PreCalculus

Edition 4*i*

based on the original work by

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Chapter 1

Introduction to Functions

1.1 Functions and their Representations

1.1.1 Functions as Mappings

Mathematics can be thought of as the study of patterns. In most disciplines, Mathematics is used as a language to express, or codify, relationships between quantities - both algebraically and geometrically - with the ultimate goal of solving real-world problems. The fact that the same algebraic equation which models the growth of bacteria in a petri dish is also used to compute the account balance of a savings account or the potency of radioactive material used in medical treatments speaks to the universal nature of Mathematics. Indeed, Mathematics is more than just about solving a specific problem in a specific situation, it's about abstracting problems and creating universal tools which can be used by a variety of scientists and engineers to solve a variety of problems.

This power of abstraction has a tendency to create a language that is initially intimidating to students. Mathematical definitions are precise and adherence to that precision is often a source of confusion and frustration. It doesn't help matters that more often than not very common words are used in Mathematics with slightly different definitions than is commonly expected. The first 'universal tool' we wish to highlight - the concept of a 'function' - is a perfect example of this phenomenon in that we redefine a word that already has multiple meanings in English.

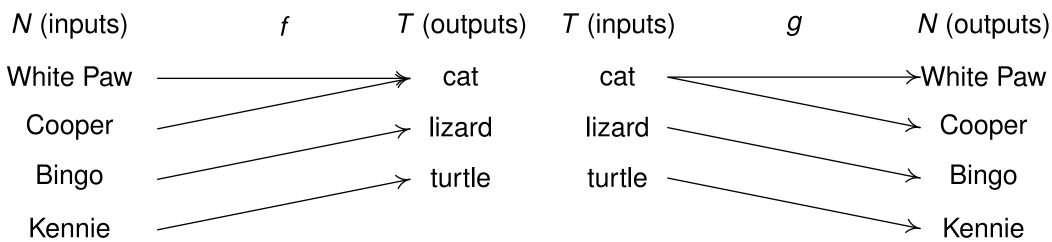
Definition 1.1. Given two sets[a] A and B, a **function** from A to B is a process by which each element of A is matched with (or 'mapped to') one and only one element of B.

[a] Please refer to Section A.1 for a review of this terminology.

The grammar here '**from** A **to** B' is important. Thinking of a function as a process, we can view the elements of the set A as our starting materials, or **inputs** to the process. The function processes these inputs according to some specified rule and the result is a set of **outputs** - elements of the set B. In terms of inputs and outputs, Definition 1.1 says that a function is a process in which each **input** is matched to one and only one **output**.

For example, let's take a look at some of the pets in the Stitz household. Taylor's pets include White Paw and Cooper (both cats), Bingo (a lizard) and Kennie (a turtle). Let N be the set of pet names: $N = \{$White Paw, Cooper, Bingo, Kennie$\}$, and let T be the set of pet types: $T = \{$cat, lizard, turtle$\}$. Let f be the process that takes each pet's name as the input and returns that pet's type as the output. Let g be the reverse of f: that is, g takes each pet type as the input and returns the names of the pets of that type as the output. Note that both f and g are codifying the **same** given information about Taylor's pets, but one of them is a function and the other is not.

To help identify which process f or g is a function and why the other is not, we create **mapping diagrams** for f and g below. In each case, we organize the inputs in a column on the left and the outputs on a column on the right. We draw an arrow connecting each input to its corresponding output(s). Note that the arrows communicate the grammatical bias: the arrow originates at the input and points to the output.



The process f is a function since f matches each of its inputs (each pet name) to just one output (the pet's type). The fact that different inputs (White Paw and Cooper) are matched to the same output (cat) is fine. On the other hand, g matches the input 'cat' to the two different outputs 'White Paw' and 'Cooper', so g is not a function. Functions are favored in mathematical circles because they are processes which produce only one answer (output) for any given query (input). In this scenario, for instance, there is only one answer to the question: 'What type of pet is White Paw?' but there is more than one answer to the question 'Which of Taylor's pets are cats?'

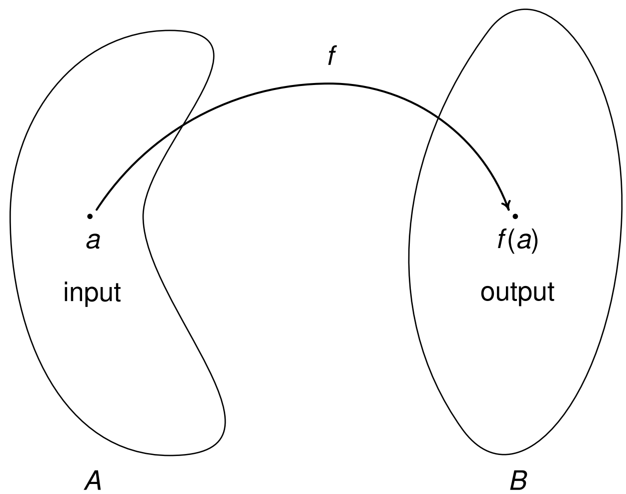
As you might expect, with functions being such an important concept in Mathematics, we need to build a vocabulary to assist us when discussing them. To that end, we have the following definitions.[1]

Definition 1.2. Suppose f is a function from A to B.

- If $a \in A$, we write $f(a)$ (read 'f of a') to denote the unique element of B to which f matches a.
  That is, if we view 'a' as the input to f, then '$f(a)$' is the output from f.
- The set A is called the **domain**.
  Said differently, the domain of a function is the set of inputs to the function.
- The set $\{f(a) \mid a \in A\}$ is called the **range** of f.
  Said differently, the range of a function is the set of outputs from the function.

[1] Please refer to Section A.1 for a review of the terminology used in these definitions.

Some remarks about Definition 1.2 are in order. First, and most importantly, the notation '$f(a)$' in Definition 1.2 introduces yet another mathematical use for parentheses. Parentheses are used in some cases as grouping symbols, to represent ordered pairs, and to delineate intervals of real numbers. More often than not, the use of parentheses in expressions like '$f(a)$' is confused with multiplication. As always, paying attention to the context is key. If f is a function and 'a' is in the domain of f, then '$f(a)$' is the output from f when you input a. The diagram below provides a nice generic picture to keep in mind when thinking of a function as a mapping process with input 'a' and output '$f(a)$'.



In the preceding pet example, the symbol f(Bingo), read 'f of Bingo', is asking what type of pet Bingo is, so f(Bingo) = lizard. The fact that f is a function means f(Bingo) is unambiguous because f matches the name 'Bingo' to only one pet type, namely 'lizard'. In contrast, if we tried to use the notation 'g(cat)' to indicate what pet name g matched to 'cat', we have **two** possibilities, White Paw and Cooper, with no way to determine which one (or both) is indicated.

Continuing to apply Definition 1.2 to our pet example, we find that the domain of the function f is N, the set of pet names. Finding the range takes a little more work, mostly because it's easy to be caught off guard by the notation used in the definition of 'range'. The description of the range as '$\{f(a) \mid a \in A\}$' is an example of 'set-builder' notation. In English, '$\{f(a) \mid a \in A\}$' reads as 'the set of $f(a)$ such that a is in A'. In other words, the range consists of all of the outputs from f - all of the $f(a)$ values - as a varies through each of the elements in the domain A. Note that while every element of the set A is, by definition, an element of the domain of f, not every element of the set B is necessarily part of the range of f.[2]

In our pet example, we can obtain the range of f by looking at the mapping diagram or by constructing the set $\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\}$ which lists all of the outputs from f as we run through all of the inputs to f. Keep in mind that we list each element of a set only once so the range of f is:[3]

$$\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\} = \{\text{cat}, \text{lizard}, \text{turtle}\} = T.$$

[2]For purposes of completeness, the set B is called the **codomain** of f. For us, the concepts of domain and range suffice since our codomain will most always be the set of real numbers, $\mathbb{R}$.

[3]If instead of mapping N into T, we could have mapped N into $U = \{\text{cat}, \text{lizard}, \text{turtle}, \text{dog}\}$ in which case the range of f would not have been the entire codomain U.

If we let n denote a generic element of N then $f(n)$ is some element t in T, so we write $t = f(n)$. In this equation, n is called the **independent variable** and t is called the **dependent variable**.[4] Moreover, we say 't is a function of n', or, more specifically, 'the type of pet is a function of the pet name' meaning that every pet name n corresponds to one, and only one, pet type t. Even though f and t are different things,[5] it is very common for the function and its outputs to become more-or-less synonymous, even in what are otherwise precise mathematical definitions.[6] We will endeavor to point out such ambiguities as we move through the text.

While the concept of a function is very general in scope, we will be focusing primarily on functions of real numbers because most disciplines use real numbers to quantify data. Our next example explores a function defined using a table of numerical values.

Example 1.1.1. Suppose Skippy records the outdoor temperature every two hours starting at 6 a.m. and ending at 6 p.m. and summarizes the data in the table below:

time (hours after 6 a.m.)	outdoor temperature in degrees Fahrenheit
0	64
2	67
4	75
6	80
8	83
10	83
12	82

1. Explain why the recorded outdoor temperature is a function of the corresponding time.
2. Is time a function of the outdoor temperature? Explain.
3. Let f be the function which matches time to the corresponding recorded outdoor temperature.
   (a) Find and interpret the following:
   - $f(2)$
   - $f(4)$
   - $f(2+4)$
   - $f(2)+f(4)$
   - $f(2)+4$

   (b) Solve and interpret $f(t) = 83$.
   (c) State the range of f. What is lowest recorded temperature of the day? The highest?

[4] These adjectives stem from the fact that the value of t **depends** entirely on our (independent) choice of n.
[5] Specifically, f is a function so it requires and domain, a range and a rule of assignment whereas t is simply the output from f.
[6] In fact, it is not uncommon to see the name of the function as the same as the dependent variable. For example, writing '$y = y(x)$' would be a way to communicate the idea that 'y is a function of x'.

Solution.

1. The outdoor temperature is a function of time because each time value is associated with only one recorded temperature.

2. Time is not a function of the outdoor temperature because there are instances when different times are associated with a given temperature. For example, the temperature 83 corresponds to both of the times 8 and 10.

3. (a)
   - To find $f(2)$, we look in the table to find the recorded outdoor temperature that corresponds to when the time is 2. We get $f(2) = 67$ which means that 2 hours after 6 a.m. (i.e., at 8 a.m.), the temperature is 67°F.
   - Per the table, $f(4) = 75$, so the recorded outdoor temperature at 10 a.m. (4 hours after 6 a.m.) is 75°F.
   - From the table, we find $f(2 + 4) = f(6) = 80$, which means that at noon (6 hours after 6 a.m.), the recorded outdoor temperature is 80°F.
   - Using results from above we see that $f(2) + f(4) = 67 + 75 = 142$. When adding $f(2) + f(4)$, we are adding the recorded outdoor temperatures at 8 a.m. (2 hours after 6 a.m.) and 10 a.m. (4 hours after 6 AM), respectively, to get 142°F.
   - We compute $f(2) + 4 = 67 + 4 = 71$. Here, we are adding 4°F to the outdoor temperature recorded at 8 a.m..

   (b) Solving $f(t) = 83$ means finding all of the input (time) values t which produce an output value of 83. From the data, we see that the temperature is 83 when the time is 8 or 10, so the solution to $f(t) = 83$ is $t = 8$ or $t = 10$. This means the outdoor temperature is 83°F at 2 p.m. (8 hours after 6 a.m.) and at 4 p.m. (10 hours after 6 a.m.).

   (c) The range of f is the set of all of the outputs from f, or in this case, the outside recorded temperatures. Based on the data, we get $\{64, 67, 75, 80, 82, 83\}$. (Here again, we list elements of a set only once.) The lowest recorded temperature of the day is 64°F and the highest recorded temperature of the day is 83°F. □

A few remarks about Example 1.1.1 are in order. First, note that $f(2 + 4)$, $f(2) + f(4)$ and $f(2) + 4$ all work out to be numerically different, and more importantly, all represent different things.[7] One of the common mistakes students make is to misinterpret expressions like these, so it's important to pay close attention to the syntax here.

Next, when solving $f(t) = 83$, the variable 't' is being used as a convenient 'dummy' variable or placeholder in the sense that solving $f(t) = 83$ produces the same solutions as solving $f(x) = 83$, $f(w) = 83$, or even $f(?) = 83$. All of these equations are asking for the same thing: what inputs to f produce an output of 83. The choice of the letter 't' here makes sense since the inputs are **t**ime values. Throughout the text, we will endeavor to use meaningful labels when working in applied situations, but the fact remains that the choice of letters (or symbols) is completely arbitrary.

[7] You may be wondering why one would ever compute these quantities. Rest assured that we will use expressions like these in examples throughout the text. For now, it suffices just to know that they are different.

Finally, given that the range in this example was a finite set of real numbers, we could find the smallest and largest elements of it. Here, they correspond to the coolest and warmest temperatures of the day, respectively, but the meaning would change if the function related different quantities. In many applications involving functions, the end goal is to find the minimum or maximum values of the outputs of those functions (called **optimizing** the function) so for that reason, we have the following definition.

Definition 1.3. Suppose f is a function whose range is a set of real numbers containing m and M.

- The value m is called the **minimum**[a] of f if $m \leq f(x)$ for all x in the domain of f.

  That is, the minimum of f is the smallest output from f, if it exists.

- The value M is called the **maximum**[b] of f if $f(x) \leq M$ for all x in the domain of f.

  That is, the maximum of f is the largest output from f, if it exists.

- Taken together, the values m and M (if they exist) are called the **extrema**[c] of f.

[a] also called ‘absolute’ or ‘global’ minimum
[b] also called ‘absolute’ or ‘global’ maximum
[c] also called the ‘absolute’ or ‘global’ extrema or the ‘extreme values’

Definition 1.3 is an example where the name of the function, f, is being used almost synonymously with its outputs in that when we speak of ‘the minimum and maximum of the **function** f’ we are really talking about the minimum and maximum values of the **outputs** $f(x)$ as x varies through the domain of f. Thus we say that the maximum of f is 83 and the minimum of f is 64 when referring to the highest and lowest recorded temperatures in the previous example.

1.1.2 Algebraic Representations of Functions

By focusing our attention to functions that involve real numbers, we gain access to all of the structures and tools from prior courses in Algebra. In this subsection, we discuss how to represent functions algebraically using formulas and begin with the following example.

Example 1.1.2.

1. Let f be the function which takes a real number and performs the following sequence of operations:

   - Step 1: add 2
   - Step 2: multiply the result of Step 1 by 3
   - Step 3: subtract 1 from the result of Step 2.

   (a) Find and simplify $f(-5)$.

   (b) Find and simplify a formula for $f(x)$.

2. Let $h(t) = -t^2 + 3t + 4$.

   (a) Find and simplify the following:

      i. $h(-1)$, $h(0)$ and $h(2)$.

      ii. $h(2x)$ and $2h(x)$.

      iii. $h(t+2)$, $h(t) + 2$ and $h(t) + h(2)$.

   (b) Solve $h(t) = 0$.

Solution.

1. (a) We take -5 and follow it through each step:

   - Step 1: adding 2 gives us $-5 + 2 = -3$.
   - Step 2: multiplying the result of Step 1 by 3 yields $(-3)(3) = -9$.
   - Step 3: subtracting 1 from the result of Step 2 produces $-9 - 1 = -10$.

   Hence, $f(-5) = -10$.

   (b) To find a formula for $f(x)$, we repeat the above process but use the variable 'x' in place of the number -5:

   - Step 1: adding 2 gives us the quantity $x + 2$.
   - Step 2: multiplying the result of Step 1 by 3 yields $(x + 2)(3) = 3x + 6$.
   - Step 3: subtracting 1 from the result of Step 2 produces $(3x + 6) - 1 = 3x + 5$.

   Hence, we have codified f using the formula $f(x) = 3x + 5$. In other words, the function f matches each real number 'x' with the value of the expression '$3x + 5$'. As a partial check of our answer, we use this formula to find $f(-5)$. We compute $f(-5)$ by substituting $x = -5$ into the formula $f(x)$ and find $f(-5) = 3(-5) + 5 = -10$ as before.

2. As before, representing the function h as $h(t) = -t^2 + 3t + 4$ means that h matches the real number t with the value of the expression $-t^2 + 3t + 4$.

   (a) To find $h(-1)$, we substitute -1 for t in the expression $-t^2 + 3t + 4$. It is highly recommended that you be generous with parentheses here in order to avoid common mistakes:

$$\begin{aligned} h(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0. \end{aligned}$$

   Similarly, $h(0) = -(0)^2 + 3(0) + 4 = 4$, and $h(2) = -(2)^2 + 3(2) + 4 = -4 + 6 + 4 = 6$.

   (b) To find $h(2x)$, we substitute $2x$ for t:

$$\begin{aligned} h(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4. \end{aligned}$$

The expression $2h(x)$ means that we multiply the expression $h(x)$ by 2. We first get $h(x)$ by substituting x for t: $h(x) = -x^2 + 3x + 4$. Hence,

$$\begin{aligned} 2h(x) &= 2\left(-x^2 + 3x + 4\right) \\ &= -2x^2 + 6x + 8. \end{aligned}$$

(c) To find $h(t+2)$, we substitute the quantity $t+2$ in place of t:

$$\begin{aligned} h(t+2) &= -(t+2)^2 + 3(t+2) + 4 \\ &= -\left(t^2 + 4t + 4\right) + (3t+6) + 4 \\ &= -t^2 - 4t - 4 + 3t + 6 + 4 \\ &= -t^2 - t + 6. \end{aligned}$$

To find $h(t) + 2$, we add 2 to the expression for $h(t)$

$$\begin{aligned} h(t) + 2 &= \left(-t^2 + 3t + 4\right) + 2 \\ &= -t^2 + 3t + 6. \end{aligned}$$

From our work above, we see that $h(2) = 6$ so

$$\begin{aligned} h(t) + h(2) &= \left(-t^2 + 3t + 4\right) + 6 \\ &= -t^2 + 3t + 10. \end{aligned}$$

3. We know $h(-1) = 0$ from above, so $t = -1$ should be one of the answers to $h(t) = 0$. In order to see if there are any more, we set $h(t) = -t^2 + 3t + 4 = 0$. Factoring[8] gives $-(t+1)(t-4) = 0$, so we get $t = -1$ (as expected) along with $t = 4$. □

A few remarks about Example 1.1.2 are in order. First, note that $h(2x)$ and $2h(x)$ are different expressions. In the former, we are multiplying the **input** by 2; in the latter, we are multiplying the **output** by 2. The same goes for $h(t+2)$, $h(t)+2$ and $h(t)+h(2)$. The expression $h(t+2)$ calls for adding 2 to the input t and then performing the function h. The expression $h(t)+2$ has us performing the process h first, then adding 2 to the output $h(t)$. Finally, $h(t)+h(2)$ directs us to first find the outputs $h(t)$ and $h(2)$ and then add the results. As we saw in Example 1.1.1, we see here again the importance paying close attention to syntax.[9]

Let us return for a moment to the function f in Example 1.1.2 which we ultimately represented using the formula $f(x) = 3x + 5$. If we introduce the dependent variable y, we get the equation $y = f(x) = 3x + 5$, or, more simply $y = 3x + 5$. To say that the equation $y = 3x + 5$ describes y as a function of x means that for each choice of x, the formula $3x + 5$ determines only one associated y-value.

We could turn the tables and ask if the equation $y = 3x + 5$ describes x as a function of y. That is, for each value we pick for y, does the equation $y = 3x + 5$ produce only one associated x value? One way to proceed is to solve $y = 3x + 5$ for x and get $x = \frac{1}{3}(y-5)$. We see that for each choice of y, the expression $\frac{1}{3}(y-5)$ evaluates to just one number, hence, x is a function of y. If we give this function a name, say g, we have $x = g(y) = \frac{1}{3}(y-5)$, where in this equation, y is the independent variable and x is the dependent variable. We explore this idea in the next example.

[8] You may need to review Section A.9.

[9] As was mentioned before, we will give meanings to the these quantities in other examples throughout the text.

Example 1.1.3.

1. Consider the equation $x^3 + y^2 = 25$.

   (a) Does this equation represent y as a function of x? Explain.

   (b) Does this equation represent x as a function of y? Explain.

2. Consider the equation $u^4 + t^3 u = 16$.

   (a) Does this equation represent t as a function of u? Explain.

   (b) Does this equation represent u as a function of t? Explain.

Solution.

1. (a) To say that $x^3 + y^2 = 25$ represents y as a function of x, we need to show that for each x we choose, the equation produces only one associated y-value. To help with this analysis, we solve the equation for y in terms of x.

$$\begin{array}{rcll} x^3 + y^2 & = & 25 & \\ y^2 & = & 25 - x^3 & \\ y & = & \pm\sqrt{25 - x^3} & \text{extract square roots. (See Section A.13 for a review, if needed.)} \end{array}$$

   The presence of the '$\pm$' indicates that there is a good chance that for some x-value, the equation will produce **two** corresponding y-values. Indeed, $x = 0$ produces $y = \pm\sqrt{25 - 0^3} = \pm 5$. Hence, $x^3 + y^2 = 25$ equation does **not** represent y as a function of x because $x = 0$ is matched with more than one y-value.

   (b) To see if $x^3 + y^2 = 25$ represents x as a function of y, we solve the equation for x in terms of y:

$$\begin{array}{rcll} x^3 + y^2 & = & 25 & \\ x^3 & = & 25 - y^2 & \\ x & = & \sqrt[3]{25 - y^2} & \text{extract cube roots. (See Section A.13 for a review, if needed.)} \end{array}$$

   In this case, each choice of y produces only **one** corresponding value for x, so $x^3 + y^2 = 25$ represents x as a function of y.

2. (a) To see if $u^4 + t^3 u = 16$ represents t as a function of u, we proceed as above and solve for t in terms of u:

$$\begin{array}{rcll} u^4 + t^3 u & = & 16 & \\ t^3 u & = & 16 - u^4 & \\ t^3 & = & \dfrac{16 - u^4}{u} & \text{assumes } u \neq 0 \\ t & = & \sqrt[3]{\dfrac{16 - u^4}{u}} & \text{extract cube roots.} \end{array}$$

Although it's a bit cumbersome, as long as $u \neq 0$ the expression $\sqrt[3]{\frac{16-u^4}{u}}$ will produce just one value of t for each value of u. What if $u = 0$? In that case, the equation $u^4 + t^3 u = 16$ reduces to $0 = 16$ - which is never true - so we don't need to worry about that case.[10] Hence, $u^4 + t^3 u = 16$ represents t as a function of u.

(b) In order to determine if $u^4 + t^3 u = 16$ represents u as a function of t, we could attempt to solve $u^4 + t^3 u = 16$ for u in terms of t, but we won't get very far.[11] Instead, we take a different approach and experiment with looking for solutions for u for specific values of t. If we let $t = 0$, we get $u^4 = 16$ which gives $u = \pm\sqrt[4]{16} = \pm 2$. Hence, $t = 0$ corresponds to more than one u-value which means $u^4 + t^3 u = 16$ does not represent u as a function of t. $\square$

We'll have more to say about using equations to describe functions in Section 5.5. For now, we turn our attention to a geometric way to represent functions.

1.1.3 Geometric Representations of Functions

In this section, we introduce how to graph functions. As we'll see in this and later sections, visualizing functions geometrically can assist us in both analyzing them and using them to solve associated application problems. Our playground, if you will, for the Geometry in this course is the Cartesian Coordinate Plane. The reader would do well to review Section A.3 as needed.

Our path to the Cartesian Plane requires ordered pairs. In general, we can represent every function as a set of ordered pairs. Indeed, given a function f with domain A, we can represent $f = \{(a, f(a)) \mid a \in A\}$. That is, we represent f as a set of ordered pairs $(a, f(a))$, or, more generally, (input, output). For example, the function f which matches Taylor's pet's names to their associated pet type can be represented as:

$$f = \{(\text{White Paw}, \text{cat}), (\text{Cooper}, \text{cat}), (\text{Bingo}, \text{lizard}), (\text{Kennie}, \text{turtle})\}$$

Moving on, we next consider the function f from Example 1.1.1 which relates time to temperature. In this case, $f = \{(0, 64), (2, 67), (4, 75), (6, 80), (8, 83), (10, 83), (12, 82)\}$. This function has numerical values for both the domain and range so we can identify these ordered pairs with points in the Cartesian Plane. The first coordinates of these points (the abscissae) represent time values so we'll use t to label the horizontal axis. Likewise, we'll use T to label the vertical axis since the second coordinates of these points (the ordinates) represent temperature values. Note that labeling these axes in this way determines our independent and dependent variable names, t and T, respectively.

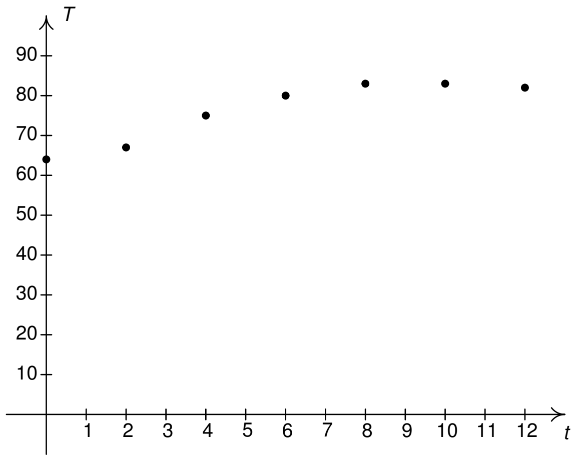
The plot of these points is called 'the **graph** of f'. More specifically, we could describe this plot as 'the graph of $f(t)$', because we have decided to name the independent variable t. Most specifically, we could describe the plot as 'the graph of $T = f(t)$', given that we have named the independent variable t and the dependent variable T.

On the next page we present two plots, both of which are graphs of the function f. In both cases, the vertical axis has been scaled in order to save space. In the graph on the left, the same increment on

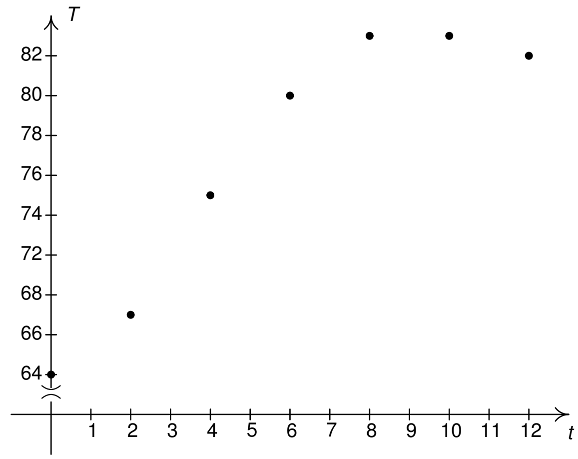
[10] Said differently, $u = 0$ is not in the domain of the function represented by the equation $u^4 + t^3 u = 16$.

[11] Try it for yourself!

the horizontal axis to measure 1 unit measures 10 units on the vertical axis whereas in the graph on the right, this ratio is 1 : 2. The '≍' symbol on the vertical axis in the graph on the right is used to indicate a jump in the vertical labeling. Both are perfectly accurate data plots, but they have different visual impacts. Note here that the extrema of f, 64 and 83, correspond to the lowest and highest points on the graph, respectively: $(0, 64)$, $(8, 83)$ and $(10, 83)$. More often than not, we will use the graph of a function to help us optimize that function.[12]



The graph of $T = f(t)$.



The graph of $T = f(t)$.

If you found yourself wanting to connect the dots in the graphs above, you're not alone. As it stands, however, the function f matches only seven inputs to seven outputs, so those seven points - and just those seven points - comprise the graph of f. That being said, common everyday experience tells us that while the data Skippy collected in his table gives some good information about the relationship between time and temperature on a given day, it is by no means a complete description of the relationship.

For example, Skippy's data cannot tell us what the temperature was at 7 a.m. or 12:13 p.m, although we are pretty sure there were outdoor temperatures at those times. Also, given that at some point it was 64°F and later on it was 83°F, it seems reasonable to assume that at some point it was 70°F or even 79.923°F.

Skippy's temperature function f is an example of a **discrete** function in the sense that each of the data points are 'isolated' with measurable gaps in between. The idea of 'filling in' those gaps is a quest to find a **continuous** function to model this same phenomenon.[13] We'll return to this example in Sections 1.2 and 1.4 in an attempt to do just that.

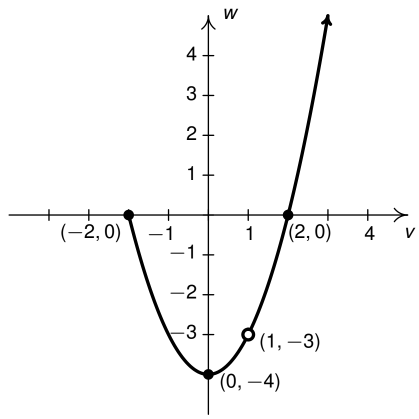
In the meantime, our next example involves a function whose domain is (almost) an **interval** of real numbers and whose graph consists of a (mostly) **connected** arc.

---

[12] One major use of Calculus is to optimize functions analytically - that is, without a graph.

[13] Roughly speaking, a **continuous variable** is a variable which takes on values over an **interval** of real numbers as opposed to values in a discrete list. In this case we would think of time as a 'continuum' - an interval of real numbers as opposed to 7 or so isolated times. A **continuous function** is a function which takes an interval of real numbers and maps it in such a way that its graph is a connected curve with no holes or gaps. This is technically a Calculus idea, but we'll need to discuss the notion of continuity a few times in the text.

Example 1.1.4. Consider the graph below.



1. (a) Explain why this graph suggests that w is a function of v, $w = F(v)$.
   (b) Find $F(0)$ and solve $F(v) = 0$.
   (c) Find the domain and range of F using interval notation.[14] Find the extrema of F, if any exist.

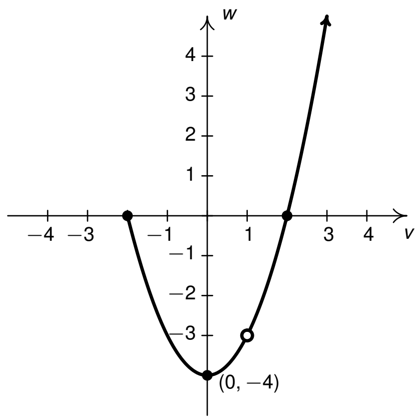
2. Does this graph suggest v is a function of w? Explain.

Solution. The challenge in working with only a graph is that unless points are specifically labeled (as some are in this case), we are forced to approximate values. In addition to the labeled points, there are other interesting features of the graph; a gap or 'hole' labeled $(1, -3)$ and an arrow on the upper right hand part of the curve. We'll have more to say about these two features shortly.

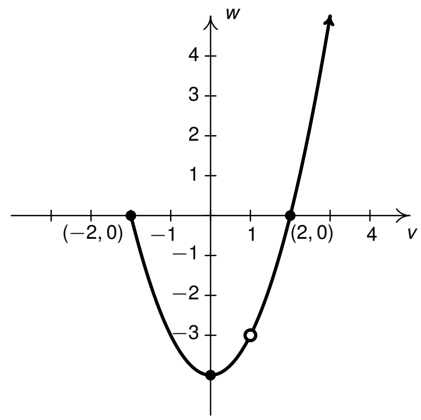
1. (a) In order for w to be a function of v, each v-value on the graph must be paired with only one w-value. What if this weren't the case? We'd have at least two points with the **same** v-coordinate with **different** w-coordinates. Graphically, we'd have two points on graph on the same vertical line, one above the other. This never happens so we may conclude that w is a function of v.
   (b) The value $F(0)$ is the output from F when $v = 0$. The points on the graph of F are of the form $(v, F(v))$ thus we are looking for the w-coordinate of the point on the graph where $v = 0$. Given that the point $(0, -4)$ is labeled on the graph, we can be sure $F(0) = -4$.

   To solve $F(v) = 0$, we are looking for the v-values where the output, or associated w value, is 0. Hence, we are looking for points on the graph with a w-coordinate of 0. We find two such points, $(-2, 0)$ and $(2, 0)$, so our solutions to $F(v) = 0$ are $v = \pm 2$. Pictures highlighting the relevant graphical features are given at the top of the next page.

[14] Please consult Section A.1 for a review of interval notation if need be.

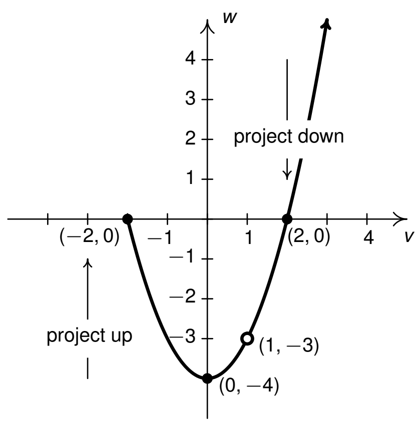


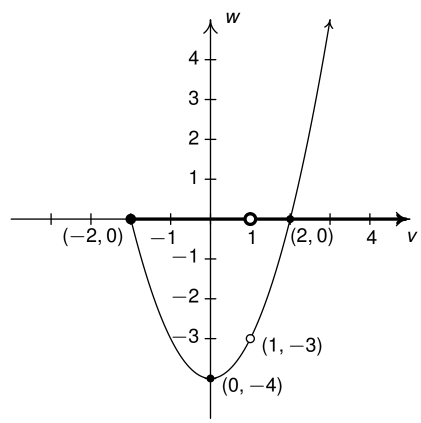
Finding $F(0) = -4$.



Solving $F(v) = 0$.

(c) The domain of F is the set of inputs to F. With v as the input here, we need to describe the set of v-values on the graph. We can accomplish this by **projecting** the graph to the v-axis and seeing what part of the v-axis is covered. The leftmost point on the graph is $(-2, 0)$, so we know that the domain starts at $v = -2$. The graph continues to the right until we encounter the 'hole' labeled at $(1, -3)$. This indicates one and only one point, namely $(1, -3)$ is missing from the curve which for us means $v = 1$ is not in the domain of F. The graph continues to the right and the arrow on the graph indicates that the graph goes upwards to the right indefinitely. Hence, our domain is $\{v \,|\, v \geq -2,\ v \neq 1\}$ which, in interval notation, is $[-2, 1) \cup (1, \infty)$. Pictures demonstrating the process of projecting the graph to the v-axis are shown below.

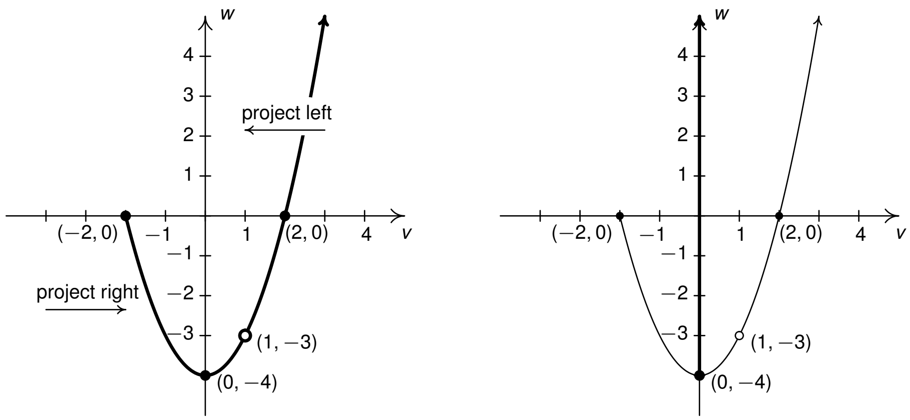




To find the range of F, we need to describe the set of outputs - in this case, the w-values on the graph. Here, we project the graph to the w-axis. Vertically, the graph starts at $(0, -4)$ so our range starts at $w = -4$. Note that even though there is a hole at $(1, -3)$, the w-value -3 is covered by what **appears** to be the point $(-1, -3)$ on the graph.[15] The arrow indicates that the graph extends upwards indefinitely so the range of F is $\{w \,|\, w \geq -4\}$ or, in interval notation, $[-4, \infty)$. Regarding extrema, F has a minimum of -4 when $v = 0$, but given that the graph extends upwards indefinitely, F has no maximum.

[15] For all we know, it could be $(-0.992, -3)$.

Pictures showing the projection of the graph onto the w-axis are given below.



2. Finally, to determine if v is a function of w, we look to see if each w-value is paired with only one v-value on the graph. We have points on the graph, namely $(-2, 0)$ and $(2, 0)$, that clearly show us that $w = 0$ is matched with the **two** v-values $v = 2$ and $v = -2$. Hence, v is not a function of w. $\square$

It cannot be stressed enough that when given a graphical representation of a function, certain assumptions must be made. In the previous example, for all we know, the minimum of the graph is at $(0.001, -4.0001)$ instead of $(0, -4)$. If we aren't given an equation or table of data, or if specific points aren't labeled, we really have no way to tell. We also are assuming that the graph depicted in the example, while ultimately made of infinitely many points, has no gaps or holes other than those noted. This allows us to make such bold claims as the existence of a point on the graph with a w-coordinate of -3.

Before moving on to our next example, it is worth noting that the geometric argument made in Example 1.1.4 to establish that w is a function of v can be generalized to any graph. This result is the celebrated Vertical Line Test and it enables us to detect functions geometrically. Note that the statement of the theorem resorts to the 'default' x and y labels on the horizontal and vertical axes, respectively.

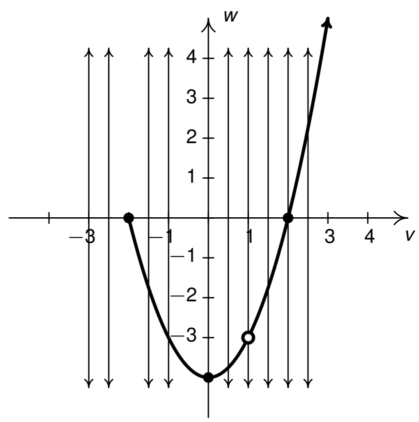
Theorem 1.1. The Vertical Line Test: A graph in the xy-plane[a] represents y as a function of x if and only if no vertical line intersects the graph more than once.

[a]That is, the horizontal axis is labeled with 'x' and the vertical axis is labeled with 'y'.

Let's take a minute to discuss the phrase 'if and only if' used in Theorem 1.1. The statement 'the graph represents y as a function of x **if and only if** no vertical line intersects the graph more than once' is actually saying two things. First, it's saying 'the graph represents y as a function of x **if** no vertical line intersects the graph more than once' and, second, 'the graph represents y as a function of x **only if** no vertical line intersects the graph more than once'.

Logically, these statements are saying two different things. The first says that if no vertical line crosses the graph more than once, then the graph represents y as a function of x. But the question remains: could a graph represent y as a function of x and yet there be a vertical line that intersects the graph more than once? The answer to this is 'no' because the second statement says that the **only** way the graph represents y as a function of x is the case when no vertical line intersects the graph more than once.

Applying the Vertical Line Test to the graph given in Example 1.1.4, we see below that all of the vertical lines meet the graph at most once (several are shown for illustration) showing w is a function of v. Notice that some of the lines ($x = -3$ and $x = 1$, for example) don't hit the graph at all. This is fine because the Vertical Line Test is looking for lines that hit the graph more than once. It does not say **exactly** once so missing the graph altogether is permitted.



There is also a geometric test to determine if the graph above represents v as a function of w. We introduce this aptly-named **Horizontal Line Test** in Exercise 57 and revisit it in Sections 5.5 and 5.6.

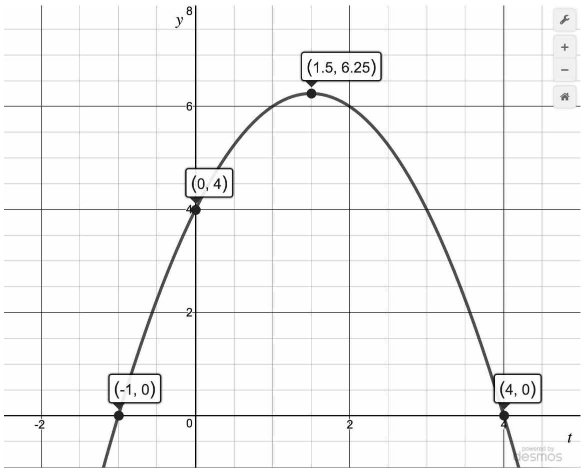
Our next example revisits the function h from Example 1.1.2 from a graphical perspective.

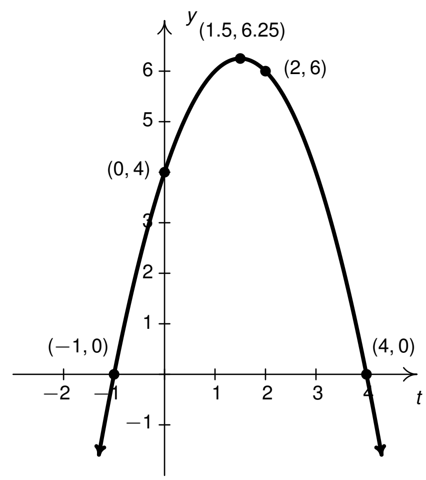
Example 1.1.5. With the help of a graphing utility graph $h(t) = -t^2 + 3t + 4$. From your graph, state the domain, range and extrema, if any exist.

Solution. The dependent variable wasn't specified so we use the default 'y' label for the vertical axis and set about graphing $y = h(t)$. From our work in Example 1.1.2, we already know $h(-1) = 0$, $h(0) = 4$, $h(2) = 6$ and $h(4) = 0$. These give us the points $(-1, 0)$, $(0, 4)$, $(2, 6)$ and $(4, 0)$, respectively. Using these as a guide, we can use desmos to produce the graph at the top of the next page on the left.[16]

As nice as the graph is, it is still technically incomplete. There is no restriction stated on the independent variable t so the domain of h is all real numbers. However, the graph as presented shows only the behavior of h between roughly $t = -2.5$ and $t = 4.25$. By zooming out, we see that the graph extends downwards indefinitely which we indicate by adding the arrows you see in the graph on the right. We find that the domain is $(-\infty, \infty)$ and the range is $(-\infty, 6.25]$. There is no minimum, but the maximum of h is 6.25 and it occurs at $t = 1.5$. The point $(1.5, 6.25)$ is shown on both graphs.

[16]The curve in this example is called a 'parabola'. In Section 1.4, we'll learn how to graph these accurately **by hand**.

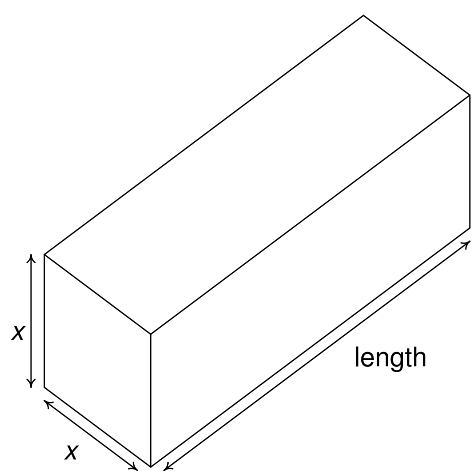




□

Our last example of the section uses the interplay between algebraic and graphical representations of a function to solve a real-world problem.

Example 1.1.6. The United States Postal Service mandates that when shipping parcels using 'Parcel Select' service, the sum of the length (the longest dimension) and the girth (the distance around the thickest part of the parcel perpendicular to the length) must not exceed 130 inches.[17] Suppose we wish to ship a rectangular box whose girth forms a square measuring x inches per side as shown below.



It turns out[18] that the volume of a box, $V(x)$, measured in cubic inches, whose length plus girth is exactly 130 inches is given by the formula: $V(x) = x^2(130 - 4x)$ for $0 < x \leq 26$.

[17] See here.

[18] We'll skip the explanation for now because we want to focus on just the different representations of the function. Rest assured, you'll be asked to construct this very model in Exercise 56a in Section 2.1.

1. Find and interpret $V(5)$.

2. Make a table of values and use these along with a graphing utility to graph $y = V(x)$.

3. What is the largest volume box that can be shipped? What value of x maximizes the volume? Round your answers to two decimal places.

Solution.

