Chapter 1

FURTHER TOPICS ON FUNCTIONS

1.1 Graphs of Functions

Up until this point in the text, we have primarily focused on studying particular *families* of functions. These families and their relationships to one another provide useful *examples* of more abstract function structures and relationships. The notions introduced in this chapter will not only provide us a more formal vocabulary with which to describe the connections between the function families we have already studied, but, more importantly, give us additional lenses through which to view new families of functions that we'll encounter.

In this section, we review of the concepts associated with the graphs of functions. We introduced the notion of the graph of a function in Section **??**, and the vast majority of the graphs we have encountered in this text were generated from an algebraic representation of a function. In this section, we define the functions geometrically from the outset and review the important concepts associated with the graphs of functions.

Recall the **domain** of a function is the set of inputs to the function and the **range** of a function is the set of outputs from the function. When graphing a function whose domain and range are subsets of real numbers, we plot the ordered pairs (input, output) on the Cartesian plane. Hence, the domain values are found on the horizontal axis while the range values are found on the vertical axis.

Recall from Definition **??** that the largest output from the function (if there is one) is called the **maximum** or, when there may be some confusion, the **absolute maximum** of the function. Likewise, the smallest output from the function (again, if there is one) is called the **minimum** or **absolute minimum**.

A concept related to 'absolute' maximum and minimum is the concept of 'local' maximum and minimum as described in Definition **??**. Here, a point (a, b) on the graph of a function f is a **local maximum** if b is the maximum function value for some open interval in the domain containing a. The notion of 'local' here meaning instead of surveying the entire domain, we instead restrict our attention to inputs 'local' or 'near' the input a. The concept of **local minimum** is defined similarly.

Next, we review the notions of **increasing**, **decreasing**, and **constant** as described in Definition **??**. Recall a function is increasing over an interval if, as the inputs increase, do the outputs. This means that, geometrically, the graph of the function rises as we move left to right. Similarly, a function is decreasing over an interval if the outputs decrease as the inputs increase. Geometrically, a decreasing function falls as we move left to right. Finally, a function is constant over an interval if the output is the same regardless of the input. If a function is constant over an interval, its graph remains 'flat' - a horizontal line.

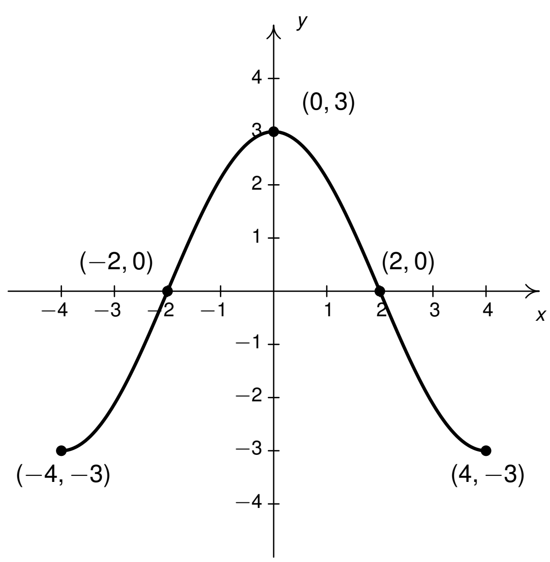
Last, and according to some[1] least, we briefly review the notion of symmetry in the graphs of functions. Recall from Definition **??** that a function f is called **even** if $f(-x) = f(x)$ for all x in the domain of f. The graphs of even functions are symmetric about the vertical (usually y-) axis. In a similar manner, Definition **??** tells us a function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f. Geometrically, the graphs of odd functions are symmetric about the origin.

The next example reviews all of the aforementioned concepts as well as many more.

EXAMPLE 1.1.1. Given the graph of $y = f(x)$ below, answer all of the following questions.

[1] Jeff

1. Find the domain of f.
2. Find the range of f.
3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the x-intercepts, if any exist.
6. List the y-intercepts, if any exist.
7. Find the zeros of f.
8. Solve $f(x) < 0$.
9. Determine $f(2)$.
10. Solve $f(x) = -3$.
11. Find the number of solutions to $f(x) = 1$.
12. Does f appear to be even, odd, or neither?
13. List the local maximums, if any exist.
14. List the local minimums, if any exist.
15. List the intervals on which f is increasing.
16. List the intervals on which f is decreasing.



Solution.

1. To find the domain of f, we proceed as in Section **??**. By projecting the graph to the x-axis, we see that the portion of the x-axis which corresponds to a point on the graph is everything from -4 to 4, inclusive. Hence, the domain is $[-4, 4]$.

2. To find the range, we project the graph to the y-axis. We see that the y values from -3 to 3, inclusive, constitute the range of f. Hence, our answer is $[-3, 3]$.

3. The maximum value of f is the largest y-coordinate which is 3.

4. The minimum value of f is the smallest y-coordinate which is -3.

5. The x-intercepts are the points on the graph with y-coordinate 0, namely $(-2, 0)$ and $(2, 0)$.

6. The y-intercept is the point on the graph with x-coordinate 0, namely $(0, 3)$.

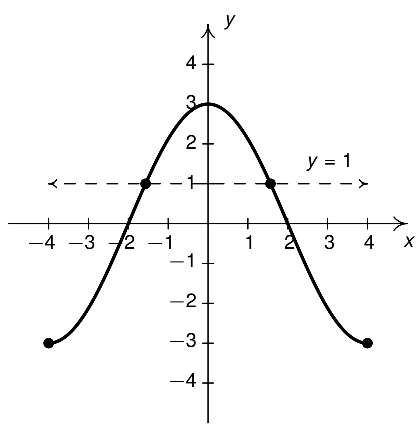
7. The zeros of f are the x-coordinates of the x-intercepts of the graph of $y = f(x)$ which are $x = -2, 2$.

8. To solve $f(x) < 0$, we look for the x values of the points on the graph where the $y = f(x)$ is negative. Graphically, we are looking for where the graph is *below* the x-axis. This happens for the x values from -4 to -2 and again from 2 to 4. So our answer is $[-4, -2) \cup (2, 4]$.

9. Since the graph of f is the graph of the equation $y = f(x)$, $f(2)$ is the y-coordinate of the point which corresponds to $x = 2$. Since the point $(2, 0)$ is on the graph, we have $f(2) = 0$.

10. To solve $f(x) = -3$, we look where $y = f(x) = -3$. We find two points with a y-coordinate of -3, namely $(-4, -3)$ and $(4, -3)$. Hence, the solutions to $f(x) = -3$ are $x = \pm 4$.

11. As in the previous problem, to solve $f(x) = 1$, we look for points on the graph where the y-coordinate is 1. If we imagine the horizontal like $y = 1$ superimposed over the graph of f as sketched below, we get two intersections. Hence, even though these points aren't specified, we know there are *two* points on the graph of f whose y-coordinate is 1. Hence, there are two solutions to $f(x) = 1$.



12. The graph appears to be symmetric about the y-axis. This suggests[2] that f is even.

13. The function has its only local maximum at $(0, 3)$.

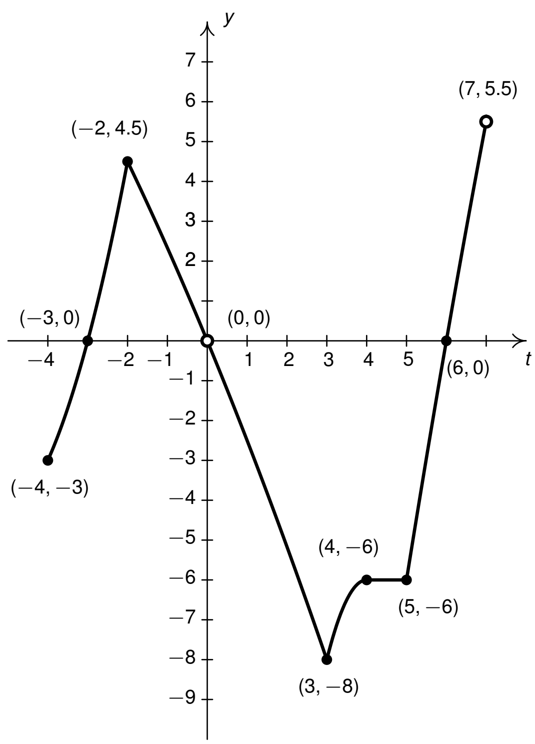
14. There are no local minimums. Why don't $(-4, -3)$ and $(4, -3)$ count? Let's consider the point $(-4, -3)$ for a moment. Recall that, in the definition of local minimum, there needs to be an open interval containing $x = -4$ which is in the domain of f. In this case, there is no open interval containing $x = -4$ which lies entirely in the domain of f, $[-4, 4]$. Because we are unable to fulfill the requirements of the definition for a local minimum, we cannot claim that f has one at $(-4, -3)$. The point $(4, -3)$ fails for the same reason $-$ no open interval around $x = 4$ stays within the domain of f.

[2] but does not prove

15. As we move from left to right, the graph rises from $(-4,-3)$ to $(0,3)$. This means f is increasing on the interval $[-4,0]$. (Remember, the answer here is an interval on the x-axis.)
16. As we move from left to right, the graph falls from $(0,3)$ to $(4,-3)$. This means f is decreasing on the interval $[0,4]$. (Again, the answer here is an interval on the x-axis.) $\square$

Our next example involves a more complicated function and asks more complicated questions.

EXAMPLE 1.1.2. Consider the graph of the function g below.



The graph of $y = g(t)$

1. Find the domain of g.
2. Find the range of g.
3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the local maximums, if any exist.
6. List the local minimums, if any exist.
7. Solve $(t^2-25)g(t)=0$.
8. Solve $\dfrac{g(t)}{t^2+t-30} \geq 0$.

Solution.

1. Projecting the graph of g to the t-axis, we see the domain contains values of t from -4 up to, but not including $t=0$ and values greater than $t=0$ up to, but not including $t=7$. Using interval notation, we write the domain as $[-4,0) \cup (0,7)$.

2. Projecting the graph of g to the y-axis, we see the range of g contains all real numbers from $y = -8$ up to, but not including, $y = 5.5$. Note that even though there is a hole in the graph at $(0, 0)$, the points $(-3, 0)$ and $(6, 0)$ put $y = 0$ in the range of g. Hence, the range of g is $[-8, 5.5)$.

3. Owing to the hole in the graph at $(7, 5.5)$, g has no maximum.[3]

4. The minimum of g is -8 which occurs at the point $(3, -8)$.

5. The point $(-2, 4.5)$ is clearly a local maximum, but there are actually infinitely many more. Per Definition **??**, all points of the form $(t, -6)$ for $4 \leq t < 5$ are also local maximums. For each of these points, we can find an open interval on the t axis within which we produce no points on the graph higher than $(t, -6)$. (You may think about 'zooming in' on the point $(4.5, -6)$ to see how this works.)

6. The local minimums of the graph are $(3, -8)$ along with points of the form $(t, -6)$ for $4 < t \leq 5$. Note the point $(-4, -3)$ is not a local minimum since there is no open interval containing $t = -4$ which lies entirely within the domain of g.

7. To solve $(t^2 - 25)g(t) = 0$, we use the zero product property of real numbers[4] to conclude either $t^2 - 25 = 0$ or $g(t) = 0$.

   From $t^2 - 25 = 0$, we get $t = \pm 5$. However, since $t = -5$ isn't in the domain of g, it cannot be regarded as a solution to the equation $(t^2 - 25)g(t) = 0$. (If we substitute $t = -5$ into the equation, we'd get $((-5)^2 - 25)g(-5) = 0 \cdot g(-5)$. Since $g(-5)$ is undefined, so is $0 \cdot g(-5)$.)

   To solve $g(t) = 0$, we look for the zeros of g which are $t = -3$ and $t = 6$. (Again, there is a hole at $(0, 0)$, so $t = 0$ doesn't count as a zero.) Our final answer to $(t^2 - 25)g(t) = 0$ is $t = -3$, 5, or 6.

8. To solve $\frac{g(t)}{t^2+t-30} \geq 0$, we employ a sign diagram as we (most recently) have done in Section **??**.[5] To that end, we define $F(t) = \frac{g(t)}{t^2+t-30}$ and we set about finding the domain of f.

   First, we note that since F is defined in terms of g, the domain of F is restricted to some subset of the domain of g, namely $[-4, 0) \cup (0, 7)$. Since $t^2 + t - 30$ is in the denominator of $F(t)$, we must also exclude the values where $t^2 + t - 30 = (t+6)(t-5) = 0$. Hence, we must exclude $t = -6$ (which isn't in the domain of g in the first place) along with $t = 5$. Hence, the domain of F is $[-4, 0) \cup (0, 5) \cup (5, 7)$.

   Next, we find the zeros of F. Setting $F(t) = \frac{g(t)}{t^2+t-30} = 0$ amounts to solving $g(t) = 0$. Graphically, we see this occurs when $t = -3$ and $t = 6$. Hence, we need to select test values in each of the following intervals: $[-4, -3)$, $(-3, 0)$, $(0, 5)$, $(5, 6)$ and $(6, 7)$.

   For the interval $[-4, -3)$, we may choose $t = -4$. $F(-4) = \frac{g(-4)}{(-4)^2+(-4)-30} = \frac{-3}{-18} > 0$ so is $(+)$. For the interval $(-3, 0)$ we choose $t = -2$ and get $F(-2) = \frac{g(-2)}{(-2)^2+(-2)-30} = \frac{4.5}{-28} < 0$ so is $(-)$. For the interval $(0, 5)$, we choose $t = 3$ and find $F(3) = \frac{g(3)}{(3)^2+(3)-30} = \frac{-8}{-18} > 0$ which is $(+)$ again.

[3] There is no real number 'right before' 5.5 ...

[4] see Section **??**, **??**

[5] Note that g is continuous on its domain, and hence, it follows that $\frac{g(t)}{t^2+t-30}$ is, too. (Thank Calculus!) This means the Intermediate Value Theorem applies so a Sign Diagram approach is valid.

For the last two intervals, $(5, 6)$ and $(6, 7)$, we do not have specific function values for g. However, all we are interested in is the *sign* of the function over these intervals, and we can get that information about g graphically.

For the interval $(5, 6)$, we choose $t = 5.5$ as our test value. Since the graph of $y = g(t)$ is *below* the t-axis when $t = 5.5$. we know $g(5.5)$ is $(-)$. Hence, $F(5.5) = \frac{g(5.5)}{(5.5)^2+(5.5)-30} = \frac{(-)}{5.75} < 0$ so is $(-)$. Similarly, when $t = 6.5$, the graph of $y = g(t)$ is *above* the t-axis so $F(6.5) = \frac{g(6.5)}{(6.5)^2+(6.5)-30} = \frac{(+)}{18.75} > 0$ so is $(+)$. Putting all of this together, we get the sign diagram for $F(t) = \frac{g(t)}{t^2+t-30}$ below:

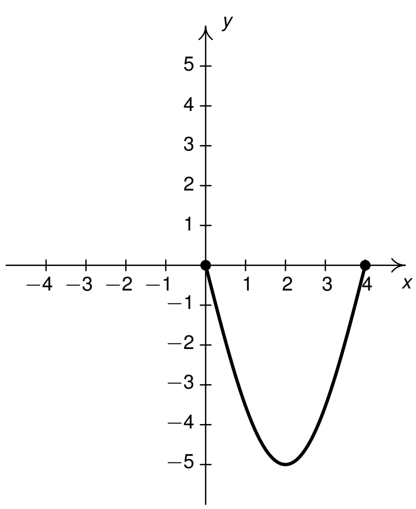
(+)	0	(−)	‽	(+)	‽	(−)	0	(+)	‽
−4	−3		0		5		6		7

Hence, $F(t) \geq 0$ on $[-4, -3] \cup (0, 5) \cup [6, 7)$. □

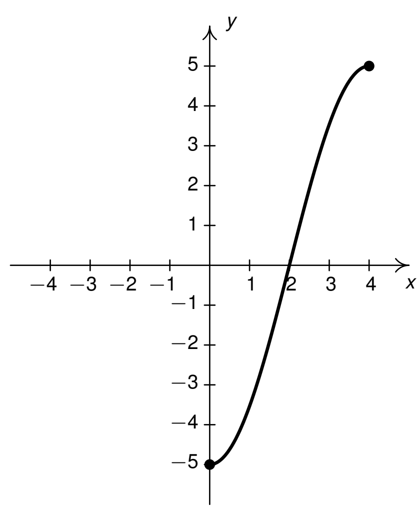
Our last example focuses on symmetry. The reader is encouraged to review the notes about symmetry as summarized on page **??** in Section **??**.

EXAMPLE 1.1.3. Below are the partial graphs of functions f and g.

1. If possible, complete the graphs of f and g assuming both functions are even.
2. If possible, complete the graphs of f and g assuming both functions are odd.



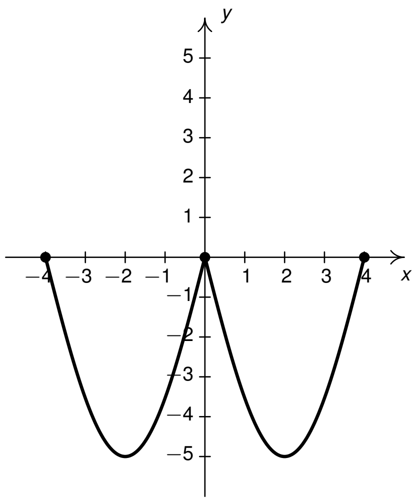
Partial graph of $y = f(x)$



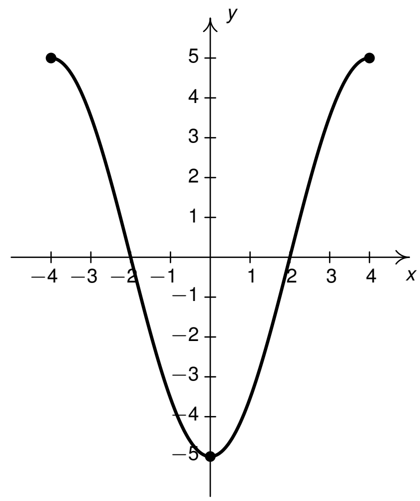
Partial graph of $y = g(x)$

Solution.

1. If f and g are even then their graphs are symmetric about the y-axis. Hence, to complete each graph, we reflect each point on the graphs of f and g about the y-axis.



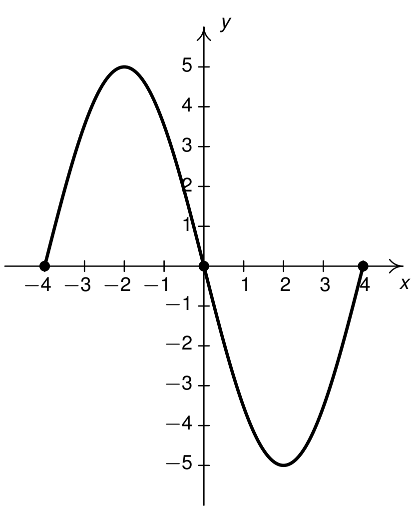
The graph of f assuming f is even.



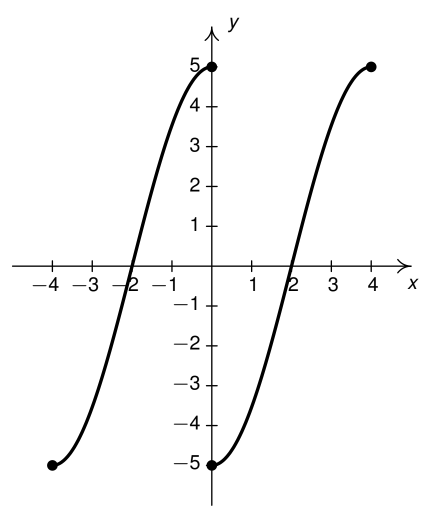
The graph of g assuming g is even.

2. If f and g are odd then their graphs are symmetric about the origin. Hence, to complete each graph, we imagine reflecting each of the points on their graphs through the origin. We complete the process on the graph of f with no issues.

   However, when attempting to do the same with the graph of the function g, we find the point $(0, -5)$ is reflected to the point $(0, 5)$. Hence, this new graph doesn't pass the vertical line test and hence is not a function. Therefore, g cannot be odd.[6]



The graph of f assuming f is odd.



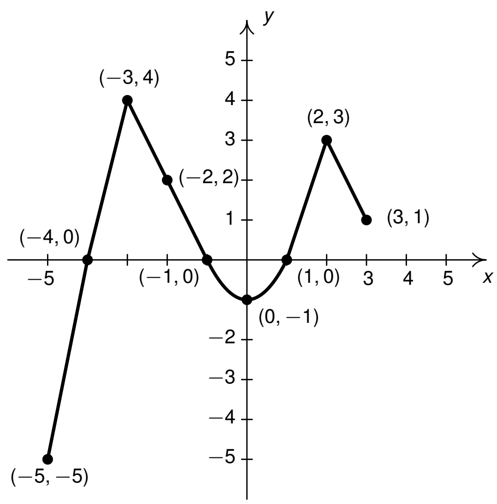
This graph fails the vertical line test.

□

[6] We leave it as an exercise to show that if a function f is odd and 0 is in the domain of f, then, necessarily, $f(0) = 0$.

1.1.1 Exercises

In Exercises 1 - 4, use the graph of $y = f(x)$ given below to answer the question.



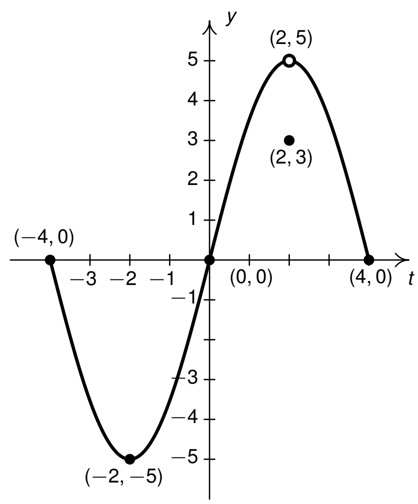
$y = f(x)$

1. Find the domain of f.
2. Find the range of f.
3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the local maximums, if any exist.
6. List the local minimums, if any exist.
7. List the intervals where f is increasing.
8. List the intervals where f is decreasing.
9. Determine $f(-2)$.
10. Solve $f(x) = 4$.
11. List the x-intercepts, if any exist.
12. List the y-intercepts, if any exist.
13. Find the zeros of f.
14. Solve $f(x) \geq 0$.
15. Find the number of solutions to $f(x) = 1$.
16. Find the number of solutions to $|f(x)| = 1$.
17. Solve $(x^2 - x - 2)f(x) = 0$
18. Solve $(x^2 - x - 2)f(x) > 0$

With help from your classmates:

19. Find the domain of $R(x) = \dfrac{1}{f(x)}$
20. Find the range of $R(x) = \dfrac{1}{f(x)}$

In Exercises 21 - 24, use the graph of $y = g(t)$ given below to answer the question.

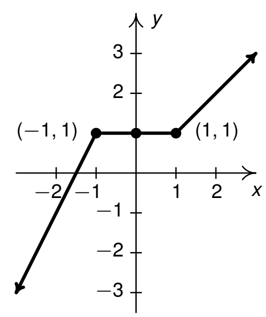


$y = g(t)$

21. Find the domain of g.
22. Find the range of g.
23. Find the maximum, if it exists.
24. Find the minimum, if it exists.
25. List the local maximums, if any exist.
26. List the local minimums, if any exist.
27. List the intervals where g is increasing.
28. List the intervals where g is decreasing.
29. Determine $g(2)$.
30. Solve $g(t) = -5$.
31. List the t-intercepts, if any exist.
32. List the y-intercepts, if any exist.
33. Find the zeros of g.
34. Solve $g(t) \leq 0$.
35. Find the domain of $G(t) = \dfrac{g(t)}{t+2}$.
36. Solve $\dfrac{g(t)}{t+2} \leq 0$.
37. How many solutions are there to $[g(t)]^2 = 9$?
38. Does g appear to be even, odd, or neither?
39. Prove that if f is an odd function and 0 is in the domain of f, then $f(0) = 0$.
40. Let $R(x)$ be the function defined as: $R(x) = 1$ if x is a rational number, $R(x) = 0$ if x is an irrational number. With help from your classmates, try to graph R. What difficulties do you encounter?

    NOTE: Between every pair of real numbers, there is both a rational and an irrational number . . .

41. Consider the graph of the function f given below.



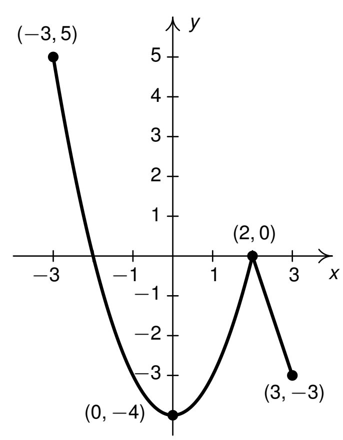
(a) Explain why f has a local maximum but not a local minimum at the point $(-1, 1)$.

(b) Explain why f has a local minimum but not a local maximum at the point $(1, 1)$.

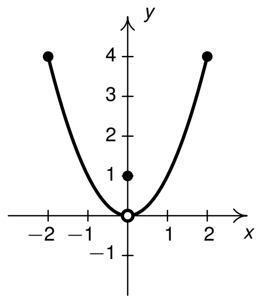
(c) Explain why f has a local maximum AND a local minimum at the point $(0, 1)$.

(d) Explain why f is constant on the interval $[-1, 1]$ and thus has both a local maximum AND a local minimum at every point $(x, f(x))$ where $-1 < x < 1$.

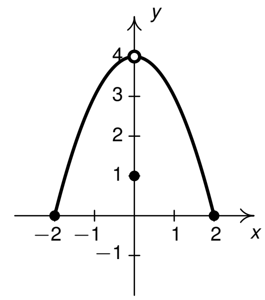
42. Explain why the function g whose graph is given below does <u>not</u> have a local maximum at $(-3, 5)$ nor does it have a local minimum at $(3, -3)$. Find its extrema, both local and absolute and find the intervals on which g is increasing and those on which g is decreasing.



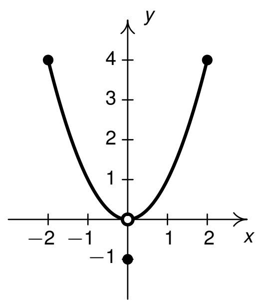
43. For each function below, find the local maximum or local minimum and list the interval over which the function is increasing and the interval over which the function is decreasing.



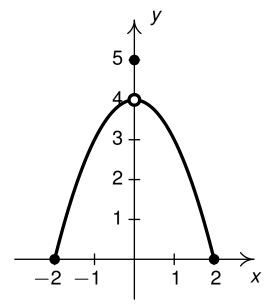
(a) Function I



(b) Function II



(c) Function III



(d) Function IV

1.1.2 Answers

1. $[-5, 3]$
2. $[-5, 4]$
3. $f(-3) = 4$
4. $f(-5) = -5$
5. $(-3, 4)$, $(2, 3)$
6. $(0, -1)$
7. $[-5, -3]$, $[0, 2]$
8. $[-3, 0]$, $[2, 3]$
9. $f(-2) = 2$
10. $x = -3$
11. $(-4, 0)$, $(-1, 0)$, $(1, 0)$
12. $(0, -1)$
13. -4, -1, 1
14. $[-4, -1]$, $[1, 3]$
15. 4
16. 6
17. $x = -4, -1, 1, 2$
18. $(-4, -1) \cup (-1, 1) \cup (2, 3)$
19. To find the domain of $R(x) = \frac{1}{f(x)}$, we start with the domain of f and exclude values where $f(x) = 0$. Hence, the domain of R is $[-5, -4) \cup (-4, -1) \cup (-1, 1) \cup (1, 3]$.
20. To find the range of $R(x) = \frac{1}{f(x)}$, we start with the range of f (excluding 0) and take reciprocals. If $-5 \leq y < 0$, then $\frac{1}{y} \leq -\frac{1}{5}$. If $0 < y \leq 4$, then $\frac{1}{y} \geq \frac{1}{4}$. Hence the range of R is $\left(-\infty, -\frac{1}{5}\right] \cup \left[\frac{1}{4}, \infty\right)$.
21. $[-4, 4]$
22. $[-5, 5)$
23. none
24. $g(-2) = -5$
25. none
26. $(-2, -5)$, $(2, 3)$
27. $[-2, 2)$
28. $[-4, -2]$, $(2, 4]$
29. $g(2) = 3$
30. $t = -2$
31. $(-4, 0)$, $(0, 0)$, $(4, 0)$
32. $(0, 0)$
33. -4, 0, 4
34. $[-4, 0] \cup \{4\}$
35. $[-4, -2) \cup (-2.4]$
36. $\{-4\} \cup (-2, 0] \cup \{4\}$
37. 5
38. Neither.
43. (a) Local maximum: $(0, 1)$, no local minimum. Increasing: $(0, 2]$, decreasing: $[-2, 0)$.
    (b) No local maximum, local minimum: $(0, 1)$. Increasing: $[-2, 0)$, decreasing: $(0, 2]$.
    (c) No local maximum, local minimum: $(0, -1)$. Increasing: $[0, 2]$, decreasing: $[-2, 0]$.
    (d) Local maximum: $(0, 5)$, no local minimum. Increasing: $[-2, 0]$, decreasing: $[0, 2]$.

1.2 Function Arithmetic

As we mentioned in Section 1.1, in this chapter, we are studying functions in a more abstract and general setting. In this section, we begin our study of what can be considered as the *algebra of functions* by defining *function arithmetic.*

Given two real numbers, we have four primary arithmetic operations available to us: addition, subtraction, multiplication, and division (provided we don't divide by 0.) Since the functions we study in this text have ranges which are sets of real numbers, it makes sense we can extend these arithmetic notions to functions.

For example, to add two functions means we add their outputs; to subtract two functions, we subtract their outputs, and so on and so forth. More formally, given two functions f and g, we *define* a new function $f + g$ whose rule is determined by adding the outputs of f and g. That is $(f + g)(x) = f(x) + g(x)$. While this looks suspiciously like some kind of distributive property, it is nothing of the sort. The '+' sign in the expression '$f + g$' is part of the *name* of the function we are defining,[1] whereas the plus sign '+' sign in the expression $f(x) + g(x)$ represents real number addition: we are adding the output from f, $f(x)$ with the output from g, $g(x)$ to determine the output from the sum function, $(f + g)(x)$.

Of course, in order to define $(f + g)(x)$ by the formula $(f + g)(x) = f(x) + g(x)$, both $f(x)$ and $g(x)$ need to be defined in the first place; that is, x must be in the domain of f *and* the domain of g. You'll recall[2] this means x must be in the *intersection* of the domains of f and g. We define the following.

DEFINITION 1.1. Suppose f and g are functions and x is in both the domain of f and the domain of g.

- The **sum** of f and g, denoted $f + g$, is the function defined by the formula

$$(f + g)(x) = f(x) + g(x)$$

- The **difference** of f and g, denoted $f - g$, is the function defined by the formula

$$(f - g)(x) = f(x) - g(x)$$

- The **product** of f and g, denoted fg, is the function defined by the formula

$$(fg)(x) = f(x)g(x)$$

- The **quotient** of f and g, denoted $\frac{f}{g}$, is the function defined by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

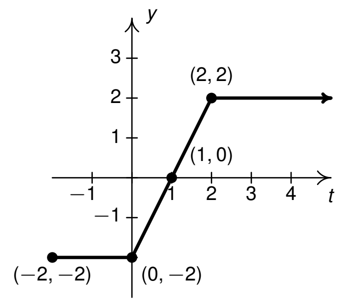
  provided $g(x) \neq 0$.

We put these definitions to work for us in the next example.

[1] We could have just as easily called this new function $S(x)$ for 'sum' of f and g and defined S by $S(x) = f(x) + g(x)$.
[2] see Section **??**.

EXAMPLE 1.2.1. Consider the following functions:

- $f(x) = 6x^2 - 2x$
- $g(t) = 3 - \dfrac{1}{t}$, $t > 0$
- $h = \{(-3, 2), (-2, 0.4), (0, \sqrt{2}), (3, -6)\}$
- s whose graph is given below:



$y = s(t)$

1. Find and simplify the following function values:

   (a) $(f + g)(1)$ (b) $(s - f)(-1)$ (c) $(fg)(2)$ (d) $\left(\dfrac{s}{h}\right)(0)$

   (e) $((s + g) + h)(3)$ (f) $(s + (g + h))(3)$ (g) $\left(\dfrac{f + h}{s}\right)(3)$ (h) $(f(g - h))(-2)$

2. Find the domain of each of the following functions:

   (a) hg (b) $\dfrac{f}{s}$

3. Find expressions for the functions below. State the domain for each.

   (a) $(fg)(x)$ (b) $\left(\dfrac{g}{f}\right)(t)$

Solution.

1. (a) By definition, $(f + g)(1) = f(1) + g(1)$. We find $f(1) = 6(1)^2 - 2(1) = 4$ and $g(1) = 3 - \frac{1}{1} = 2$. So we get $(f + g)(1) = 4 + 2 = 6$.

   (b) To find $(s - f)(-1) = s(-1) - f(-1)$, we need both $s(-1)$ and $f(-1)$. To get $s(-1)$, we look to the graph of $y = s(t)$ and look for the y-coordinate of the point on the graph with the t-coordinate of -1. While not labeled directly, we infer the point $(-1, -2)$ is on the graph which means $s(-1) = -2$. For $f(-1)$, we compute: $f(-1) = 6(-1)^2 - 2(-1) = 8$. Putting it all together, we get $(s - f)(-1) = (-2) - (8) = -10$.

   (c) Since $(fg)(2) = f(2)g(2)$, we first compute $f(2)$ and $g(2)$. We find $f(2) = 6(2)^2 - 2(2) = 20$ and $g(2) = 2 + \frac{1}{2} = \frac{5}{2}$, so $(fg)(2) = f(2)g(2) = (20)\left(\frac{5}{2}\right) = 50$.

(d) By definition, $\left(\frac{s}{h}\right)(0) = \frac{s(0)}{h(0)}$. Since $(0, -2)$ is on the graph of $y = s(t)$, so we know $s(0) = -2$. Likewise, the ordered pair $(0, \sqrt{2}) \in h$, so $h(0) = \sqrt{2}$. We get $\left(\frac{s}{h}\right)(0) = \frac{s(0)}{h(0)} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$.

(e) The expression $((s + g) + h)(3)$ involves *three* functions. Fortunately, they are grouped so that we can apply Definition 1.1 by first considering the sum of the two functions $(s + g)$ and h, then to the sum of the two functions s and g: $((s + g) + h)(3) = (s + g)(3) + h(3) = (s(3) + g(3)) + h(3)$. To get $s(3)$, we look to the graph of $y = s(t)$. We infer the point $(3, 2)$ is on the graph of s, so $s(3) = 2$. We compute $g(3) = 3 - \frac{1}{3} = \frac{8}{3}$. To find $h(3)$, we note $(3, -6) \in h$, so $h(3) = -6$. Hence, $((s + g) + h)(3) = (s + g)(3) + h(3) = (s(3) + g(3)) + h(3) = \left(2 + \frac{8}{3}\right) + (-6) = -\frac{4}{3}$.

(f) The expression $(s + (g + h))(3)$ is very similar to the previous problem, $((s + g) + h)(3)$ except that the g and h are grouped together here instead of the s and g. We proceed as above applying Definition 1.1 twice and find $(s + (g + h))(3) = s(3) + (g + h)(3) = s(3) + (g(3) + h(3))$. Substituting the values for $s(3)$, $g(3)$ and $h(3)$, we get $(s + (g + h))(3) = 2 + \left(\frac{8}{3} + (-6)\right) = -\frac{4}{3}$, which, not surprisingly, matches our answer to the previous problem.

(g) Once again, we find the expression $\left(\frac{f+h}{s}\right)(3)$ has more than two functions involved. As with all fractions, we treat '$-$' as a grouping symbol and interpret $\left(\frac{f+h}{s}\right)(3) = \frac{(f+h)(3)}{s(3)} = \frac{f(3)+h(3)}{s(3)}$. We compute $f(3) = 6(3)^2 - 2(3) = 48$ and have $h(3) = -6$ and $s(3) = 2$ from above. Hence, $\left(\frac{f+h}{s}\right)(3) = \frac{f(3)+h(3)}{s(3)} = \frac{48+(-6)}{2} = 21$.

(h) We need to need to exercise caution in parsing $(f(g - h))(-2)$. In this context, f, g, and h are all functions, so we interpret $(f(g - h))$ as the function and -2 as the argument. We view the function $f(g - h)$ as the product of f and the function $g - h$. Hence, $(f(g - h))(-2) = f(-2)[(g - h)(-2)] = f(-2)[g(-2) - h(-2)]$. We compute $f(-2) = 6(-2)^2 - 2(-2) = 28$, and $g(-2) = 3 - \frac{1}{-2} = 3 + \frac{1}{2} = \frac{7}{2} = 3.5$. Since $(-2, 0.4) \in h$, $h(-2) = 0.4$. Putting this altogether, we get $(f(g-h))(-2) = f(-2)[(g-h)(-2)] = f(-2)[g(-2)-h(-2)] = 28(3.5-0.4) = 28(3.1) = 86.8$.

2. (a) To find the domain of hg, we need to find the real numbers in both the domain of h and the domain of g. The domain of h is $\{-3, -2, 0, 3\}$ and the domain of g is $\{t \in \mathbb{R} \mid t > 0\}$ so the only real number in common here is 3. Hence, the domain of hg is $\{3\}$, which may be small, but it's better than nothing.[3]

(b) To find the domain of $\frac{f}{s}$, we first note the domain of f is all real numbers, but that the domain of s, based on the graph, is just $[-2, \infty)$. Moreover, $s(t) = 0$ when $t = 1$, so we must exclude this value from the domain of $\frac{f}{s}$. Hence, we are left with $[-2, 1) \cup (1, \infty)$.

3. (a) By definition, $(fg)(x) = f(x)g(x)$. We are given $f(x) = 6x^2 - 2x$ and $g(t) = 3 - \frac{1}{t}$ so $g(x) = 3 - \frac{1}{x}$. Hence,

[3] Since $(hg)(3) = h(3)g(3) = (-6)\left(\frac{8}{3}\right) = -16$, we can write $hg = \{(3, -16)\}$.

$$\begin{aligned}
(fg)(x) &= f(x)g(x) \\
&= \left(6x^2 - 2x\right)\left(3 - \frac{1}{x}\right) \\
&= 18x^2 - 6x^2\left(\frac{1}{x}\right) - 2x(3) + 2x\left(\frac{1}{x}\right) && \text{distribute} \\
&= 18x^2 - 6x - 6x + 2 \\
&= 18x^2 - 12x + 2
\end{aligned}$$

To find the domain of fg, we note the domain of f is all real numbers, $(-\infty, \infty)$ whereas the domain of g is restricted to $\{t \in \mathbb{R} \,|\, t > 0\} = (0, \infty)$. Hence, the domain of fg is likewise restricted to $(0, \infty)$. Note if we relied solely on the **simplified formula** for $(fg)(x) = 18x^2 - 12x + 2$, we would have obtained the *incorrect* answer for the domains of fg.

(b) To find an expression for $\left(\frac{g}{f}\right)(t) = \frac{f(t)}{g(t)}$ we first note $f(t) = 6t^2 - 2t$ and $g(t) = 3 - \frac{1}{t}$. Hence:

$$\begin{aligned}
\left(\frac{g}{f}\right)(t) &= \frac{g(t)}{f(t)} \\
&= \frac{3 - \frac{1}{t}}{6t^2 - 2t} = \frac{3 - \frac{1}{t}}{6t^2 - 2t} \cdot \frac{t}{t} && \text{simplify compound fractions} \\
&= \frac{\left(3 - \frac{1}{t}\right)t}{\left(6t^2 - 2t\right)t} = \frac{3t - 1}{\left(6t^2 - 2t\right)t} \\
&= \frac{3t - 1}{2t^2(3t - 1)} = \frac{\cancelto{1}{(3t - 1)}}{2t^2\cancel{(3t - 1)}} && \text{factor and cancel} \\
&= \frac{1}{2t^2}
\end{aligned}$$

Hence, $\left(\frac{g}{f}\right)(t) = \frac{1}{2t^2} = \frac{1}{2}t^{-2}$. To find the domain of $\frac{g}{f}$, a real number must be both in the domain of g, $(0, \infty)$, and the domain of f, $(-\infty, \infty)$ so we start with the set $(0, \infty)$. Additionally, we require $f(t) \neq 0$. Solving $f(t) = 0$ amounts to solving $6t^2 - 2t = 0$ or $2t(3t - 1) = 0$. We find $t = 0$ or $t = \frac{1}{3}$ which means we need to exclude these values from the domain. Hence, our final answer for the domain of $\frac{g}{f}$ is $\left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$. Note that, once again, using the *simplified formula* for $\left(\frac{g}{f}\right)(t)$ to determine the domain of $\frac{g}{f}$, would have produced erroneous results. □

A few remarks are in order. First, in number 1 parts 1e through 1h, we first encountered combinations of *three* functions despite Definition 1.1 only addressing combinations of *two* functions at a time. It turns out that function arithmetic inherits many of the same properties of real number arithmetic. For example, we showed above that $((s + g) + h)(3) = (s + (g + h))(3)$. In general, given any three functions f, g, and h, $(f + g) + h = f + (g + h)$ that is, function addition is *assocative*. To see this, choose an element x common to the domains of f, g, and h. Then

$$\begin{aligned}
((f+g)+h)(x) &= (f+g)(x)+h(x) && \text{definiton of } ((f+g)+h)(x) \\
&= (f(x)+g(x))+h(x) && \text{definition of } (f+g)(x) \\
&= f(x)+(g(x)+h(x)) && \text{associative property of real number addition} \\
&= f(x)+(g+h)(x) && \text{definition of } (g+h)(x) \\
&= (f+(g+h))(x) && \text{definition of } (f+(g+h))(x)
\end{aligned}$$

The key step to the argument is that $(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$ which is true courtesy of the associative property of real number addition. And just like with real number addition, because function addition is associative, we may write $f + g + h$ instead of $(f + g) + h$ or $f + (g + h)$ even though, when it comes down to computations, we can only add two things together at a time.[4]

For completeness, we summarize the properties of function arithmetic in the theorem below. The proofs of the properties all follow along the same lines as the proof of the associative property and are left to the reader. We investigate some additional properties in the exercises.

THEOREM 1.1. Suppose f, g and h are functions.

- **Commutative Law of Addition:** $f + g = g + f$
- **Associative Law of Addition:** $(f + g) + h = f + (g + h)$
- **Additive Identity:** The function $Z(x) = 0$ satisfies: $f + Z = Z + f = f$ for all functions f.
- **Additive Inverse:** The function $F(x) = -f(x)$ for all x in the domain of f satisfies:

$$f + F = F + f = Z.$$

- **Commutative Law of Multiplication:** $fg = gf$
- **Associative Law of Multiplication:** $(fg)h = f(gh)$
- **Multiplicative Identity:** The function $I(x) = 1$ satisfies: $fI = If = f$ for all functions f.
- **Multiplicative Inverse:** If $f(x) \neq 0$ for all x in the domain of f, then $F(x) = \dfrac{1}{f(x)}$ satisfies:

$$fF = Ff = I$$

- **Distributive Law of Multiplication over Addition:** $f(g + h) = fg + fh$

In the next example, we decompose given functions into sums, differences, products and/or quotients of other functions. Note that there are infinitely many different ways to do this, including some trivial ones. For example, suppose we were instructed to decompose $f(x) = x + 2$ into a sum or difference of functions. We could write $f = g + h$ where $g(x) = x$ and $h(x) = 2$ or we could choose $g(x) = 2x + 3$ and $h(x) = -x - 1$.

[4] Addition is a 'binary' operation - meaning it is defined only on two objects at once. Even though we write $1+2+3 = 6$, mentally, we add just two of numbers together at any given time to get our answer: for example, $1 + 2 + 3 = (1 + 2) + 3 = 3 + 3 = 6$.

More simply, we could write $f = g + h$ where $g(x) = x + 2$ and $h(x) = 0$. We'll call this last decomposition a 'trivial' decomposition. Likewise, if we ask for a decomposition of $f(x) = 2x$ as a product, a nontrivial solution would be $f = gh$ where $g(x) = 2$ and $h(x) = x$ whereas a trivial solution would be $g(x) = 2x$ and $h(x) = 1$. In general, non-trivial solutions to decomposition problems avoid using the additive identity, 0, for sums and differences and the multiplicative identity, 1, for products and quotients.

EXAMPLE 1.2.2. 1. For $f(x) = x^2 - 2x$, find functions g, h and k to decompose f nontrivially as:

(a) $f = g - h$ (b) $f = g + h$ (c) $f = gh$ (d) $f = g(h - k)$

2. For $F(t) = \dfrac{2t+1}{\sqrt{t^2-1}}$, find functions G, H and K to decompost F nontrivially as:

(a) $F = \dfrac{G}{H}$ (b) $F = GH$ (c) $F = G + H$ (d) $F = \dfrac{G+H}{K}$

Solution.

1. (a) To decompose $f = g - h$, we need functions g and h so $f(x) = (g - h)(x) = g(x) - h(x)$. Given $f(x) = x^2 - 2x$, one option is to let $g(x) = x^2$ and $h(x) = 2x$. To check, we find $(g - h)(x) = g(x) - h(x) = x^2 - 2x = f(x)$ as required. In addition to checking the formulas match up, we also need to check domains. There isn't much work here since the domains of g and h are all real numbers which combine to give the domain of f which is all real numbers.

   (b) In order to write $f = g + h$, we need $f(x) = (g + h)(x) = g(x) + h(x)$. One way to accomplish this is to write $f(x) = x^2 - 2x = x^2 + (-2x)$ and identify $g(x) = x^2$ and $h(x) = -2x$. To check, $(g + h)(x) = g(x) + h(x) = x^2 - 2x = f(x)$. Again, the domains for both g and h are all real numbers which combine to give f its domain of all real numbers.

   (c) To write $f = gh$, we require $f(x) = (gh)(x) = g(x)h(x)$. In other words, we need to factor $f(x)$. We find $f(x) = x^2 - 2x = x(x - 2)$, so one choice is to select $g(x) = x$ and $h(x) = x - 2$. Then $(gh)(x) = g(x)h(x) = x(x - 2) = x^2 - 2x = f(x)$, as required. As above, the domains of g and h are all real numbers which combine to give f the correct domain of $(-\infty, \infty)$.

   (d) We need to be careful here interpreting the equation $f = g(h-k)$. What we have is an equality of *functions* so the parentheses here *do not* represent function notation here, but, rather function *multiplication*. The way to parse $g(h-k)$, then, is the function g *times* the function $h-k$. Hence, we seek functions g, h, and k so that $f(x) = [g(h-k)](x) = g(x)[(h-k)(x)] = g(x)(h(x) - k(x))$. From the previous example, we know we can rewrite $f(x) = x(x - 2)$, so one option is to set $g(x) = h(x) = x$ and $k(x) = 2$ so that $[g(h - k)](x) = g(x)[(h - k)(x)] = g(x)(h(x) - k(x)) = x(x - 2) = x^2 - 2x = f(x)$, as required. As above, the domain of all constituent functions is $(-\infty, \infty)$ which matches the domain of f.

2. (a) To write $F = \frac{G}{H}$, we need $G(t)$ and $H(t)$ so $F(t) = \left(\frac{G}{H}\right)(t) = \frac{G(t)}{H(t)}$. We choose $G(t) = 2t + 1$ and $H(t) = \sqrt{t^2-1}$. Sure enough, $\left(\frac{G}{H}\right)(t) = \frac{G(t)}{H(t)} = \frac{2t+1}{\sqrt{t^2-1}} = F(t)$ as required. When it comes to the domain of F, owing to the square root, we require $t^2 - 1 \geq 0$. Since we have a denominator as well, we require $\sqrt{t^2-1} \neq 0$. The former requirement is the same restriction on H, and the

latter requirement comes from Definition 1.1. Starting with the domain of G, all real numbers, and working through the details, we arrive at the correct domain of F, $(-\infty, -1) \cup (1, \infty)$.

(b) Next, we are asked to find functions G and H so $F(t) = (GH)(t) = G(t)H(t)$. This means we need to rewrite the expression for $F(t)$ as a product. One way to do this is to convert radical notation to exponent notation:

$$F(t) = \frac{2t+1}{\sqrt{t^2-1}} = \frac{2t+1}{\left(t^2-1\right)^{\frac{1}{2}}} = (2t+1)\left(t^2-1\right)^{-\frac{1}{2}}.$$

Choosing $G(t) = 2t+1$ and $H(t) = \left(t^2-1\right)^{-\frac{1}{2}}$, we see $(GH)(t) = G(t)H(t) = (2t+1)\left(t^2-1\right)^{-\frac{1}{2}}$ as required. The domain restrictions on F stem from the presence of the square root in the denominator - both are addressed when finding the domain of H. Hence, we obtain the correct domain of F.

(c) To express F as a sum of functions G and H, we could rewrite

$$F(t) = \frac{2t+1}{\sqrt{t^2-1}} = \frac{2t}{\sqrt{t^2-1}} + \frac{1}{\sqrt{t^2-1}},$$

so that $G(t) = \frac{2t}{\sqrt{t^2-1}}$ and $H(t) = \frac{1}{\sqrt{t^2-1}}$. Indeed, $(G+H)(t) = G(t) + H(t) = \frac{2t}{\sqrt{t^2-1}} + \frac{1}{\sqrt{t^2-1}} = \frac{2t+1}{\sqrt{t^2-1}} = F(t)$, as required. Moreover, the domain restrictions for F are the same for both G and H, so we get agreement on the domain, as required.

(d) Last, but not least, to write $F = \frac{G+H}{K}$, we require $F(t) = \left(\frac{G+H}{K}\right)(t) = \frac{(G+H)(t)}{K(t)} = \frac{G(t)+H(t)}{K(t)}$. Identifying $G(t) = 2t$, $H(t) = 1$, and $K(t) = \sqrt{t^2-1}$, we get

$$\left(\frac{G+H}{K}\right)(t) = \frac{(G+H)(t)}{K(t)} = \frac{G(t)+H(t)}{K(t)} = \frac{2t+1}{\sqrt{t^2-1}} = F(t).$$

Concerning domains, the domain of both G and H are all real numbers, but the domain of K is restricted to $t^2 - 1 \geq 0$. Coupled with the restriction stated in Definition 1.1 that $K(t) \neq 0$, we recover the domain of F, $(-\infty, -1) \cup (1, \infty)$. □

1.2.1 The Arithmetic of Change

Recall the **average rate of change** of a function over the interval $[a, b]$ is the slope between the two points $(a, f(a))$ and $(b, f(b))$ and is given by

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(b)-f(a)}{b-a}.$$

For the purposes of this section, consider a function f defined over an interval containing x and $x + \Delta x$ where $\Delta x \neq 0$. The average rate of change of f over the interval $[x, x + \Delta x]$ is thus given by the formula:[5]

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}, \quad \Delta x \neq 0.$$

[5]assuming $\Delta x > 0$; otherwise, we the interval is $[x + \Delta x, x]$. We get the same formula for the difference quotient either way.

Our aim in this section is to develop formulas which relate the rate of change of arithmetic combinations of functions to the rates of change of the constituent functions. Our first step is to study the **difference operator** 'Δ' and study how it works with the standard arithmetic operations.

In general, if u is some quantity which assumes two values in a particular order, say u_1 (the 'first' or 'initial' value) and u_2 (the 'second' or 'final' value), then $\Delta u = u_2 - u_1$. For example, if u represents the temperature of an object before (u_1) heat is applied and after (u_2) heat is applied, $\Delta u = u_2 - u_1$ represents the increase in temperature of the object.

In the context of functions and rates of change, u is the function f defined on the interval $[x, x + \Delta x]$ with $u_1 = f(x)$ and $u_2 = f(x + \Delta x)$. Here, $\Delta u = u_2 - u_1 = f(x + \Delta x) - f(x) = \Delta[f(x)]$.

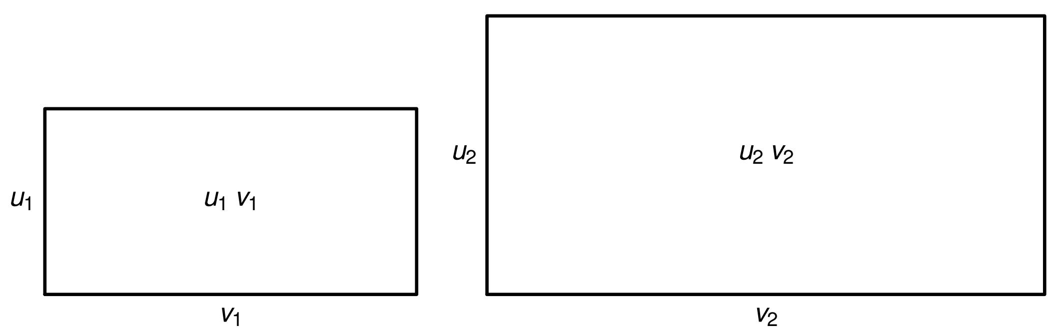
Suppose we have two quantities, u and v with $\Delta u = u_2 - u_1$ and $\Delta v = v_2 - v_1$. What do we mean by $\Delta[u + v]$? The initial value of the sum $u + v$ would be the sum of the initial values $u_1 + v_1$. Likewise, the final value of the sum would be the sum of the final values $u_2 + v_2$. Hence:

$$\begin{aligned} \Delta[u+v] &= (u_2 + v_2) - (u_1 + v_1) \\ &= u_2 + v_2 - u_1 - v_1 \\ &= (u_2 - u_1) + (v_2 - v_1) \\ &= \Delta u + \Delta v \end{aligned}$$

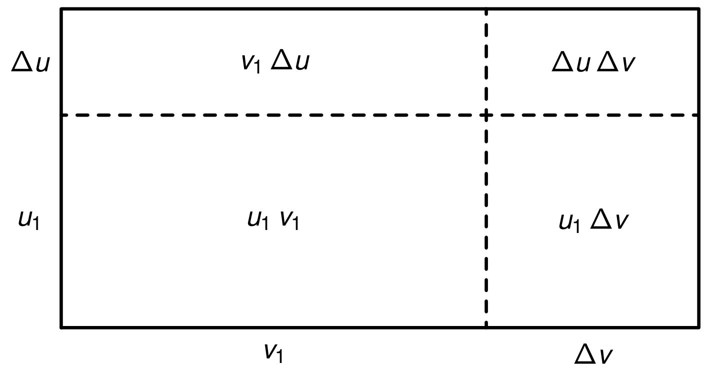
A similar calculation gives $\Delta[u - v] = \Delta u - \Delta v$.

