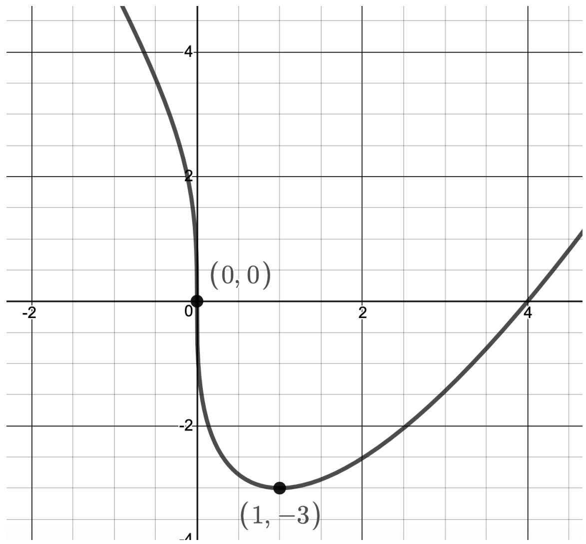


We get f is decreasing for $x < 0$ as well as from $0 < x < 1$. Since 0 is in the domain of f, we splice the two intervals together so f is decreasing from $(-\infty, 1)$. We see f is increasing from $(1, \infty)$.

2. Since f changes from decreasing just to the left of $x = 1$ to increasing just to the right of $x = 1$, the graph of f has a local minimum at $x = 1$. The local minimum value is $f(1) = (1)^{4/3} - 4(1)^{1/3} = -3$.

What is happening at $x = 0$? Since $f'(x)$ doesn't change sign on either side of 0, the graph of f doesn't have a local extremum there. The sign diagram indicates f is decreasing through that point.

A quick check using desmos reveals 'unusual steepness' at $x = 0$, a phenomenon which is called a **vertical tangent**. This means the function locally resembles a vertical line.[3]



□

Concavity and the Second Derivative

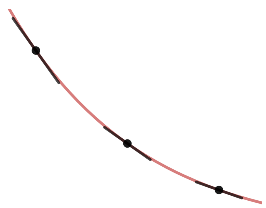
In section Section **??**, we introduced the notion of **concavity**. In that section, we described curves as being **concave up** over an interval if it resembles a portion of a '$\smile$' shape and **concave down** over an interval if resembles part of a '$\frown$' shape. Now that we've had some exposure to Calculus, we can more precisely define these notions.

DEFINITION 1.9. Let f be a differentiable function on an open interval I. Then f is said to be:

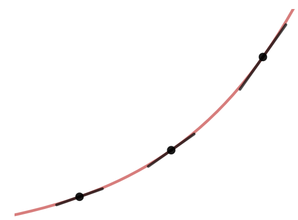
- **concave up** on I if the tangent lines lie **below** the graph on I.
- **concave down** on I if the tangent lines lie **above** the graph on I.

[3]See Example **??** for another such example and discussion.

If we take the time to study a generic concave up curve, the '$\smile$' shape can be divided into a decreasing and increasing arc:



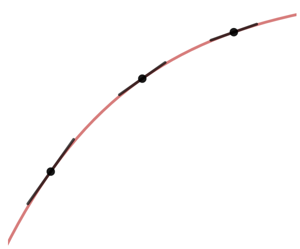
slopes are increasing towards 0



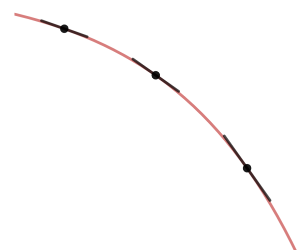
slopes are increasing away from 0

In both of these cases, the **slopes** of the tangent line are **increasing**.

Likewise, we can dissect a generic '$\frown$' shape curve into an increasing and decreasing arc:



slopes are decreasing towards 0



slopes are decreasing away from 0

Here, the **slopes** of the tangent line are **decreasing**.

We know from Theorem 1.4 that the derivative of a function can tell us where that function is increasing and decreasing. Since the function which gives us the slopes of tangent lines is the derivative, $f'(x)$, we could use the derivative of $f'(x)$ to determine where the slopes of the tangent lines were increasing and decreasing. This leads us to define the **second derivative**, $f''(x)$ as the derivative of $f'(x)$.