1. To find $V(5)$, we substitute $x = 5$ into the expression $V(x)$: $V(5) = (5)^2(130 - 4(5)) = 25(110) = 2750$. Our result means that when the length and width of the square measure 5 inches, the volume of the resulting box is 2750 cubic inches.[19]

2. The domain of V is specified by the inequality $0 < x \leq 26$, so we can begin graphing V by sampling V at finitely many x-values in this interval to help us get a sense of the range of V. This, in turn, will help us determine an adequate viewing window on our graphing utility when the time comes.

   It seems natural to start with what's happening near $x = 0$. Even though the expression $x^2(130 - 4x)$ is defined when we substitute $x = 0$ (it reduces very quickly to 0), it would be incorrect to state $V(0) = 0$ because $x = 0$ is not in the domain of V. However, there is nothing stopping us from evaluating $V(x)$ at values x 'very close' to $x = 0$. A table of such values is given below.

x	$V(x)$
0.1	1.296
0.01	0.012996
0.001	0.000129996
10^{-23}	$\approx 1.3 \times 10^{-44}$

   There is no such thing as a 'smallest' positive number,[20] so we will have points on the graph of V to the right of $x = 0$ leading to the point $(0, 0)$. We indicate this behavior by putting a hole at $(0, 0)$.[21]

   Moving forward, we start with $x = 5$ and sample V at steps of 5 in its domain. Our goal is to graph $y = V(x)$, so we plot our points $(x, V(x))$ using the domain as a guide to help us set the horizontal bounds (i.e., the bounds on x) and the sample values from the range to help us set the vertical bounds (i.e., the bounds on y). The right endpoint, $x = 26$, is included in the domain $0 < x \leq 26$ so we finish the graph by plotting the point $(26, V(26)) = (26, 17576)$. At the top of the next page on the left is the table of data and on the right is a graph produced with some help from desmos.

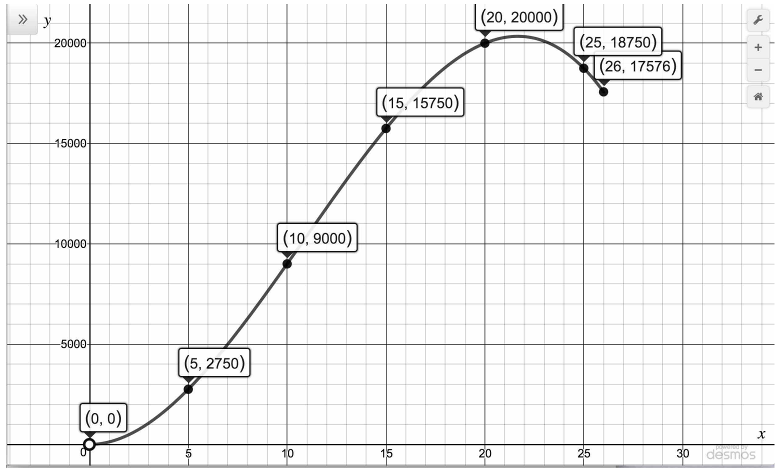
[19] Note that we have $V(5)$ and $25(110)$ in the same string of equality. The first set of parentheses is function notation and directs us to substitute 5 for x in the expression $V(x)$ while the second indicates multiplying 25 by 110. Context is key!

[20] If p is any positive real number, $0 < 0.5p < p$, so we can always find a smaller positive real number.

[21] What's really needed here is the precise definition of 'closeness' discussed in Calculus. This hand-waving will do for now.

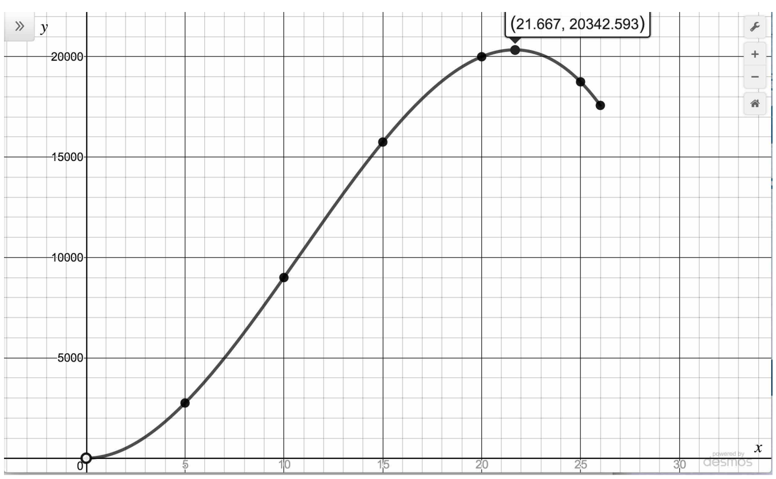
x	$V(x)$	$(x, V(x))$
≈ 0	≈ 0	hole at $(0,0)$
5	2750	$(5, 2750)$
10	9000	$(10, 9000)$
15	15,750	$(15, 15{,}750)$
20	20,000	$(20, 20{,}000)$
25	18,750	$(25, 18{,}750)$
26	17,576	$(26, 17{,}576)$

Sampling V



The graph of $y = V(x)$

3. The largest volume in this case refers to the maximum of V. The biggest y-value in our table of data is 20,000 cubic inches which occurs at $x = 20$ inches, but the graph produced by the graphing utility indicates that there are points on the graph of V with y-values (hence $V(x)$ values) greater than 20,000. Indeed, the graph continues to rise to the right of $x = 20$ and the graphing utility reports the maximum y-value to be $y \approx 20{,}342.593$ when $x \approx 21.667$. Rounding to two decimal places, we find the maximum volume obtainable under these conditions is about 20,342.59 cubic inches which occurs when the length and width of the square side of the box are approximately 21.67 inches.[22]



Finding the maximum volume using the graph of $y = V(x)$.

[22] We could also find the length of the box in this case as well. The sum of length and girth is 130 inches so the length is 130 minus the girth, or $130 - 4x \approx 130 - 4(21.67) = 43.32$ inches.

It is worth noting that while the function V has a maximum, it did not have a minimum. Even though $V(x) > 0$ for all x in its domain,[23] the presence of the hole at $(0, 0)$ means that 0 is not in the range of V. Hence, based on our model, we can never make a box with a 'smallest' volume.[24] $\square$

Example 1.1.6 typifies the interplay between Algebra and Geometry which lies ahead. Both the algebraic description of V: $V(x) = x^2(130 - 4x)$ for $0 < x \leq 26$, and the graph of $y = V(x)$ were useful in describing aspects of the physical situation at hand. Wherever possible, we'll use the algebraic representations of functions to **analytically** produce **exact** answers to certain problems and use the graphical descriptions to check the reasonableness of our answers.

That being said, we'll also encounter problems which we simply **cannot** answer analytically (such as determining the maximum volume in the previous example), so we will be forced to resort to using technology (specifically graphing technology) in order to find **approximate** solutions. The most important thing to keep in mind is that while technology may **suggest** a result, it is ultimately Mathematics that **proves** it.

We close this section with a summary of the different ways to represent functions.

Ways to Represent a Function

Suppose f is a function with domain A. Then f can be represented:

- verbally; that is, by describing how the inputs are matched with their outputs.
- using a mapping diagram.
- as a set of ordered pairs of the form (input, output): $\{(a, f(a)) \mid a \in A\}$.

If f is a function whose domain and range are subsets of real numbers, then f can be represented:

- algebraically as a formula for $f(a)$.
- graphically by plotting the points $\{(a, f(a)) \mid a \in A\}$ in the plane.

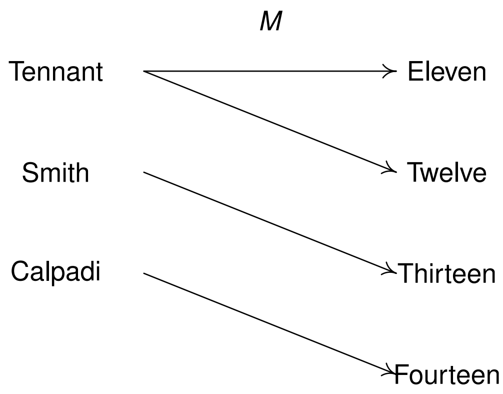
Note: An important consequence of the last bulleted item is that the point (a, b) is on the graph of $y = f(x)$ if and only if $f(a) = b$.

[23] said differently, the values of $V(x)$ are **bounded below** by 0.
[24] How realistic is this?

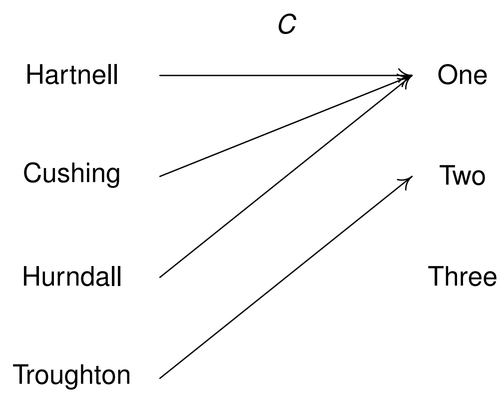
1.1.4 Exercises

In Exercises 1 - 2, determine whether or not the mapping diagram represents a function. Explain your reasoning. If the mapping does represent a function, state the domain, range, and represent the function as a set of ordered pairs.

1.



2.



In Exercises 3 - 4, determine whether or not the data in the given table represents y as a function of x. Explain your reasoning. If the mapping does represent a function, state the domain, range, and represent the function as a set of ordered pairs.

3.

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

4.

x	y
0	0
1	1
1	-1
2	2
2	-2
3	3
3	-3

5. Suppose W is the set of words in the English language and we set up a mapping from W into the set of natural numbers $\mathbb{N}$ as follows: word $\to$ number of letters in the word. Explain why this mapping is a function. What would you need to know to determine the range of the function?

6. Suppose L is the set of last names of all the people who have served or are currently serving as the President of the United States. Consider the mapping from L into $\mathbb{N}$ as follows: last name $\to$ number of their presidency. For example, Washington $\to$ 1 and Obama $\to$ 44. Is this mapping a function? What if we use full names instead of just last names? (**HINT:** Research Grover Cleveland.)

7. Under what conditions would the time of day be a function of the outdoor temperature?

For the functions f described in Exercises 8 - 13, find $f(2)$ and find and simplify an expression for $f(x)$ that takes a real number x and performs the following three steps in the order given:

8. (1) multiply by 2; (2) add 3; (3) divide by 4.
9. (1) add 3; (2) multiply by 2; (3) divide by 4.
10. (1) divide by 4; (2) add 3; (3) multiply by 2.
11. (1) multiply by 2; (2) add 3; (3) take the square root.
12. (1) add 3; (2) multiply by 2; (3) take the square root.
13. (1) add 3; (2) take the square root; (3) multiply by 2.

In Exercises 14 - 19, use the given function f to find and simplify the following:

- $f(3)$
- $f(-1)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x-4)$
- $f(x)-4$
- $f\left(x^2\right)$

14. $f(x) = 2x + 1$
15. $f(x) = 3 - 4x$
16. $f(x) = 2 - x^2$
17. $f(x) = x^2 - 3x + 2$
18. $f(x) = 6$
19. $f(x) = 0$

In Exercises 20 - 25, use the given function f to find and simplify the following:

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a) + f(2)$
- $f\left(\dfrac{2}{a}\right)$
- $\dfrac{f(a)}{2}$
- $f(a+h)$

20. $f(x) = 2x - 5$
21. $f(t) = 5 - 2t$
22. $f(w) = 2w^2 - 1$
23. $f(q) = 3q^2 + 3q - 2$
24. $f(r) = 117$
25. $f(z) = \dfrac{z}{2}$

In Exercises 26 - 29, use the given function f to find $f(0)$ and solve $f(x) = 0$

26. $f(x) = 2x - 1$
27. $f(x) = 3 - \frac{2}{5}x$
28. $f(x) = 2x^2 - 6$
29. $f(x) = x^2 - x - 12$

In Exercises 30 - 44, determine whether or not the equation represents y as a function of x.

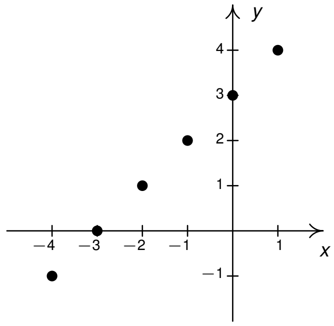
30. $y = x^3 - x$
31. $y = \sqrt{x-2}$
32. $x^3 y = -4$
33. $x^2 - y^2 = 1$
34. $y = \dfrac{x}{x^2 - 9}$
35. $x = -6$
36. $x = y^2 + 4$
37. $y = x^2 + 4$
38. $x^2 + y^2 = 4$
39. $y = \sqrt{4 - x^2}$
40. $x^2 - y^2 = 4$
41. $x^3 + y^3 = 4$
42. $2x + 3y = 4$
43. $2xy = 4$
44. $x^2 = y^2$

Exercises 45 - 56 give a set of points in the xy-plane. Determine if y is a function of x. If so, state the domain and range.

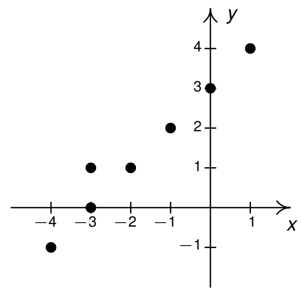
45. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$
46. $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$
47. $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$
48. $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),...\}$
49. $\{(x,y) \,|\, x$ is an odd integer, and y is an even integer$\}$
50. $\{(x,1) \,|\, x$ is an irrational number$\}$
51. $\{(1,0),(2,1),(4,2),(8,3),(16,4),(32,5),...\}$
52. $\{...\,(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9),...\}$
53. $\{(-2,y) \,|\, -3 < y < 4\}$
54. $\{(x,3) \,|\, -2 \leq x < 4\}$
55. $\{\left(x,x^2\right) \,|\, x$ is a real number$\}$
56. $\{\left(x^2,x\right) \,|\, x$ is a real number$\}$
57. The Vertical Line Test is a quick way to determine from a graph if the vertical axis variable is a function of the horizontal axis variable. If we are given a graph and asked to determine if the horizontal axis variable is a function of the vertical axis variable, we can use horizontal lines instead of vertical lines to check. Using Theorem 1.1 as a guide, formulate a 'Horizontal Line Test.' (We'll refer back to this exercise in Section 5.6.)

In Exercises 58 - 61, determine whether or not the graph suggests y is a function of x. For the ones which do, state the domain and range.

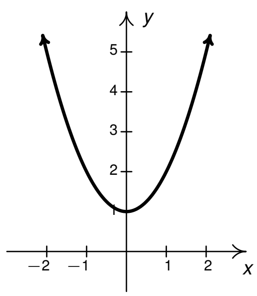
58.



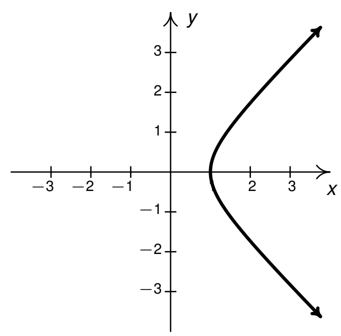
59.



60.



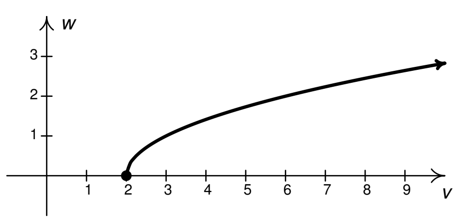
61.



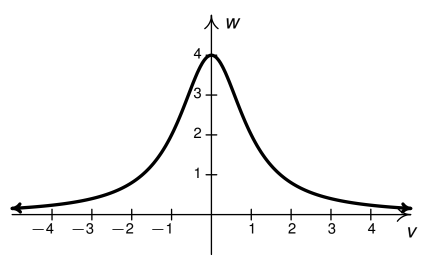
62. Determine which, if any, of the graphs in numbers 58 - 61 represent x as a function of y. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 63 - 66, determine whether or not the graph suggests w is a function of v. For the ones which do, state the domain and range.

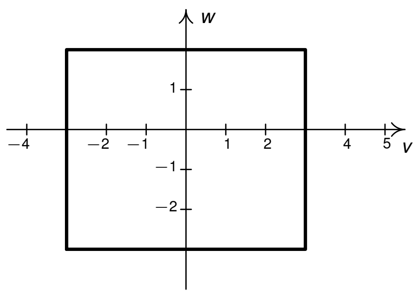
63.



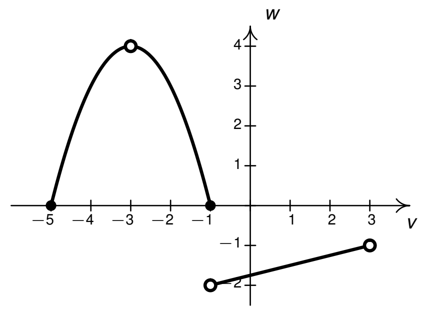
64.



65.



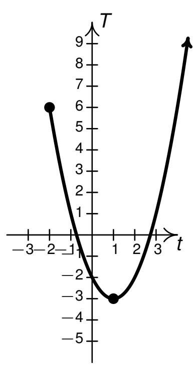
66.



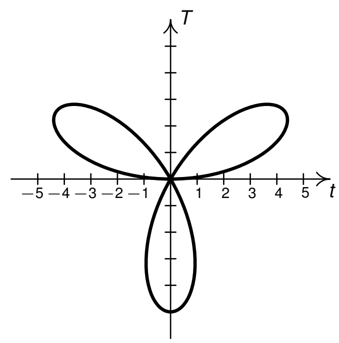
67. Determine which, if any, of the graphs in numbers 63 - 66 represent v as a function of w. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 68 - 71, determine whether or not the graph suggests T is a function of t. For the ones which do, state the domain and range.

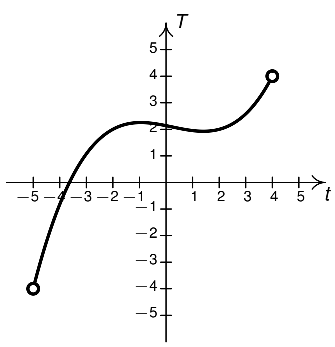
68.



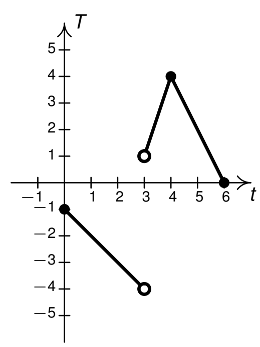
69.



70.

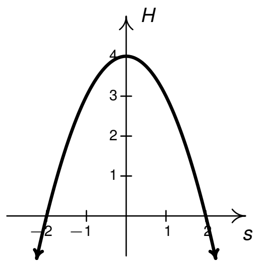


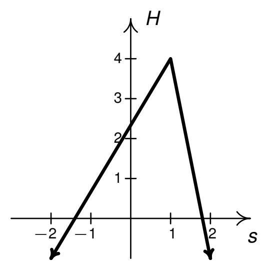
71.

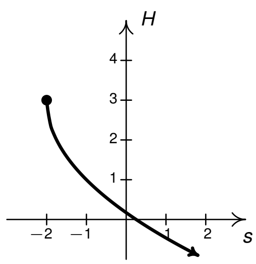


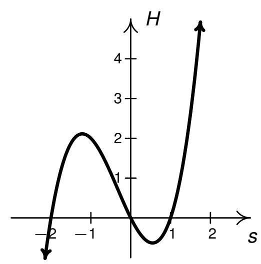
72. Determine which, if any, of the graphs in numbers 68 - 71 represent t as a function of T. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 73 - 76, determine whether or not the graph suggests H is a function of s. For the ones which do, state the domain and range.

73. 

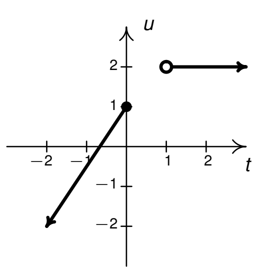
74. 

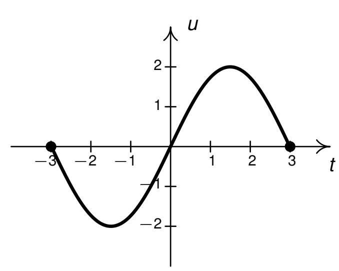
75. 

76. 

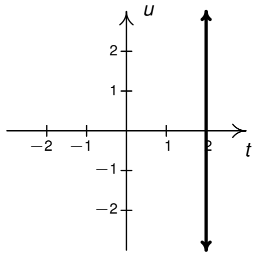
77. Determine which, if any, of the graphs in numbers 73 - 76 represent s as a function of H. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 78 - 81, determine whether or not the graph suggests u is a function of t. For the ones which do, state the domain and range.

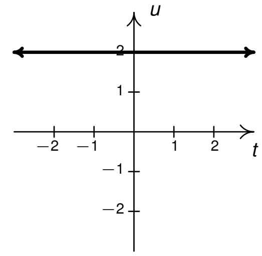
78. 

79. 

80.

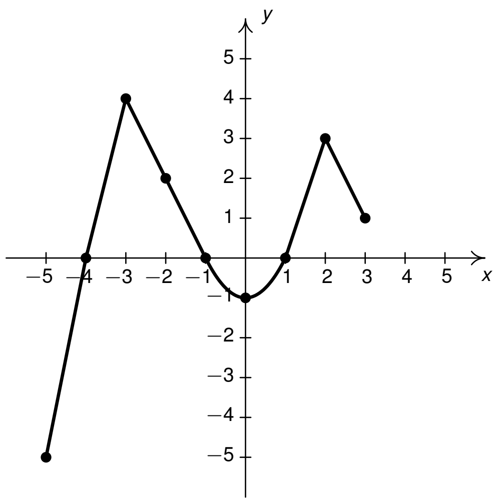


81.

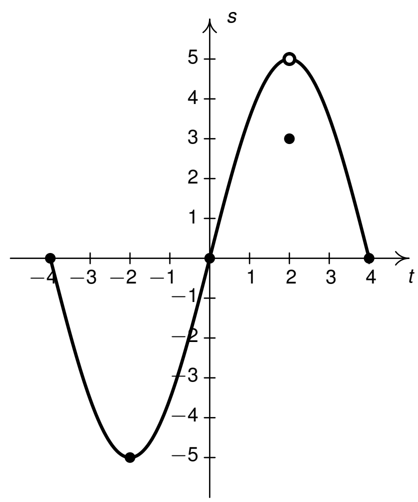


82. Determine which, if any, of the graphs in numbers 78 - 81 represent t as a function of u. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 83 - 92, use the graphs of f and g below to find the indicated values.



The graph of $y = f(x)$.



The graph of $s = g(t)$.

83. $f(-2)$

84. $g(-2)$

85. $f(2)$

86. $g(2)$

87. $f(0)$

88. $g(0)$

89. Solve $f(x) = 0$.

90. Solve $g(t) = 0$.

91. State the domain and range of f.

92. State the domain and range of g .

In Exercises 93 - 104, graph each function by making a table, plotting points, and using a graphing utility (if needed.) Use the independent variable as the horizontal axis label and the default 'y' label for the vertical axis label. State the domain and range of each function.

93. $f(x) = 2 - x$

94. $g(t) = \dfrac{t-2}{3}$

95. $h(s) = s^2 + 1$

96. $f(x) = 4 - x^2$

97. $g(t) = 2$

98. $h(s) = s^3$

99. $f(x) = x(x-1)(x+2)$

100. $g(t) = \sqrt{t-2}$

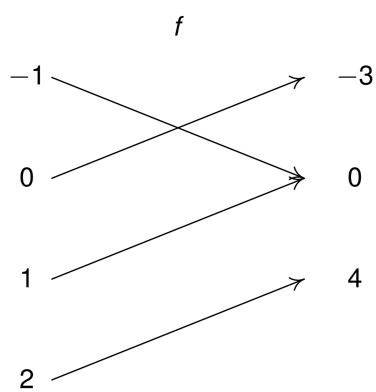
101. $h(s) = \sqrt{5-s}$

102. $f(x) = 3 - 2\sqrt{x+2}$

103. $g(t) = \sqrt[3]{t}$

104. $h(s) = \dfrac{1}{s^2+1}$

105. Consider the function f described below:



(a) State the domain and range of f.

(b) Find $f(0)$ and solve $f(x) = 0$.

(c) Write f as a set of ordered pairs.

(d) Graph f.

106. Let $g = \{(-1,4), (0,2), (2,3), (3,4)\}$

(a) State the domain and range of g.

(b) Create a mapping diagram for g.

(c) Find $g(0)$ and solve $g(x) = 0$.

(d) Graph g.

107. Let $F = \{(t, t^2) \mid t \text{ is a real number}\}$. Find $F(4)$ and solve $F(x) = 4$.

HINT: Elements of F are of the form $(x, F(x))$.

108. Let $G = \{(2t, t+5) \mid t \text{ is a real number}\}$. Find $G(4)$ and solve $G(x) = 4$.

HINT: Elements of G are of the form $(x, G(x))$.

109. The area enclosed by a square, in square inches, is a function of the length of one of its sides ℓ, when measured in inches. This function is represented by the formula $A(\ell) = \ell^2$ for $\ell > 0$. Find $A(3)$ and solve $A(\ell) = 36$. Interpret your answers to each. Why is ℓ restricted to $\ell > 0$?

110. The area enclosed by a circle, in square meters, is a function of its radius r, when measured in meters. This function is represented by the formula $A(r) = \pi r^2$ for $r > 0$. Find $A(2)$ and solve $A(r) = 16\pi$. Interpret your answers to each. Why is r restricted to $r > 0$?

111. The volume enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides s, when measured in centimeters. This function is represented by the formula $V(s) = s^3$ for $s > 0$. Find $V(5)$ and solve $V(s) = 27$. Interpret your answers to each. Why is s restricted to $s > 0$?

112. The volume enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r, when measured in feet. This function is represented by the formula $V(r) = \frac{4\pi}{3} r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r > 0$?

113. The height of an object dropped from the roof of an eight story building is modeled by the function: $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground, in feet, t seconds after the object is dropped. Find $h(0)$ and solve $h(t) = 0$. Interpret your answers to each. Why is t restricted to $0 \leq t \leq 2$?

114. The temperature in degrees Fahrenheit t hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2 + 8t + 3$ for $0 \leq t \leq 12$. Find and interpret $T(0)$, $T(6)$ and $T(12)$.

115. The function $C(x) = x^2 - 10x + 27$ models the cost, in *hundreds* of dollars, to produce x *thousand* pens. Find and interpret $C(0)$, $C(2)$ and $C(5)$.

116. Using data from the Bureau of Transportation Statistics, the average fuel economy in miles per gallon for passenger cars in the US can be modeled by $E(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Use a calculator to find $E(0)$, $E(14)$ and $E(28)$. Round your answers to two decimal places and interpret your answers to each.

117. The perimeter of a square, in centimeters, is four times the length of one if its sides, also measured in centimeters. Represent the function P which takes as its input the length of the side of a square in centimeters, s and returns the perimeter of the square in inches, $P(s)$ using a formula.

118. The circumference of a circle, in feet, is π times the diameter of the circle, also measured in feet. Represent the function C which takes as its input the length of the diameter of a circle in feet, D and returns the circumference of a circle in inches, $C(D)$ using a formula.

119. Suppose $A(P)$ gives the amount of money in a retirement account (in dollars) after 30 years as a function of the amount of the monthly payment (in dollars), P.

   (a) What does $A(50)$ mean?

   (b) What is the significance of the solution to the equation $A(P) = 250000$? .

   (c) Explain what each of the following expressions mean: $A(P + 50)$, $A(P) + 50$, and $A(P) + A(50)$.

120. Suppose $P(t)$ gives the chance of precipitation (in percent) t hours after 8 AM.

   (a) Write an expression which gives the chance of precipitation at noon.

   (b) Write an inequality which determines when the chance of precipitation is more than 50%.

121. Explain why the graph in Exercise 63 suggests that not only is v as a function of w but also w is a function of v. Suppose $w = f(v)$ and $v = g(w)$. That is, f is the name of the function which takes v values as inputs and returns w values as outputs and g is the name of the function which does vice-versa. Find the domain and range of g and compare these to the domain and range of f.

122. Sketch the graph of a function with domain $(-\infty, 3) \cup [4, 5)$ with range $\{2\} \cup (5, \infty)$.

1.1.5 Answers

1. The mapping M is not a function since 'Tennant' is matched with both 'Eleven' and 'Twelve.'

2. The mapping C is a function since each input is matched with only one output. The domain of C is $\{\text{Hartnell, Cushing, Hurndall, Troughton}\}$ and the range is $\{\text{One, Two}\}$. We can represent C as the following set of ordered pairs: $\{(\text{Hartnell, One}), (\text{Cushing, One}), (\text{Hurndall, One}), (\text{Troughton, Two})\}$

3. In this case, y is a function of x since each x is matched with only one y.

   The domain is $\{-3, -2, -1, 0, 1, 2, 3\}$ and the range is $\{0, 1, 2, 3\}$.

   As ordered pairs, this function is $\{(-3, 3), (-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2), (3, 3)\}$

4. In this case, y is not a function of x since there are x values matched with more than one y value. For instance, 1 is matched both to 1 and -1.

5. The mapping is a function since given any word, there is only one answer to 'how many letters are in the word?' For the range, we would need to know what the length of the longest word is and whether or not we could find words of all the lengths between 1 (the length of the word 'a') and it. See here.

6. Since Grover Cleveland was both the 22nd and 24th POTUS, neither mapping described in this exercise is a function.

7. The outdoor temperature could never be the same for more than two different times - so, for example, it could always be getting warmer or it could always be getting colder.

8. $f(2) = \frac{7}{4}$, $f(x) = \frac{2x+3}{4}$

9. $f(2) = \frac{5}{2}$, $f(x) = \frac{2(x+3)}{4} = \frac{x+3}{2}$

10. $f(2) = 7$, $f(x) = 2\left(\frac{x}{4} + 3\right) = \frac{1}{2}x + 6$

11. $f(2) = \sqrt{7}$, $f(x) = \sqrt{2x+3}$

12. $f(2) = \sqrt{10}$, $f(x) = \sqrt{2(x+3)} = \sqrt{2x+6}$

13. $f(2) = 2\sqrt{5}$, $f(x) = 2\sqrt{x+3}$

14. For $f(x) = 2x + 1$

    - $f(3) = 7$
    - $f(-1) = -1$
    - $f\left(\frac{3}{2}\right) = 4$
    - $f(4x) = 8x + 1$
    - $4f(x) = 8x + 4$
    - $f(-x) = -2x + 1$
    - $f(x - 4) = 2x - 7$
    - $f(x) - 4 = 2x - 3$
    - $f\left(x^2\right) = 2x^2 + 1$

15. For $f(x) = 3 - 4x$

    - $f(3) = -9$
    - $f(-1) = 7$
    - $f\left(\frac{3}{2}\right) = -3$

- $f(4x) = 3 - 16x$
- $4f(x) = 12 - 16x$
- $f(-x) = 4x + 3$
- $f(x - 4) = 19 - 4x$
- $f(x) - 4 = -4x - 1$
- $f\left(x^2\right) = 3 - 4x^2$

16. For $f(x) = 2 - x^2$

- $f(3) = -7$
- $f(-1) = 1$
- $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 2 - 16x^2$
- $4f(x) = 8 - 4x^2$
- $f(-x) = 2 - x^2$
- $f(x - 4) = -x^2 + 8x - 14$
- $f(x) - 4 = -x^2 - 2$
- $f\left(x^2\right) = 2 - x^4$

17. For $f(x) = x^2 - 3x + 2$

- $f(3) = 2$
- $f(-1) = 6$
- $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 16x^2 - 12x + 2$
- $4f(x) = 4x^2 - 12x + 8$
- $f(-x) = x^2 + 3x + 2$
- $f(x - 4) = x^2 - 11x + 30$
- $f(x) - 4 = x^2 - 3x - 2$
- $f\left(x^2\right) = x^4 - 3x^2 + 2$

18. For $f(x) = 6$

- $f(3) = 6$
- $f(-1) = 6$
- $f\left(\frac{3}{2}\right) = 6$
- $f(4x) = 6$
- $4f(x) = 24$
- $f(-x) = 6$
- $f(x - 4) = 6$
- $f(x) - 4 = 2$
- $f\left(x^2\right) = 6$

19. For $f(x) = 0$

- $f(3) = 0$
- $f(-1) = 0$
- $f\left(\frac{3}{2}\right) = 0$
- $f(4x) = 0$
- $4f(x) = 0$
- $f(-x) = 0$
- $f(x - 4) = 0$
- $f(x) - 4 = -4$
- $f\left(x^2\right) = 0$

20. For $f(x) = 2x - 5$

- $f(2) = -1$
- $f(-2) = -9$
- $f(2a) = 4a - 5$
- $2f(a) = 4a - 10$
- $f(a + 2) = 2a - 1$
- $f(a) + f(2) = 2a - 6$

- $f\left(\frac{2}{a}\right) = \frac{4}{a} - 5 = \frac{4-5a}{a}$
- $\frac{f(a)}{2} = \frac{2a-5}{2}$
- $f(a+h) = 2a + 2h - 5$

21. For $f(x) = 5 - 2x$

- $f(2) = 1$
- $f(-2) = 9$
- $f(2a) = 5 - 4a$
- $2f(a) = 10 - 4a$
- $f(a+2) = 1 - 2a$
- $f(a) + f(2) = 6 - 2a$
- $f\left(\frac{2}{a}\right) = 5 - \frac{4}{a} = \frac{5a-4}{a}$
- $\frac{f(a)}{2} = \frac{5-2a}{2}$
- $f(a+h) = 5 - 2a - 2h$

22. For $f(x) = 2x^2 - 1$

- $f(2) = 7$
- $f(-2) = 7$
- $f(2a) = 8a^2 - 1$
- $2f(a) = 4a^2 - 2$
- $f(a+2) = 2a^2 + 8a + 7$
- $f(a) + f(2) = 2a^2 + 6$
- $f\left(\frac{2}{a}\right) = \frac{8}{a^2} - 1 = \frac{8-a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{2a^2-1}{2}$
- $f(a+h) = 2a^2 + 4ah + 2h^2 - 1$

23. For $f(x) = 3x^2 + 3x - 2$

- $f(2) = 16$
- $f(-2) = 4$
- $f(2a) = 12a^2 + 6a - 2$
- $2f(a) = 6a^2 + 6a - 4$
- $f(a+2) = 3a^2 + 15a + 16$
- $f(a) + f(2) = 3a^2 + 3a + 14$
- $f\left(\frac{2}{a}\right) = \frac{12}{a^2} + \frac{6}{a} - 2 = \frac{12+6a-2a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{3a^2+3a-2}{2}$
- $f(a+h) = 3a^2 + 6ah + 3h^2 + 3a + 3h - 2$

24. For $f(x) = 117$

- $f(2) = 117$
- $f(-2) = 117$
- $f(2a) = 117$
- $2f(a) = 234$
- $f(a+2) = 117$
- $f(a) + f(2) = 234$
- $f\left(\frac{2}{a}\right) = 117$
- $\frac{f(a)}{2} = \frac{117}{2}$
- $f(a+h) = 117$

25. For $f(x) = \frac{x}{2}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = a$
- $2f(a) = a$
- $f(a+2) = \frac{a+2}{2}$
- $f(a) + f(2) = \frac{a}{2} + 1 = \frac{a+2}{2}$
- $f\left(\frac{2}{a}\right) = \frac{1}{a}$
- $\frac{f(a)}{2} = \frac{a}{4}$
- $f(a+h) = \frac{a+h}{2}$

26. For $f(x) = 2x - 1$, $f(0) = -1$ and $f(x) = 0$ when $x = \frac{1}{2}$
27. For $f(x) = 3 - \frac{2}{5}x$, $f(0) = 3$ and $f(x) = 0$ when $x = \frac{15}{2}$
28. For $f(x) = 2x^2 - 6$, $f(0) = -6$ and $f(x) = 0$ when $x = \pm\sqrt{3}$
29. For $f(x) = x^2 - x - 12$, $f(0) = -12$ and $f(x) = 0$ when $x = -3$ or $x = 4$
30. Function
31. Function
32. Function
33. Not a function
34. Function
35. Not a function
36. Not a function
37. Function
38. Not a function
39. Function
40. Not a function
41. Function
42. Function
43. Function
44. Not a function
45. Function
    domain = $\{-3, -2, -1, 0, 1, 2, 3\}$
    range = $\{0, 1, 4, 9\}$
46. Not a function
47. Function
    domain = $\{-7, -3, 3, 4, 5, 6\}$
    range = $\{0, 4, 5, 6, 9\}$
48. Function
    domain = $\{1, 4, 9, 16, 25, 36, ...\}$
    = $\{x \mid x$ is a perfect square$\}$
    range = $\{2, 4, 6, 8, 10, 12, ...\}$
    = $\{y \mid y$ is a positive even integer$\}$
49. Not a function
50. Function
    domain = $\{x \mid x$ is irrational$\}$
    range = $\{1\}$
51. Function
    domain = $\{x \mid 1, 2, 4, 8, ...\}$
    = $\{x \mid x = 2^n$ for some whole number $n\}$
    range = $\{0, 1, 2, 3, ...\}$
    = $\{y \mid y$ is any whole number$\}$
52. Function
    domain = $\{x \mid x$ is any integer$\}$
    range = $\{y \mid y$ is the square of an integer$\}$

53. Not a function

54. Function
domain = $\{x \mid -2 \leq x < 4\}$ = $[-2, 4)$,
range = $\{3\}$

55. Function
domain = $\{x \mid x$ is a real number$\}$ = $(-\infty, \infty)$
range = $\{y \mid y \geq 0\}$ = $[0, \infty)$

56. Not a function

57. **Horizontal Line Test:** A graph on the *xy*-plane represents x as a function of y if and only if no **horizontal** line intersects the graph more than once.

58. Function
domain = $\{-4, -3, -2, -1, 0, 1\}$
range = $\{-1, 0, 1, 2, 3, 4\}$

59. Not a function

60. Function
domain = $(-\infty, \infty)$
range = $[1, \infty)$

61. Not a function

62.
- Number 58 represents x as a function of y.
  domain = $\{-1, 0, 1, 2, 3, 4\}$ and range = $\{-4, -3, -2, -1, 0, 1\}$
- Number 61 represents x as a function of y.
  domain = $(-\infty, \infty)$ and range = $[1, \infty)$

63. Function
domain = $[2, \infty)$
range = $[0, \infty)$

64. Function
domain = $(-\infty, \infty)$
range = $(0, 4]$

65. Not a function

66. Function
domain = $[-5, -3) \cup (-3, 3)$
range = $(-2, -1) \cup [0, 4)$

67. Only number 63 represents v as a function of w; domain = $[0, \infty)$ and range = $[2, \infty)$

68. Function
domain = $[-2, \infty)$
range = $[-3, \infty)$

69. Not a function

70. Function
domain = $(-5, 4)$
range = $(-4, 4)$

71. Function
domain = $[0, 3) \cup (3, 6]$
range = $(-4, -1] \cup [0, 4]$

72. None of numbers 68 - 71 represent t as a function of T.

73. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

74. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

75. Function
domain = $[-2, \infty)$
range = $(-\infty, 3]$

76. Function
domain = $(-\infty, \infty)$
range = $(-\infty, \infty)$

77. Only number 75 represents s as a function of H; domain = $(-\infty, 3]$ and range = $[-2, \infty)$

78. Function
domain = $(-\infty, 0] \cup (1, \infty)$
range = $(-\infty, 1] \cup \{2\}$

79. Function
domain = $[-3, 3]$
range = $[-2, 2]$

80. Not a function

81. Function
domain = $(-\infty, \infty)$
range = $\{2\}$

82. Only number 80 represents t as a function of u; domain = $(-\infty, \infty)$ and range=$\{2\}$.

83. $f(-2) = 2$

84. $g(-2) = -5$

85. $f(2) = 3$

86. $g(2) = 3$

87. $f(0) = -1$

88. $g(0) = 0$

89. $x = -4, -1, 1$

90. $t = -4, 0, 4$

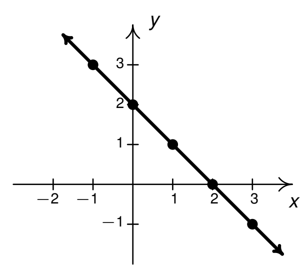
91. Domain: $[-5, 3]$, Range: $[-5, 4]$.

92. Domain: $[-4, 4]$, Range: $[-5, 5)$.

93. $f(x) = 2 - x$

Domain: $(-\infty, \infty)$

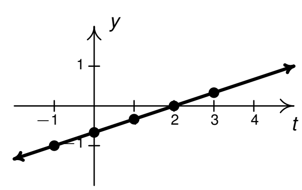
Range: $(-\infty, \infty)$



94. $g(t) = \dfrac{t-2}{3}$

Domain: $(-\infty, \infty)$

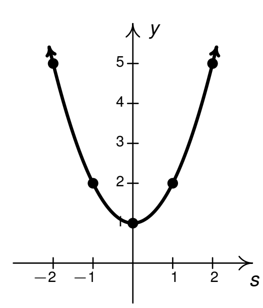
Range: $(-\infty, \infty)$



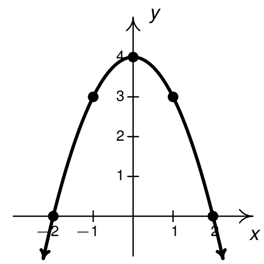
95. $h(s) = s^2 + 1$

Domain: $(-\infty, \infty)$

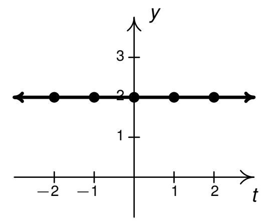
Range: $[1, \infty)$



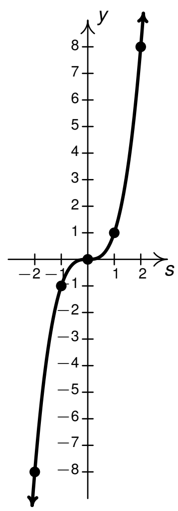
96. $f(x) = 4 - x^2$
Domain: $(-\infty, \infty)$
Range: $(-\infty, 4]$



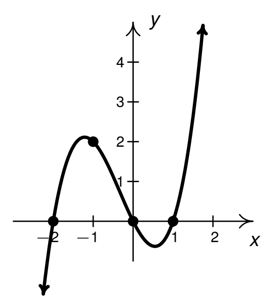
97. $g(t) = 2$
Domain: $(-\infty, \infty)$
Range: $\{2\}$



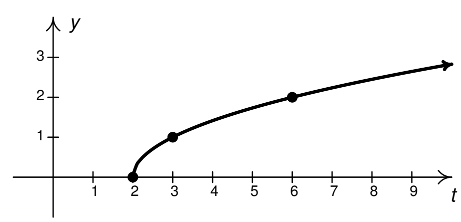
98. $h(s) = s^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



99. $f(x) = x(x-1)(x+2)$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



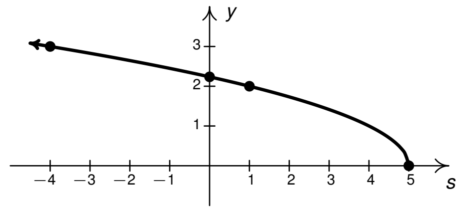
100. $g(t) = \sqrt{t-2}$
Domain: $[2, \infty)$
Domain: $[0, \infty)$



101. $h(s) = \sqrt{5 - s}$

Domain: $(-\infty, 5]$

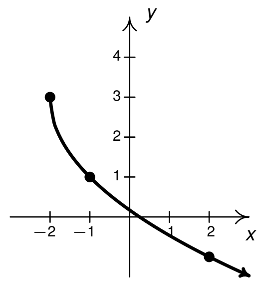
Range: $[0, \infty)$



102. $f(x) = 3 - 2\sqrt{x + 2}$

Domain: $[-2, \infty)$

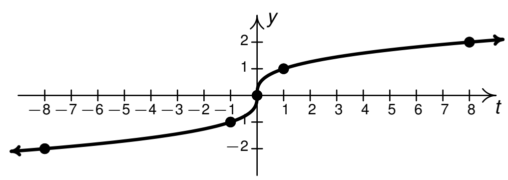
Range: $(-\infty, 3]$



103. $g(t) = \sqrt[3]{t}$

Domain: $(-\infty, \infty)$

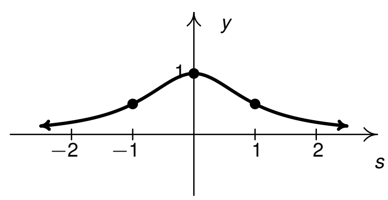
Range: $(-\infty, \infty)$



104. $h(s) = \dfrac{1}{s^2 + 1}$

Domain: $(-\infty, \infty)$

Range: $(0, 1]$

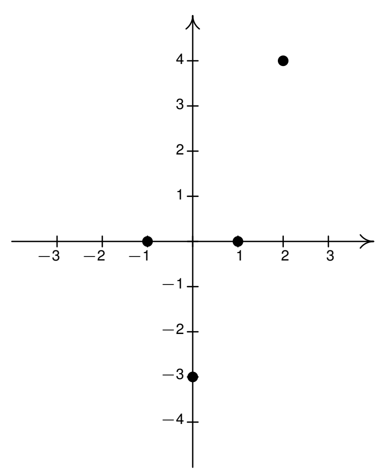


105. (a) domain = $\{-1, 0, 1, 2\}$, range = $\{-3, 0, 4\}$

(b) $f(0) = -3$, $f(x) = 0$ for $x = -1, 1$.