Let's turn our attention to products. We have $\Delta[uv] = u_2 v_2 - u_1 v_1$. We'd like to express $\Delta[uv]$ in terms of Δu and Δv and there seems to be no way obvious way to do that. We take to some geometric reasoning for inspiration. Let's assume the all the quantities we're working with are positive.

We imagine the product $u_1 v_1$ as being the area of a rectangle with width u_1 and length v_1. Likewise, the product $u_2 v_2$ is the area of a (larger) rectangle with width u_2 and length v_2.



From $\Delta u = u_2 - u_1$, we get $u_2 = u_1 + \Delta u$ and, likewise, $v_2 = v_1 + \Delta v$. Doing so allows us to decompose the larger rectangle into four smaller rectangles.



Using this schematic, we see the area $u_2\,v_2$ is the sum of the areas of four smaller rectangles:

$$u_2\,v_2 = u_1\,v_1 + v_1\,\Delta u + u_1\,\Delta v + \Delta u\,\Delta v$$

Hence, $\Delta[uv] = u_2\,v_2 - u_1\,v_1 = v_1\,\Delta u + u_1\,\Delta v + \Delta u\,\Delta v$.

To prove this formula holds in general, we can substitute $u_2 = u_1 + \Delta u$ and $v_2 = v_1 + \Delta v$ into $\Delta[uv] = u_2\,v_2 - u_1\,v_1$ and simplify. We leave the details to the reader.[6]

Next, we turn our attention to quotients. We begin with: $\Delta\left[\frac{u}{v}\right] = \frac{u_2}{v_2} - \frac{u_1}{v_1}$.

Instead of appealing to geometric reasoning here,[7] we take a cue from the previous discussion and substitute $u_2 = u_1 + \Delta u$ and $v_2 = v_1 + \Delta v$ and set about getting a common denominator:

$$\begin{aligned}
\Delta\left[\frac{u}{v}\right] &= \frac{u_2}{v_2} - \frac{u_1}{v_1} \\
&= \frac{u_1 + \Delta u}{v_1 + \Delta v} - \frac{u_1}{v_1} \\
&= \frac{(u_1 + \Delta u)\,v_1}{(v_1 + \Delta v)\,v_1} - \frac{u_1\,(v_1 + \Delta v)}{v_1\,(v_1 + \Delta v)} \\
&= \frac{u_1\,v_1 + v_1\,\Delta u - u_1\,v_1 - u_1\,\Delta v}{v_1\,(v_1 + \Delta v)} \\
&= \frac{v_1\,\Delta u - u_1\,\Delta v}{v_1\,(v_1 + \Delta v_1)}
\end{aligned}$$

We summarize these results in the following theorem.

[6] Why not do this from the start, then? Carl trained as a geometric topologist.
[7] If you come up with or know of a nice geometric argument and don't mind sharing, feel free to contact Carl.

THEOREM 1.2. Suppose $\Delta u = u_2 - u_1$ and $\Delta v = v_2 - v_1$:

- **The Sum Rule for Change:** $\Delta[u+v] = \Delta u + \Delta v$
- **The Difference Rule for Change:** $\Delta[u-v] = \Delta u - \Delta v$
- **The Product Rule for Change:** $\Delta[uv] = v_1\,\Delta u + u_1\,\Delta v + \Delta u\,\Delta v$
- **The Quotient Rule for Change:** $\Delta\left[\dfrac{u}{v}\right] = \dfrac{v_1\,\Delta u - u_1\,\Delta v}{v_1\,(v_1 + \Delta v_1)}$

In the following example, we use the Quotient Rule for Change to help approximate the **propagated error** when using measured quantities (with associated uncertainties) in calculations.[8]

EXAMPLE 1.2.3. The density of a substance, ρ, is calculated by dividing its mass, m, by its volume, V: $\rho = \frac{m}{V}$. A scientist collects 5 milliliters (mL) of a substance and determines its mass to be 68.2 grams (g). She computes the density as: $\rho = \frac{68.2\,\mathrm{g}}{5\,\mathrm{mL}} = 13.64\,\frac{\mathrm{g}}{\mathrm{mL}}$.

Since every measurement in the lab has an associated uncertainty, she notes the pipet she used to measure the volume has an uncertainty of ± 0.125 mL and the balance she used to mass the substance has an uncertainty of ± 0.01 g. This means the actual volume measurement can be anywhere from as low as $5 - 0.125 = 4.875$ mL and as high as $5 + 0.125 = 5.125$ mL. Likewise, the actual mass of the substance can be anywhere from $68.2 - 0.01 = 68.19$ g to $68.2 + 0.01 = 68.21$g. Our goal is to help estimate the associated uncertainty for the density, ρ.

1. Use Theorem 1.2 to find an expression for the uncertainty in the volume $\Delta\rho$ produced as a result in the uncertainties in the measurements of mass, Δm, and volume, ΔV.

2. Calculate $\frac{\Delta\rho}{\rho}$ and interpret your answer.

Solution.

1. In this scenario, we have two quantities, the volume, V and the mass, m. We'll take V_1 and m_1 to be the measured values of volume and mass, respectively, and use the uncertainties in each of the respective measurements as ΔV and Δm. Since $\Delta\rho = \Delta\left[\frac{m}{V}\right]$, using the Quotient Rule from Theorem 1.2 gives:
$$\Delta\rho = \frac{V_1\,\Delta m - m_1\,\Delta V}{V_1(V_1 + \Delta V)}.$$

2. Substituting $V_1 = 5$ mL, $m_1 = 68.2$ g, $\Delta V = \pm 0.125$ mL and $\Delta m = \pm 0.01$ g gives:
$$\Delta\rho = \frac{(\pm 0.01\,\mathrm{g})(5\,\mathrm{mL}) - (68.2\,\mathrm{g})(\pm 0.125\,\mathrm{mL})}{(5\,\mathrm{mL})(5\,\mathrm{mL} \pm 0.125\,\mathrm{mL})}.$$

[8] The adjective 'propagated' here means than when we use measured quantities in calculations, the uncertainties in the measured quantities will produce, or 'propagate' uncertainty in the calculated quantity.

Since we have no idea the exact value of each uncertainty, we need to make a judgement call as to which of the sign values, '$\pm$' , to use. To get the largest (most conservative) answer for $\Delta\rho$, we select the $\pm$ which generate the largest numerator and smallest denominator:

$$\Delta\rho = \frac{(+0.01\,\text{g})(5\,\text{mL}) - (68.2\,\text{g})(-0.125\,\text{mL})}{(5\,\text{mL})(5\,\text{mL} - 0.125\,\text{mL})} = \frac{343\,\text{g mL}}{975\,\text{mL}^2} \approx 0.3517\,\frac{\text{g}}{\text{mL}}$$

Hence, $\frac{\Delta\rho}{\rho} \approx \frac{0.3517\,\frac{\text{g}}{\text{mL}}}{13.64\,\frac{\text{g}}{\text{mL}}} \approx 0.0258 = 2.58\%$.

We may interpret this as the uncertainties in the measurements for mass and volume in this situation could produce up to a 2.58% error in the calculated density, $\square$

In order to establish formulas for the average rate of change for functions, we substitute $f(x)$ for u_1 and $g(x)$ for v_1 and divide each of the expressions in Theorem 1.2 by Δx. For example, if we to find an expression for the average rate of change of fg in terms of f, g, and their respective average rates of change:

$$\frac{\Delta[(fg)(x)]}{\Delta x} = \frac{\Delta[f(x)]g(x) + f(x)\Delta[g(x)] + \Delta[f(x)]\Delta[g(x)]}{\Delta x} = \frac{\Delta[f(x)]}{\Delta x}\,g(x) + f(x)\,\frac{\Delta[g(x)]}{\Delta x} + \frac{\Delta[f(x)]\Delta[g(x)]}{\Delta x}.$$

Note that with the last term, we may associate the 'Δx' with either of the factors in the numerator:

$$\frac{\Delta[f(x)]\Delta[g(x)]}{\Delta x} = \frac{\Delta[f(x)]}{\Delta x}\,\Delta[g(x)] = \Delta[f(x)]\,\frac{\Delta[g(x)]}{\Delta x}.$$

Either way, we've managed to express the average rate of change of the function fg in terms of the changes and rates of change of f and g.

In the result below, we abbreviate the average rate of change as 'ARoC' for convenience.

THEOREM 1.3. Suppose f and g are functions defined on an interval containing x and $x + \Delta x$, $\Delta x \neq 0$:

- **The Sum Rule for ARoC:** $\text{ARoC}[(f+g)(x)] = \text{ARoC}[f(x)] + \text{ARoC}[g(x)]$
- **The Difference Rule for ARoC:** $\text{ARoC}[(f-g)(x)] = \text{ARoC}[f(x)] - \text{ARoC}[g(x)]$
- **The Product Rule for ARoC:**

$$\begin{aligned} \text{ARoC}[(fg)(x)] &= \text{ARoC}[f(x)]\,g(x) + f(x)\,\text{ARoC}[g(x)] + \text{ARoC}[f(x)]\,g(x) \\ &= \text{ARoC}[f(x)]\,g(x) + f(x)\,\text{ARoC}[g(x)] + f(x)\,\text{ARoC}[g(x)] \end{aligned}$$

- **The Quotient Rule for ARoC:**

$$\text{ARoC}\left[\left(\frac{f}{g}\right)(x)\right] = \frac{\text{ARoC}[f(x)]\,g(x) - f(x)\,\text{ARoC}[g(x)]}{g(x)(g(x) + \Delta[g(x)])}$$

Our final example revisits the scenario in Exercise **??** in Section **??**.

EXAMPLE 1.2.4. The cost to produce x 'PortaBoy' handheld game systems is given by: $C(x) = .03x^3 - 4.5x^2 + 225x + 250$ and the revenue generated by selling x of the systems is given by $R(x) = -1.5x^2 + 250x$.

1. Find and interpret the average rate of change of C and R over the interval $[70, 71]$.
2. Use Theorem 1.3 to determine the average rate of change of the profit function, P over the interval $[70, 71]$ using your answers to part 1. What does your answer suggest?

Solution.

1. We find $\text{ARoC}[C(x)] = \frac{C(71)-C(70)}{71-70} = \frac{4277.83-4240}{1} = 37.83$. This means that as we move from producing 70 to 71 PortaBoy systems the cost will increase by \$37.83 per system.

   For revenue, we find $\text{ARoC}[R(x)] = \frac{R(71)-R(70)}{71-70} = \frac{10188.5-10150}{1} = 38.5$. This means that as we move from selling 70 to 71 PortaBoy systems, the revenue generated will increase by \$38.5 per system.

2. Since $P(x) = R(x) - C(x)$, the Difference Rule of Theorem 1.3 gives $\text{ARoC}[P(x)] = \text{ARoC}[R(x)] - \text{ARoC}[C(x)]$. In this case, we'd get $\text{ARoC}[P(x)] = 38.5 - 37.83 = 0.67$. This means as we move from producing and selling 70 to 71 PortaBoy systems, the profit generated will increase by just 67 cents per system. At this point, the increase in revenue is nearly balanced out by the increase in cost. Since costs typically continue to rise as the number of items is produced while the revenue falls as we try to sell more items,[9] we are likely near a maximum point with the profit. A quick check of the graph on Desmos confirms our suspicions.

□

Note that in Example 1.2.4, since $\Delta x = 1$, the average rate of change for the cost moving from producing 70 to 71 systems, $\frac{C(71)-C(70)}{71-70}$ is the same numerical value as the additional cost incurred by producing the 71st system, $C(71) - C(70)$. The same goes for the revenue and profit calculations. This is the concept of **marginal analysis** and will be explored further in the Exercises.

[9] we've seen this before: to sell more, we lower the price which, in turn, lowers revenue . . .

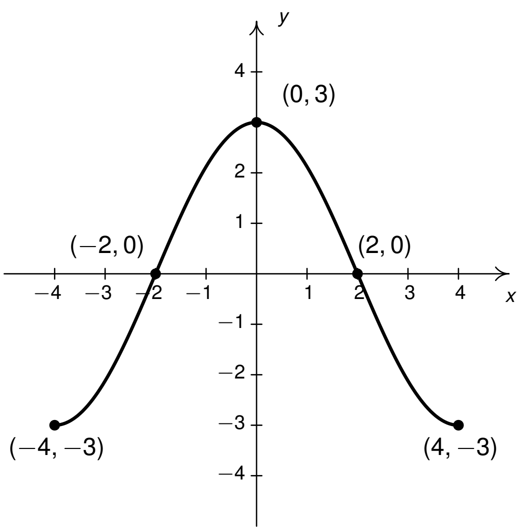
1.2.2 Exercises

In Exercises 1 - 10, use the pair of functions f and g to find the following values if they exist.

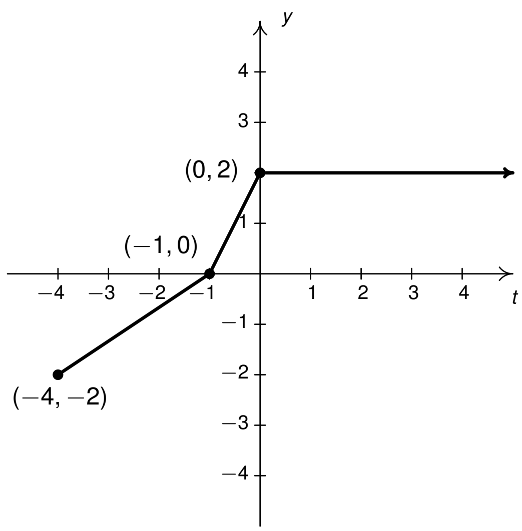
- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x+1$ and $g(t) = 4-t$
2. $f(x) = x^2$ and $g(t) = -2t+1$
3. $f(x) = x^2 - x$ and $g(t) = 12 - t^2$
4. $f(x) = 2x^3$ and $g(t) = -t^2 - 2t - 3$
5. $f(x) = \sqrt{x+3}$ and $g(t) = 2t-1$
6. $f(x) = \sqrt{4-x}$ and $g(t) = \sqrt{t+2}$
7. $f(x) = 2x$ and $g(t) = \dfrac{1}{2t+1}$
8. $f(x) = x^2$ and $g(t) = \dfrac{3}{2t-3}$
9. $f(x) = x^2$ and $g(t) = \dfrac{1}{t^2}$
10. $f(x) = x^2+1$ and $g(t) = \dfrac{1}{t^2+1}$

Exercises 11 - 20 refer to the functions f and g whose graphs are below.



$y = f(x)$



$y = g(t)$

11. $(f+g)(-4)$
12. $(f+g)(0)$
13. $(f-g)(4)$
14. $(fg)(-4)$
15. $(fg)(-2)$
16. $(fg)(4)$
17. $\left(\dfrac{f}{g}\right)(0)$
18. $\left(\dfrac{f}{g}\right)(2)$
19. $\left(\dfrac{g}{f}\right)(-1)$
20. Find the domains of $f+g$, $f-g$, fg, $\dfrac{f}{g}$ and $\dfrac{g}{f}$.

In Exercises 21 - 32, let f be the function defined by

$$f = \{(-3, 4), (-2, 2), (-1, 0), (0, 1), (1, 3), (2, 4), (3, -1)\}$$

and let g be the function defined by

$$g = \{(-3, -2), (-2, 0), (-1, -4), (0, 0), (1, -3), (2, 1), (3, 2)\}$$

Compute the indicated value if it exists.

21. $(f + g)(-3)$
22. $(f - g)(2)$
23. $(fg)(-1)$
24. $(g + f)(1)$
25. $(g - f)(3)$
26. $(gf)(-3)$
27. $\left(\frac{f}{g}\right)(-2)$
28. $\left(\frac{f}{g}\right)(-1)$
29. $\left(\frac{f}{g}\right)(2)$
30. $\left(\frac{g}{f}\right)(-1)$
31. $\left(\frac{g}{f}\right)(3)$
32. $\left(\frac{g}{f}\right)(-3)$

In Exercises 33 - 42, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f + g)(x)$
- $(f - g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

33. $f(x) = 2x + 1$ and $g(x) = x - 2$
34. $f(x) = 1 - 4x$ and $g(x) = 2x - 1$
35. $f(x) = x^2$ and $g(x) = 3x - 1$
36. $f(x) = x^2 - x$ and $g(x) = 7x$
37. $f(x) = x^2 - 4$ and $g(x) = 3x + 6$
38. $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$
39. $f(x) = \dfrac{x}{2}$ and $g(x) = \dfrac{2}{x}$
40. $f(x) = x - 1$ and $g(x) = \dfrac{1}{x - 1}$
41. $f(x) = x$ and $g(x) = \sqrt{x + 1}$
42. $f(x) = \sqrt{x - 5}$ and $g(x) = f(x) = \sqrt{x - 5}$

In Exercises 43 - 47, write the given function as a nontrivial decomposition of functions as directed.

43. For $p(z) = 4z - z^3$, find functions f and g so that $p = f - g$.
44. For $p(z) = 4z - z^3$, find functions f and g so that $p = f + g$.
45. For $g(t) = 3t|2t - 1|$, find functions f and h so that $g = fh$.
46. For $r(x) = \dfrac{3 - x}{x + 1}$, find functions f and g so $r = \dfrac{f}{g}$.
47. For $r(x) = \dfrac{3 - x}{x + 1}$, find functions f and g so $r = fg$.

48. Can $f(x) = x$ be decomposed as $f = g - h$ where $g(x) = x + \dfrac{1}{x}$ and $h(x) = \dfrac{1}{x}$?

49. Discuss with your classmates how to phrase the quantities revenue and profit in Definition **??** terms of function arithmetic as defined in Definition 1.1.

50. In this exercise, we explore decomposing a function into its positive and negative parts. Given a function f, we define the **positive part** of f, denoted f_+ and **negative part** of f, denoted f_- by:

$$f_+(x) = \frac{f(x) + |f(x)|}{2}, \qquad \text{and} \qquad f_-(x) = \frac{f(x) - |f(x)|}{2}.$$

  (a) Using a graphing utility, graph each of the functions f below along with f_+ and f_-.

  - $f(x) = x - 3$
  - $f(x) = x^2 - x - 6$
  - $f(x) = 4x - x^3$

  Why is f_+ called the 'positive part' of f and f_- called the 'negative part' of f?

  (b) Show that $f = f_+ + f_-$.

  (c) Use Definition **??** to rewrite the expressions for $f_+(x)$ and $f_-(x)$ as piecewise defined functions.

51. Let U be the unit step function defined in Exercise **??** in Section **??**. For each function $f(t)$ below:

  - Write $(Uf)(t)$ as a piecewise-defined function.
  - Graph $y = f(t)$ and $y = (Uf)(t)$.

  (a) $f(t) = t - 3$ (b) $f(t) = |t + 2|$ (c) $f(t) = (t - 1)^2$

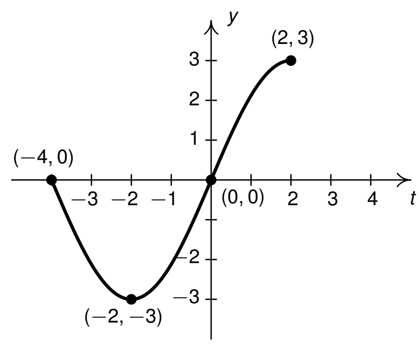
  (d) $f(t) = (t + 1)^{-1}$ (e) $f(t) = \sqrt[3]{t - 1}$ (f) $f(t) = (t - 2)^{\frac{2}{3}}$

  (g) Write a general formula for $(Uf)(t)$ for a function f. (Assume the domain of f is $(-\infty, \infty)$.)

  (h) Explain how to obtain the graph of $y = (Uf)(t)$ from $y = f(t)$.

  (i) The function $U(t)$ is used to model a change in state from 'off' to 'on' (like flipping a light switch.) How does this relate to your observations?

  (j) Use the graph of $y = f(t)$ below to graph $y = (Uf)(t)$.



$y = f(t)$

52. Use Example 1.2.3 as a guide to help find the following uncertainties.

    (a) A chemist combines the solutions from two graduated cylinders into a beaker. The volume of the first solution, V_1 is read as 101 ± 0.5 milliliters (mL). The volume of the second solution, V_2 is measured to be 16 ± 0.5 mL. Estimate the percent propagated error in calculating the volume of the combined solution as $V = V_1 + V_2 = 101 + 16 = 117$ mL.

    (b) A student measures the length, ℓ, and width, w, of a piece of paper. They find $\ell = 280 \pm 0.5$ millimeters (mm) $w = 216 \pm 0.5$ mm. Estimate the percent propagated error in calculating the area of the piece of paper as $A = \ell\, w = 280 \times 216 = 60480\,\mathrm{mm}^2$.

    (c) An airplane passenger observes a car travel a distance $d = 1320 \pm 2$ feet (ft) in time $t = 15 \pm 0.5$ seconds (s). Estimate the percent propagated error in calculating the speed of the car as $v = \frac{d}{t} = \frac{1320}{15} = 88\,\frac{\mathrm{ft}}{\mathrm{s}}$.

1.2.3 Answers

1. For $f(x) = 3x + 1$ and $g(x) = 4 - x$

   - $(f+g)(2) = 9$
   - $(f-g)(-1) = -7$
   - $(g-f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = \frac{35}{4}$
   - $\left(\frac{f}{g}\right)(0) = \frac{1}{4}$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{6}{5}$

2. For $f(x) = x^2$ and $g(x) = -2x + 1$

   - $(f+g)(2) = 1$
   - $(f-g)(-1) = -2$
   - $(g-f)(1) = -2$
   - $(fg)\left(\frac{1}{2}\right) = 0$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{5}{4}$

3. For $f(x) = x^2 - x$ and $g(x) = 12 - x^2$

   - $(f+g)(2) = 10$
   - $(f-g)(-1) = -9$
   - $(g-f)(1) = 11$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{47}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{4}{3}$

4. For $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$

   - $(f+g)(2) = 5$
   - $(f-g)(-1) = 0$
   - $(g-f)(1) = -8$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{17}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{3}{16}$

5. For $f(x) = \sqrt{x+3}$ and $g(x) = 2x - 1$

   - $(f+g)(2) = 3 + \sqrt{5}$
   - $(f-g)(-1) = 3 + \sqrt{2}$
   - $(g-f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = 0$
   - $\left(\frac{f}{g}\right)(0) = -\sqrt{3}$
   - $\left(\frac{g}{f}\right)(-2) = -5$

6. For $f(x) = \sqrt{4-x}$ and $g(x) = \sqrt{x+2}$

   - $(f+g)(2) = 2 + \sqrt{2}$
   - $(f-g)(-1) = -1 + \sqrt{5}$
   - $(g-f)(1) = 0$
   - $(fg)\left(\frac{1}{2}\right) = \frac{\sqrt{35}}{2}$
   - $\left(\frac{f}{g}\right)(0) = \sqrt{2}$
   - $\left(\frac{g}{f}\right)(-2) = 0$

7. For $f(x) = 2x$ and $g(x) = \frac{1}{2x+1}$

   - $(f+g)(2) = \frac{21}{5}$
   - $(f-g)(-1) = -1$
   - $(g-f)(1) = -\frac{5}{3}$
   - $(fg)\left(\frac{1}{2}\right) = \frac{1}{2}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$

8. For $f(x) = x^2$ and $g(x) = \frac{3}{2x-3}$

   - $(f+g)(2) = 7$
   - $(f-g)(-1) = \frac{8}{5}$
   - $(g-f)(1) = -4$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{3}{8}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{3}{28}$

9. For $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$

   - $(f+g)(2) = \frac{17}{4}$
   - $(f-g)(-1) = 0$
   - $(g-f)(1) = 0$
   - $(fg)\left(\frac{1}{2}\right) = 1$
   - $\left(\frac{f}{g}\right)(0)$ is undefined.
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$

10. For $f(x) = x^2 + 1$ and $g(x) = \frac{1}{x^2+1}$

   - $(f+g)(2) = \frac{26}{5}$
   - $(f-g)(-1) = \frac{3}{2}$
   - $(g-f)(1) = -\frac{3}{2}$
   - $(fg)\left(\frac{1}{2}\right) = 1$
   - $\left(\frac{f}{g}\right)(0) = 1$
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{25}$

11. $(f+g)(-4) = -5$
12. $(f+g)(0) = 5$
13. $(f-g)(4) = -5$
14. $(fg)(-4) = 6$
15. $(fg)(-2) = 0$
16. $(fg)(4) = -6$
17. $\left(\dfrac{f}{g}\right)(0) = \dfrac{3}{2}$
18. $\left(\dfrac{f}{g}\right)(2) = 0$
19. $\left(\dfrac{g}{f}\right)(-1) = 0$
20. The domains of $f+g$, $f-g$ and fg are all $[-4, 4]$. The domain of $\frac{f}{g}$ is $[-4, -1) \cup (-1, 4]$ and the domain of $\frac{g}{f}$ is $[-4, -2) \cup (-2, 2) \cup (2, 4]$.
21. $(f+g)(-3) = 2$
22. $(f-g)(2) = 3$
23. $(fg)(-1) = 0$
24. $(g+f)(1) = 0$
25. $(g-f)(3) = 3$
26. $(gf)(-3) = -8$
27. $\left(\frac{f}{g}\right)(-2)$ does not exist
28. $\left(\frac{f}{g}\right)(-1) = 0$
29. $\left(\frac{f}{g}\right)(2) = 4$
30. $\left(\frac{g}{f}\right)(-1)$ does not exist
31. $\left(\frac{g}{f}\right)(3) = -2$
32. $\left(\frac{g}{f}\right)(-3) = -\frac{1}{2}$

33. For $f(x) = 2x + 1$ and $g(x) = x - 2$

- $(f + g)(x) = 3x - 1$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x + 3$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 2x^2 - 3x - 2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}$
  Domain: $(-\infty, 2) \cup (2, \infty)$

34. For $f(x) = 1 - 4x$ and $g(x) = 2x - 1$

- $(f + g)(x) = -2x$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = 2 - 6x$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = -8x^2 + 6x - 1$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{1-4x}{2x-1}$
  Domain: $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

35. For $f(x) = x^2$ and $g(x) = 3x - 1$

- $(f + g)(x) = x^2 + 3x - 1$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 3x + 1$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3 - x^2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}$
  Domain: $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

36. For $f(x) = x^2 - x$ and $g(x) = 7x$

- $(f + g)(x) = x^2 + 6x$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 8x$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 7x^3 - 7x^2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-1}{7}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

37. For $f(x) = x^2 - 4$ and $g(x) = 3x + 6$

- $(f + g)(x) = x^2 + 3x + 2$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 3x - 10$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3 + 6x^2 - 12x - 24$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-2}{3}$
  Domain: $(-\infty, -2) \cup (-2, \infty)$

38. For $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$

- $(f + g)(x) = x - 3$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = -2x^2 + x + 15$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = -x^4 + x^3 + 15x^2 - 9x - 54$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = -\frac{x+2}{x+3}$
  Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

39. For $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$

- $(f + g)(x) = \frac{x^2+4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $(f - g)(x) = \frac{x^2-4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $(fg)(x) = 1$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

40. For $f(x) = x - 1$ and $g(x) = \frac{1}{x-1}$

- $(f + g)(x) = \frac{x^2-2x+2}{x-1}$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $(f - g)(x) = \frac{x^2-2x}{x-1}$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $(fg)(x) = 1$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $\left(\frac{f}{g}\right)(x) = x^2 - 2x + 1$
  Domain: $(-\infty, 1) \cup (1, \infty)$

41. For $f(x) = x$ and $g(x) = \sqrt{x + 1}$

- $(f + g)(x) = x + \sqrt{x + 1}$
  Domain: $[-1, \infty)$
- $(f - g)(x) = x - \sqrt{x + 1}$
  Domain: $[-1, \infty)$
- $(fg)(x) = x\sqrt{x + 1}$
  Domain: $[-1, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}}$
  Domain: $(-1, \infty)$

42. For $f(x) = \sqrt{x - 5}$ and $g(x) = f(x) = \sqrt{x - 5}$

- $(f + g)(x) = 2\sqrt{x - 5}$
  Domain: $[5, \infty)$
- $(f - g)(x) = 0$
  Domain: $[5, \infty)$
- $(fg)(x) = x - 5$
  Domain: $[5, \infty)$
- $\left(\frac{f}{g}\right)(x) = 1$
  Domain: $(5, \infty)$

43. One solution is $f(z) = 4z$ and $g(z) = z^3$.

44. One solution is $f(z) = 4z$ and $g(z) = -z^3$.

45. One solution is $f(t) = 3t$ and $h(t) = |2t - 1|$

46. One solution is $f(x) = 3 - x$ and $g(x) = x + 1$.

47. One solution is $f(x) = 3 - x$ and $g(x) = (x + 1)^{-1}$.

48. No. The equivalence does not hold when $x = 0$.

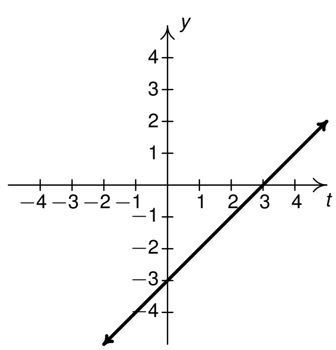
50. (b) $(f_+ + f_-)(x) = f_+(x) + f_-(x) = \dfrac{f(x) + |f(x)|}{2} + \dfrac{f(x) - |f(x)|}{2} = \dfrac{2f(x)}{2} = f(x)$.

(c)

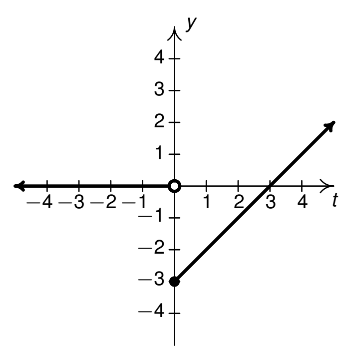
$$f_+(x) = \begin{cases} 0 & \text{if } f(x) < 0 \\ f(x) & \text{if } f(x) \geq 0 \end{cases}, \qquad f_-(x) = \begin{cases} f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geq 0 \end{cases}$$

51.

(a) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ t - 3 & \text{if } t \geq 0. \end{cases}$

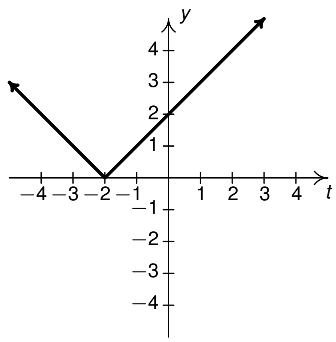


$y = f(t)$

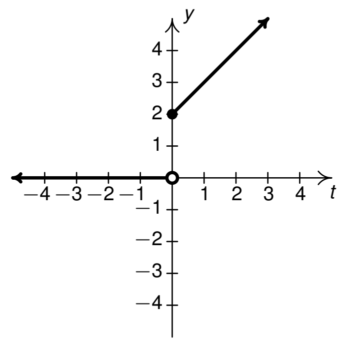


$y = (Uf)(t)$

(b) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ |t+2| = t+2 & \text{if } t \geq 0. \end{cases}$

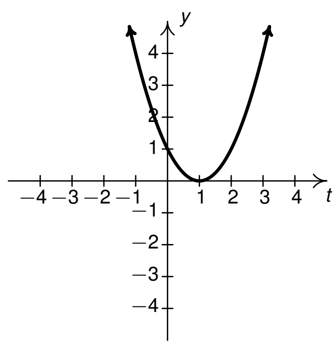


$y = f(t)$

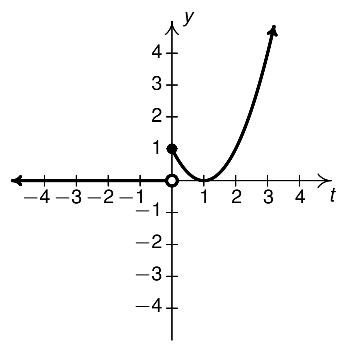


$y = (Uf)(t)$

(c) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ (t-1)^2 & \text{if } t \geq 0. \end{cases}$

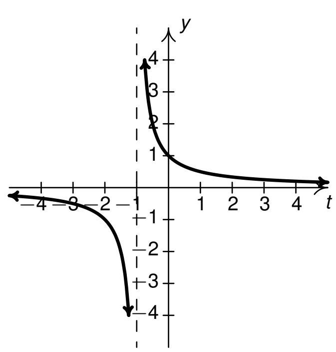


$y = f(t)$

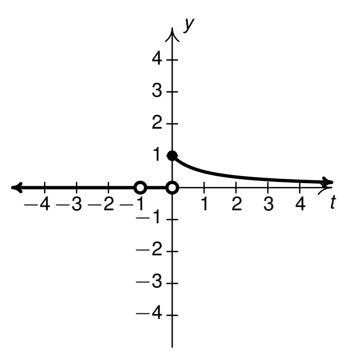


$y = (Uf)(t)$

(d) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0,\ t \neq -1 \\ (t+1)^{-1} & \text{if } t \geq 0. \end{cases}$

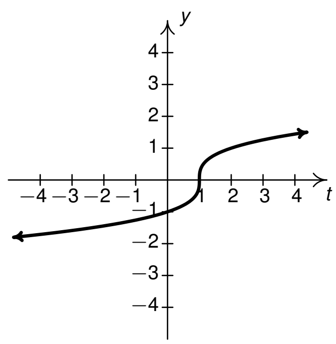


$y = f(t)$

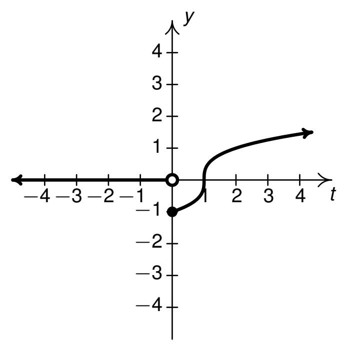


$y = (Uf)(t)$ - note the hole at $(-1, 0)$

(e) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ \sqrt[3]{t-1} & \text{if } t \geq 0. \end{cases}$

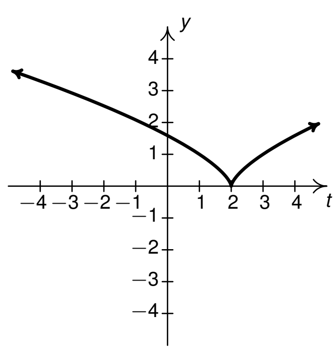


$y = f(t)$

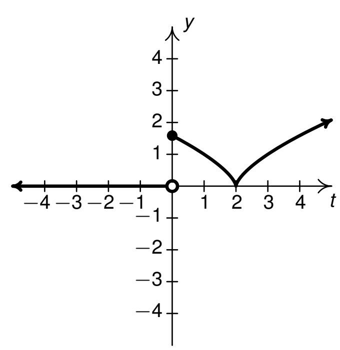


$y = (Uf)(t)$

(f) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ (t-2)^{\frac{2}{3}} & \text{if } t \geq 0. \end{cases}$



$y = f(t)$



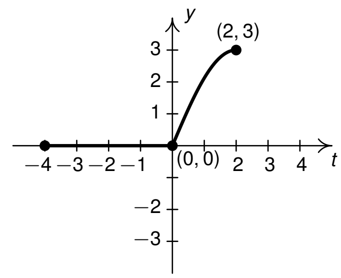
$y = (Uf)(t)$

(g) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ f(t) & \text{if } t \geq 0 \end{cases}$

(h) The graph of $(Uf)(t)$ is $y = 0$ for $t < 0$ and $y = f(t)$ for $t \geq 0$.

(i) The unit step function keeps the function ‘off’ until $t = 0$ then turns the function ‘on’ for $t \geq 0$.

(j)



$y = (Uf)(t)$

52. (a) $\Delta V = \Delta[V_1 + V_2] = \Delta V_1 + \Delta V_2 = \pm 0.5\,\text{mL} + \pm 0.5\,\text{mL} = \pm 1\text{ mL}$. $\frac{DeltaV}{V} = \pm\frac{1}{117} \approx 0.85\,\%$.

(b) $\Delta A = \Delta[\ell w] = w\Delta\ell + \ell\Delta w + \Delta ell \Delta w = (216\,\text{mm})(\pm 0.5\,\text{mm}) + (280\,\text{mm})(\pm 0.5\,\text{mm}) + (\pm 0.5\,\text{mm})(\pm 0.5\,\text{mm}) = \pm 248.26\,\text{mm}^2$. $\frac{DeltaA}{A} = \pm\frac{248.25}{60480} \approx 0.41\,\%$

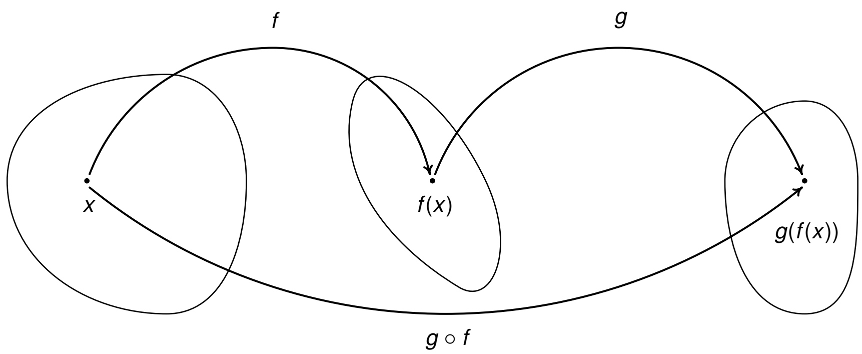
(c) $\Delta v = \Delta\left[\frac{d}{t}\right] = \frac{t\,\Delta s - s\,\Delta t}{t(t+\Delta t)} = \frac{(15\,\text{s})(\pm 2\,\text{ft}) - (1320\,\text{ft})(\pm 0.5\,\text{s})}{(15\,\text{s})(15 \pm 0.5\,\text{s})} = \pm\frac{92}{29}\,\frac{\text{ft}}{\text{s}} \approx 3.17\,\frac{\text{ft}}{\text{s}}$. $\frac{Deltav}{v} \approx \pm\frac{3.17}{88} \approx 3.60\,\%$

1.3 Function Composition

In Section 1.2, we saw how the arithmetic of real numbers carried over into an arithmetic of functions. In this section, we discuss another way to combine functions which is unique to functions and isn't shared with real numbers - function **composition**.

DEFINITION 1.2. Let f and g be functions where the real number x is in the domain of f and the real number $f(x)$ is in the domain of g. The **composite** of g with f, denoted $g \circ f$, and read 'g composed with f' is defined by the formula: $(g \circ f)(x) = g(f(x))$.

To compute $(g \circ f)(x)$, we use the formula given in Defintion 1.2: $(g \circ f)(x) = g(f(x))$. However, from a procedural viewpoint, Defintion 1.2 tells us the output from $g \circ f$ is found by taking the output from f, $f(x)$, and then making that the input to g. From this perspective, we see $g \circ f$ as a two step process taking an input x and first applying the procedure f then applying the procedure g. Abstractly, we have



In the expression $g(f(x))$, the function f is often called the 'inside' function while g is often called the 'outside' function. When evaluating composite function values we present two methods in the example below: the 'inside out' and 'outside in' methods.

EXAMPLE 1.3.1. Let $f(x) = x^2 - 4x$, $g(t) = 2 - \sqrt{t+3}$, and $h(s) = \dfrac{2s}{s+1}$.
In numbers 1 - 3, find the indicated function value.

1. $(g \circ f)(1)$
2. $(f \circ g)(1)$
3. $(g \circ g)(6)$

In numbers 4 - 10, find and simplify the indicated composite functions. State the domain of each.

4. $(g \circ f)(x)$
5. $(f \circ g)(t)$
6. $(g \circ h)(s)$
7. $(h \circ g)(t)$
8. $(h \circ h)(x)$
9. $(h \circ (g \circ f))(x)$
10. $((h \circ g) \circ f)(x)$

Solution.

1. Using Definition 1.2, $(g \circ f)(1) = g(f(1))$. Since $f(1) = (1)^2 - 4(1) = -3$ and $g(-3) = 2 - \sqrt{(-3)+3} = 2$, we have $(g \circ f)(1) = g(f(1)) = g(-3) = 2$.

2. By definition, $(f \circ g)(1) = f(g(1))$. We find $g(1) = 2 - \sqrt{1+3} = 0$, and $f(0) = (0)^2 - 4(0) = 0$, so $(f \circ g)(1) = f(g(1)) = f(0) = 0$. Comparing this with our answer to the last problem, we see that $(g \circ f)(1) \neq (f \circ g)(1)$ which tells us function composition is not commutative,[1]

3. Since $(g \circ g)(6) = g(g(6))$, we 'iterate' the process g: that is, we apply the process g to 6, then apply the process g again. We find $g(6) = 2 - \sqrt{6+3} = -1$, and $g(-1) = 2 - \sqrt{(-1)+3} = 2 - \sqrt{2}$, so $(g \circ g)(6) = g(g(6)) = g(-1) = 2 - \sqrt{2}$.

4. By definition, $(g \circ f)(x) = g(f(x))$. We now illustrate *two* ways to approach this problem.

   - *inside out*: We substitute $f(x) = x^2 - 4x$ in for t in the expression $g(t)$ and get

   $$(g \circ f)(x) = g(f(x)) = g\left(x^2 - 4x\right) = 2 - \sqrt{\left(x^2 - 4x\right) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

     Hence, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   - *outside in*: We use the formula for g first to get

   $$(g \circ f)(x) = g(f(x)) = 2 - \sqrt{f(x) + 3} = 2 - \sqrt{\left(x^2 - 4x\right) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

     We get the same answer as before, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   To find the domain of $g \circ f$, we need to find the elements in the domain of f whose outputs $f(x)$ are in the domain of g. Since the domain of f is all real numbers, we focus on finding the range elements compatible with g. Owing to the presence of the square root in the formula $g(t) = 2 - \sqrt{t+3}$ we require $t \geq -3$. Hence, we need $f(x) \geq -3$ or $x^2 - 4x \geq -3$. To solve this inequality we rewrite as $x^2 - 4x + 3 \geq 0$ and use a sign diagram. Letting $r(x) = x^2 - 4x + 3$, we find the zeros of r to be $x = 1$ and $x = 3$ and obtain

   $$\begin{array}{ccccc} (+) & 0 & (-) & 0 & (+) \\ \longleftarrow & | & \text{———} & | & \longrightarrow \\ & 1 & & 3 & \end{array}$$

   Our solution to $x^2 - 4x + 3 \geq 0$, and hence the domain of $g \circ f$, is $(-\infty, 1] \cup [3, \infty)$.

5. To find $(f \circ g)(t)$, we find $f(g(t))$.

   - *inside out*: We substitute the expression $g(t) = 2 - \sqrt{t+3}$ in for x in the formula $f(x)$ and get

   $$\begin{aligned} (f \circ g)(t) &= f(g(t)) = f\left(2 - \sqrt{t+3}\right) \\ &= \left(2 - \sqrt{t+3}\right)^2 - 4\left(2 - \sqrt{t+3}\right) \end{aligned}$$

[1] That is, in general, $g \circ f \neq f \circ g$. This shouldn't be too surprising, since, in general, the order of processes matters: adding eggs to a cake batter then baking the cake batter has a much different outcome than baking the cake batter then adding eggs.

$$\begin{aligned} &= 4 - 4\sqrt{t+3} + \left(\sqrt{t+3}\right)^2 - 8 + 4\sqrt{t+3} \\ &= 4 + t + 3 - 8 \\ &= t - 1 \end{aligned}$$

- *outside in*: We use the formula for $f(x)$ first to get

$$\begin{aligned} (f \circ g)(t) &= f(g(t)) = (g(t))^2 - 4\,(g(t)) \\ &= \left(2 - \sqrt{t+3}\right)^2 - 4\left(2 - \sqrt{t+3}\right) \\ &= t - 1 && \text{same algebra as before} \end{aligned}$$

Thus we get $(f \circ g)(t) = t - 1$. To find the domain of $f \circ g$, we look for the elements t in the domain of g whose outputs, $g(t)$ are in the domain of f. As mentioned previously, the domain of g is limited by the presence of the square root to $\{t \in \mathbb{R} \,|\, t \geq -3\}$ while the domain of f is all real numbers. Hence, the domain of $f \circ g$ is restricted only by the domain of g and is $\{t \in \mathbb{R} \,|\, t \geq -3\}$ or, using interval notation, $[-3, \infty)$. Note that as with Example 1.2.1 in Section 1.2, had we used the simplified formula for $(f \circ g)(t) = t - 1$ to determine domain, we would have arrived at the incorrect answer.

6. To find $(g \circ h)(s)$, we compute $g(h(s))$.

- *inside out*: We substitute $h(s)$ in for t in the expression $g(t)$ to get

$$\begin{aligned} (g \circ h)(s) &= g(h(s)) = g\left(\frac{2s}{s+1}\right) \\ &= 2 - \sqrt{\left(\frac{2s}{s+1}\right) + 3} \\ &= 2 - \sqrt{\frac{2s}{s+1} + \frac{3(s+1)}{s+1}} && \text{get common denominators} \\ &= 2 - \sqrt{\frac{5s+3}{s+1}} \end{aligned}$$

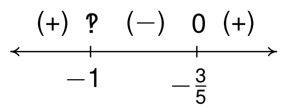
- *outside in*: We use the formula for $g(t)$ first to get

$$\begin{aligned} (g \circ h)(s) &= g(h(s)) = 2 - \sqrt{h(s) + 3} \\ &= 2 - \sqrt{\left(\frac{2s}{s+1}\right) + 3} \\ &= 2 - \sqrt{\frac{5s+3}{s+1}} && \text{get common denominators as before} \end{aligned}$$

To find the domain of $g \circ h$, we need the elements in the domain of h so that $h(s)$ is in the domain of g. Owing to the $s + 1$ in the denominator of the expression $h(s)$, we require $s \neq -1$. Once again, because of the square root in $g(t) = 2 - \sqrt{t+3}$, we need $t \geq -3$ or, in this case $h(s) \geq -3$. To use a sign diagram to solve, we rearrange this inequality:

$$\begin{aligned}
\frac{2s}{s+1} &\geq -3 \\
\frac{2s}{s+1} + 3 &\geq 0 \\
\frac{5s+3}{s+1} &\geq 0 \qquad \text{get common denominators as before}
\end{aligned}$$

Defining $r(s) = \frac{5s+3}{s+1}$, we see r is undefined at $s = -1$ (a carry over from the domain restriction of h) and $r(s) = 0$ at $s = -\frac{3}{5}$. Our sign diagram is



hence our domain is $(-\infty, -1) \cup \left[-\frac{3}{5}, \infty\right)$.

7. We find $(h \circ g)(t)$ by finding $h(g(t))$.

   - *inside out*: We substitute the expression $g(t)$ for s in the formula $h(s)$

$$\begin{aligned}
(h \circ g)(t) &= h(g(t)) = h\left(2 - \sqrt{t+3}\right) \\
&= \frac{2\left(2 - \sqrt{t+3}\right)}{\left(2 - \sqrt{t+3}\right) + 1} \\
&= \frac{4 - 2\sqrt{t+3}}{3 - \sqrt{t+3}}
\end{aligned}$$

   - *outside in*: We use the formula for $h(s)$ first to get

$$\begin{aligned}
(h \circ g)(t) &= h(g(t)) = \frac{2\,(g(t))}{(g(t)) + 1} \\
&= \frac{2\left(2 - \sqrt{t+3}\right)}{\left(2 - \sqrt{t+3}\right) + 1} \\
&= \frac{4 - 2\sqrt{t+3}}{3 - \sqrt{t+3}}
\end{aligned}$$

To find the domain of $h \circ g$, we need the elements of the domain of g so that $g(t)$ is in the domain of h. As we've seen already, for t to be in the domain of g, $t \geq -3$. For s to be in the domain of h, $s \neq -1$, so we require $g(t) \neq -1$. Hence, we solve $g(t) = 2 - \sqrt{t+3} = -1$ with the intent of excluding the solutions. Isolating the radical expression gives $\sqrt{t+3} = 3$ or $t = 6$. Sure enough, we check $g(6) = -1$ so we exclude $t = 6$ from the domain of $h \circ g$. Our final answer is $[-3, 6) \cup (6, \infty)$.

8. To find $(h \circ h)(s)$ we find $h(h(s))$:

   - *inside out*: We substitute the expression $h(s)$ for s in the expression $h(s)$ into h to get

$$
\begin{aligned}
(h \circ h)(s) &= h(h(s)) = h\left(\frac{2s}{s+1}\right) \\
&= \frac{2\left(\dfrac{2s}{s+1}\right)}{\left(\dfrac{2s}{s+1}\right)+1} \\
&= \frac{\dfrac{4s}{s+1}}{\dfrac{2s}{s+1}+1} \cdot \frac{(s+1)}{(s+1)} \\
&= \frac{\dfrac{4s}{s+1} \cdot (s+1)}{\left(\dfrac{2s}{s+1}\right) \cdot (s+1) + 1 \cdot (s+1)} \\
&= \frac{\dfrac{4s}{\cancelto{1}{(s+1)}} \cdot \cancel{(s+1)}}{\dfrac{2s}{\cancelto{1}{(s+1)}} \cdot \cancel{(s+1)} + s + 1} \\
&= \frac{4s}{3s+1}
\end{aligned}
$$

   - *outside in*: This approach yields

$$
\begin{aligned}
(h \circ h)(s) &= h(h(s)) = \frac{2(h(s))}{h(s)+1} \\
&= \frac{2\left(\dfrac{2s}{s+1}\right)}{\left(\dfrac{2s}{s+1}\right)+1} \\
&= \frac{4s}{3s+1} \qquad \text{same algebra as before}
\end{aligned}
$$

   To find the domain of $h \circ h$, we need to find the elements in the domain of h so that the outputs, $h(s)$ are also in the domain of h. The only domain restriction for h comes from the denominator: $s \neq -1$, so in addition to this, we also need $h(s) \neq -1$. To this end, we solve $h(s) = -1$ and exclude the answers. Solving $\frac{2s}{s+1} = -1$ gives $s = -\frac{1}{3}$. The domain of $h \circ h$ is $(-\infty, -1) \cup \left(-1, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, \infty\right)$.

9. The expression $(h \circ (g \circ f))(x)$ indicates that we first find the composite, $g \circ f$ and compose the function h with the result. We know from number 4 that $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$ with domain $(-\infty, 1] \cup [3, \infty)$. We now proceed as usual.

   - *inside out*: We substitute the expression $(g \circ f)(x)$ for s in the expression $h(s)$ first to get

$$\begin{aligned}
(h \circ (g \circ f))(x) &= h((g \circ f)(x)) = h\left(2 - \sqrt{x^2 - 4x + 3}\right) \\
&= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\
&= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}
\end{aligned}$$

   - *outside in*: We use the formula for $h(s)$ first to get

$$\begin{aligned}
(h \circ (g \circ f))(x) &= h((g \circ f)(x)) = \frac{2\left((g \circ f)(x)\right)}{((g \circ f)(x)) + 1} \\
&= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\
&= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}
\end{aligned}$$

   To find the domain of $h \circ (g \circ f)$, we need the domain elements of $g \circ f$, $(-\infty, 1] \cup [3, \infty)$, so that $(g \circ f)(x)$ is in the domain of h. As we've seen several times already, the only domain restriction for h is $s \neq -1$, so we set $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3} = -1$ and exclude the solutions. We get $\sqrt{x^2 - 4x + 3} = 3$, and, after squaring both sides, we have $x^2 - 4x + 3 = 9$. We solve $x^2 - 4x - 6 = 0$ using the quadratic formula and obtain $x = 2 \pm \sqrt{10}$. The reader is encouraged to check that both of these numbers satisfy the original equation, $2 - \sqrt{x^2 - 4x + 3} = -1$ and also belong to the domain of $g \circ f$, $(-\infty, 1] \cup [3, \infty)$, and so must be excluded from our final answer.[2] Our final domain for $h \circ (f \circ g)$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$.

10. The expression $((h \circ g) \circ f)(x)$ indicates that we first find the composite $h \circ g$ and then compose that with f. From number 7, we have

$$(h \circ g)(t) = \frac{4 - 2\sqrt{t + 3}}{3 - \sqrt{t + 3}}$$

    with domain $[-3, 6) \cup (6, \infty)$.

    - *inside out*: We substitute the expression $f(x)$ for t in the expression $(h \circ g)(t)$ to get

[2] We can approximate $\sqrt{10} \approx 3$ so $2 - \sqrt{10} \approx -1$ and $2 + \sqrt{10} \approx 5$.

$$\begin{aligned}((h\circ g)\circ f)(x) &= (h\circ g)(f(x)) = (h\circ g)\left(x^2-4x\right)\\ &= \frac{4-2\sqrt{\left(x^2-4x\right)+3}}{3-\sqrt{\left(x^2-4x\right)+3}}\\ &= \frac{4-2\sqrt{x^2-4x+3}}{3-\sqrt{x^2-4x+3}}\end{aligned}$$

- *outside in*: We use the formula for $(h\circ g)(t)$ first to get

$$\begin{aligned}((h\circ g)\circ f)(x) &= (h\circ g)(f(x)) = \frac{4-2\sqrt{(f(x))+3}}{3-\sqrt{f(x))+3}}\\ &= \frac{4-2\sqrt{\left(x^2-4x\right)+3}}{3-\sqrt{\left(x^2-4x\right)+3}}\\ &= \frac{4-2\sqrt{x^2-4x+3}}{3-\sqrt{x^2-4x+3}}\end{aligned}$$

Since the domain of f is all real numbers, the challenge here to find the domain of $(h\circ g)\circ f$ is to determine the values $f(x)$ which are in the domain of $h\circ g$, $[-3,6)\cup(6,\infty)$. At first glance, it appears as if we have two (or three!) inequalities to solve: $-3\leq f(x)<6$ and $f(x)>6$. Alternatively, we could solve $f(x)=x^2-4x\geq-3$ and exclude the solutions to $f(x)=x^2-4x=6$ which is not only easier from a procedural point of view, but also easier since we've already done both calculations. In number 4, we solved $x^2-4x\geq-3$ and obtained the solution $(-\infty,1]\cup[3,\infty)$ and in number 9, we solved $x^2-4x-6=0$ and obtained $x=2\pm\sqrt{10}$. Hence, the domain of $(h\circ g)\circ f$ is $(-\infty,2-\sqrt{10})\cup(2-\sqrt{10},1]\cup\left[3,2+\sqrt{10}\right)\cup\left(2+\sqrt{10},\infty\right)$. $\square$

As previously mentioned, it should be clear from Example 1.3.1 that, in general, $g\circ f\neq f\circ g$, in other words, function composition is not *commutative*. However, numbers 9 and 10 demonstrate the **associative** property of function composition. That is, when composing three (or more) functions, as long as we keep the order the same, it doesn't matter which two functions we compose first. We summarize the important properties of function composition in the theorem below.