We present the following theorem without proof, but hopefully sufficiently motivated.

THEOREM 1.6. Suppose f is twice differentiable on an open interval I:

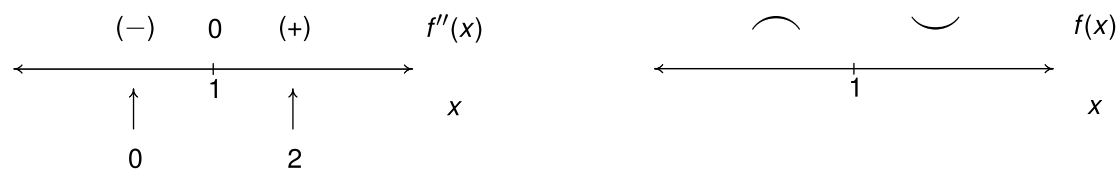
- If $f''(x) > 0$ for all x in I, then **slopes** are **increasing** and f is **concave up** on I.
- If $f''(x) < 0$ for all x in I, then **slopes** are **decreasing** and f is **concave down** on I.

EXAMPLE 1.3.3. Let $f(x) = x^3 - 3x^2 - 9x + 5$. Use the fact that $f''(x) = 6x - 6$ to find the intervals over which the graph of f is concave up and concave down.

Solution. To analyze the concavity of the graph of f, we need to make a sign diagram for $f''(x)$.

Solving $f''(x) = 6x - 6 = 0$ gives $x = 1$. We find $f''(0) = 6(0) - 6 = (-)$ and $f''(2) = 6(2) - 6 = (+)$.

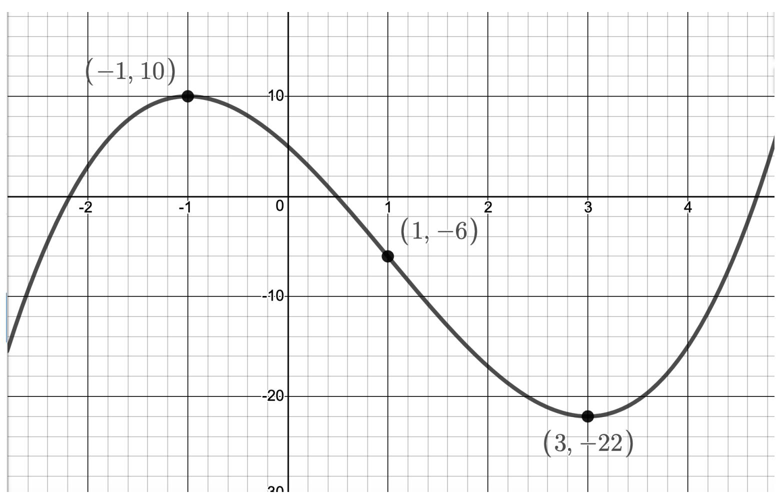
We have our sign diagram below on the left and our interpretation below on the right.



We find f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

At $x = 1$, the concavity changes. We find $f(1) = (1)^3 - 3(1)^2 - 9(1) + 5 = -6$ and we call the point $(1, -6)$ an **inflection point**. In this case since the concavity changes from concave down to concave up, the point $(1, -6)$ is the point on the graph of $y = f(x)$ were the slopes stop decreasing and start to increase.

A quick check using desmos confirms our results.



$\square$

Note that we can use concavity to help us distinguish local extrema.

For the function above, both $f'(-1) = 0$ and $f'(3) = 0$. Note that $f''(-1) < 0$ which means f is concave down there. This forces f to have a local maximum at $(-1, 6)$. Likewise, $f''(3) > 0$ which means f is concave up there. This forces f to have a local minimum at $(3, -22)$. We generalize this observation below.

THEOREM 1.7. **The Second[a] Derivative Test for Local Extrema:** Suppose f is differentiable on an open interval I containing c and $f'(c) = 0$:

- If $f''(c) > 0$ then f has a local minimum at $x = c$.
- If $f''(c) < 0$ then f has a local maximum at $x = c$.
- If $f''(c) = 0$ then the test is inconclusive. f may or may not have a local extremum at $x = c$. (In this case, we would appeal to the first derivative test.)

[a]Now you know why we titled Theorem 1.5 the 'First' Derivative Test for Local Extrema.