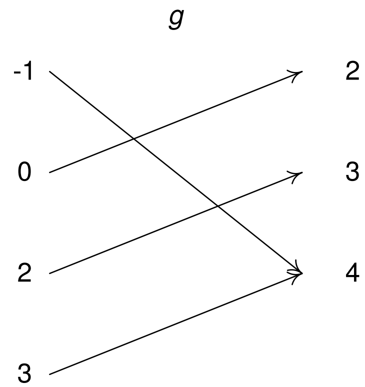
(c) $f = \{(-1, 0), (0, -3), (1, 0), (2, 4)\}$

(d)



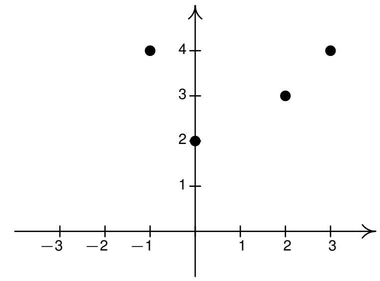
106. (a) domain = $\{-1, 0, 2, 3\}$, range = $\{2, 3, 4\}$

(b)



(c) Find $g(0) = 2$ and $g(x) = 0$ has no solutions.

(d)



107. $F(4) = 4^2 = 16$ (when $t = 4$), the solutions to $F(x) = 4$ are $x = \pm 2$ (when $t = \pm 2$).

108. $G(4) = 7$ (when $t = 2$), the solution to $G(t) = 4$ is $x = -2$ (when $t = -1$)

109. $A(3) = 9$, so the area enclosed by a square with a side of length 3 inches is 9 square inches. The solutions to $A(\ell) = 36$ are $\ell = \pm 6$. Since ℓ is restricted to $\ell > 0$, we only keep $\ell = 6$. This means for the area enclosed by the square to be 36 square inches, the length of the side needs to be 6 inches. Since ℓ represents a length, $\ell > 0$.

110. $A(2) = 4\pi$, so the area enclosed by a circle with radius 2 meters is 4π square meters. The solutions to $A(r) = 16\pi$ are $r = \pm 4$. Since r is restricted to $r > 0$, we only keep $r = 4$. This means for the area enclosed by the circle to be 16π square meters, the radius needs to be 4 meters. Since r represents a radius (length), $r > 0$.

111. $V(5) = 125$, so the volume enclosed by a cube with a side of length 5 centimeters is 125 cubic centimeters. The solution to $V(s) = 27$ is $s = 3$. This means for the volume enclosed by the cube to be 27 cubic centimeters, the length of the side needs to 3 centimeters. Since x represents a length, $x > 0$.

112. $V(3) = 36\pi$, so the volume enclosed by a sphere with radius 3 feet is 36π cubic feet. The solution to $V(r) = \frac{32\pi}{3}$ is $r = 2$. This means for the volume enclosed by the sphere to be $\frac{32\pi}{3}$ cubic feet, the radius needs to 2 feet. Since r represents a radius (length), $r > 0$.

113. $h(0) = 64$, so at the moment the object is dropped off the building, the object is 64 feet off of the ground. The solutions to $h(t) = 0$ are $t = \pm 2$. Since we restrict $0 \le t \le 2$, we only keep $t = 2$. This means 2 seconds after the object is dropped off the building, it is 0 feet off the ground. Said differently, the object hits the ground after 2 seconds. The restriction $0 \le t \le 2$ restricts the time to be between the moment the object is released and the moment it hits the ground.

114. $T(0) = 3$, so at 6 AM (0 hours after 6 AM), it is 3° Fahrenheit. $T(6) = 33$, so at noon (6 hours after 6 AM), the temperature is 33° Fahrenheit. $T(12) = 27$, so at 6 PM (12 hours after 6 AM), it is 27° Fahrenheit.

115. $C(0) = 27$, so to make 0 pens, it costs[25] \$2700. $C(2) = 11$, so to make 2000 pens, it costs \$1100. $C(5) = 2$, so to make 5000 pens, it costs \$2000.

116. $E(0) = 16.00$, so in 1980 (0 years after 1980), the average fuel economy of passenger cars in the US was 16.00 miles per gallon. $E(14) = 20.81$, so in 1994 (14 years after 1980), the average fuel economy of passenger cars in the US was 20.81 miles per gallon. $E(28) = 22.64$, so in 2008 (28 years after 1980), the average fuel economy of passenger cars in the US was 22.64 miles per gallon.

117. $P(s) = 4s$, $s > 0$.

118. $C(D) = \pi D$, $D > 0$.

119. (a) The amount in the retirement account after 30 years if the monthly payment is \$50.
   (b) The solution to $A(P) = 250000$ is what the monthly payment needs to be in order to have \$250,000 in the retirement account after 30 years.
   (c) $A(P + 50)$ is how much is in the retirement account in 30 years if \$50 is added to the monthly payment P. $A(P) + 50$ represents the amount of money in the retirement account after 30 years if \$$P$ is invested each month plus an additional \$50. $A(P) + A(50)$ is the sum of money from two retirement accounts after 30 years: one with monthly payment \$$P$ and one with monthly payment \$50.

120. (a) Since noon is 4 hours after 8 AM, $P(4)$ gives the chance of precipitation at noon.
   (b) We would need to solve $P(t) \geq 50\%$ or $P(t) \geq 0.5$.

121. The graph in question passes the horizontal line test meaning for each w there is only one v. The domain of g is $[0, \infty)$ (which is the range of f) and the range of g is $[2, \infty)$ which is the domain of f.

122. Answers vary.

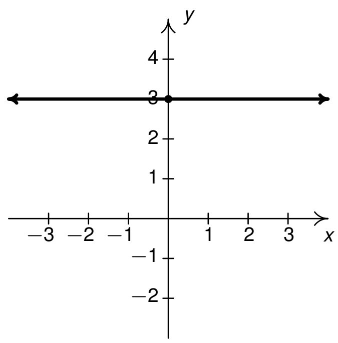
[25]This is called the 'fixed' or 'start-up' cost. We'll revisit this concept in Example 1.2.3 in Section 1.2.

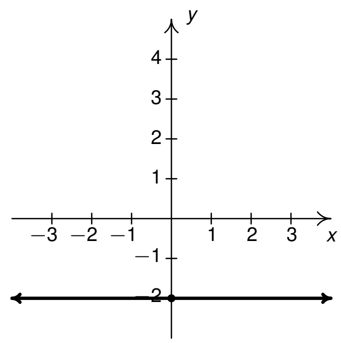
1.2 Constant and Linear Functions

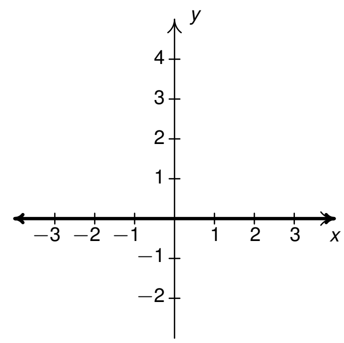
1.2.1 Constant Functions

Now that we have defined the concept of a function, we'll spend the rest of Chapter 1 revisiting families of curves from prior courses in Algebra by viewing them through a 'function lens'. We start with lines and refer the reader to Section A.5 for a review of the basic properties of lines. The simplest lines are vertical and horizontal lines. We leave it to the reader (see Exercise 58) to think about why we eschew vertical lines in our discussion here, and begin with a functional description of horizontal lines.

Consider the horizontal lines graphed in the xy-plane as shown below. The Vertical Line Test, Theorem 1.1, tells us that each describes y as a function of x so the question becomes how to represent these functions algebraically. The key here is to remember that the equation relating the independent variable x, the dependent variable y, and the function f is given by $y = f(x)$.







In the graph on the left, y always equals 3 so we have $f(x) = 3$. Procedurally, '$f(x) = 3$' says that the rule f takes the input x, and, regardless of that input, gives the output 3. This is an example of what is called a **constant** function - a function which returns the **same** value regardless of the input. Likewise, the function represented by the graph in the middle is $f(x) = -2$, and the graph on the right (the x-axis) is the graph of $f(x) = 0$. In general, we have the following definition:

Definition 1.4. A **constant function** is a function of the form

$$f(x) = b$$

where b is real number. The domain of a constant function is $(-\infty, \infty)$.

Some remarks about Definition 1.4 are in order. First, note that we are using 'x' as the independent variable, 'f' as the function name, and the letter 'b' as a **parameter**. In this context, a parameter is a fixed, but arbitrary, constant used to describe a **family** of functions. Different values of b determine different constant functions. For example, $b = 3$ gives $f(x) = 3$, $b = -2$ gives $f(x) = -2$, and so on. Once b is chosen, however, it does not change as the independent variable, x, changes.

Also note that we are using the generic defaults for function names and independent variables, namely f and x, respectively. The functions $G(t) = \sqrt{\pi}$ and $Z(\rho) = 0$ are also fine examples of constant functions.

Recall that inherent in the definition of a function is the notion of domain, so we record (as part of the definition) that a constant function has domain $(-\infty, \infty)$. The range of a constant function is the set $\{b\}$. The value b in this case is both the maximum and minimum of f, attained at each value in its domain.[1]

The next example showcases an application of constant functions and introduces the notion of a **piecewise-defined** function.

Example 1.2.1. The price of admission to see a matinee showing at a local movie theater is a function of the age of the ticket holder. If a person is aged A years, the price per ticket is $p(A)$ dollars and is given by:

$$p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50 \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$$

1. Find and interpret $p(3)$, $p(6)$ and $p(62)$.
2. Explain the pricing structure verbally.
3. Graph p.

Solution. The function p described above is an example of a **piecewise-defined** function because the **rule** to determine outputs, not just the value of the output, changes depending on the inputs.

1. To find $p(3)$, we note that the value $A = 3$ satisfies the inequality $0 \leq A < 6$ so we use the rule $p(A) = 5.75$. Hence, $p(3) = 5.75$ which means a ticket for a 3 year old is \$5.75. The next age, $A = 6$, just barely satisfies the inequality $6 \leq A < 50$ so we use the rule $p(A) = 7.25$, This yields $p(6) = 7.25$ which means a ticket for a 6 year old is \$7.25. Lastly, $A = 62$ satisfies the inequality $A \geq 50$, so we are back to the rule $p(A) = 5.75$. Thus $p(62) = 5.75$ which means someone 62 years young gets in for \$5.75.

2. Now that we've had some practice interpreting function values, we can begin to verbalize what the function is really saying. In the first 'piece' of the function, the inequality $0 \leq A < 6$ describes ticket holders under the age of 6 years and the inequality $A \geq 50$ describes ticket holders fifty years old or or older. For folks in these two age demographics, $p(A) = 5.75$ so the price per ticket is \$5.75. For everyone else, that is for folks at least 6 but younger than 50, the price is \$7.25 per ticket.

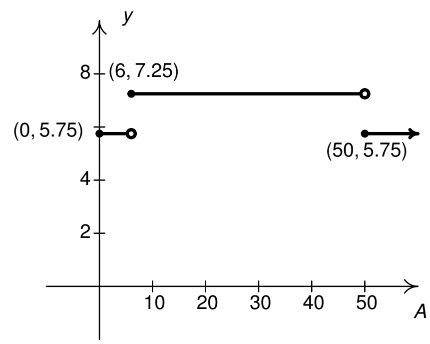
3. The independent variable here is specified as A, so we'll label our horizontal axis that way. The dependent variable remains unspecified so we can use the default y. The graph of $y = p(A)$ consists of three horizontal line pieces: the first is $y = 5.75$ for $0 \leq A < 6$, the second piece is $y = 7.25$ for $6 \leq A < 50$, and the last piece is $y = 5.75$ for $A \geq 50$.

   For the first piece, note that $A = 0$ is included in the inequality $0 \leq A < 6$ but $A = 6$ is not. For this reason, we have a point indicated at $(0, 5.75)$ but leave a hole[2] at $(6, 5.75)$. Similarly, to graph

[1] It gets much weirder than that as we explore other more complicated functions. The key is to pay attention to the precision in the definitions of the terms involved in the discussion. Stay tuned!

[2] See our discussion about holes in graphs in Example 1.1.6 in Section 1.1.

the second piece, we begin with a point at $(6, 7.25)$ and continue the horizontal line to a hole at $(50, 7.25)$. Lastly, we finish the graph with a point at $(50, 5.75)$ and continue to the right indefinitely.[3] Note the scaling on the horizontal axis compared to the vertical axis.



$$y = p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50 \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$$

□

One of the favorite piecewise-defined functions in mathematical circles is the **greatest integer of *x***, denoted by $\lfloor x \rfloor$. In Section A.1.2 we defined the set of **integers** as $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.[4] The value $\lfloor x \rfloor$ is defined to be the largest integer k with $k \leq x$. That is, $\lfloor x \rfloor$ is the unique integer k such that $k \leq x < k + 1$. Said differently, given any real number x, if x is an integer, then $\lfloor x \rfloor = x$. If not, then x lies in an interval between two integers, k and $k + 1$ and we choose $\lfloor x \rfloor = k$, the left endpoint.

Example 1.2.2. Let $\lfloor x \rfloor$ denote the greatest integer function.

1. Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$ and $\lfloor \pi + 6 \rfloor$

2. Explain how we can view $\lfloor x \rfloor$ as a piecewise-defined function and use this to graph $y = \lfloor x \rfloor$.

Solution.

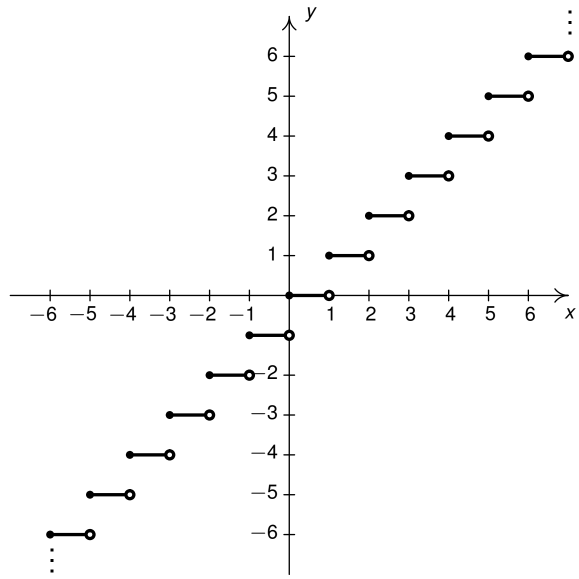
1. To find $\lfloor 0.785 \rfloor$, we note that $0 \leq 0.785 < 1$ so $\lfloor 0.785 \rfloor = 0$. Given that 117 is an integer, we have $\lfloor 117 \rfloor = 117$. To find $\lfloor -2.001 \rfloor$, we note that $-3 \leq -2.001 < -2$, so $\lfloor -2.001 \rfloor = -3$. Finally, with $\pi \approx 3.14$, we get $\pi + 6 \approx 9.14$ and $9 \leq \pi + 6 < 10$ so $\lfloor \pi + 6 \rfloor = 9$.

2. The first step in evaluating $\lfloor x \rfloor$ is to determine the interval $[k, k + 1)$ containing x so it seems reasonable that these are the intervals which produce the 'pieces'. In this case, there happen to be infinitely many pieces. The inequality '$k \leq x < k + 1$' includes the left endpoint but excludes the right endpoint, so we have points at the left endpoints of our horizontal line segments while we have holes at the right endpoints.

   A partial description of $\lfloor x \rfloor$ is given alongside a partial graph at the top of the next page. (A full description or a complete graph would require infinitely large paper!) We use the vertical dots $\vdots$ to indicate that both the rule and the graph continue indefinitely following the established pattern.[5]

[3] The domain of p is $[0, \infty)$ by definition, even though few 327 year olds are out and about these days.
[4] The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word **zahlen** means 'to count'.
[5] It is always dangerous to leave the rest of the pattern to the reader. See, for instance, this paper.

$$\lfloor x \rfloor = \begin{cases} \vdots & \\ -5 & \text{if } -5 \leq x < -4 \\ -4 & \text{if } -4 \leq x < -3 \\ -3 & \text{if } -3 \leq x < -2 \\ -2 & \text{if } -2 \leq x < -1 \\ -1 & \text{if } -1 \leq x < 0 \\ 0 & \text{if } \quad 0 \leq x < 1 \\ 1 & \text{if } \quad 1 \leq x < 2 \\ 2 & \text{if } \quad 2 \leq x < 3 \\ 3 & \text{if } \quad 3 \leq x < 4 \\ 4 & \text{if } \quad 4 \leq x < 5 \\ 5 & \text{if } \quad 5 \leq x < 6 \\ \vdots & \end{cases}$$



The graph of $y = \lfloor x \rfloor$.

$\square$

1.2.2 Linear Functions

Now that we've discussed the functions which correspond to horizontal lines, $y = b$, we move to discussing the functions which can be represented by lines of the form $y = mx + b$ where $m \neq 0$. These functions are called **linear** functions and are described below.

Definition 1.5. A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

As with Definition 1.4, in Definition 1.5, x is the independent variable, f is the function name, and both m and b are parameters. Notice that m is restricted by $m \neq 0$ for if $m = 0$ then the function $f(x) = mx + b$ would reduce to the constant function $f(x) = b$. The domain of linear functions, like that of constant functions, is specified as $(-\infty, \infty)$

Recall[6] that the form of the line $y = mx + b$ is called the slope-intercept form of the line and the slope, m, and the y-intercept $(0, b)$, are easily determined when the line is written this way. Likewise, the form of the function in Definition 1.5, $f(x) = mx + b$, is often called the **slope-intercept form** of a linear function.

The graph of a linear function is the graph of the line $y = mx + b$. Lines are uniquely determined by two points, and two points of geometric interest are the axis intercepts. We've already reminded you of the y-intercept, $(0, b)$, which is obtained by setting $x = 0$. Similarly, to find the x-intercept, we set $y = 0$

[6]or see Section A.5

and solve $mx + b = 0$ for x. We leave this to the reader in Exercise 38. In addition to having special graphical significance, axis intercepts quite often play important roles in applications involving both linear and non-linear functions. For that reason, we take the time to define them here using function notation.

Definition 1.6. Suppose f is a function represented by the graph of $y = f(x)$.

- If 0 is in the domain of f then the point $(0, f(0))$ is the y**-intercept** of the graph of $y = f(x)$.

  That is, $(0, f(0))$ is where the graph meets the y-axis.

- If 0 is in the range of f then the solutions to $f(x) = 0$ are called the **zeros** of f. If c is a zero of f then the point $(c, 0)$ is an x**-intercept** of the graph of $y = f(x)$.

  That is, $(c, 0)$ is where the graph meets the x-axis.

As is customary in this text, Definition 1.6 uses the default independent variable x, function name f, and dependent variable y, so these letters will change depending on the context. Also note that the 'zeros' of a function are the solutions to $f(x) = 0$ - so they are **real numbers**. The x-intercepts are, on the other hand, **points** on the graph. As a quick example, consider $f(x) = x - 3$. The zeros of f are found by solving $f(x) = 0$, or $x - 3 = 0$. We get one solution, $x = 3$. Therefore, $x = 3$ is the **zero** of f that corresponds graphically to the x-**intercept** $(3, 0)$.

We now turn our attention to slope. The role of slope, or more generally a 'rate of change', in Science and Mathematics cannot be overstated.[7] As you may recall, or quickly read about on page 1381, the slope of a line that has been graphed in the xy-plane is defined geometrically as follows:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x},$$

where the capital Greek letter 'Δ' denotes 'change in.'[8] In this course, it is vital that we regard the slope of a linear function as a rate of change of **function outputs** to **function inputs**. That is, given the graph of a linear function $y = f(x) = mx + b$:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{\Delta[f(x)]}{\Delta x} = \frac{\Delta\text{outputs}}{\Delta\text{inputs}}.$$

What is important to note here is that for linear functions, the rate of change m is constant for all values in the domain.[9] We'll see the importance of this statement in the upcoming examples.

Geometrically, the sign of the slope has a profound impact on the graph of the line. Recall that if the slope $m > 0$, the line rises as we read from left to right; if $m < 0$, the line falls as we read from left to right; if $m = 0$, we have a horizontal line and the graph plateaus. We define these notions more precisely for general functions in the following definition.

[7] The first half of any introductory Calculus course is about slope.

[8] More specifically, if (x_0, y_0) and (x_1, y_1) are two distinct points in the plane, then $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$.

[9] See Exercise 57 for more details.

Definition 1.7. Let f be a function defined on an interval I. Then f is said to be:

- **increasing** on I if, whenever $a < b$, then $f(a) < f(b)$. (i.e., as inputs increase, outputs **increase**.)

  NOTE: The graph of an increasing function **rises** as one moves from left to right.

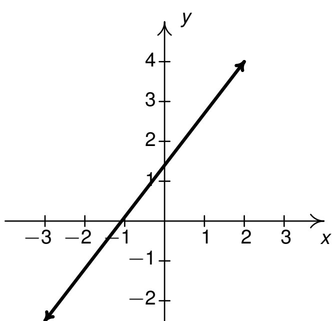
- **decreasing** on I if, whenever $a < b$, then $f(a) > f(b)$. (i.e., as inputs increase, outputs **decrease**.)

  NOTE: The graph of a decreasing function **falls** as one moves from left to right.

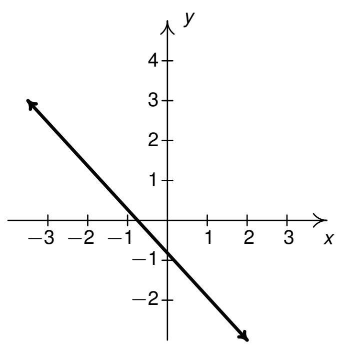
- **constant** on I if $f(a) = f(b)$ for all a, b in I. (i.e., outputs don't change with inputs.)

  NOTE: The graph of a function that is constant over an interval is a horizontal line.

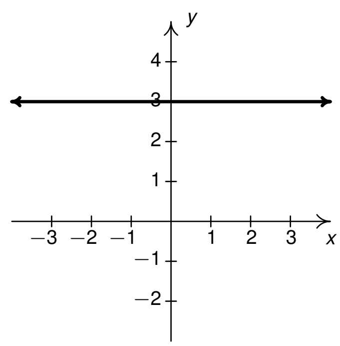
Again, as with Definition 1.6, Definition 1.7 applies to any function, not just linear and constant functions. Also, note that, like Definition 1.3, Definition 1.7 blurs the line between the function, f, and its outputs, $f(x)$, because the verbiage 'f is increasing' is really a statement about the outputs, $f(x)$. Finally, when we ask 'where' a function is increasing, decreasing or constant, we are looking for an interval of **inputs**. We'll have more to say about this in later sections, but for now, we summarize these ideas graphically below.



'increasing', $m > 0$



'decreasing', $m < 0$



'constant', $m = 0$

From the graphs above, we see that regardless if $m > 0$ or $m < 0$, the range of linear functions is $(-\infty, \infty)$. Therefore, linear functions have no maximum or minimum.[10]

Example 1.2.3. The cost, in dollars, to produce x PortaBoy[11] game systems for a local retailer is given by $C(x) = 80x + 150$ for $x \geq 0$.

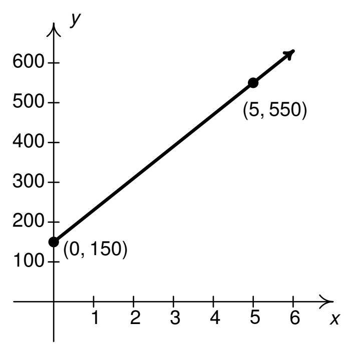
1. Find and interpret $C(0)$ and $C(5)$ and use these to graph $y = C(x)$.
2. Explain the significance of the restriction on the domain, $x \geq 0$.
3. Interpret the slope of $y = C(x)$ geometrically and as a rate of change.
4. How many PortaBoys can be produced for \$15,000?

[10] This is one of the more pedantic reasons why we distinguish between constant and linear functions. See the discussion concerning the range of a constant function on page 41.

[11] The similarity of this name to PortaJohn is deliberate.

Solution.

1. To find $C(0)$, we substitute 0 for x in the formula $C(x)$ and obtain: $C(0) = 80(0) + 150 = 150$. Given that x represents the number of PortaBoys produced and $C(x)$ represents the cost to produce said PortaBoys, $C(0) = 150$ means it costs \$150 even if we don't produce any PortaBoys at all. At first, this may not seem realistic, but that \$150 is often called the **fixed** or **start-up** cost of the venture. Things like re-tooling equipment, leasing space, or any other 'up front' costs get lumped into the fixed cost. To find $C(5)$, we substitute 5 for x in the formula $C(x)$: $C(5) = 80(5) + 150 = 550$. This means it costs \$550 to produce 5 PortaBoys for the local retailer. These two computations give us two points on the graph: $(0, C(0))$ and $(5, C(5))$. Along with the domain restriction $x \geq 0$, we get:



$y = C(x)$

2. In this context, x represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems,[12] so $x \geq 0$.

3. The cost function $C(x) = 80x + 150$ is in slope-intercept form so we recognize the slope as the coefficient of x, $m = 80$. With $m > 0$, the function C is always increasing. This means that it costs more money to make more game systems. To interpret the slope as a rate of change, we note that the output, $C(x)$, is the cost in dollars, while the input, x, is the number of PortaBoys produced:

$$m = 80 = \frac{80}{1} = \frac{\Delta[C(x)]}{\Delta x} = \frac{\$80}{1 \text{ PortaBoy produced}}.$$

   Hence, the cost to produce PortaBoys is increasing at a rate of \$80 per PortaBoy produced. This is often called the **variable cost** for the venture.

4. To find how many PortaBoys can be produced for \$15,000, we solve $C(x) = 15000$, which means $80x + 150 = 15000$. This yields $x = 185.625$. We can produce only a whole number amount of PortaBoys so we are left with two options: produce 185 or 186 PortaBoys. Given that $C(185) = 14950$ and $C(186) = 15030$, we would be over budget if we produced 186 PortaBoys. Hence, we can produce 185 PortaBoys for \$15,000 (with \$50 to spare). □

[12] Actually, it makes no sense to produce a fractional part of a game system, either, which we'll discuss later in this example.

A couple of remarks about Example 1.2.3 are in order. First, if x represents the number of PortaBoy game systems being produced, then x can really only take on whole number values. We will revisit this scenario in Section 1.4 where we will see how the approach presented here allows us to use more elegant techniques when analyzing the situation than a discrete data set would allow.[13]

Second, once we know that the variable cost is \$80 per PortaBoy, we can revisit a computation we did earlier in the example. We computed $C(185) = 14950$ and needed to compute $C(186)$. With 186 being just one more PortaBoy than 185, we can use the variable cost to get

$$C(186) = C(185) + 80(1) = 14950 + 80 = 15030,$$

which agrees with our earlier computation.[14] If we wanted to find $C(300)$, we could do something similar. Using $300 - 185 = 115$, we can find $C(300)$ as follows:

$$C(300) = C(185) + 80(115) = 14950 + 9200 = 24150.$$

In general, we could rewrite $C(x) = C(185) + 80(x - 115)$. This same reasoning shows that for any x_0 in the domain of C, we have $C(x) = C(x_0) + 80(x - x_0)$ - a fact we invite the reader to verify.[15]

Indeed, the computations above are at the heart of what it means to be a linear function: linear functions change at a constant rate known as the slope. To better see this algebraically, recall that given a point (x_0, y_0) on a line along with the slope, m, the **point-slope form of the line** is: $y - y_0 = m(x - x_0)$.[16] Rewriting, we get $y = y_0 + m(x - x_0)$ and setting $y = f(x)$ and $y_0 = f(x_0)$ yields:

Equation 1.1. The **point-slope form** of a linear function is

$$f(x) = f(x_0) + m(x - x_0)$$

A few remarks are in order. First note that if the point $(x_0, f(x_0))$ is the y-intercept $(0, b)$, Equation 1.1 immediately reduces to the slope-intercept form of the line: $f(x) = f(x_0) + m(x - x_0) = b + m(x - 0) = mx + b$, so you can use Equation 1.1 exclusively from this point forward.[17]

Second, if we write $\Delta x = x - x_0$, then $x = x_0 + \Delta x$ so we can rewrite Equation 1.1 as follows:

$$\begin{array}{ccccc} f(x_0 + \Delta x) & = & f(x_0) & + & m\Delta x \\ \text{(new output)} & = & \text{(known output)} & + & \text{(change in outputs)} \end{array}$$

In other words, changing the **input** by Δx results in changing the **output** by $m\Delta x$. This tracks since

$$m\Delta x = \frac{\Delta[f(x)]}{\Delta x}\Delta x = \Delta[f(x)] = \Delta\text{outputs}.$$

[13] This is an example of using a 'continuous' variable to model a 'discrete' scenario. Contrast this with the discussion following Example 1.1.1 in Section 1.1.

[14] The cost to produce 'just one more item' is called the **marginal cost**. The difference between variable and marginal costs in this case are the units used: the variable cost is \$80 per Portaboy whereas the marginal cost is simply \$80.

[15] In the case $x_0 = 0$, this formula reduces to $C(x) = C(0) + 80(x - 0) = 150 + 80x = 80x + 150$. To show the formula in general, consider $C(x_0) = 80x_0 + 150 \ldots$

[16] See Section A.5 for a review of this form.

[17] In other words, the slope intercept form of a line is just a special case of the point-slope form.

The fact that we can write $\Delta\text{outputs} = m\Delta x$ for any choice of x_0 is another way to see that for linear functions, the rate of change is constant. That is, the rate of change, m, is the same for all values x_0 in the domain. We'll put Equation 1.1 to good use in the next example.

Example 1.2.4. The local retailer in Example 1.2.3 is trying to mathematically model the relationship between the number of PortaBoy systems sold and the price per system. Suppose 20 systems were sold when the price was \$220 per system but when the systems went on sale for \$190 each, sales doubled.

1. Find a formula for a linear function p which represents the price $p(x)$ as a function of the number of systems sold, x. Graph $y = p(x)$, find and interpret the intercepts, and determine a reasonable domain for p.

2. Interpret the slope of $p(x)$ in terms of price and game system sales.

3. If the retailer wants to sell 150 PortaBoys next week, what should the price be?

4. How many systems would sell if the price per system were set at \$150?

Solution.

1. We are asked to find a linear function $p(x)$ ostensibly because the retailer has only two data points and two points are all that is needed to determine a unique line. We know that 20 PortaBoys were sold when the price was 220 dollars and double that, so 40 units, were sold when the price was 190 dollars. Using the language of function notation, these statements translate to $p(20) = 220$ and $p(40) = 190$, respectively. We first find the slope

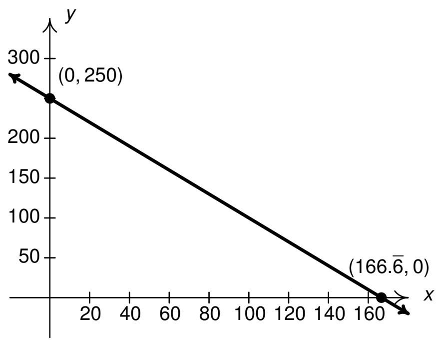
$$m = \frac{\Delta[p(x)]}{\Delta x} = \frac{190 - 220}{40 - 20} = \frac{-30}{20} = -1.5$$

and then substitute it and a pair $(x_0, p(x_0))$ into the point-slope formula. We have two choices: $x_0 = 20$ and $p(x_0) = 220$ or $x_0 = 40$ and $p(x_0) = 190$. We'll choose the former and invite the reader to use the latter - both will result in the same simplified expression. The point-slope formula yields

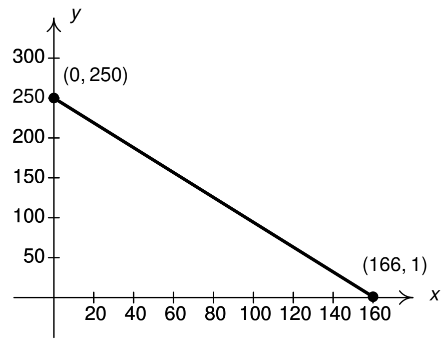
$$p(x) = p(x_0) + m(x - x_0) = 220 + (-1.5)(x - 40)$$

which simplifies to $p(x) = -1.5x + 250$. (To check this algebraically, we can verify that $p(20) = 220$ and $p(40) = 190$.) To find the y-intercept of the graph, we substitute $x = 0$ and find $p(0) = 250$. Hence our y-intercept is $(0, 250)$. To find the x-intercept, we set $p(x) = 0$. Solving $-1.5x + 250 = 0$ gives $x = 166.\overline{6}$, so our x-intercept is $(166.\overline{6}, 0)$.[18] The graph on the left is that of the line $y = -1.5x + 250$.

[18]The exact value is $x = \frac{500}{3}$. Recall that the bar over the 6 indicates that the decimal repeats. See page 1322 for details.



$y = -1.5x + 250$



$y = p(x)$

To determine a reasonable domain for p, we certainly require $x \geq 0$, because we can't sell a negative number of game systems.[19] Next, we require $p(x) \geq 0$, otherwise we'd be **paying** customers to 'buy' PortaBoys. Solving $-1.5x + 250 \geq 0$ results in $x \leq 166.\overline{6}$. This shouldn't be too surprising since our graph passes through the x-axis at $(166.\overline{6}, 0)$, going from positive y-values (hence, positive $p(x)$ values) to negative y (hence negative $p(x)$ values).[20]

Given that x represents the number of PortaBoys sold, we need to choose to end the domain at either $x = 166$ or $x = 167$. We have that $p(166) = 1 > 0$ but $p(167) = -0.5 < 0$ so we settle on the domain $[0, 166]$. Our final answer is $p(x) = -1.5x + 250$ restricted to $0 \leq x \leq 166$ which is graphed above on the right.

2. The slope $m = -1.5$ represents the rate of change of the price of a system with respect to sales of PortaBoys. The slope is negative so we have that the price is **decreasing** at a rate of \$1.50 per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every \$1.50 drop in price.)

3. To determine the price which will move 150 PortaBoys, we find $p(150) = -1.5(150) + 250 = 25$. That is, the price would have to be \$25 per system.

4. If the price of a PortaBoy were set at \$150, we'd have $p(x) = 150$, or $-1.5x + 250 = 150$. This yields $-1.5x = -100$ or $x = 66.\overline{6}$. Again our algebraic solution lies between two whole numbers, so we find $p(66) = 151$ and $p(67) = 149.5$. If the price were set at \$150, we'd sell 66 systems, since to sell 67 systems, we'd have to drop the price just under \$150. □

The function p in Example 1.2.4 is called the **price-demand** function (or, sometimes called more simply a 'demand function') because it returns the price $p(x)$ associated with a certain demand x - that is, how many products will sell.[21] These functions, along with cost functions like the one in Example 1.2.3, will be revisited in Example 1.4.3.

Our next two examples focus on writing formulas for piecewise-defined functions, the second of which models a real-world situation.

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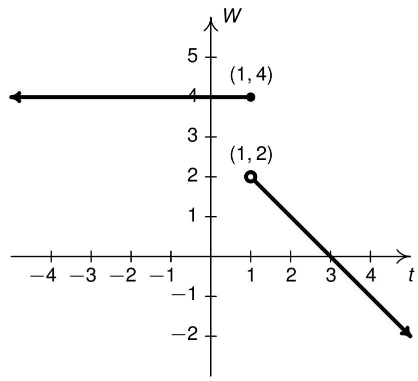
[19] ignoring returns, that is.

[20] We'll discuss these sorts of connections in greater depth in Section 1.3.

[21] It may seem counter-intuitive to express price as a function of demand. Shouldn't the price determine how many systems people will buy? We will address this issue later.

Example 1.2.5. Find a formula for the function L graphed below.



$W = L(t)$

Solution. From the graph of $W = L(t)$ we see that there are two distinct pieces. Taking note of the point at $(1, 4)$, we get $L(t) = 4$ for $t \leq 1$. To represent L for $t > 1$, we use the point-slope form of a linear function: $L(t) = L(t_0) + m(t - t_0)$. The only 'point' labeled with this part of the graph is the hole at $(1, 2)$ and it isn't technically part of the graph, so we will avoid using it.[22] Instead, we infer from the graph two other points: $(2, 1)$ and $(3, 0)$. We get the slope to be

$$m = \frac{\Delta W}{\Delta t} = \frac{\Delta[L(t)]}{\Delta t} = \frac{3 - 2}{0 - 1} = -1.$$

Next, we choose a point to plug into $L(t) = L(t_0) + m(t - t_0)$. We have two options: $t_0 = 2$ and $L(t_0) = 1$ or $t_0 = 3$ and $L(t_0) = 0$. Using the latter, we get $L(t) = 0 + (-1)(t - 3)$, or $L(t) = -t + 3$. Putting this together with the first part, we get:

$$L(t) = \begin{cases} 4 & \text{if } t \leq 1 \\ -t + 3 & \text{if } t > 1 \end{cases}$$

Note that when $t = 1$ is substituted into the expression $-t + 3$, we get 2, so the hole at $(1, 2)$ checks.[23] □

Example 1.2.6. A popular Fōn-i smartphone carrier offers the following smartphone data plan: use any amount of data up to and including 4 gigabytes for \$60 per month with an 'overage' charge of \$5 per gigabyte. Determine a formula that computes the cost in dollars as a function of using g gigabytes of data per month. Graph your answer.

Solution. It is clear from context that we are to use the variable g (for 'g'igabytes) as the independent variable. We are asked to compute the cost so it seems natural to name the function C. Hence, we are after a formula for $C(g)$. Knowing that g represents the amount of data used each month, we must have $g \geq 0$. In order to get a feel for the formula for $C(g)$, we can choose some specific values for g and determine the cost, $C(g)$. For example, if we use no data at all, 1 gigabyte of data, or 3.796 gigabytes

[22] We actually could use the point $(1, 2)$ to find the equation of the **line** containing $(1, 2)$ and, say $(3, 0)$, which is $y = -t + 3$. It's just that the graph of $L(t)$ and the line $y = -t + 3$ only agree for $t > 1$, so it would be incorrect to write $L(1) = 2$.

[23] Alternatively, for t values larger than 1 but getting closer and closer to 1, $L(t) \approx 2$.

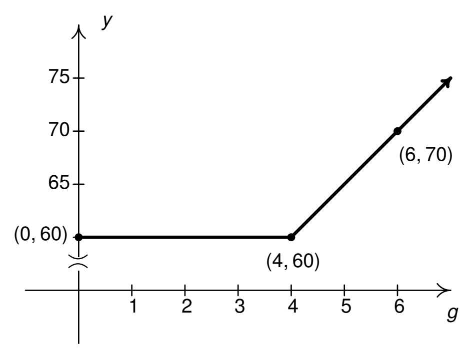
of data, the cost is the same: \$60. Indeed, per the plan, for any amount of data up to and including 4 gigabytes, the cost is \$60.

Translating this to function notation means $C(0) = 60$, $C(1) = 60$, $C(3.796) = 60$, and, in general, $C(g) = 60$ for $0 \leq g \leq 4$. What happens if we use more than 4 gigabytes? Let's say we use 6 gigabytes. Per the plan, we are charged \$60 for the first 4 and then \$5 for each gigabyte over 4. Using 6 gigabytes means that we are 2 gigabytes over and our overage charge is (\$5)(2) = \$10. The total cost is the base plus the overages or \$60 + \$10 = \$70. In general, if $g > 4$, the expression $(g - 4)$ computes the amount of data used over 4 gigabytes. Our base plus overage then comes to: $60 + 5(g - 4) = 5g + 40$. Putting this together with our previous work, we get

$$C(g) = \begin{cases} 60 & \text{if } 0 \leq g \leq 4 \\ 5g + 40 & \text{if } g > 4 \end{cases}$$

To graph C, we graph $y = C(g)$. For $0 \leq g \leq 4$, we have the horizontal line $y = 60$ from $(0, 60)$ to $(4, 60)$. For $g > 4$, we have the line $y = 5g + 40$. Even though the inequality $g > 4$ is strict, we nevertheless substitute $g = 4$ into the formula $y = 5g + 40$ and get $y = 60$. Normally, this would produce a hole at $(4, 60)$, but in this case, the point $(4, 60)$ is already on the graph from the first piece of the function. Essentially, the point $(4, 60)$ from $C(g) = 60$ for $0 \leq g \leq 4$ 'plugs' the hole from $C(g) = 5g + 40$ when $g > 4$.

We are graphing a line so we need to plot just one more point to determine the graph. From our work above, we know $C(6) = 70$, so we use $(6, 70)$ as our second point. Our graph is below. As with the graphs shown on page 11 from Example 1.1.1, we use '$\asymp$' to denote a break in the vertical axis in order to better display the graph.



$y = C(g)$

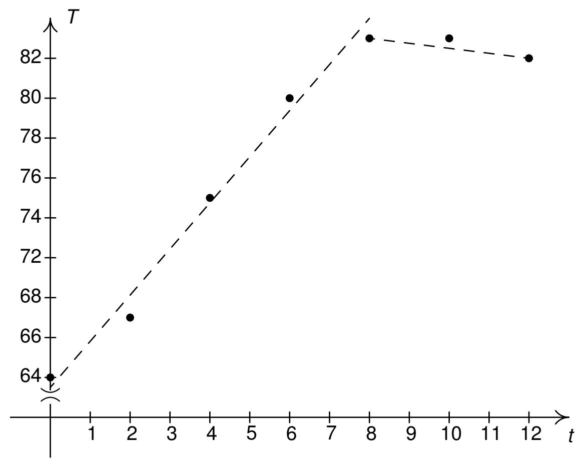
□

1.2.3 Linear Regression

We have demonstrated in this section that constant, linear, and piecewise combinations of these two function types can be used to model a variety of phenomena inspired by real-world situations. What happens, as if often the case in real-world situations, when we are given data sets that are not precisely linear, but still have a definite linear trend? An example of this is Skippy's time and temperature data from Example 1.1.1 in Section 1.1.

In that example, t represented the time (number of hours after 6 a.m.) and T represented the outdoor temperature in degrees Fahrenheit. The data Skippy collected along with a plot of the function $T = f(t)$ are given below. Even though the data points as t varies from $t = 0$ to $t = 8$ do not all lie on the same line - a fact we could prove analytically by checking slopes - there does appear to be a linear **trend** evident. The same can be said for the data as t varies from $t = 8$ to $t = 12$. As we'll see, there are statistical methods which can produce linear functions that are in some sense 'closest' to all of the data, and they are represented below by the dashed lines below on the right.

t: hours after 6 a.m.	T: temperature °F
0	64
2	67
4	75
6	80
8	83
10	83
12	82



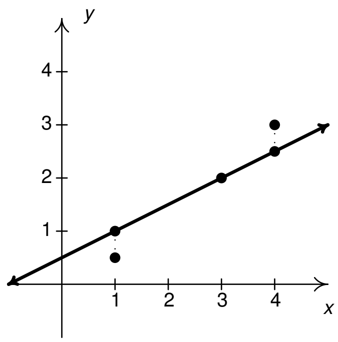
The graph of $T = f(t)$.