THEOREM 1.4. **Properties of Function Composition:** Suppose f, g, and h are functions.

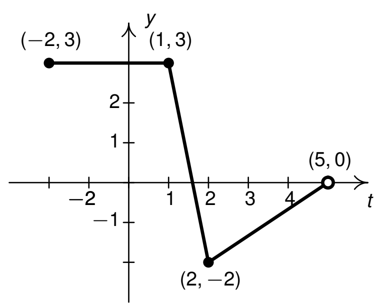
- **Associative Law of Composition:** $h\circ(g\circ f)=(h\circ g)\circ f$, provided the composite functions are defined.
- **Composition Identity:** The function $I(x)=x$ satisfies: $I\circ f=f\circ I=f$ for all functions, f.

By repeated applications of Definition 1.2, we find $(h\circ(g\circ f))(x)=h((g\circ f)(x))=h(g(f(x)))$. Similarly, $((h\circ g)\circ f)(x)=(h\circ g)(f(x))=h(g(f(x)))$. This establishes that the formulas for the two functions are the same. We leave it to the reader to think about why the domains of these two functions are identical, too. These two facts establish the equality $h \circ (g \circ f) = (h \circ g) \circ f$. A consequence of the associativity of function composition is that there is no need for parentheses when we write $h \circ g \circ f$. The second property can also be verified using Definition 1.2. Recall that the function $I(x) = x$ is called the *identity function* and was introduced in Exercise **??** in Section **??**. If we compose the function I with a function f, then we have $(I \circ f)(x) = I(f(x)) = f(x)$, and a similar computation shows $(f \circ I)(x) = f(I(x)) = f(x)$. This establishes that we have an identity for function composition much in the same way the function $I(x) = 1$ is an identity for function multiplication.

As we know, not all functions are described by formulas, and, moreover, not all functions are described by just *one* formula. The next example applies the concept of function composition to functions represented in various and sundry ways.

EXAMPLE 1.3.2. Consider the following functions:

- $f(x) = 6x - x^2$
- $g(t) \begin{cases} 2t-1 & \text{if } -1 \leq t < 3, \\ t^2 & \text{if } t \geq 3. \end{cases}$
- $h = \{(-3,1), (-2,6), (0,-2), (1,5), (3,-1)\}$
- s whose graph is given below:



$y = s(t)$

1. Find and simplify the following function values:

   (a) $(g \circ f)(2)$ (b) $(h \circ g)(-1)$ (c) $(h \circ s)(-2)$ (d) $(f \circ s)(0)$

2. Find and simplify a formula for $(g \circ f)(x)$.

3. Write $s \circ h$ as a set of ordered pairs.

Solution.

1. (a) To find $(g \circ f)(2) = g(f(2))$ we first find $f(2) = 6(2) - (2)^2 = 8$. Since $8 \geq 3$, we use the rule $g(t) = t^2$ so $g(8) = (8)^2 = 64$. Hence, $(g \circ f)(3) = g(f(3)) = g(8) = 64$.

   (b) Since $(h \circ g)(-1) = h(g(-1))$ we first need $g(-1)$. Since $-1 \leq -1 < 3$, we use the rule $g(t) = 2t - 1$ and find $g(-1) = 2(-1) - 1 = -3$. Next, we need $h(-3)$. Since $(-3,1) \in h$, we have that $h(-3) = 1$. Putting it all together, we find $(h \circ g)(-1) = h(g(-1)) = h(-3) = 1$.

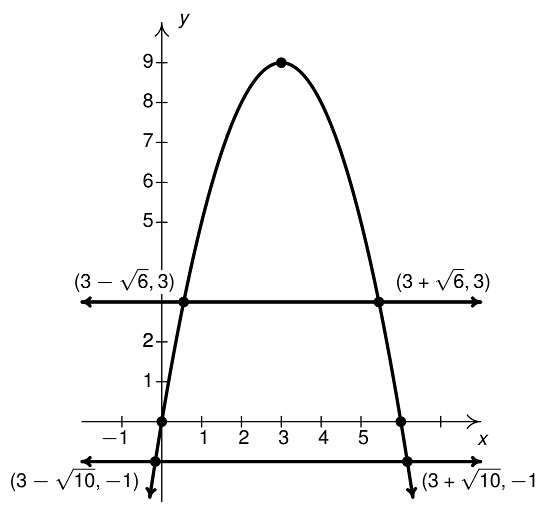
(c) To find $(h \circ s)(-2) = h(s(-2))$, we first need $s(-2)$. We see the point $(-2, 3)$ is on the graph of s, so $s(-2) = 3$. Next, we see $(3, -1) \in h$, so $h(3) = -1$. Hence, $(h \circ s)(-2) = h(s(-2)) = h(3) = -1$.

(d) To find $(f \circ s)(0) = f(s(0))$ we infer from the graph of s that it contains the point $(0, 3)$, so $s(0) = 3$. Sine $f(3) = 6(3) - (3)^2 = 9$, we have $(f \circ s)(0) = f(s(0)) = f(3) = 9$.

2. To find a formula for $(g \circ f)(x) = g(f(x))$, we substitute $f(x) = 6x - x^2$ in for t in the formula for $g(t)$:

$$(g \circ f)(x) = g(f(x)) = g(6x - x^2) = \begin{cases} 2(6x - x^2) - 1 & \text{if } -1 \leq 6x - x^2 < 3, \\ (6x - x^2)^2 & \text{if } 6x - x^2 \geq 3. \end{cases}$$

Simplifying each expression, we get $2(6x - x^2) - 1 = -2x^2 + 12x - 1$ for the first piece and $(6x - x^2)^2 = x^4 - 12x^3 + 36x^2$ for the second piece. The real challenge comes in solving the inequalities $-1 \leq 6x - x^2 < 3$ and $6x - x^2 \geq 3$. While we could solve each individually using a sign diagram, a graphical approach works best here. We graph the parabola $y = 6x - x^2$, finding the vertex is $(3, 9)$ with intercepts $(0, 0)$ and $(6, 0)$ along with the horizontal lines $y = -1$ and $y = 3$ below. We determine the intersection points by solving $6x - x^2 = -1$ and $6x - x^2 = 3$. Using the quadratic formula, we find the solutions to each equation are $x = 3 \pm \sqrt{10}$ and $x = 3 \pm \sqrt{6}$, respectively.

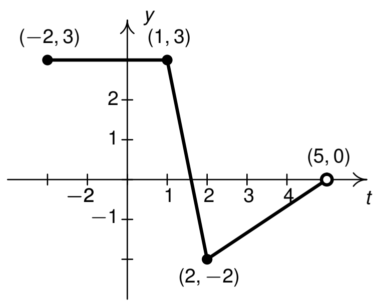


$y = 6x - x^2$, $y = -1$, $y = 3$

From the graph, we see the parabola $y = 6x - x^2$ is between the lines $y = -1$ and $y = 3$ from $x = 3 - \sqrt{10}$ to $x = 3 - \sqrt{6}$ and again from $x = 3 + \sqrt{6}$ to $x = 3 + \sqrt{10}$. Hence the solution to $-1 \leq 6x - x^2 < 3$ is $[3 - \sqrt{10}, 3 - \sqrt{6}) \cup (3 + \sqrt{6}, 3 + \sqrt{10}]$. We also note $y = 6x - x^2$ is above the line $y = 3$ for all x between $x = 3 - \sqrt{6}$ and $3 + \sqrt{6}$. Hence, the solution to $6x - x^2 \geq 3$ is $[3 - \sqrt{6}, 3 + \sqrt{6}]$. Hence,

$$(g \circ f)(x) = \begin{cases} -2x^2 + 12x - 1 & \text{if } x \in [3 - \sqrt{10}, 3 - \sqrt{6}) \cup (3 + \sqrt{6}, 3 + \sqrt{10}], \\ x^4 - 12x^3 + +36x^2 & \text{if } x \in [3 - \sqrt{6}, 3 + \sqrt{6}]. \end{cases}$$

3. Last but not least, we are tasked with representing $s \circ h$ as a set of ordered pairs. Since h is described by the discrete set of points $h = \{(-3, 1), (-2, 6), (0, -2), (1, 5), (3, -1)\}$, we find $s \circ h$ point by point. We keep the graph of s handy and construct the table below to help us organize our work.



$y = s(t)$

x	$h(x)$	$s(h(x))$
-3	1	3
-2	6	undefined
0	-2	3
1	5	undefined
3	-1	3

Since neither 6 nor 5 are in the domain of s, -2 and 1 are not in the domain of $s \circ h$. Hence, we get $s \circ h = \{(-3, 3), (0, 3), (3, 3)\}$. □

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions which our last example illustrates. As with Example 1.2.2, we want to avoid trivial decompositions, which, when it comes to function composition, are those involving the identity function $I(x) = x$ as described in Theorem 1.4.

EXAMPLE 1.3.3.

1. Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

   (a) $F(x) = |3x - 1|$ (b) $G(t) = \dfrac{2}{t^2 + 1}$ (c) $H(s) = \dfrac{\sqrt{s} + 1}{\sqrt{s} - 1}$

2. For $F(x) = \sqrt{\dfrac{2x - 1}{x^2 + 4}}$, find functions f, g, and h to decompose F nontrivially as $F = f \circ \left(\dfrac{g}{h}\right)$.

Solution. There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. (a) Our goal is to express the function F as $F = g \circ f$ for functions g and f. From Definition 1.2, we know $F(x) = g(f(x))$, and we can think of $f(x)$ as being the 'inside' function and g as being the 'outside' function. Looking at $F(x) = |3x - 1|$ from an 'inside versus outside'

perspective, we can think of $3x - 1$ being inside the absolute value symbols. Taking this cue, we define $f(x) = 3x - 1$. At this point, we have $F(x) = |f(x)|$. What is the outside function? The function which takes the absolute value of its input, $g(x) = |x|$. Sure enough, this checks: $(g \circ f)(x) = g(f(x)) = |f(x)| = |3x - 1| = F(x)$.

(b) We attack deconstructing G from an operational approach. Given an input t, the first step is to square t, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write G as a composite of *three* functions: f, g and h. Our first function, f, is the function that squares its input, $f(t) = t^2$. The next function is the function that adds 1 to its input, $g(t) = t + 1$. Our last function takes its input and divides it into 2, $h(t) = \frac{2}{t}$. The claim is that $G = h \circ g \circ f$ which checks:

$$(h \circ g \circ f)(t) = h(g(f(t))) = h(g\left(t^2\right)) = h\left(t^2 + 1\right) = \frac{2}{t^2 + 1} = G(x).$$

(c) If we look $H(s) = \frac{\sqrt{s}+1}{\sqrt{s}-1}$ with an eye towards building a complicated function from simpler functions, we see the expression $\sqrt{s}$ is a simple piece of the larger function. If we define $f(s) = \sqrt{s}$, we have $H(s) = \frac{f(s)+1}{f(s)-1}$. If we want to decompose $H = g \circ f$, then we can glean the formula for $g(s)$ by looking at what is being done to $f(s)$. We take $g(s) = \frac{s+1}{s-1}$, and check below:

$$(g \circ f)(s) = g(f(s)) = \frac{f(s) + 1}{f(s) - 1} = \frac{\sqrt{s} + 1}{\sqrt{s} - 1} = H(s).$$

$\square$

2. To write $F = f \circ \left(\frac{g}{h}\right)$ means

$$F(x) = \sqrt{\frac{2x - 1}{x^2 + 4}} = \left(f \circ \left(\frac{g}{h}\right)\right)(x) = f\left(\left(\frac{g}{h}\right)(x)\right) = f\left(\frac{g(x)}{h(x)}\right).$$

Working from the inside out, we have a rational expression with numerator $g(x)$ and denominator $h(x)$. Looking at the formula for $F(x)$, one choice is $g(x) = 2x - 1$ and $h(x) = x^2 + 4$. Making these identifications, we have

$$F(x) = \sqrt{\frac{2x - 1}{x^2 + 4}} = \sqrt{\frac{g(x)}{h(x)}}.$$

Since F takes the square root of $\frac{g(x)}{h(x)}$, the our last function f is the function that takes the square root of its input, i.e., $f(x) = \sqrt{x}$. We leave it to the reader to check that, indeed, $F = f \circ \left(\frac{g}{h}\right)$. $\square$

We close this section of a real-world application of function composition.

EXAMPLE 1.3.4. The surface area of a sphere is a function of its radius r and is given by the formula $S(r) = 4\pi r^2$. Suppose a spherical balloon being inflated so that the radius of the sphere is increasing according to the formula $r(t) = 2t$, where t is measured in minutes (min), $t \geq 0$, and r is measured in centimeters (cm). Find and interpret $(S \circ r)(t)$.

Solution. The function $S(r)$ gives the surface area of the sphere and $r(t)$ gives the radius of the sphere at a given time. Given a specific time, t, we find the radius at that time, $r(t)$ and feed that into $S(r)$ to find the surface area. Hence, the surface area S is ultimately a function of time t and we find $(S \circ r)(t) = S(r(t)) = 4\pi(r(t))^2 = 4\pi\,(2t)^2 = 16\pi t^2$. This formula allows us to compute the surface area directly given the time without going through the 'intermediary variable' r. □

1.3.1 Related Rates

In Section 1.2.1, we studied the difference operator, Δ and showed how average rates of change operate with the basic function arithmetic. In this section, we explore how rates of change of composite functions are related to the rates of change of their constituent functions. As in that section, we'll use the formulation:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0,$$

adjusting the names of functions and independent variables as needed.

As a motivational example, we revisit the scenario in Example 1.3.4.

Example 1.3.5. The surface area of a sphere is a function of its radius r and is given by the formula $S(r) = 4\pi r^2$. Suppose a spherical balloon being inflated so that the radius of the sphere is increasing according to the formula $r(t) = 2t$, where t is measured in minutes (min), $t \geq 0$, and r is measured in centimeters (cm).

1. Find and simplify an expression for the average rate of change of S with respect to r. Find and interpret the average rate of change of S withrespect to r over the interval $[1, 3]$.

2. Find, simplify, and interpret an expression for the average rate of change of r with respect to t.

3. Find and simplify an expression for the average rate of change of S with respect to t. Find and interpret the average rate of change of S with respect to t over the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$.

4. Multiply your answers to 1 and 2 and compare those to your answer in 3.

Solution. It is important to note that as we work through the expressions below, the variables r and Δr as well as t and Δt are distinct. That is, they do not combine as 'like terms.'

1. We start by simplifying $\dfrac{\Delta[S(r)]}{\Delta r} = \dfrac{S(r+\Delta r) - S(r)}{\Delta r}$:

$$\begin{aligned} \frac{\Delta[S(r)]}{\Delta r} &= \frac{S(r+\Delta r) - S(r)}{\Delta r} \\ &= \frac{4\pi(r+\Delta r)^2 - 4\pi r^2}{\Delta r} \end{aligned}$$

$$
\begin{aligned}
&= \frac{4\pi\left[r^2 + 2r\,\Delta r + (\Delta r)^2\right] - 4\pi r^2}{\Delta r} \\
&= \frac{4\pi r^2 + 8\pi r\,\Delta r + 4\pi(\Delta r)^2 - 4\pi r^2}{\Delta r} \\
&= \frac{8\pi r\,\Delta r + 4\pi(\Delta r)^2}{\Delta r} \\
&= \frac{(\Delta r)\,(8\pi r + 4\pi\,\Delta r)}{\Delta r} \\
&= \frac{\cancel{(\Delta r)}\,(8\pi r + 4\pi\,\Delta r)}{\cancelto{1}{(\Delta r)}} \\
&= 8\pi r + 4\pi\,\Delta r
\end{aligned}
$$

To find the average rate of change of S over the interval $[1, 3]$, we take $r = 1$ and $\Delta r = 3 - 1 = 2$:

$$\frac{\Delta[S(r)]}{\Delta r} = 8\pi(1) + 4\pi(2) = 16\pi.$$

This means as the radius of the balloon increases from 1 centimeter to 3 centimeters, the surface area is increasing at an average rate of $16\pi\,\frac{\text{cm}^2}{\text{cm}}$.

Note that the units here, cm, do cancel and we could write the average rate of change as 16π cm. This somewhat hides the fact this number represents a ratio. Any time area and length are measured in compatible units, the ratio of units $\frac{\text{area}}{\text{length}}$ will simplify to units of length.[3]

2. Next, we simplify $\dfrac{\Delta[r(t)]}{\Delta t} = \dfrac{r(t+\Delta t) - r(t)}{\Delta t}$:

$$
\begin{aligned}
\frac{\Delta[r(t)]}{\Delta t} &= \frac{r(t+\Delta t) - r(t)}{\Delta t} \\
&= \frac{2(t+\Delta t) - 2t}{\Delta t} \\
&= \frac{2t + 2\Delta t - 2t}{\Delta t} \\
&= \frac{2\cancel{\Delta t}}{\cancelto{1}{(\Delta t)}} \\
&= 2
\end{aligned}
$$

The fact that the average rate of change here is constant shouldn't be too surprising. $r(t) = 2t$ is a linear function whose slope, 2 is the constant rate of change.[4] This means that the radius of the balloon is increasing at a constant rate of $2\,\frac{\text{cm}}{\text{min}}$.

[3] As always, context is key!

[4] We could probably have lead with that and avoided some tedious computations ...

3. To find $\dfrac{\Delta[S(t)]}{\Delta t} = \dfrac{S(t+\Delta t) - S(r)}{\Delta t}$, we start with our answer from Example 1.3.4: $S(t) = 16\pi t^2$:

$$\begin{aligned}
\frac{\Delta[S(t)]}{\Delta t} &= \frac{S(t+\Delta t) - S(t)}{\Delta t} \\
&= \frac{16\pi(t+\Delta t)^2 - 16\pi t^2}{\Delta t} \\
&= \frac{16\pi\left[t^2 + 2t\,\Delta t + (\Delta t)^2\right] - 16\pi t^2}{\Delta t} \\
&= \frac{16\pi t^2 + 32\pi t\,\Delta t + 16\pi(\Delta t)^2 - 16\pi t^2}{\Delta t} \\
&= \frac{32\pi t\,\Delta t + 16\pi(\Delta t)^2}{\Delta t} \\
&= \frac{(\Delta t)(32\pi t + 16\pi\,\Delta t)}{\Delta t} \\
&= \frac{\cancel{(\Delta t)}(32\pi t + 16\pi\,\Delta t)}{\cancelto{1}{(\Delta t)}} \\
&= 32\pi t + 16\pi\,\Delta t
\end{aligned}$$

To find the average rate of change of S over the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$, we take $t = \frac{1}{2}$ and $\Delta t = \frac{3}{2} - \frac{1}{2} = 1$:

$$\frac{\Delta[S(t)]}{\Delta t} = 32\pi\left(\frac{1}{2}\right) + 16\pi(1) = 32\pi.$$

This means the surface area of the balloon is increasing at an average rate of $32\pi\,\frac{\text{cm}^2}{\text{min}}$ over the time span of $\frac{1}{2}$ minute (30 seconds) after the start of inflation to $\frac{3}{2}$ (90 seconds) after the start of inflation.

4. We begin with:

$$\frac{\Delta[S(r)]}{\Delta r}\cdot\frac{\Delta[r(t)]}{\Delta t} = (8\pi r + 4\pi\,\Delta r)(2),$$

which doesn't look like much unless we substitute $r = 2t$ and $\Delta r = 2\Delta t$. We get:

$$\begin{aligned}
\frac{\Delta[S(r)]}{\Delta r}\cdot\frac{\Delta[r(t)]}{\Delta t} &= (8\pi r + 4\pi\,\Delta r)(2) \\
&= (8\pi(2t) + 4\pi\,(2\Delta t)(2) \\
&= (16\pi t + 8\pi\,\Delta t)(2) \\
&= 32\pi t + 16\pi\,\Delta t
\end{aligned}$$

We find in this case,

$$\frac{\Delta[S(r)]}{\Delta r} \cdot \frac{\Delta[r(t)]}{\Delta t} = \frac{\Delta[S(r)]}{\Delta t}.$$

Moreover, we note that the time interval $\frac{1}{2} \leq t \leq \frac{3}{2}$ corresponds to the interval $1 \leq r \leq 3$ so it makes sense to multiply our numerical answers as well:

$$\frac{\Delta[S(r)]}{\Delta r} \cdot \frac{\Delta[r(t)]}{\Delta t} = 16\pi \text{ cm} \cdot 2\,\frac{\text{cm}}{\text{min}} = 32\pi\,\frac{\text{cm}^2}{\text{min}}.$$

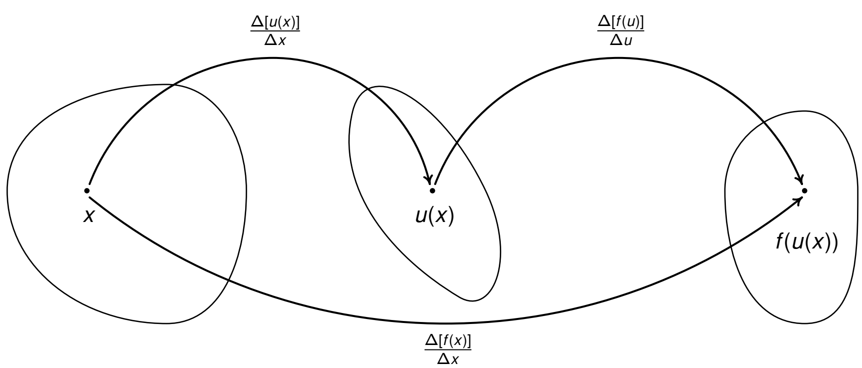
□

Example 1.3.5 verifies a property of rates we formalize below.

THEOREM 1.5. **Related Rates:** Let f and u be functions where u is defined over an interval containing x and $x + \Delta x$ and f is defined over an interval containing $u(x)$ and $u(x + \Delta x)$. Then:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{\Delta[f(u)]}{\Delta u} \cdot \frac{\Delta[u(x)]}{\Delta x}, \quad \Delta x \neq 0, \Delta u \neq 0.$$

If we think of u as being an 'intermediary' variable, Theorem 1.5 allows us to determine the rate of change of f with respect to x by multiplying the rate of change of f with respect to this 'intermediary' u by the rate of change of the 'intermediary' u with respect to x. That is, we are looking for rate information on how f depends on x by decomposing the rates into two rates as visualized below.



We close the section with one last example.

EXAMPLE 1.3.6. The drag force F, in Newtons (N), of a perfectly fine OER Precalculus Textbook which has been discarded off of a cliff because it didn't have enough Calculus in it is given by: $F(v) = 0.6\,v^2$, where v is the speed of the book as it hurtles towards the Earth. If the speed is increasing at a constant rate of 9.8 meters per second per second, $\frac{\text{m/s}}{\text{s}}$, determine the rate of change of F with respect to time as the speed changes from $5\,\frac{\text{m}}{\text{s}}$ to $6\,\frac{\text{m}}{\text{s}}$

Solution. In this scenario, the drag force, F depends directly on the speed, v and the speed, v depends directly on time. (The longer the book falls, the faster it falls.[5]) By Theorem 1.5, we know

$$\frac{\Delta[F(t)]}{\Delta t} = \frac{\Delta[F(v)]}{\Delta v} \cdot \frac{\Delta v}{\Delta t}.$$

We are told that the speed is increasing at a constant rate of 9.8 meters per second per second, so we know $\frac{\Delta v}{\Delta t} = 9.8\,\frac{\text{m/s}}{\text{s}}$ for all time (and hence, speeds.).

To $\frac{\Delta[F(v)]}{\Delta v}$ as the speed changes from 5 $\frac{\text{m}}{\text{s}}$ to 6 $\frac{\text{m}}{\text{s}}$, we calculate:

$$\frac{\Delta[F(v)]}{\Delta v} = \frac{F(6) - F(5)}{6 - 5} = \frac{0.6(6)^2 - 0.6(5)^2}{1} = 6.6.$$

The units on $\frac{\Delta[F(v)]}{\Delta v}$ would be the units of F, N, divided by the units of v, $\frac{\text{m}}{\text{s}}$ which works out[6] to $\frac{\text{N s}}{\text{m}}$.

Hence,

$$\frac{\Delta[F(t)]}{\Delta t} = \frac{\Delta[F(v)]}{\Delta v} \cdot \frac{\Delta v}{\Delta t} = \left(6.6\,\frac{\text{N s}}{\text{m}}\right)\left(9.8\,\frac{\text{m/s}}{\text{s}}\right) = 64.68\,\frac{\text{N}}{\text{s}}.$$

The force is increasing at an average rate of 64.68 Newtons per second. □

Note that we never needed to know explicitly how the speed, v directly depended on time in order to answer the question posed in Example 1.3.6. All we needed was the rate.

[5]Well until it reaches terminal velocity . . .

[6]Note: we could simplify this a bit farther. A Newton, N, has units $\frac{\text{kg m}}{\text{s}^2}$, so some units could cancel to give $\frac{\text{kg}}{\text{s}}$. We leave things as they are for now for a more simple calculation later.

1.3.2 Exercises

In Exercises 1 - 12, use the given pair of functions to find the following values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

1. $f(x) = x^2$, $g(t) = 2t + 1$
2. $f(x) = 4 - x$, $g(t) = 1 - t^2$
3. $f(x) = 4 - 3x$, $g(t) = |t|$
4. $f(x) = |x - 1|$, $g(t) = t^2 - 5$
5. $f(x) = 4x + 5$, $g(t) = \sqrt{t}$
6. $f(x) = \sqrt{3 - x}$, $g(t) = t^2 + 1$
7. $f(x) = 6 - x - x^2$, $g(t) = t\sqrt{t + 10}$
8. $f(x) = \sqrt[3]{x + 1}$, $g(t) = 4t^2 - t$
9. $f(x) = \dfrac{3}{1 - x}$, $g(t) = \dfrac{4t}{t^2 + 1}$
10. $f(x) = \dfrac{x}{x + 5}$, $g(t) = \dfrac{2}{7 - t^2}$
11. $f(x) = \dfrac{2x}{5 - x^2}$, $g(t) = \sqrt{4t + 1}$
12. $f(x) = \sqrt{2x + 5}$, $g(t) = \dfrac{10t}{t^2 + 1}$

In Exercises 13 - 24, use the given pair of functions to find and simplify expressions for the following functions and state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(t)$
- $(f \circ f)(x)$

13. $f(x) = 2x + 3$, $g(t) = t^2 - 9$
14. $f(x) = x^2 - x + 1$, $g(t) = 3t - 5$
15. $f(x) = x^2 - 4$, $g(t) = |t|$
16. $f(x) = 3x - 5$, $g(t) = \sqrt{t}$
17. $f(x) = |x + 1|$, $g(t) = \sqrt{t}$
18. $f(x) = 3 - x^2$, $g(t) = \sqrt{t + 1}$
19. $f(x) = |x|$, $g(t) = \sqrt{4 - t}$
20. $f(x) = x^2 - x - 1$, $g(t) = \sqrt{t - 5}$
21. $f(x) = 3x - 1$, $g(t) = \dfrac{1}{t + 3}$
22. $f(x) = \dfrac{3x}{x - 1}$, $g(t) = \dfrac{t}{t - 3}$
23. $f(x) = \dfrac{x}{2x + 1}$, $g(t) = \dfrac{2t + 1}{t}$
24. $f(x) = \dfrac{2x}{x^2 - 4}$, $g(t) = \sqrt{1 - t}$

In Exercises 25 - 30, use $f(x) = -2x$, $g(t) = \sqrt{t}$ and $h(s) = |s|$ to find and simplify expressions for the following functions and state the domain of each using interval notation.

25. $(h \circ g \circ f)(x)$
26. $(h \circ f \circ g)(t)$
27. $(g \circ f \circ h)(s)$
28. $(g \circ h \circ f)(x)$
29. $(f \circ h \circ g)(t)$
30. $(f \circ g \circ h)(s)$

In Exercises 31 - 43, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

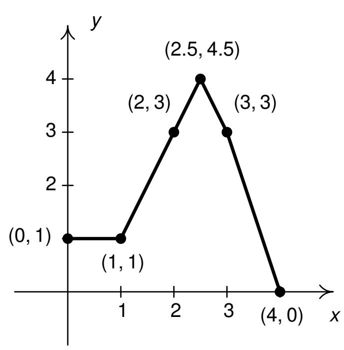
and let g be the function defined by

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}.$$

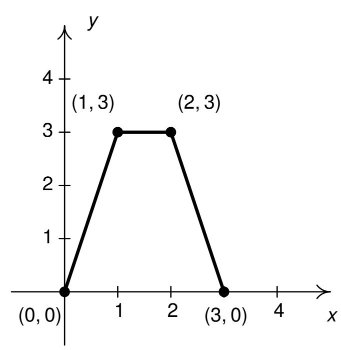
Find the following, if it exists.

31. $(f \circ g)(3)$
32. $f(g(-1))$
33. $(f \circ f)(0)$
34. $(f \circ g)(-3)$
35. $(g \circ f)(3)$
36. $g(f(-3))$
37. $(g \circ g)(-2)$
38. $(g \circ f)(-2)$
39. $g(f(g(0)))$
40. $f(f(f(-1)))$
41. $f(f(f(f(f(1)))))$
42. $\underbrace{(g \circ g \circ \cdots \circ g)}_{n \text{ times}}(0)$
43. Find the domain and range of $f \circ g$ and $g \circ f$.

In Exercises 44 - 50, use the graphs of $y = f(x)$ and $y = g(x)$ below to find the following if it exists.



$y = f(x)$



$y = g(x)$

44. $(g \circ f)(1)$
45. $(f \circ g)(3)$
46. $(g \circ f)(2)$
47. $(f \circ g)(0)$
48. $(f \circ f)(4)$
49. $(g \circ g)(1)$
50. Find the domain and range of $f \circ g$ and $g \circ f$.

In Exercises 51 - 60, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

51. $p(x) = (2x+3)^3$

52. $P(x) = \left(x^2 - x + 1\right)^5$

53. $h(t) = \sqrt{2t-1}$

54. $H(t) = |7-3t|$

55. $r(s) = \dfrac{2}{5s+1}$

56. $R(s) = \dfrac{7}{s^2-1}$

57. $q(z) = \dfrac{|z|+1}{|z|-1}$

58. $Q(z) = \dfrac{2z^3+1}{z^3-1}$

59. $v(x) = \dfrac{2x+1}{3-4x}$

60. $w(x) = \dfrac{x^2}{x^4+1}$

61. Write the function $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}}$ as a composition of three or more non-identity functions.

62. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?

63. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?

64. Let $f(x) = 3x+1$ and let $g(x) = \begin{cases} 2x-1 & \text{if } x \leq 3 \\ 4-x & \text{if } x > 3 \end{cases}$. Find expressions for $(f \circ g)(x)$ and $(g \circ f)(x)$.

65. The volume V of a cube is a function of its side length x. Let's assume that $x = t+1$ is also a function of time t, where x is measured in inches and t is measured in minutes. Find a formula for V as a function of t.

66. Suppose a local vendor charges \$2 per hot dog and that the number of hot dogs sold per hour x is given by $x(t) = -4t^2 + 20t + 92$, where t is the number of hours since 10 AM, $0 \leq t \leq 4$.

    (a) Find an expression for the revenue per hour R as a function of x.

    (b) Find and simplify $(R \circ x)(t)$. What does this represent?

    (c) What is the revenue per hour at noon?

67. Discuss with your classmates how 'real-world' processes such as filling out federal income tax forms or computing your final course grade could be viewed as a use of function composition. Find a process for which composition with itself (iteration) makes sense.

1.3.3 Answers

1. For $f(x) = x^2$ and $g(t) = 2t + 1$,
   - $(g \circ f)(0) = 1$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = 16$
   - $(g \circ f)(-3) = 19$
   - $(f \circ g)\left(\frac{1}{2}\right) = 4$
   - $(f \circ f)(-2) = 16$
2. For $f(x) = 4 - x$ and $g(t) = 1 - t^2$,
   - $(g \circ f)(0) = -15$
   - $(f \circ g)(-1) = 4$
   - $(f \circ f)(2) = 2$
   - $(g \circ f)(-3) = -48$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{13}{4}$
   - $(f \circ f)(-2) = -2$
3. For $f(x) = 4 - 3x$ and $g(t) = |t|$,
   - $(g \circ f)(0) = 4$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = 10$
   - $(g \circ f)(-3) = 13$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{5}{2}$
   - $(f \circ f)(-2) = -26$
4. For $f(x) = |x - 1|$ and $g(t) = t^2 - 5$,
   - $(g \circ f)(0) = -4$
   - $(f \circ g)(-1) = 5$
   - $(f \circ f)(2) = 0$
   - $(g \circ f)(-3) = 11$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{23}{4}$
   - $(f \circ f)(-2) = 2$
5. For $f(x) = 4x + 5$ and $g(t) = \sqrt{t}$,
   - $(g \circ f)(0) = \sqrt{5}$
   - $(f \circ g)(-1)$ is not real
   - $(f \circ f)(2) = 57$
   - $(g \circ f)(-3)$ is not real
   - $(f \circ g)\left(\frac{1}{2}\right) = 5 + 2\sqrt{2}$
   - $(f \circ f)(-2) = -7$
6. For $f(x) = \sqrt{3 - x}$ and $g(t) = t^2 + 1$,
   - $(g \circ f)(0) = 4$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = \sqrt{2}$
   - $(g \circ f)(-3) = 7$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt{7}}{2}$
   - $(f \circ f)(-2) = \sqrt{3 - \sqrt{5}}$
7. For $f(x) = 6 - x - x^2$ and $g(t) = t\sqrt{t + 10}$,
   - $(g \circ f)(0) = 24$
   - $(f \circ g)(-1) = 0$
   - $(f \circ f)(2) = 6$
   - $(g \circ f)(-3) = 0$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{27 - 2\sqrt{42}}{8}$
   - $(f \circ f)(-2) = -14$

8. For $f(x) = \sqrt[3]{x+1}$ and $g(t) = 4t^2 - t$,

- $(g \circ f)(0) = 3$
- $(f \circ g)(-1) = \sqrt[3]{6}$
- $(f \circ f)(2) = \sqrt[3]{\sqrt[3]{3}+1}$
- $(g \circ f)(-3) = 4\sqrt[3]{4} + \sqrt[3]{2}$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt[3]{12}}{2}$
- $(f \circ f)(-2) = 0$

9. For $f(x) = \frac{3}{1-x}$ and $g(t) = \frac{4t}{t^2+1}$,

- $(g \circ f)(0) = \frac{6}{5}$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = \frac{3}{4}$
- $(g \circ f)(-3) = \frac{48}{25}$
- $(f \circ g)\left(\frac{1}{2}\right) = -5$
- $(f \circ f)(-2)$ is undefined

10. For $f(x) = \frac{x}{x+5}$ and $g(t) = \frac{2}{7-t^2}$,

- $(g \circ f)(0) = \frac{2}{7}$
- $(f \circ g)(-1) = \frac{1}{16}$
- $(f \circ f)(2) = \frac{2}{37}$
- $(g \circ f)(-3) = \frac{8}{19}$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{8}{143}$
- $(f \circ f)(-2) = -\frac{2}{13}$

11. For $f(x) = \frac{2x}{5-x^2}$ and $g(t) = \sqrt{4t+1}$,

- $(g \circ f)(0) = 1$
- $(f \circ g)(-1)$ is not real
- $(f \circ f)(2) = -\frac{8}{11}$
- $(g \circ f)(-3) = \sqrt{7}$
- $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{3}$
- $(f \circ f)(-2) = \frac{8}{11}$

12. For $f(x) = \sqrt{2x+5}$ and $g(t) = \frac{10t}{t^2+1}$,

- $(g \circ f)(0) = \frac{5\sqrt{5}}{3}$
- $(f \circ g)(-1)$ is not real
- $(f \circ f)(2) = \sqrt{11}$
- $(g \circ f)(-3)$ is not real
- $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{13}$
- $(f \circ f)(-2) = \sqrt{7}$

13. For $f(x) = 2x + 3$ and $g(t) = t^2 - 9$

- $(g \circ f)(x) = 4x^2 + 12x$, domain: $(-\infty, \infty)$
- $(f \circ g)(t) = 2t^2 - 15$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = 4x + 9$, domain: $(-\infty, \infty)$

14. For $f(x) = x^2 - x + 1$ and $g(t) = 3t - 5$

- $(g \circ f)(x) = 3x^2 - 3x - 2$, domain: $(-\infty, \infty)$
- $(f \circ g)(t) = 9t^2 - 33t + 31$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = x^4 - 2x^3 + 2x^2 - x + 1$, domain: $(-\infty, \infty)$

15. For $f(x) = x^2 - 4$ and $g(t) = |t|$
    - $(g \circ f)(x) = |x^2 - 4|$, domain: $(-\infty, \infty)$
    - $(f \circ g)(t) = |t|^2 - 4 = t^2 - 4$, domain: $(-\infty, \infty)$
    - $(f \circ f)(x) = x^4 - 8x^2 + 12$, domain: $(-\infty, \infty)$
16. For $f(x) = 3x - 5$ and $g(t) = \sqrt{t}$
    - $(g \circ f)(x) = \sqrt{3x - 5}$, domain: $\left[\frac{5}{3}, \infty\right)$
    - $(f \circ g)(t) = 3\sqrt{t} - 5$, domain: $[0, \infty)$
    - $(f \circ f)(x) = 9x - 20$, domain: $(-\infty, \infty)$
17. For $f(x) = |x + 1|$ and $g(t) = \sqrt{t}$
    - $(g \circ f)(x) = \sqrt{|x + 1|}$, domain: $(-\infty, \infty)$
    - $(f \circ g)(t) = |\sqrt{t} + 1| = \sqrt{t} + 1$, domain: $[0, \infty)$
    - $(f \circ f)(x) = ||x + 1| + 1| = |x + 1| + 1$, domain: $(-\infty, \infty)$
18. For $f(x) = 3 - x^2$ and $g(t) = \sqrt{t + 1}$
    - $(g \circ f)(x) = \sqrt{4 - x^2}$, domain: $[-2, 2]$
    - $(f \circ g)(t) = 2 - t$, domain: $[-1, \infty)$
    - $(f \circ f)(x) = -x^4 + 6x^2 - 6$, domain: $(-\infty, \infty)$
19. For $f(x) = |x|$ and $g(t) = \sqrt{4 - t}$
    - $(g \circ f)(x) = \sqrt{4 - |x|}$, domain: $[-4, 4]$
    - $(f \circ g)(t) = |\sqrt{4 - t}| = \sqrt{4 - t}$, domain: $(-\infty, 4]$
    - $(f \circ f)(x) = ||x|| = |x|$, domain: $(-\infty, \infty)$
20. For $f(x) = x^2 - x - 1$ and $g(t) = \sqrt{t - 5}$
    - $(g \circ f)(x) = \sqrt{x^2 - x - 6}$, domain: $(-\infty, -2] \cup [3, \infty)$
    - $(f \circ g)(t) = t - 6 - \sqrt{t - 5}$, domain: $[5, \infty)$
    - $(f \circ f)(x) = x^4 - 2x^3 - 2x^2 + 3x + 1$, domain: $(-\infty, \infty)$
21. For $f(x) = 3x - 1$ and $g(t) = \frac{1}{t+3}$
    - $(g \circ f)(x) = \frac{1}{3x+2}$, domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
    - $(f \circ g)(t) = -\frac{t}{t+3}$, domain: $(-\infty, -3) \cup (-3, \infty)$
    - $(f \circ f)(x) = 9x - 4$, domain: $(-\infty, \infty)$

22. For $f(x) = \frac{3x}{x-1}$ and $g(t) = \frac{t}{t-3}$

    - $(g \circ f)(x) = x$, domain: $(-\infty, 1) \cup (1, \infty)$
    - $(f \circ g)(t) = t$, domain: $(-\infty, 3) \cup (3, \infty)$
    - $(f \circ f)(x) = \frac{9x}{2x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 1\right) \cup (1, \infty)$

23. For $f(x) = \frac{x}{2x+1}$ and $g(t) = \frac{2t+1}{t}$

    - $(g \circ f)(x) = \frac{4x+1}{x}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0), \cup(0, \infty\right)$
    - $(f \circ g)(t) = \frac{2t+1}{5t+2}$, domain: $\left(-\infty, -\frac{2}{5}\right) \cup \left(-\frac{2}{5}, 0\right) \cup (0, \infty)$
    - $(f \circ f)(x) = \frac{x}{4x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$

24. For $f(x) = \frac{2x}{x^2-4}$ and $g(t) = \sqrt{1-t}$

    - $(g \circ f)(x) = \sqrt{\frac{x^2-2x-4}{x^2-4}}$, domain: $(-\infty, -2) \cup \left[1-\sqrt{5}, 2\right) \cup \left[1+\sqrt{5}, \infty\right)$
    - $(f \circ g)(t) = -\frac{2\sqrt{1-t}}{t+3}$, domain: $(-\infty, -3) \cup (-3, 1]$
    - $(f \circ f)(x) = \frac{4x-x^3}{x^4-9x^2+16}$, domain: $\left(-\infty, -\frac{1+\sqrt{17}}{2}\right) \cup \left(-\frac{1+\sqrt{17}}{2}, -2\right) \cup \left(-2, \frac{1-\sqrt{17}}{2}\right) \cup \left(\frac{1-\sqrt{17}}{2}, \frac{-1+\sqrt{17}}{2}\right) \cup \left(\frac{-1+\sqrt{17}}{2}, 2\right) \cup \left(2, \frac{1+\sqrt{17}}{2}\right) \cup \left(\frac{1+\sqrt{17}}{2}, \infty\right)$

25. $(h \circ g \circ f)(x) = |\sqrt{-2x}| = \sqrt{-2x}$, domain: $(-\infty, 0]$

26. $(h \circ f \circ g)(t) = |-2\sqrt{t}| = 2\sqrt{t}$, domain: $[0, \infty)$

27. $(g \circ f \circ h)(s) = \sqrt{-2|s|}$, domain: $\{0\}$

28. $(g \circ h \circ f)(x) = \sqrt{|-2x|} = \sqrt{2|x|}$, domain: $(-\infty, \infty)$

29. $(f \circ h \circ g)(t) = -2|\sqrt{t}| = -2\sqrt{t}$, domain: $[0, \infty)$

30. $(f \circ g \circ h)(s) = -2\sqrt{|s|}$, , domain: $(-\infty, \infty)$

31. $(f \circ g)(3) = f(g(3)) = f(2) = 4$

32. $f(g(-1)) = f(-4)$ which is undefined

33. $(f \circ f)(0) = f(f(0)) = f(1) = 3$

34. $(f \circ g)(-3) = f(g(-3)) = f(-2) = 2$

35. $(g \circ f)(3) = g(f(3)) = g(-1) = -4$

36. $g(f(-3)) = g(4)$ which is undefined

37. $(g \circ g)(-2) = g(g(-2)) = g(0) = 0$

38. $(g \circ f)(-2) = g(f(-2)) = g(2) = 1$

39. $g(f(g(0))) = g(f(0)) = g(1) = -3$

40. $f(f(f(-1))) = f(f(0)) = f(1) = 3$

41. $f(f(f(f(f(1))))) = f(f(f(f(3)))) = f(f(f(-1))) = f(f(0)) = f(1) = 3$

42. $\underbrace{(g \circ g \circ \cdots \circ g)}_{n \text{ times}}(0) = 0$

43.
 - The domain of $f \circ g$ is $\{-3, -2, 0, 1, 2, 3\}$ and the range of $f \circ g$ is $\{1, 2, 3, 4\}$.
 - The domain of $g \circ f$ is $\{-2, -1, 0, 1, 3\}$ and the range of $g \circ f$ is $\{-4, -3, 0, 1, 2\}$.

44. $(g \circ f)(1) = 3$

45. $(f \circ g)(3) = 1$

46. $(g \circ f)(2) = 0$

47. $(f \circ g)(0) = 1$

48. $(f \circ f)(4) = 1$

49. $(g \circ g)(1) = 0$

50.
 - The domain of $f \circ g$ is $[0, 3]$ and the range of $f \circ g$ is $[1, 4.5]$.
 - The domain of $g \circ f$ is $[0, 2] \cup [3, 4]$ and the range is $[0, 3]$.

51. Let $f(x) = 2x + 3$ and $g(x) = x^3$, then $p(x) = (g \circ f)(x)$.

52. Let $f(x) = x^2 - x + 1$ and $g(x) = x^5$, $P(x) = (g \circ f)(x)$.

53. Let $f(t) = 2t - 1$ and $g(t) = \sqrt{t}$, then $h(t) = (g \circ f)(t)$.

54. Let $f(t) = 7 - 3t$ and $g(t) = |t|$, then $H(t) = (g \circ f)(t)$.

55. Let $f(s) = 5s + 1$ and $g(s) = \frac{2}{s}$, then $r(s) = (g \circ f)(s)$.

56. Let $f(s) = s^2 - 1$ and $g(s) = \frac{7}{s}$, then $R(s) = (g \circ f)(s)$.

57. Let $f(z) = |z|$ and $g(z) = \frac{z+1}{z-1}$, then $q(z) = (g \circ f)(z)$.

58. Let $f(z) = z^3$ and $g(z) = \frac{2z+1}{z-1}$, then $Q(z) = (g \circ f)(z)$.

59. Let $f(x) = 2x$ and $g(x) = \frac{x+1}{3-2x}$, then $v(x) = (g \circ f)(x)$.

60. Let $f(x) = x^2$ and $g(x) = \frac{x}{x^2+1}$, then $w(x) = (g \circ f)(x)$.

61. $F(x) = \sqrt{\frac{x^3+6}{x^3-9}} = (h(g(f(x)))$ where $f(x) = x^3$, $g(x) = \frac{x+6}{x-9}$ and $h(x) = \sqrt{x}$.

62. $F(x) = 3\sqrt{-x+2} - 4 = k(j(f(h(g(x)))))$

63. One solution is $F(x) = -\frac{1}{2}(2x-7)^3 + 1 = k(j(f(h(g(x)))))$ where $g(x) = 2x$, $h(x) = x - 7$, $j(x) = -\frac{1}{2}x$ and $k(x) = x + 1$. You could also have $F(x) = H(f(G(x)))$ where $G(x) = 2x - 7$ and $H(x) = -\frac{1}{2}x + 1$.

64. $(f \circ g)(x) = \begin{cases} 6x - 2 & \text{if } x \leq 3 \\ 13 - 3x & \text{if } x > 3 \end{cases}$ and $(g \circ f)(x) = \begin{cases} 6x + 1 & \text{if } x \leq \dfrac{2}{3} \\ 3 - 3x & \text{if } x > \dfrac{2}{3} \end{cases}$

65. $V(x) = x^3$ so $V(x(t)) = (t+1)^3$

66.
 (a) $R(x) = 2x$
 (b) $(R \circ x)(t) = -8t^2 + 40t + 184$, $0 \leq t \leq 4$. This gives the revenue per hour as a function of time.
 (c) Noon corresponds to $t = 2$, so $(R \circ x)(2) = 232$. The hourly revenue at noon is \$232 per hour.

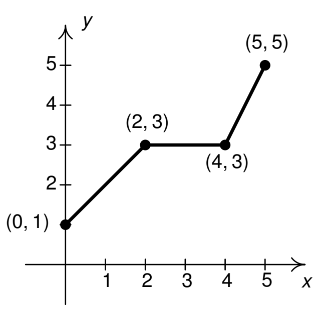
1.4 Transformations of Graphs

Theorems **??**, **??**, **??**, **??**, **??** and **??** all describe ways in which the graph of a function can changed, or 'transformed' to obtain the graph of a related function. The results and proofs of each of these theorems are virtually identical, and with the language of function composition, we can see better why.

Consider, for instance, Theorem **??**, in which we describe how to transform the graph of $f(x) = x^r$ to $F(x) = a(bx - h)^r + k$. We may think of F as being build up from f by composing f with linear functions. Specifically, if we let $i(x) = bx - h$, then $(f \circ i)(x) = f(i(x)) = f(bx - h) = (bx - h)^r$. If, additionally, we let and $j(x) = ax + k$, then $(j \circ (f \circ i))(x) = j((f \circ i)(x)) = j((bx - h)^r) = a(bx - h)^r + k = F(x)$. Hence, we can view $F = j \circ f \circ i$.

In this section, our goal is to generalize the aforementioned theorems to the graphs of *all* functions. Along the way, you'll see some very familiar arguments, but, additionally, we hope this section affords the reader an opportunity to not only see *how* these transformations work they way they do, but *why*.

Our motivational example for the results in this section is the graph of $y = f(x)$ below. While we could formulate an expression for $f(x)$ as a piecewise-defined function consisting of linear and constant parts, we wish to focus more on the geometry here. That being said, we do record some of the function values - the 'key points' if you will - to track through each transformation.



$y = f(x)$

x	$(x, f(x))$	$f(x)$
0	$(0, 1)$	1
2	$(2, 3)$	3
4	$(4, 3)$	3
5	$(5, 5)$	5

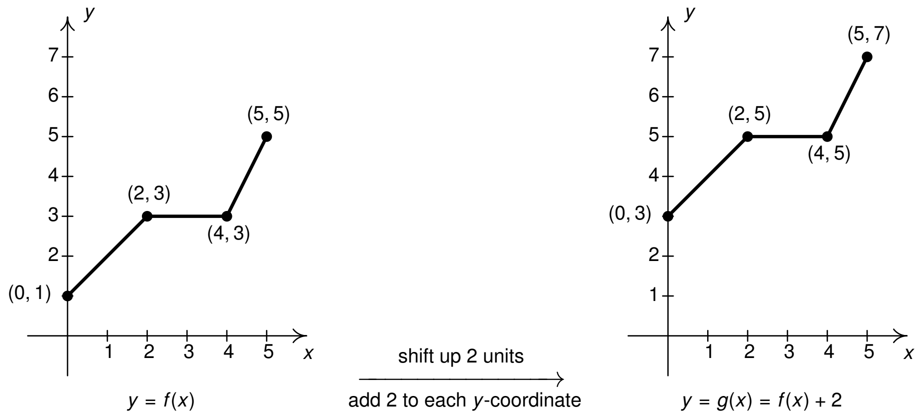
1.4.1 Vertical and Horizontal Shifts

Suppose we wished to graph $g(x) = f(x) + 2$. From a procedural point of view, we start with an input x to the function f and we obtain the output $f(x)$. The function g takes the output $f(x)$ and adds 2 to it. Using the sample values for f from the table above we can create a table of values for g below, hence generating points on the graph of g.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0, 1)$	1	$1 + 2 = 3$	$(0, 3)$
2	$(2, 3)$	3	$3 + 2 = 5$	$(2, 5)$
4	$(4, 3)$	3	$3 + 2 = 5$	$(4, 5)$
5	$(5, 5)$	5	$5 + 2 = 7$	$(5, 7)$

In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Hence, $g(a) = f(a) + 2 = b + 2$, so the point $(a, b+2)$ is on the graph of g. In other words, to obtain the graph of g, we add 2 to the y-coordinate of each point on the graph of f.

Geometrically, adding 2 to the y-coordinate of a point moves the point 2 units above its previous location. Adding 2 to every y-coordinate on a graph *en masse* is moves or 'shifts' the entire graph of f up 2 units. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four 'key points' we moved in the same manner in which they were connected before.



You'll note that the domain of f and the domain of g are the same, namely $[0, 5]$, but that the range of f is $[1, 5]$ while the range of g is $[3, 7]$. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range.

You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of f, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of j is the same as f, but the range of j is $[-1, 3]$. In general, we have:

THEOREM 1.6. **Vertical Shifts.** Suppose f is a function and k is a real number.
To graph $F(x) = f(x) + k$, add k to each of the y-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a vertical shift up k units if $k > 0$ or down k units if $k < 0$.

To prove Theorem 1.6, we first note that f and F have the same domain (why?) Let c be an element in the domain of F and, hence, the domain of f. The fact that f and F are *functions* guarantees there is *exactly one* point on each of their graphs corresponding to $x = c$. On $y = f(x)$, this point is $(c, f(c))$; on $y = F(x)$, this point is $(c, F(c)) = (c, f(c)+k)$. This sets up a nice correspondence between the two graphs and shows that each of the points on the graph of F can be obtained to by adding k to each of the y-coordinates of the corresponding point on the graph of f. This proves Theorem 1.6. In the language of 'inputs' and 'outputs', Theorem 1.6 says adding to the *output* of a function causes the graph to shift *vertically*.