EXAMPLE 1.3.4. Let $f(x) = x^{4/3} - 4x^{1/3}$. Use the fact that $f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3}$ to help you find:

1. the open intervals over which the graph of f is concave up and concave down.
2. the inflection points in the graph.

Solution.

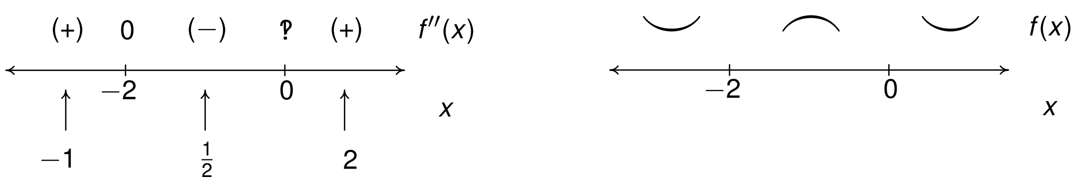
1. As in Example 1.3.2, our first step is to rewrite $f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3}$ as a single fraction:

$$f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4}{9x^{2/3}} + \frac{8}{9x^{5/3}} = \frac{4}{9x^{2/3}} \cdot \frac{x^{3/3}}{x^{3/3}} + \frac{8}{9x^{5/3}} = \frac{4x}{9x^{5/3}} + \frac{8}{9x^{5/3}} = \frac{4x+8}{9x^{5/3}}$$

We see $f''(x) = \frac{4x+8}{9x^{5/3}}$ is undefined when $9x^{5/3} = 0$, that is, when $x = 0$.

Solving $f''(x) = \frac{4x+8}{9x^{5/3}}$ gives $4x + 8 = 0$ so $x = -2$.

Going through the usual routine, we obtain our sign diagram for $f''(x)$ is below.

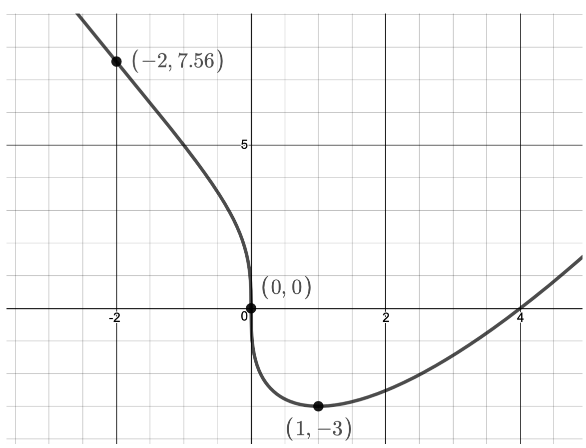


We see f is concave up on $(-\infty, -2)$ and again from $(0, \infty)$. f is concave down on $(-2, 0)$.

2. Since f changes concavity at both $x = -2$ and $x = 0$, we have inflection point at both of these values.

   We find: $f(-2) = (-2)^{4/3} - 4(-2)^{1/3} = 2(2)^{1/3} + 4(2)^{1/3} = 6(2)^{1/3}$. So $\left(-2, 6(2)^{1/3}\right)$ is one inflection point. When $x = 0$, $f(0) = (0)^{4/3} - 4(0)^{1/3} = 0$, so $(0, 0)$ is the other inflection point.

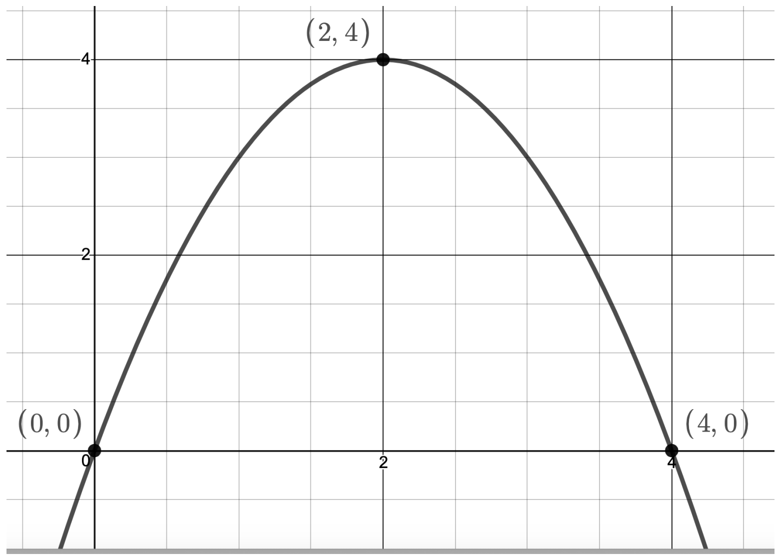
   Checking with desmos, it's not apparent that the graph of $y = f(x)$ is concave up for $x < -2$. We invite the reader to graph f and zoom out to see that characteristic of the graph.