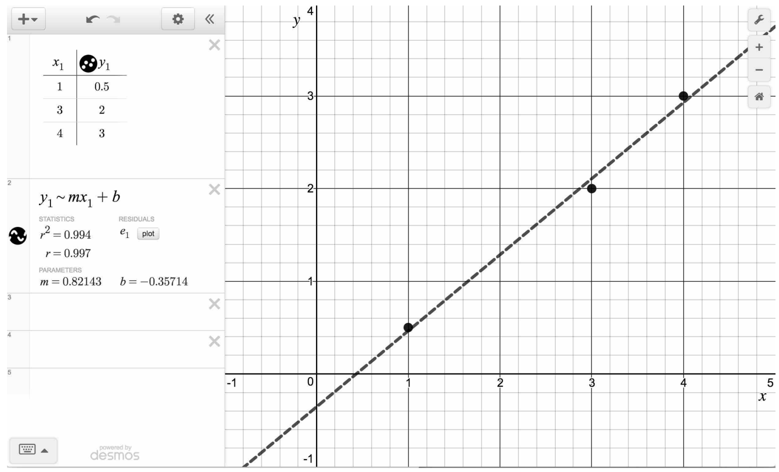
How do we measure how 'close' a set of points is to a given line? Let's leave Skippy's data for the moment and focus on a smaller data set. Suppose we collected three data points: $\{(1, 0.5), (3, 2), (4, 3)\}$. At the top of the next page (on the left) we plot these points along with the line $y = 0.5x + 0.5$. The way we measure how close the line is to these points is by computing the **total squared (vertical) error** between the data points and the line as follows. For each of our data points, we find the vertical distance between the point and the line. To accomplish this, we need to find a point on the line directly above or below each data point. In other words, we need a point on the line with the same x-coordinate as our data point.

For example, to find the point on the line directly above $(1, 0.5)$, we plug $x = 1$ into $y = 0.5x + 0.5$ and we get the point $(1, 1)$. Similarly, we find $(3, 2)$ is on the line already and $(4, 2.5)$ is the point on the line directly beneath $(4, 3)$. We find the total squared error E by taking the sum of the squares of the differences of the y-coordinates of each data point and its corresponding point on the line. For the data and line in this discussion $E = (0.5 - 1)^2 + (2 - 2)^2 + (3 - 2.5)^2 = 0.5$.

Using advanced mathematical machinery,[24] it is possible to find the line which results in the lowest value of E. This line is called the **least squares regression line**, or sometimes the 'line of best fit'. The formula for the line of best fit requires notation we won't present until Chapter 10, so we will revisit it then. Most graphing utilities have a built-in regression feature, so at this point we turn the computations over to the technology. A screenshot from desmos is given on the right at the top of the next page.

[24] like Calculus or Linear Algebra . . .





Our graphing utility produces the model[25] $y = mx + b$ where the slope is $m \approx 0.821$ and the y-coordinate of the y-intercept is $b \approx -0.357$. The value r is the **correlation coefficient** and is a measure of how close the data is to being on the same line. The closer $|r|$ is to 1, the better the linear fit.[26] Having $r \approx 0.997$ tells us that the points have a strong, positive correlation - that is, they are very close to being on a line with a positive slope, namely $y = 0.821x - 0.357$. Indeed, the total squared error between our data set and this line is $E \approx 0.018$. The mathematics tells us that this is the smallest we can get E by modifying the parameters m and b, even though none of the data points actually lie on the line.

Now that we have this new mathematical machinery, let's revisit Skippy's time and temperature data.

Example 1.2.7.

1. Use a graphing utility to find best fit linear models for each of the data sets below. Comment on the fit and interpret the slope of each.

t: hours after 6 a.m.	T: temperature °F
0	64
2	67
4	75
6	80
8	83

t: hours after 6 a.m.	T: temperature °F
8	83
10	83
12	82

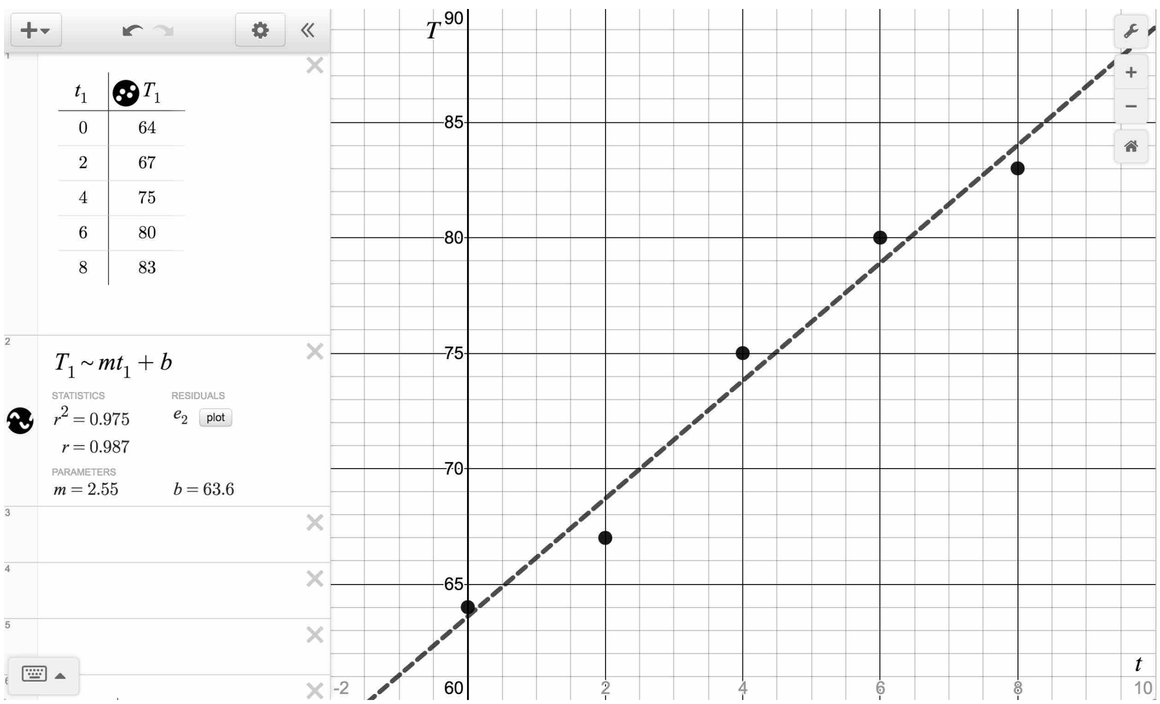
2. Use your models to predict the temperature at 7 a.m. and 3 p.m., rounded to one decimal place.

[25]We chose to use three decimal places for the approximations in this demonstration. How many you get to use in reality varies from one application to another.
[26]The value r^2 is called the **coefficient of determination** and is also a measure of the goodness of fit. We refer the interested reader to a course in Statistics to explore the significance of r and r^2.

Solution.

1. For our first set of data, we get the line $T = F(t) = 2.55t + 63.6$. The value $r = 0.987$ tells us that it is a fairly good fit and we see this graphically, too.[27] Thus we can be confident in using this model to predict the temperature during between the hours of 6 a.m. and 2 p.m. with reasonable accuracy.

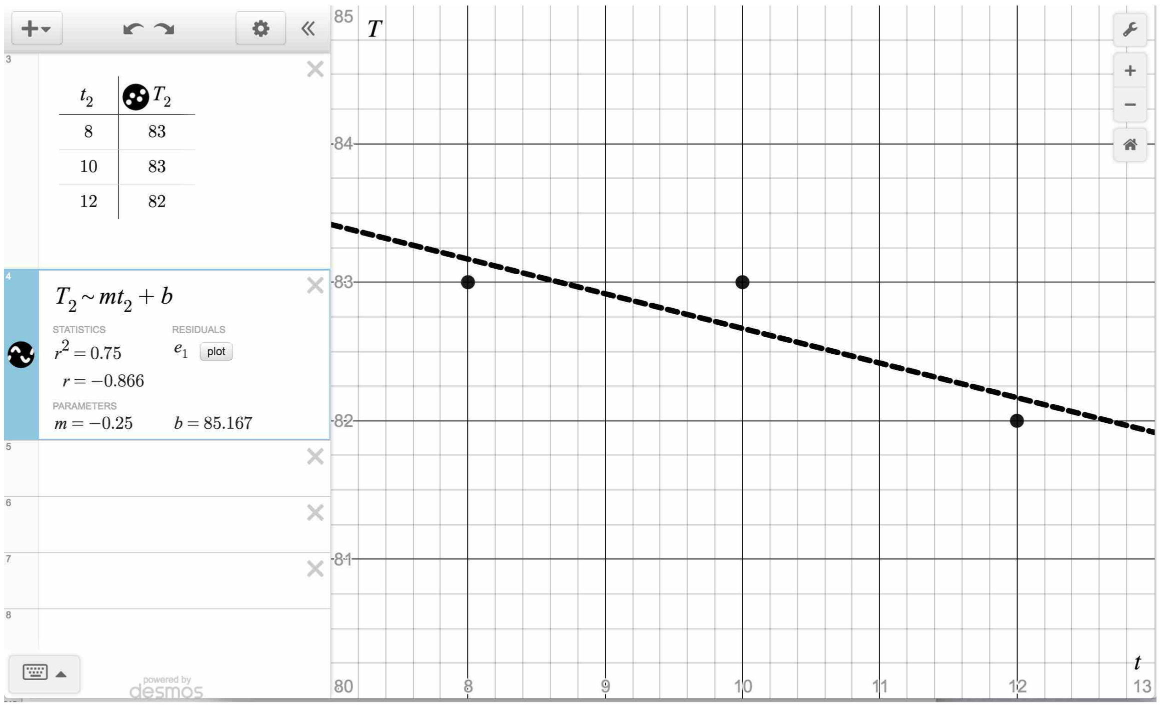
   To interpret the slope, we recognize t as the independent variable (input) and T as the dependent variable (output), so the slope $m = \frac{\Delta T}{\Delta t}$ is the rate of change of temperature with respect to time. In this case, $m = 2.55$ means that the temperature is increasing (getting warmer) at a rate of 2.55°F per hour. A screenshot from Desmos is given below.



   For the second set of data, we get $T = G(t) = -0.25t + 85.167$ and we have $r = -0.866$. Here, the negative sign on r indicates a negative correlation which means our line has a negative slope.[28] While the fit looks OK, it certainly isn't as strong as with the first data set, so using this model to predict the temperature between 2 p.m. and 6 p.m. (let alone beyond) is a bit risky.

   The slope in this case is $m = -0.25$ which corresponds to the temperature decreasing (getting cooler) at a rate of 0.25°F per hour. That's what a negative correlation means - an increase in input (more time passes) yields a decrease in output (cooler temperatures). As screenshot from Desmos is given at the top of the next page.

[27] We use F as the name of the function here to distinguish it from f - the function determined solely by the given set of data.
[28] We use G as the function name here to distinguish it from the given function f and the regression for the first data set, F.



2. The time 7 a.m. corresponds to $t = 1$. This falls between $t = 0$ and $t = 8$ so we use our first model. Substituting $t = 1$ gives $T = F(1) = 2.55(1) + 63.6 \approx 66.2$. Therefore, the model predicts the temperature to be 66.2°F at 7 a.m.. Likewise, 3 p.m. corresponds to $t = 9$. This is greater than 8, so we use the second model: $T = G(9) = -0.25(9) + 85.167 \approx 82.9$. The model predicts the temperature at 3 p.m. to be 82.9°F. Based on the goodness of fit of each model, we have more confidence in the former prediction than in the latter. □

Examples 1.2.3, 1.2.4 and 1.2.7 (among others) represent three different levels of mathematical modeling. In Example 1.2.3, the mathematical model (the cost function) was provided and our task was to use the model to **interpret** the mathematics in that context. In Example 1.2.4, we were given a minimal amount of information, namely, two data points, and then asked to **construct** a model which fit those data exactly. Lastly, in Example 1.2.7, we were given several data points and we used statistical methods to construct a **best fit** model to the data.

The validity of the models rests on the validity of the underlying assumptions used to create the models. For instance, is there any reason to assume a price-demand function would be linear? Is it reasonable to assume that the temperature changes at a constant rate? These are questions for economists and scientists. Mathematicians often take on a role of equal parts translator and prophet: they codify ideas into formulas and then use them to make predictions about yet-to-be observed phenomena.

1.2.4 The Average Rate of Change of a Function

As mentioned earlier in the section, the concepts of slope and the more general rates of change are important concepts not just in Mathematics, but also in other fields. Many important phenomena are modeled using non-linear functions, and while the rates of change of these functions are not constant, we can sample the function at two points and compute what is known as an **average rate of change** between them to give some sense as to the function's behavior over that interval.[29]

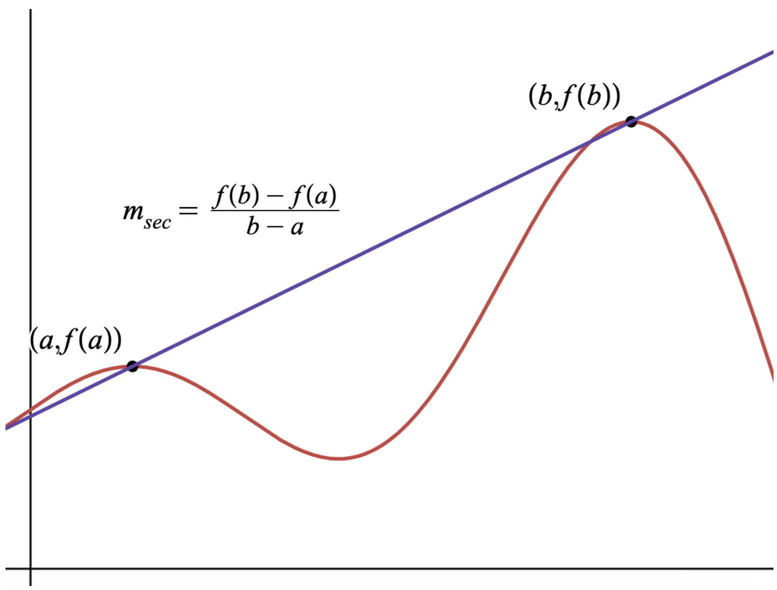
Definition 1.8. Let f be a function defined on the interval $[a, b]$.

The **average rate of change** of f over $[a, b]$ is defined as:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Geometrically, the average rate of change is the slope of the line[a] containing $(a, f(a))$ and $(b, f(b))$.

[a]This line is called a **secant line**.



The graph of a function f along with the secant line through the points $(a, f(a))$ and $(b, f(b))$.

As with Definitions 1.3 and 1.7, the wording in Definition 1.8, while referring to the function f, is really making a statement about its outputs $f(x)$.

If f is increasing over $[a, b]$, then the average rate of change will be positive. Likewise, if f is decreasing or constant, the average rate of change will be negative or 0, respectively. (Think about this for a moment.) However, as the next example demonstrates, the converses of these statements aren't always true.[30]

Example 1.2.8. The formula $s(t) = -5t^2 + 100t$ for $0 \leq t \leq 20$ gives the height, $s(t)$, measured in feet, of a model rocket above the Moon's surface as a function of the time t, in seconds after lift-off.

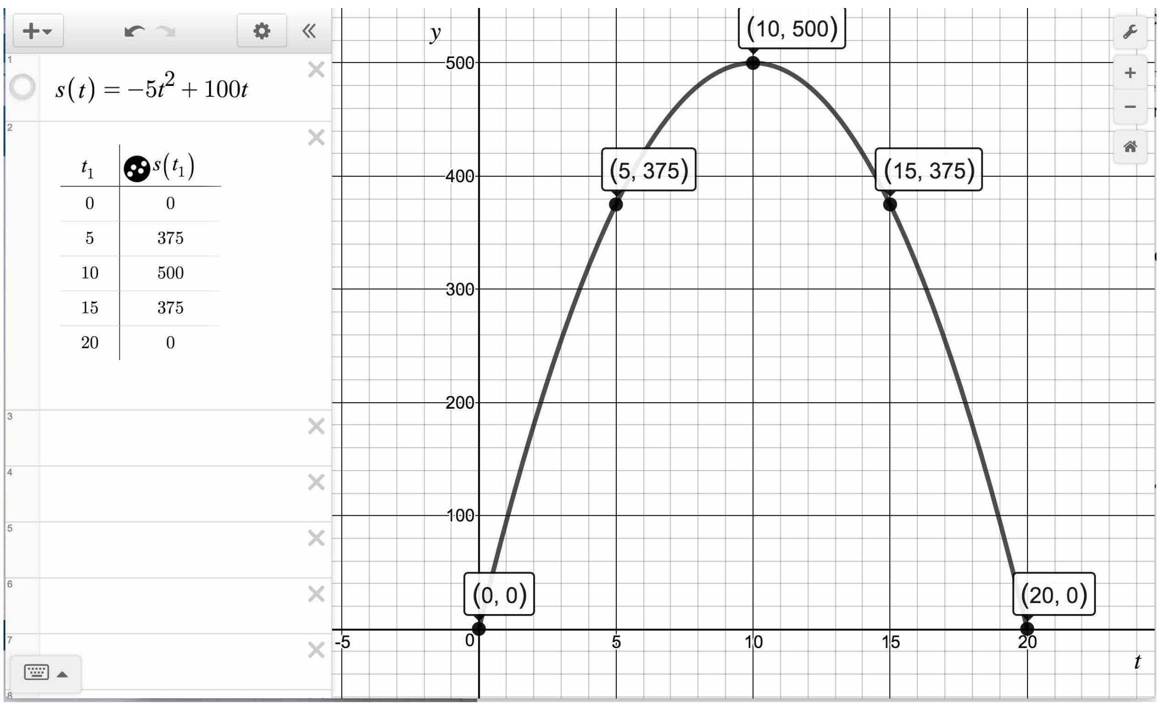
[29]We are basically pretending that the function is linear on a short interval to see what we can say about its behavior.

[30]For example, the average rate of change over an interval could be positive yet the function could decrease over part of that interval and then increase on a different part.

1. Find $s(0)$, $s(5)$, $s(10)$, $s(15)$ and $s(20)$ and use these along with a graphing utility to graph $y = s(t)$.
2. State the range of s and interpret the extrema, if any exist.
3. Find and interpret the t- and y-intercepts.
4. Find and interpret the interval(s) over which s is increasing, decreasing or constant.
5. Find and interpret the average rate of change of s over the intervals $[0, 5]$, $[5, 10]$, $[10, 20]$ and $[5, 15]$.

Solution.

1. To find $s(0)$, we substitute $t = 0$ into the formula for $s(t)$: $s(0) = -5(0)^2 + 100(0) = 0$. Similarly, $s(5) = -5(5)^2 + 100(5) = -5(25) + 500 = -125 + 500 = 375$. Continuing, we obtain: $s(10) = 500$, $s(15) = 375$ and $s(20) = 0$. We construct a table of values and with a graphing utility we obtain:



2. Projecting the graph to the y-axis, we see that the range of s is $[0, 500]$ so the minimum of s is 0 and the maximum is 500. This means that the rocket at some point is on the surface of the Moon and reaches its highest altitude of 500 feet above the lunar surface.
3. The first intercept we see is $(0, 0)$ which is both a t- and a y-intercept. Since t is the time after lift-off and $y = s(t)$ is the height above the Moon's surface, the point $(0, 0)$ means that the model rocket was launched ($t = 0$) from the Moon's surface ($s(t) = 0$). The remaining intercept, $(20, 0)$, is another t-intercept. This means that 20 seconds after lift-off ($t = 20$), the model rocket returns to the Moon's surface ($s(t) = 0$). That is, the 'time of flight' of the model rocket is 20 seconds.

4. Referring to Definition 1.7, s increases over the interval $[0, 10]$, since for those values of t, as we read from left to right, the graph of the function is rising meaning the y values (hence $s(t)$ values) are getting larger. Thus the model rocket is heading upwards for the first 10 seconds of its flight. We find that s decreases over the interval $[10, 20]$, indicating once it has reached its highest altitude of 500 feet 10 seconds into the flight, the rocket begins to fall back to the surface of the Moon, landing 20 seconds after lift-off.

5. To find the average rate of change of s over the interval $[0, 5]$ we compute

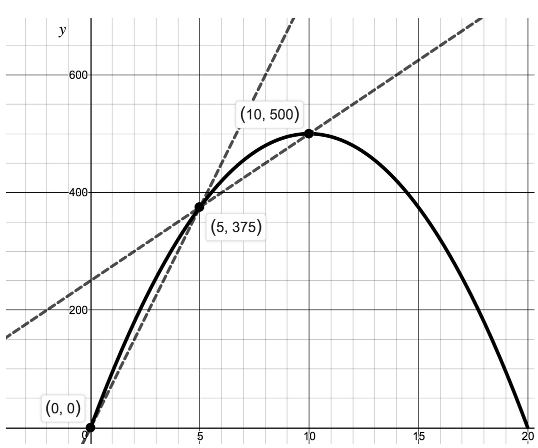
$$\frac{\Delta[s(t)]}{\Delta t} = \frac{s(5) - s(0)}{5 - 0} = \frac{375 \text{ feet}}{5 \text{ seconds}} = 75 \text{ feet per second.}$$

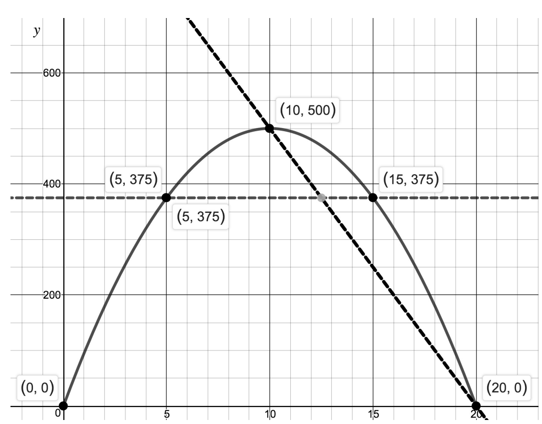
   In other words, the height is **increasing** at an **average rate** of 75 feet per second during the first 5 seconds of flight. The rate here is called the **average velocity** of the rocket over this interval. Velocity differs from speed in that velocity comes with a direction. In this case, a positive velocity indicates that the rocket is traveling **upwards**, since when s is increasing, the model rocket is climbing higher.

   Similarly, the average rate of change of s over the interval $[5, 10]$ works out to be 25. This means that the average velocity over the next 5 seconds of the flight has slowed to 25 feet per second. The model rocket is still, on average, traveling upwards, albeit more slowly than before.

   Over the interval $[10, 20]$, the average rate of change of s works out to be -50. This means that, on average, the rocket is **falling** at a rate of 50 feet per second. The rocket has managed to fall from its highest point 500 feet above the surface of the Moon back to the Moon's surface in 10 seconds so this makes sense. Finally, the average rate of change of s over $[5, 15]$ is 0. This means that the model is the same height above the ground after 5 seconds (375 feet) as it is after 15 seconds.

   Geometrically, the average rate of change of a function over an interval can be interpreted as the slope of a secant line. Below on the left is a dotted line containing $(0, 0)$ and $(5, 375)$ (which has slope 75) along with a dotted line containing the points $(5, 375)$ and $(10, 500)$ (which has slope 25). Visually, the lines help demonstrate that, while s is increasing over $[0, 10]$, the rate of increase is slowing down as t nears 10.





The graph above on the right depicts a dotted line through $(10, 500)$ and $(20, 0)$ indicating a net decrease over that interval. We also have a horizontal line (0 slope) containing the points $(5, 375)$ and $(15, 375)$, which shows no net change between those two points, despite the fact that the rocket rose to its maximum height then began its descent during the interval $[5, 15]$. $\square$

An important lesson from the last example is that average rates of change give us a snapshot of what is happening **at the endpoints** of an interval, but not necessarily what happens **over the course** of the interval. Calculus gives us tools to compute slopes **at** points which correspond to **instantaneous** rates of changes. While we don't quite have the machinery to properly express these ideas, we can hint at them in the Exercises. Speaking of exercises . . .

1.2.5 Exercises

In Exercises 1 - 6, graph the function. Find the slope and axis intercepts, if any.

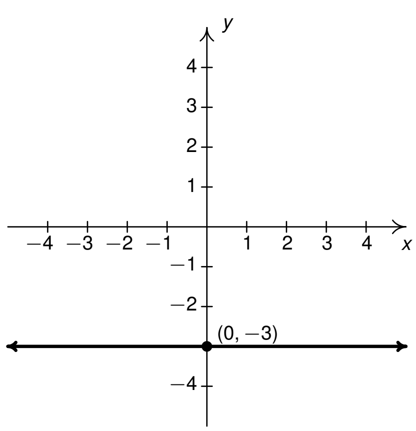
1. $f(x) = 2x - 1$
2. $g(t) = 3 - t$
3. $F(w) = 3$
4. $G(s) = 0$
5. $h(t) = \frac{2}{3}t + \frac{1}{3}$
6. $j(w) = \dfrac{1-w}{2}$

In Exercises 7 - 10, graph the function. Find the domain, range, and axis intercepts, if any.

7. $f(x) = \begin{cases} 4 - x & \text{if} \quad x \leq 3 \\ 2 & \text{if} \quad x > 3 \end{cases}$
8. $g(x) = \begin{cases} 2 - x & \text{if} \quad x < 2 \\ x - 2 & \text{if} \quad x \geq 2 \end{cases}$
9. $F(t) = \begin{cases} -2t - 4 & \text{if} \quad t < 0 \\ 3t & \text{if} \quad t \geq 0 \end{cases}$
10. $G(t) = \begin{cases} -3 & \text{if} \quad t < 0 \\ 2t - 3 & \text{if} \quad 0 < t < 3 \\ 3 & \text{if} \quad t > 3 \end{cases}$
11. The **unit step function** is defined as $U(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$
    (a) Graph $y = U(t)$.
    (b) State the domain and range of U.
    (c) List the interval(s) over which U is increasing, decreasing, and/or constant.
    (d) Write $U(t-2)$ as a piecewise defined function and graph.

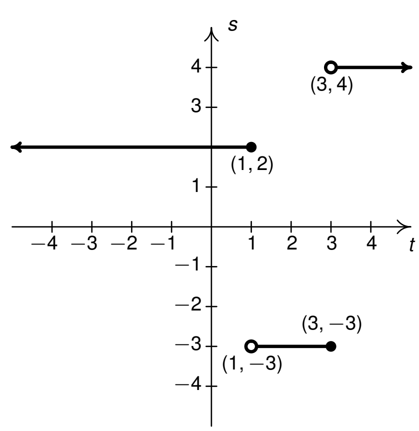
In Exercises 12 - 15, find a formula for the function.

12.



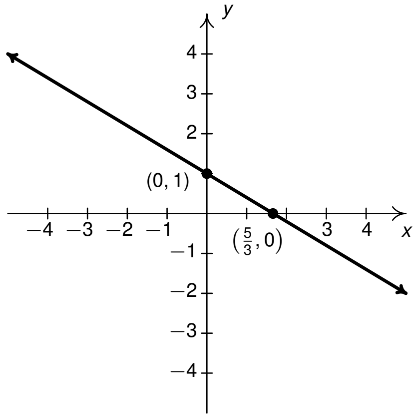
$y = f(x)$

13.



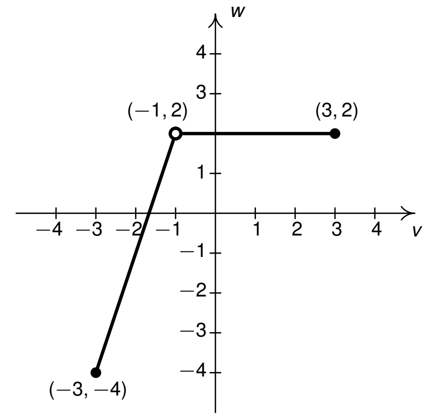
$s = F(t)$

14.



$y = L(x)$

15.



$w = g(v)$

16. For n copies of the book *Me and my Sasquatch*, a print on-demand company charges $C(n)$ dollars, where $C(n)$ is determined by the formula

$$C(n) = \begin{cases} 15n & \text{if} \quad 1 \leq n \leq 25 \\ 13.50n & \text{if} \quad 25 < n \leq 50 \\ 12n & \text{if} \quad n > 50 \end{cases}$$

   (a) Find and interpret $C(20)$.
   (b) How much does it cost to order 50 copies of the book? What about 51 copies?
   (c) Your answer to 16b should get you thinking. Suppose a bookstore estimates it will sell 50 copies of the book. How many books can, in fact, be ordered for the same price as those 50 copies? (Round your answer to a whole number of books.)

17. An on-line comic book retailer charges shipping costs according to the following formula

$$S(n) = \begin{cases} 1.5n + 2.5 & \text{if} \quad 1 \leq n \leq 14 \\ 0 & \text{if} \quad n \geq 15 \end{cases}$$

   where n is the number of comic books purchased and $S(n)$ is the shipping cost in dollars.

   (a) What is the cost to ship 10 comic books?
   (b) What is the significance of the formula $S(n) = 0$ for $n \geq 15$?

18. The cost in dollars $C(m)$ to talk m minutes a month on a mobile phone plan is modeled by

$$C(m) = \begin{cases} 25 & \text{if} \quad 0 \leq m \leq 1000 \\ 25 + 0.1(m - 1000) & \text{if} \quad m > 1000 \end{cases}$$

   (a) How much does it cost to talk 750 minutes per month with this plan?
   (b) How much does it cost to talk 20 hours a month with this plan?
   (c) Explain the terms of the plan verbally.

19. Jeff can walk comfortably at 3 miles per hour. Find an expression for a linear function $d(t)$ that represents the total distance Jeff can walk in t hours, assuming he doesn't take any breaks.

20. Carl can stuff 6 envelopes per *minute*. Find an expression for a linear function $E(t)$ that represents the total number of envelopes Carl can stuff after t *hours*, assuming he doesn't take any breaks.

21. A landscaping company charges \$45 per cubic yard of mulch plus a delivery charge of \$20. Find an expression for a linear function $C(x)$ which computes the total cost in dollars to deliver x cubic yards of mulch.

22. A plumber charges \$50 for a service call plus \$80 per hour. If she spends no longer than 8 hours a day at any one site, find an expression for a linear function $C(t)$ that computes her total daily charges in dollars as a function of the amount of time spent in hours, t at any one given location.

23. A salesperson is paid \$200 per week plus 5% commission on her weekly sales of x dollars. Find an expression for a linear function $W(x)$ which computes her total weekly pay in dollars as a function of x. What must her weekly sales be in order for her to earn \$475.00 for the week?

24. An on-demand publisher charges \$22.50 to print a 600 page book and \$15.50 to print a 400 page book. Find an expression for a linear function which models the cost of a book in dollars $C(p)$ as a function of the number of pages p. Find and interpret both the slope of the linear function and $C(0)$.

25. The Topology Taxi Company charges \$2.50 for the first fifth of a mile and \$0.45 for each additional fifth of a mile. Find an expression for a linear function which models the taxi fare $F(m)$ as a function of the number of miles driven, m. Find and interpret both the slope of the linear function and $F(0)$.

26. Water freezes at 0° Celsius and 32° Fahrenheit and it boils at 100°C and 212°F.

   (a) Find an expression for a linear function $F(T)$ that computes temperature in the Fahrenheit scale as a function of the temperature T given in degrees Celsius. Use this function to convert 20°C into Fahrenheit.
   (b) Find an expression for a linear function $C(T)$ that computes temperature in the Celsius scale as a function of the temperature T given in degrees Fahrenheit. Use this function to convert 110°F into Celsius.
   (c) Is there a temperature T such that $F(T) = C(T)$?

27. Legend has it that a bull Sasquatch in rut will howl approximately 9 times per hour when it is $40°F$ outside and only 5 times per hour if it's $70°F$. Assuming that the number of howls per hour, N, can be represented by a linear function of temperature Fahrenheit, find the number of howls per hour he'll make when it's only $20°F$ outside. What troubles do you encounter when trying to determine a reasonable applied domain?

28. Economic forces have changed the cost function for PortaBoys to $C(x) = 105x + 175$. Rework Example 1.2.3 with this new cost function.

29. In response to the economic forces in Exercise 28 above, the local retailer sets the selling price of a PortaBoy at \$250. Remarkably, 30 units were sold each week. When the systems went on sale for \$220, 40 units per week were sold. Rework Example 1.2.4 with this new data.

30. A local pizza store offers medium two-topping pizzas delivered for \$6.00 per pizza plus a \$1.50 delivery charge per order. On weekends, the store runs a 'game day' special: if six or more medium two-topping pizzas are ordered, they are \$5.50 each with no delivery charge. Write a piecewise-defined linear function which calculates the cost in dollars $C(p)$ of p medium two-topping pizzas delivered during a weekend.

31. A restaurant offers a buffet which costs \$15 per person. For parties of 10 or more people, a group discount applies, and the cost is \$12.50 per person. Write a piecewise-defined linear function which calculates the total bill $T(n)$ of a party of n people who all choose the buffet.

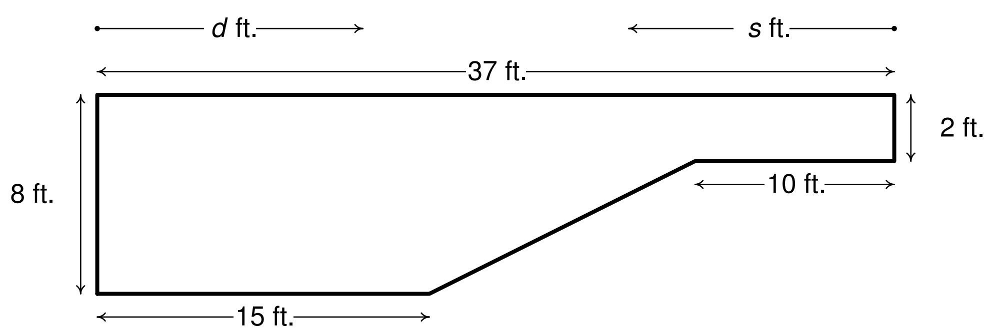
32. A mobile plan charges a base monthly rate of \$10 for the first 500 minutes of air time plus a charge of 15¢ for each additional minute. Write a piecewise-defined linear function which calculates the monthly cost in dollars $C(m)$ for using m minutes of air time.

    HINT: You may wish to refer to number 18 for inspiration.

33. The local pet shop charges 12¢ per cricket up to 100 crickets, and 10¢ per cricket thereafter. Write a piecewise-defined linear function which calculates the price in dollars $P(c)$ of purchasing c crickets.

34. The cross-section of a swimming pool is below. Write a piecewise-defined linear function which describes the depth of the pool, D (in feet) as a function of:

    (a) the distance (in feet) from the edge of the shallow end of the pool, d.
    (b) the distance (in feet) from the edge of the deep end of the pool, s.
    (c) Graph each of the functions in (a) and (b). Discuss with your classmates how to transform one into the other and how they relate to the diagram of the pool.



35. The function defined by $I(x) = x$ is called the Identity Function. Thinking from a procedural perspective, explain a possible origin of this name.

36. Why must the graph of a function $y = f(x)$ have at most one y-intercept?

    HINT: Consider what would happen graphically if there were more than one . . .

37. Why is a discussion of vertical lines omitted when discussing functions?

38. Find a formula for the x-intercept of the graph of $f(x) = mx + b$. Assume $m \neq 0$.

39. Suppose $(c, 0)$ is the x-intercept of a linear function f. Use the point-slope form of a liner function, Equation 1.1 to show $f(x) = m(x - c)$. This is the 'slope x-intercept' form of the linear function.

40. Prove that for all linear functions L with with slope 3, $L(120) = L(100) + 60$.

41. Find the slopes between the following points from the data set given in Example 1.2.7 and compare them with the slope of the corresponding regression line:

    (a) $(0, 64)$, $(4, 75)$ (b) $(4, 75)$, $(8, 83)$ (c) $(8, 83)$, $(10, 83)$ (d) $(10, 83)$, $(12, 82)$

42. According to this website[31], the census data for Lake County, Ohio is:

Year	1970	1980	1990	2000
Population	197200	212801	215499	227511

    (a) Find the least squares regression line for these data and comment on the goodness of fit.[32] Interpret the slope of the line of best fit.
    (b) Use the regression line to predict the population of Lake County in 2010. (The recorded figure from the 2010 census is 230,041)
    (c) Use the regression line to predict when the population of Lake County will reach 250,000.

43. According to this website[33], the census data for Lorain County, Ohio is:

Year	1970	1980	1990	2000
Population	256843	274909	271126	284664

    (a) Find the least squares regression line for these data and comment on the goodness of fit. Interpret the slope of the line of best fit.
    (b) Use the regression line to predict the population of Lorain County in 2010. (The recorded figure from the 2010 census is 301,356)

[31] http://www.ohiobiz.com/census/Lake.pdf

[32] We'll develop more sophisticated models for the growth of populations in Chapter 7. For the moment, we use a theorem from Calculus to approximate those functions with lines.

[33] http://www.ohiobiz.com/census/Lorain.pdf

(c) Use the regression line to predict when the population of Lake County will reach 325,000.

44. The chart below contains a portion of the fuel consumption information for a 2002 Toyota Echo that Jeffrey used to own. The first row is the cumulative number of gallons of gasoline that I had used and the second row is the odometer reading when I refilled the gas tank. So, for example, the fourth entry is the point (28.25, 1051) which says that I had used a total of 28.25 gallons of gasoline when the odometer read 1051 miles.

Gasoline Used (Gallons)	0	9.26	19.03	28.25	36.45	44.64	53.57	62.62	71.93	81.69	90.43
Odometer (Miles)	41	356	731	1051	1347	1631	1966	2310	2670	3030	3371

Find the least squares line for this data. Is it a good fit? What does the slope of the line represent? Do you and your classmates believe this model would have held for ten years had I not crashed the car on the Turnpike a few years ago?

45. Using the energy production data given below

Year	1950	1960	1970	1980	1990	2000
Production (in Quads)	35.6	42.8	63.5	67.2	70.7	71.2

(a) Plot the data using a graphing utility and explain why it does not appear to be linear.

(b) Discuss with your classmates why ignoring the first two data points may be justified from a historical perspective.

(c) Find the least squares regression line for the last four data points and comment on the goodness of fit. Interpret the slope of the line of best fit.

(d) Use the regression line to predict the annual US energy production in the year 2010.

(e) Use the regression line to predict when the annual US energy production will reach 100 Quads.

In Exercises 46 - 51, compute the average rate of change of the function over the specified interval.

46. $f(x) = x^3$, $[-1, 2]$

47. $g(x) = \dfrac{1}{x}$, $[1, 5]$

48. $f(t) = \sqrt{t}$, $[0, 16]$

49. $g(t) = x^2$, $[-3, 3]$

50. $F(s) = \dfrac{s+4}{s-3}$, $[5, 7]$

51. $G(s) = 3s^2 + 2s - 7$, $[-4, 2]$

52. The height of an object dropped from the roof of a building is modeled by: $h(t) = -16t^2 + 64$, for $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground in feet t seconds after the object is dropped. Find and interpret the average rate of change of h over the interval $[0, 2]$.

53. Using data from Bureau of Transportation Statistics, the average fuel economy $F(t)$ in miles per gallon for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Find and interpret the average rate of change of F over the interval $[0, 28]$.

54. The temperature $T(t)$ in degrees Fahrenheit t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

   (a) Find and interpret $T(4)$, $T(8)$ and $T(12)$.
   (b) Find and interpret the average rate of change of T over the interval $[4, 8]$.
   (c) Find and interpret the average rate of change of T from $t = 8$ to $t = 12$.
   (d) Find and interpret the average rate of temperature change between 10 AM and 6 PM.

55. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. Find and interpret the average rate of change as production is increased from making 3000 to 5000 pens.

56. Recall from Example 1.2.8 The formula $s(t) = -5t^2 + 100t$ for $0 \leq t \leq 20$ gives the height, $s(t)$, measured in feet, of a model rocket above the Moon's surface as a function of the time after lift-off, t, in seconds.

   (a) Find and interpret the average rate of change of s over the following intervals:

      i. $[14.9, 15]$ ii. $[15, 15.1]$ iii. $[14.99, 15]$ iv. $[15, 15.01]$

   (b) What value does the average rate of change appear to be approaching as the interval shrinks closer to the value $t = 15$?
   (c) Find the equation of the line containing $(15, 375)$ with slope $m = -50$ and graph it along with s on the same set of axes using a graphing utility. What happens as you zoom in near $(15, 375)$?

57. Show the average rate of change of a function of the form $f(x) = mx + b$ over *any* interval is m.

58. Why doesn't the graph of the vertical line $x = b$ in the xy-plane represent y as a function of x?

59. With help from a graphing utility, graph the following pairs of functions on the same set of axes:[34]

   - $f(x) = 2 - x$ and $g(x) = \lfloor 2 - x \rfloor$
   - $f(x) = x^2 - 4$ and $g(x) = \lfloor x^2 - 4 \rfloor$
   - $f(x) = x^3$ and $g(x) = \lfloor x^3 \rfloor$
   - $f(x) = \sqrt{x} - 4$ and $g(x) = \lfloor \sqrt{x} - 4 \rfloor$

   Choose more functions $f(x)$ and graph $y = f(x)$ alongside $y = \lfloor f(x) \rfloor$ until you can explain how, in general, one would obtain the graph of $y = \lfloor f(x) \rfloor$ given the graph of $y = f(x)$.

[34] See Example 1.2.2 for the definition of $\lfloor x \rfloor$.

60. The Lagrange Interpolate function L for two points (x_0, y_0) and (x_1, y_1) where $x_0 \neq x_1$ is given by:

$$L(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

(a) For each of the following pairs of points, find $L(x)$ using the formula above and verify each of the points lies on the graph of $y = L(x)$.

i. $(-1, 3)$, $(2, 3)$ ii. $(-3, -2)$, $(5, -2)$ iii. $(-3, -2)$, $(0, 1)$ iv. $(-1, 5)$, $(2, -1)$

(b) Verify that, in general, $L(x_0) = y_0$ and $L(x_1) = y_1$.

(c) Show the point-slope form of a linear function, Equation 1.1 is equivalent to the formula given for $L(x)$ after making the identifications: $f(x_0) = y_0$ and $m = \dfrac{y_1 - y_0}{x_1 - x_0}$.