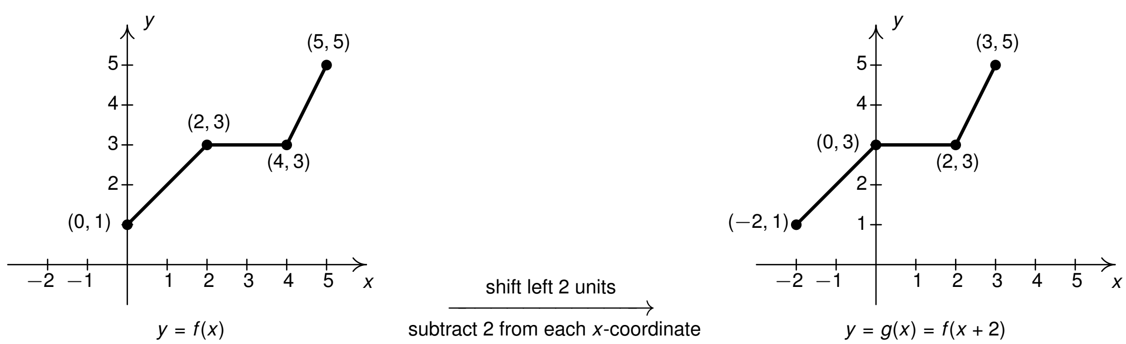
Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x + 2)$. In other words, we are looking to see what happens when we add 2 to the input of the function. Let's try to generate a table of values of g based on those we know for f. We quickly find that we run into some difficulties. For instance, when we substitute $x = 4$ into the formula $g(x) = f(x + 2)$, we are asked to find $f(4 + 2) = f(6)$ which doesn't exist because the domain of f is only $[0, 5]$. The same thing happens when we attempt to find $g(5)$.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x+2)$	$(x, g(x))$
0	$(0,1)$	1	$g(0) = f(0+2) = f(2) = 3$	$(0,3)$
2	$(2,3)$	3	$g(2) = f(2+2) = f(4) = 3$	$(2,3)$
4	$(4,3)$	3	$g(4) = f(4+2) = f(6) = ?$	
5	$(5,5)$	5	$g(5) = f(5+2) = f(7) = ?$	

What we need here is a new strategy. We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of g, we need to figure out what value of x we must substitute into $g(x) = f(x+2)$ so that the quantity $x+2$, works out to be 0. Solving $x + 2 = 0$ gives $x = -2$, and $g(-2) = f((-2)+2) = f(0) = 1$ so $(-2, 1)$ on the graph of g. To use the fact $f(2) = 3$, we set $x + 2 = 2$ to get $x = 0$. Substituting gives $g(0) = f(0+2) = f(2) = 3$. Continuing in this fashion, we produce the table below.

x	$x+2$	$g(x) = f(x+2)$	$(x, g(x))$
-2	0	$g(-2) = f(-2+2) = f(0) = 1$	$(-2,1)$
0	2	$g(0) = f(0+2) = f(2) = 3$	$(0,3)$
2	4	$g(2) = f(2+2) = f(4) = 3$	$(2,3)$
3	5	$g(3) = f(3+2) = f(5) = 5$	$(3,5)$

In summary, the points $(0,1)$, $(2,3)$, $(4,3)$ and $(5,5)$ on the graph of $y = f(x)$ give rise to the points $(-2,1)$, $(0,3)$, $(2,3)$ and $(3,5)$ on the graph of $y = g(x)$, respectively. In general, if (a,b) is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x+2 = a$ gives $x = a-2$ so that $g(a-2) = f((a-2)+2) = f(a) = b$. As such, $(a-2, b)$ is on the graph of $y = g(x)$. The point $(a-2, b)$ is exactly 2 units to the *left* of the point (a,b) so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units, as pictured below.



Note that while the ranges of f and g are the same, the domain of g is $[-2, 3]$ whereas the domain of f is $[0, 5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph $j(x) = f(x-2)$, we would find ourselves *adding* 2 to all of the x values of the points on the graph of $y = f(x)$ to effect a shift to the *right* 2 units. Generalizing these notions produces the following result.

THEOREM 1.7. **Horizontal Shifts.** Suppose f is a function and h is a real number.
To graph $F(x) = f(x - h)$, add h to each of the x-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a horizontal shift right h units if $h > 0$ or left h units if $h < 0$.

To prove Theorem 1.7, we first note the domains of f and F may be different. If c is in the domain of f, then the only number we know for sure is in the domain of F is $c + h$, since $F(c + h) = f((c + h) - h) = f(c)$. This sets up a nice correspondence between the domain of f and the domain of F which spills over to a correspondence between their graphs, The point $(c, f(c))$ is the one and only point on the graph of $y = f(x)$ corresponding to $x = c$ just as the point $(c + h, F(c + h)) = (c + h, f(c))$ is the one and only point on the graph of $y = F(x)$ corresponding to $x = c + h$. This correspondence shows we may obtain the graph of F by adding h to each x-coordinate of each point on the graph of f, which establishes the theorem. In words, Theorem 1.7 says that subtracting from the *input* to a function amounts to shifting the graph *horizontally*.

Theorems 1.6 and 1.7 present a theme which will run common throughout the section: changes to the *outputs* from a function result in some kind of *vertical change*; changes to the *inputs* to a function result in some kind of *horizontal* change. We demonstrate Theorems 1.6 and 1.7 in the example below.

EXAMPLE 1.4.1. Use Theorems 1.6 and 1.7 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1, 3)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (a) $y = f(x) + 5$ (b) $y = f(x + 5)$ (c) $f(x - 7) + 4$

2. Find a formula for a function $g(t)$ whose graph is the same as $f(t) = |t| - 2t$ but is shifted:

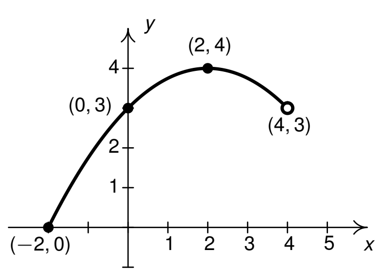
   (a) to the right 4 units. (b) down 2 units.

3. Predict how the graph of $F(x) = \dfrac{(x-2)^{\frac{2}{3}}}{x}$ relates to the graph of $f(x) = \dfrac{x^{\frac{2}{3}}}{x+2}$.

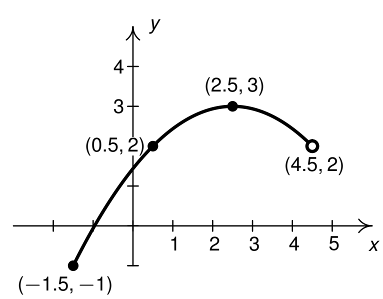
4. Below on the left is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = f(x - 2)$ (b) $F(x) = f(x) + 1$ (c) $F(x) = f(x + 1) - 2$

5. Below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.



$y = f(x)$



$y = g(x)$

Solution.

1. (a) To apply Theorem 1.6, we identify $f(x) + 5 = f(x) + k$ so $k = 5$. Hence, we add 5 to the y-coordinate of $(-1, 3)$ and get $(-1, 3 + 5) = (-1, 8)$. To check our answer note since $(-1, 3)$ is on the graph of f this means $f(-1) = 3$. Substituting $x = -1$ into the formula $y = f(x) + 5$, we get $y = f(-1) + 5 = 3 + 5 = 8$. Hence, $(-1, 8)$ is on the graph of $f(x) + 5$.

   (b) We note that $f(x + 5)$ can be written as $f(x - (-5)) = f(x - h)$ so we apply Theorem 1.7 with $h = -5$. Adding -5 to (subtracting 5 from) the x-coordinate of $(-1, 3)$ gives $(-1 + (-5), 3) = (-6, 3)$. To check our answer, since $(-1, 3)$ is on the graph of f, $f(-1) = 3$. Substituting $x = -6$ into $y = f(x + 5)$ gives $y = f(-6 + 5) = f(-1) = 3$, proving $(-6, 3)$ is on the graph of $y = f(x + 5)$.

   (c) Note that the expression $f(x - 7) + 4$ differs from $f(x)$ in *two* ways indicating two different transformations. In situations like this, its best if we handle each transformation in turn, starting with the graph of $y = f(x)$ and 'building up' to the graph of $y = f(x - 7) + 4$.

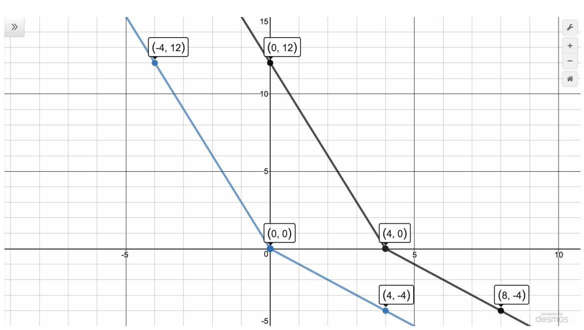
   We choose to work from the 'inside' (argument) out and use Theorem 1.7 to first get a point on the graph of $y = f(x - 7) = f(x - h)$. Identifying $h = 7$, we add 7 to the x-coordinate of $(-1, 3)$ to get $(-1 + 7, 3) = (6, 3)$. Hence, $(6, 3)$ is a point on the graph of $y = f(x - 7)$.

   Next, we apply Theorem 1.6 to graph $y = f(x - 7) + 4$ starting with $y = f(x - 7)$. Viewing $f(x - 7) + 4 = f(x - 7) + k$, we identify $k = 4$ and add 4 to the y-coordinate of $(6, 3)$ to get $(6, 3 + 4) = (6, 7)$. To check, we note that if we substitute $x = 6$ into $y = f(x - 7) + 4$, we get $y = f(6 - 7) + 4 = f(-1) + 4 = 3 + 4 = 7$.

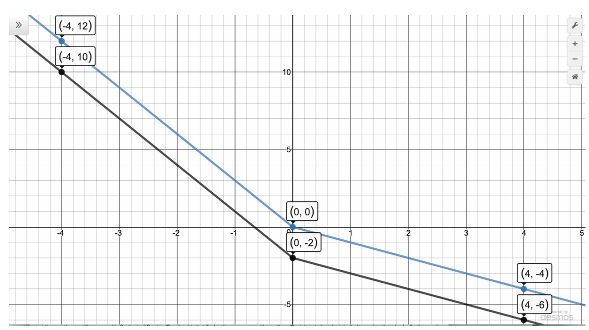
2. Here the independent variable is t instead of x which doesn't affect the geometry in any way since our convention is the independent variable is used to label the horizontal axis and the dependent variable is used to label the vertical axis.

   (a) Per Theorem 1.7, the graph of $g(t) = f(t - 4) = |t - 4| - 2(t - 4) = |t - 4| - 2t + 8$ should be the graph of $f(t) = |t| - 2t$ shifted to the right 4 units. Our check is below on the left.

   (b) Per Theorem 1.6, the graph of $g(t) = f(t) + (-2) = |t| - 2t + (-2) = |t| - 2t - 2$ should be the graph of $f(t) = |t| - 2t$ shifted down 2 units. Our check is below on the right.



$y = |t| - 2t$ (lighter color) and $y = |t - 4| - 2t + 8$

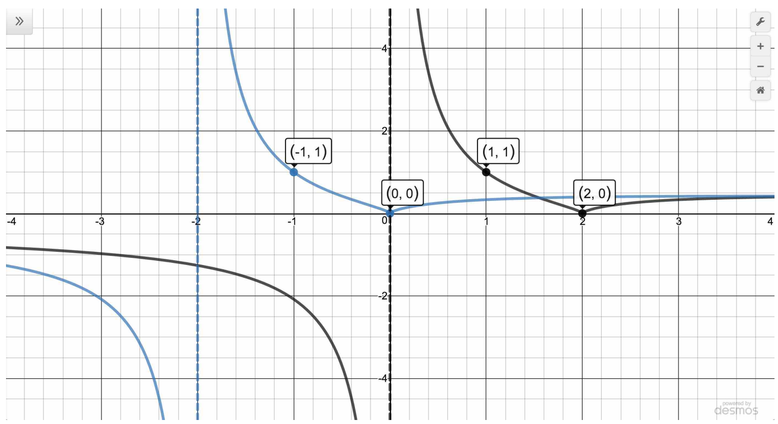


$y = |t| - 2t$ (lighter color) and $y = |t| - 2t - 2$

3. Comparing *formulas*, it appears as if $F(x) = f(x-2)$. We check:

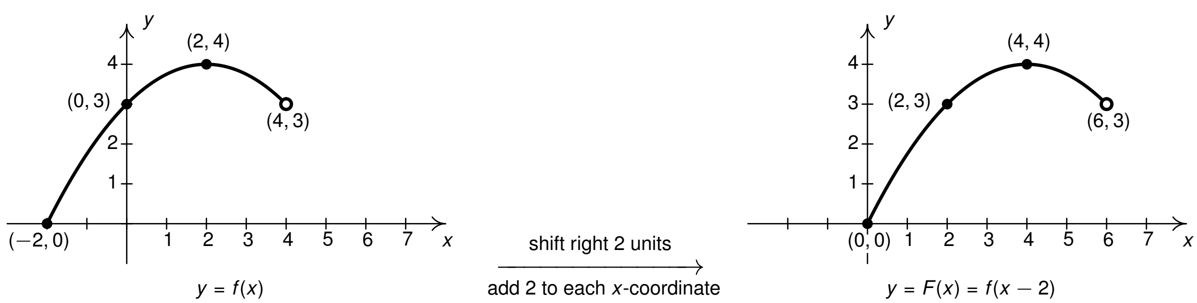
$$f(x-2) = \frac{(x-2)^{\frac{2}{3}}}{(x-2)+2} = \frac{(x-2)^{\frac{2}{3}}}{x} = F(x),$$

so, per Theorem 1.7, the graph of $y = F(x)$ should be the graph of $y = f(x)$ but shifted to the right 2 units. We graph both functions below to confirm our answer.



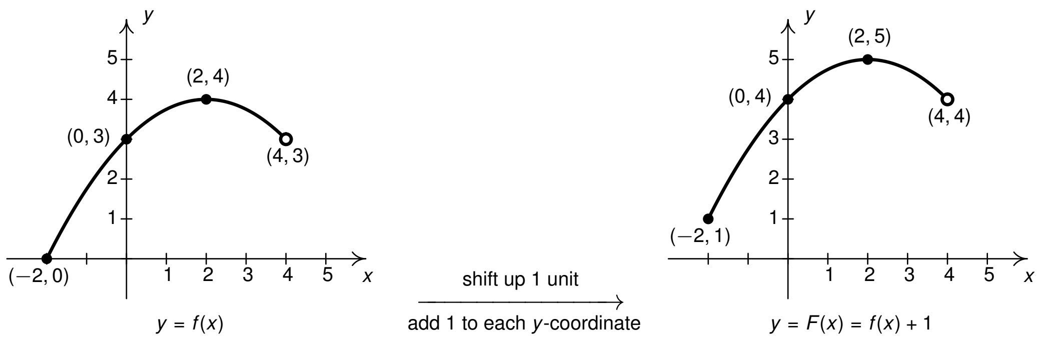
$y = \dfrac{x^{\frac{2}{3}}}{x+2}$ (lighter color) and $y = \dfrac{(x-2)^{\frac{2}{3}}}{x}$

4. (a) We recognize $F(x) = f(x-2) = f(x-h)$. With $h = 2$, Theorem 1.7 tells us to add 2 to each of the x-coordinates of the points on the graph of f, moving the graph of f to the *right* two units.



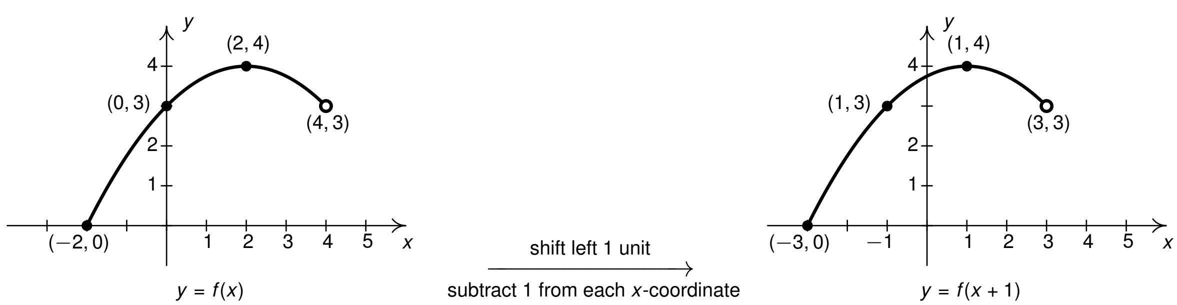
We can check our answer by showing each ordered pair (x, y) listed on our final graph satisfies the equation $y = f(x-2)$. Starting with $(0,0)$, we substitute $x = 0$ into $y = f(x-2)$ and get $y = f(0-2) = f(-2)$. Since $(-2,0)$ is on the graph of f, we know $f(-2) = 0$. Hence, $y = f(0-2) = f(-2) = 0$, showing the point $(0,0)$ is on the graph of $y = f(x-2)$. We invite the reader to check the remaining points.

(b) We have $F(x) = f(x) + 1 = f(x) + k$ where $k = 1$, so Theorem 1.6 tells us to move the graph of f *up* 1 unit by adding 1 to each of the y-coordinates of the points on the graph of f.

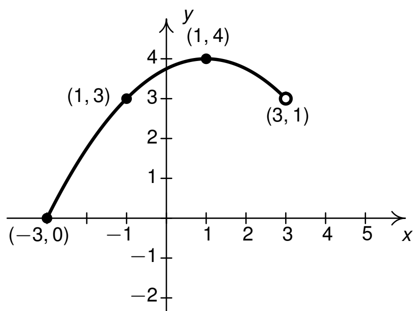


To check our answer, we proceed as above. Starting with the point $(-2, 1)$, we substitute $x = -2$ into $y = f(-2) + 1$ to get $y = f(-2) + 1$. Since $(-2, 0)$ is on the graph of f, we know $f(-2) = 0$. Hence, $y = f(-2) + 1 = 0 + 1 = 1$. This proves $(-2, 1)$ is on the graph of $y = f(x) + 1$. We encourage the reader to check the remaining points in kind.

(c) We are asked to graph $F(x) = f(x+1) - 2$. As above, when we have more than one modification to do, we work from the inside out and build up to $F(x) = f(x + 1) - 2$ from $f(x)$ in stages. First, we apply Theorem 1.7 to graph $y = f(x + 1)$ from $y = f(x)$. Rewriting $f(x + 1) = f(x - (-1))$, we identify $h = -1$, so we add -1 to (subtract 1 from) each of the x-coordinates on the graph of f, shifting it to the *left* 1 unit.

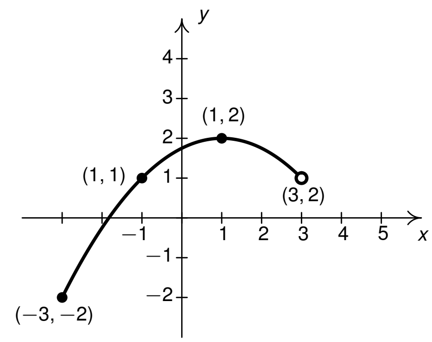


Next, we apply Theorem 1.6 to graph $y = f(x + 1) - 2$ starting with the graph of $y = f(x + 1)$. Writing $f(x + 1) - 2 = f(x + 1) + (-2) = f(x + 1) + k$, we identify $k = -2$ so Theorem 1.6 instructs us to add -2 to (subtract 2 from) each of the y-coordinates on the graph of $y = f(x+1)$, thereby shifting the graph *down* two units.



$y = f(x + 1)$

shift down 2 units
$\longrightarrow$
subtract 2 from each y-coordinate



$y = f(x + 1) - 2 = F(x)$

To check, we start with the point $(-3, -2)$. We find when we substitute $x = -3$ into the equation $y = f(x + 1) - 2$ we get $y = f(-3 + 1) - 2 = f(-2) - 2$. Since $(-2, 0)$ is on the graph of f, we know $f(-2) = 0$, so $y = f(-3 + 1) - 2 = f(-2) - 2 = 0 - 2 = -2$. This proves $(-3, -2)$ is on the graph of $y = f(x + 1) - 2$. We leave the checks of the remaining points to the reader.

5. To write $g(x)$ in terms of $f(x)$, we note that based on points which are labeled, it appears as if the graph of g can be obtained from the graph of f by shifting the graph of f to the right 0.5 units and down 1 unit.

   Per Theorems 1.7 and 1.6, $g(x)$ must take the form $g(x) = f(x - h) + k$. Since the horizontal shift is to the *right* 0.5 units, $h = 0.5$ and since the vertical shift is *down* 1 unit, $k = -1$. Hence, we get $g(x) = f(x - 0.5) - 1$.

   We can check our answer by working through both transformations, in sequence, as in the previous example. To write $f(x)$ in terms of $g(x)$, we need to reverse the process - that is, we need to shift the graph of g *left* one half of a unit and *up* one unit. Theorems 1.7 and 1.6 suggest the formula $f(x) = g(x + 0.5) + 1$. We leave it to the reader to check. □

1.4.2 Reflections about the Coordinate Axes

We now turn our attention to reflections. We know from Section **??** that to reflect a point (x, y) across the x-axis, we replace y with $-y$. If (x, y) is on the graph of f, then $y = f(x)$, so replacing y with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y = -f(x)$ is the graph of f reflected across the x-axis. Similarly, the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis.[1]

THEOREM 1.8. **Reflections.** Suppose f is a function.
To graph $F(x) = -f(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by -1.
NOTE: This results in a reflection across the x-axis.
To graph $F(x) = f(-x)$, multiply each of the x-coordinates of the points on the graph of $y = f(x)$ by -1.
NOTE: This results in a reflection across the y-axis.

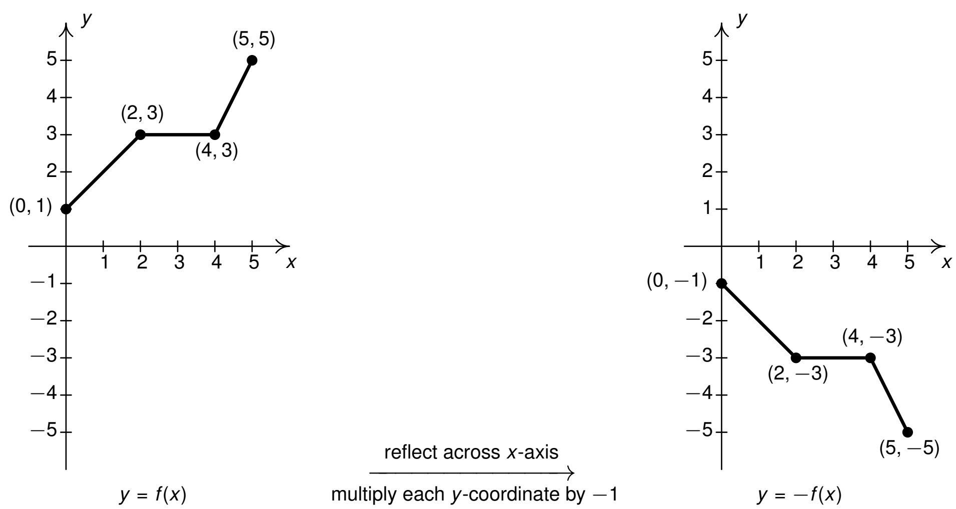
[1] The expressions $-f(x)$ and $f(-x)$ should look familiar - they are the quantities we used in Section **??** to determine if a function was even, odd or neither. We explore impact of symmetry on reflections in Exercise 74.

The proof of Theorem 1.8 follows in much the same way as the proofs of Theorems 1.6 and 1.7. If c is an element of the domain of f and $F(x) = -f(x)$, then the point $(c, f(c))$ corresponds to the point $(c, F(c)) = (c, -f(c))$. Comparing the corresponding points $(c, f(c))$ and $(c, -f(c))$, we see they only difference is the y-coordinates are the exact opposite - indicating they are mirror-images across the x-axis. Similarly, if c is an element in the domain of f, then c corresponds to the element $-c$ in the domain of $F(x) = f(-x)$ since $F(-c) = f(-(-c)) = f(c)$. Hence, the corresponding points here are $(c, f(c))$ and $(-c, F(-c)) = (-c, f(c))$. Comparing $(c, f(c))$ with $(-c, f(c))$, we see they are reflections about the y-axis.

Using the language of inputs and outputs, Theorem 1.8 says that multiplying the *outputs* from a function by -1 reflects its graph across the *horizontal* axis, while multiplying the *inputs* to a function by -1 reflects the graph across the *vertical* axis.

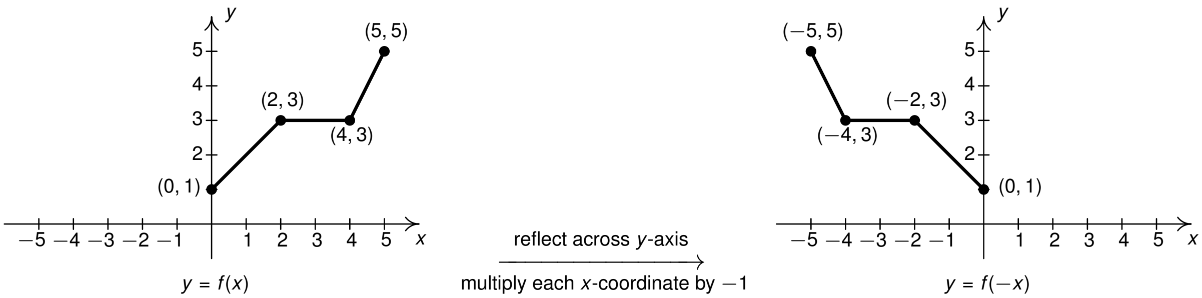
Applying Theorem 1.8 to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of f about the x-axis.

x	$(x, f(x))$	$f(x)$	$g(x) = -f(x)$	$(x, g(x))$
0	$(0, 1)$	1	-1	$(0, -1)$
2	$(2, 3)$	3	-3	$(2, -3)$
4	$(4, 3)$	3	-3	$(4, -3)$
5	$(5, 5)$	5	-5	$(5, -5)$



By reflecting the graph of f across the y-axis, we obtain the graph of $y = f(-x)$.

x	$-x$	$g(x) = f(-x)$	$(x, g(x))$
0	0	$g(0) = f(-(-0)) = f(0) = 1$	$(0, 1)$
-2	2	$g(-2) = f(-(-2)) = f(2) = 3$	$(-2, 3)$
-4	4	$g(-4) = f(-(-4)) = f(4) = 3$	$(-4, 3)$
-5	5	$g(-5) = f(-(-5)) = f(5) = 5$	$(-5, 5)$



EXAMPLE 1.4.2. Use Theorems 1.6, 1.7 and 1.8 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(2, -5)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (a) $y = f(-x)$ (b) $y = -f(x+2)$ (c) $f(8-x)$

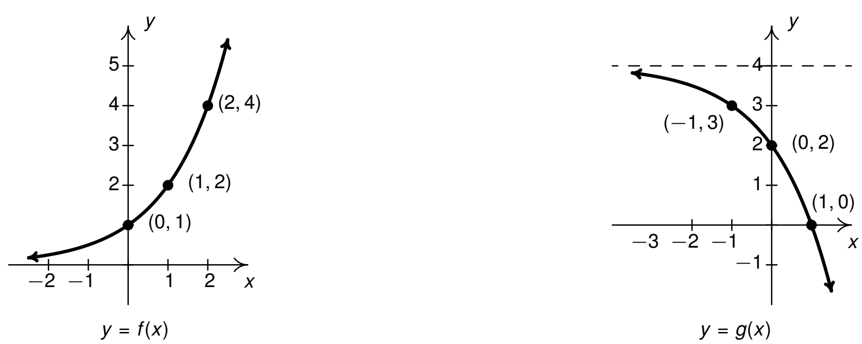
2. Find a formula for a function $H(s)$ whose graph is the same as $t = h(s) = s^3 - s^2$ but is reflected across the t-axis.

3. Predict how the graph of $G(t) = \dfrac{t+4}{t-3}$ relates to the graph of $g(t) = \dfrac{t+4}{3-t}$.

4. Below on the left is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = f(-x) + 1$ (b) $F(x) = 1 - f(2-x)$

5. Below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.



NOTE: The x-axis, $y = 0$, is a horizontal asymptote to the graph of $y = f(x)$ and the line $y = 4$ is a horizontal asymptote to the graph of $y = g(x)$.

Solution.

1. (a) To find a point on the graph of $y = f(-x)$, Theorem 1.8 tells us to multiply the x-coordinate of the point on the graph of $y = f(x)$ by -1: $((-1)2, -5) = (-2, -5)$.

    To check, since $(2, -5)$ is on the graph of f, we know $f(2) = -5$. Hence, when we substitute $x = -2$ into $y = f(-x)$, we get $y = f(-(-2)) = f(2) = -5$, proving $(-2, -5)$ is on the graph of $y = f(-x)$.

    (b) To find a point on the graph of $y = -f(x+2)$, we first note we have two transformations at work here, so we work our way from the inside out and build $f(x)$ to $-f(x+2)$.

    First, we find a point on the graph of $y = f(x+2)$. Writing $f(x+2) = f(x-(-2))$, we apply Theorem 1.7 with $h = -2$ and add -2 to (or subtract 2 from) the x-coordinate of the point we know is on $y = f(x)$: $(2-2, -5) = (0, -5)$.

    Next we apply Theorem 1.8 to the graph of $y = f(x+2)$ to get a point on the graph of $y = -f(x+2)$ by multiplying the y-coordinate of $(0, -5)$ by -1: $(0, (-1)(-5)) = (0, 5)$.

    To check, recall $f(2) = -5$ so that when we substitute $x = 0$ into the equation $y = -f(x+2)$, we get $y = -f(0+2) = -f(2) = -(-5) = 5$, as required.

    (c) Rewriting $f(8-x) = f(-x+8)$ we see we have two transformations at play here: a reflection across the y-axis and a horizontal shift. Since both of these transformations affect the x-coordinates of the graph, the question becomes which transformation to address first. To help us with this decision, we attack the problem algebraically.

    Recall that since $(2, -5)$ is on the graph of f, we know $f(2) = -5$. Hence, to get a point on the graph of $y = f(-x+8)$, we need to match up the arguments of $f(-x+8)$ and $f(2)$: $-x+8 = 2$.

    To solve this equation, we first subtract 8 from both sides to get $-x = -6$. Geometrically, subtracting 8 from the x-coordinate of $(2, -5)$, shifts the point $(2, -5)$ left 8 units to get the point $(-6, -5)$.

    Next, we multiply both sides of the equation $-x = -6$ by -1 to get $x = 6$. Geometrically, multiplying the x-coordinate of $(-6, -5)$ by -1 reflects the point $(-6, -5)$ across the y-axis to $(6, -5)$.

    To check we substitute $x = 6$ into $y = f(-x+8)$, and obtain $y = f(-6+8) = f(2) = -5$.

    Even though we have found our answer, we re-examine this process from a 'build' perspective. We began with a point on the graph of $y = f(x)$ and first shifted the graph to the left 8 units. Per Theorem 1.7, this point is on the graph of $y = f(x+8)$.

    Next we took a point on the graph of $y = f(x+8)$ and reflected it about the y-axis. Per Theorem 1.8, this put the point on the graph of $y = f(-x+8)$.

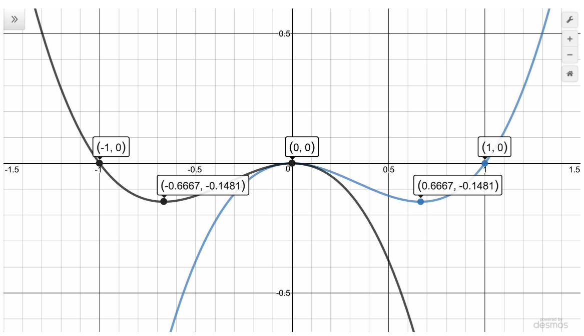
    In general, when faced with graphing functions in which there is both a horizontal shift and a reflection about the y-axis, we'll deal with the shift first.

2. In this example, the independent variable is s and the dependent variable is t. We are asked to reflect the graph of h about the t-axis, which in this case is the *vertical* axis. Hence, $H(s) = h(-s) = (-s)^3 - (-s)^2 = -s^3 - s^2$. Our confirmation is below on the left.

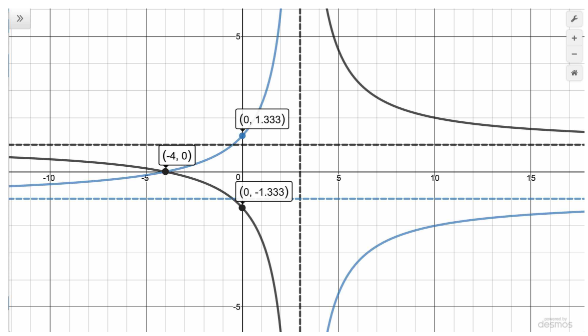
3. Comparing the formulas for $G(t) = \frac{t+4}{t-3}$ and $g(t) = \frac{t+4}{3-t}$, we have the same numerators, but in the denominator, we have $(t-3) = -(3-t)$:

$$G(t) = \frac{t+4}{t-3} = \frac{t+4}{-(3-t)} = -\frac{t+4}{3-t} = -g(t).$$

Hence, the graph of $y = G(t)$ should be the graph of $y = g(t)$ reflected across the t-axis. We check our answer below on the right.

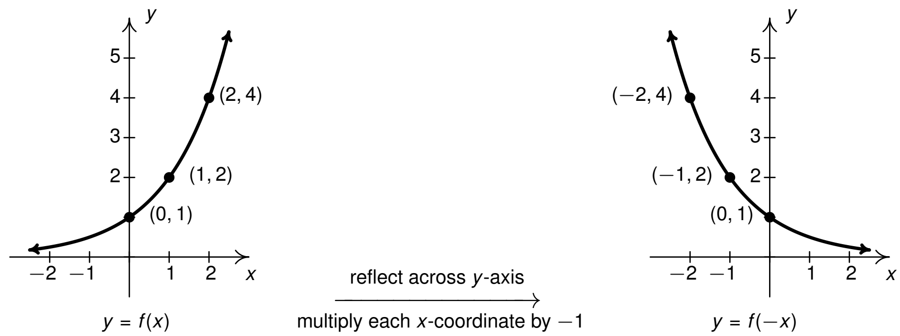


$t = s^3 - s^2$ (lighter color) and $t = -s^3 - s^2$

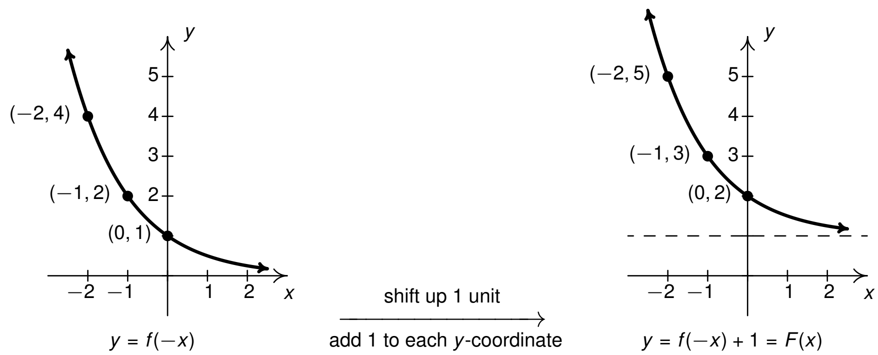


$y = \dfrac{t+4}{3-t}$ (lighter color) and $y = \dfrac{t+4}{t-3}$

4. (a) We have two transformations indicated with the formula $F(x) = f(-x) + 1$: a reflection across the y-axis and a vertical shift. Working from the inside out, we first tackle the reflection. Per Theorem 1.8, to obtain the graph of $y = f(-x)$ from $y = f(x)$, we multiply each of the x-coordinates of each of the points on the graph of $y = f(x)$ by (-1).



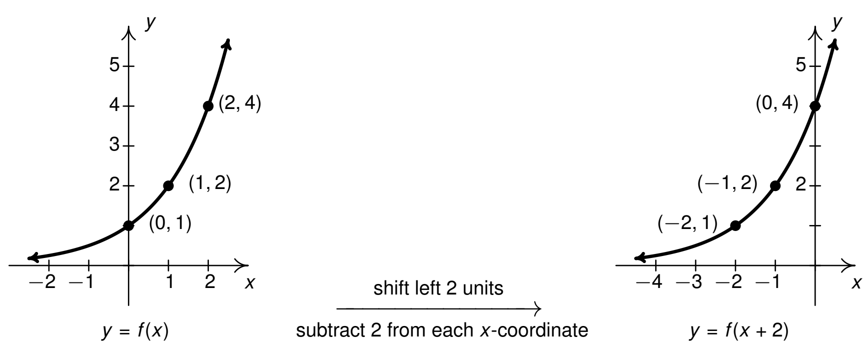
Next, we use Theorem 1.6 to obtain the graph of $y = f(-x) + 1$ from the graph of $y = f(-x)$ by adding 1 to each of the y-coordinates of each of the points on the graph of $y = f(-x)$. This shifts the graph of $y = f(-x)$ up one unit. Note, the horizontal asymptote $y = 0$ is also shifted up 1 unit to $y = 1$.



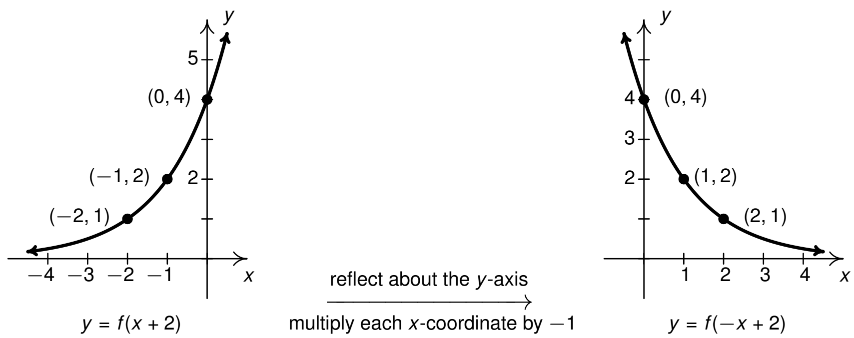
To check our answer, we begin with the point $(0, 2)$. Substituting $x = 0$ into $y = f(-x) + 1$, we get $y = f(-0) + 1 = f(0) + 1$. Since the point $(0, 1)$ is on the graph of f, we know $f(0) = 1$. Hence, $y = f(0) + 1 = 1 + 1 = 2$, so $(0, 2)$ is, indeed, on the graph of $y = f(-x) + 1$. We leave it to the reader to check the remaining points.

(b) In order to graph $F(x) = 1 - f(2 - x)$, we first rewrite as $F(x) = -f(-x + 2) + 1$ and note there are *four* modifications to the formula $f(x)$ indicated here.

Working from the inside out, we see we have a reflection about the y-axis indicated as well as a horizontal shift. From our work above, we know we first handle the shift: that is, we apply Theorem 1.7 to graph $y = f(x + 2) = f(x - (-2))$ by adding -2 to (subtracting 2 from) the x-coordinates of the points on the graph of $y = f(x)$.



Next, we use Theorem 1.8 to graph $y = f(-x + 2)$ starting with the graph of $y = f(x + 2)$ by multiplying each of the x-coordinates of the points of the graph of $y = f(x + 2)$ by -1. This reflects the graph of $f(x + 2)$ about the y-axis.

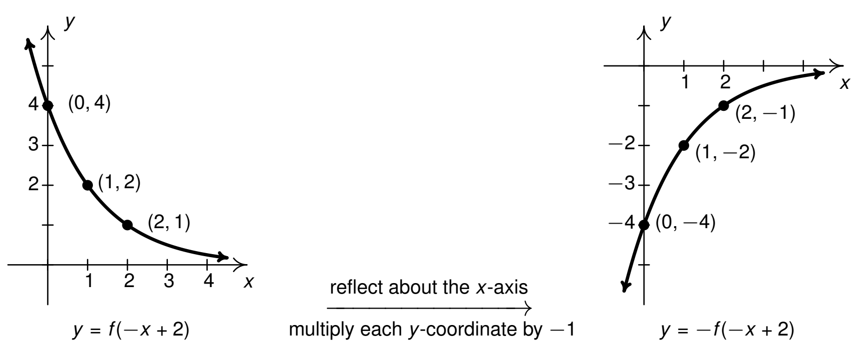


We have the graph of $y = f(-x+2)$ and need to build towards the graph of $y = -f(-x+2)+1$. The transformations that remain are a reflection about the x-axis and a vertical shift. The question is which to do first.

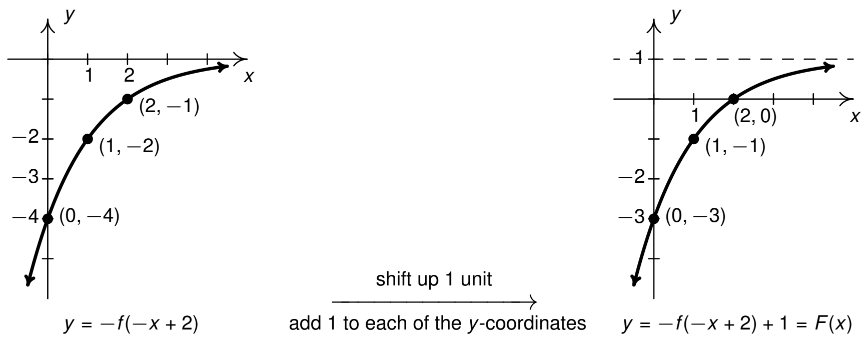
Once again, we can think algebraically about the problem. We know the point $(0,1)$ is on the graph of f which means $f(0) = 1$. This point corresponds to the point $(2,1)$ on the graph of $f(-x+2)$. Indeed, when we substitute $x = 2$ into $y = f(-x+2)$, we get $y = f(-2+2) = f(0) = 1$.

If we substitute $x = 2$ into the formula $y = -f(-x+2)+1$, we get $y = -f(-2+2)+1 = -f(0)+1 = -1(1)+1 = 0$. That is, we first multiply the y-coordinate of $(2,1)$ by -1 then add 1. This suggests we take care of the reflection about the x-axis first, then the vertical shift.

We proceed below to obtain the graph of $y = -f(-x+2)$ from $y = f(-x+2)$ by multiplying each of the y-coordinates on the graph of $y = f(-x+2)$ by -1. Note the horizontal asymptote remains unchanged: $y = (-1)(0) = 0$.



Finally, we take care of the vertical shift. Per Theorem 1.6, we graph $y = -f(-x+2)+1$ by adding 1 to the y-coordinates of each of the points on the graph of $y = -f(-x+2)$. This moves the graph up one unit, including the horizontal asymptote: $y = 0 + 1 = 1$.



To check, we begin with the point $(2, 0)$. Substituting $x = 2$ into $y = 1 - f(2 - x)$, we obtain $y = 1 - f(2 - 2) = 1 - f(0)$. Since $(0, 1)$ is on the graph of f, we know $f(0) = 1$. This means $y = 1 - f(2 - 2) = 1 - f(0) = 1 - 1 = 0$. This proves $(2, 0)$ is on the graph of $y = 1 - f(2 - x)$, and we recommend the reader check the remaining points.

5. With the transformations at our disposal, our task amounts to finding values of h and k and choosing between signs $\pm$ so that $g(x) = \pm f(\pm x - h) + k$.

   Based on the horizontal asymptote, $y = 4$, we choose $k = 4$. Note, however, in the graph of $y = f(x) + 4$, the entire graph is *above* the line $y = 4$. Since the graph of g approaches the asymptote from below, we know $y = -f(\pm x - h) + 4$.

   Hence, two of transformations applied to the graph of f are a reflection across the x-axis followed by a shift up 4 units. This means the point $(0, 1)$ on the graph of f must correspond to the point $(-1, 3)$ on the graph of g, since these are the points closest to the asymptote on each graph.

   Likewise, the points $(1, 2)$ and $(2, 4)$ on the graph of f must correspond to $(0, 2)$ and $(1, 0)$, respectively, on the graph of g. Looking at the x-coordinates only, we have $x = 0$ moves to $x = -1$, $x = 1$ moves to $x = 0$, and $x = 2$ moves to $x = 1$. Hence, the net effect on the x-values is a shift left 1 unit. Hence, we guess the formula for $g(x)$ to be $g(x) = -f(x + 1) + 4$.

   We can readily check by going through the transformations: first, shift left 1 unit; next, reflect across the x-axis; finally, shift up 4. We leave it to the reader to verify that tracking each of the points on the graph of f along with the horizontal asymptote through this sequence of transformations results in the graph of g.

   One way to recover the graph of f from the graph of g is to reverse the process by which we obtained g from f. The challenge here comes from the fact that two different operations were done which affected the y-values: reflection and shifting - and the order in which these are done matters.

   To motivate our methodology, let's consider a more down-to-earth example like putting on socks and then putting on shoes. Unless we're very talented, to reverse this process, we take off the shoes first, then the socks - that is, we undo each step in the reverse order.[2] In the same way, when we

[2] We'll have more to say about this sort of thing in Section 1.6.

think about reversing the steps transforming the graph of f to the graph of g, we need to undo each transformation in the opposite order.

To review, we obtained the graph of g from the graph of f by first shifting the graph to the left 1 unit, then reflecting the graph about the x-axis, then, finally, shifting the graph up 4 units. Hence, we first undo the vertical shift. Instead of shifting the graph *up* four units, we shift the graph *down* four units. This takes the graph of $y = g(x)$ to $y = g(x) - 4$.

Next, we have to undo the refection across the x-axis. Thinking at the level of points, to recover the point (a, b) from its reflection across the x-axis, $(a, -b)$, we simply reflect across the x-axis again: $(a, -(-b)) = (a, b)$. Per Theorem 1.8, this takes the graph the graph of $y = g(x) - 4$ to the graph of $y = -[g(x) - 4] = -g(x) + 4$.[3]

Last, to undo moving the graph to the *left* 1 unit, we move the graph of $y = -g(x) + 4$ to the *right* 1 unit. Per Theorem 1.7, we accomplish this by graphing $y = -g(x - 1) + 4$. We leave it to the reader to start with the graph of $y = g(x)$ and graph $y = -g(x - 1) + 4$ and show it matches the graph of $y = f(x)$. □

Some remarks about Example 1.4.2 are in order. In number 1c above, to find a point on the graph of $y = f(-x+8)$, we took the given x-coordinate on our starting graph, 2, and subtracted 8 first then multiplied by -1. If this seems somehow 'backwards' it should.

When *evaluating* the expression $-x + 8$, the order of operations mandates we multiply by -1 first then add 8. Here, however, we weren't *evaluating* an expression - we were *solving* an equation: $-x + 8 = 2$, which meant we did the exact opposite steps in the opposite order.[4] This exemplifies a larger theme with transformations: when adjusting inputs, the resulting points on the graph are obtained by applying the opposite operations indicated by the formula in the opposite order of operations.

On the other hand, when it came to multiple transformations involving the y-coordinates, we followed the order of operations. As in 4b above, when it came to applying a reflection about the x-axis and a vertical shift, we applied the reflection first, then the shift. This is because instead of *solving* an *equation* to find the new y-coordinates, we were *simplifying* an *expression*. Again, this is an example of a much larger theme: when adjusting outputs, the resulting points on the graph are obtained by applying the stated operations in the usual order.

Last but not least, in number 5, to find f in terms of g, we reversed the steps used to transform f into g. Another tact is to approach the problem in the same way we approached transforming f into g: namely, starting with the graph of g, determine values h and k and signs $\pm$ so that $f(x) = \pm g(\pm x - h) + k$. We leave this to the reader.

[3]To see this better, let us temporarily write $F(x) = g(x) - 4$. Theorem 1.8 tells us to reflect the graph of F about the x-axis, graph $y = -F(x) = -[g(x) - 4] = -g(x) + 4$.

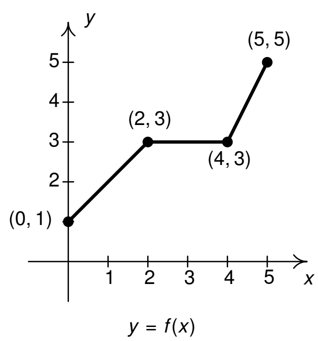
[4]Note that dividing by -1 is the same as multiplying by -1, so to keep with the 'opposite steps in opposite order' theme, we could more precisely say we subtracted 8 and *divided* by -1.

1.4.3 Scalings

We now turn our attention to our last class of transformations: **scalings**. A thorough discussion of scalings can get complicated because they are not as straight-forward as the previous transformations. A quick review of what we've covered so far, namely vertical shifts, horizontal shifts and reflections, will show you why those transformations are known as **rigid transformations**.

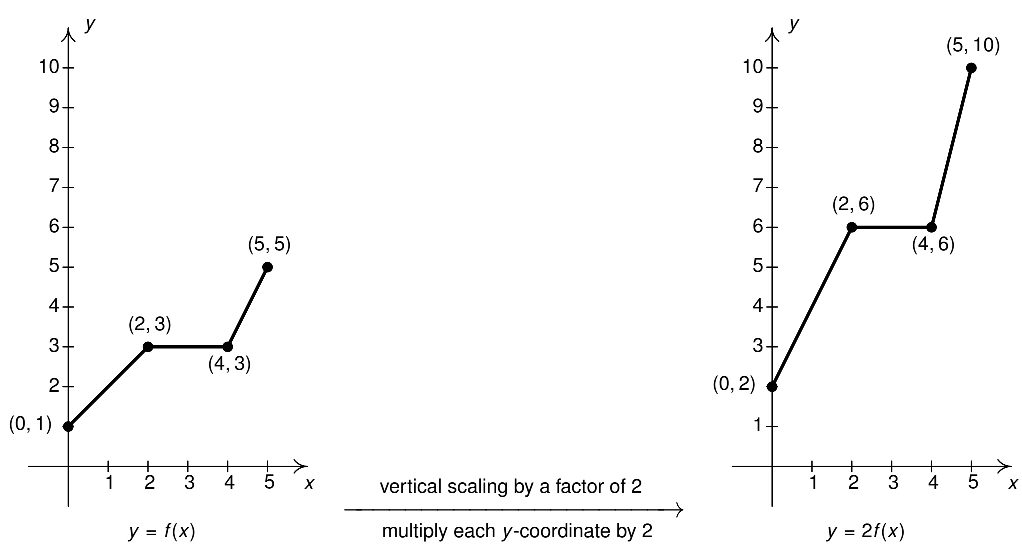
Simply put, rigid transformations preserve the distances between points on the graph - only their position and orientation in the plane change.[5] If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be affecting the distance between points. These sorts of transformations are hence called **non-rigid**. As always, we motivate the general theory with an example.

Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given at the beginning of the section. From its graph, we can build a table of values for g as before.



x	$(x, f(x))$	$f(x)$	$g(x) = 2f(x)$	$(x, g(x))$
0	$(0, 1)$	1	2	$(0, 2)$
2	$(2, 3)$	3	6	$(2, 6)$
4	$(4, 3)$	3	6	$(4, 6)$
5	$(5, 5)$	5	10	$(5, 10)$

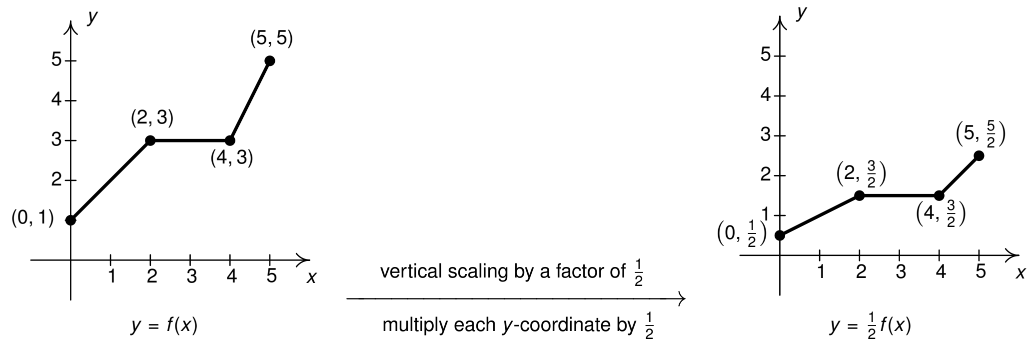
Graphing, we get:



[5]Another word that can be used here instead of 'rigid transformation' is 'isometry' - meaning 'same distance.'

In general, if (a, b) is on the graph of f, then $f(a) = b$ so that $g(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of g. In other words, to obtain the graph of g, we multiply all of the y-coordinates of the points on the graph of f by 2. Multiplying all of the y-coordinates of all of the points on the graph of f by 2 causes what is known as a 'vertical scaling[6] by a factor of 2.'

If we wish to graph $y = \frac{1}{2}f(x)$, we multiply the all of the y-coordinates of the points on the graph of f by $\frac{1}{2}$. This creates a 'vertical scaling[7] by a factor of $\frac{1}{2}$' as seen below.



These results are generalized in the following theorem.

THEOREM 1.9. **Vertical Scalings.** Suppose f is a function and $a > 0$ is a real number.
To graph $F(x) = af(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by a.

- If $a > 1$, we say the graph of f has undergone a vertical stretch[a] by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrink[b] by a factor of $\frac{1}{a}$.

[a]expansion, dilation
[b]compression, contraction

The proof of Theorem 1.9 mimics the proofs of Theorems 1.6 and 1.8. If c is in the domain of f, then $(c, f(c))$ is on the graph of f and the corresponding point on the graph of $F(x) = af(x)$ is $(c, F(c)) = (c, af(c))$. Comparing the points $(c, f(c))$ and $(c, af(c))$ proves the theorem.

A few remarks about Theorem 1.9 are in order. First, a note about the verbiage. To the authors, the words 'stretch', 'expansion', and 'dilation' all indicate something getting bigger. Hence, 'stretched by a factor of 2' makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrink', 'compression' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it 'shrinks by a factor of 2' - not 'shrinks by a factor of $\frac{1}{2}$'. This is why we have written the descriptions 'stretch by a factor of a' and 'shrink by a factor of $\frac{1}{a}$' in the statement of the theorem.

Second, in terms of inputs and outputs, Theorem 1.9 says multiplying the *outputs* from a function by positive number a causes the graph to be vertically scaled by a factor of a. It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

[6]Also called a 'vertical stretch,' 'vertical expansion' or 'vertical dilation' by a factor of 2.
[7]Also called 'vertical shrink,' 'vertical compression' or 'vertical contraction' by a factor of 2.

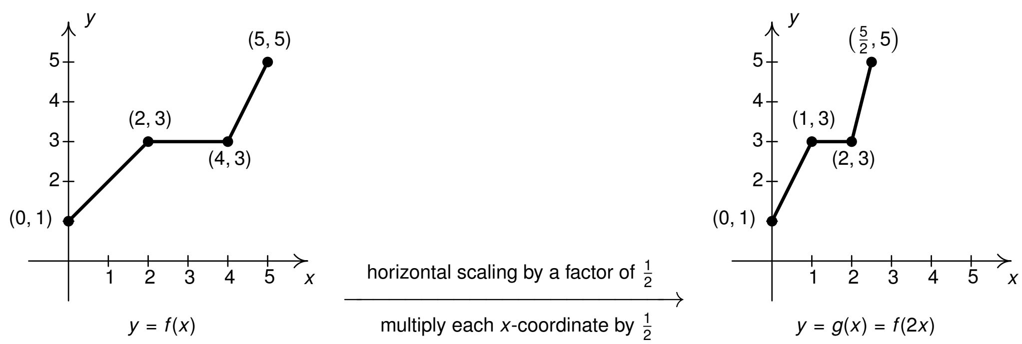
Referring to the graph of f given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to f by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem 1.7, as seen in the table on the left below.