□

Our last example offers a twist on these sorts of curve-sketching problems.

EXAMPLE 1.3.5. Below is the graph of the **derivative** of a function. Assume as $x \to \pm\infty$, $f'(x) \to -\infty$.



The graph of $y = f'(x)$

1. Use the graph of $y = f'(x)$ to determine the open intervals where f is increasing and decreasing.

   Find the x-coordinates of the local extrema.

2. Use the graph of $y = f'(x)$ make a sign diagram for $y = f''(x)$.

3. List the open intervals over which the graph of f is concave up and concave down.

   Find the x-coordinates of the inflection points.

4. Sketch a plausible graph of $y = f(x)$.

Solution.

1. Recall from algebra, the solutions to $f'(x) < 0$ are the x-values where the graph of $y = f'(x)$ is below the x-axis. This happens on the intervals $(-\infty, 0)$ and $(4, \infty)$, so this means f is decreasing here.

   Likewise, the solutions to $f'(x) > 0$ are the x-values where $y = f'(x)$ is above the x-axis. This happens on the interval $(0, 4)$, so f is increasing here.

   Since f goes from decreasing to the left of $x = 0$ to increasing to the right of $x = 0$, f has a local minimum at $x = 0$. Since f goes from increasing to the left of $x = 4$ to decreasing to the right of $x = 4$, f has a local maximum at $x = 4$.

2. Since $f''(x)$ is the derivative of $f'(x)$, we know $f''(x) > 0$ on $(-\infty, 2)$ since $f'(x)$ is increasing there. We see $f''(2) = 0$ since $f'(x)$ is locally flat at $(2, 4)$. Lastly, we see $f''(x) < 0$ on $(2, \infty)$ since $f'(x)$ is decreasing there. We put all this together in a sign diagram below.

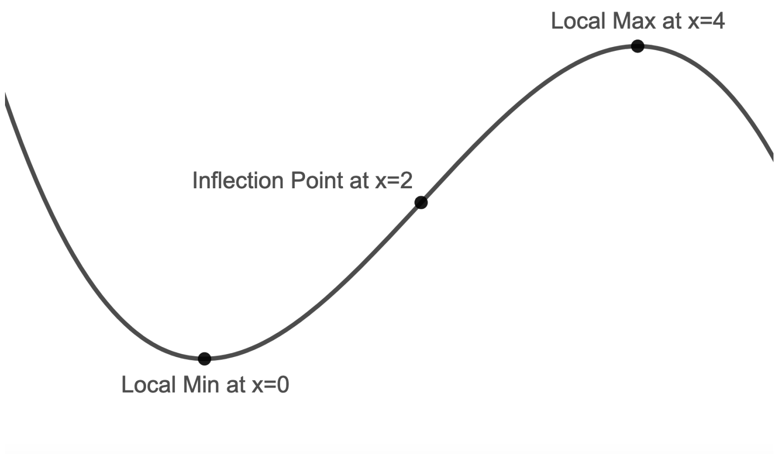
(+) 0 (−) $f''(x)$
AppDerivatives.9
2
x

$f(x)$
AppDerivatives.10
2
x

3. We have f is concave up on $(-\infty, 2)$ and concave down on $(2, \infty)$.

   Since f changes concavity at $x = 2$, there is an inflection point there.

4. A plausible graph of $y = f(x)$ is below. We cannot determine any y-coordinates (do you see why not?)



A possible graph of $y = f(x)$

□

1.3.2 Related Rates

1.3.3 Marginal Analysis

1.3.4 Exercises

To incude:

- Limits from graphs
- Graphs from limits
- Using continuity to find limits
- Piecewise-defined functions
- Simplify to find limits
- One-sided continuity
- Explorations with formal definitions involving infinite limits - prelude to epsilonics