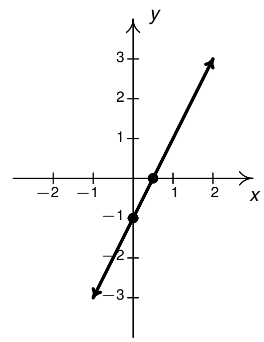
1.2.6 Answers

1. $f(x) = 2x - 1$

   slope: $m = 2$

   y-intercept: $(0, -1)$

   x-intercept: $\left(\frac{1}{2}, 0\right)$

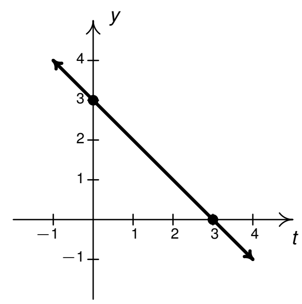


2. $g(t) = 3 - t$

   slope: $m = -1$

   y-intercept: $(0, 3)$

   t-intercept: $(3, 0)$

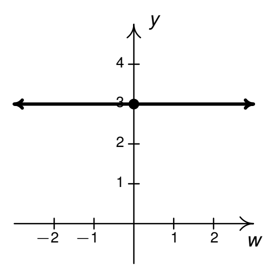


3. $F(w) = 3$

   slope: $m = 0$

   y-intercept: $(0, 3)$

   w-intercept: none

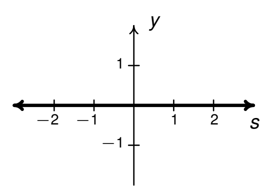


4. $G(s) = 0$

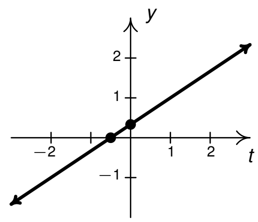
   slope: $m = 0$

   y-intercept: $(0, 0)$

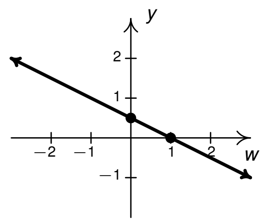
   s-intercept: $\{(s, 0) \mid s \text{ is a real number}\}$



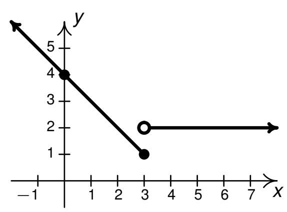
5. $h(t) = \frac{2}{3}x + \frac{1}{3}$
   slope: $m = \frac{2}{3}$
   y-intercept: $\left(0, \frac{1}{3}\right)$
   t-intercept: $\left(-\frac{1}{2}, 0\right)$



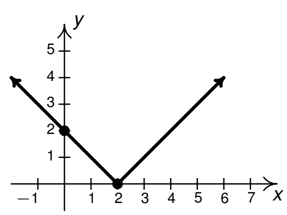
6. $j(w) = \dfrac{1-w}{2}$
   slope: $m = -\frac{1}{2}$
   y-intercept: $\left(0, \frac{1}{2}\right)$
   w-intercept: $(1, 0)$



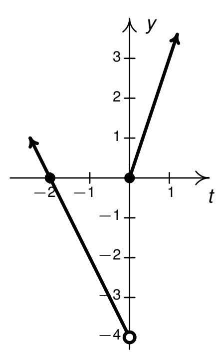
7.
   domain: $(-\infty, \infty)$
   range: $[1, \infty)$
   y-intercept: $(0, 4)$
   x-intercept: none



8.
   domain: $(-\infty, \infty)$
   range: $[0, \infty)$
   y-intercept: $(0, 2)$
   x-intercept: $(2, 0)$



9.
   domain: $(-\infty, \infty)$
   range: $(-4, \infty)$
   y-intercept: $(0, 0)$
   t-intercepts: $(-2, 0)$, $(0, 0)$



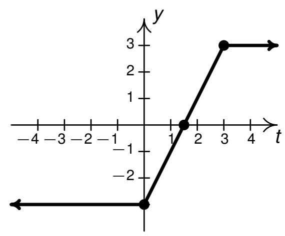
10.

domain: $(-\infty, \infty)$

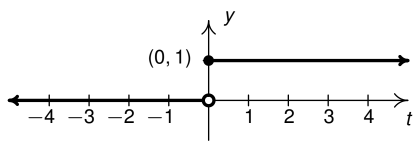
range: $[-3, 3]$

y-intercept: $(0, -3)$

t-intercept: $\left(\frac{3}{2}, 0\right) = (1.5, 0)$



11. (a)

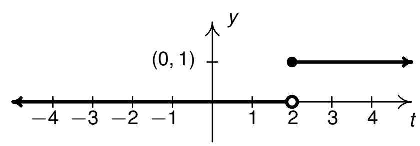


$y = U(t)$

(b) domain: $(-\infty, \infty)$, range: $\{0, 1\}$

(c) U is constant on $(-\infty, 0)$ and $[0, \infty)$.

(d) $U(t-2) = \begin{cases} 0 & \text{if } t < 2, \\ 1 & \text{if } t \geq 2. \end{cases}$



$y = U(t-2)$

12. $f(x) = -3$

13. $F(t) = \begin{cases} 2 & \text{if } t \leq 1, \\ -3 & \text{if } 1 < t \leq 3, \\ 4 & \text{if } t > 3. \end{cases}$

14. $L(x) = -\frac{3}{5}x + 1$

15. $g(v) = \begin{cases} 3v + 5 & \text{if } -3 \leq v < -1, \\ 2 & \text{if } -1 < v \leq 3, \end{cases}$

16. (a) $C(20) = 300$. It costs \$300 for 20 copies of the book.
    (b) $C(50) = 675$, \$675. $C(51) = 612$, \$612.
    (c) 56 books.

17. (a) $S(10) = 17.5$, \$17.50.
    (b) There is free shipping on orders of 15 or more comic books.

18. (a) $C(750) = 25$, \$25.
    (b) $C(1200) = 45$, \$45.
    (c) It costs \$25 for up to 1000 minutes and 10 cents per minute for each minute over 1000 minutes.

19. $d(t) = 3t$, $t \geq 0$.

20. $E(t) = 360t$, $t \geq 0$.

21. $C(x) = 45x + 20$, $x \geq 0$.

22. $C(t) = 80t + 50$, $0 \leq t \leq 8$.

23. $W(x) = 200 + .05x$, $x \geq 0$ She must make \$5500 in weekly sales.

24. $C(p) = 0.035p + 1.5$ The slope 0.035 means it costs 3.5¢ per page. $C(0) = 1.5$ means there is a fixed, or start-up, cost of \$1.50 to make each book.

25. $F(m) = 2.25m + 2.05$ The slope 2.25 means it costs an additional \$2.25 for each mile beyond the first 0.2 miles. $F(0) = 2.05$, so according to the model, it would cost \$2.05 for a trip of 0 miles. Would this ever really happen? Depends on the driver and the passenger, we suppose.

26. (a) $F(T) = \frac{9}{5}T + 32$

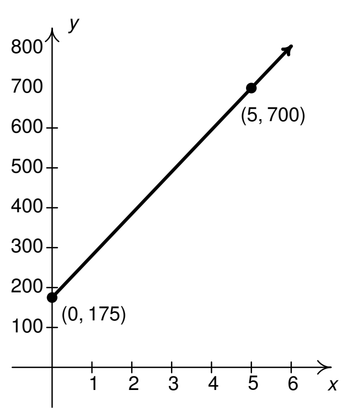
    (b) $C(T) = \frac{5}{9}(T - 32) = \frac{5}{9}T - \frac{160}{9}$

    (c) $F(-40) = -40 = C(-40)$.

27. $N(T) = -\frac{2}{15}T + \frac{43}{3}$ and $N(20) = \frac{35}{3} \approx 12$ howls per hour.

    Having a negative number of howls makes no sense and since $N(107.5) = 0$ we can put an upper bound of $107.5^{\circ}F$ on the domain. The lower bound is trickier because there's nothing other than common sense to go on. As it gets colder, he howls more often. At some point it will either be so cold that he freezes to death or he's howling non-stop. So we're going to say that he can withstand temperatures no lower than $-42^{\circ}F$ so that the applied domain is $[-42, 107.5]$.

28. (a) $C(0) = 175$, so our start-up costs are \$175. $C(5) = 700$, so to produce 5 systems, it costs \$700.



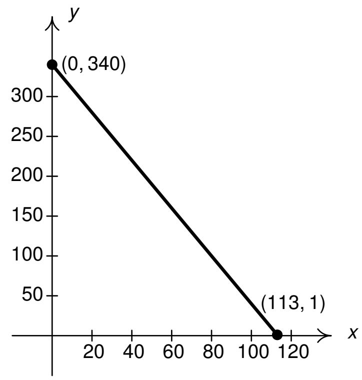
$y = C(x)$

    (b) Since we can't make a negative number of game systems, $x \geq 0$.

    (c) The slope is $m = 105$ so for each additional system produced, it costs an additional \$105.

    (d) Solving $C(x) = 15000$ gives $x \approx 141.19$ so 141 can be produced for \$15,000.

29. (a) $p(x) = -3x + 340$, $0 \leq x \leq 113$.



$y = p(x)$

(b) The slope is $m = -3$ so for each \$3 drop in price, we sell one additional game system.

(c) Since $x = 150$ is not in the domain of p, $p(150)$ is not defined. (In other words, under these conditions, it is impossible to sell 150 game systems.)

(d) Solving $p(x) = 150$ gives $x \approx 63.33$ so if the price \$150 per system, we would sell 63 systems.

30. $C(p) = \begin{cases} 6p + 1.5 & \text{if} \quad 1 \leq p \leq 5 \\ 5.5p & \text{if} \quad p \geq 6 \end{cases}$

31. $T(n) = \begin{cases} 15n & \text{if} \quad 1 \leq n \leq 9 \\ 12.5n & \text{if} \quad n \geq 10 \end{cases}$

32. $C(m) = \begin{cases} 10 & \text{if} \quad 0 \leq m \leq 500 \\ 10 + 0.15(m - 500) & \text{if} \quad m > 500 \end{cases}$

33. $P(c) = \begin{cases} 0.12c & \text{if} \quad 1 \leq c \leq 100 \\ 12 + 0.1(c - 100) & \text{if} \quad c > 100 \end{cases}$

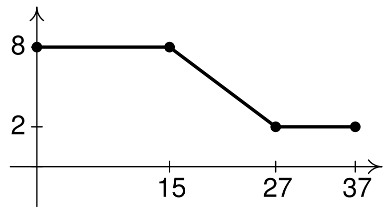
34. (a)

$$D(d) = \begin{cases} 8 & \text{if} \quad 0 \leq d \leq 15 \\ -\frac{1}{2}d + \frac{31}{2} & \text{if} \quad 15 \leq d \leq 27 \\ 2 & \text{if} \quad 27 \leq d \leq 37 \end{cases}$$

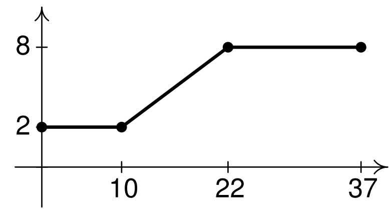
(b)

$$D(s) = \begin{cases} 2 & \text{if} \quad 0 \leq s \leq 10 \\ \frac{1}{2}s - 3 & \text{if} \quad 10 \leq s \leq 22 \\ 8 & \text{if} \quad 22 \leq s \leq 37 \end{cases}$$

(c)



$y = D(d)$



$y = D(s)$

35. Since $I(x) = x$ for all real numbers x, the function I doesn't change the 'identity' of the input at all.

36. If a graph contains more than one y-intercept, it would violate the Vertical Line Test since $x = 0$ would be matched with (at least) two different y-values.

37. Vertical Lines fail the Vertical Line Test.

38. $\left(-\frac{b}{m}, 0\right)$. (Note the importance here of $m \neq 0$.)

39. Plugging in $(c, 0)$ for $(x_0, f(x_0))$, we get $f(x) = f(x_0) + m(x - x_0) = 0 + m(x - c)$ or $f(x) = m(x - c)$.

40. Since L is linear with slope 3, $L(x) = L(x_0) + m\Delta x = L(100) + (3)(120 - 100) = L(100) + 60$.

41. (a) $m = \frac{75-64}{4-0} = 2.75$
    (b) $m = \frac{83-75}{8-4} = 2$
    (c) $m = \frac{83-83}{10-8} = 0$
    (d) $m = \frac{82-83}{12-10} = -0.5$

    The first two points contributed to a regression line slope of $m = 2.55$; the last two points contributed to a regression line slope of $m = -0.25$.

42. (a) $y = 936.31x - 1645322.6$ with $r = 0.9696$ which indicates a good fit. The slope 936.31 indicates Lake County's population is increasing at a rate of (approximately) 936 people per year.
    (b) According to the model, the population in 2010 will be 236,660.
    (c) According to the model, the population of Lake County will reach 250,000 sometime between 2024 and 2025.

43. (a) $y = 796.8x - 1309762.5$ with $r = 0.8916$ which indicates a reasonable fit. The slope 796.8 indicates Lorain County's population is increasing at a rate of (approximately) 797 people per year.
    (b) According to the model, the population in 2010 will be 291,805.
    (c) According to the model, the population of Lake County will reach 325,000 sometime between 2051 and 2052.

44. The regression line is $y = 36.8x + 16.39$ with $r = .99987$, so this is an excellent fit. The slope 36.8 represents mileage in miles per gallon.

45. (c) $y = 0.266x - 459.86$ with $r = 0.9607$ which indicates a good fit. The slope 0.266 indicates the country's energy production is increasing at a rate of 0.266 Quad per year.
(d) According to the model, the production in 2010 will be 74.8 Quad.
(e) According to the model, the production will reach 100 Quad in the year 2105.

46. $\frac{2^3-(-1)^3}{2-(-1)} = 3$

47. $\frac{\frac{1}{5}-\frac{1}{1}}{5-1} = -\frac{1}{5}$

48. $\frac{\sqrt{16}-\sqrt{0}}{16-0} = \frac{1}{4}$

49. $\frac{3^2-(-3)^2}{3-(-3)} = 0$

50. $\dfrac{\frac{7+4}{7-3} - \frac{5+4}{5-3}}{7-5} = -\frac{7}{8}$

51. $\frac{(3(2)^2+2(2)-7)-(3(-4)^2+2(-4)-7)}{2-(-4)} = -4$

52. The average rate of change is $\frac{h(2)-h(0)}{2-0} = -32$. During the first two seconds after it is dropped, the object has fallen at an average rate of 32 feet per second.

53. The average rate of change is $\frac{F(28)-F(0)}{28-0} = 0.2372$. From 1980 to 2008, the average fuel economy of passenger cars in the US increased, on average, at a rate of 0.2372 miles per gallon per year.

54. (a) $T(4) = 56$, so at 10 AM (4 hours after 6 AM), it is 56°F. $T(8) = 64$, so at 2 PM (8 hours after 6 AM), it is 64°F. $T(12) = 56$, so at 6 PM (12 hours after 6 AM), it is 56°F.
(b) The average rate of change is $\frac{T(8)-T(4)}{8-4} = 2$. Between 10 AM and 2 PM, the temperature increases, on average, at a rate of 2°F per hour.
(c) The average rate of change is $\frac{T(12)-T(8)}{12-8} = -2$. Between 2 PM and 6 PM, the temperature decreases, on average, at a rate of 2°F per hour.
(d) The average rate of change is $\frac{T(12)-T(4)}{12-4} = 0$. Between 10 AM and 6 PM, the temperature, on average, remains constant.

55. The average rate of change is $\frac{C(5)-C(3)}{5-3} = -2$. As production is increased from 3000 to 5000 pens, the cost decreases at an average rate of \$200 per 1000 pens produced (20¢ per pen.)

56. (a) i. -49.5 so the average velocity of the rocket between 14.9 and 15 seconds after lift off is -49.5 feet per second (49.5 feet per second directed *downwards*.)
ii. -50.5 so the average velocity of the rocket between 14 and 15.1 seconds after lift off is -50.5 feet per second. (50.5 feet per second directed *downwards*.)
iii. -49.95 so the average velocity of the rocket between 14.99 and 15 seconds after lift off is -49.95 feet per second. (49.95 feet per second directed *downwards*.)
iv. -50.05 so the average velocity of the rocket between 15.01 and 15 seconds after lift off is -50.05 feet per second. (50.05 feet per second directed *downwards*.)
(b) The average rate of change seem to be approaching -50.
(c) Line: $y = -50(t-15)+375$ or $y = -50t+1125$. Graphing this line along with the s on a graphing utility we find the two graphs become indistinguishable as we zoom in near $(15, 375)$.

60. (a) i. $L(x) = 3$ ii. $L(x) = -2$ iii. $L(x) = x+1$ iv. $L(x) = -2x+3$

1.3 Absolute Value Functions

1.3.1 Graphs of Absolute Value Functions

In Section 1.2, we revisited lines in a function context. In this section, we revisit the absolute value in a similar manner, so it may be useful to refresh yourself with the basics in Section A.7. Recall that the absolute value of a real number x, denoted $|x|$, can be defined as the distance from x to 0 on the real number line.[1] This definition is very useful for several applications, and lends itself well to solving equations and inequalities such as $|x-2|+1=5$ or $2|t+1|>4$.

We now wish to explore solving more complicated equations and inequalities, such as $|x-2|+1=x$ and $2|t+1|\geq t+4$. We'll approach these types of problems from a function standpoint and use the interplay between the graphical and analytical representations of these functions to obtain solutions. The key to this section is understanding the absolute value from that function (or procedural) standpoint.

Consider a real number $x\geq 0$ such as $x=0$, $x=\pi$ or $x=117.42$. When computing absolute values, we find $|0|=0$, $|\pi|=\pi$ and $|117.42|=117.42$. In general, if $x\geq 0$, the absolute value function does nothing to change the input, so $|x|=x$. On the other hand, if $x<0$, say $x=-1$, $x=-\sqrt{42}$ or $x=-117.42$, we get $|-1|=1$, $|-\sqrt{42}|=\sqrt{42}$ and $|-117.42|=117.42$. That is, if $x<0$, $|x|$ returns the exact **opposite** of the input x, so $|x|=-x$.

Putting these two observations together, we have the following.

Definition 1.9. The **absolute value** of a real number x, denoted $|x|$, is given by

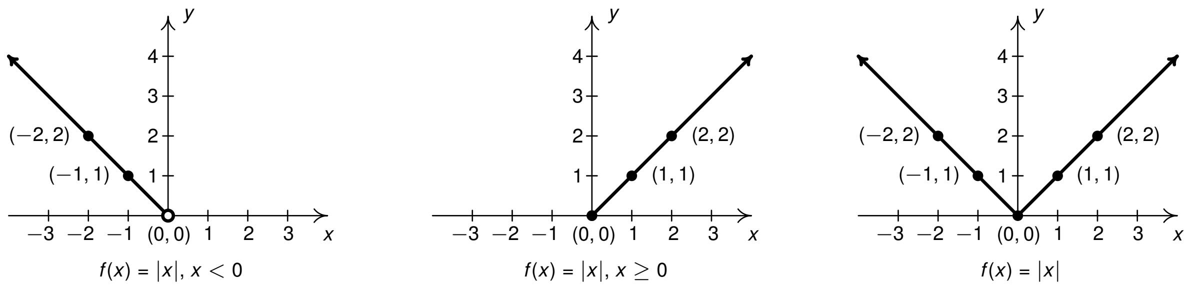
$$|x|=\begin{cases}-x & \text{if } x<0\\ x & \text{if } x\geq 0\end{cases}$$

In Definition 1.9, it is **absolutely** essential to read '$-x$' as 'the **opposite** of x' as **opposed** to 'negative x' in order to avoid serious errors later. To see that this description agrees with our previous experience, consider $|117.42|$. Given that $117.42\geq 0$, we use the rule $|x|=x$. Hence, $|117.42|=117.42$. Likewise, $|0|=0$. To compute $|-\sqrt{42}|$, we note that $-\sqrt{42}<0$ we use the rule $|x|=-x$ in this case. We get $|-\sqrt{42}|=-(-\sqrt{42})$ (the opposite of $-\sqrt{42}$), so $|-\sqrt{42}|=-(-\sqrt{42})=\sqrt{42}$.

Another way to view Definition 1.9 is to think of $-x=(-1)x$ and $x=(1)x$. That is, $|x|$ multiplies negative inputs by -1 and non-negative inputs by 1. This viewpoint is especially useful in graphing $f(x)=|x|$. For $x<0$, $|x|=(-1)x$, so the graph of $y=|x|$ is the graph of $y=-x=(-1)x$: a line with slope -1 and y-intercept $(0,0)$. Likewise, for $x\geq 0$, $|x|=x$, so the graph of $y=|x|$ is the graph of $y=x=(1)x$: a line with slope 1 and y-intercept $(0,0)$.

At the top of the next page we graph each piece and then put them together. Note that when graphing $f(x)=|x|$ for $x<0$, we have a hole at $(0,0)$ because the inequality $x<0$ is strict. However, the point $(0,0)$ is included in the graph of $f(x)=|x|$ for $x\geq 0$, so there is no hole in our final graph.

[1]More generally, $|x-c|$ is the distance from x to c on the number line.



The graph of $f(x) = |x|$ is a very distinctive '$\vee$' shape and is worth remembering. The point $(0, 0)$ on the graph is called the **vertex**. This terminology makes sense from a geometric viewpoint because $(0, 0)$ is the point where two lines meet to form an angle. We will also see this term used in Section 1.4 where, more generally, it corresponds to the graphical location of the sole maximum or minimum of a quadratic function.

We put Definition 1.9 to good use in the next example and review the basics of graphing along the way.

Example 1.3.1. For each of the functions below, analytically find the zeros of the function and the axis intercepts of the graph, if any exist. Rewrite the function using Definition 1.9 as a piecewise-defined function and sketch its graph. From the graph, determine the vertex, find the range of the function and any extrema, and then list the intervals over which the function is increasing, decreasing or constant.

1. $f(x) = |x - 3|$ 2. $g(t) = |t| - 3$ 3. $h(u) = |2u - 1| - 3$ 4. $i(w) = 4 - 2|3w - 1|$

Solution. In what follows below, we will be doing quite a bit of substitution. As we have mentioned before, when substituting one expression in for another, the use of parentheses or other grouping symbols is highly recommended. Also, the dependent variable wasn't specified so we use the default y in each case.

1. To find the zeros of f, we solve $f(x) = 0$ or $|x - 3| = 0$. We get $x = 3$ so the sole x-intercept of the graph of f is $(3, 0)$. To find the y-intercept, we compute $f(0) = |0 - 3| = 3$ and obtain $(0, 3)$. Using Definition 1.9 to rewrite the expression for $f(x)$ means that we substitute the expression $x - 3$ in for x and simplify. Note that when substituting the $x - 3$ in for x, we do so for **every** instance of x – both in the formula (output) as well as the inequality (input).

$$f(x) = |x - 3| = \begin{cases} -(x-3) & \text{if } (x-3) < 0 \\ (x-3) & \text{if } (x-3) \geq 0 \end{cases} \quad \longrightarrow \quad f(x) = \begin{cases} -x+3 & \text{if } x < 3 \\ x-3 & \text{if } x \geq 3 \end{cases}$$

As both pieces of the graph of f are lines, we need just two points for each piece. We already have two points for the graph: $(0, 3)$ and $(3, 0)$. These two points both lie on the line $y = -x + 3$ but the strictness of the inequality means $f(x) = -x + 3$ only for $x < 3$, not $x = 3$, so we would have a hole at $(3, 0)$ instead of a point there. For $x \geq 3$, $f(x) = x - 3$, so the hole we thought we had at $(3, 0)$ gets plugged because $f(3) = 3 - 3 = 0$. We need just one more point for $f(x)$ where $x \geq 3$ and choose somewhat arbitrarily $x = 6$. We find $f(6) = |6 - 3| = 3$ so our final point on the graph is $(6, 3)$. Now that we have a complete graph,[2] we see that the vertex is $(3, 0)$ and the range is $[0, \infty)$.

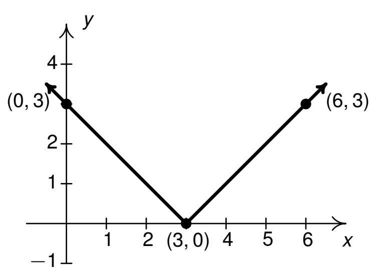
[2]We know it's complete because we did the Math - no trusting technology on this example!

The minimum of f is 0 when $x = 3$ and f has no maximum. Also, f is decreasing over $(-\infty, 3]$ and increasing on $[3, \infty)$. The graph is given below on the left.

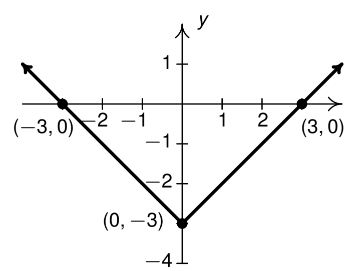
2. To find the zeros of g, we solve $g(t) = |t| - 3 = 0$ and get $|t| = 3$ or $t = \pm 3$. Hence, the t-intercepts of the graph of g are $(-3, 0)$ and $(3, 0)$. To find the y-intercept, we compute $g(0) = |0| - 3 = -3$ and get $(0, -3)$. To rewrite $g(t)$ has a piecewise defined function, we first substitute t in for x in Definition 1.9 to get a piecewise definition of $|t|$. This breaks the domain into two pieces: $t < 0$ and $t \geq 0$. For $t < 0$, $|t| = -t$, so $g(t) = |t| - 3 = (-t) - 3 = -t - 3$. Likewise, for $t \geq 0$, $|t| = t$ so $g(t) = |t| - 3 = t - 3$.

$$|t| = \begin{cases} -t & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases} \quad \longrightarrow \quad g(t) = |t| - 3 = \begin{cases} -t - 3 & \text{if } t < 0 \\ t - 3 & \text{if } t \geq 0 \end{cases}$$

Once again, we have two lines to graph, but in this case we have three points: $(-3, 0)$, $(0, -3)$ and $(3, 0)$. Both $(-3, 0)$ and $(0, -3)$ lie on $y = -t - 3$, but $g(t) = -t - 3$ only for $t < 0$. This would yield a hole at $(0, -3)$, but, just like in the previous example, the hole is plugged thanks to the second piece of the function because $g(0) = 0 - 3 = -3$. We also pick up the second t-intercept, $(3, 0)$ and this helps us complete our graph. We see that the vertex is $(0, -3)$ and the range is $[-3, \infty)$. The minimum of g is -3 at $t = 0$ and there is no maximum. Also, g is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. The graph of g is shown below on the right.



$f(x) = |x - 3|$



$g(t) = |t| - 3$

3. Solving $h(u) = |2u - 1| - 3 = 0$ gives $|2u - 1| = 3$ or $2u - 1 = \pm 3$. We get two zeros: $u = -1$ and $u = 2$ which correspond to two u-intercepts: $(-1, 0)$ and $(2, 0)$. We find $h(0) = |2(0) - 1| - 3 = -2$ so our y-intercept is $(0, -2)$. To rewrite $h(u)$ as a piecewise defined function, we first rewrite $|2u - 1|$ as a piecewise function. Substituting the expression $2u - 1$ in for x in Definition 1.9 gives:

$$|2u - 1| = \begin{cases} -(2u - 1) & \text{if } 2u - 1 < 0 \\ 2u - 1 & \text{if } 2u - 1 \geq 0 \end{cases} \quad \longrightarrow \quad |2u - 1| = \begin{cases} -2u + 1 & \text{if } u < \frac{1}{2} \\ 2u - 1 & \text{if } u \geq \frac{1}{2} \end{cases}$$

Hence, for $u < \frac{1}{2}$, $|2u - 1| = -2u + 1$ so $h(u) = |2u - 1| - 3 = (-2u + 1) - 3 = -2u - 2$. Likewise, for $u \geq \frac{1}{2}$, $|2u - 1| = 2u - 1$ so $h(u) = |2u - 1| - 3 = (2u - 1) - 3 = 2u - 4$.

$$h(u) = |2u - 1| - 3 = \begin{cases} (-2u + 1) - 3 & \text{if } u < \frac{1}{2} \\ (2u - 1) - 3 & \text{if } u \geq \frac{1}{2} \end{cases} \quad \longrightarrow \quad h(u) = \begin{cases} -2u - 2 & \text{if } u < \frac{1}{2} \\ 2u - 4 & \text{if } u \geq \frac{1}{2} \end{cases}$$

We have three points to help us graph $y = h(u)$: $(-1, 0)$, $(0, -2)$ and $(2, 0)$. Unlike in the last two examples, these points do not give us information at the value $u = \frac{1}{2}$ where the rule for $h(u)$ changes. Substituting $u = \frac{1}{2}$ into the expression $-2u - 2$ gives -3, so from $h(u) = -2u - 2$, $u < \frac{1}{2}$, we get a hole at $\left(\frac{1}{2}, -3\right)$. However, this hole is filled because $h\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) - 4 = -3$ and this produces the vertex at $\left(\frac{1}{2}, -3\right)$. The range of h is $[-3, \infty)$, with the minimum of h being -3 at $t = \frac{1}{2}$. Moreover, h is decreasing on $\left(-\infty, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, \infty\right)$. The graph of h is given below on the left.

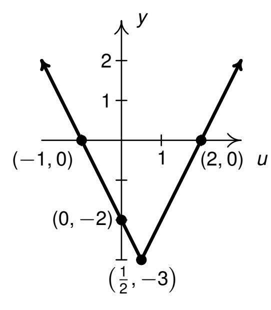
4. Solving $i(w) = 4 - 2|3w - 1| = 0$ yields $|3w - 1| = 2$ or $3w - 1 = \pm 2$. This gives two zeros, $w = -\frac{1}{3}$ and $w = 1$, which correspond to two w-intercepts, $\left(-\frac{1}{3}, 0\right)$ and $(1, 0)$. Also, $i(0) = 4 - 2|3(0) - 1| = 2$, so the y-intercept of the graph is $(0, 2)$. As in the previous example, the first step in rewriting $i(w)$ as a piecewise defined function is to rewrite $|3w - 1|$ as a piecewise function. Once again, we substitute the expression $3w - 1$ in for every occurrence of x in Definition 1.9:

$$|3w - 1| = \begin{cases} -(3w - 1) & \text{if } 3w - 1 < 0 \\ 3w - 1 & \text{if } 3w - 1 \geq 0 \end{cases} \longrightarrow \quad |3w - 1| = \begin{cases} -3w + 1 & \text{if } w < \frac{1}{3} \\ 3w - 1 & \text{if } w \geq \frac{1}{3} \end{cases}$$

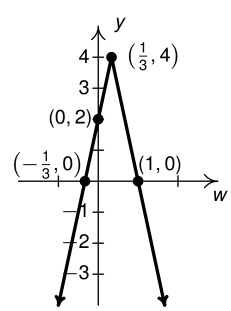
Thus for $w < \frac{1}{3}$, $|3w - 1| = -3w + 1$, so $i(w) = 4 - 2|3w - 1| = 4 - 2(-3w + 1) = 6w + 2$. Likewise, for $w \geq \frac{1}{3}$, $|3w - 1| = 3w - 1$ so $i(w) = 4 - 2|3w - 1| = 4 - 2(3w - 1) = -6w + 6$.

$$i(w) = 4 - 2|3w - 1| = \begin{cases} 4 - 2(-3w + 1) & \text{if } w < \frac{1}{3} \\ 4 - 2(3w - 1) & \text{if } w \geq \frac{1}{3} \end{cases} \longrightarrow \quad i(w) = \begin{cases} 6w + 2 & \text{if } w < \frac{1}{3} \\ -6w + 6 & \text{if } w \geq \frac{1}{3} \end{cases}$$

As with the previous example, we have three points on the graph of i: $\left(-\frac{1}{3}, 0\right)$, $(0, 2)$ and $(1, 0)$, but no information about happens at $w = \frac{1}{3}$. Substituting this value of w into the formula $6w + 2$ would produce a hole at $\left(\frac{1}{3}, 4\right)$. As we've seen several times already, however, $i\left(\frac{1}{3}\right) = 4$ so we don't have a hole at $\left(\frac{1}{3}, 4\right)$ but, rather, the vertex. From the graph we see that the range of i is $(-\infty, 4]$ with the maximum of i being 4 when $w = \frac{1}{3}$. Also, i is increasing over $\left(-\infty, \frac{1}{3}\right]$ and decreasing on $\left[\frac{1}{3}, \infty\right)$. Its graph is given below on the right.



$h(u) = |2u - 1| - 3$



$i(w) = 4 - 2|3w - 1|$ □

As we take a step back and look at the graphs produced in Example 1.3.1, some patterns begin to emerge. Indeed, each of the graphs has the common '$\vee$' shape (in the case of the function i it's a '$\wedge$') with the vertex located at the x-value where the rule for each function changes from one formula to the other. It turns out that, independent variable labels aside, each and every function in Example 1.3.1 can be rewritten in the form $F(x) = a|x - h| + k$ for real number parameters a, h and k.

Each of the functions from Example 1.3.1 is rewritten in this form below and we record the vertex along with the slopes of the lines in the graph.

- $f(x) = |x-3| = (1)|x-3| + 0$: $a = 1$, $h = 3$, $k = 0$; vertex $(3, 0)$; slopes ± 1
- $g(t) = |t| - 3 = (1)|t-0| + (-3)$: $a = 1$, $h = 0$, $k = -3$; vertex $(0, -3)$; slopes ± 1
- $h(u) = |2u-1| - 3 = 2\left|u - \frac{1}{2}\right| + (-3)$: $a = 2$, $h = \frac{1}{2}$, $k = -3$; vertex $\left(\frac{1}{2}, -3\right)$; slopes ± 2
- $i(w) = 4 - 2|3w-1| = -6\left|w - \frac{1}{3}\right| + 4$: $a = -6$, $h = \frac{1}{3}$, $k = 4$; vertex $\left(\frac{1}{3}, 4\right)$; slopes ± 6

These specific examples suggest the following theorem.

Theorem 1.2. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a|x-h| + k$ consists of parts of two lines with slopes $\pm a$ which meet at a vertex (h, k). If $a > 0$, the shape resembles '$\vee$'. If $a < 0$, the shape resembles '$\wedge$'. Moreover, the graph is symmetric about the line $x = h$.

Proof. What separates Mathematics from the other sciences is its ability to actually **prove** patterns like the one stated in the theorem above as opposed to just **verifying** it by working more examples. The proof of Theorem 1.2 uses the exact same concepts as were used in Example 1.3.1, just in a more general context by which we mean using letters as parameters instead of numbers.

The first step is to rewrite $|x-h|$ as a piecewise function.

$$|x-h| = \begin{cases} -(x-h) & \text{if } x-h < 0 \\ x-h & \text{if } x-h \geq 0 \end{cases} \quad \longrightarrow \quad |x-h| = \begin{cases} -x+h & \text{if } x < h \\ x-h & \text{if } x \geq h \end{cases}$$

We plug that work into $F(x)$ to rewrite it as a piecewise function. For $x < h$, we have $|x-h| = -x+h$, so

$$F(x) = a|x-h| + k = a(-x+h) + k = -ax + ah + k = -ax + (ah+k)$$

Similarly, for $x \geq h$, we have $|x-h| = x-h$, so

$$F(x) = a|x-h| + k = a(x-h) + k = ax - ah + k = ax + (-ah+k)$$

Hence,

$$F(x) = a|x-h| + k = \begin{cases} a(-x+h) + k & \text{if } x < h \\ a(x-h) + k & \text{if } x \geq h \end{cases} \quad \longrightarrow \quad F(x) = \begin{cases} -ax + (ah+k) & \text{if } x < h \\ ax + (-ah+k) & \text{if } x \geq h \end{cases}$$

All three parameters, a, h and k, are fixed (but arbitrary) real numbers. Thus, for any given choice of a, h and k the numbers $ah+k$ and $-ah+k$ are also just numbers as opposed to variables. This shows that the graph of F is comprised of pieces of two lines, $y = -ax + (ah+k)$ and $y = ax + (-ah+k)$, the former with slope $-a$ and the latter with slope a. Note that substituting $x = h$ into $y = -ax + (ah+k)$ produces $y = -ah + (ah+k) = k$ and substituting $x = h$ into $y = ax + (-ah+k)$ also produces $y = ah + (-ah+k) = k$. This tells us that the two linear pieces meet at the point (h, k).

If $a > 0$ then $-a < 0$ so the line $y = -ax + (ah + k)$, hence F, is decreasing on $(-\infty, h]$. Similarly, the line $y = ax+(-ah+k)$, hence F, is increasing on $[h, \infty)$. This produces a '$\vee$' shape. On the other hand, if $a < 0$ then $-a > 0$ which produces a '$\wedge$' shape because F is increasing on $(-\infty, h]$ followed by decreasing on $[h, \infty)$. (Said another way, $-a > 0$ means that the first linear piece has a positive slope and $a < 0$ means that the second piece has a negative slope.)

To show that the graph is symmetric about the line $x = h$, we need to show that if we move left or right the same distance away from $x = h$, then we get the same y-value on the graph. Suppose we move Δx to the right or left of h. The y-values are the function values so we need to show that $F(a + \Delta x) = F(a - \Delta x)$. Given that

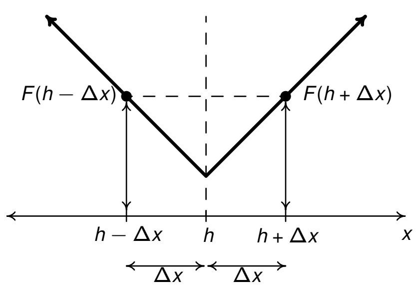
$$F(a + \Delta x) = a|a + \Delta x - a| + k = a|\Delta x| + k$$

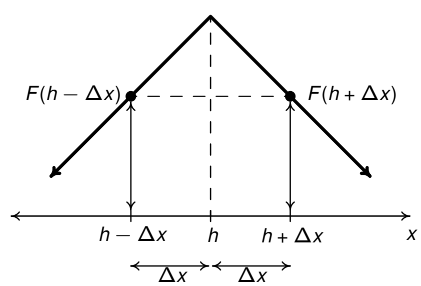
and

$$F(a - \Delta x) = a|a - \Delta x - a| + k = a|-\Delta x| + k = a|\Delta x| + k$$

we see that $F(a + \Delta x) = F(a - \Delta x)$. Thus we have shown that the y-values on the graph on either side of $x = h$ are equal provided we move the same distance away from $x = a$. This completes the proof. $\square$

The line $x = a$ in Theorem 1.2 is called the **axis of symmetry** of the graph of $y = F(x)$. This language is consistent with the basics of symmetry discussed in Section A.3 and we will build upon our work here in several upcoming sections. For now, we simply present two graphs illustrating the concept of the axis of symmetry below.





While Theorem 1.2 and its proof are specific to the particular family of absolute value functions, there are ideas here that apply to all functions. Thus we wish to take a slight detour away from the main narrative to argue this result again from an even more generalized viewpoint. Our goal is to 'build' the formula $F(x) = a|x - h| + k$ from $f(x) = |x|$ in three stages, each corresponding to the role of one of the parameters a, h and k, and track the geometric changes that go along with each stage. We will revisit all of the ideas described below in complete generality in Section 5.4.

The graph of $f(x) = |x|$ consists of the points $\{(c, |c|) \mid c \in \mathbb{R}\}$.[3] Consider $F_1(x) = |x - h|$. The graph of F_1 is the set of points $\{(x, |x - h|) \mid x \in \mathbb{R}\}$. If we relabel $x - h = c$, then $x = c + h$, and as x varies through all of the real numbers, so does c and vice-versa.[4]

[3]See the box on page 19. Also, we use 'c' as our dummy variable to avoid the confusion that would arise by over-using 'x'.
[4]That is, every real number c can be written as $x - h$ for some x, and every real number x can be written as $c + h$ for some c.

Hence, we can write $\{(x, |x-h|) \mid x \in \mathbb{R}\} = \{(c+h, |c|) \mid c \in \mathbb{R}\}$. If we fix a y-coordinate, $|c|$, we see that the corresponding points on the graph of f and F_1, $(c, |c|)$ and $(c+h, |c|)$, respectively, differ only in that one is horizontally shifted by h. In other words, to get the graph of F_1, we simply take the graph of f and shift each point horizontally by adding h to the x-coordinate. Translating the graph in this manner preserves the '$\vee$' shape and symmetry, but moves the vertex from $(0,0)$ to $(h,0)$.

Next, we examine $F_2(x) = a|x-h|$ and compare its graph to that of $F_1(x) = |x-h|$. The graph of F_2 consists of the points $\{(x, a|x-h|) \mid x \in \mathbb{R}\}$ whereas the graph of F_1 consists of the points $\{(x, |x-h|) \mid x \in \mathbb{R}\}$. The only difference between the points $(x, |x-h|)$ and $(x, a|x-h|)$ is that the y-coordinate of one is a times the y-coordinate of the other. If $a > 0$, all we are doing is scaling the y-axis by a factor of a. As we've seen when plotting points and graphing functions, the scaling of the y-axis affects only the relative vertical displacement of points[5] and not the overall shape.

If $a < 0$, then in addition to scaling the vertical axis, we are reflecting the points across the x-axis.[6] Such a transformation doesn't change the '$\vee$' shape except for flipping it upside-down to make it a '$\wedge$'. In either case, the vertex $(h,0)$ stays put at $(h,0)$ because the y-value of the vertex is 0 and $a \cdot 0 = 0$ regardless if $a > 0$ or $a < 0$.

Last, we examine the graph of $F(x) = a|x-h|+k$ to see how it relates to the graph of $F_2(x) = a|x-h|$. The graph of F consists of the points $\{(x, a|x-h|+k) \mid x \in \mathbb{R}\}$ whereas the graph of F_2 consists of the points $\{(x, a|x-h|) \mid x \in \mathbb{R}\}$. The difference between the corresponding points $(x, a|x-h|)$ and $(x, a|x-h|+k)$ is the addition of k in the y-coordinate of the latter. Adding k to each of the y-values translates the graph of F_2 vertically by k units. The basic shape doesn't change but the vertex goes from $(h,0)$ to (h,k).

In summary, the graph of $F(x) = a|x-h|+k$ can be obtained from the graph of $f(x) = |x|$ in three steps: first, add h to each of the x-coordinates; second, multiply each y-coordinate by a; and third, add k to each y-coordinate. Geometrically, these steps mean that we first move the graph left or right, then scale the y-axis by a factor of a (and reflect across the x-axis if $a < 0$), and then move the graph up or down. Throughout all of these **transformations**, the graph maintains its '$\vee$' or '$\wedge$' shape.

Of course, not every function involving absolute values can be written in the form given in Theorem 1.2. A good example of this is $G(x) = |x-2| - x$. However recognizing the ones that can be rewritten will greatly simplify the graphing process. In the next example, we graph four more absolute value functions, two using Theorem 1.2 and two using Definition 1.9.

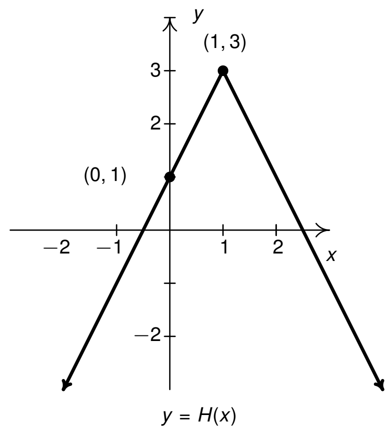
Example 1.3.2.

1. Graph each of the functions below using Theorem 1.2 or by rewriting it as a piecewise defined function using Definition 1.9. Find the zeros, axis-intercepts and the extrema (if any exist) and then list the intervals over which the function is increasing, decreasing or constant.

   (a) $F(x) = |x+3| + 2$ (b) $f(t) = \dfrac{4 - |5-3t|}{2}$ (c) $G(x) = |x-2| - x$ (d) $g(t) = |t-2| - |t|$

[5] See the discussion following Example 1.1.1 regarding the plot of Skippy's data.
[6] See the box on page 1357 in Section A.3.

2. Use Theorem 1.2 to write a possible formula for $H(x)$ whose graph is given below:



Solution.

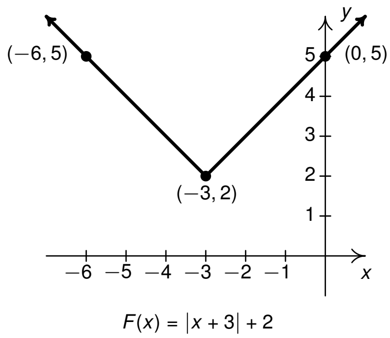
1. (a) Rewriting $F(x) = |x+3|+2 = (1)|x-(-3)|+2$, we have $F(x)$ in the form stated in Theorem 1.2 with $a = 1$, $h = -3$ and $k = 2$. The vertex is $(-3, 2)$ and the graph will be a '$\vee$' shape. Seeing as the vertex is already above the x-axis and the graph opens upwards, there are no x-intercepts on the graph of F, hence there are no zeros.[7] With $F(0) = 5$, the y-intercept is $(0, 5)$. To get a third point, we can pick an arbitrary x-value to the left of the vertex or we could use symmetry: three units to the **right** of the vertex the y-value is 5, so the same must be true three units to the **left** of the vertex, at $x = -6$. Sure enough, $F(-6) = |-6+3|+2 = |-3|+2 = 5$. The range of F is $[2, \infty)$ with its minimum of 2 when $x = -3$ and F decreasing on $(-\infty, -3]$ then increasing on $[-3, \infty)$. The graph is in the middle of the next page on the left.

   (b) We see in the formula for $f(t)$ that t appears only once to the first power inside the absolute values, so we proceed to rewrite it in the form $a|t-h|+k$:

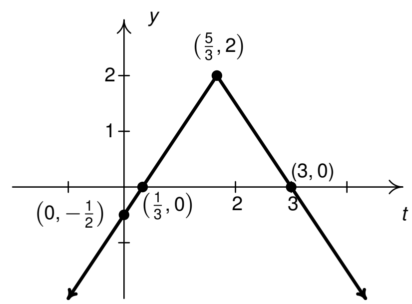
$$\begin{aligned} f(x) &= \frac{4-|5-3t|}{2} \\ &= -\frac{|5-3t|}{2}+\frac{4}{2} \\ &= \left(-\frac{1}{2}\right)\left|(-3)\left(t-\frac{5}{3}\right)\right|+2 \\ &= \left(-\frac{1}{2}\right)|-3|\left|t-\frac{5}{3}\right|+2 \\ &= -\frac{3}{2}\left|t-\frac{5}{3}\right|+2. \end{aligned}$$

[7] Alternatively, setting $|x+3|+2=0$ gives $|x+3|=-2$. Absolute values are never negative thus we have no solution.