We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on g which corresponds to the point $(2, 3)$ on the graph of f, we set $2x = 2$ so that $x = 1$. Substituting $x = 1$ into $g(x)$, we obtain $g(1) = f(2 \cdot 1) = f(2) = 3$, so that $(1, 3)$ is on the graph of g. Continuing in this fashion, we obtain the table on the lower right.

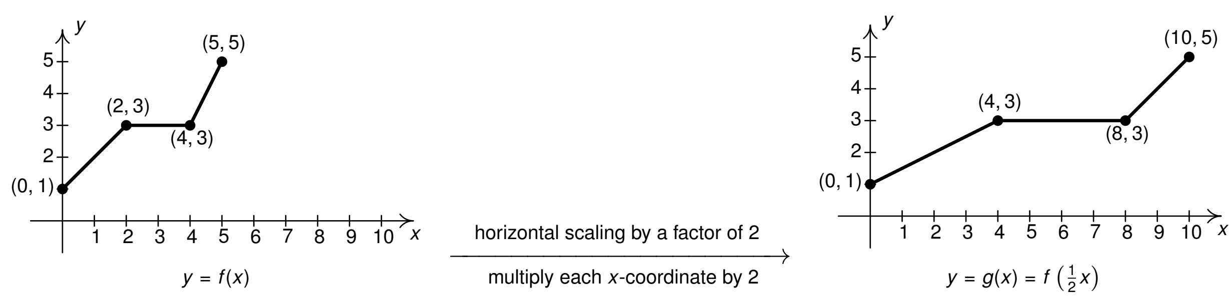
x	$(x, f(x))$	$f(x)$	$g(x) = f(2x)$	$(x, g(x))$
0	$(0, 1)$	1	$f(2 \cdot 0) = f(0) = 1$	$(0, 1)$
2	$(2, 3)$	3	$f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(2 \cdot 4) = f(8) = ?$	
5	$(5, 5)$	5	$f(2 \cdot 5) = f(10) = ?$	

x	$2x$	$g(x) = f(2x)$	$(x, g(x))$
0	0	$g(0) = f(2 \cdot 0) = f(0) = 1$	$(0, 0)$
1	2	$g(1) = f(2 \cdot 1) = f(2) = 3$	$(1, 3)$
2	4	$g(2) = f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f\left(2 \cdot \frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$

In general, if (a, b) is on the graph of f, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2 \cdot \frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of g. In other words, to graph g we divide the x-coordinates of the points on the graph of f by 2. This results in a horizontal scaling[8] by a factor of $\frac{1}{2}$.



If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling[9] by a factor of 2, as demonstrated below.



We have the following theorem.

[8] Also called 'horizontal shrink,' 'horizontal compression' or 'horizontal contraction' by a factor of 2.
[9] Also called 'horizontal stretch,' 'horizontal expansion' or 'horizontal dilation' by a factor of 2.

THEOREM 1.10. **Horizontal Scalings.** Suppose f is a function and $b > 0$ is a real number. To graph $F(x) = f(bx)$, divide each of the x-coordinates of the points on the graph of $y = f(x)$ by b.

- If $0 < b < 1$, we say the graph of f has undergone a horizontal stretch[a] by a factor of $\frac{1}{b}$.
- If $b > 1$, we say the graph of f has undergone a horizontal shrink[b] by a factor of b.

[a]expansion, dilation
[b]compression, contraction

The proof of Theorem 1.10 follows closely the spirit of the proof of Theorems 1.7 and 1.8. If c is an element of the domain of f, them the number $\frac{c}{b}$ corresponds to a domain element of $F(x) = f(bx)$ since $F\left(\frac{c}{b}\right) = f\left(b \cdot \frac{c}{b}\right) = f(c)$. Hence, there is a correspondence between the point $(c, f(c))$ on the graph of f and the point $\left(\frac{c}{b}, F\left(\frac{c}{b}\right)\right) = \left(\frac{c}{b}, f(c)\right)$ on the graph of F. We can obtain $\left(\frac{c}{b}, f(c)\right)$ by dividing the x-coordinate of $(c, f(c))$ by b and the result follows.

Theorem 1.10 tells us that if we multiply the input to a function by b, the resulting graph is scaled horizontally by a factor of $\frac{1}{b}$. The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

EXAMPLE 1.4.3. Use Theorems 1.6, 1.7, 1.8, 1.9 and 1.10 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1, 4)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (a) $y = 3f(x-2)$ (b) $y = f\left(-\frac{1}{2}x\right)$ (c) $f(2x-3)+1$

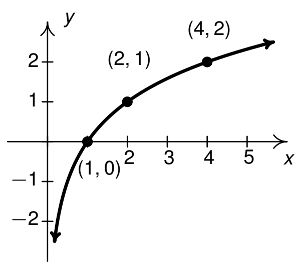
2. Find a formula for a function $G(t)$ whose graph is the same as $y = g(t) = \frac{2t+1}{t-1}$ but is vertically stretched by a factor of 4.

3. Predict how the graph of $H(s) = 8s^3 - 12s^2$ relates to the graph of $h(s) = s^3 - 3s^2$.

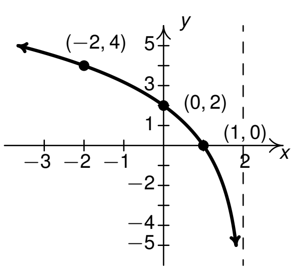
4. Below on the left is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = \dfrac{1 - f(x)}{2}$ (b) $F(x) = f\left(\dfrac{1-x}{2}\right)$

5. Below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.



$y = f(x)$



$y = g(x)$

NOTE: The y-axis, $x = 0$, is a vertical asymptote to the graph of $y = f(x)$ and the line $x = 2$ is a vertical asymptote to the graph of $y = g(x)$.

Solution.

1. (a) As we examine the formula $y = 3f(x-2)$, we note two modifications from $y = f(x)$. Building from the inside out, we start with obtaining a point on the graph of $y = f(x-2)$.

   Per Theorem 1.7, this shifts all of the points on the graph of $y = f(x)$ 2 units to the right. Hence, the point $(-1, 4)$ on the graph of $y = f(x)$ moves to the point $(-1+2, 4) = (1, 4)$ on the graph of $y = f(x-2)$.

   To get a point on the graph of $y = 3f(x-2) = af(x-3)$, we apply Theorem 1.9 with $a = 3$ to the point $(1, 4)$ on the graph of $y = f(x-2)$ to get the point $(1, 3(4)) = (1, 12)$ on the graph of $y = 3f(x-2)$.

   To check, we note that since $(-1, 4)$ is on the graph of $y = f(x)$, we know $f(-1) = 4$. Hence, when we substitute $x = 1$ into the $y = 3f(x-2)$, we get $y = 3f(1-2) = 3f(-1) = 3(4) = 12$.

   (b) The formula $y = f\left(-\frac{1}{2}x\right)$ also indicates two transformations: a horizontal scaling, indicated by $\frac{1}{2}$ factor, as well as a reflection across the y-axis. The question before us is which to do first.

   If we return to algebra for inspiration, we know $f(-1) = 4$, so we match up the arguments of $f\left(-\frac{1}{2}x\right)$ and $f(-1)$ and get the equation $-\frac{1}{2}x = -1$. We solve this equation by multiplying both sides by -2: $x = (-2)(-1) = 2$. That is, we take the original x-value on the graph of $y = f(x)$ and multiply it by -2.

   If we think of $-2 = (-1)(2)$ then multiplying by the '2' in '$(-1)(2)$' produces a horizontal stretch by a factor of 2 while multiplying by the '-1' reflects the point across the y-axis.

   Applying the horizontal stretch first, we use Theorem 1.10 and start with the point $(-1, 4)$ on the graph of $y = f(x)$ and multiply the x-coordinate by 2 to obtain a point on the graph of $y = f\left(\frac{1}{2}x\right)$: $(-1(2), 4) = (-2, 4)$.

   Next, we take care of the reflection about the y-axis using Theorem 1.8 . Starting with $(-2, 4)$ on the graph of $y = f\left(\frac{1}{2}x\right)$, we multiply the x-coordinate by -1 to obtain a point on the graph of $y = f\left(\frac{1}{2}(-x)\right) = f\left(-\frac{1}{2}x\right)$: $((-1)(-2), 4) = (2, 4)$.

   To check, note when $x = 2$ is substituted into $y = f\left(-\frac{1}{2}x\right)$, we get $y = f\left(-\frac{1}{2}(2)\right) = f(-1) = 4$.

   Of course, we could have equally written the multiple $-2 = (2)(-1)$ and reversed these steps: doing the reflection first, then the horizontal scaling.

   Proceeding this way, we start with the point $(-1, 4)$ on the graph of $y = f(x)$ and reflect across the y-axis to obtain the point $((-1)(-1), 4) = (1, 4)$ on the graph of $y = f(-x)$.

   Next, we stretch the graph of $y = f(-x)$ by a factor of 2 by multiplying the x-coordinates of the points on the graph by 2 and obtain $(2(1), 4) = (2, 4)$ on the graph of $y = f\left(-\frac{1}{2}x\right)$.

   In general when it comes to reflections and scalings, whether horizontal or, as we'll see soon, vertical, either order will produce the same results.

   (c) The formula $f(2x-3)+1$ indicates *three* transformations: a horizontal shift, a horizontal scaling, and a vertical shift. As usual, we appeal to algebra to give us guidance on which horizontal transformation to apply first.

Since we know $f(-1) = 4$, we set $2x - 3 = -1$ and solve. Our first step is to add 3 to both sides: $2x = (-1) + 3 = 2$. Since we are adding 3 to the given x-value -1, this corresponds to a shift to the right 3 units, so the point $(-1, 4)$ is moved to the point $(2, 4)$.

Next, to solve $2x = 2$, we divide this new x-coordinate 2 by 2 and get $x = \frac{2}{2} = 1$ which corresponds to a horizontal compression by a factor of 2. This moves the point $(2, 4)$ to $(1, 4)$.

Hence, the algebra suggests we use Theorem 1.7 first and follow it up with Theorem 1.10. Starting with $(-1, 4)$ on the graph of $y = f(x)$, we shift to the right 3 units to obtain the point $(-1 + 3, 4) = (2, 4)$ on the graph of $y = f(x - 3)$.

Next, we start with the point $(2, 4)$ on the graph of $y = f(x-3)$ and horizontally shrink the x-axis by a factor of 2 to get the point $\left(\frac{2}{2}, 4\right) = (1, 4)$ on the graph of $y = f(2x - 3)$.

Last, but not least, we take care of the vertical shift using Theorem 1.6. Starting with the point $(1, 4)$ on the graph of $y = f(2x-3)$, we add 1 to the y-coordinate to get the point $(1, 4+1) = (1, 5)$ on the graph of $y = f(2x - 3) + 1$.

To check, we substitute $x = 1$ into the formula $y = f(2x - 3) + 1$ and get $y = f(2(1) - 3) + 1 = f(-1) + 1 = 4 + 1 = 5$, as required.

2. To vertically stretch the graph of $y = g(t)$ by 4, we use Theorem 1.9 with $a = 4$ to get

$$G(t) = 4g(t) = 4\frac{2t + 1}{t - 1} = \frac{4(2t + 1)}{t - 1} = \frac{8t + 4}{t - 1}.$$

We check our answer below on the left.

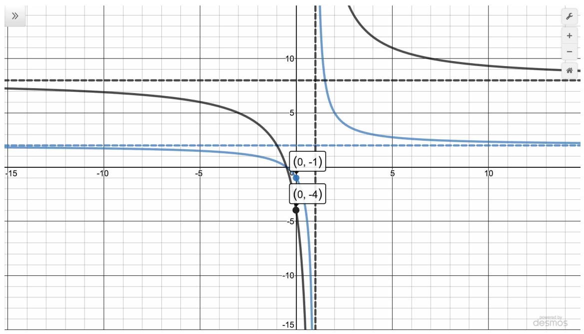
3. When comparing the formulas for $H(s) = 8s^3 - 12s^2$ and $h(s) = s^3 - 3s^2$, it doesn't appear as if any shifting or reflecting is going on (why not?)

We also note that since the coefficient of s^3 in the expression of $H(s)$ is 8 times that of the coefficient of s^3 in $h(s)$, but the coefficient of s^2 in $H(s)$ is only 4 times the coefficient of s^2 in $h(s)$, the change is not the result of a vertical scaling (again, why not?)

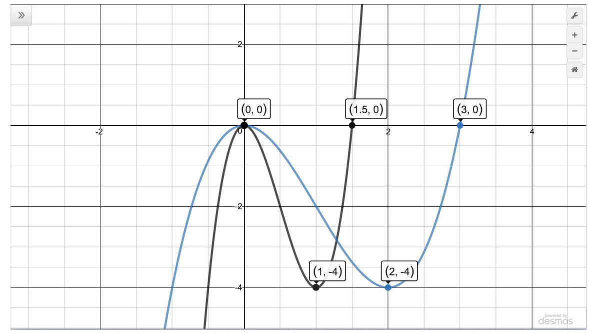
Hence, if anything, we are looking for a horizontal scaling. In other words, we are looking for a real number $b > 0$ so $h(bs) = H(s)$, that is, $(bs)^3 - 3(bs)^2 = b^3s^3 - 3b^2s^2 = 8s^3 - 12s^2$.

Matching up coefficients of s^3 gives $b^3 = 8$ so $b = 2$ which checks with the coefficients of s^2 : $3b^2 = 3(2)^2 = 12$.

Hence, we predict the graph of $y = H(s)$ to be the same as $y = h(s)$ except horizontally compressed by a factor of 2. Our check is below on the right.



$y = g(t) = \frac{2t+1}{t-1}$ (lighter color) and $y = 4g(t) = \dfrac{8t+4}{t-1}$



$y = h(s) = s^3 - 3s^2$ (lighter color) and $y = H(s) = 8s^3 - 12s^2$

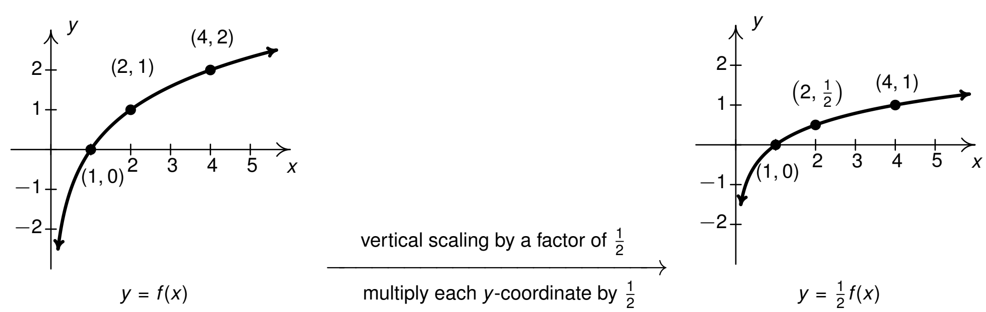
4. (a) We first rewrite the expression for $F(x) = \frac{1-f(x)}{2} = -\frac{1}{2}f(x) + \frac{1}{2}$ in order to use the theorems available to us. Note we have two modifications to the formula of $f(x)$ which correspond to three transformations.

Multiplying $f(x)$ by $-\frac{1}{2}$ indicates a vertical compression by a factor of 2 along with a reflection about the x-axis. Adding $\frac{1}{2}$ indicates a vertical shift up $\frac{1}{2}$ units.

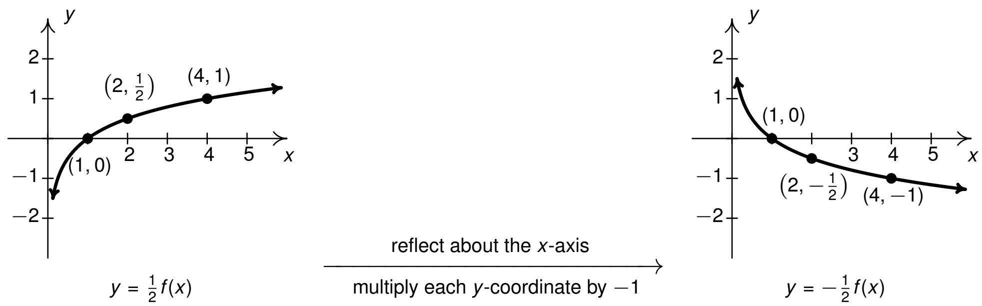
As always the question is which to do first. Once again, we look to algebra for the answer. Picking the point $(1, 0)$ on the graph of $f(x)$, we know $f(1) = 0$. To see which point this corresponds to on the graph of $y = F(x)$, we find $F(1) = -\frac{1}{2}f(1) + \frac{1}{2} = -\frac{1}{2}(0) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$.

Hence, we first multiplied the y-value 0 by $-\frac{1}{2}$. As above, we can think of $-\frac{1}{2} = (-1)\frac{1}{2}$ so that multiplying by $-\frac{1}{2}$ amounts to a vertical compression by a factor of 2 first, then the refection about the x-axis second. Lastly, adding the $\frac{1}{2}$ is the vertical shift up $\frac{1}{2}$ unit.

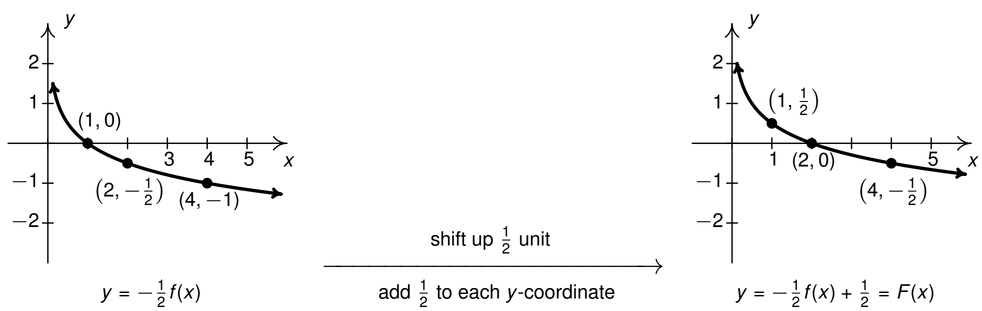
Beginning with the vertical scaling by a factor of $\frac{1}{2}$, we use Theorem 1.9 to graph $y = \frac{1}{2}f(x)$ starting from $y = f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = f(x)$ by $\frac{1}{2}$.



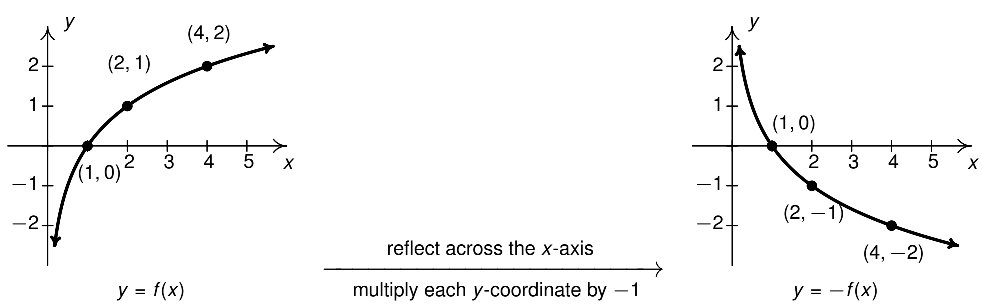
Next, we reflect the graph of $y = \frac{1}{2}f(x)$ across the x-axis to produce the graph of $y = -\frac{1}{2}f(x)$ by multiplying each of the y-coordinates of the points on the graph of $y = \frac{1}{2}f(x)$ by -1:



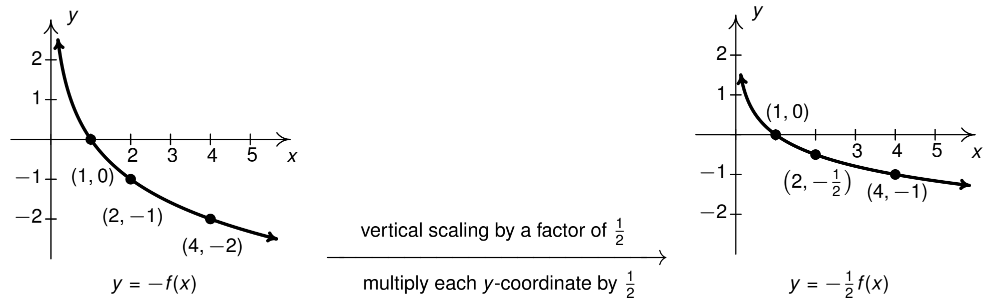
Finally, we shift the graph of $y = -\frac{1}{2}f(x)$ vertically up $\frac{1}{2}$ unit by adding $\frac{1}{2}$ to each of the y-coordinates of each of the points to obtain the graph of $y = -\frac{1}{2}f(x) + \frac{1}{2} = F(x)$.



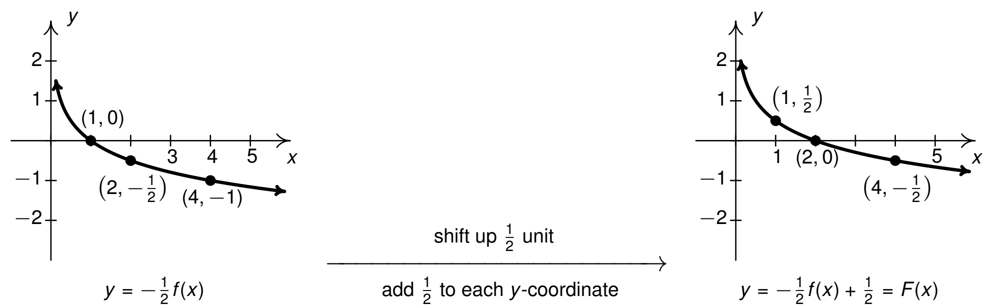
Note that as with horizontal scalings and reflections about the y-axis, the order of vertical scalings and reflections across the x-axis is interchangeable. Had we decided to think of the factor $-\frac{1}{2} = \frac{1}{2} \cdot (-1)$, we could have just as well started with the graph of $y = f(x)$ and produced the graph of $y = -f(x)$ first:



Next, we vertically scale the graph of $y = -f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = -f(x)$ by $\frac{1}{2}$ to obtain the graph of $y = -\frac{1}{2}f(x)$:



Notice we've reached the same graph of $y = -\frac{1}{2}f(x)$ that we had before, and, hence we arrive at the same final answer as before:



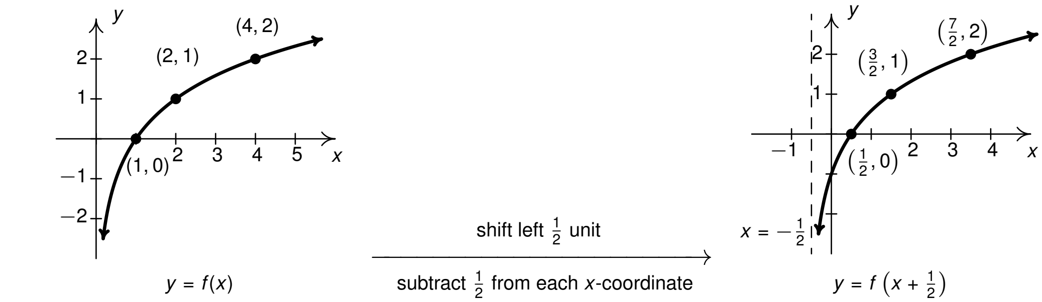
We check our answer as we have so many times before. We start with the point $\left(1, \frac{1}{2}\right)$ and substitute $x = 1$ into $y = \frac{1-f(x)}{2}$ to get $y = \frac{1-f(1)}{2}$. From the graph of f, we know $f(1) = 0$, so we get $y = \frac{1-f(1)}{2} = \frac{1-0}{2} = \frac{1}{2}$. This proves $\left(1, \frac{1}{2}\right)$ is on the graph of $y = \frac{1-f(x)}{2}$. We invite the reader to check the remaining points.

Note that in the preceding example, since none of the transformations included adjusting the x-coordinates of points, the vertical asymptote, $x = 0$ remained in place.

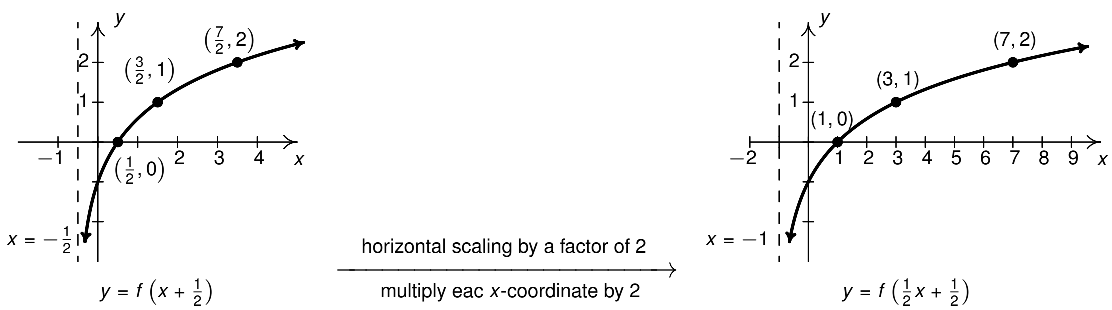
(b) As with the previous example, we first rewrite $F(x) = f\left(\frac{1-x}{2}\right) = F\left(-\frac{1}{2}x + \frac{1}{2}\right)$. Here again, we have two modifications to the formula $f(x)$, the $-\frac{1}{2}$ multiple indicating a horizontal scaling and a reflection across the y-axis and a horizontal shift.

Based on our experience from previous examples, we do the horizontal shift first, with the order of the scaling and reflection more or less irrelevant.

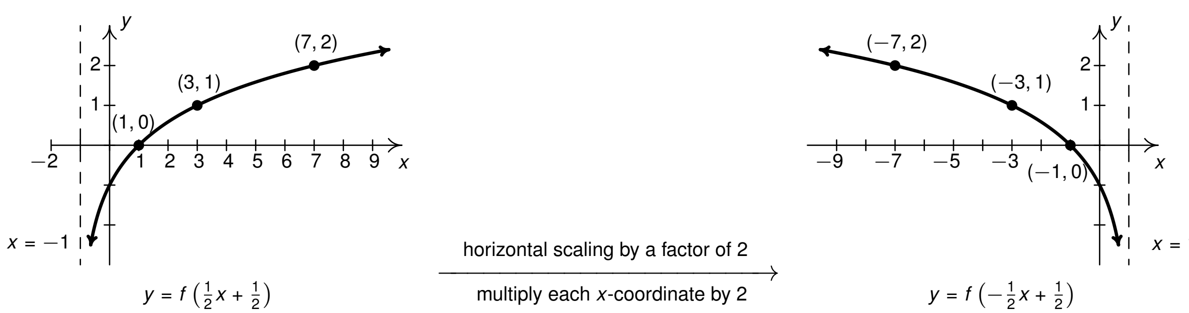
To produce the graph of $y = f\left(x + \frac{1}{2}\right)$ we subtract $\frac{1}{2}$ from each of the x-coordinates of each of the points on the graph of $y = f(x)$. This moves the graph to the left $\frac{1}{2}$ unit, including the vertical asymptote $x = 0$ which moves to $x = -\frac{1}{2}$.



Next, we graph $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ starting with $y = f\left(x + \frac{1}{2}\right)$ by horizontally expanding the graph by a factor of 2. That is, we multiply each x-coordinates on the graph of $y = f\left(x + \frac{1}{2}\right)$ by 2, including the vertical asymptote, $x = -\frac{1}{2}$ which moves to $x = 2\left(-\frac{1}{2}\right) = -1$.



Finally, we reflect the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ about the y-axis to graph $y = f\left(-\frac{1}{2}x + \frac{1}{2}\right)$. We accomplish this by multiplying each of the x-coordinates of each of the points on the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ by -1. This includes the vertical asymptote which is moved to $x = (-1)(-1) = 1$.



To check our answer, we begin with the point $(-1, 0)$ and substitute $x = -1$ into $y = f\left(\frac{1-x}{2}\right)$. We get $y = f\left(\frac{1-(-1)}{2}\right) = f\left(\frac{2}{2}\right) = f(1)$. From the graph of f, we know $f(1) = 0$, hence we have $y = f(1) = 0$, proving $(-1, 0)$ is on the graph of $y = f\left(\frac{1-x}{2}\right)$. The reader is encouraged to check the remaining points.

As mentioned previously, instead of doing the horizontal scaling first, then the reflection, we could have done the reflection first, then the scaling. We leave this to the reader to check.

5. To write $g(x)$ in terms of $f(x)$, we assume we can find real numbers a, b, h, and k and choose signs $\pm$ so that $g(x) = \pm af(\pm bx - h) + k$.

   The most notable change we see is the vertical asymptote $x = 0$ has moved to $x = 2$. Moreover, instead of the graph increasing off to the right, it is decreasing coming in from the left. This suggests a horizontal shift of 2 units as well as a reflection across the y-axis.

   Since we always shift first then reflect, we have a shift *left* of 2 units followed by a reflection about the y-axis. In other words, $g(x) = \pm af(-x + 2) + k$.

   Comparing y-values, the y-values on the graph of g appear to be exactly twice the corresponding values on the graph of f, indicating a vertical stretch by a factor of 2. Hence, we get $g(x) = 2f(-x+2)$. We leave it to the reader to check the graph of $y = 2f(-x + 2)$ matches the graph of $y = g(x)$.

   To write $f(x)$ in terms of $g(x)$, we reverse the steps done in obtaining the graph of $g(x)$ from $f(x)$ in the reverse order.

   Since to get from the graph of f to the graph of g, we: first, shifted left 2 units; second reflected across the y-axis; third, vertically stretched by a factor of 2, our first step in taking g back to f is to implement a vertical compression by a factor of 2. Hence, starting with the graph of $y = g(x)$, our first step results in the formula $y = \frac{1}{2}g(x)$.

   Next, we need to undo the reflection abut the y-axis. If the point (a, b) is reflected about the y-axis, we obtain the point $(-a, b)$. To return to the point (a, b), we reflect $(-a, b)$ across the y-axis again: $(-(-a), b) = (a, b)$. Hence, we take the graph of $y = \frac{1}{2}g(x)$ and reflect it across the y-axis to obtain $y = \frac{1}{2}g(-x)$.

   Our last step is to undo a horizontal shift to the left 2 units. The reverse of this process is shifting the graph to the *right* two units, so we get $y = \frac{1}{2}g(-(x - 2)) = \frac{1}{2}g(-x + 2)$.[10]

   We leave it to the reader to start with the graph of $y = g(x)$ and check the graph of $y = \frac{1}{2}g(-x + 2)$ matches the graph of $y = f(x)$. $\square$

1.4.4 Transformations in Sequence

Now that we have studied three basic classes of transformations: shifts, reflections, and scalings, we present a result below which provides one algorithm to follow to transform the graph of $y = f(x)$ into the graph of $y = af(bx - h) + k$ without the need of using Theorems 1.6, 1.7, 1.8, 1.9 and 1.10 individually.

Theorem 1.11 is the ultimate generalization of Theorems **??**, **??**, **??**, **??**, **??** and **??**. We note the underlying assumption here is that regardless of the order or number of shifts, reflections and scalings applied to the graph of a function f, we can always represent the final result in the form $g(x) = af(bx - h) + k$.

[10] To see this better, let $F(x) = \frac{1}{2}g(-x)$. Per Theorem 1.7, the graph of $F(x - 2) = \frac{1}{2}g(-(x - 2)) = \frac{1}{2}g(-x + 2)$ is the same as the graph of F but shifted 2 units to the right.

Since each of these transformations can ultimately be traced back to composing f with linear functions,[11] this fact is verified by showing compositions of linear functions results in a linear function.[12]

THEOREM 1.11. **Transformations in Sequence.** Suppose f is a function. If $a, b \neq 0$, then to graph $g(x) = af(bx - h) + k$ start with the graph of $y = f(x)$ and follow the steps below.

1. Add h to each of the x-coordinates of the points on the graph of f.

   NOTE: This results in a horizontal shift to the left if $h < 0$ or right if $h > 0$.

2. Divide the x-coordinates of the points on the graph obtained in Step 1 by b.

   NOTE: This results in a horizontal scaling, but includes a reflection about the y-axis if $b < 0$.

3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by a.

   NOTE: This results in a vertical scaling, but includes a reflection about the x-axis if $a < 0$.

4. Add k to each of the y-coordinates of the points on the graph obtained in Step 3.

   NOTE: This results in a vertical shift up if $k > 0$ or down if $k < 0$.

Theorem 1.11 can be established by generalizing the techniques developed in this section. Suppose $(c, f(c))$ is on the graph of f. To match up the inputs of $f(bx - h)$ and $f(c)$, we solve $bx - h = c$ and solve.

We first add the h (causing the horizontal shift) and then divide by b. If b is a positive number, this induces only a horizontal scaling by a factor of $\frac{1}{b}$. If $b < 0$, then we have a factor of -1 in play, and dividing by it induces a reflection about the y-axis. So we have $x = \frac{c+h}{b}$ as the input to g which corresponds to the input $x = c$ to f.

We now evaluate $g\left(\frac{c+h}{b}\right) = af\left(b \cdot \frac{c+h}{b} - h\right) + K = af(c + h - h) = af(c) + k$. We notice that the output from f is first multiplied by a. As with the constant b, if $a > 0$, this induces only a vertical scaling. If $a < 0$, then the -1 induces a reflection across the x-axis. Finally, we add k to the result, which is our vertical shift.

A less precise, but more intuitive way to paraphrase Theorem 1.11 is to think of the quantity $bx - h$ is the 'inside' of the function f. What's happening inside f affects the inputs or x-coordinates of the points on the graph of f. To find the x-coordinates of the corresponding points on g, we undo what has been done to x in the same way we would solve an equation.

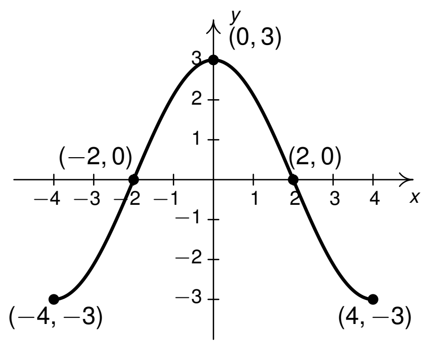
What's happening to the output can be thought of as things happening 'outside' the function, f. Things happening outside affect the outputs or y-coordinates of the points on the graph of f. Here, we follow the usual order of operations to simplify the new y-value: we first multiply by a then add k to find the corresponding y-coordinates on the graph of g.

It needs to be stressed that our approach to handling multiple transformations, as summarized in Theorem 1.11 is only one approach. Your instructor may have a different algorithm. As always, the more you understand, the less you'll ultimately need to memorize, so whatever algorithm you choose to follow, it is worth thinking through each step both algebraically and geometrically.

[11] See the remarks at the beginning of the section.
[12] See Exercise 72.

We make good use of Theorem 1.11 in the following example.

EXAMPLE 1.4.4. Below is the complete graph of $y = f(x)$. Use Theorem 1.11 to graph $g(x) = \dfrac{4 - 3f(1 - 2x)}{2}$.



Solution. We use Theorem 1.11 to track the five 'key points' $(-4, -3)$, $(-2, 0)$, $(0, 3)$, $(2, 0)$ and $(4, -3)$ indicated on the graph of f to their new locations.

We first rewrite $g(x)$ in the form presented in Theorem 1.11, $g(x) = -\frac{3}{2}f(-2x + 1) + 2$. We set $-2x + 1$ equal to the x-coordinates of the key points and solve.

For example, solving $-2x + 1 = -4$, we first subtract 1 to get $-2x = -5$ then divide by -2 to get $x = \frac{5}{2}$. Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by -2 can be thought of as a two step process: dividing by 2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by -1 which causes a reflection across the y-axis. We summarize the results in a table below on the left.

Next, we take each of the x values and substitute them into $g(x) = -\frac{3}{2}f(-2x+1)+2$ to get the corresponding y-values. Substituting $x = \frac{5}{2}$, and using the fact that $f(-4) = -3$, we get

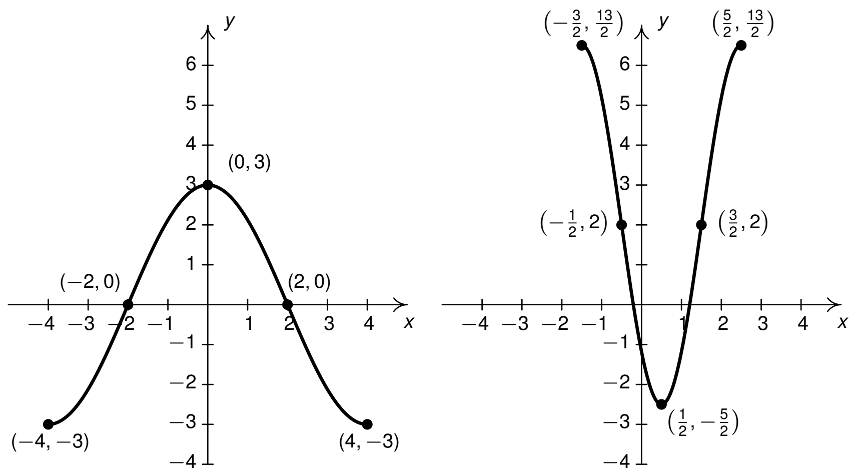
$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right) + 1\right) + 2 = -\frac{3}{2}f(-4) + 2 = -\frac{3}{2}(-3) + 2 = \frac{9}{2} + 2 = \frac{13}{2}$$

We see that the output from f is first multiplied by $-\frac{3}{2}$. Thinking of this as a two step process, multiplying by $\frac{3}{2}$ then by -1, we have a vertical stretching by a factor of $\frac{3}{2}$ followed by a reflection across the x-axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get the table below on the right.

$(c, f(c))$	c	$-2x + 1 = c$	x
$(-4, -3)$	-4	$-2x + 1 = -4$	$x = \frac{5}{2}$
$(-2, 0)$	-2	$-2x + 1 = -2$	$x = \frac{3}{2}$
$(0, 3)$	0	$-2x + 1 = 0$	$x = \frac{1}{2}$
$(2, 0)$	2	$-2x + 1 = 2$	$x = -\frac{1}{2}$
$(4, -3)$	4	$-2x + 1 = 4$	$x = -\frac{3}{2}$

x	$g(x)$	$(x, g(x))$
$\frac{5}{2}$	$\frac{13}{2}$	$\left(\frac{5}{2}, \frac{13}{2}\right)$
$\frac{3}{2}$	2	$\left(\frac{3}{2}, 2\right)$
$\frac{1}{2}$	$-\frac{5}{2}$	$\left(\frac{1}{2}, -\frac{5}{2}\right)$
$-\frac{1}{2}$	2	$\left(-\frac{1}{2}, 2\right)$
$-\frac{3}{2}$	$\frac{13}{2}$	$\left(-\frac{3}{2}, \frac{13}{2}\right)$

To graph g, we plot each of the points in the table above and connect them in the same order and fashion as the points to which they correspond. Plotting f and g side-by-side gives



□

The reader is strongly encouraged to graph the series of functions which shows the gradual transformation of the graph of f into the graph of g in Example 1.4.4. We have outlined the sequence of transformations in the above exposition; all that remains is to plot the five intermediate stages. Our next example turns the tables and asks for the formula of a function given a desired sequence of transformations.

EXAMPLE 1.4.5. Let $f(x) = x^2 - |x|$. Find and simplify the formula of the function $g(x)$ whose graph is the result of the graph of $y = f(x)$ undergoing the following sequence of transformations. Check your answer to each step using a graphing utility.

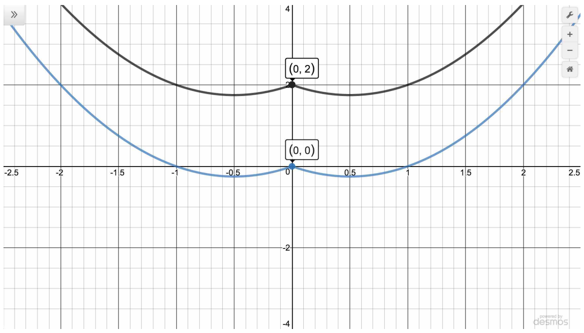
1. Vertical shift up 2 units.
2. Reflection across the x-axis.
3. Horizontal shift right 1 unit.
4. Horizontal compression by a factor of 2.
5. Vertical shift up 3 units.
6. Reflection across the y-axis.

Solution. To help keep us organized we will label each intermediary function. The function g_1 will be the result of applying the first transformation to f. The function g_2 will be the result of applying the first two transformations to f - which is also the result of applying the second transformation to g_1, and so on.[13]

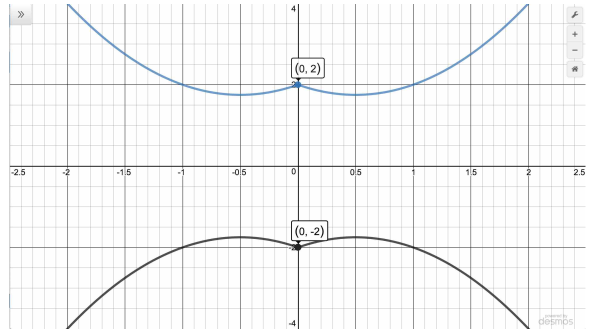
1. Per Theorem 1.6, $g_1(x) = f(x) + 2 = x^2 - |x| + 2$.

2. Per Theorem 1.8, $g_2(x) = -g_1(x) = -[x^2 - |x| + 2] = -x^2 + |x| - 2$.

[13] So, we can think of $g_0 = f$ and $g_6 = g$.



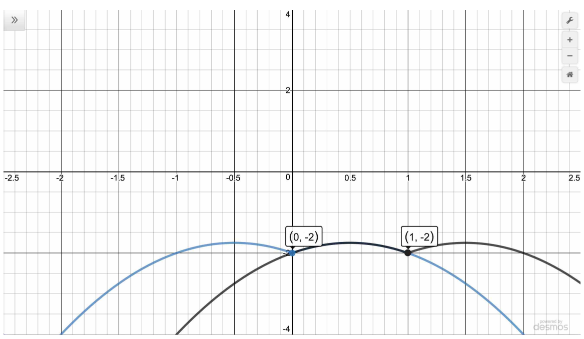
$y = f(x)$ (lighter color) and $y = g_1(x) = f(x) + 2$



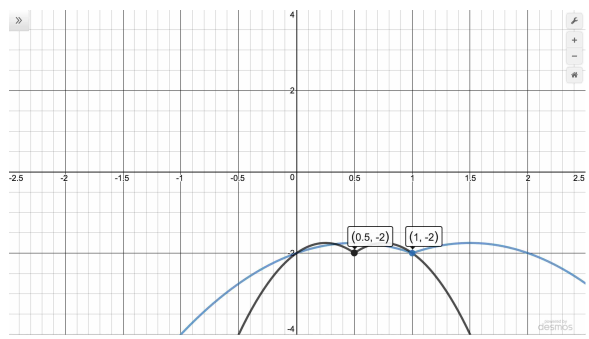
$y = g_1(x)$ (lighter color) and $y = g_2(x) = -g_1(x)$

3. Per Theorem 1.7, $g_3(x) = g_2(x-1) = -(x-1)^2 + |x-1| - 2$.

4. Per Theorem 1.10, $g_4(x) = g_3(2x) = -(2x-1)^2 + |2x-1| - 2$.



$y = g_2(x)$ (lighter color) and $y = g_3(x) = g_2(x-1)$

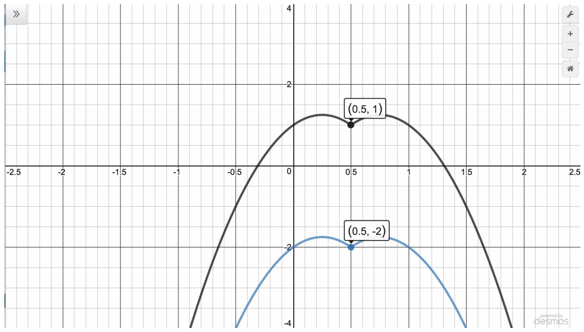


$y = g_3(x)$ (lighter color) and $y = g_4(x) = g_3(2x)$

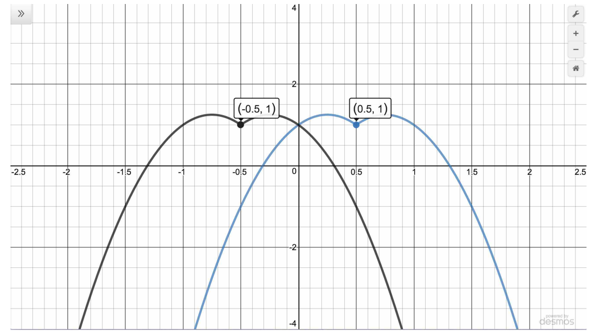
5. Per Theorem 1.6, $g_5(x) = g_4(x) + 3 = -(2x-1)^2 + |2x-1| - 2 + 3 = -(2x-1)^2 + |2x-1| + 1$.

6. Per Theorem 1.8, $g_6(x) = g_5(-x)$:

$$
\begin{aligned}
g_6(x) &= g_5(-x) \\
&= -(2(-x)-1)^2 + |2(-x)-1| + 1 \\
&= -(-2x-1)^2 + |-2x-1| + 1 \\
&= -[(-1)(2x+1)]^2 + |[(-1)(2x+1)| + 1 \\
&= -(-1)^2(2x+1)^2 + |-1||2x+1| + 1 \\
&= -(2x+1)^2 + |2x+1| + 1
\end{aligned}
$$



$y = g_4(x)$ (lighter color) and $y = g_5(x) = g_4(x) + 3$



$y = g_5(x)$ (lighter color) and $y = g_6(x) = g_5(-x)$

Hence, $g(x) = g_6(x) = -(2x+1)^2 + |2x+1| + 1$. $\square$

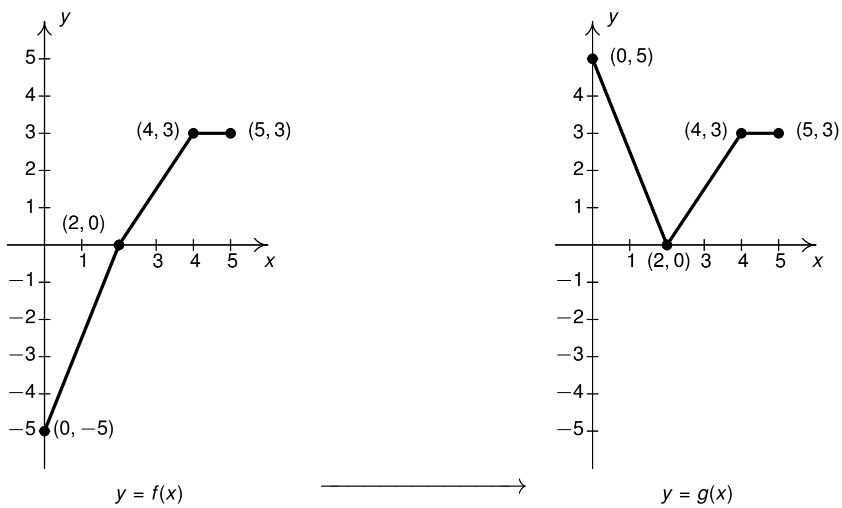
It is instructive to show that the expression $g(x)$ in Example 1.4.4 can be written as $g(x) = af(bx - h) + k$. One way is to compare the graphs of f and g and work backwards. A more methodical way is to repeat the work of Example 1.4.4, but never substitute the formula for $f(x)$ as follows:

1. Per Theorem 1.6, $g_1(x) = f(x) + 2$.
2. Per Theorem 1.8, $g_2(x) = -g_1(x) = -[f(x) + 2] = -f(x) - 2$.
3. Per Theorem 1.7, $g_3(x) = g_2(x - 1) = -f(x - 1) - 2$.
4. Per Theorem 1.10, $g_4(x) = g_3(2x) = -f(2x - 1) - 2$.
5. Per Theorem 1.6, $g_5(x) = g_4(x) + 3 = -f(2x - 1) - 2 + 3 = -f(2x - 1) + 1$.
6. Per Theorem 1.8, $g_6(x) = g_5(-x) = -f(2(-x) - 1) + 1 = -f(-2x - 1) + 1$.

Hence $g(x) = -f(-2x - 1) + 1$. Note we can show f is even,[14] so $f(-2x - 1) = f(-(2x + 1)) = f(2x + 1)$ and obtain $g(x) = -f(2x + 1) + 1$.
At the beginning of this section, we discussed how all of the transformations we'd be discussing are the result of composing given functions with linear functions. Not all transformations, not even all rigid transformations,[15] fall into these categories.

For example, consider the graphs of $y = f(x)$ and $y = g(x)$ below.

[14] Recall this means $f(-x) = f(x)$.
[15] See Section **??**.



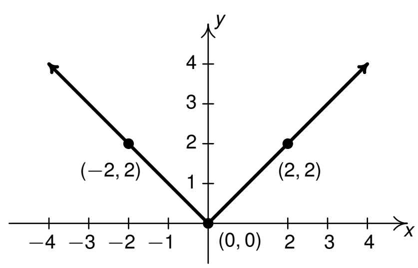
In Exercise 76, we explore a non-linear transformation and revisit the pair of functions f and g then.

1.4.5 Exercises

Suppose $(2,-3)$ is on the graph of $y=f(x)$. In Exercises 1 - 18, use Theorem 1.11 to find a point on the graph of the given transformed function.

1. $y=f(x)+3$
2. $y=f(x+3)$
3. $y=f(x)-1$
4. $y=f(x-1)$
5. $y=3f(x)$
6. $y=f(3x)$
7. $y=-f(x)$
8. $y=f(-x)$
9. $y=f(x-3)+1$
10. $y=2f(x+1)$
11. $y=10-f(x)$
12. $y=3f(2x)-1$
13. $y=\frac{1}{2}f(4-x)$
14. $y=5f(2x+1)+3$
15. $y=2f(1-x)-1$
16. $y=f\left(\dfrac{7-2x}{4}\right)$
17. $y=\dfrac{f(3x)-1}{2}$
18. $y=\dfrac{4-f(3x-1)}{7}$

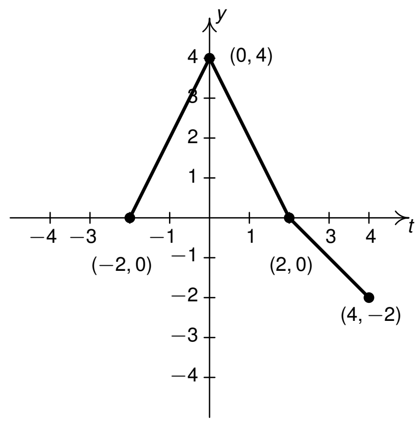
The complete graph of $y=f(x)$ is given below. In Exercises 19 - 27, use it and Theorem 1.11 to graph the given transformed function.



The graph of $y=f(x)$ for Ex. 19 - 27

19. $y=f(x)+1$
20. $y=f(x)-2$
21. $y=f(x+1)$
22. $y=f(x-2)$
23. $y=2f(x)$
24. $y=f(2x)$
25. $y=2-f(x)$
26. $y=f(2-x)$
27. $y=2-f(2-x)$
28. Some of the answers to Exercises 19 - 27 above should be the same. Which ones match up? What properties of the graph of $y=f(x)$ contribute to the duplication?
29. The function f used in Exercises 19 - 27 should look familiar. What is $f(x)$? How does this this explain some of the duplication in the answers to Exercises 19 - 27 mentioned in Exercise 28?

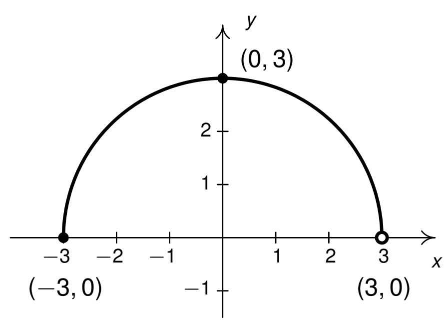
The complete graph of $y = g(t)$ is given below. In Exercises 30 - 38, use it and Theorem 1.11 to graph the given transformed function.



The graph of $y = g(t)$ for Ex. 30 - 38

30. $y = g(t) - 1$
31. $y = g(t + 1)$
32. $y = \frac{1}{2}g(t)$
33. $y = g(2t)$
34. $y = -g(t)$
35. $y = g(-t)$
36. $y = g(t + 1) - 1$
37. $y = 1 - g(t)$
38. $y = \frac{1}{2}g(t + 1) - 1$

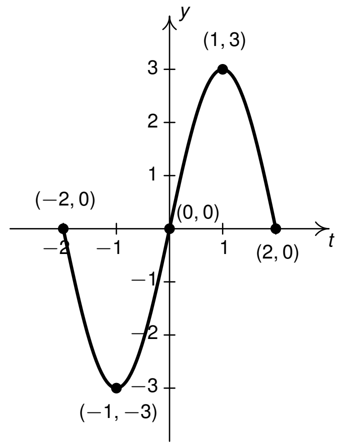
The complete graph of $y = f(x)$ is given below. In Exercises 39 - 50, use it and Theorem 1.11 to graph the given transformed function.



The graph of $y = f(x)$ for Ex. 39 - 50

39. $g(x) = f(x) + 3$
40. $h(x) = f(x) - \frac{1}{2}$
41. $j(x) = f\left(x - \frac{2}{3}\right)$
42. $a(x) = f(x + 4)$
43. $b(x) = f(x + 1) - 1$
44. $c(x) = \frac{3}{5}f(x)$
45. $d(x) = -2f(x)$
46. $k(x) = f\left(\frac{2}{3}x\right)$
47. $m(x) = -\frac{1}{4}f(3x)$
48. $n(x) = 4f(x - 3) - 6$
49. $p(x) = 4 + f(1 - 2x)$
50. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3$

The complete graph of $y = S(t)$ is given below.