Matching up the constants in the formula $f(t)$ to the parameters of $F(x)$ in Theorem 1.2, we identify $a = -\frac{3}{2}$, $h = \frac{5}{3}$ and $k = 2$. Hence the vertex is $\left(\frac{5}{3}, 2\right)$, and the graph is shaped like '$\wedge$' comprised of pieces of lines with slopes $\pm\frac{3}{2}$. To find the zeros of f, we set $f(t) = 0$. (We can use either expression here.) Solving $-\frac{3}{2}\left|t - \frac{5}{3}\right| + 2 = 0$, we get $\left|t - \frac{5}{3}\right| = \frac{4}{3}$, so $t - \frac{5}{3} = \pm\frac{4}{3}$. Hence our zeros are $t = \frac{1}{3}$ and $t = 3$, producing the t-intercepts $\left(\frac{1}{3}, 0\right)$ and $(3, 0)$. Using either formula gives $f(0) = -\frac{1}{2}$, so our y-intercept is $\left(0, -\frac{1}{2}\right)$. Plotting the vertex, along with the intercepts, gives us enough information to produce the graph below on the right. The range is $(-\infty, 2]$ with a maximum of 2 at $t = \frac{5}{3}$ and f is increasing on $\left(-\infty, \frac{5}{3}\right]$ then decreasing on $\left[\frac{5}{3}, \infty\right)$.



$F(x) = |x + 3| + 2$



$f(t) = \dfrac{4 - |5 - 3t|}{2}$

(c) We are unable to apply Theorem 1.2 to $G(x) = |x - 2| - x$ because there is an x both inside and outside of the absolute value. We can, however, rewrite the function as a piecewise function using Definition 1.9. Our first step is to rewrite $|x - 2|$ as a piecewise function:

$$|x-2| = \begin{cases} -(x-2) & \text{if } x - 2 < 0 \\ x - 2 & \text{if } x - 2 \geq 0 \end{cases} \longrightarrow |x-2| = \begin{cases} -x+2 & \text{if } x < 2 \\ x - 2 & \text{if } x \geq 2 \end{cases}$$

Hence, for $x < 2$, $|x - 2| = -x + 2$ so $G(x) = |x - 2| - x = (-x + 2) - x = -2x + 2$. Likewise, for $x \geq 2$, $|x - 2| = x - 2$ so $G(x) = |x - 2| - x = x - 2 - x = -2$.

$$G(x) = |x-2| - x = \begin{cases} (-x+2) - x & \text{if } x < 2 \\ (x-2) - x & \text{if } x \geq 2 \end{cases} \longrightarrow G(x) = \begin{cases} -2x+2 & \text{if } x < 2 \\ -2 & \text{if } x \geq 2 \end{cases}$$

To find the zeros of G, we set $G(x) = 0$. Solving $|x - 2| - x = 0$ can be problematic, given that x is both inside and outside of the absolute values.[8] We can, however, use the piecewise description of $G(x)$. With $G(x) = -2x + 2$ for $x < 2$, we solve $-2x + 2 = 0$ to get $x = 1$. This works because $1 < 2$, so we have $x = 1$ as the zero of G corresponding to the x-intercept $(1, 0)$. The other piece of $G(x)$ is $G(x) = -2$ which is never 0. For the y-intercept, we find $G(0) = 2$, and get $(0, 2)$.

To graph $y = G(x)$, we have the line $y = -2x + 2$ which contains $(0, 2)$ and $(1, 0)$ and continues to a hole at $(2, -2)$. At this point, $G(x) = -2$ takes over and we have a horizontal line containing

[8] We'll return to this momentarily.

$(2, -2)$ extending indefinitely to the right. The range of G is $[-2, \infty)$ with a minimum value of -2 attained for all $x \geq 2$. Moreover, G is decreasing on $(-\infty, 2]$ and then constant on $[2, \infty)$. The graph is below on the left.

(d) Once again we are unable to use Theorem 1.2 because $g(t) = |t - 2| - |t|$ has two absolute values with no apparent way to combine them. Thus we proceed by re-writing the function g with two separate applications of Definition 1.9 to remove each instance of the absolute values. To start with we have:

$$|t| = \begin{cases} -t & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases} \quad \text{and} \quad |t-2| = \begin{cases} -t+2 & \text{if } t < 2 \\ t-2 & \text{if } t \geq 2 \end{cases}$$

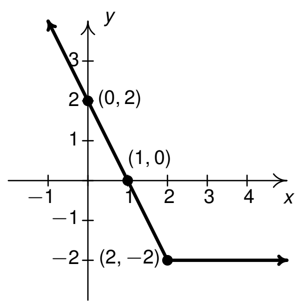
Taken together, these break the domain into **three** pieces: $t < 0$, $0 \leq t < 2$ and $t \geq 2$. For $t < 0$, $|t| = -t$ and $|t-2| = -t+2$. Therefore $g(t) = |t-2| - |t| = (-t+2) - (-t) = 2$ for $t < 0$. For $0 \leq t < 2$, $|t| = t$ and $|t - 2| = -t + 2$, so $g(t) = |t - 2| - |t| = (-t + 2) - t) = -2t + 2$.

Last, for $t \geq 2$, $|t| = t$ and $|t - 2| = t - 2$, so $g(t) = |t - 2| - |t| = (t - 2) - (t) = -2$. Putting all three parts together yields:

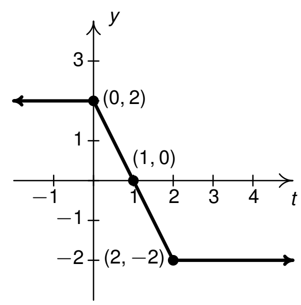
$$g(t) = |t-2| - |t| = \begin{cases} (-t+2) - (-t) & \text{if } t < 0 \\ (-t+2) - (t) & \text{if } 0 \leq t < 2 \\ (t-2) - (t) & \text{if } t \geq 2 \end{cases} = \begin{cases} 2 & \text{if } t < 0 \\ -2t+2 & \text{if } 0 \leq t < 2 \\ -2 & \text{if } t \geq 2 \end{cases}$$

As with the previous example, we'll delay discussing the absolute value algebra needed to find the zeros of g and use the piecewise description instead. To graph g, we have the horizontal line $y = 2$ up to, but not including, the point $(0, 2)$. For $0 \leq t < 2$, we have the line $y = -2t + 2$ which has a y-intercept at $(0, 2)$ (thus picking up where the first part left off) and a t-intercept at $(1, 0)$. This piece ends with a hole at $(2, -2)$ which is promptly plugged by the horizontal line $y = -2$ for $t \geq 2$. Hence the only zero of t is $t = 1$.

The range of g is $[-2, 2]$ with a minimum of -2 achieved for all $t \geq 2$, and a maximum of 2 for $t \leq 0$. We note that g is constant on $(-\infty, 0]$ and $[2, \infty)$, but with different values, and g is decreasing on $[0, 2]$. The graph is given below on the right.



$G(x) = |x - 2| - x$



$g(t) = |t - 2| - |t|$

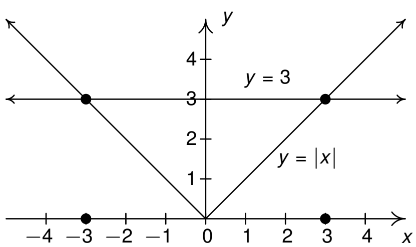
2. We are told to use Theorem 1.2 to find a formula for $H(x)$ so we start with $H(x) = a|x - h| + k$ and look for real numbers a, h and k that make sense. The vertex is labeled as $(1, 3)$, meaning $h = 1$ and $k = 3$. Hence we know $H(x) = a|x - 1| + 3$, so all that is left for us to find is the value of a. The only other point labeled for us is $(0, 1)$, meaning $H(0) = 1$. Substituting $x = 0$ into our formula for $H(x)$ gives: $H(0) = a|0 - 1| + 3 = a + 3$. Given that $H(0) = 1$, we have $a + 3 = 1$, so $a = -2$. Our final answer is $H(x) = -2|x - 1| + 3$. □

If nothing else, Example 1.3.2 demonstrates the value of **changing forms** of functions and the utility of the interplay between algebraic and graphical descriptions of functions. These themes resonate time and time again in this and later courses in Mathematics.

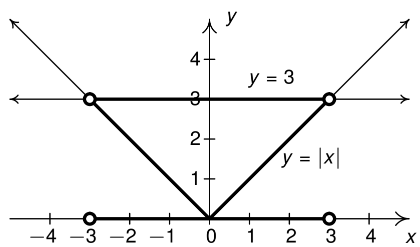
1.3.2 Graphical Solution Techniques for Equations and Inequalities

Consider the basic equation and related inequalities: $|x| = 3$, $|x| < 3$ and $|x| > 3$. At some point you learned how to solve these using properties of the absolute value inspired by the distance definition. (If not, see Section A.7.) While there is nothing wrong with this understanding, we wish to use these problems to motivate powerful graphical techniques which we'll use to solve more complicated equations and inequalities in this section, and in many other sections of the textbook.

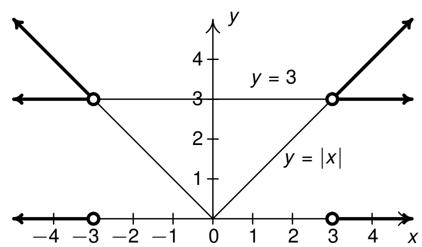
To that end, let's call $f(x) = |x|$ and $g(x) = 3$. If we graph $y = f(x)$ and $y = g(x)$ on the same set of axes then, by looking for x values where $f(x) = g(x)$, we are looking for x-values which have the same y-value on both graphs. That is, the solutions to $f(x) = g(x)$ are the x-coordinates of the **intersection points** of the two graphs. We graph $y = f(x) = |x|$ (the characteristic '$\vee$') along with $y = g(x) = 3$ (the horizontal line) below on the far left. Indeed, the two graphs intersect at $(-3, 3)$ and $(3, 3)$ so our solutions to $f(x) = g(x)$ are the x-values of these points, $x = \pm 3$.



Graphically solving $|x| = 3$.



Graphically solving $|x| < 3$.



Graphically solving $|x| > 3$.

Likewise, if we wish to solve $|x| < 3$, we can view this as a functional inequality $f(x) < g(x)$ which means we are looking for the x-values where the $f(x)$ values are less than the corresponding $g(x)$ values. On the graphs, this means we'd be looking for the x-values where the y-values of $y = f(x)$ are less than, hence **below**, those on the graph of $y = g(x)$.

In the middle picture above we see that the graph of f is below the graph of g between $x = -3$ and $x = 3$, so our solution is $-3 < x < 3$, or in interval notation, $(-3, 3)$. Finally, the inequality $|x| > 3$ is equivalent to $f(x) > g(x)$ so we are looking for the x-values where the graph of f is **above** the graph of g.[9] The picture

[9]Solving $f(x) > g(x)$ is equivalent to solving $g(x) < f(x)$ - that is, finding where the graph of g is below the graph of f.

on the far right on the previous page shows that this is true for all $x < -3$ or for all $x > 3$. In interval notation, the solution set is $(-\infty, -3) \cup (3, \infty)$.

The methodology and reasoning behind solving the above equation and inequalities extend to any pair of functions f and g, since when graphed on the same set of axes, function outputs are always the dependent variable or the ordinate (second coordinate) of the ordered pairs which comprise the graph. In general:

Graphical Interpretation of Equations and Inequalities

Suppose f and g are functions whose domains and ranges are sets of real numbers.

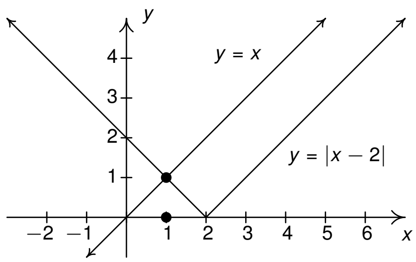
- The solutions to $f(x) = g(x)$ are the x-values where the graphs of f and g intersect.
- The solution to $f(x) < g(x)$ is the set of x-values where the graph of f is **below** the graph of g.
- The solution to $f(x) > g(x)$ is the set of x-values where the graph of f **above** the graph of g.

Let's return to Example 1.3.2 where were asked to find the zeros of the functions $G(x) = |x-2| - x$ and $g(t) = |t-2| - |t|$. In that Example, instead of tackling the algebra involving the absolute values head on we rewrote each function as a piecewise-defined function and obtained our solutions that way.

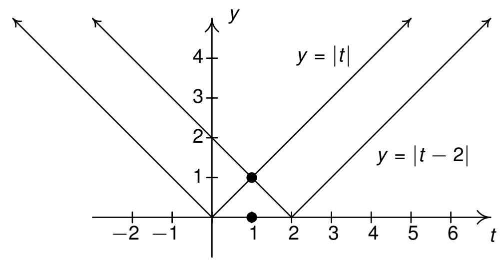
Let's see what this looks like graphically. Note that solving $|x-2| - x = 0$ is equivalent to solving $|x-2| = x$. We graphed $y = |x-2|$ and $y = x$ on the same set of axes on the left of the top of the next page and it appears as if we have just one point of intersection, corresponding to just one solution.

Indeed, we can **show** that there is just one point of intersection. The graph of $y = |x-2|$ is comprised of parts of two lines, $y = -(x-2)$ and $y = x - 2$. The first line has a slope of -1 and the second has slope 1. The line $y = x$ also has a slope 1 meaning it and the 'right half' of $y = |x-2|$ are parallel, so they never intersect. If our graphs are accurate enough, we may even be able to guess that the solution is $x = 1$, which we can verify by substituting $x = 1$ into $|x-2| = x$ and seeing that it checks.

Likewise, solving $|t-2| - |t| = 0$ is equivalent to solving $|t-2| = |t|$. We graphed $y = |t-2|$ and $y = |t|$ on the right at the top of the next page and used the same arguments to get the solution $t = 1$ here as well.



Graphically solving $|x-2| = x$.



Graphically solving $|t-2| = |t|$.

There is more to see here. Consider solving $|x-2| - x = 0$ algebraically using the techniques from a previous Algebra course (or Section A.7). Our first step would be to isolate the absolute value quantity: $|x-2| = x$. We then 'drop' the absolute value by paying the price of a '$\pm$': $x - 2 = \pm x$. This gives us two equations: $x - 2 = x$ and $x - 2 = -x$. The first equation, $x - 2 = x$ reduces to $-2 = 0$ which has no solution. The second equation, $x - 2 = -x$, does have a solution, namely $x = 1$.

How does the algebra tie into the graphs above? Instead of 'dropping' the absolute value and tagging the right hand side with a $\pm$, we can think about the piecewise definition of $|x-2|$ and write $|x-2| = \pm(x-2)$ depending on if $x < 2$ or if $x \geq 2$. That is, $|x - 2| = x$ is more precisely equivalent to the two equations: $-(x - 2) = x$ which is valid for $x < 2$ or $x - 2 = x$ which is valid for $x \geq 2$.

Graphically, the first equation is looking for intersection points between the 'left half' of the '$\vee$' of $y = |x-2|$ and the line $y = x$. Indeed, $-(x - 2) = x$ is equivalent to $x - 2 = -x$ from which we obtain our solution $x = 1$. Likewise, the second equation, $x - 2 = x$ is looking for intersection points of the 'right half' of the '$\vee$' and the line $y = x$, but there is none. The equation $-2 = 0$ is telling us that for us to have any solutions, the lines $y = x - 2$ and $y = x$, which have the same slope, must also have the same y-intercepts: that is, -2 would have to equal 0 and that's just silly.

Similarly, when solving $|t - 2| - |t| = 0$ or $|t - 2| = |t|$, we can use our graphs to prove that the only intersection point is when the 'left half' of $y = |t - 2|$ intersects the 'right half' of $y = |t|$ - that is, when $-(t-2) = t$. The moral of the story is this: careful graphs can help us simplify the algebra, because we can narrow down the cases. This is especially useful in solving inequalities, as we'll see in our next example.

Example 1.3.3. Solve the following equations and inequalities.

1. $4 - |x| = 0.9x - 3.6$
2. $|t - 3| - |t| = 3$
3. $|x + 1| \geq \dfrac{x + 4}{2}$
4. $2 < |t - 1| \leq 5$

Solution.

1. We begin by graphing $y = 4 - |x|$ and $y = 0.9x - 3.6$ to look for intersection points. Using Theorem 1.2, we know that the graph of $y = 4 - |x| = -|x| + 4$ has a vertex at $(0, 4)$ and is a '$\wedge$' shape, so there are x-intercepts to find. Solving $4 - |x| = 0$, we get $|x| = 4$, or $x = \pm 4$. Hence, we have two x-intercepts: $(-4, 0)$ and $(4, 0)$.

   We know from Section A.5 that the graph of $y = 0.9x - 3.6$ is a line with slope 0.9 and y-intercept $(0, -3.6)$. To find the x-intercept here we solve $0.9x - 3.6 = 0$ and get $x = 4$. Hence, $(4, 0)$ is an x-intercept here as well, and we have stumbled upon one solution to $4 - |x| = 0.9x - 3.6$, namely $x = 4$. The question is if there are any other solutions. Our graph (below on the left) certainly looks as if there is just one intersection point, but we know from Theorem 1.2 that the slopes of the linear parts of $y = 4 - |x|$ are ± 1. The slope of $y = 0.9x - 3.6$ is 0.9 and $0.9 \neq 1$ so we know that the left hand side of the'$\wedge$' must meet up with the graph of the line because they are not parallel.[10]

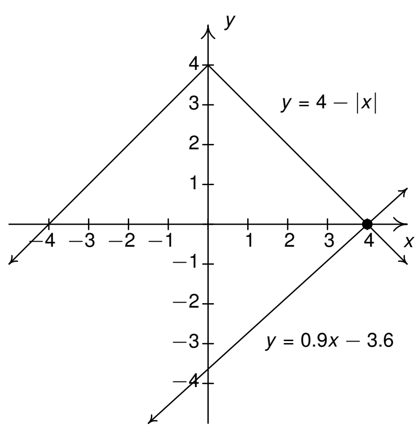
   Definition 1.9 tells us that when $x < 0$, $|x| = -x$, so $4 - |x| = 4 - (-x) = 4 + x$. Hence we set about solving $4 + x = 0.9x - 3.6$ and get $x = -76$. Both $x = -76$ and $x = 4$ check in our original equation, $4 - |x| = 0.9x - 3.6$, so we have found our two solutions.[11]

[10]See Theorem A.3.

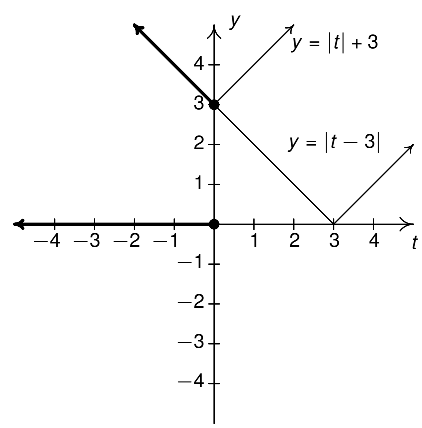
[11]Our picture shows only one of the solutions. We encourage you to take the time with a graphing utility to get the picture to show both points of intersection.

2. While we could graph $y = |t-3| - |t|$ and $y = 3$ to help us find solutions, we choose to rewrite the equation as $|t-3| = |t| + 3$. This way, we have somewhat easier graphs to deal with, namely $y = |t-3|$ and $y = |t| + 3$. The first graph, $y = |t-3|$, has a vertex at $(3, 0)$ and is shaped like a '$\vee$' with slopes ± 1 and a y-intercept of $(0, 3)$. The second graph, $y = |t| + 3$, has a vertex at $(0, 3)$ and is also shaped like a '$\vee$,' with slopes ± 1, and has no t-intercepts.

   To our surprise and delight, the graphs (below on the right) appear to overlap for $t \leq 0$. Indeed, for $t \leq 0$, $|t-3| = -(t-3) = -t + 3$ and $|t| + 3 = -t + 3$. Since the formulas are **identical** for these values of t, our solutions are all values of t with $t \leq 0$. Using interval notation, we state our solution as $(-\infty, 0]$. (The other parts of the graphs are non-intersecting parallel lines so we ignored them.)



Solving $4 - |x| = 0.9x - 3.6$.



Solving $|t-3| - |t| = 3$.

3. To solve $|x+1| \geq \frac{x+4}{2}$, we first graph $y = |x+1|$ and $y = \frac{x+4}{2} = \frac{1}{2}x + 2$. The former is '$\vee$' shaped with a vertex at $(-1, 0)$ and a y-intercept of $(0, 1)$. The latter is a line with y-intercept $(0, 2)$, slope $m = \frac{1}{2}$ and x-intercept $(-4, 0)$. The picture in the middle of the next page on the right shows two intersection points. To find these, we solve the equations: $-(x+1) = \frac{x+4}{2}$, obtaining $x = -2$, and $x + 1 = \frac{x+4}{2}$ obtaining $x = 2$.

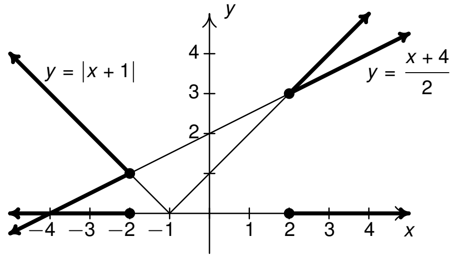
   Graphically, the inequality $|x+1| \geq \frac{x+4}{2}$ is looking for where the graph of $y = |x+1|$, the '$\vee$,' intersects $(=)$ or is above $(>)$ the line $y = \frac{x+4}{2}$. The graph shows this happening whenever $x \leq -2$ or $x \geq 2$. Using interval notation, our solution is $(-\infty, -2] \cup [2, \infty)$. While we cannot check every single x value individually, choosing test values $x < -2$, $x = 2$, $-2 < x < 2$, $x = 2$, and $x > 2$ to see if the original inequality $|x+1| \geq \frac{x+4}{2}$ holds would help us verify our solution.

4. Recall that the inequality $2 < |t-1| \leq 5$ is an example of a 'compound' inequality in that is two inequalities in one.[12] The values of t in the solution set need to satisfy $2 < |t-1|$ **and** $|t-1| \leq 5$. To help us sort through the cases, we graph the horizontal lines $y = 2$ and $y = 5$ along with the '$\vee$' shaped $y = |t-1|$ with vertex $(1, 0)$ and y-intercept $(0, 1)$.

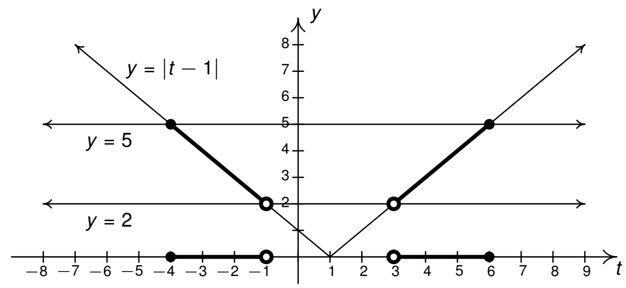
[12] See Example A.4.2 for examples of linear compound inequalities.

Geometrically, we are looking for where $y = |t-1|$ is strictly **above** the line $y = 2$ but **below** (or meets) the line $y = 5$. Solving $|t-1| = 2$ gives $t = -1$ and $t = 3$ whereas solving $|t-1| = 5$ gives $t = -4$ or $t = 6$. Per the graph (below on the right), we see that $y = |t-1|$ lies between $y = 2$ and $y = 5$ when $-4 \leq t < -1$ and again when $3 < t \leq 6$.

In interval notation, our solution is $[-4, -1) \cup (3, 6]$. As with the previous example, it is impossible to check each and every one of these solutions, but choosing t values both in and around the solution intervals would give us some numerical confidence we have the correct and complete solution.



Solving $|x+1| \geq \frac{x+4}{2}$



Solving $2 < |t-1| \leq 5$

□

We will see the interplay of Algebra and Geometry throughout the rest of this course. In the Exercises, do not hesitate to use whatever mix of algebraic and graphical methods you deem necessary to solve the given equation or inequality. Indeed, there is great value in checking your algebraic answers graphically and vice-versa.

One of the classic applications of inequalities involving absolute values is the notion of tolerances.[13] Recall that for real numbers x and c, the quantity $|x-c|$ may be interpreted as the distance from x to c. Solving inequalities of the form $|x-c| \leq d$ for $d > 0$ can then be interpreted as finding all numbers x which lie within d units of c. We can think of the number d as a 'tolerance' and our solutions x as being within an accepted tolerance of c. We use this principle in the next example.

Example 1.3.4. Suppose a manufacturer needs to produce a 24 inch by 24 inch square piece of particle board as part of a home office desk kit. How close does the side of the piece of particle board need to be cut to 24 inches to guarantee that the area of the piece is within a tolerance of 0.25 square inches of the target area of 576 square inches?

Solution. Let x denote the length of the side of the square piece of particle board so that the area of the board is x^2 square inches. Our tolerance specifies that the area of the board, x^2, needs to be within 0.25 square inches of 576. Mathematically, this translates to $|x^2 - 576| \leq 0.25$. Rewriting, we get $-0.25 \leq x^2 - 576 \leq 0.25$, or $575.75 \leq x^2 \leq 576.25$. At this point, we take advantage of the fact that the square root is increasing.[14] Therefore, taking square roots preserves the inequality. When simplifying, we keep in mind that since x represents a length, $x > 0$.

[13] The underlying concept of Calculus can be phrased in terms of tolerances, so this is well worth your attention.

[14] This means that for $a, b \geq 0$, if $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$.

$$\begin{array}{rcll} 575.75 \leq & x^2 & \leq 576.25 & \\ \sqrt{575.75} \leq & \sqrt{x^2} & \leq \sqrt{576.25} & \text{(take square roots.)} \\ \sqrt{575.75} \leq & |x| & \leq \sqrt{576.25} & (\sqrt{x^2} = |x|) \\ \sqrt{575.75} \leq & x & \leq \sqrt{576.25} & (|x| = x \text{ since } x > 0) \end{array}$$

The side of the piece of particle board must be between $\sqrt{575.75} \approx 23.995$ and $\sqrt{576.25} \approx 24.005$ inches, a tolerance of (approximately) 0.005 inches of the target length of 24 inches, to ensure that the area is within 0.25 square inches of 576. □

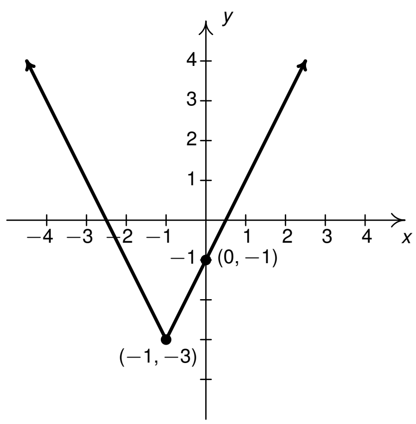
1.3.3 Exercises

In Exercises 1 - 6, graph the function using Theorem 1.2. Find the axis intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, the maximum and minimum of each function, if they exist, and list the intervals on which the function is increasing, decreasing or constant.

1. $f(x) = |x + 4|$
2. $f(x) = |x| + 4$
3. $f(x) = |4x|$
4. $g(t) = -3|t|$
5. $g(t) = 3|t + 4| - 4$
6. $g(t) = \dfrac{1}{3}|2t - 1|$

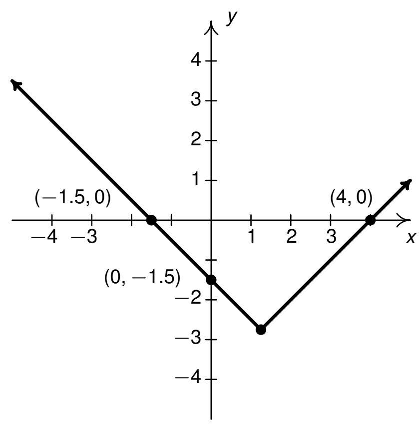
In Exercises 7 - 10, find a formula for each function below in the form $F(x) = a|x - h| + k$.

7.



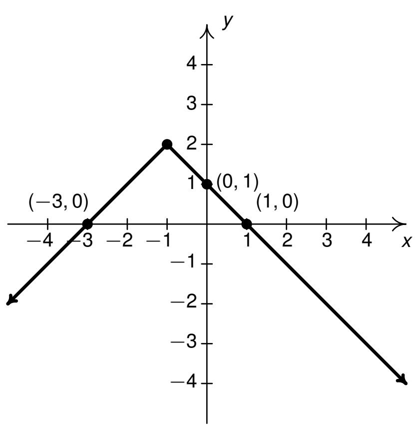
$y = F(x)$

8.



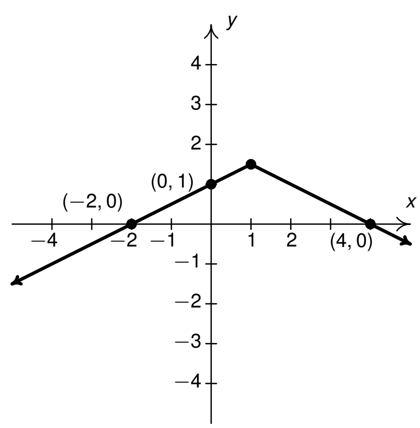
$y = F(x)$

9.



$y = F(x)$

10.



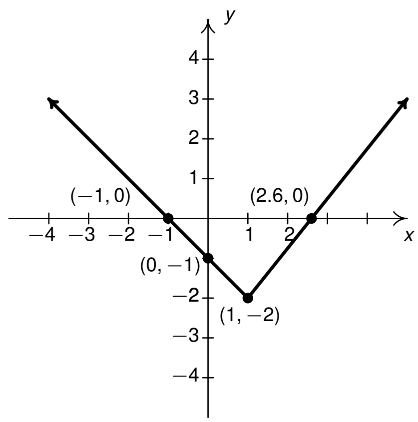
$y = F(x)$

11. With help from a graphing utility, graph the following pairs of functions on the same set of axes:

- $f(x) = 2 - x$ and $g(x) = |2 - x|$
- $f(x) = x^2 - 4$ and $g(x) = |x^2 - 4|$
- $f(x) = x^3$ and $g(x) = |x^3|$
- $f(x) = \sqrt{x} - 4$ and $g(x) = |\sqrt{x} - 4|$

Choose more functions $f(x)$ and graph $y = f(x)$ alongside $y = |f(x)|$ until you can explain how, in general, one would obtain the graph of $y = |f(x)|$ given the graph of $y = f(x)$. How does your explanation tie in with with Definition 1.9?

12. Explain why the function below cannot be written in the form $F(x) = a|x - h| + k$. Write $F(x)$ as a piecewise-defined linear function.

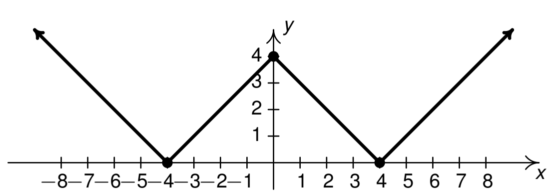


$y = F(x)$

In Exercises 13 - 18, graph the function by rewriting each function as a piecewise defined function using Definition 1.9. Find the axis intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, the maximum and minimum of each function, if they exist, and list the intervals on which the function is increasing, decreasing or constant.

13. $f(x) = x + |x| - 3$
14. $f(x) = |x + 2| - x$
15. $f(x) = |x + 2| - |x|$
16. $g(t) = |t + 4| + |t - 2|$
17. $g(t) = \dfrac{|t + 4|}{t + 4}$
18. $g(t) = \dfrac{|2 - t|}{2 - t}$

19. With the help of your classmates, find an absolute value function whose graph is given below.



In Exercises 20 - 31, solve the equation.

20. $|x| = 6$
21. $|3x - 1| = 10$
22. $|4 - x| = 7$
23. $4 - |t| = 3$
24. $2|5t + 1| - 3 = 0$
25. $|7t - 1| + 2 = 0$
26. $\dfrac{5 - |w|}{2} = 1$
27. $\frac{2}{3}|5 - 2w| - \frac{1}{2} = 5$
28. $|w| = w + 3$
29. $|2x - 1| = x + 1$
30. $4 - |x| = 2x + 1$
31. $|x - 4| = x - 5$

Solve the equations in Exercises 32 - 37 using the property that if $|a| = |b|$ then $a = \pm b$.

32. $|3x - 2| = |2x + 7|$
33. $|3x + 1| = |4x|$
34. $|1 - 2x| = |x + 1|$
35. $|4 - t| - |t + 2| = 0$
36. $|2 - 5t| = 5|t + 1|$
37. $3|t - 1| = 2|t + 1|$

In Exercises 38 - 53, solve the inequality. Write your answer using interval notation.

38. $|3x - 5| \leq 4$
39. $|7x + 2| > 10$
40. $|2t + 1| - 5 < 0$
41. $|2 - t| - 4 \geq -3$
42. $|3w + 5| + 2 < 1$
43. $2|7 - w| + 4 > 1$
44. $2 \leq |4 - x| < 7$
45. $1 < |2x - 9| \leq 3$
46. $|t + 3| \geq |6t + 9|$
47. $|t - 3| - |2t + 1| < 0$
48. $|1 - 2x| \geq x + 5$
49. $x + 5 < |x + 5|$
50. $x \geq |x + 1|$
51. $|2x + 1| \leq 6 - x$
52. $t + |2t - 3| < 2$
53. $|3 - t| \geq t - 5$

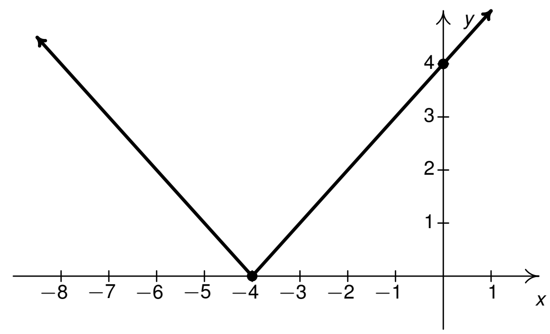
54. Show that if δ is a real number with $\delta > 0$, the solution to $|x - a| < \delta$ is the interval: $(a - \delta, a + \delta)$. That is, an interval centered at a with 'radius' δ.

55. The Triangle Inequality for real numbers states that for all real numbers x and a, $|x + a| \leq |x| + |a|$ and, moreover, $|x + a| = |x| + |a|$ if and only if x and a are both positive, both negative, or one or the other is 0. Graph each pair of functions below on the same pair of axes and use the graphs to verify the triangle inequality in each instance.

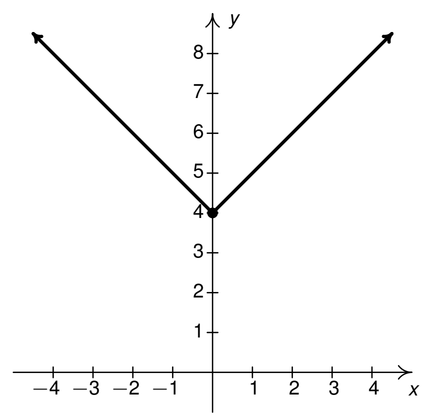
   - $f(x) = |x + 2|$ and $g(x) = |x| + 2$.
   - $f(x) = |x + 4|$ and $g(x) = |x| + 4$.

1.3.4 Answers

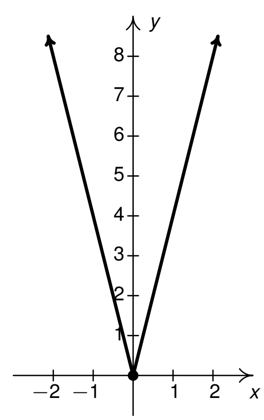
1. $f(x) = |x+4|$
   x-intercept $(-4, 0)$
   y-intercept $(0, 4)$
   Domain $(-\infty, \infty)$
   Range $[0, \infty)$
   Decreasing on $(-\infty, -4]$
   Increasing on $[-4, \infty)$
   Minimum is 0 at $(-4, 0)$
   No maximum



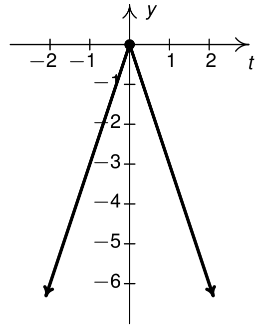
2. $f(x) = |x| + 4$
   No x-intercepts
   y-intercept $(0, 4)$
   Domain $(-\infty, \infty)$
   Range $[4, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Minimum is 4 at $(0, 4)$
   No maximum



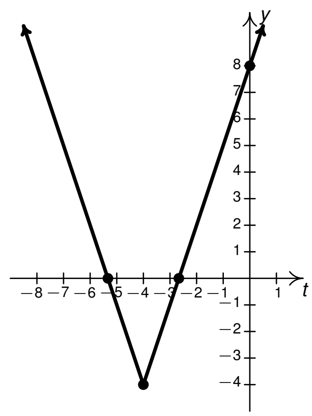
3. $f(x) = |4x|$
   x-intercept $(0, 0)$
   y-intercept $(0, 0)$
   Domain $(-\infty, \infty)$
   Range $[0, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Minimum is 0 at $(0, 0)$
   No maximum



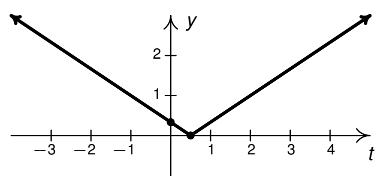
4. $g(t) = -3|t|$
   t-intercept $(0, 0)$
   y-intercept $(0, 0)$
   Domain $(-\infty, \infty)$
   Range $(-\infty, 0]$
   Increasing on $(-\infty, 0]$
   Decreasing on $[0, \infty)$
   Maximum is 0 at $(0, 0)$
   No minimum



5. $g(t) = 3|t + 4| - 4$
   t-intercepts $\left(-\frac{16}{3}, 0\right)$, $\left(-\frac{8}{3}, 0\right)$
   y-intercept $(0, 8)$
   Domain $(-\infty, \infty)$
   Range $[-4, \infty)$
   Decreasing on $(-\infty, -4]$
   Increasing on $[-4, \infty)$
   Minimum is -4 at $(-4, -4)$
   No maximum



6. $g(t) = \frac{1}{3}|2t - 1|$
   t-intercepts $\left(\frac{1}{2}, 0\right)$
   y-intercept $\left(0, \frac{1}{3}\right)$
   Domain $(-\infty, \infty)$
   Range $[0, \infty)$
   Decreasing on $\left(-\infty, \frac{1}{2}\right]$
   Increasing on $\left[\frac{1}{2}, \infty\right)$
   Minimum is 0 at $\left(\frac{1}{2}, 0\right)$
   No maximum



7. $F(x) = 2|x + 1| - 3$

8. $F(x) = |x - 1.25| - 2.75$

9. $F(x) = -|x + 1| + 2$

10. $F(x) = -\frac{1}{2}|x + 1| + \frac{3}{2}$

11. In each case, the graph of g can be obtained from the graph of f by reflecting the portion of the graph of f which lies below the x-axis about the x-axis. This meshes with Definition 1.9 since what we are doing algebraically is making the negative y-values positive.

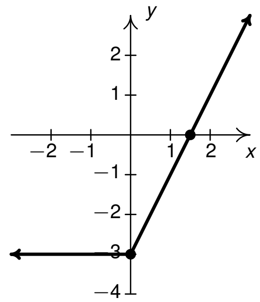
12. If $F(x) = a|x-h|+k$, then for the vertex to be at $(1,-2)$, $h=1$ and $k=-2$ so $F(x) = a|x-1|-2$. Since $(0,-1)$ is on the graph, $F(0) = -1$ so $-1 = a|0-1|-2$ which means $a = 1$. This means $F(x) = |x-1|-2$. However, $(2.6, 0)$ is also on the graph, so it should work out that $F(2.6) = 0$. However, we find $F(2.6) = |2.6-1|-2 = -0.4 \neq 0$.

$$F(x) = \begin{cases} -x-1 & \text{if } x \leq 1, \\ \frac{5}{4}x - \frac{13}{4} & \text{if } x \geq 1, \end{cases}$$

13. Re-write $f(x) = x + |x| - 3$ as

$$f(x) = \begin{cases} -3 & \text{if} \quad x < 0 \\ 2x-3 & \text{if} \quad x \geq 0 \end{cases}$$

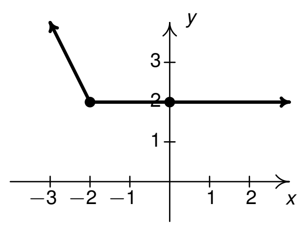
x-intercept $\left(\frac{3}{2}, 0\right)$
y-intercept $(0,-3)$
Domain $(-\infty,\infty)$
Range $[-3,\infty)$
Increasing on $[0,\infty)$
Constant on $(-\infty,0]$
Minimum is -3 at $(x,-3)$ where $x \leq 0$
No maximum



14. Re-write $f(x) = |x+2| - x$ as

$$f(x) = \begin{cases} -2x-2 & \text{if} \quad x < -2 \\ 2 & \text{if} \quad x \geq -2 \end{cases}$$

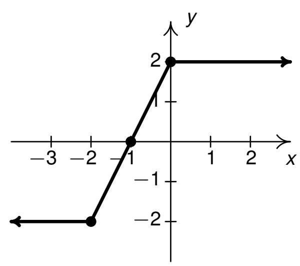
No x-intercepts
y-intercept $(0,2)$
Domain $(-\infty,\infty)$
Range $[2,\infty)$
Decreasing on $(-\infty,-2]$
Constant on $[-2,\infty)$
Minimum is 2 at every point $(x,2)$ where $x \geq -2$
No maximum



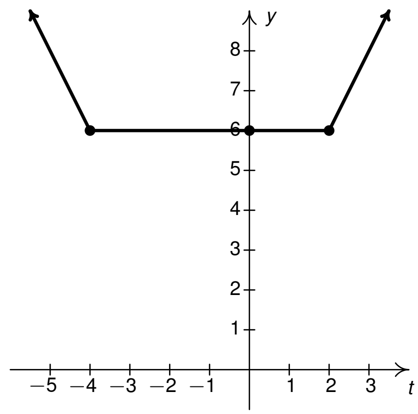
15. Re-write $f(x) = |x+2| - |x|$ as

$$f(x) = \begin{cases} -2 & \text{if} \quad x < -2 \\ 2x+2 & \text{if} \quad -2 \leq x < 0 \\ 2 & \text{if} \quad x \geq 0 \end{cases}$$

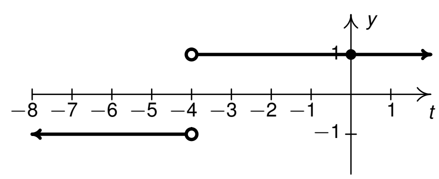
x-intercept $(-1,0)$
y-intercept $(0,2)$
Domain $(-\infty,\infty)$
Range $[-2,2]$
Increasing on $[-2,0]$
Constant on $(-\infty,-2]$
Constant on $[0,\infty)$
Minimum is -2 at $(x,-2)$ where $x \leq -2$
Maximum is 2 at $(x,2)$ where $x \geq 0$



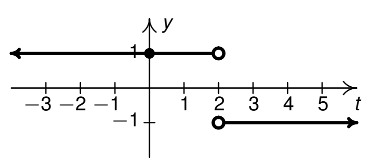
16. Re-write $g(t) = |t+4| + |t-2|$ as
$$g(t) = \begin{cases} -2t-2 & \text{if} \quad t < -4 \\ 6 & \text{if} \quad -4 \leq t < 2 \\ 2t+2 & \text{if} \quad t \geq 2 \end{cases}$$
No t-intercept
y-intercept $(0, 6)$
Domain $(-\infty, \infty)$
Range $[6, \infty)$
Decreasing on $(-\infty, -4]$
Constant on $[-4, 2]$
Increasing on $[2, \infty)$
Minimum is 6 at $(t, 6)$ where $-4 \leq t \leq 2$
No maximum



17. Re-write $g(t) = \dfrac{|t+4|}{t+4}$ as
$$g(t) = \begin{cases} -1 & \text{if} \quad t < -4 \\ 1 & \text{if} \quad t > -4 \end{cases}$$
No t-intercept
y-intercept $(0, 1)$
Domain $(-\infty, -4) \cup (-4, \infty)$
Range $\{-1, 1\}$
Constant on $(-\infty, -4)$
Constant on $(-4, \infty)$
Minimum is -1 at every point $(t, -1)$ where $t < -4$
Maximum is 1 at $(t, 1)$ where $t > -4$



18. Re-write $g(t) = \dfrac{|2-t|}{2-t}$ as
$$g(t) = \begin{cases} 1 & \text{if} \quad t < 2 \\ -1 & \text{if} \quad t > 2 \end{cases}$$
No t-intercept
y-intercept $(0, 1)$
Domain $(-\infty, 2) \cup (2, \infty)$
Range $\{-1, 1\}$
Constant on $(-\infty, 2)$
Constant on $(2, \infty)$
Minimum is -1 at $(t, -1)$ where $t > 2$
Maximum is 1 at every point $(t, 1)$ where $t < 2$



19. $f(x) = ||x| - 4|$

20. $x = -6$ or $x = 6$

21. $x = -3$ or $x = \frac{11}{3}$

22. $x = -3$ or $x = 11$

23. $t = -1$ or $t = 1$

24. $t = -\frac{1}{2}$ or $t = \frac{1}{10}$

25. no solution

26. $w = -3$ or $w = 3$

27. $w = -\frac{13}{8}$ or $w = \frac{53}{8}$

28. $w = -\frac{3}{2}$

29. $x = 0$ or $x = 2$

30. $x = 1$

31. no solution

32. $x = -1$ or $x = 9$
33. $x = -\frac{1}{7}$ or $x = 1$
34. $x = 0$ or $x = 2$
35. $t = 1$
36. $t = -\frac{3}{10}$
37. $t = \frac{1}{5}$ or $t = 5$
38. $\left[\frac{1}{3}, 3\right]$
39. $\left(-\infty, -\frac{12}{7}\right) \cup \left(\frac{8}{7}, \infty\right)$
40. $(-3, 2)$
41. $(-\infty, 1] \cup [3, \infty)$
42. No solution
43. $(-\infty, \infty)$
44. $(-3, 2] \cup [6, 11)$
45. $[3, 4) \cup (5, 6]$
46. $\left[-\frac{12}{7}, -\frac{6}{5}\right]$
47. $(-\infty, -4) \cup \left(\frac{2}{3}, \infty\right)$
48. $\left(-\infty, -\frac{4}{3}\right] \cup [6, \infty)$
49. $(-\infty, -5)$
50. No Solution.
51. $\left[-7, \frac{5}{3}\right]$
52. $\left(1, \frac{5}{3}\right)$
53. $(-\infty, \infty)$

1.4 Quadratic Functions

1.4.1 Graphs of Quadratic Functions

You may recall studying quadratic equations in a previous Algebra course. If not, you may wish to refer to Section A.10 to revisit this topic. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

Definition 1.10. A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

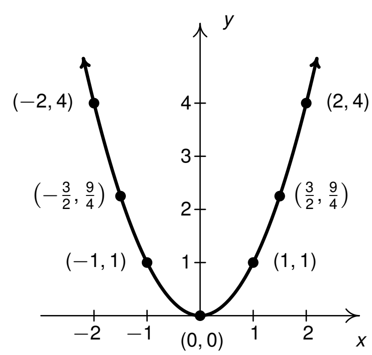
where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

As in Definitions 1.4 and 1.5, the independent variable in Definition 1.10 is x while the values a, b and c are parameters. Note that $a \neq 0$ - otherwise we would have a linear function (see Definition 1.5).

The most basic quadratic function is $f(x) = x^2$, the squaring function, whose graph appears below along with a corresponding table of values. Its shape may look familiar from your previous studies in Algebra – it is called a **parabola**. The point $(0, 0)$ is called the **vertex** of the parabola because it is the sole point where the function obtains its extreme value, in this case, a minimum of 0 when $x = 0$.

Indeed, the range of $f(x) = x^2$ appears to be $[0, \infty)$ from the graph. We can substantiate this algebraically since for all x, $f(x) = x^2 \geq 0$. This tells us that the range of f is a subset of $[0, \infty)$. To show that the range of f actually equals $[0, \infty)$, we need to show that every real number c in $[0, \infty)$ is in the range of f. That is, for every $c \geq 0$, we have to show c is an output from f. In other words, we have to show there is a real number x so that $f(x) = x^2 = c$. Choosing $x = \sqrt{c}$, we find $f(x) = f(\sqrt{c}) = (\sqrt{c})^2 = c$, as required.[1]

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



$f(x) = x^2$

The techniques we used to graph many of the absolute value functions in Section 1.3 can be applied to quadratic functions, too. In fact, knowing the graph of $f(x) = x^2$ enables us to graph **every** quadratic function, but there's some extra work involved. We start with the following theorem:

[1] We'll revisit this argument (with an inequality) in Section 6.1.

Theorem 1.3. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a(x-h)^2 + k$ is a parabola with vertex (h, k). If $a > 0$, the graph resembles '$\smile$.' If $a < 0$, the graph resembles '$\frown$.' Moreover, the vertical line $x = h$ is the **axis of symmetry** of the graph of $y = F(x)$.

To prove Theorem 1.3 the reader is encouraged to revisit the discussion following the proof of Theorem 1.2, replacing every occurrence of absolute value notation with the squared exponent.[2] Alternatively, the reader can skip ahead and read the statement and proof of Theorem 2.1 in Section 2.1. In the meantime we put Theorem 1.3 to good use in the next example.

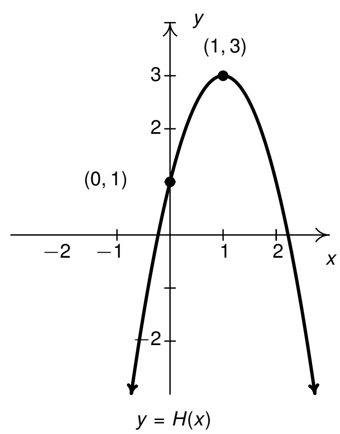
Example 1.4.1.

1. Graph the following functions using Theorem 1.3. Find the vertex, zeros and axis-intercepts (if any exist). Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

   (a) $f(x) = \dfrac{(x-3)^2}{2}$ (b) $g(x) = (x+2)^2 - 3$

   (c) $h(t) = -2(t-3)^2 + 1$ (d) $i(t) = \dfrac{(3-2t)^2 + 1}{2}$

2. Use Theorem 1.3 to write a possible formula for $H(x)$ whose graph is given below:

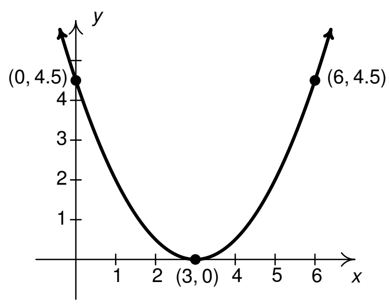


Solution.

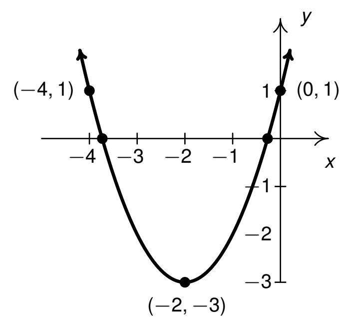
1. (a) For $f(x) = \frac{(x-3)^2}{2} = \frac{1}{2}(x-3)^2 + 0$, we identify $a = \frac{1}{2}$, $h = 3$ and $k = 0$. Thus the vertex is $(3, 0)$ and the parabola opens upwards. The only x-intercept is $(3, 0)$. Since $f(0) = \frac{1}{2}(0-3)^2 = \frac{9}{2}$, our y-intercept is $\left(0, \frac{9}{2}\right)$. To help us graph the function, it would be nice to have a third point and we'll use symmetry to find it. The y-value three units to the **left** of the vertex is 4.5, so the y-value must be 4.5 three units to the **right** of the vertex as well. Hence, we have our third point: $\left(6, \frac{9}{2}\right)$. From the graph, we get that the range is $[0, \infty)$ and see that f has the minimum value of 0 at $x = 3$ and no maximum. Also, f is decreasing on $(-\infty, 3]$ and increasing on $[3, \infty)$. The graph is the one on the left of the two on the next page.

[2] i.e., replace $|x|$ with x^2, $|c|$ with c^2, $|x-h|$ with $(x-h)^2$.

(b) For $g(x) = (x+2)^2 - 3 = (1)(x-(-2))^2 + (-3)$, we identify $a = 1$, $h = -2$ and $k = -3$. This means that the vertex is $(-2, -3)$ and the parabola opens upwards. Thus we have two x-intercepts. To find them, we set $y = g(x) = 0$ and solve. Doing so yields the equation $(x+2)^2 - 3 = 0$, or $(x+2)^2 = 3$. Extracting square roots gives us the two zeros of g: $x + 2 = \pm\sqrt{3}$, or $x = -2 \pm \sqrt{3}$. Our x-intercepts are $(-2-\sqrt{3}, 0) \approx (-3.73, 0)$ and $(-2+\sqrt{3}, 0) \approx (-0.27, 0)$. We find $g(0) = (0+2)^2 - 3 = 1$ so our y-intercept is $(0, 1)$. Using symmetry, we get $(-4, 1)$ as another point to help us graph. The range of g is $[-3, \infty)$. The minimum of g is -3 at $x = -2$, and g has no maximum. Moreover, g is decreasing on $(-\infty, -2]$ and g is increasing on $[-2, \infty)$. The graph is below on the right.



$f(x) = \frac{(x-3)^2}{2}$



$g(x) = (x+2)^2 - 3$

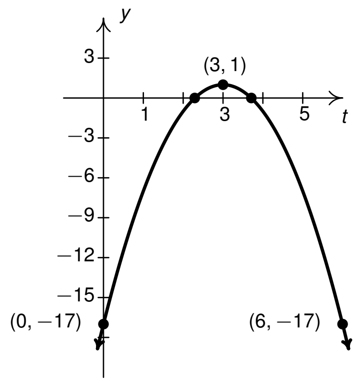
(c) Given $h(t) = -2(t-3)^2 + 1$, we identify $a = -2$, $h = 3$ and $k = 1$. Hence the vertex of the graph is $(3, 1)$ and the parabola opens downwards. Solving $h(t) = -2(t-3)^2 + 1 = 0$ gives $(t-3)^2 = \frac{1}{2}$. Extracting square roots[3] gives $t - 3 = \pm\frac{\sqrt{2}}{2}$, so that when we add 3 to each side,[4] we get $t = \frac{6 \pm \sqrt{2}}{2}$. Hence, our t-intercepts are $\left(\frac{6-\sqrt{2}}{2}, 0\right) \approx (2.29, 0)$ and $\left(\frac{6+\sqrt{2}}{2}, 0\right) \approx (3.71, 0)$. To find the y-intercept, we compute $h(0) = -2(0-3)^2 + 1 = -17$. Thus the y-intercept is $(0, -17)$. Using symmetry, we also have that $(6, -17)$ is on the graph which we show on the left side at the top of the next page.

(d) We have some work ahead of us to put $i(t)$ into a form we can use to exploit Theorem 1.3:

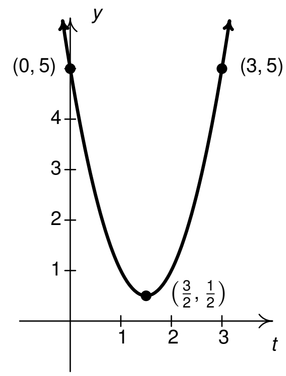
$$\begin{aligned} i(t) = \frac{(3-2t)^2+1}{2} &= \tfrac{1}{2}(-2t+3)^2 + \tfrac{1}{2} &&= \tfrac{1}{2}\left[-2\left(t - \tfrac{3}{2}\right)\right]^2 + \tfrac{1}{2} \\ &= \tfrac{1}{2}(-2)^2\left(t - \tfrac{3}{2}\right)^2 + \tfrac{1}{2} &&= 2\left(t - \tfrac{3}{2}\right)^2 + \tfrac{1}{2} \end{aligned}$$

We identify $a = 2$, $h = \frac{3}{2}$ and $k = \frac{1}{2}$. Hence our vertex is $\left(\frac{3}{2}, \frac{1}{2}\right)$ and the parabola opens upwards, meaning there are no t-intercepts. Since $i(0) = \frac{(3-2(0))^2+1}{2} = 5$, we get $(0, 5)$ as the y-intercept. Using symmetry, this means we also have $(3, 5)$ on the graph. The range is $\left[\frac{1}{2}, \infty\right)$ with the minimum of i, $\frac{1}{2}$, occurring when $t = \frac{3}{2}$. Also, i is decreasing on $\left(-\infty, \frac{3}{2}\right]$ and increasing on $\left[\frac{3}{2}, \infty\right)$. The graph is given on the right at the top of the next page.

[3] and rationalizing denominators!
[4] and get common denominators!



$h(t) = -2(t-3)^2 + 1$



$i(t) = \frac{(3-2t)^2+1}{2}$

2. We are instructed to use Theorem 1.3, so we know $H(x) = a(x-h)^2+k$ for some choice of parameters a, h and k. The vertex is $(1,3)$ so we know $h = 1$ and $k = 3$, and hence $H(x) = a(x-1)^2 + 3$. To find the value of a, we use the fact that the y-intercept, as labeled, is $(0,1)$. This means $H(0) = 1$, or $a(0-1)^2+3 = 1$. This reduces to $a+3 = 1$ or $a = -2$. Our final answer[5] is $H(x) = -2(x-1)^2+3$. □

A few remarks about Example 1.4.1 are in order. First note that none of the functions are in the form of Definition 1.10. However, if we took the time to perform the indicated operations and simplify, we'd find:

- $f(x) = \frac{(x-3)^2}{2} = \frac{1}{2}x^2 - 3x + \frac{9}{2}$
- $g(x) = (x+2)^2 - 3 = x^2 + 4x + 1$
- $h(t) = -2(t-3)^2 + 1 = -2t^2 + 12t - 17$
- $i(t) = \frac{(3-2t)^2+1}{2} = 2t^2 - 6t + 5$

While the y-intercepts of the graphs of the each of the functions are easier to see when the formulas for the functions are written in the form of Definition 1.10, the vertex is not. For this reason, the form of the functions presented in Theorem 1.3 are given a special name.

Definition 1.11. Standard and General Form of Quadratic Functions:

- The **general form** of the quadratic function f is $f(x) = ax^2 + bx + c$, where a, b and c are real numbers with $a \neq 0$.
- The **standard form** of the quadratic function f is $f(x) = a(x-h)^2 + k$, where a, h and k are real numbers with $a \neq 0$.

If we proceed as in the remarks following Example 1.4.1, we can convert any quadratic function given to us in standard form and convert to general form by performing the indicated operation and simplifying:

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= a\left(x^2 - 2hx + h^2\right) + k \\ &= ax^2 - 2ahx + ah^2 + k \\ &= ax^2 + (-2ah)x + (ah^2 + k). \end{aligned}$$

[5]The reader is encouraged to compare this example with number 2 of Example 1.3.2.

With the identifications $b = -2ah$ and $c = ah^2 + k$, we have written $f(x)$ in the form $f(x) = ax^2 + bx + c$. Likewise, through a process known as 'completing the square', we can take any quadratic function written in general form and rewrite it in standard form. We briefly review this technique in the following example – for a more thorough review the reader should see Section A.10.

Example 1.4.2. Graph the following functions. Find the vertex, zeros and axis-intercepts, if any exist. Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

1. $f(x) = x^2 - 4x + 3$.
2. $g(t) = 6 - 4t - 2t^2$

Solution.

1. We follow the procedure for completing the square in Section A.10. The only difference here is instead of the quadratic equation being set to 0, it is equal to $f(x)$. This means when we are finished completing the square, we need to solve for $f(x)$.

$$\begin{array}{rcll} f(x) & = & x^2 - 4x + 3 & \\ f(x) - 3 & = & x^2 - 4x & \text{Subtract 3 from both sides.} \\ f(x) - 3 + (-2)^2 & = & x^2 - 4x + (-2)^2 & \text{Add } \left(\frac{1}{2}(-4)\right)^2 \text{ to both sides.} \\ f(x) + 1 & = & (x-2)^2 & \text{Factor the perfect square trinomial.} \\ f(x) & = & (x-2)^2 - 1 & \text{Solve for } f(x). \end{array}$$

   The reader is encouraged to start with $f(x) = (x - 2)^2 - 1$, perform the indicated operations and simplify the result to $f(x) = x^2 - 4x + 3$. From the standard form, $f(x) = (x - 2)^2 - 1$, we see that the vertex is $(2, 1)$ and that the parabola opens upwards. To find the zeros of f, we set $f(x) = 0$.

   We have two equivalent expressions for $f(x)$ so we could use either the general form or standard form. We solve the former and leave it to the reader to solve the latter to see that we get the same results either way. To solve $x^2 - 4x + 3 = 0$, we factor: $(x - 3)(x - 1) = 0$ and obtain $x = 1$ and $x = 3$. We get two x-intercepts, $(1, 0)$ and $(3, 0)$.

   To find the y-intercept, we need $f(0)$. We use the general form and find that the y-intercept is $(0, 3)$. From symmetry, we know the point $(4, 3)$ is also on the graph. The range of f is $[-1, \infty)$ with the minimum -1 at $x = 2$. Finally, f is decreasing on $(-\infty, 2]$ and increasing from $[2, \infty)$. The graph is given on the left at the bottom the next page.

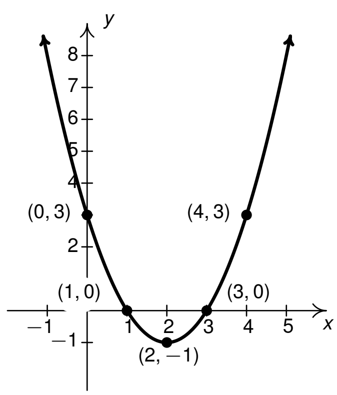
2. We rewrite $g(t) = 6 - 4t - 2t^2$ as $g(t) = -2t^2 - 4t + 6$ and proceed to complete the square:

$$\begin{array}{rcll} g(t) & = & -2t^2 - 4t + 6 & \\ g(t) - 6 & = & -2t^2 - 4t & \text{Subtract 6 from both sides.} \end{array}$$

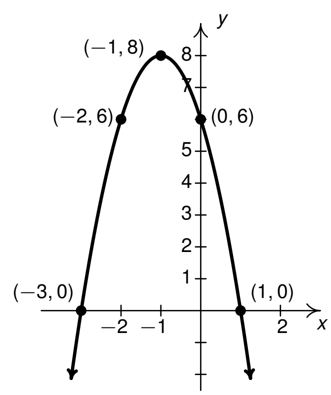
$$\begin{array}{rcll}
\dfrac{g(t)-6}{-2} & = & \dfrac{-2t^2-4t}{-2} & \text{Divide both sides by } -2. \\
\dfrac{g(t)-6}{-2}+(1)^2 & = & t^2+2t+(1)^2 & \text{Add } \left(\frac{1}{2}(2)\right)^2 \text{ to both sides.} \\
\dfrac{g(t)-6}{-2}+1 & = & (t+1)^2 & \text{Factor the prefect square trinomial.} \\
\dfrac{g(t)-6}{-2} & = & (t+1)^2-1 & \\
g(t)-6 & = & -2\left[(t+1)^2-1\right] & \\
g(t) & = & -2(t+1)^2+2+6 & \\
g(t) & = & -2(t+1)^2+8 & \text{Solve for } g(t).
\end{array}$$

We can check our answer by expanding $-2(t+1)^2+8$ and show that it simplifies to $-2t^2-4t+6$. From the standard form, we find that the vertex is $(-1,8)$ and that the parabola opens downwards. Setting $g(t)=-2t^2-4t+6=0$, we factor to get $-2(t-1)(t+3)=0$ so $t=-3$ and $t=1$. Hence, our two t-intercepts are $(-3,0)$ and $(1,0)$.

Since $g(0)=6$, we get the y-intercept to be $(0,6)$. Using symmetry, we also have the point $(-2,6)$ on the graph. The range is $(-\infty,8]$ with a maximum of 8 when $t=-1$. Finally we note that g is increasing on $(-\infty,-1]$ and decreasing on $[-1,\infty)$. The graph is below on the right.



$f(x)=x^2-4x+3$



$g(x)=6-4t-2t^2$

□

We now generalize the procedure demonstrated in Example 1.4.2. Let $f(x)=ax^2+bx+c$ for $a\neq 0$:

$$\begin{array}{rcll}
f(x) & = & ax^2+bx+c & \\
f(x)-c & = & ax^2+bx & \text{Subtract } c \text{ from both sides.}
\end{array}$$

$$\begin{aligned}
\frac{f(x)-c}{a} &= \frac{ax^2+bx}{a} && \text{Divide both sides by } a \neq 0. \\
\frac{f(x)-c}{a} &= x^2+\frac{b}{a}x && \\
\frac{f(x)-c}{a}+\left(\frac{b}{2a}\right)^2 &= x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2 && \text{Add } \left(\frac{b}{2a}\right)^2 \text{ to both sides.} \\
\frac{f(x)-c}{a}+\frac{b^2}{4a^2} &= \left(x+\frac{b}{2a}\right)^2 && \text{Factor the perfect square trinomial.} \\
\frac{f(x)-c}{a} &= \left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2} && \text{Solve for } f(x). \\
f(x)-c &= a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}\right] && \\
f(x)-c &= a\left(x+\frac{b}{2a}\right)^2-a\frac{b^2}{4a^2} && \\
f(x) &= a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c && \\
f(x) &= a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} && \text{Get a common denominator.}
\end{aligned}$$

By setting $h = -\frac{b}{2a}$ and $k = \frac{4ac-b^2}{4a}$, we have written the function in the form $f(x) = a(x-h)^2+k$. This establishes the fact that every quadratic function can be written in standard form.[6] Moreover, writing a quadratic function in standard form allows us to identify the vertex rather quickly, and so our work also shows us that the vertex of $f(x) = ax^2+bx+c$ is $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$. It is not worth memorizing the expression $\frac{4ac-b^2}{4a}$ especially since we can write this as $f\left(-\frac{b}{2a}\right)$. (This about this last statement for a moment.)

We summarize the information detailed above in the following:

> **Equation 1.2. Vertex Formulas for Quadratic Functions**:
> Suppose a, b, c, h and k are real numbers where $a \neq 0$.
>
> - If $f(x) = a(x-h)^2+k$ then the vertex of the graph of $y = f(x)$ is the point (h, k).
> - If $f(x) = ax^2+bx+c$ then the vertex of the graph of $y = f(x)$ is the point $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

Completing the square is also the means by which we may derive the celebrated Quadratic Formula, a formula which returns the solutions to $ax^2+bx+c = 0$ for $a \neq 0$. Before we state it here for reference, we wish to encourage the reader to pause a moment and read the derivation if the Quadratic Formula found in Section A.10. The work presented in this section transforms the general form of a quadratic **function** into

[6]To avoid completing the square, we could solve the equations $b = -2ah$ and $c = ah^2+k$ for h and k. See Exercise 58.

the standard form whereas the work in Section A.10 finds a formula to solve an **equation**. There is great value in understanding the similarities and differences between the two approaches.

Equation 1.3. The Quadratic Formula: The zeros of the quadratic function $f(x) = ax^2 + bx + c$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is worth pointing out the symmetry inherent in Equation 1.3. We may rewrite the zeros as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

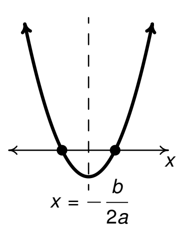
so that, if there are real zeros, they (like the rest of the parabola) are symmetric about the line $x = -\frac{b}{2a}$. Another way to view this symmetry is that the x-coordinate of the vertex is the average of the zeros. We encourage the reader to verify this fact in all of the preceding examples, where applicable.

Next, recall that if the quantity $b^2 - 4ac$ is strictly negative then we do not have any real zeros. This quantity is called the **discriminant** and is useful in determining the number and nature of solutions to a quadratic equation. We remind the reader of this below.

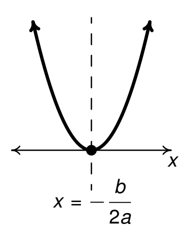
Equation 1.4. The Discriminant of a Quadratic Function: Given a quadratic function in general form $f(x) = ax^2 + bx + c$, the **discriminant** is the quantity $b^2 - 4ac$.

- If $b^2 - 4ac > 0$ then f has two unequal (distinct) real zeros.
- If $b^2 - 4ac = 0$ then f has one (repeated) real zero.
- If If $b^2 - 4ac < 0$ then f has two unequal (distinct) non-real zeros.

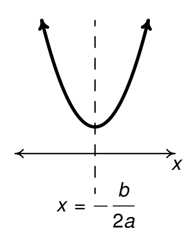
We'll talk more about what we mean by a 'repeated' zero and how to compute 'non-real' zeros in Chapter 2. For us, the discriminant has the graphical implication that if $b^2 - 4ac > 0$ then we have two x-intercepts; if $b^2 - 4ac = 0$ then we have just one x-intercept, namely, the vertex; and if $b^2 - 4ac < 0$ then we have no x-intercepts because the parabola lies entirely above or below the x-axis. We sketch each of these scenarios below assuming $a > 0$. (The sketches for $a < 0$ are similar - see Exercise 53.)



$b^2 - 4ac > 0$



$b^2 - 4ac = 0$



$b^2 - 4ac < 0$

We now revisit the economic scenario first described in Examples 1.2.3 and 1.2.4 where we were producing and selling PortaBoy game systems. Recall that the cost to produce x PortaBoys is denoted by $C(x)$ and the price-demand function, that is, the price to charge in order to sell x systems is denoted by $p(x)$. We introduce two more related functions below: the **revenue** and **profit** functions.

Definition 1.12. Revenue and Profit: Suppose $C(x)$ represents the cost to produce x units and $p(x)$ is the associated price-demand function. Under the assumption that we are producing the same number of units as are being sold:

- The **revenue** obtained by selling x units is $R(x) = x\,p(x)$.

  That is, revenue = (number of items sold) · (price per item).

- The **profit** made by selling x units is $P(x) = R(x) - C(x)$.

  That is, profit = (revenue) − (cost).

Said differently, the **revenue** is the amount of money **collected** by selling x items whereas the **profit** is how much money is **left over** after the costs are paid.

Example 1.4.3. In Example 1.2.3 the cost to produce x PortaBoy game systems for a local retailer was given by $C(x) = 80x + 150$ for $x \geq 0$ and in Example 1.2.4 the price-demand function was found to be $p(x) = -1.5x + 250$, for $0 \leq x \leq 166$.

1. Find formulas for the associated revenue and profit functions; include the domain of each.

2. Find and interpret $P(0)$.

3. Find and interpret the zeros of P.

4. Graph $y = P(x)$. Find the vertex and axis intercepts.

5. Interpret the vertex of the graph of $y = P(x)$.

6. What should the price per system be in order to maximize profit?

7. Find and interpret the average rate of change of P over the interval $[0, 57]$.

Solution.

1. The formula for the revenue function is $R(x) = x\,p(x) = x(-1.5x + 250) = -1.5x^2 + 250x$. Since the domain of p is restricted to $0 \leq x \leq 166$, so is the domain of R. To find the profit function $P(x)$, we subtract $P(x) = R(x) - C(x) = \left(-1.5x^2 + 250x\right) - (80x + 150) = -1.5x^2 + 170x - 150$. The cost function formula is valid for $x \geq 0$, but the revenue function is valid when $0 \leq x \leq 166$. Hence, the domain of P is likewise restricted to $[0, 166]$.

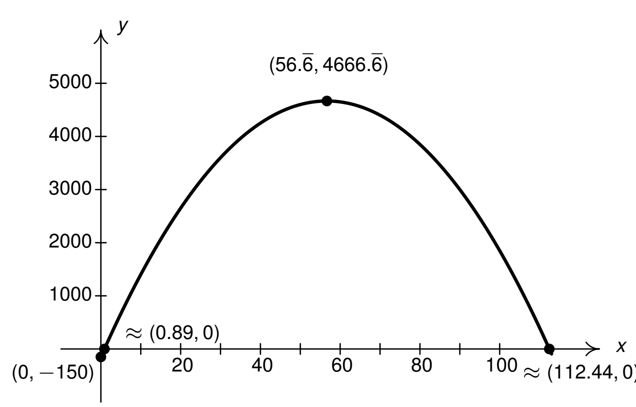
2. We find $P(0) = -1.5(0)^2 + 170(0) - 150 = -150$. This means that if we produce and sell 0 PortaBoy game systems, we have a profit of $-\$150$. Since profit = (revenue) − (cost), this means our costs exceed our revenue by \$150. This makes perfect sense, since if we don't sell any systems, our revenue is \$0 but our fixed costs (see Example 1.2.3) are \$150.

3. To find the zeros of P, we set $P(x) = 0$ and solve $-1.5x^2 + 170x - 150 = 0$. Factoring here would be challenging to say the least, so we use the Quadratic Formula, Equation 1.3. Identifying $a = -1.5$, $b = 170$ and $c = -150$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} \\ &= \frac{-170 \pm \sqrt{28000}}{-3} \\ &= \frac{170 \pm 20\sqrt{70}}{3} \\ &\approx 0.89,\ 112.44. \end{aligned}$$

Given that profit = (revenue) − (cost), if profit = 0, then revenue = cost. Hence, the zeros of P are called the 'break-even' points - where just enough product is sold to recover the cost spent to make the product. Also, x represents a number of game systems, which is a whole number, so instead of using the exact values of the zeros, or even their approximations, we consider $x = 0$ and $x = 1$ along with $x = 112$ and $x = 113$. We find $P(0) = -150$, $P(1) = 18.5$, $P(112) = 74$ and $P(113) = -93.5$. These data suggest that, in order to be profitable, at least 1 but not more than 112 systems should be produced and sold, as borne out in the graph below.

4. Knowing the zeros of P, we have two x-intercepts: $\left(\frac{170-20\sqrt{70}}{3}, 0\right) \approx (0.89, 0)$ and $\left(\frac{170+20\sqrt{70}}{3}, 0\right) \approx (112.44, 0)$. Since $P(0) = -150$, we get the y-intercept is $(0, -150)$. To find the vertex, we appeal to Equation 1.2. Substituting $a = -1.5$ and $b = 170$, we get $x = -\frac{170}{2(-1.5)} = \frac{170}{3} = 56.\overline{6}$. To find the y-coordinate of the vertex, we compute $P\left(\frac{170}{3}\right) = \frac{14000}{3} = 4666.\overline{6}$. Hence, the vertex is $(56.\overline{6}, 4666.\overline{6})$. The domain is restricted $0 \leq x \leq 166$ and we find $P(166) = -13264$. Attempting to plot all of these points on the same graph to any sort of scale is challenging. Instead, we present a portion of the graph for $0 \leq x \leq 113$. Even with this, the intercepts near the origin are crowded.



$y = P(x)$

5. From the vertex, we see that the maximum of P is $4666.\overline{6}$ when $x = 56.\overline{6}$. As before, x represents the number of PortaBoy systems produced and sold, so we cannot produce and sell $56.\overline{6}$ systems. Hence, by comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude that we will make a maximum profit of \$4666.50 if we sell 57 game systems.

6. We've determined that we need to sell 57 PortaBoys to maximize profit, so we substitute $x = 57$ into the price-demand function to get $p(57) = -1.5(57) + 250 = 164.5$. In other words, to sell 57 systems, and thereby maximize the profit, we should set the price at \$164.50 per system.

7. To find the average rate of change of P over $[0, 57]$, we compute

$$\frac{\Delta[P(x)]}{\Delta x} = \frac{P(57) - P(0)}{57 - 0} = \frac{4666.5 - (-150)}{57} = 84.5.$$

This means that as the number of systems produced and sold ranges from 0 to 57, the average profit per system is increasing at a rate of \$84.50. In other words, for each additional system produced and sold, the profit increased by \$84.50 on average. □

We hope Example 1.4.3 shows the value of using a continuous model to describe a discrete situation. True, we could have 'run the numbers' and computed $P(1)$, $P(2)$, ..., $P(166)$ to eventually determine the maximum profit, but the vertex formula made much quicker work of the problem.

Along these same lines, in our next example we revisit Skippy's temperature data from Example 1.1.1 in Section 1.1. We found a piecewise-linear model in Section 1.2 to model the temperature over the course the day and now we seek a quadratic function to do the job. The methodology used here is similar to that of the least squares regression line discussed in Section 1.2.3 but instead of finding the line closest to the data points, we want the **parabola** closest to them that comes from a function of the form $f(x) = ax^2 + bx + c$. The Mathematics required to find the desired quadratic function is beyond the scope of this text, but most graphing utilities can do these quickly. In the quadratic case, the machine will return a value of R^2 such that $0 \leq R^2 \leq 1$. The closer R^2 is to 1, the better the fit. (Again, how R^2 is computed is beyond this text.)

Example 1.4.4.

1. Use a graphing utility to fit a quadratic model to the time and temperature data in Example 1.1.1. Comment on the goodness of fit.

2. Use your model to predict the temperature at 7 AM and 3 PM. Round your answers to one decimal place. How do your results compare with those from Example 1.2.7?

3. According to the model, what was the warmest temperature of the day? When did that occur? Round your answers to one decimal place.

Solution.

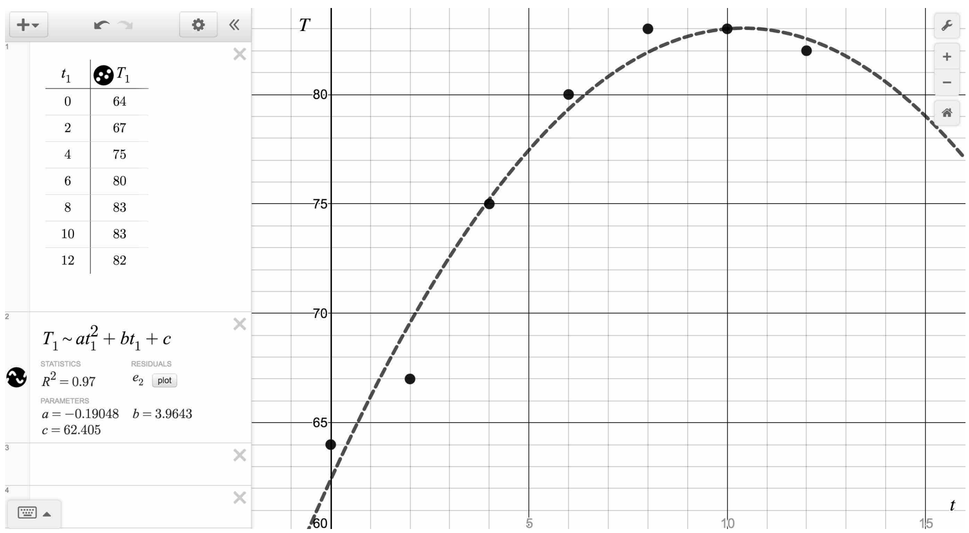
1. Entering the data in Desmos we find $T = F(t) = -0.1905t^2 + 3.9643t + 62.405$ with an R^2 value of 0.97, indicating a pretty strong fit.

2. Since 7 AM corresponds to $t = 1$, we find $T = F(1) \approx 66.18$. Hence our quadratic model predicts a temperature of 66.2° F at 7 AM - identical (when rounded) to the 66.2° F predicted in Example 1.2.7. Similarly, 3 PM corresponds to $t = 9$, so we find $T = F(9) \approx 82.65$. Thus the model predicts an outdoor temperature of 82.6° F which is very close to the 82.9°F prediction from Example 1.2.7.

3. The model is quadratic with $a < 0$ so the maximum (warmest) temperature can be determined by finding the vertex. We get

$$t = -\frac{b}{2a} = -\frac{3.9643}{2(-0.1905)} \approx -10.40, \qquad T = F(-10.40) \approx 83.03,$$

or, in other words, the warmest temperature is 83.0° F at 4:24 PM (10.40 hours after 6 AM.)



□

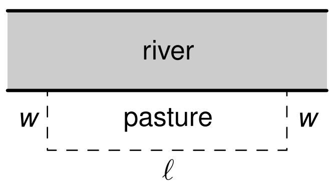
It is interesting how close the predictions from Examples 1.2.7 and 1.4.4 despite one using linear models and one using a quadratic model. Which model is the 'better' model? We leave that discussion to the reader and their classmates.

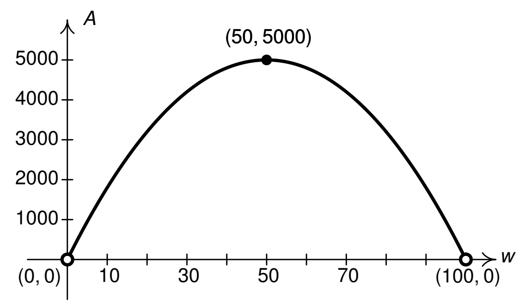
Our next example is classic application of optimizing a quadratic function.

Example 1.4.5. Much to Donnie's surprise and delight, he inherits a large parcel of land in Ashtabula County from one of his (e)strange(d) relatives so the time is right for him to pursue his dream of raising alpaca. He wishes to build a rectangular pasture and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a river (so that no fencing is required on that side), what are the dimensions of the pasture which maximize the area? What is the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpaca can Donnie keep in his pasture?

Solution. We are asked to find the dimensions of the pasture which would give a maximum area, so we begin by sketching the diagram seen below on the left. We let w denote the width of the pasture and we let ℓ denote the length of the pasture. The units given to us in the statement of the problem are feet, so we assume that w and ℓ are measured in feet. The area of the pasture, which we'll call A, is related to w and ℓ by the equation $A = w\ell$. Since w and ℓ are both measured in feet, A has units of feet^2, or square feet.

We are also told that the total amount of fencing available is 200 feet, which means $w + \ell + w = 200$, or, $\ell + 2w = 200$. We now have two equations, $A = w\ell$ and $\ell + 2w = 200$. In order to use the tools given to us in this section to **maximize** A, we need to use the information given to write A as a function of just **one** variable, either w or ℓ. This is where we use the equation $\ell + 2w = 200$. Solving for ℓ, we find $\ell = 200 - 2w$, and we substitute this into our equation for A. We get $A = w\ell = w(200 - 2w) = 200w - 2w^2$. We now have A as a function of w, $A = f(w) = 200w - 2w^2 = -2w^2 + 200w$.





$A = f(w)$

Before we go any further, we need to find the applied domain of f so that we know what values of w make sense in this situation.[7] Given that w represents the width of the pasture we need $w > 0$. Likewise, ℓ represents the length of the pasture, so $\ell = 200 - 2w > 0$. Solving this latter inequality yields $w < 100$. Hence, the function we wish to maximize is $f(w) = -2w^2 + 200w$ for $0 < w < 100$. We know two things about the quadratic function f: the graph of $A = f(w)$ is a parabola and (since the coefficient of w^2 is -2) the parabola opens downwards.

This means that there is a maximum value to be found, and we know it occurs at the vertex. Using the vertex formula, we find $w = -\frac{200}{2(-2)} = 50$, and $A = f(50) = -2(50)^2 + 200(50) = 5000$. Since $w = 50$ lies in the applied domain, $0 < w < 100$, we have that the area of the pasture is maximized when the width is 50 feet. To find the length, we use $\ell = 200 - 2w$ and find $\ell = 200 - 2(50) = 100$, so the length of the pasture is 100 feet. The maximum area is $A = f(50) = 5000$, or 5000 square feet. If an average alpaca requires 25 square feet of pasture, Donnie can raise $\frac{5000}{25} = 200$ average alpaca. $\square$

The function f in Example 1.4.5 is called the **objective function** for this problem - it's the function we're trying to optimize. In the case above, we were trying to maximize f. The equation $\ell + 2w = 200$ along with the inequalities $w > 0$ and $\ell > 0$ are called the **constraints**. As we saw in this example, and as we'll see again and again, the constraint equation is used to rewrite the objective function in terms of just one of the variables where constraint inequalities, if any, help determine the applied domain.

[7] Donnie would be very upset if, for example, we told him the width of the pasture needs to be -50 feet.

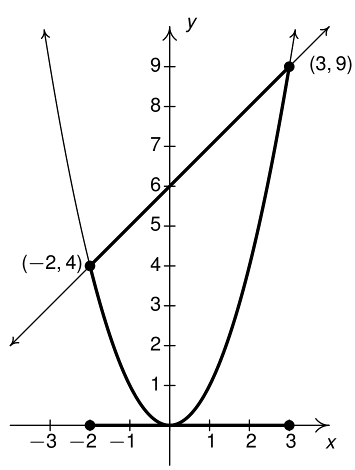
1.4.2 Inequalities involving Quadratic Functions

We now turn our attention to solving inequalities involving quadratic functions. Consider the inequality $x^2 \leq 6$. We could use the fact that the square root is increasing[8] to get: $\sqrt{x^2} \leq \sqrt{6}$, or $|x| \leq \sqrt{6}$. This reduces to $-\sqrt{6} \leq x \leq \sqrt{6}$ or, using interval notation, $[-\sqrt{6}, \sqrt{6}]$. If, however, we had to solve $x^2 \leq x+6$, things are more complicated. One approach is to complete the square:

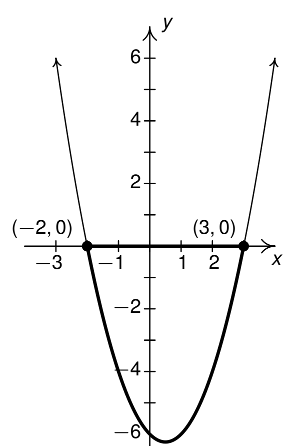
$$\begin{array}{rcl}
x^2 & \leq & x+6 \\
x^2 - x & \leq & 6 \\
x^2 - x + \frac{1}{4} & \leq & 6 + \frac{1}{4} \\
\left(x - \frac{1}{2}\right)^2 & \leq & \frac{25}{4} \\
\sqrt{\left(x - \frac{1}{2}\right)^2} & \leq & \sqrt{\frac{25}{4}} \\
\left|x - \frac{1}{2}\right| & \leq & \frac{5}{2} \\
-\frac{5}{2} \leq & x - \frac{1}{2} & \leq \frac{5}{2} \\
-2 & \leq x \leq & 3
\end{array}$$

We get the solution $[-2, 3]$. While there is nothing wrong with this approach, we seek methods here that will generalize to higher degree polynomials such as those we'll see in Chapter 2.

To that end, we look at the inequality $x^2 \leq x+6$ graphically. Identifying $f(x) = x^2$ and $g(x) = x+6$, we graph f and g on the same set of axes below on the left and look for where the graph of f (the parabola) meets or is below the graph of g (the line). There are two points of intersection which we determine by solving $f(x) = g(x)$ or $x^2 = x+6$. As usual, we rewrite this equation as $x^2 - x - 6 = 0$ in order to use the primary tools we've developed to handle these types[9] of quadratic equations: factoring, or failing that, the Quadratic Formula. We find $x^2 - x - 6 = (x+2)(x-3)$ so we get two solutions to $(x+2)(x-3) = 0$, namely $x = -2$ and $x = 3$. Putting these together with the graph, we obtain the same solution: $[-2, 3]$.



Solving $x^2 \leq x+6$.



Solving $x^2 - x - 6 \leq 0$.

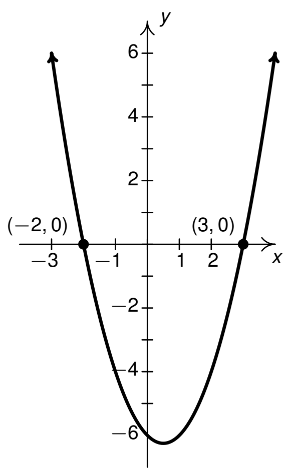
[8] That is, if $a < b$, then $\sqrt{a} < \sqrt{b}$.

[9] Namely ones with a nonzero coefficient of 'x'.

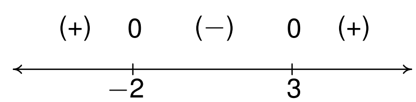
Yet a third way to attack $x^2 \leq x + 6$ is to rewrite the inequality as $x^2 - x - 6 \leq 0$. Here, we graph $f(x) = x^2 - x - 6$ to look for where the graph meets or is below the graph of $g(x) = 0$, a.k.a. the x-axis. Doing so requires us to find the zeros of f, that is, solve $f(x) = x^2 - x - 6 = 0$ from which we obtain $x = -2$ and $x = 3$ as before. We find the same solution, $[-2, 3]$ as is showcased in the graph at the bottom of the previous page on the right.