The graph of $y = S(t)$

The purpose of Exercises 51 - 54 is to build up to the graph of $y = \frac{1}{2}S(-t+1)+1$ one step at a time.

51. $y = S_1(t) = S(t+1)$

52. $y = S_2(t) = S_1(-t) = S(-t+1)$

53. $y = S_3(t) = \frac{1}{2}S_2(t) = \frac{1}{2}S(-t+1)$

54. $y = S_4(t) = S_3(t) + 1 = \frac{1}{2}S(-t+1)+1$

Let $f(x) = \sqrt{x}$. Find a formula for a function g whose graph is obtained from f from the given sequence of transformations.

55. (1) shift right 2 units; (2) shift down 3 units

56. (1) shift down 3 units; (2) shift right 2 units

57. (1) reflect across the x-axis; (2) shift up 1 unit

58. (1) shift up 1 unit; (2) reflect across the x-axis

59. (1) shift left 1 unit; (2) reflect across the y-axis; (3) shift up 2 units

60. (1) reflect across the y-axis; (2) shift left 1 unit; (3) shift up 2 units

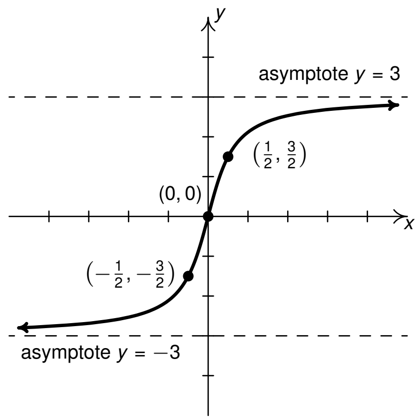
61. (1) shift left 3 units; (2) vertical stretch by a factor of 2; (3) shift down 4 units

62. (1) shift left 3 units; (2) shift down 4 units; (3) vertical stretch by a factor of 2

63. (1) shift right 3 units; (2) horizontal shrink by a factor of 2; (3) shift up 1 unit

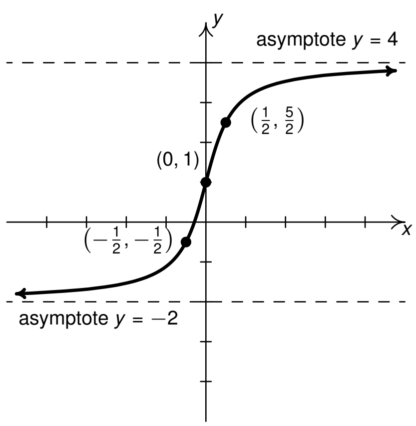
64. (1) horizontal shrink by a factor of 2; (2) shift right 3 units; (3) shift up 1 unit

For Exercises 65 - 70, use the given of $y = f(x)$ to write each function in terms of $f(x)$.

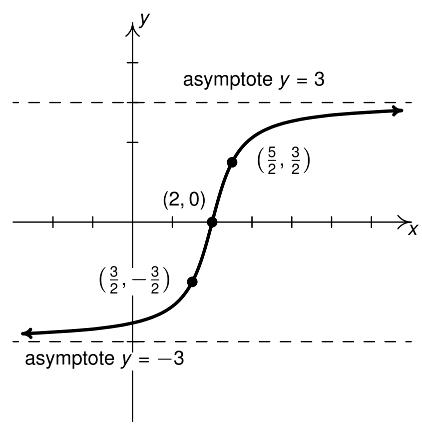


The graph of $y = f(x)$ for Ex. 65 - 70.

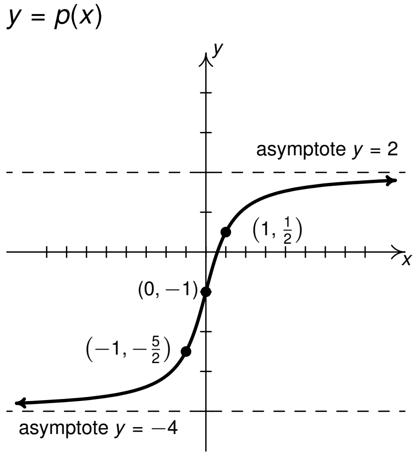
65. $y = g(x)$



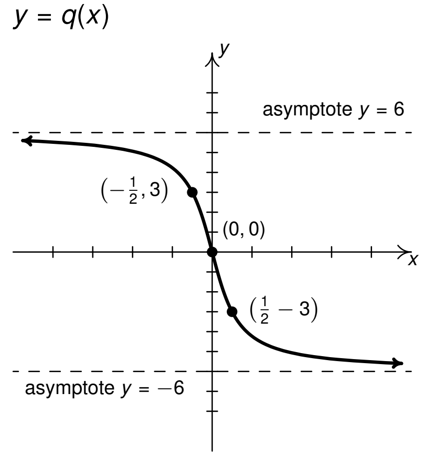
66. $y = h(x)$



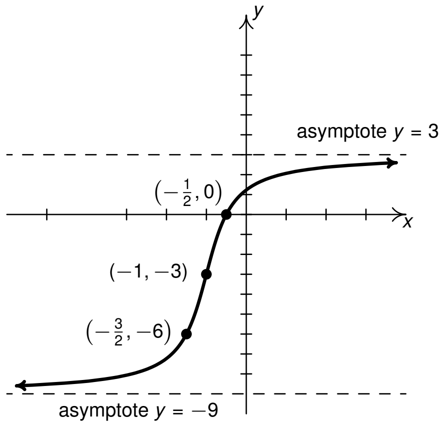
67. $y = p(x)$



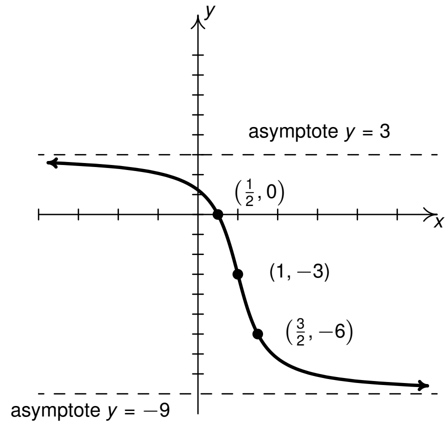
68. $y = q(x)$



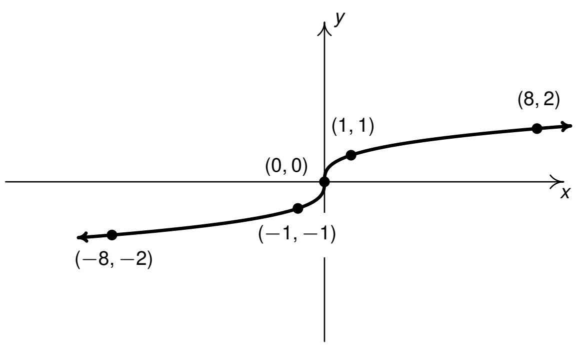
69. $y = r(x)$



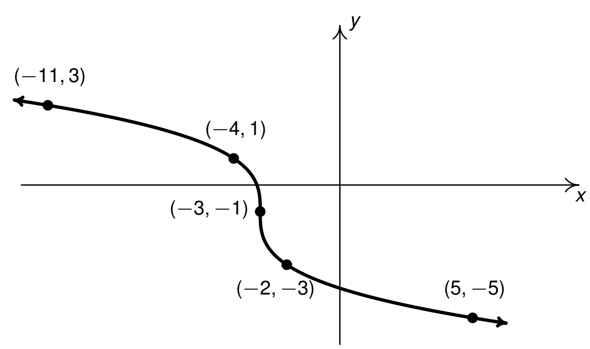
70. $y = s(x)$



71. The graph of $y = f(x) = \sqrt[3]{x}$ is given below on the left and the graph of $y = g(x)$ is given on the right. Find a formula for g based on transformations of the graph of f. Check your answer by confirming that the points shown on the graph of g satisfy the equation $y = g(x)$.



$y = \sqrt[3]{x}$



$y = g(x)$

72. Show that the composition of two linear functions is a linear function. Hence any (finite) sequence of transformations discussed in this section can be combined into the form given in Theorem 1.11.

    (HINT: Let $f(x) = ax + b$ and $g(x) = cx + d$. Find $(f \circ g)(x)$.)

73. For many common functions, the properties of Algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, $\sqrt{9x} = 3\sqrt{x}$, so a horizontal compression of $y = \sqrt{x}$ by a factor of 9 results in the same graph as a vertical stretch of $y = \sqrt{x}$ by a factor of 3.

    With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings $y = (2x)^3$, $y = |5x|$, $y = \sqrt[3]{27x}$ and $y = \left(\frac{1}{2}x\right)^2$.

    What about $y = (-2x)^3$, $y = |-5x|$, $y = \sqrt[3]{-27x}$ and $y = \left(-\frac{1}{2}x\right)^2$?

74. Discuss the following questions with your classmates.

    - If f is even, what happens when you reflect the graph of $y = f(x)$ across the y-axis?
    - If f is odd, what happens when you reflect the graph of $y = f(x)$ across the y-axis?
    - If f is even, what happens when you reflect the graph of $y = f(x)$ across the x-axis?
    - If f is odd, what happens when you reflect the graph of $y = f(x)$ across the x-axis?
    - How would you describe symmetry about the origin in terms of reflections?

75. We mentioned earlier in the section that, in general, the order in which transformations are applied matters, yet in our first example with two transformations the order did not matter. (You could perform the shift to the left followed by the shift down or you could shift down and then left to achieve the same result.) With the help of your classmates, determine the situations in which order does matter and those in which it does not.

76. This Exercise is a follow-up to Exercise **??** in Section **??**.

    (a) Fill in the table below.

$f(x)$	$\lvert f(x)\rvert$	$f\left(\lvert x\rvert\right)$
$x+2$		
x^2-4x		
x^3-3x^2		
$(x+1)^{-1}$		
$\sqrt{x+2}-3$		

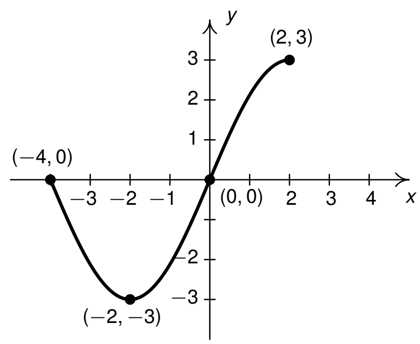
    (b) For each function f above, graph $y = f(x)$ and $y = \lvert f(x)\rvert$ using a graphing utility.

        i. Write a sentence (or two!) explaining how to obtain the graph of $y = \lvert f(x)\rvert$ from $y = f(x)$.
        ii. How does your explanation relate to Definition **??**?

    (c) For each function f above, graph $y = f(x)$ and $y = f(\lvert x\rvert)$ using a graphing utility.

        i. Write a sentence (or two!) explaining how to obtain the graph of $y = f(\lvert x\rvert)$ from $y = f(x)$.
        ii. How does your explanation relate to Definition **??**?

(d) Use the graph of $y = f(x)$ below to graph $y = |f(x)|$ and $y = f(|x|)$.



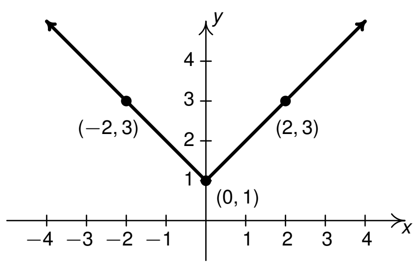
$y = f(x)$

(e) Referring to the functions f and g graphed on page 94, write g in terms of f.

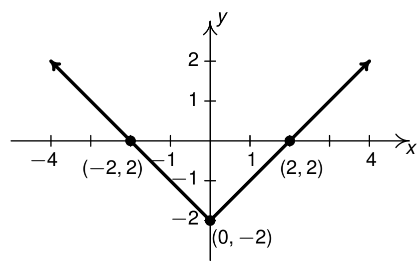
1.4.6 Answers

1. $(2, 0)$
2. $(-1, -3)$
3. $(2, -4)$
4. $(3, -3)$
5. $(2, -9)$
6. $\left(\frac{2}{3}, -3\right)$
7. $(2, 3)$
8. $(-2, -3)$
9. $(5, -2)$
10. $(1, -6)$
11. $(2, 13)$
12. $y = (1, -10)$
13. $\left(2, -\frac{3}{2}\right)$
14. $\left(\frac{1}{2}, -12\right)$
15. $(-1, -7)$
16. $\left(-\frac{1}{2}, -3\right)$
17. $\left(\frac{2}{3}, -2\right)$
18. $(1, 1)$

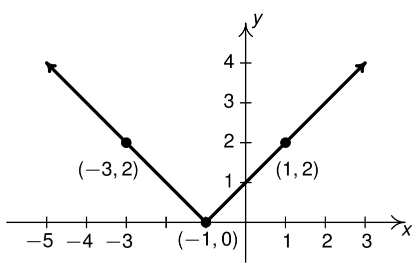
19. $y = f(x) + 1$



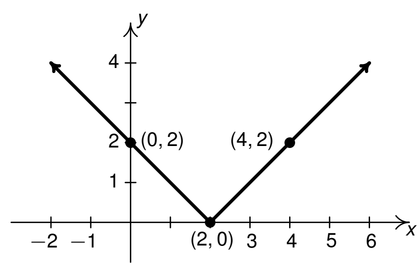
20. $y = f(x) - 2$



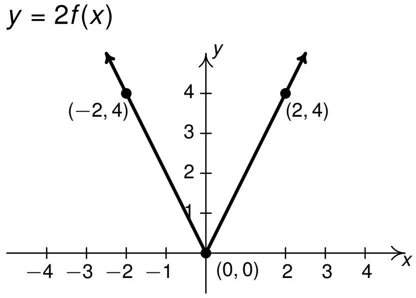
21. $y = f(x + 1)$



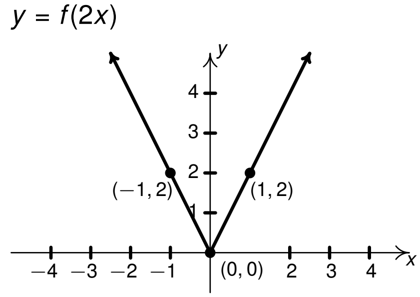
22. $y = f(x - 2)$



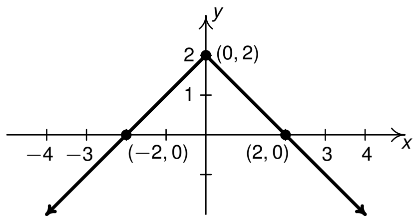
23. $y = 2f(x)$



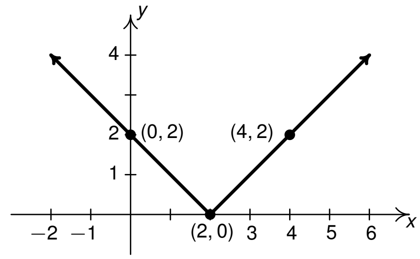
24. $y = f(2x)$



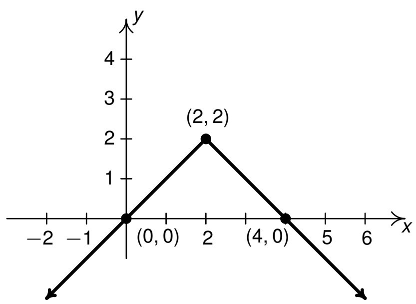
25. $y = 2 - f(x)$



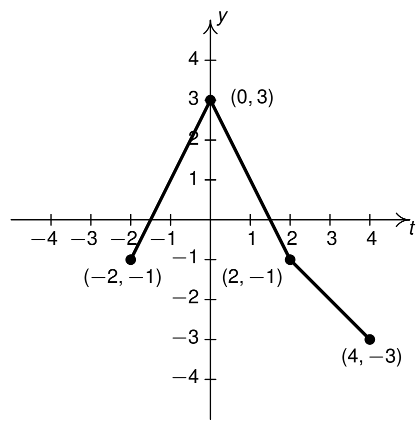
26. $y = f(2 - x)$



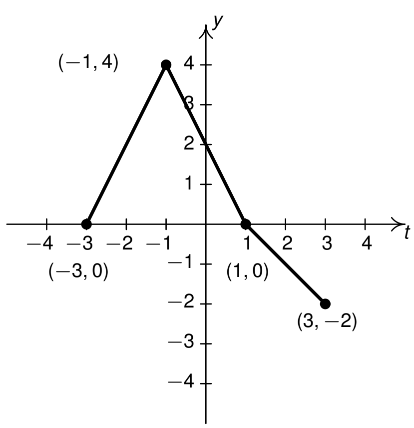
27. $y = 2 - f(2 - x)$



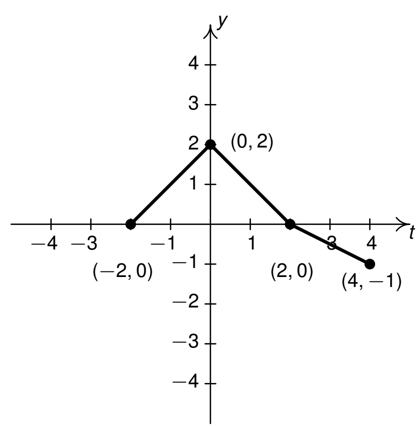
30. $y = g(t) - 1$



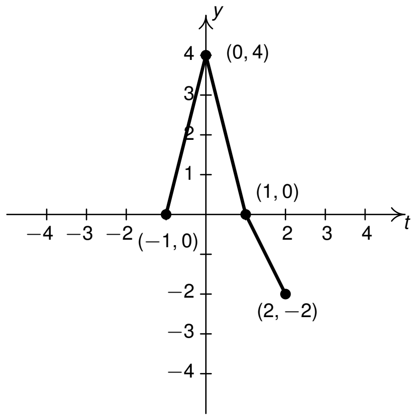
31. $y = g(t + 1)$



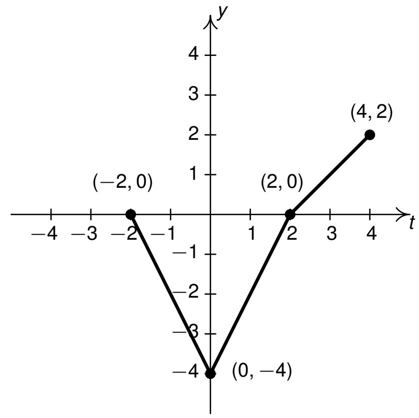
32. $y = \frac{1}{2}g(t)$



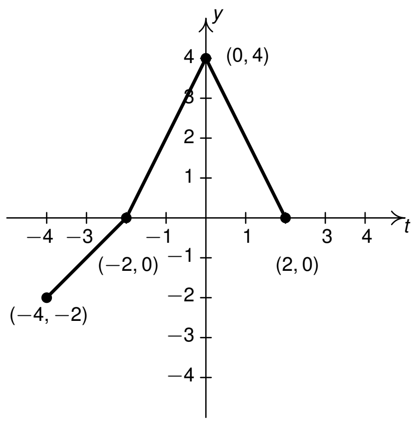
33. $y = g(2t)$



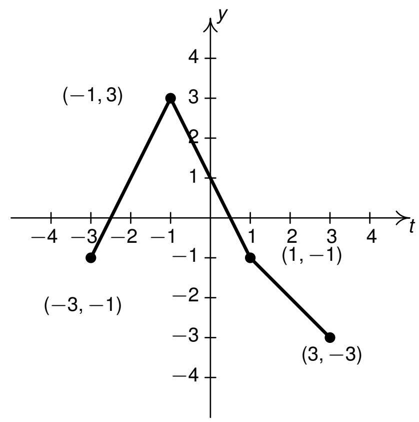
34. $y = -g(t)$



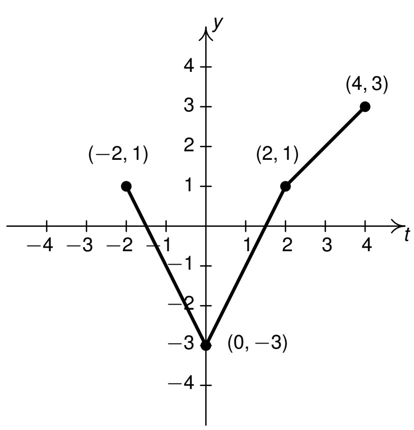
35. $y = g(-t)$



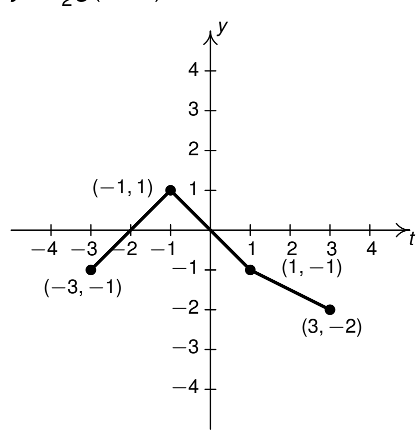
36. $y = g(t + 1) - 1$



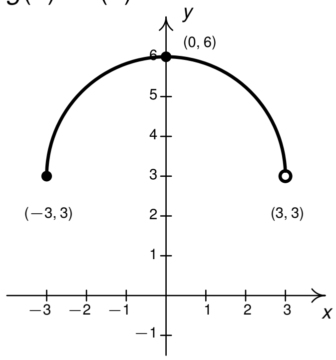
37. $y = 1 - g(t)$



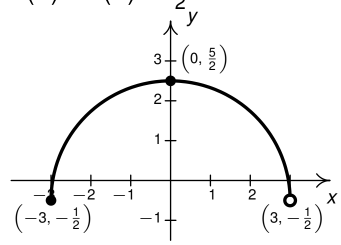
38. $y = \frac{1}{2}g(t + 1) - 1$



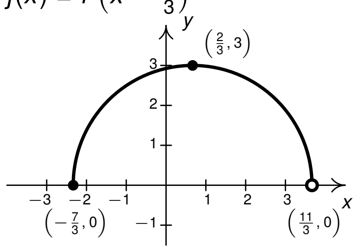
39. $g(x) = f(x) + 3$



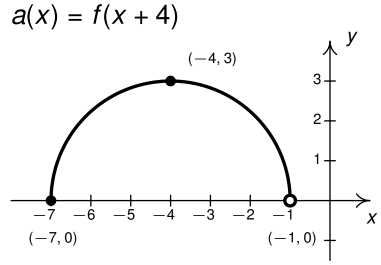
40. $h(x) = f(x) - \frac{1}{2}$



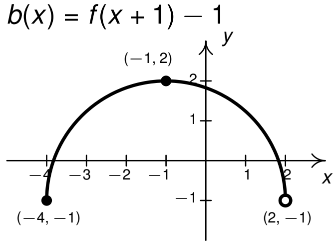
41. $j(x) = f\left(x - \frac{2}{3}\right)$



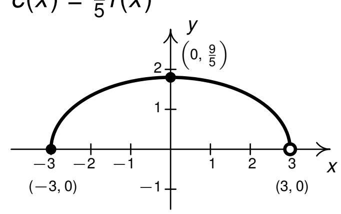
42. $a(x) = f(x + 4)$



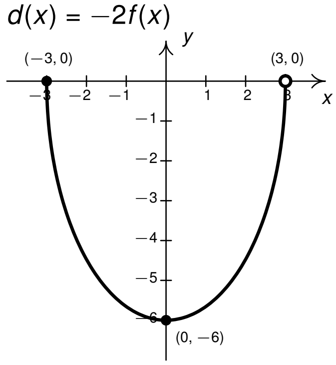
43. $b(x) = f(x + 1) - 1$



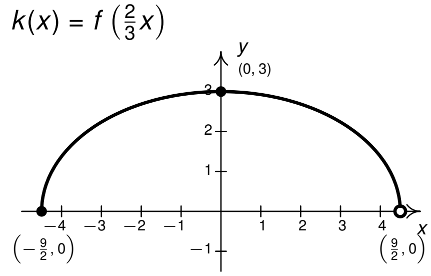
44. $c(x) = \frac{3}{5}f(x)$



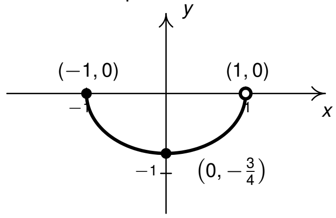
45. $d(x) = -2f(x)$



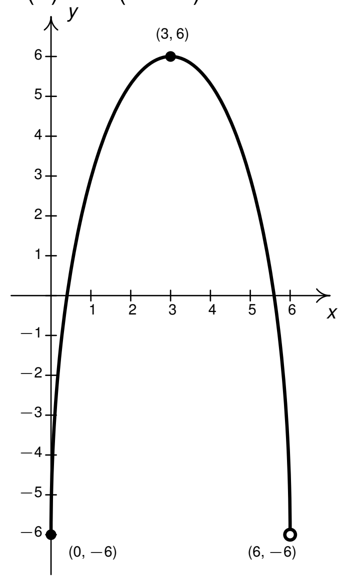
46. $k(x) = f\left(\frac{2}{3}x\right)$



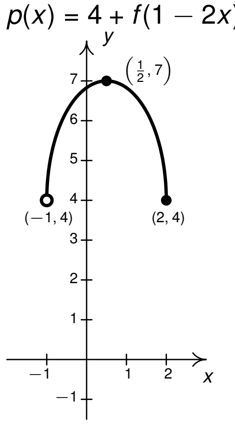
47. $m(x) = -\frac{1}{4}f(3x)$



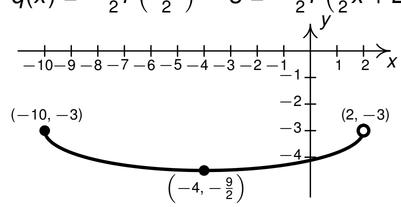
48. $n(x) = 4f(x-3) - 6$



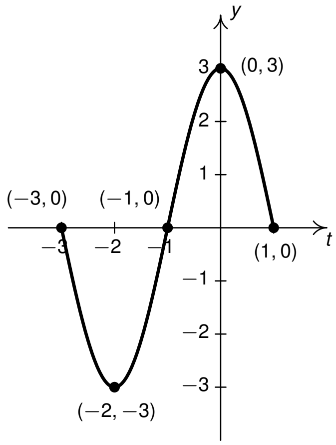
49. $p(x) = 4 + f(1-2x) = f(-2x+1) + 4$



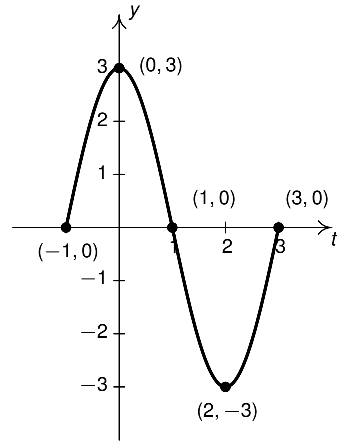
50. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3 = -\frac{1}{2}f\left(\frac{1}{2}x+2\right) - 3$



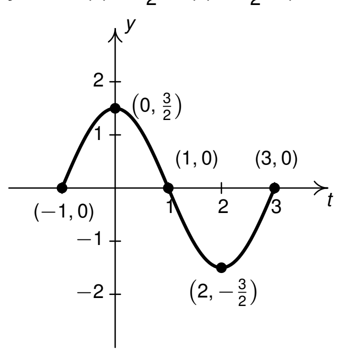
51. $y = S_1(t) = S(t+1)$



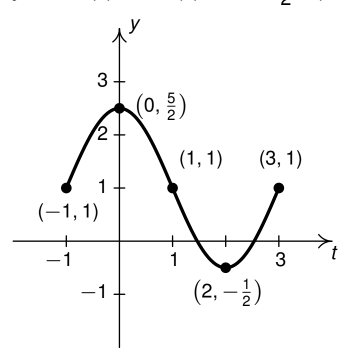
52. $y = S_2(t) = S_1(-t) = S(-t+1)$



53. $y = S_3(t) = \frac{1}{2}S_2(t) = \frac{1}{2}S(-t+1)$



54. $y = S_4(t) = S_3(t) + 1 = \frac{1}{2}S(-t+1) + 1$



55. $g(x) = \sqrt{x-2} - 3$

56. $g(x) = \sqrt{x-2} - 3$

57. $g(x) = -\sqrt{x} + 1$

58. $g(x) = -(\sqrt{x} + 1) = -\sqrt{x} - 1$

59. $g(x) = \sqrt{-x+1} + 2$

60. $g(x) = \sqrt{-(x+1)} + 2 = \sqrt{-x-1} + 2$

61. $g(x) = 2\sqrt{x+3} - 4$

62. $g(x) = 2\left(\sqrt{x+3} - 4\right) = 2\sqrt{x+3} - 8$

63. $g(x) = \sqrt{2x-3} + 1$

64. $g(x) = \sqrt{2(x-3)} + 1 = \sqrt{2x-6} + 1$

65. $g(x) = f(x) + 1$

66. $h(x) = f(x-2)$

67. $p(x) = f\left(\frac{x}{2}\right) - 1$

68. $q(x) = -2f(x) = 2f(-x)$

69. $r(x) = 2f(x+1) - 3$

70. $s(x) = 2f(-x+1) - 3 = -2f(x-1) + 3$

71. $g(x) = -2\sqrt[3]{x+3} - 1$ or $g(x) = 2\sqrt[3]{-x-3} - 1$

76. (a)

$f(x)$	$\lvert f(x)\rvert$	$f\left(\lvert x\rvert\right)$
$x+2$	$\lvert x+2\rvert$	$\lvert x\rvert+2$
x^2-4x	$\lvert x^2-4x\rvert$	$\lvert x\rvert^2-3\lvert x\rvert = x^2-4\lvert x\rvert$
x^3-3x^2	$\lvert x^3-3x^2\rvert$	$\lvert x\rvert^3-3\lvert x\rvert^2 = \lvert x\rvert^3-3x^2$
$(x+1)^{-1}$	$\lvert (x+1)^{-1}\rvert$	$(\lvert x\rvert+1)^{-1}$
$\sqrt{x+2}-3$	$\lvert\sqrt{x+2}-3\rvert$	$\sqrt{\lvert x\rvert+2}-3$

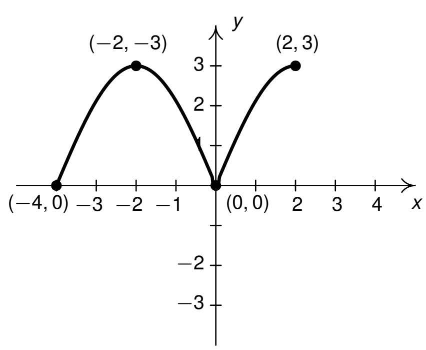
(b) i. To graph $y = |f(x)|$ from the graph of $y = f(x)$, reflect about the x-axis any portion of the graph of $y = f(x)$ which is below the x-axis.

ii. If the graph is below the x-axis, then $f(x) < 0$. Since $|f(x)| = -f(x)$ if $f(x) < 0$, we are graphing $y = -f(x)$ for these values of x which is a reflection across the x-axis.

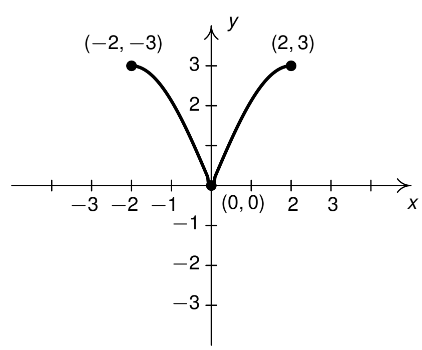
(c) i. To graph $y = f(|x|)$ from the graph of $y = f(x)$, replace the graph of $y = f(x)$ for $x \leq 0$ with the reflection about the y-axis of the graph of $y = f(x)$ for $x \geq 0$.

ii. If $x < 0$, then $|x| = -x$, so $f(|x|) = f(-x)$. Since if $x < 0$, $-x > 0$, this means we reflect the graph of $y = f(x)$ about the y-axis for $x > 0$ only.

(d)



$y = |f(x)|$



$y = f(|x|)$ - note the domain

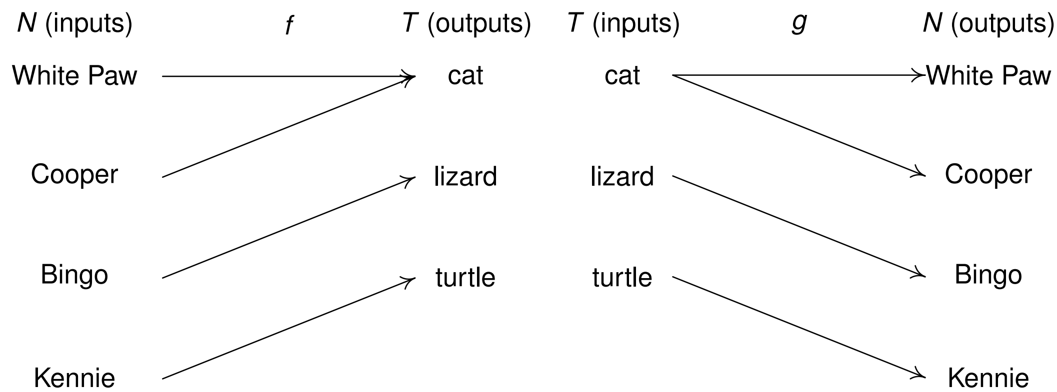
(e) $g(x) = |f(x)|$.

1.5 Relations and Implicit Functions

Up until now in this text, we have been exclusively special kinds of mappings called *functions*. In this section, we broaden our horizons to study more general mappings called *relations*. The reader is encouraged to revisit Definition **??** in Section **??** before proceeding with the definition of *relation* below.

DEFINITION 1.3. Given two sets A and B, a **relation** from A to B is a process by which elements of A are matched with (or 'mapped to') elements of B.

Unlike Definition **??**, Definition 1.3 puts no conditions on the process which maps elements of A to elements of B. This means that while all functions are relations, not all relations need be functions. For example, consider the mappings f and g below from Section **??**.



Both f and g are relations. More specifically, f is a *function* from N to T while g is merely *relation* from T to N. As with functions, we may describe general relations in a variety of different ways: verbally, as mapping diagrams, or a set of ordered pairs. For example, just as we may describe the function f above as

$$f = \{(\text{White Paw}, \text{cat}), (\text{Cooper}, \text{cat}), (\text{Bingo}, \text{lizard}), (\text{Kennie}, \text{turtle})\},$$

we may represent g as

$$g = \{(\text{cat}, \text{White Paw}), (\text{cat}, \text{Cooper}), (\text{lizard}, \text{Bingo}), (\text{turtle}, \text{Kennie})\}.$$

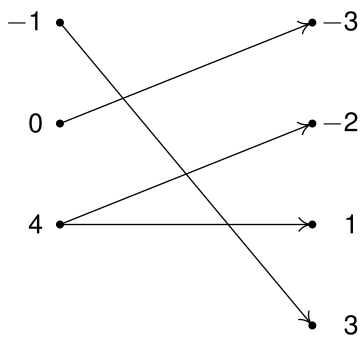
Note here the grammar 'g is a relation *from T to N*' is evidenced by the elements of T being listed first in the ordered pairs (i.e., the abscissae) and the elements of N being listed second (i.e., the ordinates.)

Unlike functions, we do not use function notation when describing the input/output relationship for general relations. For example, we may write 'f(White Paw) = cat' since f maps the input 'White Paw' to only one output, 'cat.' However, g(cat) is ambiguous since it could mean 'White Paw' or 'Cooper.'[1]

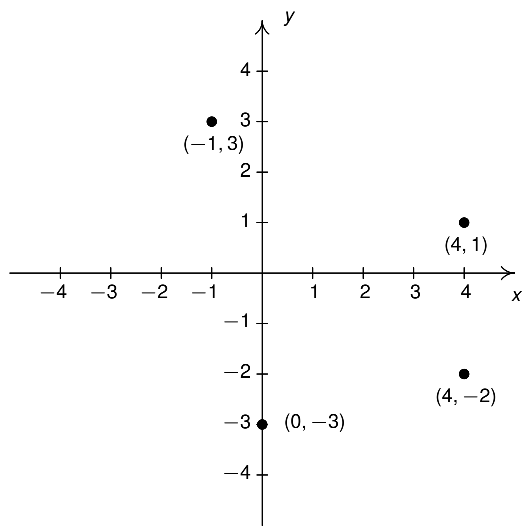
As with functions, our focus in this course will rest with relations of real numbers. Consider the relation R described as follows: $R = \{(-1, 3), (0, -3), (4, -2), (4, 1)\}$. Below on the left is a mapping diagram of R.

[1] In more advanced texts, we would write 'cat g White Paw' and 'cat g Cooper' to indicate g maps 'cat' to both 'White Paw' and 'Cooper.' Our study of relations, however, isn't deep enough to necessitate introducing and using this notation. Similarly, we won't introduce the notions of 'domain,' 'codomain,' and 'range' for relations, either.

However, since R relates real numbers, we can also create the graph of R in the same way we graphed functions - by interpreting the ordered pairs which comprise R as points in the plane. Since we have no context, we use the default labels 'x' for the horizontal axis and 'y' for the vertical axis.



A Mapping Diagram of R.



The graph of R.

Our next example focuses on using relations to describe sets of points in the plane and vice-versa.

EXAMPLE 1.5.1.

1. Graph the following relations.

   (a) $S = \{(k, 2^k) \mid k = 0, \pm 1, \pm 2\}$

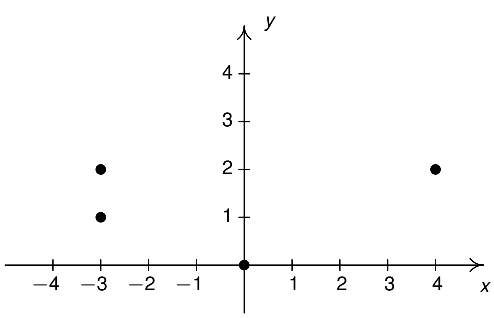
   (b) $P = \{(j, j^2) \mid j \text{ is an integer}\}$

   (c) $V = \{(3, y) \mid y \text{ is a real number}\}$

   (d) $R = \{(x, y) \mid x \text{ is a real number}, 1 < y \leq 3\}$

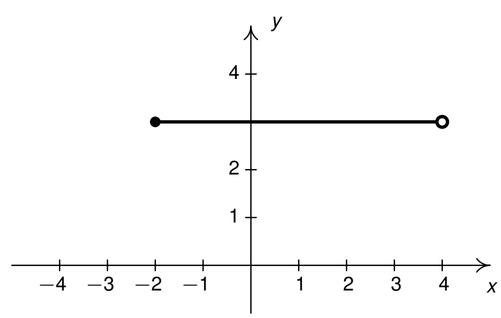
2. Find a roster or set-builder description for each of the relations below.

   (a)



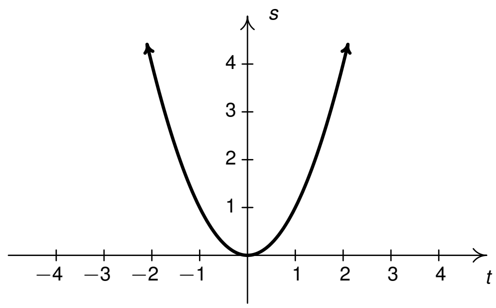
The graph of A

   (b)



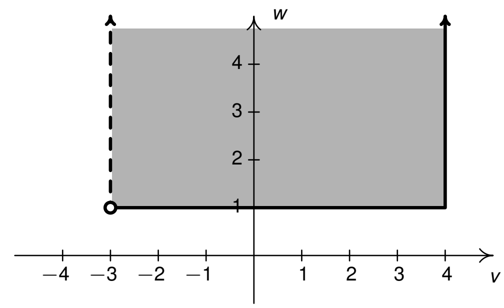
The graph of H

(c)



The graph of Q

(d)



The graph of T

Solution.

1. (a) The relation S is described using *set-builder notation*.[2] To generate the ordered pairs which belong to S, we substitute the given values of k, $k = 0, \pm 1, \pm 2$, into the formula $\left(k, 2^k\right)$.

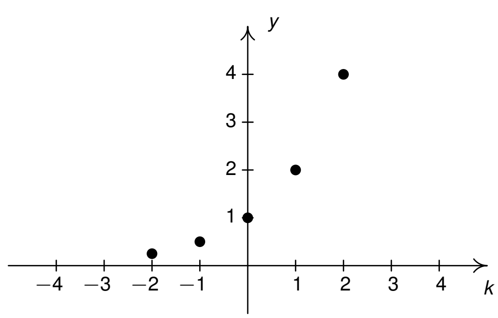
   Starting with $k = 0$, we get $\left(0, 2^0\right) = (0,1)$. For $k = 1$, we get $\left(1, 2^1\right) = (1,2)$, and for $k = -1$, we get $\left(-1, 2^{-1}\right) = \left(-1, \frac{1}{2}\right)$. Continuing, we get $\left(2, 2^2\right) = (2,4)$ for $k = 2$ and, finally $\left(-2, 2^{-2}\right) = \left(-2, \frac{1}{4}\right)$ for $k = -2$. Hence, a roster description of S is $S = \left\{\left(-2, \frac{1}{4}\right), \left(-1, \frac{1}{2}\right), (0,1), (1,2), (2,4)\right\}$.

   When we graph S, we label the horizontal axis as the k-axis, since 'k' was the variable chosen used to generate the ordered pairs and keep the default label 'y' for the vertical axis. The graph of S is below on the left.

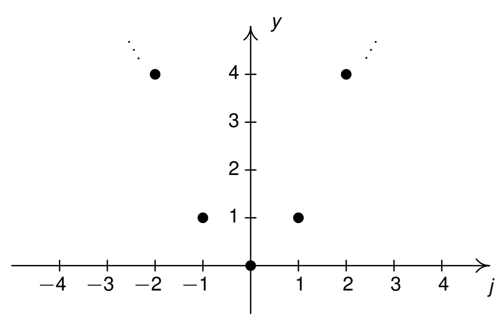
   (b) To graph the relation $P = \left\{\left(j, j^2\right) \mid j \text{ is an integer}\right\}$, we proceed as above when we graphed the relation S. Here, j is restricted to being an integer, which means $j = 0$, ± 1, ± 2, etc.

   Plugging in these sample values for j, we obtain the ordered pairs $(0,0)$, $(1,1)$, $(-1,1)$, $(2,4)$, $(-2,4)$, etc. Since the variable j takes on only integer values, we could write P using the roster notation: $P = \{(0,0), (\pm 1, 1), (\pm 2, 4), \ldots\}$.

   We plot a few of these points and use some periods of ellipsis to indicate the complete graph contains additional points not in the current field of view. The graph of P is below on the right.



The graph of S



The graph of P

[2]See Section **??** to review this, if needed.

(c) Next, we come to the relation V, described, once again, using set-builder notation. In this case, V consists of all ordered pairs of the form $(3, y)$ where y is free to be whatever real number we like, without any restriction.[3] For example, $(3, 0)$, $(3, -1)$, and $(3, 117)$ all belong to V as do $\left(3, \frac{1}{2}\right)$, $(3, -1.0342)$, $(3, \sqrt{2})$, etc.

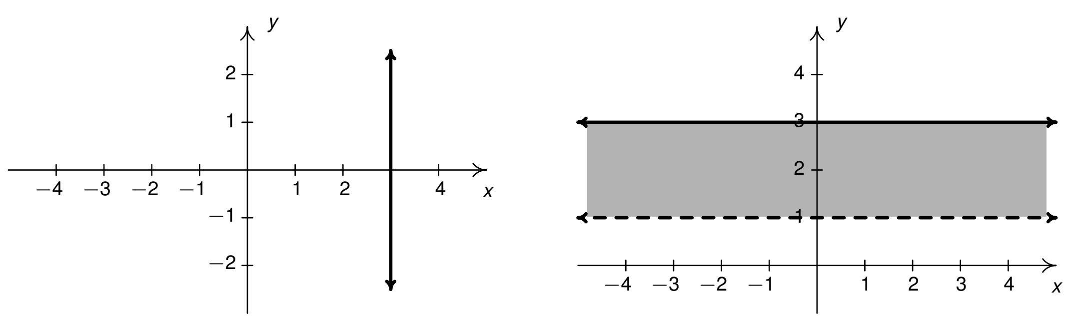
After plotting some sample points, becomes apparent that the ordered pairs which belong to V correspond to points which lie on the vertical line $x = 3$, and vice-versa. That is, every point on the line $x = 3$ has coordinates which correspond to an ordered pair belonging to V. The graph of V is below on the left.

(d) In the relation $R = \{(x, y) \,|\, 1 < y \leq 3\}$, we see y is restricted by the inequality $1 < y \leq 3$, but x is free to be whatever it likes.

Since x is unrestricted, this means whatever the graph of R is, it will extend indefinitely off to the right and left. The restriction $y > 1$ means all points on the graph of R have a y-coordinate larger than one, so they are *above* the horizontal line $y = 1$. The restriction $y \leq 3$, on the other hand, means all the points on the graph of R have a y-coordinate less than or equal to 3, meaning they are either *on* or *below* the horizontal line $y = 3$.

In other words, the graph of R is the region in the plane between $y = 1$ and $y = 3$, including $y = 3$ but not $y = 1$. We signify this by *shading* the region between these two horizontal lines.

How do we communicate $y = 1$ is not part of the graph? One way is to visualize putting 'holes' all along the line $y = 1$ to indicate this is not part of the graph. In practice, however, this looks cluttered and could be confusing. Instead, we 'dash' the line $y = 1$ as seen below on the right.



The graph of V The graph of R

2. (a) Since A consists of finitely many points, we can describe A using the roster method:

$$A = \{(-3, 2), (-3, 1), (0, 0), (4, 2)\}.$$

(b) The graph of H appears to be a portion of the horizontal line $y = 3$ from $x = -2$ (including $x = -2$) up to, but not including $x = 4$. Since it is impossible[4] to *list* each and every one of these points, we'll opt to describe H using set-builder as opposed to the roster method. Taking a cue from the description of the relations V and R above, we write $H = \{(x, 3) \,|\, -2 \leq x < 4\}$.

[3] We'll revisit the concept of a 'free variable' in Section **??**.

[4] Really impossible. The interested reader is encouraged to research **countable** versus **uncountable** sets.

(c) The graph of Q appears to be the graph of the function $s = f(t) = t^2$. Again, as the graph consists of infinitely many points, we will use set-builder notation to describe Q out of necessity.

There are a couple of different ways to do this. Taking a cue from the relation P above, we could write $Q = \{(t, t^2) \mid t \text{ is a real number}\}$. Alternatively, we could introduce the dependent variable, s into the description by writing $Q = \{(t, s) \mid s = t^2\}$ where here the assumption is x takes in all real number values.

(d) As with the relation R above, the relation T describes a region in the plane, The v-values appear to range between -3 (not including -3) and up to, and including, $v = 4$. The only restriction on the w-values is that $w \geq 1$, so we have $T = \{(v, w) \mid -3 < v \leq 4, w \geq 1\}$. □

As with functions, we can describe relations algebraically using equations. For example, the equation $v^2 + w^3 = 1$ relates two variables v and w each of which represent real numbers. More formally, we can express this sentiment by defining the relation $R = \{(v, w) \mid v^2 + w^3 = 1\}$. An ordered pair $(v, w) \in R$ means v and w are *related* by the equation $v^2 + w^3 = 1$; that is, the pair (v, w) *satisfy* the equation.

For example, to show $(3, -2) \in R$, we check that when we substitute $v = 3$ and $w = -2$, the equation $v^2 + w^3 = 1$ is true. Sure enough, $(3)^2 + (-2)^3 = 9 - 8 = 1$. Hence, R maps 3 to -2. Note, however, that $(-2, 3) \notin R$ since $(-2)^2 + (3)^3 = -8 + 27 \neq 1$ which means R does not map -2 to 3.

When asked to 'graph the equation' $v^2 + w^3 = 1$, we really have two options. We could graph the relation R above. In this case, we would be graphing $v^2 + w^3 = 1$ on the vw-plane.[5] Alternatively, we could define $S = \{(w, v) \mid v^2 + w^3 = 1\}$ and graph S. This is equivalent to graphing $v^2 + w^3 = 1$ on the wv-plane. We do both in our next example.

EXAMPLE 1.5.2. Graph the equation $v^2 + w^3 = 1$ in the vw- and wv-planes. Include the axis-intercepts.
Solution.

- *graphing in the vw-plane:* We begin by finding the axis intercepts of the graph. To obtain a point on the v-axis, we require $w = 0$. To see if we have any v-intercepts on the graph of the equation $v^2 + w^3 = 1$, we substitute $w = 0$ into the equation and solve for v: $v^2 + (0)^3 = 1$. We get $v^2 = 1$ or $v = \pm 1$ so our two v-intercepts, as described in the vw-plane, are $(1, 0)$ and $(-1, 0)$.

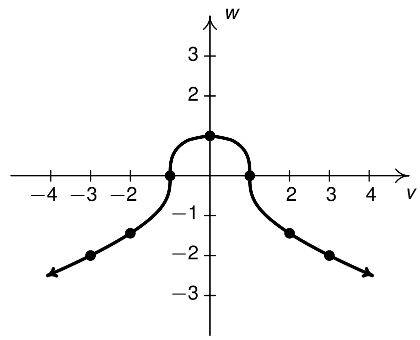
  Likewise, to find w-intercepts of the graph, we substitute $v = 0$ into the equation $v^2 + w^3 = 1$ and get $w^3 = 1$ or $w = 1$. Hence, he have only one w-intercept, $(0, 1)$.

  One way to efficiently produce additional points is to solve the equation $v^2 + w^3 = 1$ for one of the variables, say w, in terms of the other, v. In this way, we are treating w as the dependent variable and v as the independent variable. From $v^2 + w^3 = 1$, we get $w^3 = 1 - v^2$ or $w = \sqrt[3]{1 - v^2}$.

  We now substitute a value in for v, determine the corresponding value w, and plot the resulting point (v, w). We summarize our results below on the left. By plotting additional points (or getting help from a graphing utility), we produce the graph below on the right.

[5]Recall this means the horizontal axis is labeled 'v' and the vertical axis is labeled 'w.'

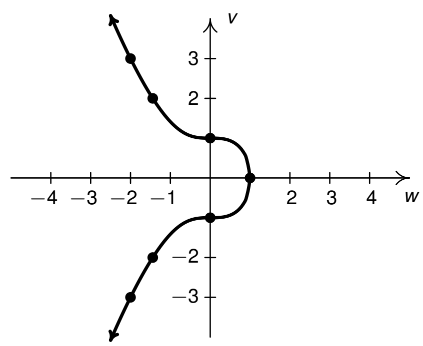
v	w	(v, w)
-3	-2	$(-3, -2)$
-2	$-\sqrt[3]{3}$	$(-2, -\sqrt[3]{3})$
-1	0	$(-1, 0)$
0	1	$(0, 1)$
1	0	$(1, 0)$
2	$-\sqrt[3]{3}$	$(2, -\sqrt[3]{3})$
3	-2	$(3, -2)$



$v^2 + w^3 = 1$

- *graphing in the wv-plane:* To graph $v^2 + w^3 = 1$ in the wv-plane, all we need to do is reverse the coordinates of the ordered pairs we obtained for our graph in the vw-plane. In particular, the v-intercepts are written $(0, 1)$ and $(0, -1)$ and the w-intercept is written $(1, 0)$. Using the table below on the left we produce the graph below on the right.

v	w	(w, v)
-3	-2	$(-2, -3)$
-2	$-\sqrt[3]{3}$	$(-\sqrt[3]{3}, -2)$
-1	0	$(0, -1)$
0	1	$(1, 0)$
1	0	$(0, 1)$
2	$-\sqrt[3]{3}$	$(-\sqrt[3]{3}, 2)$
3	-2	$(-2, 3)$



$v^2 + w^3 = 1$

□

Note that regardless of which geometric depiction we choose for $v^2 + w^3 = 1$, the graph appears to be symmetric about the w-axis. To prove this is the case, consider a generic point (v, w) on the graph of $v^2 + w^3 = 1$ in the vw-plane.

To show the point symmetric about the w-axis, $(-v, w)$ is also on the graph of $v^2 + w^3 = 1$, we need to show that the coordinates of the point $(-v, w)$ satisfy the equation $v^2 + w^3 = 1$. That is, we need to show $(-v)^2 + w^3 = 1$. Since $(-v)^2 + w^3 = v^2 + w^3$, and we know by assumption $v^2 + w^3 = 1$, we get $(-v)^2 + w^3 = v^2 + w^3 = 1$, proving $(-v, w)$ is also on the graph of the equation.

The key reason our proof above is successful is that algebraically, the equation $v^2 + w^3 = 1$ is unchanged if v is replaced with $-v$. Geometrically, this means the graph is the same if it undergoes a reflection across the w-axis. We generalize this reasoning in the following result. Note that, as usual, we default to the more common x and y-axis labels.

THEOREM 1.12. **Testing the Graph of an Equation for Symmetry:**
To test the graph of an equation in the xy-plane for symmetry:

- about the x-axis: substitute $(x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the x-axis.
- about the y-axis: substitute $(-x, y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the y-axis.
- about the origin: substitute $(-x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the origin.

Parts of Theorem 1.12 should look familiar from our work with even and odd functions. Indeed if a function f is even, $f(-x) = f(x)$. Hence, the equation $y = f(-x)$ reduces to the equation $y = f(x)$, so the graph of f is symmetric about the y-axis.

Likewise if f is odd, then $f(-x) = -f(x)$. In this case, the equation $-y = f(-x)$ reduces to $-y = -f(x)$, or $y = f(x)$, proving the graph is symmetric about the origin.

When it comes to symmetry about the x-axis, most of the time this indicates a violation of the Vertical Line Test, which is why we haven't discussed that particular kind of symmetry until now.

We put Theorem 1.12 to good use in the following example.

EXAMPLE 1.5.3. Graph each of the equations below in the xy-plane. Find the axis intercepts, if any, and prove any symmetry suggested by the graphs.

1. $x^2 - y^2 = 4$
2. $(x - 1)^2 + 4y^2 = 16$

Solution.

1. We begin graphing $x^2 - y^2 = 4$ by checking for axis intercepts. To check for x-intercepts, we set $y = 0$ and solve $x^2 - (0)^2 = 4$. We get $x = \pm 2$ and obtain two x-intercepts $(-2, 0)$ and $(2, 0)$.

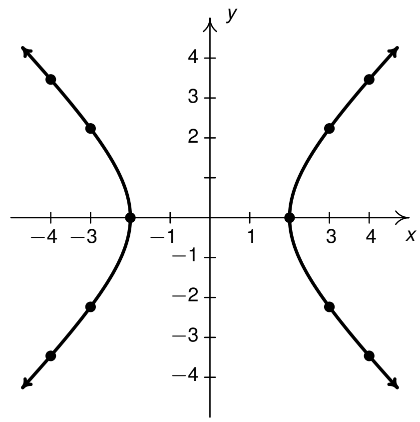
   When looking for y-intercepts, we set $x = 0$ and get $(0)^2 - y^2 = 4$ or $y^2 = -4$. Since this equation has no real number solutions, we have no y-intercepts.

   In order to produce more points on the graph, we solve $x^2 - y^2 = 4$ for y and obtain $y = \pm\sqrt{x^2 - 4}$. Since we know $x^2 - 4 \geq 0$ in order to produce real number results for y, we restrict our attention to $x \leq -2$ and $x \geq 2$. Doing so produces the table below on the left. Using these, we construct the graph below the right.

   The graph certainly appears to be symmetric about both axes and the origin. To prove this, we note that the equation $x^2 - (-y)^2 = 4$ quickly reduces to $x^2 - y^2 = 4$, proving the graph is symmetric about the x-axis.

   Likewise, the equations $(-x)^2 - y^2 = 4$ and $(-x)^2 - (-y)^2 = 4$ also reduce to $x^2 - y^2 = 4$, proving the graph is, indeed, symmetric about the y-axis and origin, respectively.

x	y	(x,y)
-4	$\pm 2\sqrt{3}$	$(-4, \pm 2\sqrt{3})$
-3	$\pm\sqrt{5}$	$(-3, \pm\sqrt{5})$
-2	0	$(-2, 0)$
2	0	$(2, 0)$
3	$\pm\sqrt{5}$	$(3, \pm\sqrt{5})$
4	$\pm 2\sqrt{3}$	$(4, \pm 2\sqrt{3})$



$x^2 - y^2 = 4$

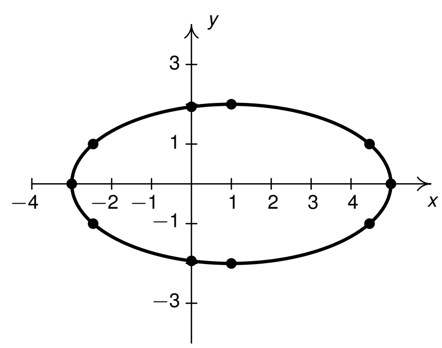
2. To determine if there are any x-intercepts on the graph of $(x-1)^2 + 4y^2 = 16$, we set $y = 0$ and solve $(x-1)^2 + 4(0)^2 = 16$. This reduces to $(x-1)^2 = 16$ which gives $x = -3$ and $x = 5$. Hence, we have two x-intercepts, $(-3, 0)$ and $(5, 0)$.

   Looking for y-intercepts, we set $x = 0$ and solve $(0-1)^2 + 4y^2 = 16$ or $1 + 4y^2 = 16$. This gives $y^2 = \frac{15}{4}$ so $y = \pm\frac{\sqrt{15}}{2}$. Hence, we have two y-intercepts: $\left(0, \pm\frac{\sqrt{15}}{2}\right)$.

   In this case, it is slightly easier[6] to solve for x in terms of y. From $(x-1)^2 + 4y^2 = 16$ we get $(x-1)^2 = 16 - 4y^2$ which gives $x = 1 \pm \sqrt{16 - 4y^2}$.

   Since we know $16 - 4y^2 \geq 0$ to produce real number results for x, we require $-2 \leq y \leq 2$. Selecting values in that range produces the table below on the left. Plotting these points, along with the y-intercepts produces the graph on the right.

y	x	(x,y)
-2	1	$(1, -2)$
-1	$1 \pm 2\sqrt{3}$	$(1 \pm 2\sqrt{3}, -1)$
0	$1 \pm 4 = -3, 5$	$(-3, 0), (5, 0)$
1	$1 \pm 2\sqrt{3}$	$(1 \pm 2\sqrt{3}, 1)$
2	1	$(1, 2)$



$(x-1)^2 + 4y^2 = 16$

   The graph certainly appears to be symmetric about the x-axis. To check, we substitute $(-y)$ in for y and get $(x-1)^2 + 4(-y)^2 = 16$ which reduces to $(x-1)^2 + 4y^2 = 16$.

   Owing to the placement of the x-intercepts, $(-3, 0)$ and $(5, 0)$, the graph is most certainly not symmetric about the y-axis nor about the origin. □

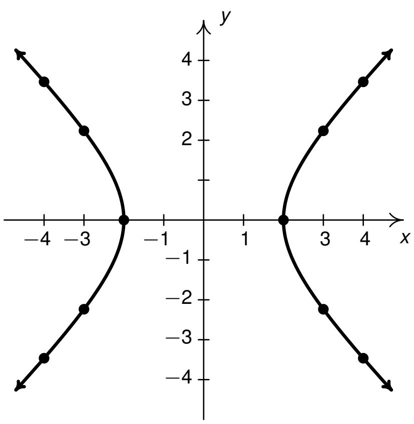
[6] Read this as we're avoiding fractions.

Looking at the graphs of the equations $x^2 - y^2 = 4$ and $(x-1)^2 + 4y^2 = 16$ in Example 1.5.3, it is evident neither of these equations represents y as a function of x nor x as a function of y. (Do you see why?)

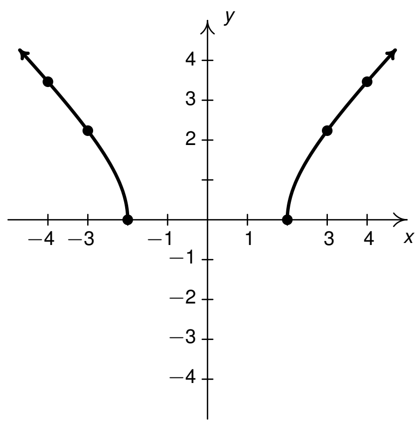
With the concept of 'function' being touted in the opening remarks of Section **??** as being one of the 'universal tools' with which scientists and engineers solve a wide variety of problems, you may well wonder if we can't somehow apply what we know about functions to these sorts of relations. It turns out that while, taken all at once, these equations do not describe functions, taken in parts, they do.

For example, consider the equation $x^2 - y^2 = 4$. Solving for y, we obtained $y = \pm\sqrt{x^2-4}$. Defining $f_1(x) = \sqrt{x^2-4}$ and $f_2(x) = -\sqrt{x^2-4}$, we get a functional description for the upper and lower halves, or *branches* of the curve, respectively.[7]

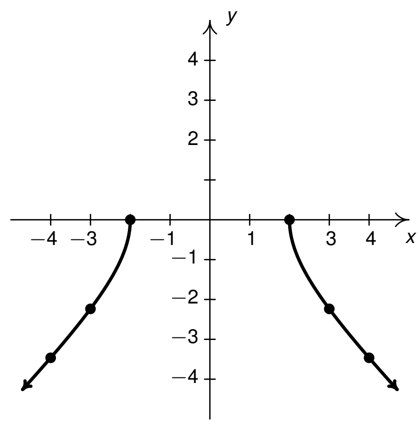
If, for instance, we wanted to analyze this curve near $(3, -\sqrt{5})$, we could use the *function* f_2 and all the associated function tools[8] to do just that.



$x^2 - y^2 = 4$



$f_1(x) = \sqrt{x^2-4}$



$f_2(x) = -\sqrt{x^2-4}$

In this way we say the equation $x^2 - y^2 = 4$ *implicitly* describes y as a function of x meaning that given any point (x_0, y_0) on $x^2 - y^2 = 4$, we can find a function f defined (on an interval) containing x_0 so that $f(x_0) = y_0$ and whose graph lies on the curve $x^2 - y^2 = 4$.

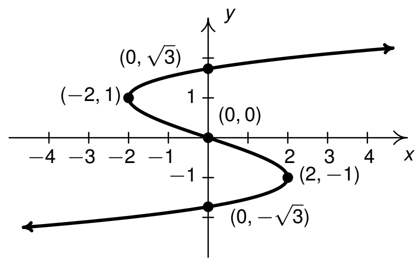
Note that in this case, we are fortunate to have two *explicit* formulas for functions that cover the entire curve, namely $f_1(x) = \sqrt{x^2-4}$ and $f_2(x) = -\sqrt{x^2-4}$. We explore this concept further in the next example.

EXAMPLE 1.5.4. Consider the graph of the relation R below.

1. Explain why this curve does not represent y as a function of x.
2. Resolve the graph of R into two or more graphs of implicitly defined functions.
3. Explain why this curve represents x as a function of y and find a formula for $x = g(y)$.

[7]There are many more ways to break this relation into functional parts. We could, for instance, go piecewise and take portions of the graph which lie in Quadrants I and III as one function and leave the parts in Quadrants II and IV as the other; we could look at this as being comprised of *four* functions, and so on.

[8]including, when the time comes, Calculus



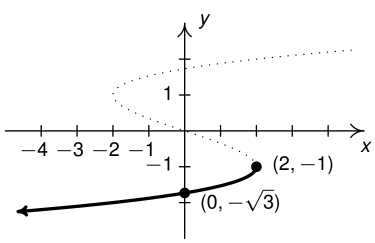
The graph of R

Solution.

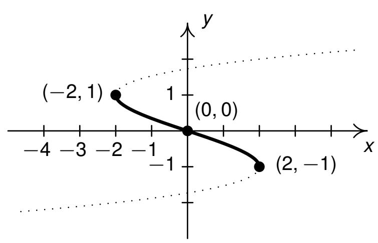
1. Using the Vertical Line Test, Theorem **??**, we find several instances where vertical lines intersect the graph of R more than once. The y-axis, $x = 0$ is one such line. We have $x = 0$ matched with *three* different y-values: $-\sqrt{3}$, 0, and $\sqrt{3}$.

2. Since the maximum number of times a vertical line intersects the graph of R is three, it stands to reason we need to resolve the graph of R into at least three pieces.

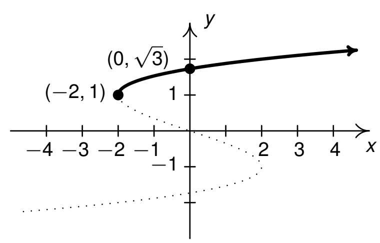
   One strategy is to begin at the far left and begin tracing the graph until it begins to 'double back' and repeat y-coordinates. Doing so we get three functions (represented by the bold solid lines) below.



$y = f_1(x)$



$y = f_2(x)$



$y = f_3(x)$

3. To verify that R represents x as a function of y, we check to see if any y-value has more than one x associated with it. One way to do this is to employ the the Horizontal Line Test (Exercise **??** in Section **??**.) Since every horizontal line intersects the graph at most once, x is a function of y.

   Using Theorem **??** from Chapter **??**, we get $x = (1)y(y - \sqrt{3})(y + \sqrt{3}) = y^3 - 3y$, a fact we can readily check using a graphing utility. □

Not all equations implicitly define y as a function of x. For a quick example, take $x = 117$ or any other vertical line. Even if an equation implicitly describes y as a function of x near one point, there's no guarantee we can find an explicit algebraic representation for that function.[9]

While the theory of implicit functions is well beyond the scope of this text, we will nevertheless see this concept come into play in Section 1.6. For our purposes, it suffices to know that just because a relation is not a function doesn't mean we cannot find a way to apply what we know about functions to analyze the relation locally through a functional lens.

[9] An example of this is $y^5 - y - x = 1$ near $(-1, 0)$.

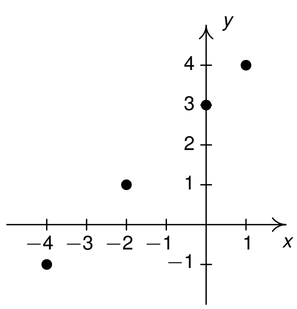
1.5.1 Exercises

In Exercises 1 - 20, graph the given relation in the xy-plane.

1. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$
2. $\{(-2,0),\ (-1,1),\ (-1,-1),\ (0,2),\ (0,-2),\ (1,3),\ (1,-3)\}$
3. $\{(m,2m) \mid m = 0, \pm1, \pm2\}$
4. $\left\{\left(\frac{6}{k}, k\right) \mid k = \pm1, \pm2, \pm3, \pm4, \pm5, \pm6\right\}$
5. $\left\{\left(n, 4-n^2\right) \mid n = 0, \pm1, \pm2\right\}$
6. $\left\{\left(\sqrt{j}, j\right) \mid j = 0, 1, 4, 9\right\}$
7. $\{(x,-2) \mid x > -4\}$
8. $\{(x,3) \mid x \leq 4\}$
9. $\{(-1,y) \mid y > 1\}$
10. $\{(2,y) \mid y \leq 5\}$
11. $\{(-2,y) \mid -3 < y \leq 4\}$
12. $\{(3,y) \mid -4 \leq y < 3\}$
13. $\{(x,2) \mid -2 \leq x < 3\}$
14. $\{(x,-3) \mid -4 < x \leq 4\}$
15. $\{(x,y) \mid x > -2\}$
16. $\{(x,y) \mid x \leq 3\}$
17. $\{(x,y) \mid y < 4\}$
18. $\{(x,y) \mid x \leq 3,\ y < 2\}$
19. $\{(x,y) \mid x > 0,\ y < 4\}$
20. $\{(x,y) \mid -\sqrt{2} \leq x \leq \frac{2}{3},\ \pi < y \leq \frac{9}{2}\}$

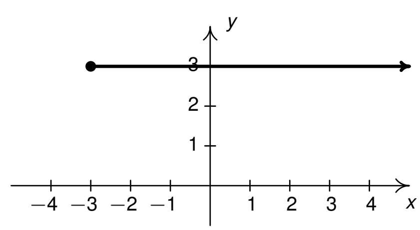
In Exercises 21 - 30, describe the given relation using either the roster or set-builder method.

21.



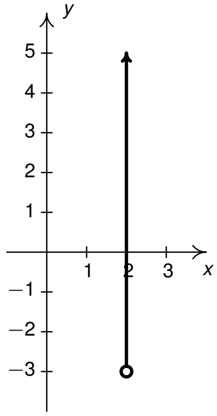
Relation A

22.



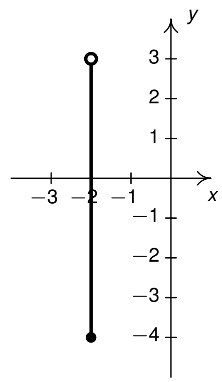
Relation B

23.



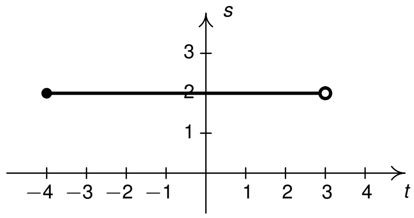
Relation C

24.



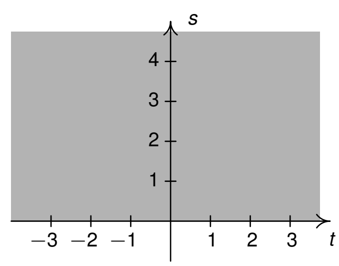
Relation D

25.



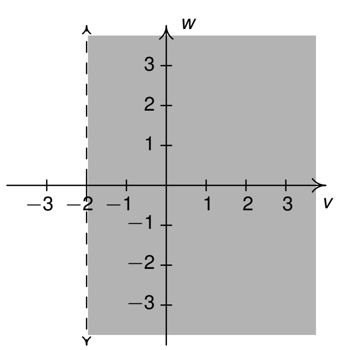
Relation E

26.



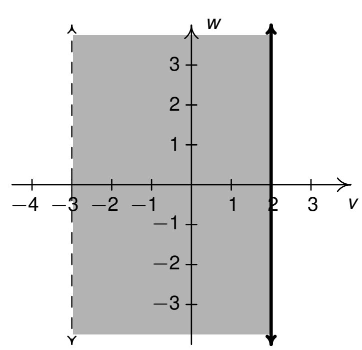
Relation F

27.



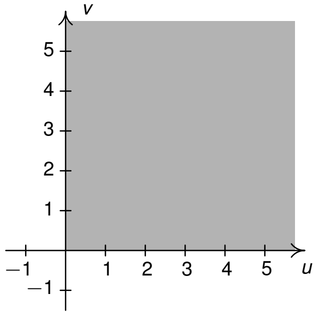
Relation G

28.



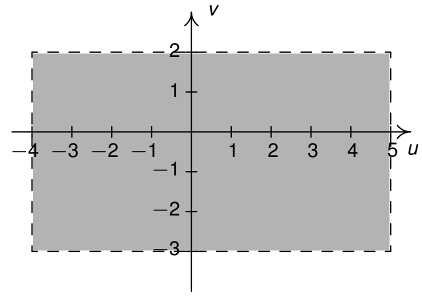
Relation H

29.



Relation I

30.



Relation J

Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. Discuss with your classmates how you might graph the relations given in Exercises 31 - 34. Note that in the notation below we are using the ellipsis, '. . . ,' to denote that the list does not end, but rather, continues to follow the established pattern indefinitely.

For the relations in Exercises 31 and 32, give two examples of points which belong to the relation and two points which do not belong to the relation.

31. $\{(x, y) \mid x$ is an odd integer, and y is an even integer.$\}$

32. $\{(x, 1) \mid x$ is an irrational number $\}$

33. $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), ...\}$

34. $\{... , (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), ...\}$

For each equation given in Exercises 35 - 38:

- Graph the equation in the xy-plane by creating a table of points.
- Find the axis intercepts, if they exist.
- Test the equation for symmetry. If the equation fails a symmetry test, find a point on the graph of the equation whose symmetric point is not on the graph of the equation.
- Determine if the equation describes y as a function of x. If not, describe the graph of the equation using two or more explicit functions of x. Check your answers using a graphing utility.

35. $(x + 2)^2 + y^2 = 16$

36. $x^2 - y^2 = 1$

37. $4y^2 - 9x^2 = 36$

38. $x^3y = -4$

For each equation given in Exercises 39 - 42:

- Graph the equation in the vw-plane by creating a table of points.
- Find the axis intercepts, if they exist.
- Test the equation for symmetry. If the equation fails a symmetry test, find a point on the graph of the equation whose symmetric point is not on the graph of the equation.
- Determine if the equation describes w as a function of v. If not, describe the graph of the equation using two or more explicit functions of v. Check your answers using a graphing utility.

39. $v + w^2 = 4$

40. $v^3 + w^3 = 8$

41. $v^2w^3 = 8$

42.[10] $v^4 - 2v^2w + w^2 = 16$

The procedures which we have outlined in the Examples of this section and used in Exercises 35 - 38 all rely on the fact that the equations were "well-behaved". Not everything in Mathematics is quite so tame, as the following equations will show you. Discuss with your classmates how you might approach graphing the equations given in Exercises 43 - 46. What difficulties arise when trying to apply the various tests and procedures given in this section? For more information, including pictures of the curves, each curve name is a link to its page at www.wikipedia.org. For a much longer list of fascinating curves, click here.

43. $x^3 + y^3 - 3xy = 0$ Folium of Descartes

44. $x^4 = x^2 + y^2$ Kampyle of Eudoxus

45. $y^2 = x^3 + 3x^2$ Tschirnhausen cubic

46. $(x^2 + y^2)^2 = x^3 + y^3$ Crooked egg

47. With the help of your classmates, find examples of equations whose graphs possess

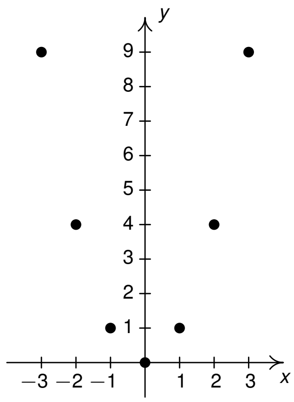
    - symmetry about the x-axis only
    - symmetry about the y-axis only
    - symmetry about the origin only
    - symmetry about the x-axis, y-axis, and origin

    Can you find an example of an equation whose graph possesses exactly *two* of the symmetries listed above? Why or why not?

[10] HINT: $v^4 - 2v^2w + w^2 = \left(v^2 - w\right)^2 \ldots$

1.5.2 Answers

1.



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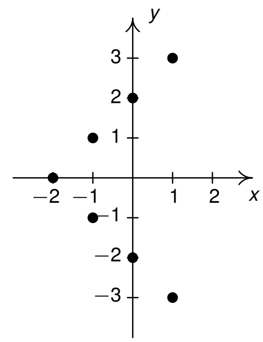

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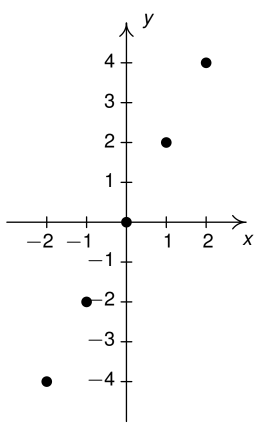

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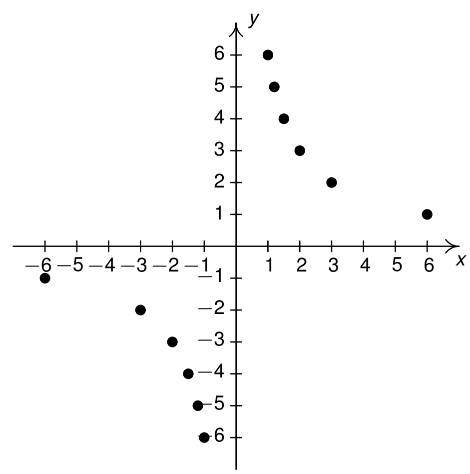

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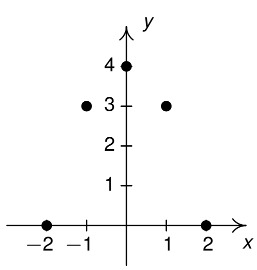

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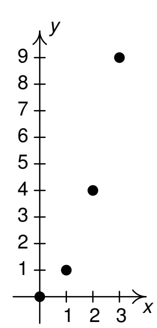

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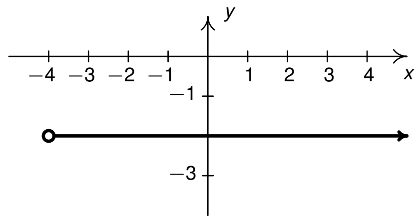
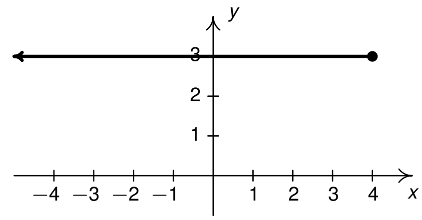

y
x
−4 −3 −2 −1
1 2 3 4
−1
−3

8.


y
x
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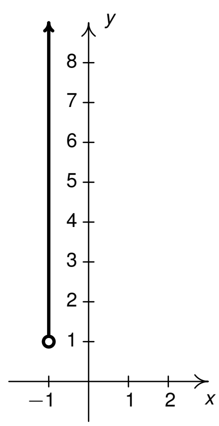
−4 −3 −2 −1
1 2 3 4

9.


y
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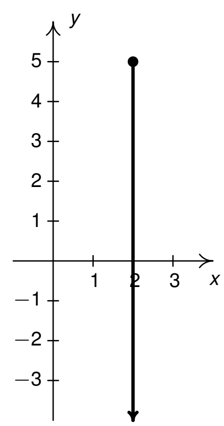
10.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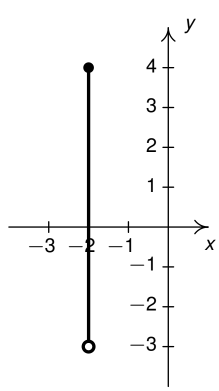
11.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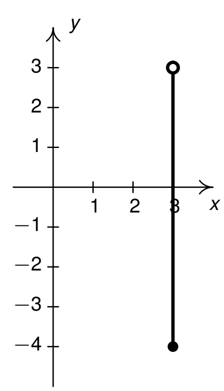
12.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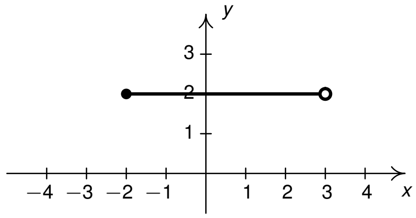
−3
−4

13.



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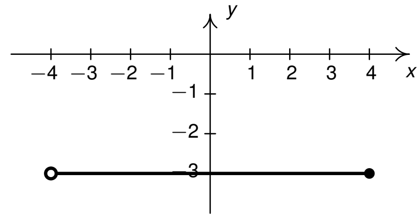

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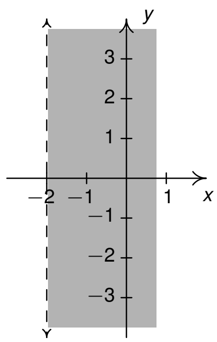

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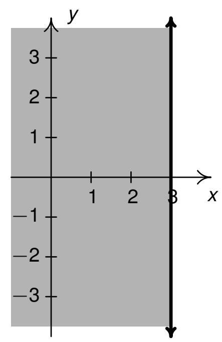

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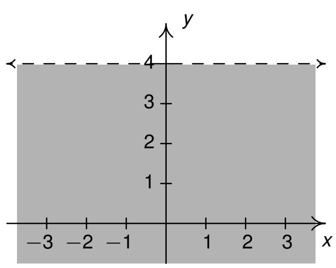

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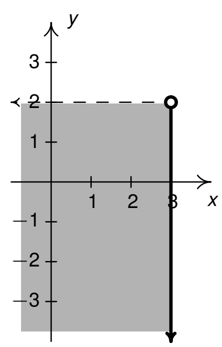

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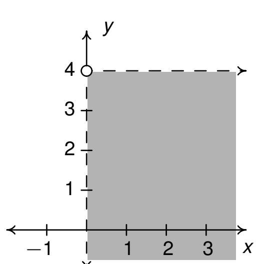

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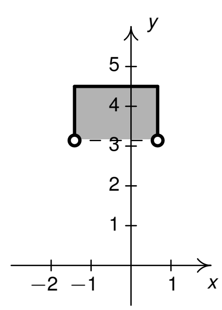


21. $A = \{(-4,-1),(-2,1),(0,3),(1,4)\}$

22. $B = \{(x,3) \mid x \geq -3\}$

23. $C = \{(2,y) \mid y > -3\}$

24. $D = \{(-2,y) \mid -4 \leq y < 3\}$

25. $E = \{(t,2) \mid -4 < t \leq 3\}$

26. $F = \{(t,s) \mid s \geq 0\}$

27. $G = \{(v,w) \mid v > -2\}$

28. $H = \{(v,w) \mid -3 < v \leq 2\}$

29. $I = \{(u,v) \mid u \geq 0, v \geq 0\}$

30. $J = \{(u,v) \mid -4 < u < 5,\ -3 < v < 2\}$

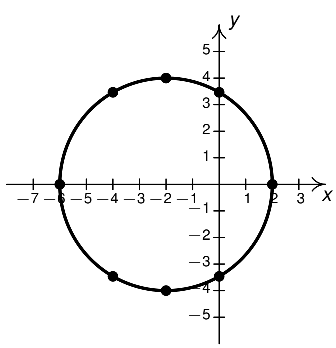
35. $(x+2)^2 + y^2 = 16$
Re-write as $y = \pm\sqrt{16-(x+2)^2}$.

The graph of the equation is the graphs of $f_1(x) = \sqrt{16-(x+2)^2}$ together with $f_2(x) = -\sqrt{16-(x+2)^2}$.

x-intercepts: $(-6,0)$, $(2,0)$

y-intercepts: $\left(0, \pm 2\sqrt{3}\right)$

x	y	(x,y)
-6	0	$(-6,0)$
-4	$\pm 2\sqrt{3}$	$\left(-4, \pm 2\sqrt{3}\right)$
-2	± 4	$(-2, \pm 4)$
0	$\pm 2\sqrt{3}$	$\left(0, \pm 2\sqrt{3}\right)$
2	0	$(2,0)$



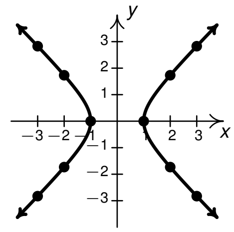
The graph is symmetric about the x-axis

The graph is not symmetric about the y-axis: $(-6,0)$ is on the graph but $(6,0)$ is not.

The graph is not symmetric about the origin: $(-6,0)$ is on the graph but $(6,0)$ is not.

The equation does not describe y as a function of x.

36. $x^2 - y^2 = 1$
Re-write as: $y = \pm\sqrt{x^2 - 1}$.



x-intercepts: $(-1, 0), (1, 0)$

The graph has no y-intercepts

x	y	(x, y)
-3	$\pm\sqrt{8}$	$(-3, \pm\sqrt{8})$
-2	$\pm\sqrt{3}$	$(-2, \pm\sqrt{3})$
-1	0	$(-1, 0)$
1	0	$(1, 0)$
2	$\pm\sqrt{3}$	$(2, \pm\sqrt{3})$
3	$\pm\sqrt{8}$	$(3, \pm\sqrt{8})$

The graph is symmetric about the x-axis.

The graph is symmetric about the y-axis.

The graph is symmetric about the origin.

The equation does not describe y as a function of x.

The graph of the equation is the graphs of $f_1(x) = \sqrt{x^2 - 1}$ together with $f_2(x) = -\sqrt{x^2 - 1}$.

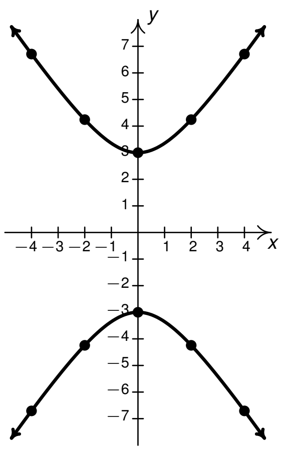
37. $4y^2 - 9x^2 = 36$

Re-write as: $y = \pm\dfrac{\sqrt{9x^2+36}}{2}$.

The graph has no x-intercepts

y-intercepts: $(0, \pm 3)$

x	y	(x, y)
-4	$\pm 3\sqrt{5}$	$\left(-4, \pm 3\sqrt{5}\right)$
-2	$\pm 3\sqrt{2}$	$\left(-2, \pm 3\sqrt{2}\right)$
0	± 3	$(0, \pm 3)$
2	$\pm 3\sqrt{2}$	$\left(2, \pm 3\sqrt{2}\right)$
4	$\pm 3\sqrt{5}$	$\left(4, \pm 3\sqrt{5}\right)$



The graph is symmetric about the x-axis.

The graph is symmetric about the y-axis.

The graph is symmetric about the origin.

The equation does not describe y as a function of x.

The graph of the equation is the graphs of $f_1(x) = \dfrac{\sqrt{9x^2+36}}{2}$ together with $f_2(x) = -\dfrac{\sqrt{9x^2+36}}{2}$.

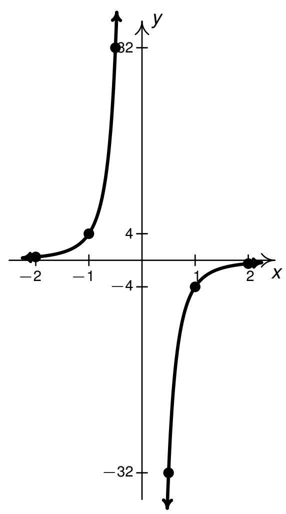
38. $x^3y = -4$

Re-write as: $y = -\dfrac{4}{x^3} = -4x^{-3}$.

The graph has no x-intercepts

The graph has no y-intercepts

x	y	(x, y)
-2	$\frac{1}{2}$	$(-2, \frac{1}{2})$
-1	4	$(-1, 4)$
$-\frac{1}{2}$	32	$(-\frac{1}{2}, 32)$
$\frac{1}{2}$	-32	$(\frac{1}{2}, -32)$
1	-4	$(1, -4)$
2	$-\frac{1}{2}$	$(2, -\frac{1}{2})$



The graph is not symmetric about the x-axis: $(1, -4)$ is on the graph but $(1, 4)$ is not.

The graph is not symmetric about the y-axis: $(1, -4)$ is on the graph but $(-1, -4)$ is not.

The graph is symmetric about the origin.

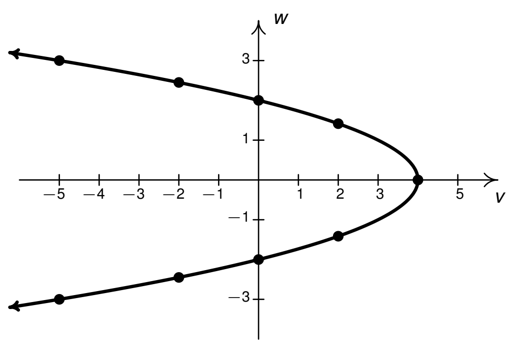
The equation does describe y as a function of x, namely $y = f(x) = -4x^{-3}$.

39. $v + w^2 = 4$
Re-write as $w = \pm\sqrt{4 - v}$.

v-intercept: $(4, 0)$

w-intercepts: $(0, \pm 2)$

v	w	(x, y)
-5	± 3	$(-5, \pm 3)$
-2	$\pm\sqrt{6}$	$(-2, \pm\sqrt{6})$
0	± 2	$(0, \pm 2)$
2	$\pm\sqrt{2}$	$(1, \pm\sqrt{3})$
4	0	$(4, 0)$



The graph is symmetric about the v-axis

The graph is not symmetric about the w-axis: $(4, 0)$ is on the graph but $(-4, 0)$ is not.

The graph is not symmetric about the origin: $(4, 0)$ is on the graph but $(-4, 0)$ is not.

The equation does not describe w as a function of v.

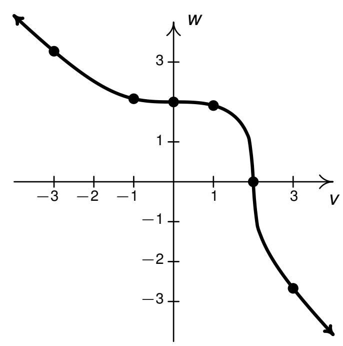
The graph of the equation is the graphs of $f_1(v) = \sqrt{4 - v}$ together with $f_2(v) = -\sqrt{4 - v}$.

40. $v^3 + w^3 = 8$
Re-write as: $w = \sqrt[3]{8 - v^3}$.

v-intercept: $(2, 0)$

w-intercept: $(0, 2)$

v	w	(v, w)
-3	$\sqrt[3]{35}$	$(-3, \sqrt[3]{35})$
-1	$\sqrt[3]{9}$	$(-1, \sqrt[3]{9})$
0	2	$(0, 2)$
1	$\sqrt[3]{7}$	$(1, \sqrt[3]{7})$
2	0	$(2, 0)$
3	$-\sqrt[3]{19}$	$(3, -\sqrt[3]{19})$



The graph is not symmetric about the v-axis: $(0, 2)$ is on the graph but $(0, -2)$ is not.

The graph is not symmetric about the w-axis: $(2, 0)$ is on the graph but $(-2, 0)$ is not.

The graph is not symmetric about the origin: $(0, 2)$ is on the graph but $(0, -2)$ is not.

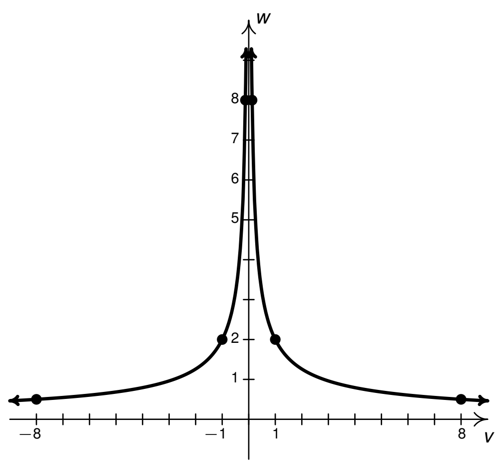
The equation does describe w as a function of v, namely $w = f(v) = \sqrt[3]{8 - v^3}$.

41. $v^2w^3 = 8$
Re-write as $w = \dfrac{2}{\sqrt[3]{v^2}} = 2v^{-\frac{2}{3}}$.

The graph has no v-intercepts.

The graph has no w-intercepts.

v	w	(x, y)
-8	$\frac{1}{2}$	$\left(-8, \frac{1}{2}\right)$
-1	2	$(-1, 2)$
$-\frac{1}{8}$	8	$\left(-\frac{1}{8}, 8\right)$
$\frac{1}{8}$	8	$\left(\frac{1}{8}, 8\right)$
1	2	$(1, 2)$
8	$\frac{1}{2}$	$\left(8, \frac{1}{2}\right)$



The graph is not symmetric about the v-axis: $(-1, 2)$ is on the graph but $(-1, -2)$ is not.

The graph is symmetric about the w-axis.

The graph is not symmetric about the origin: $(-1, 2)$ is on the graph but $(-1, -2)$ is not.

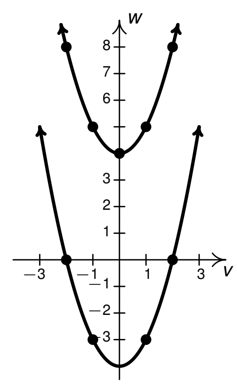
The equation does describe w as a function of v, namely $w = f(v) = 2v^{-\frac{2}{3}}$.

42. $v^4 - 2v^2w + w^2 = 16$
Re-write as: $\left(v^2 - w\right)^2 = 16$
Extracting square roots gives:
$w = v^2 + 4$ and $w = v^2 - 4$

v-intercepts: $(-2, 0), (2, 0)$.

w-intercepts: $(0, -4), (0, 4)$

v	w	(v, w)
-2	8	$(-2, 8)$
-2	0	$(-2, 0)$
-1	5	$(-1, 5)$
-1	-3	$(-1, -3)$
0	± 4	$(0, \pm 4)$
1	5	$(1, 5)$
1	-3	$(1, -3)$
2	8	$(2, 8)$
2	0	$(2, 0)$



The graph is not symmetric about the v-axis: $(1, 5)$ is on the graph but $(1, -5)$ is not.

The graph is symmetric about the w-axis.

The graph is not symmetric about the origin: $(1, 5)$ is on the graph but $(-1, -5)$ is not.

The equation does not describe w as a function of v.

The graph of the equation is the graphs of $f_1(v) = v^2 + 4$ together with $f_2(v) = v^2 - 4$.

1.6 Inverse Functions

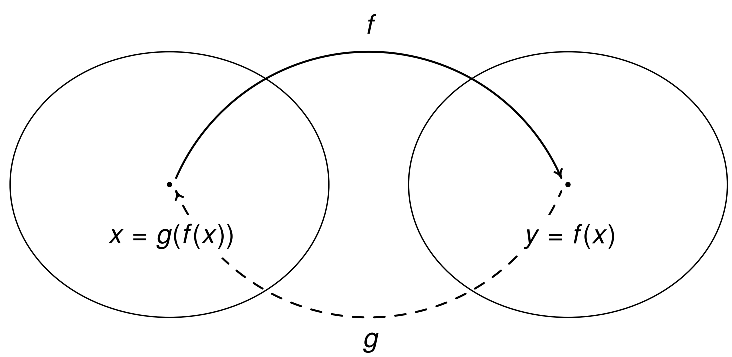
In Section **??**, we defined functions as processes. In this section, we seek to reverse, or 'undo' those processes. As in real life, we will find that some processes (like putting on socks and shoes) are reversible while some (like baking a cake) are not.

Consider the function $f(x) = 3x + 4$. Starting with a real number input x, we apply two steps in the following sequence: first we multiply the input by 3 and, second, we add 4 to the result.

To reverse this process, we seek a function g which will undo each of these steps and take the output from f, $3x + 4$, and return the input x. If we think of the two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes and then we take off the socks. In much the same way, the function g should undo each step of f but in the opposite order. That is, the function g should first *subtract* 4 from the input x then *divide* the result by 3. This leads us to the formula $g(x) = \frac{x-4}{3}$.

Let's check to see if the function g does the job. If $x = 5$, then $f(5) = 3(5) + 4 = 15 + 4 = 19$. Taking the output 19 from f, we substitute it into g to get $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$, which is our original input to f. To check that g does the job for all x in the domain of f, we take the generic output from f, $f(x) = 3x + 4$, and substitute that into g. That is, we simplify $g(f(x)) = g(3x + 4) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$, which is our original input to f. If we carefully examine the arithmetic as we simplify $g(f(x))$, we actually see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3.

Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x) = \frac{x-4}{3}$, and substitute that into f, we get $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right) + 4 = (x - 4) + 4 = x$. Using the language of function composition developed in Section 1.3, the statements $g(f(x)) = x$ and $f(g(x)) = x$ can be written as $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$, respectively.[1] Abstractly, we can visualize the relationship between f and g in the diagram below.



The main idea to get from the diagram is that g takes the outputs from f and returns them to their respective inputs, and conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the central definition of the section.

[1] At the level of functions, $g \circ f = f \circ g = I$, where I is the identity function as defined as $I(x) = x$ for all real numbers, x.

DEFINITION 1.4. Suppose f and g are two functions such that

1. $(g \circ f)(x) = x$ for all x in the domain of f

   and

2. $(f \circ g)(x) = x$ for all x in the domain of g

then f and g are **inverses** of each other and the functions f and g are said to be **invertible**.

If we abstract one step further, we can express the sentiment in Definition 1.4 by saying that f and g are inverses if and only if $g \circ f = I_1$ and $f \circ g = I_2$ where I_1 is the identity function restricted[2] to the domain of f and I_2 is the identity function restricted to the domain of g.

In other words, $I_1(x) = x$ for all x in the domain of f and $I_2(x) = x$ for all x in the domain of g. Using this description of inverses along with the properties of function composition listed in Theorem 1.4, we can show that function inverses are unique.[3]

Suppose g and h are both inverses of a function f. By Theorem 1.13, the domain of g is equal to the domain of h, since both are the range of f. This means the identity function I_2 applies both to the domain of h and the domain of g. Thus $h = h \circ I_2 = h \circ (f \circ g) = (h \circ f) \circ g = I_1 \circ g = g$, as required.

We summarize the important properties of invertible functions in the following theorem.[4] Apart from introducing notation, each of the results below are immediate consequences of the idea that inverse functions map the outputs from a function f back to their corresponding inputs.

THEOREM 1.13. **Properties of Inverse Functions:** Suppose f is an invertible function.

- There is exactly one inverse function for f, denoted f^{-1} (read 'f-inverse')
- The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1}
- $f(a) = c$ if and only if $a = f^{-1}(c)$

  NOTE: In particular, for all y in the range of f, the solution to $f(x) = y$ is $x = f^{-1}(y)$.
- (a, c) is on the graph of f if and only if (c, a) is on the graph of f^{-1}

  NOTE: This means graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across $y = x$.[a]
- f^{-1} is an invertible function and $(f^{-1})^{-1} = f$.

[a] See Example **??** in Section **??** and Example **??** in Section **??**.

[2] The identity function I, first introduced in Exercise **??** in Section **??** and mentioned in Theorem 1.4, has a domain of all real numbers. Since the domains of f and g may not be all real numbers, we need the restrictions listed here.

[3] In other words, invertible functions have exactly one inverse.

[4] In the interests of full disclosure, the authors would like to admit that much of the discussion in the previous paragraphs could have easily been avoided had we appealed to the description of a function as a set of ordered pairs. We make no apology for our discussion from a function composition standpoint, however, since it exposes the reader to more abstract ways of thinking of functions and inverses. We will revisit this concept again in Chapter **??**.

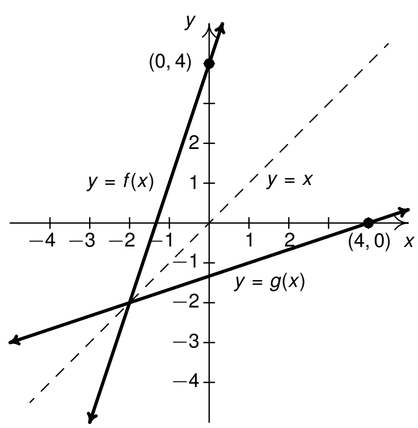
The notation f^{-1} is an unfortunate choice since you've been programmed since Elementary Algebra to think of this as $\frac{1}{f}$. This is most definitely *not* the case since, for instance, $f(x) = 3x + 4$ has as its inverse $f^{-1}(x) = \frac{x-4}{3}$, which is certainly different than $\frac{1}{f(x)} = \frac{1}{3x+4}$.

Why does this confusing notation persist? As we mentioned in Section 1.3, the identity function I is to function composition what the real number 1 is to real number multiplication. The choice of notation f^{-1} alludes to the property that $f^{-1} \circ f = I_1$ and $f \circ f^{-1} = I_2$, in much the same way as $3^{-1} \cdot 3 = 1$ and $3 \cdot 3^{-1} = 1$.

Before we embark on an example, we demonstrate the pertinent parts of Theorem 1.13 to the inverse pair $f(x) = 3x + 4$ and $g(x) = f^{-1}(x) = \frac{x-4}{3}$. Suppose we wanted to solve $3x + 4 = 7$. Going through the usual machinations, we obtain $x = 1$.

If we view this equation as $f(x) = 7$, however, then we are looking for the input x corresponding to the output $f(x) = 7$. This is exactly the question f^{-1} was built to answer. In other words, the solution to $f(x) = 7$ is $x = f^{-1}(7) = 1$. In other words, the formula $f^{-1}(x)$ encodes all of the algebra required to 'undo' what the formula $f(x)$ does to x. More generally, any time you have ever solved an equation, you have really been working through an inverse problem.

We also note the graphs of $f(x) = 3x + 4$ and $g(x) = f^{-1}(x) = \frac{x-4}{3}$ are easily seen to be reflections across the line $y = x$ as seen below. In particular, note that the y-intercept $(0, 4)$ on the graph of $y = f(x)$ corresponds to the x-intercept on the graph of $y = f^{-1}(x)$. Indeed, the point $(0, 4)$ on the graph of $y = f(x)$ can be interpreted as $(0, 4) = (0, f(0)) = (f^{-1}(4), 4)$ just as the point $(4, 0)$ on the graph of $y = f^{-1}(x)$ can be interpreted as $(4, 0) = (4, f^{-1}(4)) = (f(0), 0)$.



Graphs of inverse functions $y = f(x) = 3x + 4$ and $y = f^{-1}(x) = \dfrac{x-4}{3}$.

EXAMPLE 1.6.1. For each pair of functions f and g below:

1. Verify each pair of functions f and g are inverses: (a) algebraically and (b) graphically.
2. Use the fact f and g are inverses to solve $f(x) = 5$ and $g(x) = -3$

- $f(x) = \sqrt[3]{x-1} + 2$ and $g(x) = (x-2)^3 + 1$
- $f(t) = \dfrac{2t}{t+1}$ and $g(t) = \dfrac{t}{2-t}$

Solution.

Solution for $f(x) = \sqrt[3]{x-1} + 2$ *and* $g(x) = (x-2)^3 + 1$.

1. (a) To verify $f(x) = \sqrt[3]{x-1} + 2$ and $g(x) = (x-2)^3 + 1$ are inverses, we appeal to Definition 1.4 and show $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$ for all real numbers, x.

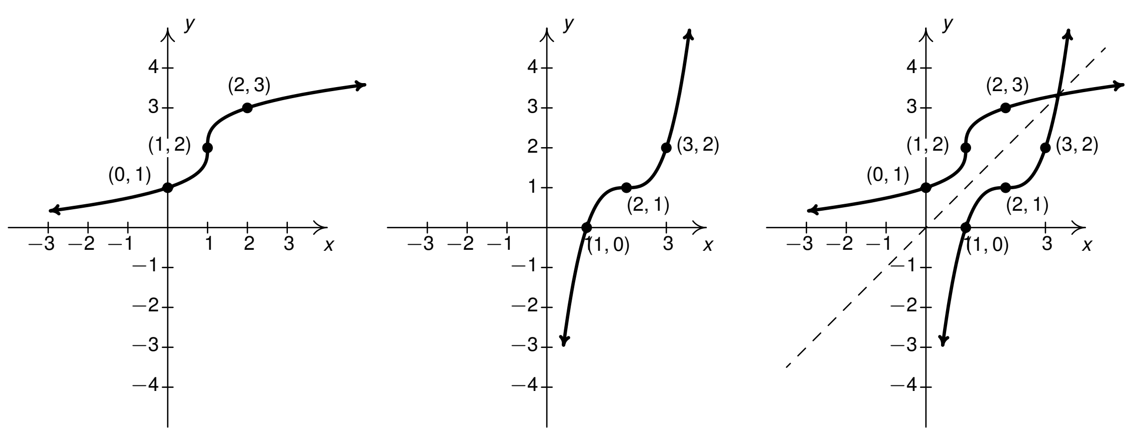
$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(\sqrt[3]{x-1} + 2) \\ &= [(\sqrt[3]{x-1} + 2) - 2]^3 + 1 \\ &= (\sqrt[3]{x-1})^3 + 1 \\ &= x - 1 + 1 \\ &= x \checkmark \end{aligned} \qquad \begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f((x-2)^3 + 1) \\ &= \sqrt[3]{[(x-2)^3 + 1] - 1} + 2 \\ &= \sqrt[3]{(x-2)^3} + 2 \\ &= x - 4 + 4 \\ &= x \checkmark \end{aligned}$$

Since the root here, 3, is odd, Theorem **??** gives $(\sqrt[3]{x-1})^3 = x - 1$ and $\sqrt[3]{(x-2)^3} = x - 2$.

(b) To show f and g are inverses graphically, we graph $y = f(x)$ and $y = g(x)$ on the same set of axes and check to see if they are reflections about the line $y = x$.

The graph of $y = f(x) = \sqrt[3]{x-1} + 2$ appears below on the left courtesy of Theorem **??** in Section **??**. The graph of $y = g(x) = (x-2)^3 + 1$ appears below in the middle thanks to Theorem **??** in Section **??**.

We can immediately see three pairs of corresponding points: $(0, 1)$ and $(1, 0)$, $(1, 2)$ and $(2, 1)$, $(2, 3)$ and $(3, 2)$. When graphed on the same pair of axes, the two graphs certainly appear to be symmetric about the line $y = x$, as required.



$y = f(x)$ $\qquad$ $y = g(x)$ $\qquad$ $y = f(x)$, $y = g(x)$, $y = x$

2. Since f and g are inverses, the solution to $f(x) = 5$ is $x = f^{-1}(5) = g(5) = (5-2)^3 + 1 = 28$. To check, we find $f(28) = \sqrt[3]{28-1} + 2 = \sqrt[3]{27} + 2 = 3 + 2 = 5$, as required.

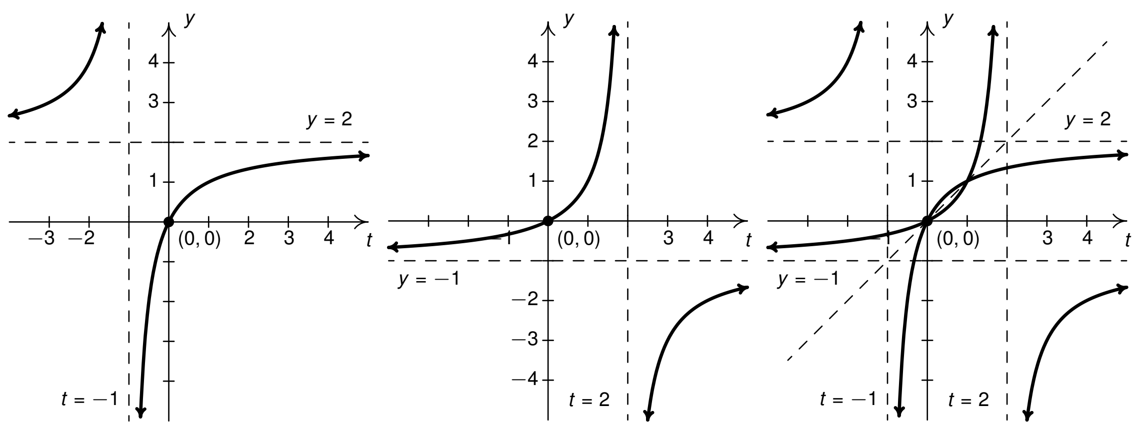
   Likewise, the solution to $g(x) = -3$ is $x = g^{-1}(-3) = f(-3) = \sqrt[3]{(-3)-1} + 2 = 2 - \sqrt[3]{4}$. Once again, to check, we find $g(2 - \sqrt[3]{4}) = (2 - \sqrt[3]{4} - 2)^3 + 1 = (-\sqrt[3]{4})^3 + 1 = -4 + 1 = -3$.

Solution for $f(t) = \dfrac{2t}{t+1}$ *and* $g(t) = \dfrac{t}{2-t}$.

1. (a) Note the domain of f excludes $t = -1$ and the domain of g excludes $t = 2$. Hence, when simplifying $(g \circ f)(t)$ and $(f \circ g)(t)$, we tacitly assume $t \neq -1$ and $t \neq 2$, respectively.

$$
\begin{aligned}
(g \circ f)(t) &= g(f(t)) \\
&= g\left(\frac{2t}{t+1}\right) \\
&= \frac{\frac{2t}{t+1}}{2 - \frac{2t}{t+1}} \\
&= \frac{\frac{2t}{t+1}}{2 - \frac{2t}{t+1}} \cdot \frac{(t+1)}{(t+1)} \\
&= \frac{2t}{2(t+1) - 2t} \\
&= \frac{2t}{2t + 2 - 2t} \\
&= \frac{2t}{2} \\
&= t \checkmark
\end{aligned}
\qquad
\begin{aligned}
(f \circ g)(t) &= f(g(t)) \\
&= f\left(\frac{t}{2-t}\right) \\
&= \frac{2\left(\frac{t}{2-t}\right)}{\left(\frac{t}{2-t}\right) + 1} \\
&= \frac{2\left(\frac{t}{2-t}\right)}{\left(\frac{t}{2-t}\right) + 1} \cdot \frac{(2-t)}{(2-t)} \\
&= \frac{2t}{t + (1)(2-t)} \\
&= \frac{2t}{t + 2 - t)} \\
&= \frac{2t}{2} \\
&= t \checkmark
\end{aligned}
$$

(b) We graph $y = f(t)$ and $y = g(t)$ using the techniques discussed in Sections **??** and **??**.



$y = f(t)$ $\quad$ $y = g(t)$ $\quad$ $y = f(t)$, $y = g(t)$, and $y = x$

We find the graph of f has a vertical asymptote $t = -1$ and a horizontal asymptote $y = 2$. Corresponding to the *vertical* asymptote $t = -1$ on the graph of f, we find the graph of g has a *horizontal* asymptote $y = -1$.

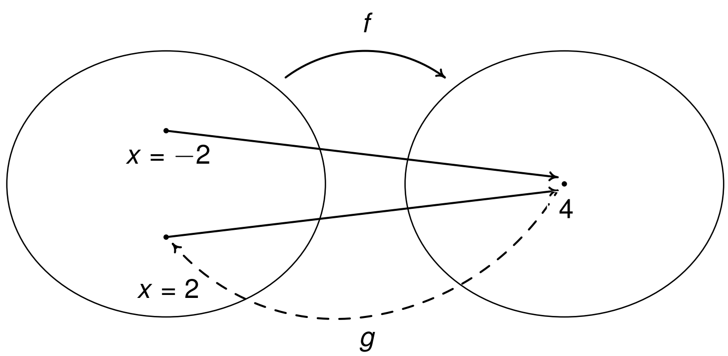
Likewise, the *horizontal* asymptote $y = 2$ on the graph of f corresponds to the *vertical* asymptote $t = 2$ on the graph of g. Both graphs share the intercept $(0, 0)$. When graphed together on the same set of axes, the graphs of f and g do appear to be symmetric about the line $y = t$.

2. Don't let the fact that f and g in this case were defined using the independent variable, 't' instead of 'x' deter you in your efforts to solve $f(x) = 5$. Remember that, ultimately, the function f here is the *process* represented by the formula $f(t)$, and is the same process (with the same inverse!) regardless of the letter used as the independent variable. Hence, the solution to $f(x) = 5$ is $x = f^{-1}(1) = g(5)$. We get $g(5) = \frac{5}{2-5} = -\frac{5}{3}$.

   To check, we find $f\left(-\frac{5}{3}\right) = \left(-\frac{10}{3}\right) / \left(-\frac{2}{3}\right) = 5$. Similarly, we solve $g(x) = -3$ by finding $x = g^{-1}(-3) = f(-3) = \frac{-6}{-2} = 3$. Sure enough, we find $g(3) = \frac{3}{2-3} = -3$. □

We now investigate under what circumstances a function is invertible. As a way to motivate the discussion, we consider $f(x) = x^2$. A likely candidate for the inverse is the function $g(x) = \sqrt{x}$. However, $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$, which is not equal to x unless $x \geq 0$.

For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$. That is, g failed to return the input -2 from its output 4. Instead, g matches the output 4 to a *different* input, namely 2, which satisfies $f(2) = 4$. Schematically:



We see from the diagram that since both $f(-2)$ and $f(2)$ are 4, it is impossible to construct a *function* which takes 4 back to *both* $x = 2$ and $x = -2$ since, by definition, a function can match 4 with only *one* number.

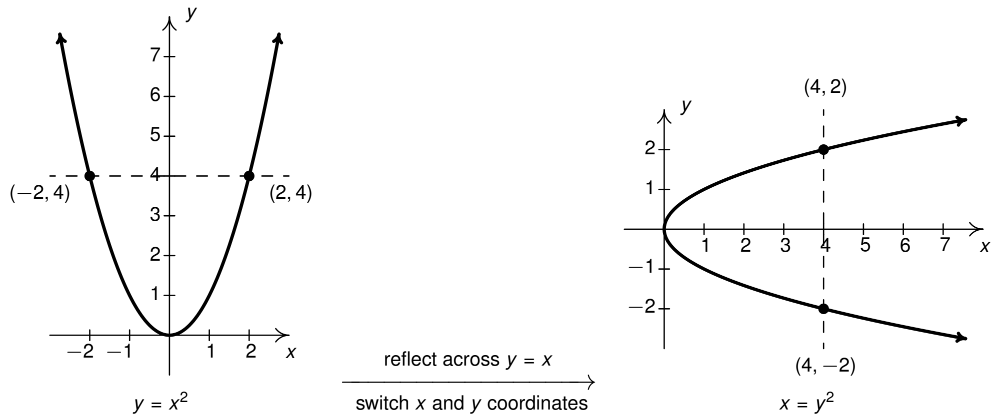
In general, in order for a function to be invertible, each output can come from only *one* input. Since, by definition, a function matches up each input to only *one* output, invertible functions have the property that they match one input to one output and vice-versa. We formalize this concept below.

DEFINITION 1.5. A function f is said to be **one-to-one** if whenever $f(a) = f(b)$, then $a = b$.

Note that an equivalent way to state Definition 1.5 is that a function is one-to-one if *different* inputs go to *different* outputs. That is, if $a \neq b$, then $f(a) \neq f(b)$.

Before we solidify the connection between invertible functions and one-to-one functions, we take a moment to see what goes wrong graphically when trying to find the inverse of $f(x) = x^2$.

Per Theorem 1.13, the graph of $y = f^{-1}(x)$, if it exists, is obtained from the graph of $y = x^2$ by reflecting $y = x^2$ about the line $y = x$. Procedurally, this is accomplished by interchanging the x and y coordinates of each point on the graph of $y = x^2$. Algebraically, we are swapping the variables 'x' and 'y' which results in the equation $x = y^2$ whose graph is below on the right.



We see immediately the graph of $x = y^2$ fails the Vertical Line Test, Theorem **??**. In particular, the vertical line $x = 4$ intersects the graph at two points, $(4, -2)$ and $(4, 2)$ meaning the relation described by $x = y^2$ matches the x-value 4 with two different y-values, -2 and 2.

Note that the *vertical* line $x = 4$ and the points $(4, \pm 2)$ on the graph of $x = y^2$ correspond to the *horizontal* line $y = 4$ and the points $(\pm 2, 4)$ on the graph of $y = x^2$ which brings us right back to the concept of one-to-one. The fact that both $(-2, 4)$ and $(2, 4)$ are on the graph of f means $f(-2) = f(2) = 4$. Hence, f takes different inputs, -2 and 2, to the same output, 4, so f is not one-to-one.

Recall the Horizontal Line Test from Exercise **??** in Section **??**. Applying that result to the graph of f we say the graph of f 'fails' the Horizontal Line Test since the horizontal line $y = 4$ intersects the graph of $y = x^2$ more than once. This means that the equation $y = x^2$ does not represent x is not a function of y.

Said differently, the Horizontal Line Test detects when there is at least one y-value (4) which is matched to more than one x-value (± 2). In other words, the Horizontal Line Test can be used to detect whether or not a function is one-to-one.

So, to review, $f(x) = x^2$ is not invertible, not one-to-one, and its graph fails the Horizontal Line Test. It turns out that these three attributes: being invertible, one-to-one, and having a graph that passes the Horizontal Line Test are mathematically equivalent. That is to say if one if these things is true about a function, then they all are; it also means that, as in this case, if one of these things *isn't* true about a function, then *none* of them are. We summarize this result in the following theorem.

THEOREM 1.14. **Equivalent Conditions for Invertibility:**
For a function f, either all of the following statements are true or none of them are:

- f is invertible.
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.[a]

[a] i.e., no horizontal line intersects the graph more than once.

To prove Theorem 1.14, we first suppose f is invertible. Then there is a function g so that $g(f(x)) = x$ for all x in the domain of f. If $f(a) = f(b)$, then $g(f(a)) = g(f(b))$. Since $g(f(x)) = x$, the equation $g(f(a)) = g(f(b))$ reduces to $a = b$. We've shown that if $f(a) = f(b)$, then $a = b$, proving f is one-to-one.

Next, assume f is one-to-one. Suppose a horizontal line $y = c$ intersects the graph of $y = f(x)$ at the points (a, c) and (b, c). This means $f(a) = c$ and $f(b) = c$ so $f(a) = f(b)$. Since f is one-to-one, this means $a = b$ so the points (a, c) and (b, c) are actually one in the same. This establishes that each horizontal line can intersect the graph of f at most once, so the graph of f passes the Horizontal Line Test.

Last, but not least, suppose the graph of f passes the Horizontal Line Test. Let c be a real number in the range of f. Then the horizontal line $y = c$ intersects the graph of $y = f(x)$ just *once*, say at the point $(a, c) = (a, f(a))$. Define the mapping g so that $g(c) = g(f(a)) = a$. The mapping g is a *function* since each horizontal line $y = c$ where c is in the range of f intersects the graph of f only *once*. By construction, we have the domain of g is the range of f and that for all x in the domain of $f, g(f(x)) = x$. We leave it to the reader to show that for all x in the domain of g, $f(g(x)) = x$, too.

Hence, we've shown: first, if f invertible, then f is one-to-one; second, if f is one-to-one, then the graph of f passes the Horizontal Line Test; and third, if f passes the Horizontal Line Test, then f is invertible. Hence if f is satisfies any one of these three conditions, we can show f must satisfy the other two.[5]

We put this result to work in the next example.

EXAMPLE 1.6.2. Determine if the following functions are one-to-one: (a) analytically using Definition 1.5 and (b) graphically using the Horizontal Line Test. For the functions that are one-to-one, graph the inverse.

1. $f(x) = x^2 - 2x + 4$
2. $g(t) = \dfrac{2t}{1 - t}$
3. $F = \{(-1, 1), (0, 2), (1, -3), (2, 1)\}$
4. $G = \{(t^3 + 1, 2t) \mid t \text{ is a real number.}\}$

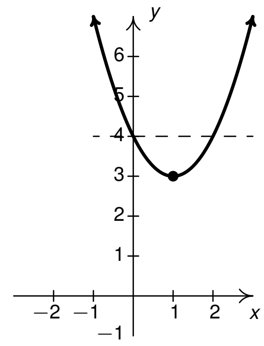
Solution.

1. (a) To determine whether or not f is one-to-one analytically, we assume $f(a) = f(b)$ and work to see if we can deduce $a = b$. As we work our way through the problem below on the left, we encounter a quadratic equation. We rewrite the equation so it equals 0 and factor by grouping. We get $a = b$ as one possibility, but we also get the possibility that $a = 2 - b$. This suggests that f may not be one-to-one. Taking $b = 0$, we get $a = 0$ or $a = 2$. Since $f(0) = 4$ and $f(2) = 4$, we have two different inputs with the same output, proving f is neither one-to-one nor invertible.

   (b) We note that f is a quadratic function and we graph $y = f(x)$ using the techniques presented in Section **??** below on the right. We see the graph fails the Horizontal Line Test quite often - in particular, crossing the line $y = 4$ at the points $(0, 4)$ and $(2, 4)$.

[5] For example, if we know f is one-to-one, we showed the graph of f passes the HLT which, in turn, guarantees f is invertible.

$$\begin{aligned}
f(a) &= f(b) \\
a^2 - 2a + 4 &= b^2 - 2b + 4 \\
a^2 - 2a &= b^2 - 2b \\
a^2 - b^2 - 2a + 2b &= 0 \\
(a+b)(a-b) - 2(a-b) &= 0 \\
(a-b)((a+b)-2) &= 0 \\
a - b = 0 \quad &\text{or} \quad a + b - 2 = 0 \\
a = b \quad &\text{or} \quad a = 2 - b
\end{aligned}$$

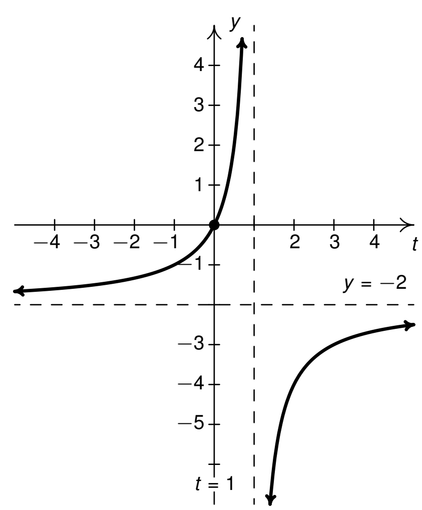


$y = f(x)$

2. (a) We begin with the assumption that $g(a) = g(b)$ for a, b in the domain of g (That is, we assume $a \neq 1$ and $b \neq 1$.) Through our work below on the left, we deduce $a = b$, proving g is one-to-one.

   (b) We graph $y = g(t)$ below on the right using the procedure outlined in Section **??**. We find the sole intercept is $(0,0)$ with asymptotes $t = 1$ and $y = -2$. Based on our graph, the graph of g appears to pass the Horizontal Line Test, verifying g is one-to-one.

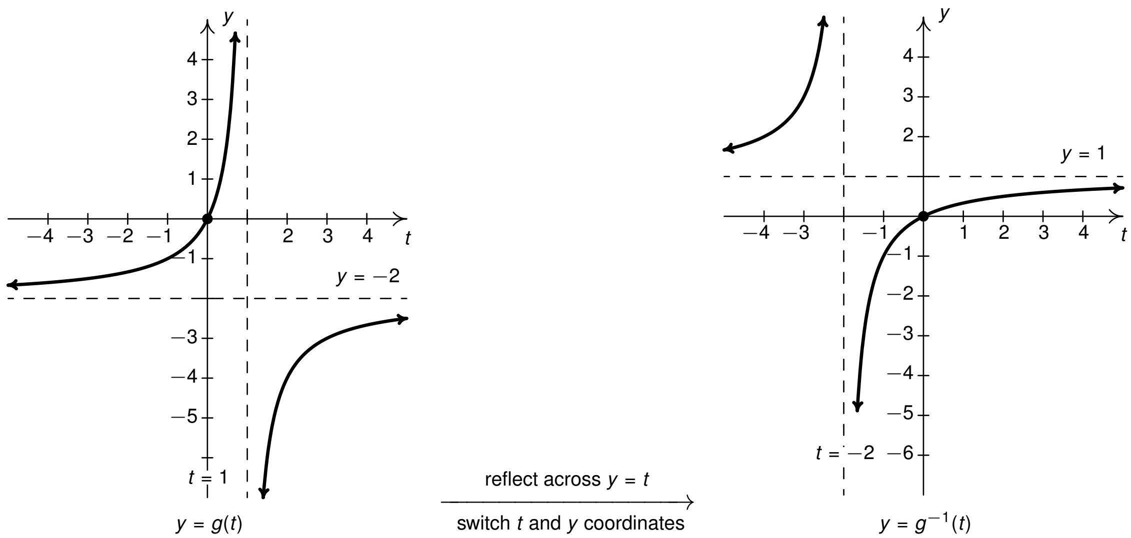
$$\begin{aligned}
g(a) &= g(b) \\
\frac{2a}{1-a} &= \frac{2b}{1-b} \\
2a(1-b) &= 2b(1-a) \\
2a - 2ab &= 2b - 2ba \\
2a &= 2b \\
a &= b \ \checkmark
\end{aligned}$$



$y = g(t)$

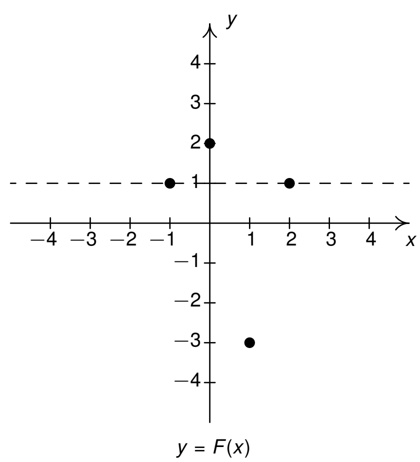
Since g is one-to-one, g is invertible. Even though we do not have a *formula* for $g^{-1}(t)$, we can nevertheless sketch the graph of $y = g^{-1}(t)$ by reflecting the graph of $y = g(t)$ across $y = t$.

Corresponding to the *vertical* asymptote $t = 1$ on the graph of g, the graph of $y = g^{-1}(t)$ will have a *horizontal* asymptote $y = 1$. Similarly, the *horizontal* asymptote $y = -2$ on the graph of g corresponds to a *vertical* asymptote $t = -2$ on the graph of g^{-1}. The point $(0,0)$ remains unchanged when we switch the t and y coordinates, so it is on both the graph of g and g^{-1}.



3. (a) The function F is given to us as a set of ordered pairs. Recall each ordered pair is of the form $(a, F(a))$. Since $(-1, 1)$ and $(2, 1)$ are both elements of F, this means $F(-1) = 1$ and $F(2) = 1$. Hence, we have two distinct inputs, -1 an 2 with the same output, 1, so F is not one-to-one and, hence, not invertible.

   (b) To graph F, we plot the points in F below on the left. We see the horizontal line $y = 1$ crosses the graph more than once. Hence, the graph of F fails the Horizontal Line Test.

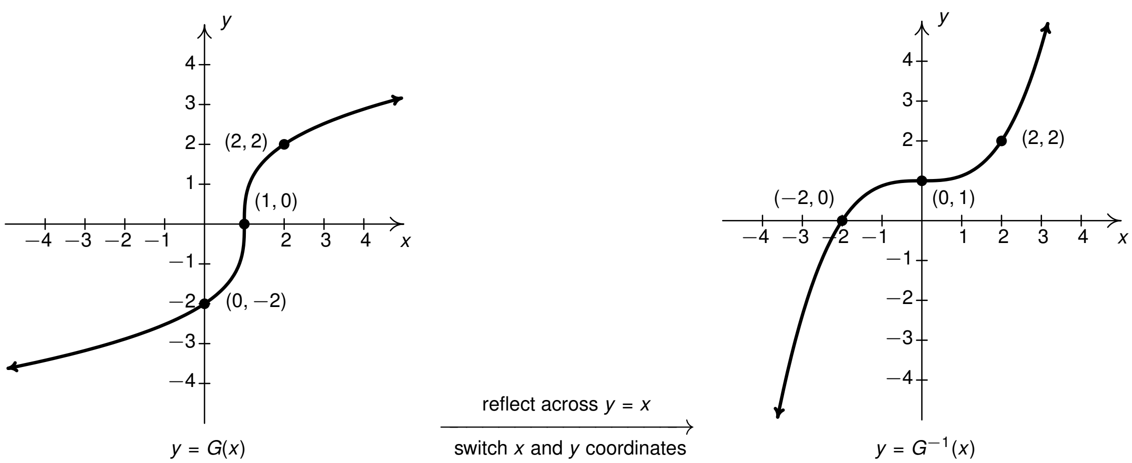


4. Like the function F above, the function G is described as a set of ordered pairs. Before we set about determining whether or not G is one-to-one, we take a moment to show G is, in fact, a function. That is, we must show that each real number input to G is matched to only one output.

   We are given $G = \left\{(t^3 + 1, 2t) \mid t \text{ is a real number.}\right\}$. and we know that when represented in this way, each ordered pair is of the form (input, output). Hence, the inputs to G are of the form $t^3 + 1$ and the outputs from G are of the form $2t$. To establish G is a function, we must show that each input produces only one output. If it should happen that $a^3 + 1 = b^3 + 1$, then we must show $2a = 2b$. The equation $a^3 + 1 = b^3 + 1$ gives $a^3 = b^3$, or $a = b$. From this it follows that $2a = 2b$ so G is a function.

(a) To show G is one-to-one, we must show that if two outputs from G are the same, the corresponding inputs must also be the same. That is, we must show that if $2a = 2b$, then $a^3 + 1 = b^3 + 1$. We see almost immediately that if $2a = 2b$ then $a = b$ so $a^3 + 1 = b^3 + 1$ as required. This shows G is one-to-one and, hence, invertible.

(b) We graph G below on the left by plotting points in the default xy-plane by choosing different values for t. For instance, $t = 0$ corresponds to the point $(0^3+1, 2(0)) = (1, 0)$, $t = 1$ corresponds to the point $(1^3 + 1, 2(1)) = (2, 2)$, $t = -1$ corresponds to the point $((-1)^3 + 1, 2(-1)) = (0, -2)$, etc.[6] Our graph appears to pass the Horizontal Line Test, confirming G is one-to-one. We obtain the graph of G^{-1} below on the right by reflecting the graph of G about the line $y = x$.



□

In Example 1.6.2, we showed the functions G and g are invertible and graphed their inverses. While graphs are perfectly fine representations of functions, we have seen where they aren't the most accurate. Ideally, we would like to represent G^{-1} and g^{-1} in the same manner in which G and g are presented to us. The key to doing this is to recall that inverse functions take outputs back to their associated inputs.

Consider $G = \{(t^3 + 1, 2t) \mid t \text{ is a real number.}\}$. As mentioned in Example 1.6.2, the ordered pairs which comprise G are in the form (input, output). Hence to find a compatible description for G^{-1}, we simply interchange the expressions in each of the coordinates to obtain $G^{-1} = \{(2t, t^3 + 1) \mid t \text{ is a real number.}\}$.

Since the function g was defined in terms of a formula we would like to find a formula representation for g^{-1}. We apply the same logic as above. Here, the input, represented by the independent variable t, and the output, represented by the dependent variable y, are related by the equation $y = g(t)$. Hence, to

[6]Foreshadowing Section **??**, we could let $x = t^3 + 1$ so that $t = \sqrt[3]{x-1}$. Hence, $y = 2t = 2\sqrt[3]{x-1}$.

exchange inputs and outputs, we interchange the 't' and 'y' variables. Doing so, we obtain the equation $t = g(y)$ which is an *implicit* description for g^{-1}. Solving for y gives an explicit formula for g^{-1}, namely $y = g^{-1}(t)$. We demonstrate this technique below.

$$\begin{aligned}
y &= g(t) \\
y &= \frac{2t}{1-t} \\
t &= \frac{2y}{1-y} && \text{interchange variables: } t \text{ and } y \\
t(1-y) &= 2y \\
t - ty &= 2y \\
t &= ty + 2y \\
t &= y(t+2) && \text{factor} \\
y &= \frac{t}{t+2}
\end{aligned}$$

We claim $g^{-1}(t) = \frac{t}{t+2}$, and leave the algebraic verification of this to the reader.

We generalize this approach below. As always, we resort to the default 'x' and 'y' labels for the independent and dependent variables, respectively.

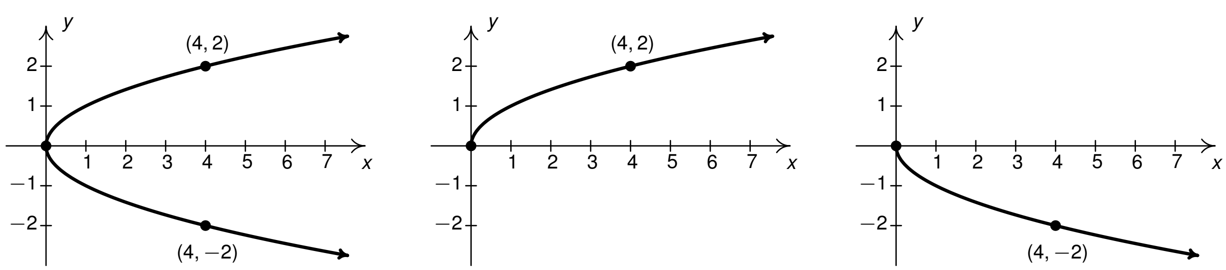
Steps for finding a formula for the Inverse of a one-to-one function

1. Write $y = f(x)$
2. Interchange x and y
3. Solve $x = f(y)$ for y to obtain $y = f^{-1}(x)$

We now return to $f(x) = x^2$. We know that f is not one-to-one, and thus, is not invertible, but our goal here is to see what way to see what goes wrong algebraically.

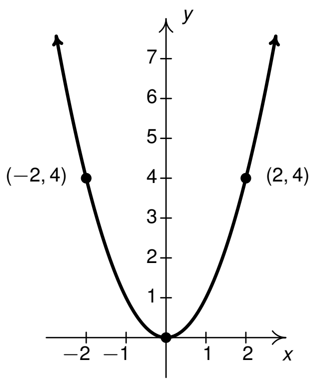
If we attempt to follow the algorithm above to find a formula for $f^{-1}(x)$, we start with the equation $y = x^2$ and interchange the variables 'x' and 'y' to produce the equation $x = y^2$. Solving for y gives $y = \pm\sqrt{x}$. It's this '$\pm$' which is causing the problem for us since this produces *two* y-values for any $x > 0$.

Using the language of Section 1.5, the equation $x = y^2$ implicitly defines *two* functions, $g_1(x) = \sqrt{x}$ and $g_2(x) = -\sqrt{x}$, each of which represents the top and bottom halves, respectively, of the graph of $x = y^2$.

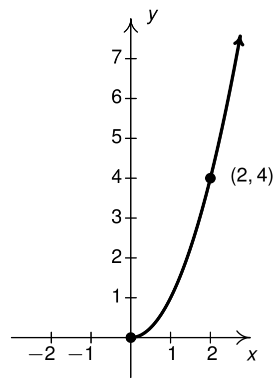


$x = y^2$ $\quad$ $y = g_1(x) = \sqrt{x}$ $\quad$ $y = g_2(x) = -\sqrt{x}$

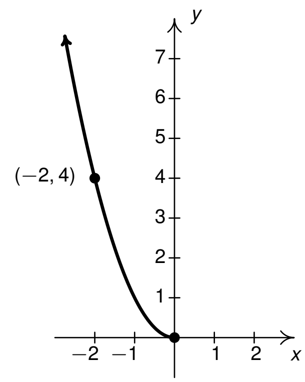
Hence, in some sense, we have two *partial* inverses for $f(x) = x^2$: $g_1(x) = \sqrt{x}$ returns the *positive* inputs from f and $g_2(x) = -\sqrt{x}$ returns the *negative* inputs to f. In order to view each of these functions as strict inverses, however, we need to split f into two parts: $f_1(x) = x^2$ for $x \geq 0$ and $f_2(x) = x^2$ for $x \leq 0$.



$y = f(x) = x^2$



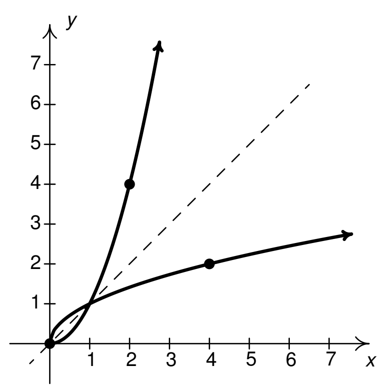
$y = f_1(x) = x^2,\ x \geq 0$



$y = f_2(x) = x^2,\ x \leq 0$

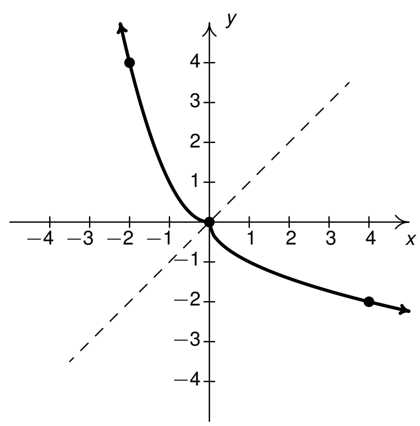
We claim that f_1 and g_1 are an inverse function pair as are f_2 and g_2. Indeed, we find:

$$\begin{aligned} (g_1 \circ f_1)(x) &= g_1(f_1(x)) \\ &= g_1(x^2) \\ &= \sqrt{x^2} \\ &= |x| = x, \text{ as } x \geq 0. \end{aligned} \qquad \begin{aligned} (f_1 \circ g_1)(x) &= f_1(g_1(x)) \\ &= f_1(\sqrt{x}) \\ &= (\sqrt{x})^2 \\ &= x \end{aligned}$$



$y = f_1(x) = x^2,\ x \geq 0$ and $y = g_1(x) = \sqrt{x}$

$$\begin{aligned} (g_2 \circ f_2)(x) &= g_2(f_2(x)) \\ &= g_2(x^2) \\ &= -\sqrt{x^2} \\ &= -|x| \\ &= -(-x) = x, \text{ as } x \leq 0. \end{aligned} \qquad \begin{aligned} (f_2 \circ g_2)(x) &= f_2(g_2(x)) \\ &= f_2(-\sqrt{x}) \\ &= (-\sqrt{x})^2 \\ &= (\sqrt{x})^2 \\ &= x \end{aligned}$$



$y = f_2(x) = x^2,\ x \leq 0$ and $y = g_2(x) = -\sqrt{x}$

Hence, by restricting the domain of f we are able to produce invertible functions. Said differently, in much the same way the equation $x = y^2$ implicitly describes a pair of *functions*, the equation $y = x^2$ implicitly describes a pair of *invertible* functions.

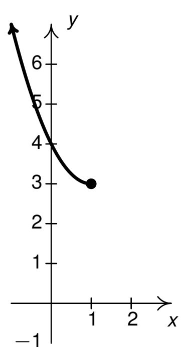
Our next example continues the theme of restricting the domain of a function to find inverse functions.

EXAMPLE 1.6.3. Graph the following functions to show they are one-to-one and find their inverses. Check your answers analytically using function composition and graphically.

1. $j(x) = x^2 - 2x + 4$, $x \leq 1$.
2. $k(t) = \sqrt{t+2} - 1$

Solution.

1. The function j is a restriction of the function f from Example 1.6.2. Since the domain of j is restricted to $x \leq 1$, we are selecting only the 'left half' of the parabola. Hence, the graph of j, seen below on the left, passes the Horizontal Line Test and thus j is invertible. Below on the right, we find an explicit formula for $j^{-1}(x)$ using our standard algorithm.[7]



$y = j(x)$

$$
\begin{aligned}
y &= j(x) \\
y &= x^2 - 2x + 4, \quad x \leq 1 \\
x &= y^2 - 2y + 4, \quad y \leq 1 && \text{switch } x \text{ and } y \\
0 &= y^2 - 2y + 4 - x \\
y &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} && \text{quadratic formula, } c = 4 - x \\
y &= \frac{2 \pm \sqrt{4x - 12}}{2} \\
y &= \frac{2 \pm \sqrt{4(x-3)}}{2} \\
y &= \frac{2 \pm 2\sqrt{x-3}}{2} \\
y &= \frac{2\left(1 \pm \sqrt{x-3}\right)}{2} \\
y &= 1 \pm \sqrt{x-3} \\
y &= 1 - \sqrt{x-3} && \text{since } y \leq 1.
\end{aligned}
$$

Hence, $j^{-1}(x) = 1 - \sqrt{x-3}$.

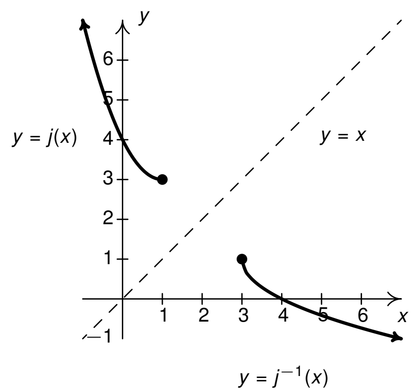
To check our answer algebraically, we simplify $(j^{-1} \circ j)(x)$ and $(j \circ j^{-1})(x)$ Note the importance of the domain restriction $x \leq 1$ when simplifying $(j^{-1} \circ j)(x)$.

[7] Here, we use the Quadratic Formula to solve for y. For 'completeness,' we note you can (and should!) also consider solving for y by 'completing' the square.

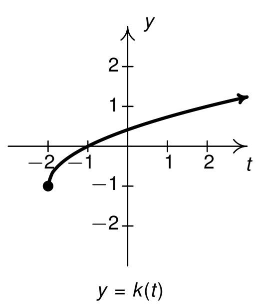
$$\begin{aligned}
\left(j^{-1} \circ j\right)(x) &= j^{-1}(j(x)) \\
&= j^{-1}\left(x^2 - 2x + 4\right), \quad x \leq 1 \\
&= 1 - \sqrt{\left(x^2 - 2x + 4\right) - 3} \\
&= 1 - \sqrt{x^2 - 2x + 1} \\
&= 1 - \sqrt{(x-1)^2} \\
&= 1 - |x - 1| \\
&= 1 - (-(x-1)) \text{ since } x \leq 1 \\
&= x \checkmark
\end{aligned}$$

$$\begin{aligned}
\left(j \circ j^{-1}\right)(x) &= j\left(j^{-1}(x)\right) \\
&= j\left(1 - \sqrt{x-3}\right) \\
&= \left(1 - \sqrt{x-3}\right)^2 - 2\left(1 - \sqrt{x-3}\right) + 4 \\
&= 1 - 2\sqrt{x-3} + \left(\sqrt{x-3}\right)^2 - 2 \\
&\quad + 2\sqrt{x-3} + 4 \\
&= 1 + x - 3 - 2 + 4 \\
&= x \checkmark
\end{aligned}$$

We graph both j and j^{-1} on the axes below. They appear to be symmetric about the line $y = x$.



2. Graphing $y = k(t) = \sqrt{t+2} - 1$, we see k is one-to-one, so we proceed to find an formula for k^{-1}.



$$\begin{aligned}
y &= k(t) \\
y &= \sqrt{t+2} - 1 \\
t &= \sqrt{y+2} - 1 \quad \text{switch } t \text{ and } y \\
t + 1 &= \sqrt{y+2} \\
(t+1)^2 &= \left(\sqrt{y+2}\right)^2 \\
t^2 + 2t + 1 &= y + 2 \\
y &= t^2 + 2t - 1
\end{aligned}$$

We have $k^{-1}(t) = t^2 + 2t - 1$. Based on our experience, we know something isn't quite right. We determined k^{-1} is a quadratic function, and we have seen several times in this section that these are not one-to-one unless their domains are suitably restricted.

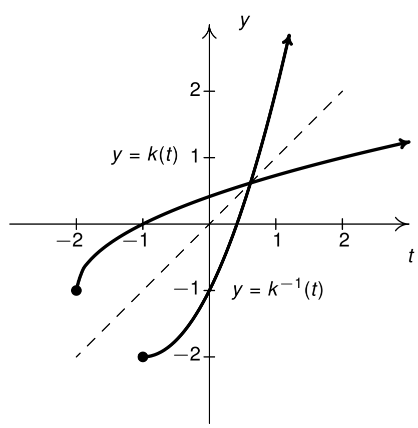
Theorem 1.13 tells us that the domain of k^{-1} is the range of k. From the graph of k, we see that the range is $[-1, \infty)$, which means we restrict the domain of k^{-1} to $t \geq -1$.

We now check that this works in our compositions. Note the importance of the domain restriction, $t \geq -1$ when simplifying $(k \circ k^{-1})(t)$.

$$\begin{aligned}
\left(k^{-1} \circ k\right)(t) &= k^{-1}(k(t)) \\
&= k^{-1}\left(\sqrt{t+2}-1\right) \\
&= \left(\sqrt{t+2}-1\right)^2 + 2\left(\sqrt{t+2}-1\right) - 1 \\
&= \left(\sqrt{t+2}\right)^2 - 2\sqrt{t+2} + 1 \\
&\quad + 2\sqrt{t+2} - 2 - 1 \\
&= t + 2 - 2 \\
&= t \ \checkmark
\end{aligned}$$

$$\begin{aligned}
\left(k \circ k^{-1}\right)(t) &= k\left(t^2 + 2t - 1\right), \quad t \geq -1 \\
&= \sqrt{\left(t^2+2t-1\right)+2} - 1 \\
&= \sqrt{t^2+2t+1} - 1 \\
&= \sqrt{(t+1)^2} - 1 \\
&= |t+1| - 1 \\
&= t + 1 - 1, \text{ since } t \geq -1 \\
&= t \ \checkmark
\end{aligned}$$

Graphically, everything checks out, provided that we remember the domain restriction on k^{-1} means we take the right half of the parabola.



□

Our last example of the section gives an application of inverse functions. Recall in Example **??** in Section **??**, we modeled the demand for PortaBoy game systems as the price per system, $p(x)$ as a function of the number of systems sold, x. In the following example, we find $p^{-1}(x)$ and interpret what it means.

EXAMPLE 1.6.4. Recall the price-demand function for PortaBoy game systems is modeled by the formula $p(x) = -1.5x + 250$ for $0 \leq x \leq 166$ where x represents the number of systems sold (the demand) and $p(x)$ is the price per system, in dollars.

1. Explain why p is one-to-one and find a formula for $p^{-1}(x)$. State the restricted domain.
2. Find and interpret $p^{-1}(220)$.
3. Recall from Section **??** that the profit P, in dollars, as a result of selling x systems is given by $P(x) = -1.5x^2 + 170x - 150$. Find and interpret $\left(P \circ p^{-1}\right)(x)$.

4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example **??**.

Solution.

1. Recall the graph of $p(x) = -1.5x + 250$, $0 \leq x \leq 166$, is a line segment from $(0, 250)$ to $(166, 1)$, and as such passes the Horizontal Line Test. Hence, p is one-to-one. We find the expression for $p^{-1}(x)$ as usual and get $p^{-1}(x) = \frac{500-2x}{3}$. The domain of p^{-1} should match the range of p, which is $[1, 250]$, and as such, we restrict the domain of p^{-1} to $1 \leq x \leq 250$.

2. We find $p^{-1}(220) = \frac{500-2(220)}{3} = 20$. Since the function p took as inputs the number of systems sold and returned the price per system as the output, p^{-1} takes the price per system as its input and returns the number of systems sold as its output. Hence, $p^{-1}(220) = 20$ means 20 systems will be sold in if the price is set at \$220 per system.

3. We compute $\left(P \circ p^{-1}\right)(x) = P\left(p^{-1}(x)\right) = P\left(\frac{500-2x}{3}\right) = -1.5\left(\frac{500-2x}{3}\right)^2 + 170\left(\frac{500-2x}{3}\right) - 150$. After a hefty amount of Elementary Algebra,[8] we obtain $\left(P \circ p^{-1}\right)(x) = -\frac{2}{3}x^2 + 220x - \frac{40450}{3}$.

   To understand what this means, recall that the original profit function P gave us the profit as a function of the number of systems sold. The function p^{-1} gives us the number of systems sold as a function of the price. Hence, when we compute $(P \circ p^{-1})(x) = P(p^{-1}(x))$, we input a price per system, x into the function p^{-1}.

   The number $p^{-1}(x)$ is the number of systems sold at that price. This number is then fed into P to return the profit obtained by selling $p^{-1}(x)$ systems. Hence, $\left(P \circ p^{-1}\right)(x)$ gives us the profit (in dollars) as a function of the price per system, x.

4. We know from Section **??** that the graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards. The maximum profit is realized at the vertex. Since we are concerned only with the price per system, we need only find the x-coordinate of the vertex. Identifying $a = -\frac{2}{3}$ and $b = 220$, we get, by the Vertex Formula, Equation **??**, $x = -\frac{b}{2a} = 165$.

   Hence, weekly profit is maximized if we set the price at \$165 per system. Comparing this with our answer from Example **??**, there is a slight discrepancy to the tune of \$0.50. We leave it to the reader to balance the books appropriately. $\square$

[8] It is good review to actually do this!

1.6.1 Exercises

In Exercises 1 - 8, verify the given pairs of functions are inverses algebraically and graphically.

1. $f(x) = 2x + 7$ and $g(x) = \dfrac{x-7}{2}$
2. $f(x) = \dfrac{5-3x}{4}$ and $g(x) = -\dfrac{4}{3}x + \dfrac{5}{3}$.
3. $f(t) = \dfrac{5}{t-1}$ and $g(t) = \dfrac{t+5}{t}$
4. $f(t) = \dfrac{t}{t-1}$ and $g(t) = f(t) = \dfrac{t}{t-1}$
5. $f(x) = \sqrt{4-x}$ and $g(x) = -x^2 + 4$, $x \geq 0$
6. $f(x) = 1 - \sqrt{x+1}$ and $g(x) = x^2 - 2x$, $x \leq 1$.
7. $f(t) = (t-1)^3 + 5$ and $g(t) = \sqrt[3]{t-5} + 1$
8. $f(t) = -\sqrt[4]{t-2}$ and $g(t) = t^4 + 2$, $t \leq 0$.

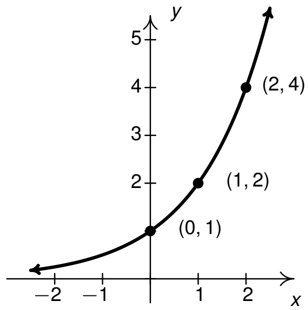
In Exercises 9 - 28, show that the given function is one-to-one and find its inverse. Check your answers algebraically and graphically. Verify the range of the function is the domain of its inverse and vice-versa.

9. $f(x) = 6x - 2$
10. $f(x) = 42 - x$
11. $g(t) = \dfrac{t-2}{3} + 4$
12. $g(t) = 1 - \dfrac{4+3t}{5}$
13. $f(x) = \sqrt{3x-1} + 5$
14. $f(x) = 2 - \sqrt{x-5}$
15. $g(t) = 3\sqrt{t-1} - 4$
16. $g(t) = 1 - 2\sqrt{2t+5}$
17. $f(x) = \sqrt[5]{3x-1}$
18. $f(x) = 3 - \sqrt[3]{x-2}$
19. $g(t) = t^2 - 10t$, $t \geq 5$
20. $g(t) = 3(t+4)^2 - 5$, $t \leq -4$
21. $f(x) = x^2 - 6x + 5$, $x \leq 3$
22. $f(x) = 4x^2 + 4x + 1$, $x < -1$
23. $g(t) = \dfrac{3}{4-t}$
24. $g(t) = \dfrac{t}{1-3t}$
25. $f(x) = \dfrac{2x-1}{3x+4}$
26. $f(x) = \dfrac{4x+2}{3x-6}$
27. $g(t) = \dfrac{-3t-2}{t+3}$
28. $g(t) = \dfrac{t-2}{2t-1}$
29. Explain why each set of ordered pairs below represents a one-to-one function and find the inverse.
    (a) $F = \{(0,0), (1,1), (2,-1), (3,2), (4,-2), (5,3), (6,-3)\}$
    (b) $G = \{(0,0), (1,1), (2,-1), (3,2), (4,-2), (5,3), (6,-3), \ldots\}$
    NOTE: The difference between F and G is the '....'
    (c) $P = \{(2t^5, 3t-1) \mid t \text{ is a real number.}\}$
    (d) $Q = \{(n, n^2) \mid n \text{ is a } \textit{natural} \text{ number.}\}$[9]

[9] Recall this means $n = 0, 1, 2, \ldots$.

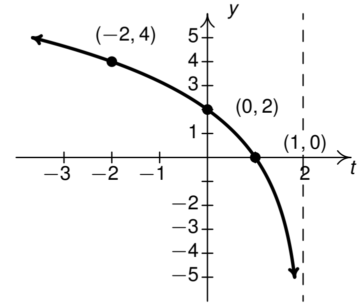
In Exercises 30 - 33, explain why each graph represents[10] a one-to-one function and graph its inverse.

30. $y = f(x)$



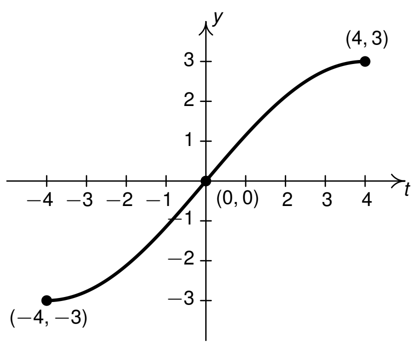
Asymptote: $y = 0$.

31. $y = g(t)$



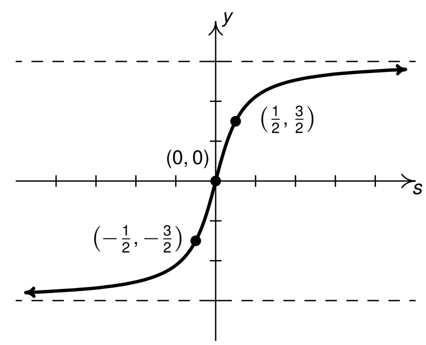
Asymptote: $t = 2$.

32. $y = S(t)$



Domain: $[-4, 4]$.

33. $y = R(s)$



Asymptotes: $y = \pm 3$.

34. The price of a dOpi media player, in dollars per dOpi, is given as a function of the weekly sales x according to the formula $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.

 (a) Find $p^{-1}(x)$ and state its domain.
 (b) Find and interpret $p^{-1}(105)$.
 (c) The profit (in dollars) made from producing and selling x dOpis per week is given by the formula $P(x) = -15x^2 + 350x - 2000$, for $0 \leq x \leq 30$. Find $\left(P \circ p^{-1}\right)(x)$ and determine what price per dOpi would yield the maximum profit. What is the maximum profit? How many dOpis need to be produced and sold to achieve the maximum profit?

35. Show that the Fahrenheit to Celsius conversion function found in Exercise **??** in Section **??** is invertible and that its inverse is the Celsius to Fahrenheit conversion function.

36. Analytically show that the function $f(x) = x^3 + 3x + 1$ is one-to-one. Use Theorem 1.13 to help you compute $f^{-1}(1)$, $f^{-1}(5)$, and $f^{-1}(-3)$. What happens when you attempt to find a formula for $f^{-1}(x)$?

[10] or, more precisely, *appears* to represent . . .

37. Let $f(x) = \dfrac{2x}{x^2 - 1}$.

    (a) Graph $y = f(x)$ using the techniques in Section **??**. Check your answer using a graphing utility.

    (b) Verify that f is one-to-one on the interval $(-1, 1)$.

    (c) Use the procedure outlined on Page 142 to find the formula for $f^{-1}(x)$ for $-1 < x < 1$.

    (d) Since $f(0) = 0$, it should be the case that $f^{-1}(0) = 0$. What goes wrong when you attempt to substitute $x = 0$ into $f^{-1}(x)$? Discuss with your classmates how this problem arose and possible remedies.

38. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.

39. If f is odd and invertible, prove that f^{-1} is also odd.

40. Let f and g be invertible functions. With the help of your classmates show that $(f \circ g)$ is one-to-one, hence invertible, and that $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

With help from your classmates, find the inverses of the functions in Exercises 41 - 44.

41. $f(x) = ax + b, \; a \neq 0$

42. $f(x) = a\sqrt{x - h} + k, \; a \neq 0, x \geq h$

43. $f(x) = ax^2 + bx + c$ where $a \neq 0, \; x \geq -\dfrac{b}{2a}$.

44. $f(x) = \dfrac{ax + b}{cx + d}$, (See Exercise 45 below.)

45. What conditions must you place on the values of a, b, c and d in Exercise 44 in order to guarantee that the function is invertible?

46. The function given in number 4 is an example of a function which is its own inverse.

    (a) Algebraically verify every function of the form: $f(x) = \dfrac{ax + b}{cx - a}$ is its own inverse. What assumptions do you need to make about the values of a, b, and c?

    (b) Under what conditions is $f(x) = mx + b$, $m \neq 0$ its own inverse? Prove your answer.

1.6.2 Answers

9. $f^{-1}(x) = \dfrac{x+2}{6}$

10. $f^{-1}(x) = 42 - x$

11. $g^{-1}(t) = 3t - 10$

12. $g^{-1}(t) = -\frac{5}{3}t + \frac{1}{3}$

13. $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3}$, $x \geq 5$

14. $f^{-1}(x) = (x-2)^2 + 5$, $x \leq 2$

15. $g^{-1}(t) = \frac{1}{9}(t+4)^2 + 1$, $t \geq -4$

16. $g^{-1}(t) = \frac{1}{8}(t-1)^2 - \frac{5}{2}$, $t \leq 1$

17. $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$

18. $f^{-1}(x) = -(x-3)^3 + 2$

19. $g^{-1}(t) = 5 + \sqrt{t+25}$

20. $g^{-1}(t) = -\sqrt{\frac{t+5}{3}} - 4$

21. $f^{-1}(x) = 3 - \sqrt{x+4}$

22. $f^{-1}(x) = -\frac{\sqrt{x}+1}{2}$, $x > 1$

23. $g^{-1}(t) = \dfrac{4t-3}{t}$

24. $g^{-1}(t) = \dfrac{t}{3t+1}$

25. $f^{-1}(x) = \dfrac{4x+1}{2-3x}$

26. $f^{-1}(x) = \dfrac{6x+2}{3x-4}$

27. $g^{-1}(t) = \dfrac{-3t-2}{t+3}$

28. $g^{-1}(t) = \dfrac{t-2}{2t-1}$

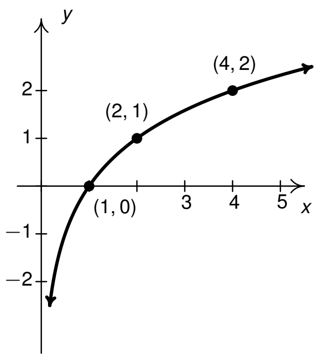
29. (a) None of the first coordinates of the ordered pairs in F are repeated, so F is a function and none of the second coordinates of the ordered pairs of F are repeated, so F is one-to-one. $F^{-1} = \{(0,0),(1,1),(-1,2),(2,3),(-2,4),(3,5),(-3,6)\}$

(b) Because of the '...' it is helpful to determine a formula for the matching. For the even numbers n, $n = 0, 2, 4, ...$, the ordered pair $\left(n, -\frac{n}{2}\right)$ is in G. For the odd numbers $n = 1, 3, 5, ...$, the ordered pair $\left(n, \frac{n+1}{2}\right)$ is in G. Hence, given any input to G, n, whether it be even or odd, there is only one output from G, either $-\frac{n}{2}$ or $\frac{n+1}{2}$, both of which are functions of n. To show G is one to one, we note that if the output from G is 0 or less, then it must be of the form $-\frac{n}{2}$ for an even number n. Moreover, if $-\frac{n}{2} = -\frac{m}{2}$, then $n = m$. In the case we are looking at outputs from G which are greater than 0, then it must be of the form $\frac{n+1}{2}$ for an odd number n. In this, too, if $\frac{n+1}{2} = \frac{m+1}{2}$, then $n = m$. Hence, in any case, if the outputs from G are the same, then the inputs to G had to be the same so G is one-to-one and $G^{-1} = \{(0,0),(1,1),(-1,2),(2,3),(-2,4),(3,5),(-3,6),...\}$

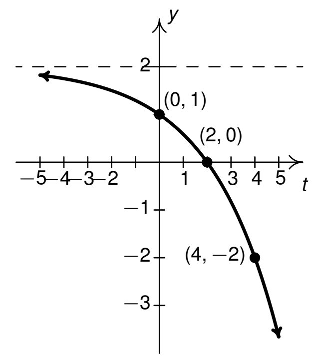
(c) To show P is a function we note that if we have the same inputs to P, say $2t^5 = 2u^5$, then $t = u$. Hence the corresponding outputs, $2t-1$ and $3u-1$, are equal, too. To show P is one-to-one, we note that if we have the same outputs from P, $3t-1 = 3u-1$, then $t = u$. Hence, the corresponding inputs $2t^5$ and $2u^5$ are equal, too. Hence P is one-to-one and $P^{-1} = \{(3t-1, 2t^5) \mid t \text{ is a real number.}\}$

(d) To show Q is a function, we note that if we have the same inputs to Q, say $n = m$, then the outputs from Q, namely n^2 and m^2 are equal. To show Q is one-to-one, we note that if we get the same output from Q, namely $n^2 = m^2$, then $n = \pm m$. However since n and m are *natural* numbers, both n and m are positive so $n = m$. Hence Q is one-to-one and $Q^{-1} = \{(n^2, n) \mid n \text{ is a } \textit{natural} \text{ number.}\}$.

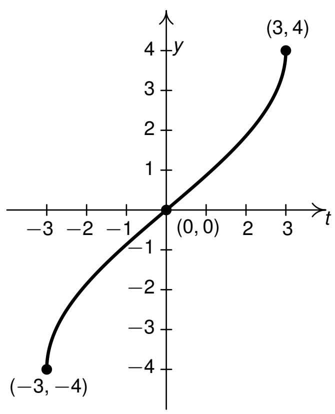
30. $y = f^{-1}(x)$. Asymptote: $x = 0$.



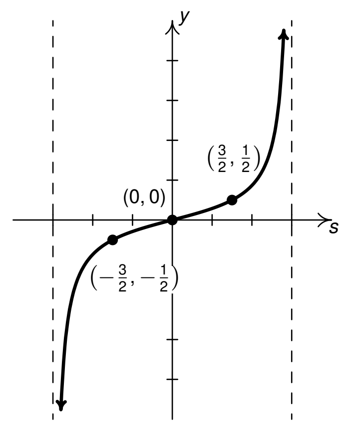
31. $y = g^{-1}(t)$. Asymptote: $y = 2$.



32. $y = S^{-1}(t)$. Domain $[-3, 3]$.



33. $y = R^{-1}(s)$. Asymptotes: $s = \pm 3$.



34. (a) $p^{-1}(x) = \frac{450-x}{15}$. The domain of p^{-1} is the range of p which is $[0, 450]$

(b) $p^{-1}(105) = 23$. This means that if the price is set to \$105 then 23 dOpis will be sold.

(c) $\left(P \circ p^{-1}\right)(x) = -\frac{1}{15}x^2 + \frac{110}{3}x - 5000$, $0 \leq x \leq 450$.

The graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards with vertex $\left(275, \frac{125}{3}\right) \approx (275, 41.67)$. This means that the maximum profit is a whopping \$41.67 when the price per dOpi is set to \$275. At this price, we can produce and sell $p^{-1}(275) = 11.\overline{6}$ dOpis. Since we cannot sell part of a system, we need to adjust the price to sell either 11 dOpis or 12 dOpis. We find $p(11) = 285$ and $p(12) = 270$, which means we set the price per dOpi at either \$285 or \$270, respectively. The profits at these prices are $\left(P \circ p^{-1}\right)(285) = 35$ and $\left(P \circ p^{-1}\right)(270) = 40$, so it looks as if the maximum profit is \$40 and it is made by producing and selling 12 dOpis a week at a price of \$270 per dOpi.

36. Given that $f(0) = 1$, we have $f^{-1}(1) = 0$. Similarly $f^{-1}(5) = 1$ and $f^{-1}(-3) = -1$

46. (b) If $b = 0$, then $m = \pm 1$. If $b \neq 0$, then $m = -1$ and b can be any real number.