One advantage to using this last approach is that we are essentially concerned with one function and its **zeros**. This approach can be generalized to all functions - not just quadratics, so we take the time to develop this method more thoroughly now.

Consider the graph of $f(x) = x^2 - x - 6$ below The zeros of f are $x = -2$ and $x = 3$ and they divide the domain (the x-axis) into three intervals: $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$. For every number in $(-\infty, -2)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty, -2)$. Similarly, $f(x) < 0$ for all x in $(-2, 3)$, and $f(x) > 0$ for all x in $(3, \infty)$. We represent this schematically with the **sign diagram** below.



$f(x) = x^2 - x - 6$.



A sign diagram for $f(x) = x^2 - x - 6$

The (+) above a portion of the number line indicates $f(x) > 0$ for those values of x and the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of f, so we place 0 above them. For the inequality $f(x) = x^2 - x - 6 \leq 0$, we read from the sign diagram that the solution is $[-2, 3]$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function. While parabolas aren't that bad to graph knowing what we know, our sights are set on more general functions whose graphs are more complicated.

An important property of parabolas is that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis at some point in between. Said differently, if the function is positive at one point and negative at another, the function must have at least one zero in between. This property is a consequence of quadratic functions being **continuous**. A precise definition of 'continuous' requires the language of Calculus, but it suffices for us to know that the graph of a continuous function has no gaps or holes. This allows us to determine the sign of **all** of the function values on a given interval by testing the function at just **one** value in the interval.

The result below applies to all continuous functions defined on an interval of real numbers, but we restrict our attention to quadratic functions for the time being,

Steps for Creating A Sign Diagram for A Quadratic Function

Suppose f is a quadratic function.

1. Find the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine and record the sign of $f(x)$ for each test value in step 2.

To use a sign diagram to solve an inequality, we must always remember to compare the function to 0.

Solving Inequalities using Sign Diagrams

To solve an inequality using a sign diagram:

1. Rewrite the inequality so some function $f(x)$ is being compared to '0.'
2. Make a sign diagram for f.
3. Record the solution.

We practice this approach in the following example.

Example 1.4.6. Solve the following inequalities analytically[10] and check your solutions graphically.

1. $2x^2 \leq 3 - x$
2. $t^2 - 2t > 1$
3. $x^2 + 1 \leq 2x$
4. $2t - t^2 \geq |t - 1| - 1$

Solution.

1. To solve $2x^2 \leq 3 - x$, we rewrite it as $2x^2 + x - 3 \leq 0$. We find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. We place these values on the number line with 0 above them and choose test values in the intervals $\left(-\infty, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, 1\right)$ and $(1, \infty)$. For the interval $\left(-\infty, -\frac{3}{2}\right)$, we choose[11] $x = -2$; for $\left(-\frac{3}{2}, 1\right)$, we pick $x = 0$; and for $(1, \infty)$, $x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$, so we place $(+)$ above $\left(-\infty, -\frac{3}{2}\right)$; $f(0) = -3 < 0$, so $(-)$ goes above the interval $\left(-\frac{3}{2}, 1\right)$; and, $f(2) = 7$, which means $(+)$ is placed above $(1, \infty)$.

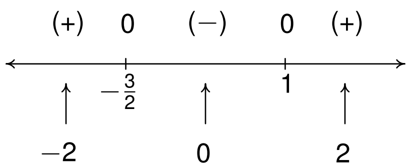
   We are solving $2x^2 + x - 3 \leq 0$ so we need solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a $(-)$ above them. The

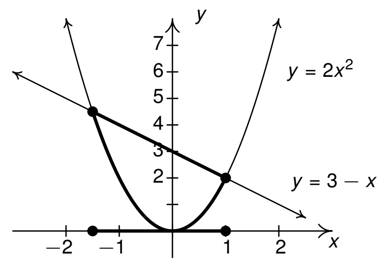
[10] By 'solve analytically' we mean 'algebraically' using a sign diagram.

[11] We have to choose **something** in each interval. If you don't like our choices, please feel free to choose different numbers. You'll get the same sign chart.

sign diagram shows only one: $\left(-\frac{3}{2}, 1\right)$. Also, we know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so our final answer is $\left[-\frac{3}{2}, 1\right]$.

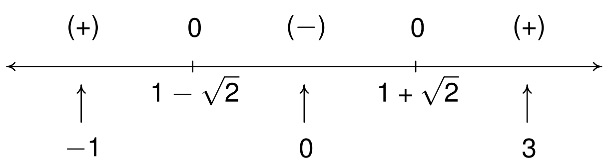
To verify our solution graphically, we refer to the original inequality, $2x^2 \leq 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the x values where the graph of g is below that of h (the solution to $g(x) < h(x)$) as well as the points of intersection (the solutions to $g(x) = h(x)$). The graphs of g and h are given on the right with the sign chart on the left.

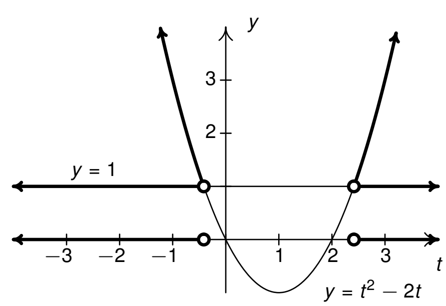




2. Once again, we re-write $t^2 - 2t > 1$ as $t^2 - 2t - 1 > 0$ and we identify $f(t) = t^2 - 2t - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $t^2 - 2t - 1$ doesn't factor nicely. Hence, we resort to the Quadratic Formula and find $t = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $t = -1$, $t = 0$ and $t = 3$ as our test values and find $f(-1) = 2$, which is (+); $f(0) = -1$ which is $(-)$; and $f(3) = 2$ which is (+) again. Our solution to $t^2 - 2t - 1 > 0$ is where we have (+), so, in interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$.

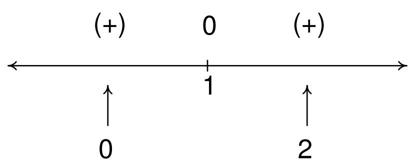
   To check the inequality $t^2 - 2t > 1$ graphically, we set $g(t) = t^2 - 2t$ and $h(t) = 1$. We are looking for the t values where the graph of g is above the graph of h. As before we present the graphs on the right and the sign chart on the left.

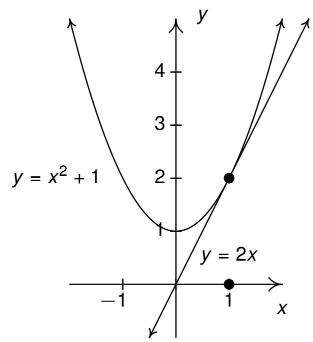




3. To solve $x^2 + 1 \leq 2x$, as before, we solve $x^2 - 2x + 1 \leq 0$. Setting $f(x) = x^2 - 2x + 1 = 0$, we find only one zero of f: $x = 1$. This one x value divides the number line into two intervals, from which we choose $x = 0$ and $x = 2$ as test values. We find $f(0) = 1 > 0$ and $f(2) = 1 > 0$. Since we are looking for solutions to $x^2 - 2x + 1 \leq 0$, we are looking for x values where $x^2 - 2x + 1 < 0$ as well as where $x^2 - 2x + 1 = 0$. Looking at our sign diagram, there are no places where $x^2 - 2x + 1 < 0$ (there are no $(-)$), so our solution is only $x = 1$ (where $x^2 - 2x + 1 = 0$). We write this as $\{1\}$.

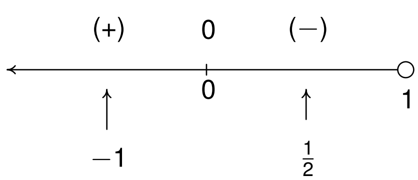
Graphically, we solve $x^2 + 1 \leq 2x$ by graphing $g(x) = x^2 + 1$ and $h(x) = 2x$. We are looking for the x values where the graph of g is below the graph of h (for $x^2 + 1 < 2x$) and where the two graphs intersect ($x^2 + 1 = 2x$). Notice that the line and the parabola touch at $(1, 2)$, but the parabola is always above the line otherwise.[12]



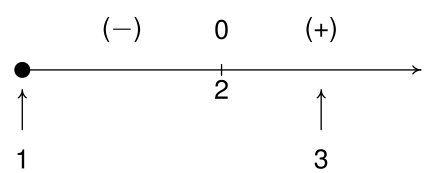


4. To solve $2t - t^2 \geq |t - 1| - 1$ analytically we first rewrite the absolute value using cases. For $t < 1$, $|t - 1| = -(t - 1) = -t + 1$, so we get $2t - t^2 \geq (-t + 1) - 1$ which simplifies to $t^2 - 3t \leq 0$. Finding the zeros of $f(t) = t^2 - 3t$, we get $t = 0$ and $t = 3$. However, we are concerned only with the portion of the number line where $t < 1$, so the only zero that we deal with is $t = 0$. This divides the interval $t < 1$ into two intervals: $(-\infty, 0)$ and $(0, 1)$. We choose $t = -1$ and $t = \frac{1}{2}$ as our test values. We find $f(-1) = 4$ and $f\left(\frac{1}{2}\right) = -\frac{5}{4}$. Hence, our solution to $t^2 - 3t \leq 0$ for $t < 1$ is $[0, 1)$.

   Next, we turn our attention to the case $t \geq 1$. Here, $|t - 1| = t - 1$, so our original inequality becomes $2t - t^2 \geq (t - 1) - 1$, or $t^2 - t - 2 \leq 0$. Setting $g(t) = t^2 - t - 2$, we find the zeros of g to be $t = -1$ and $t = 2$. Of these, only $t = 2$ lies in the region $t \geq 1$, so we ignore $t = -1$. Our test intervals are now $[1, 2)$ and $(2, \infty)$. We choose $t = 1$ and $t = 3$ as our test values and find $g(1) = -2$ and $g(3) = 4$. Hence, our solution to $g(t) = t^2 - t - 2 \leq 0$, in this region is $[1, 2)$.



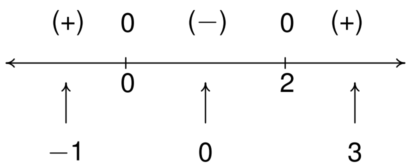
Solving $2t - t^2 \geq |t - 1| - 1$ for $t < 1$.

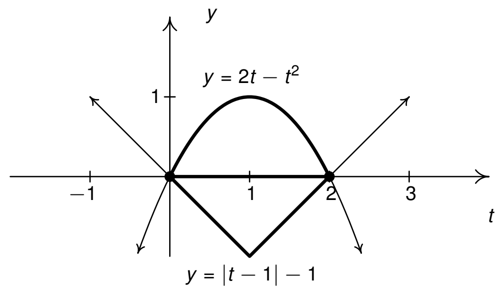


Solving $2t - t^2 \geq |t - 1| - 1$ for $t \geq 1$.

   Combining these into one sign diagram, we have that our solution is $[0, 2]$. Graphically, to check $2t - t^2 \geq |t - 1| - 1$, we set $h(t) = 2t - t^2$ and $i(t) = |t - 1| - 1$ and look for the t values where the graph of h intersects or is above the the graph of i The combined sign chart is given on the left and the graphs are on the right.

[12]In this case, we say the line $y = 2x$ is **tangent** to $y = x^2 + 1$ at $(1, 2)$. Finding tangent lines to arbitrary functions is a fundamental problem solved, in general, with Calculus.





□

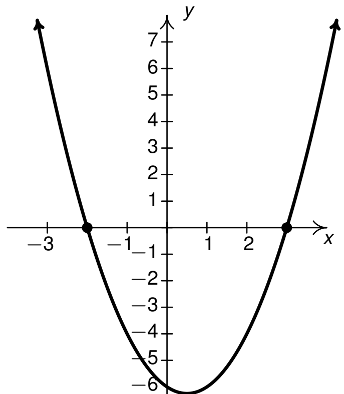
We end this section with an example that combines quadratic inequalities with piecewise functions.

Example 1.4.7. Rewrite $g(x) = |x^2 - x - 6|$ as a piecewise function and graph.

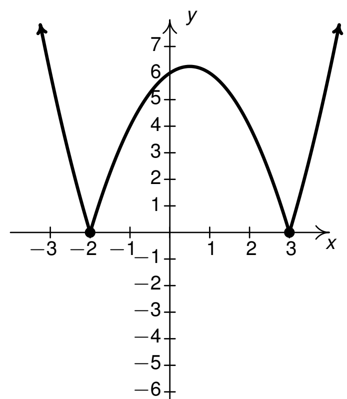
Solution. Using the definition of absolute value, Definition 1.9 and the sign diagram we constructed for $f(x) = x^2 - x - 6$ near the beginning of the subsection, we get:

$$g(x) = |x^2 - x - 6| = \begin{cases} -(x^2 - x - 6) & \text{if } (x^2 - x - 6) < 0, \\ (x^2 - x - 6) & \text{if } (x^2 - x - 6) \geq 0. \end{cases} \longrightarrow \quad g(x) = \begin{cases} -x^2 + x + 6 & \text{if } -2 < x < 3, \\ x^2 - x - 6 & \text{if } x \leq -2 \text{ or } x \geq 3. \end{cases}$$

Going through the usual machinations results on the graph below on the right. Compare it to the graph below on the left. Notice anything?



$y = f(x) = x^2 - x - 6$



$y = g(x) = |x^2 - x - 6|$

If we take a step back and look at the graphs of f and g, we notice that to obtain the graph of g from the graph of f, we reflect a **portion** of the graph of f about the x-axis. In general, if $g(x) = |f(x)|$, then:

$$g(x) = |f(x)| = \begin{cases} -f(x) & \text{if } f(x) < 0, \\ f(x) & \text{if } f(x) \geq 0. \end{cases}$$

The function g is defined so that when $f(x)$ is negative (i.e., when its graph is below the x-axis), the graph of g is the refection of the graph of f across the x-axis. This is a general method to graph functions of the form $g(x) = |f(x)|$. Indeed, the graph of $g(x) = |x|$ can be obtained by reflection the portion of the line $f(x) = x$ which is below the x-axis back above the x-axis creating the characteristic '$\vee$' shape.[13] □

[13]See Exercise 11 in Section 1.3.

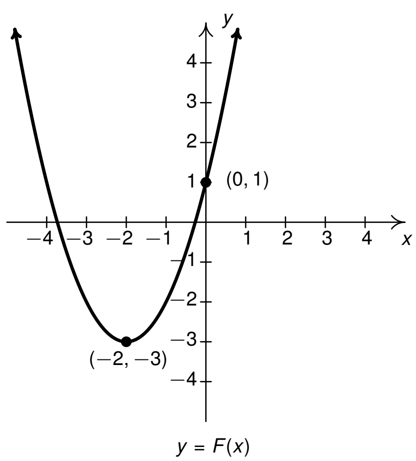
1.4.3 Exercises

In Exercises 1 - 9, graph the quadratic function. Find the vertex and axis intercepts of each graph, if they exist. State the domain and range, identify the maximum or minimum, and list the intervals over which the function is increasing or decreasing. If the function is given in general form, convert it into standard form; if it is given in standard form, convert it into general form.

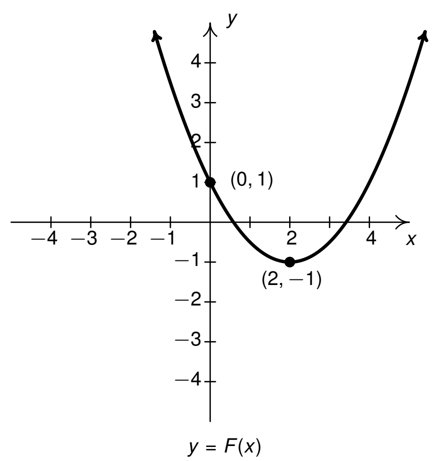
1. $f(x) = x^2 + 2$
2. $f(x) = -(x + 2)^2$
3. $f(x) = x^2 - 2x - 8$
4. $g(t) = -2(t + 1)^2 + 4$
5. $g(t) = 2t^2 - 4t - 1$
6. $g(t) = -3t^2 + 4t - 7$
7. $h(s) = s^2 + s + 1$
8. $h(s) = -3s^2 + 5s + 4$
9. $h(s) = s^2 - \frac{1}{100}s - 1$

In Exercises 10 - 13, find a formula for each function below in the form $F(x) = a(x - h)^2 + k$.

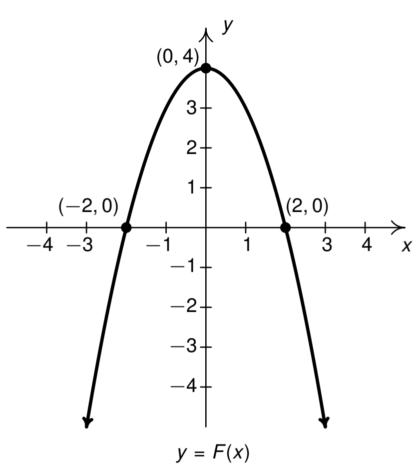
10.



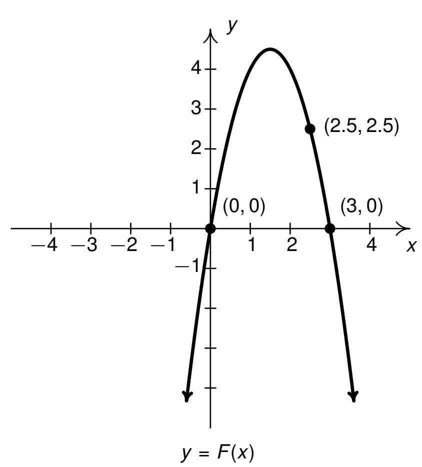
11.



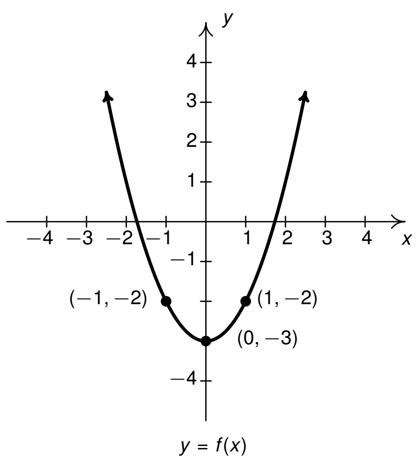
12.

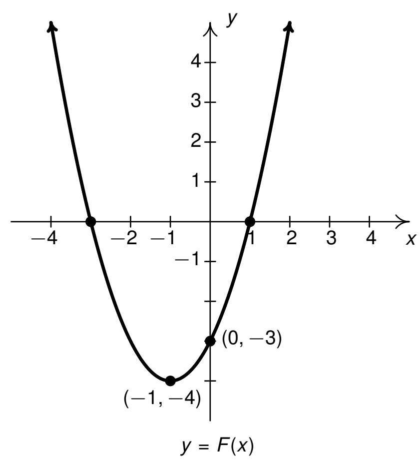


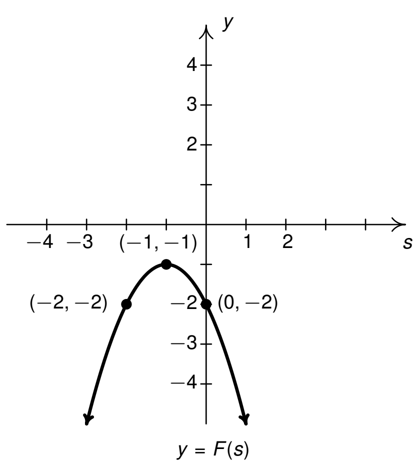
13.

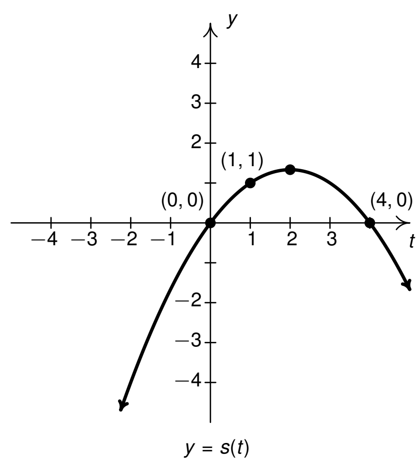


In Exercises 14 - 17 Find both the standard and general form of the quadratic functions below.

14. 

15. 

16. 

17. 

In Exercises 18 - 33, solve the inequality. Write your answer using interval notation.

18. $x^2 + 2x - 3 \geq 0$

19. $16x^2 + 8x + 1 > 0$

20. $t^2 + 9 < 6t$

21. $9t^2 + 16 \geq 24t$

22. $u^2 + 4 \leq 4u$

23. $u^2 + 1 < 0$

24. $3x^2 \leq 11x + 4$

25. $x > x^2$

26. $2t^2 - 4t - 1 > 0$

27. $5t + 4 \leq 3t^2$

28. $2 \leq |x^2 - 9| < 9$

29. $x^2 \leq |4x - 3|$

30. $t^2 + t + 1 \geq 0$

31. $t^2 \geq |t|$

32. $x|x + 5| \geq -6$

33. $x|x - 3| < 2$

In Exercises 34 - 38, cost and price-demand functions are given. For each scenario,

- Find the profit function $P(x)$.
- Find the number of items which need to be sold in order to maximize profit.
- Find the maximum profit.
- Find the price to charge per item in order to maximize profit.
- Find and interpret break-even points.

34. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, for $0 \leq x \leq 15$.

35. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, for $0 \leq x \leq 35$.

36. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, for $0 \leq x \leq 30$.

37. The daily cost, in dollars, to produce x Sasquatch Berry Pies is $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, for $0 \leq x \leq 24$.

38. The monthly cost, in *hundreds* of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in *hundreds* of dollars per scooter, is $p(x) = 140 - 2x$, for $0 \leq x \leq 70$.

39. The International Silver Strings Submarine Band holds a bake sale each year to fund their trip to the National Sasquatch Convention. It has been determined that the cost in dollars of baking x cookies is $C(x) = 0.1x + 25$ and that the demand function for their cookies is $p = 10 - .01x$ for $0 \leq x \leq 1000$. How many cookies should they bake in order to maximize their profit?

40. Using data from Bureau of Transportation Statistics, the average fuel economy $F(t)$ in miles per gallon for passenger cars in the US t years after 1980 can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$. Find and interpret the coordinates of the vertex of the graph of $y = F(t)$.

41. The temperature T, in degrees Fahrenheit, t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

What is the warmest temperature of the day? When does this happen?

42. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. How many pens should be produced to minimize the cost? What is this minimum cost?

43. Skippy wishes to plant a vegetable garden along one side of his house. In his garage, he found 32 linear feet of fencing. Since one side of the garden will border the house, Skippy doesn't need fencing along that side. What are the dimensions of the garden which will maximize the area of the garden? What is the maximum area of the garden?

44. In the situation of Example 1.4.5, Donnie has a nightmare that one of his alpaca fell into the river. To avoid this, he wants to move his rectangular pasture *away* from the river so that all four sides of the pasture require fencing. If the total amount of fencing available is still 200 linear feet, what dimensions maximize the area of the pasture now? What is the maximum area? Assuming an average alpaca requires 25 square feet of pasture, how many alpaca can he raise now?

45. What is the largest rectangular area one can enclose with 14 inches of string?

46. The height of an object dropped from the roof of an eight story building is modeled by by the function $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground, in feet, t seconds after the object is dropped. How long before the object hits the ground?

47. The height $h(t)$ in feet of a model rocket above the ground t seconds after lift-off is given by the function $h(t) = -5t^2 + 100t$, for $0 \leq t \leq 20$. When does the rocket reach its maximum height above the ground? What is its maximum height?

48. Carl's friend Jason participates in the Highland Games. In one event, the hammer throw, the height $h(t)$ in feet of the hammer above the ground t seconds after Jason lets it go is modeled by the function $h(t) = -16t^2 + 22.08t + 6$. What is the hammer's maximum height? What is the hammer's total time in the air? Round your answers to two decimal places.

49. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time t of a falling object is given by $s(t) = -4.9t^2 + v_0t + s_0$ where s is in meters, t is in seconds, v_0 is the object's initial velocity in meters per second and s_0 is its initial position in meters.

    (a) What is the applied domain of this function?

    (b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.

    (c) Come up with a scenario in which $s_0 < 0$.

    (d) Let's say a slingshot is used to shoot a marble straight up from the ground ($s_0 = 0$) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?

(e) If the marble is shot from the top of a 25 meter tall tower, when does it hit the ground?

(f) What would the height function be if instead of shooting the marble up off of the tower, you were to shoot it straight DOWN from the top of the tower?

50. The two towers of a suspension bridge are 400 feet apart. The parabolic cable[14] attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

51. On New Year's Day, Jeff started weighing himself every morning in order to have an interesting data set for this section of the book. (Discuss with your classmates if that makes him a nerd or a geek. Also, the professionals in the field of weight management strongly discourage weighing yourself every day. When you focus on the number and not your overall health, you tend to lose sight of your objectives. Jeff was making a noble sacrifice for science, but you should not try this at home.) The whole chart would be too big to put into the book neatly, so we've decided to give only a small portion of the data to you. This then becomes a Civics lesson in honesty, as you shall soon see. There are two charts given below. One has Jeff's weight for the first eight Thursdays of the year (January 1, 2009 was a Thursday and we'll count it as Day 1.) and the other has Jeff's weight for the first 10 Saturdays of the year.

Day # (Thursday)	1	8	15	22	29	36	43	50
My weight in pounds	238.2	237.0	235.6	234.4	233.0	233.8	232.8	232.0

Day # (Saturday)	3	10	17	24	31	38	45	52	59	66
My weight in pounds	238.4	235.8	235.0	234.2	236.2	236.2	235.2	233.2	236.8	238.2

(a) Find the least squares line for the Thursday data and comment on its goodness of fit.

(b) Find the least squares line for the Saturday data and comment on its goodness of fit.

(c) Use Quadratic Regression to find a parabola which models the Saturday data and comment on its goodness of fit.

(d) Compare and contrast the predictions the three models make for Jeff's weight on January 1, 2010 (Day #366). Can any of these models be used to make a prediction of Jeff's weight 20 years from now? Explain your answer.

(e) Why is this a Civics lesson in honesty? Well, compare the two linear models you obtained above. One was a good fit and the other was not, yet both came from careful selections of real

[14] The weight of the bridge deck forces the bridge cable into a parabola and a free hanging cable such as a power line does not form a parabola. We shall see in Exercise 37 in Section 7.6 what shape a free hanging cable makes.

data. In presenting the tables to you, we've not lied about Jeff's weight, nor have you used any bad math to falsify the predictions. The word we're looking for here is 'disingenuous'. Look it up and then discuss the implications this type of data manipulation could have in a larger, more complex, politically motivated setting.

52. (Data that is neither linear nor quadratic.) We'll close this exercise set with two data sets that, for reasons presented later in the book, cannot be modeled correctly by lines or parabolas. It is a good exercise, though, to see what happens when you attempt to use a linear or quadratic model when it's not appropriate.

    (a) This first data set came from a Summer 2003 publication of the Portage County Animal Protective League called "Tattle Tails". They make the following statement and then have a chart of data that supports it. "It doesn't take long for two cats to turn into 80 million. If two cats and their surviving offspring reproduced for ten years, you'd end up with 80,399,780 cats." We assume $N(0) = 2$.

Year x	1	2	3	4	5	6	7	8	9	10
Number of Cats $N(x)$	12	66	382	2201	12680	73041	420715	2423316	13968290	80399780

    Use Quadratic Regression to find a parabola which models this data and comment on its goodness of fit. (Spoiler Alert: Does anyone know what type of function we need here?)

    (b) This next data set comes from the U.S. Naval Observatory. That site has loads of awesome stuff on it, but for this exercise I used the sunrise/sunset times in Fairbanks, Alaska for 2009 to give you a chart of the number of hours of daylight they get on the 21$^{\text{st}}$ of each month. We'll let $x = 1$ represent January 21, 2009, $x = 2$ represent February 21, 2009, and so on.

Month Number	1	2	3	4	5	6	7	8	9	10	11	12
Hours of Daylight	5.8	9.3	12.4	15.9	19.4	21.8	19.4	15.6	12.4	9.1	5.6	3.3

    Use Quadratic Regression to find a parabola which models this data and comment on its goodness of fit. (Spoiler Alert: Does anyone know what type of function we need here?)

53. Redraw the three scenarios discussed in the discriminant box for $a < 0$.

54. Graph $f(x) = |1 - x^2|$

55. Find all of the points on the line $y = 1 - x$ which are 2 units from $(1, -1)$.

56. Let L be the line $y = 2x + 1$. Find a function $D(x)$ which measures the distance *squared* from a point on L to $(0, 0)$. Use this to find the point on L closest to $(0, 0)$.

57. With the help of your classmates, show that if a quadratic function $f(x) = ax^2 + bx + c$ has two real zeros then the x-coordinate of the vertex is the midpoint of the zeros.

58. On page 102, we argued that any quadratic function in standard form $f(x) = a(x-h)^2 + k$ can be converted to a quadratic function in general form $f(x) = ax^2 + bx + c$ by making the identifications $b = -2ah$ and $c = ah^2 + k$. In this exercise, we use same identifications to show every parabola given in general form can be converted to standard form without completing the square.

    Solve $b = -2ah$ for h and substitute the result into the equation $c = ah^2 + k$ and then solve for k. Show $h = -\frac{b}{2a}$ and $k = \frac{4ac-b^2}{4a}$ so that

    $$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}.$$

In Exercises 59 - 64, solve the quadratic equation for the indicated variable.

59. $x^2 - 10y^2 = 0$ for x
60. $y^2 - 4y = x^2 - 4$ for x
61. $x^2 - mx = 1$ for x
62. $y^2 - 3y = 4x$ for y
63. $y^2 - 4y = x^2 - 4$ for y
64. $-gt^2 + v_0 t + s_0 = 0$ for t (Assume $g \neq 0$.)
65. (This is a follow-up to Exercise 60 in Section 1.2.) The Lagrange Interpolate function L for three points (x_0, y_0), (x_1, y_1), and (x_2, y_2) where x_0, x_1, and x_2 are three distinct real numbers is given by:

    $$L(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

    (a) For each of the following sets of points, find $L(x)$ using the formula above and verify each of the points lies on the graph of $y = L(x)$.

        i. $(-1, 1)$, $(1, 1)$, $(2, 4)$
        ii. $(1, 3)$, $(2, 10)$, $(3, 21)$
        iii. $(0, 1)$, $(1, 5)$, $(2, 7)$

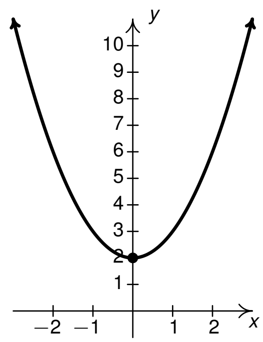
    (b) Verify that, in general, $L(x_0) = y_0$, $L(x_1) = y_1$, and $L(x_2) = y_2$.

    (c) Find $L(x)$ for the points $(-1, 6)$, $(1, 4)$ and $(3, 2)$. What happens?

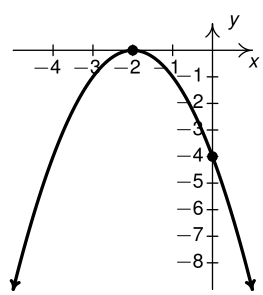
    (d) Under what conditions will $L(x)$ produce a quadratic function? Make a conjecture, test some cases, and prove your answer.

1.4.4 Answers

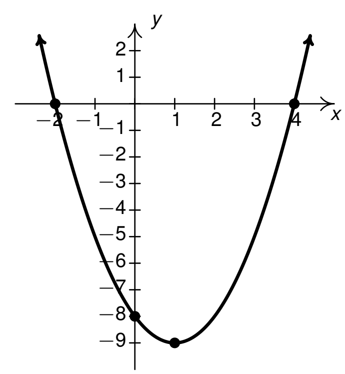
1. $f(x) = x^2 + 2$ (this is both forms!)
   No x-intercepts
   y-intercept $(0, 2)$
   Domain: $(-\infty, \infty)$
   Range: $[2, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Vertex $(0, 2)$ is a minimum
   Axis of symmetry $x = 0$



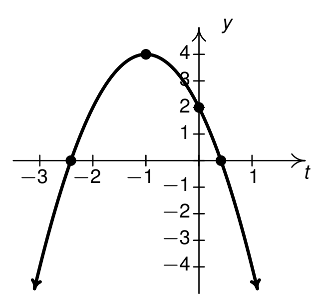
2. $f(x) = -(x + 2)^2 = -x^2 - 4x - 4$
   x-intercept $(-2, 0)$
   y-intercept $(0, -4)$
   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 0]$
   Increasing on $(-\infty, -2]$
   Decreasing on $[-2, \infty)$
   Vertex $(-2, 0)$ is a maximum
   Axis of symmetry $x = -2$



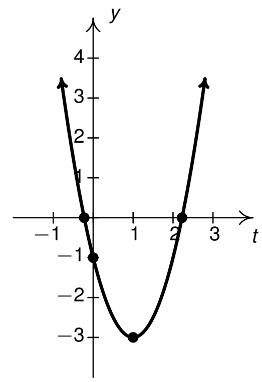
3. $f(x) = x^2 - 2x - 8 = (x - 1)^2 - 9$
   x-intercepts $(-2, 0)$ and $(4, 0)$
   y-intercept $(0, -8)$
   Domain: $(-\infty, \infty)$
   Range: $[-9, \infty)$
   Decreasing on $(-\infty, 1]$
   Increasing on $[1, \infty)$
   Vertex $(1, -9)$ is a minimum
   Axis of symmetry $x = 1$



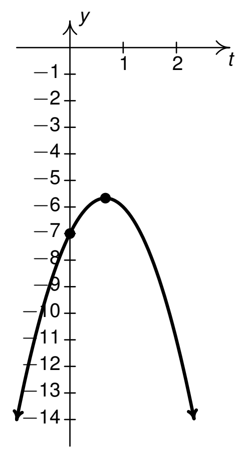
4. $g(t) = -2(t + 1)^2 + 4 = -2t^2 - 4t + 2$
   t-intercepts $(-1 - \sqrt{2}, 0)$ and $(-1 + \sqrt{2}, 0)$
   y-intercept $(0, 2)$
   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 4]$
   Increasing on $(-\infty, -1]$
   Decreasing on $[-1, \infty)$
   Vertex $(-1, 4)$ is a maximum
   Axis of symmetry $t = -1$



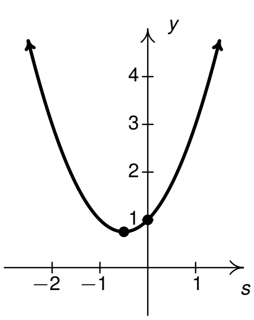
5. $g(t) = 2t^2 - tx - 1 = 2(t-1)^2 - 3$
   t-intercepts $\left(\frac{2-\sqrt{6}}{2}, 0\right)$ and $\left(\frac{2+\sqrt{6}}{2}, 0\right)$
   y-intercept $(0, -1)$
   Domain: $(-\infty, \infty)$
   Range: $[-3, \infty)$
   Increasing on $[1, \infty)$
   Decreasing on $(-\infty, 1]$
   Vertex $(1, -3)$ is a minimum
   Axis of symmetry $t = 1$



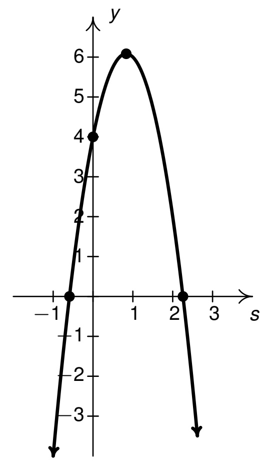
6. $g(t) = -3t^2 + 4t - 7 = -3\left(t - \frac{2}{3}\right)^2 - \frac{17}{3}$
   No t-intercepts
   y-intercept $(0, -7)$
   Domain: $(-\infty, \infty)$
   Range: $\left(-\infty, -\frac{17}{3}\right]$
   Increasing on $\left(-\infty, \frac{2}{3}\right]$
   Decreasing on $\left[\frac{2}{3}, \infty\right)$
   Vertex $\left(\frac{2}{3}, -\frac{17}{3}\right)$ is a maximum
   Axis of symmetry $t = \frac{2}{3}$



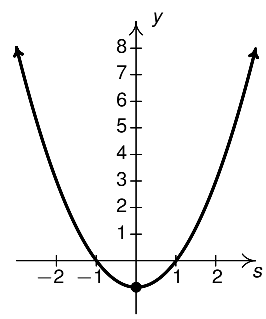
7. $h(s) = s^2 + s + 1 = \left(s + \frac{1}{2}\right)^2 + \frac{3}{4}$
   No s-intercepts
   y-intercept $(0, 1)$
   Domain: $(-\infty, \infty)$
   Range: $\left[\frac{3}{4}, \infty\right)$
   Increasing on $\left[-\frac{1}{2}, \infty\right)$
   Decreasing on $\left(-\infty, -\frac{1}{2}\right]$
   Vertex $\left(-\frac{1}{2}, \frac{3}{4}\right)$ is a minimum
   Axis of symmetry $s = -\frac{1}{2}$



8. $h(s) = -3s^2 + 5s + 4 = -3\left(s - \frac{5}{6}\right)^2 + \frac{73}{12}$
   s-intercepts $\left(\frac{5-\sqrt{73}}{6}, 0\right)$ and $\left(\frac{5+\sqrt{73}}{6}, 0\right)$
   y-intercept $(0, 4)$
   Domain: $(-\infty, \infty)$
   Range: $\left(-\infty, \frac{73}{12}\right]$
   Increasing on $\left(-\infty, \frac{5}{6}\right]$
   Decreasing on $\left[\frac{5}{6}, \infty\right)$
   Vertex $\left(\frac{5}{6}, \frac{73}{12}\right)$ is a maximum
   Axis of symmetry $s = \frac{5}{6}$



9. $h(s) = s^2 - \frac{1}{100}s - 1 = \left(s - \frac{1}{200}\right)^2 - \frac{40001}{40000}$
   s-intercepts $\left(\frac{1+\sqrt{40001}}{200}\right)$ and $\left(\frac{1-\sqrt{40001}}{200}\right)$
   y-intercept $(0, -1)$
   Domain: $(-\infty, \infty)$
   Range: $\left[-\frac{40001}{40000}, \infty\right)$
   Decreasing on $\left(-\infty, \frac{1}{200}\right]$
   Increasing on $\left[\frac{1}{200}, \infty\right)$
   Vertex $\left(\frac{1}{200}, -\frac{40001}{40000}\right)$ is a minimum[15]
   Axis of symmetry $s = \frac{1}{200}$



10. $F(x) = (x + 2)^2 - 3$
11. $F(x) = \frac{1}{2}(x - 2)^2 - 1$
12. $F(x) = -x^2 + 4$
13. $F(x) = -2(x - 1.5)^2 + 4.5$
14. $f(x) = x^2 - 3$
15. $F(x) = (x + 1)^2 - 4 = x^2 + 2x - 3$
16. $F(s) = -(s + 1)^2 - 1 = -s^2 - 2s - 2$
17. $s(t) = -\frac{1}{3}(t - 2)^2 + \frac{4}{3} = -\frac{1}{3}t^2 + \frac{4}{3}t$
18. $(-\infty, -3] \cup [1, \infty)$
19. $\left(-\infty, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$
20. No solution
21. $(-\infty, \infty)$
22. $\{2\}$
23. No solution
24. $\left[-\frac{1}{3}, 4\right]$
25. $(0, 1)$

[15] You'll need to use your calculator to zoom in far enough to see that the vertex is not the y-intercept.

26. $\left(-\infty, 1 - \frac{\sqrt{6}}{2}\right) \cup \left(1 + \frac{\sqrt{6}}{2}, \infty\right)$

27. $\left(-\infty, \frac{5-\sqrt{73}}{6}\right] \cup \left[\frac{5+\sqrt{73}}{6}, \infty\right)$

28. $\left(-3\sqrt{2}, -\sqrt{11}\right] \cup \left[-\sqrt{7}, 0\right) \cup \left(0, \sqrt{7}\right] \cup \left[\sqrt{11}, 3\sqrt{2}\right)$

29. $\left[-2 - \sqrt{7}, -2 + \sqrt{7}\right] \cup [1, 3]$

30. $(-\infty, \infty)$

31. $(-\infty, -1] \cup \{0\} \cup [1, \infty)$

32. $[-6, -3] \cup [-2, \infty)$

33. $(-\infty, 1) \cup \left(2, \frac{3+\sqrt{17}}{2}\right)$

34.
- $P(x) = -2x^2 + 28x - 26$, for $0 \leq x \leq 15$.
- 7 T-shirts should be made and sold to maximize profit.
- The maximum profit is \$72.
- The price per T-shirt should be set at \$16 to maximize profit.
- The break even points are $x = 1$ and $x = 13$, so to make a profit, between 1 and 13 T-shirts need to be made and sold.

35.
- $P(x) = -x^2 + 25x - 100$, for $0 \leq x \leq 35$
- Since the vertex occurs at $x = 12.5$, and it is impossible to make or sell 12.5 bottles of tonic, maximum profit occurs when either 12 or 13 bottles of tonic are made and sold.
- The maximum profit is \$56.
- The price per bottle can be either \$23 (to sell 12 bottles) or \$22 (to sell 13 bottles.) Both will result in the maximum profit.
- The break even points are $x = 5$ and $x = 20$, so to make a profit, between 5 and 20 bottles of tonic need to be made and sold.

36.
- $P(x) = -3x^2 + 72x - 240$, for $0 \leq x \leq 30$
- 12 cups of lemonade need to be made and sold to maximize profit.
- The maximum profit is 192¢ or \$1.92.
- The price per cup should be set at 54¢ per cup to maximize profit.
- The break even points are $x = 4$ and $x = 20$, so to make a profit, between 4 and 20 cups of lemonade need to be made and sold.

37.
- $P(x) = -0.5x^2 + 9x - 36$, for $0 \leq x \leq 24$
- 9 pies should be made and sold to maximize the daily profit.
- The maximum daily profit is \$4.50.
- The price per pie should be set at \$7.50 to maximize profit.
- The break even points are $x = 6$ and $x = 12$, so to make a profit, between 6 and 12 pies need to be made and sold daily.

38.
- $P(x) = -2x^2 + 120x - 1000$, for $0 \leq x \leq 70$

- 30 scooters need to be made and sold to maximize profit.
- The maximum monthly profit is 800 hundred dollars, or \$80,000.
- The price per scooter should be set at 80 hundred dollars, or \$8000 per scooter.
- The break even points are $x = 10$ and $x = 50$, so to make a profit, between 10 and 50 scooters need to be made and sold monthly.

39. 495 cookies

40. The vertex is (approximately) $(29.60, 22.66)$, which corresponds to a maximum fuel economy of 22.66 miles per gallon, reached sometime between 2009 and 2010 (29 – 30 years after 1980.) Unfortunately, the model is only valid up until 2008 (28 years after 1908.) So, at this point, we are using the model to *predict* the maximum fuel economy.

41. 64° at 2 PM (8 hours after 6 AM.)

42. 5000 pens should be produced for a cost of \$200.

43. 8 feet by 16 feet; maximum area is 128 square feet.

44. 50 feet by 50 feet; maximum area is 2500 feet; he can raise 100 average alpacas.

45. The largest rectangle has area 12.25 square inches.

46. 2 seconds.

47. The rocket reaches its maximum height of 500 feet 10 seconds after lift-off.

48. The hammer reaches a maximum height of approximately 13.62 feet. The hammer is in the air approximately 1.61 seconds.

49. (a) The applied domain is $[0, \infty)$.
    (d) The height function is this case is $s(t) = -4.9t^2 + 15t$. The vertex of this parabola is approximately $(1.53, 11.48)$ so the maximum height reached by the marble is 11.48 meters. It hits the ground again when $t \approx 3.06$ seconds.
    (e) The revised height function is $s(t) = -4.9t^2 + 15t + 25$ which has zeros at $t \approx -1.20$ and $t \approx 4.26$. We ignore the negative value and claim that the marble will hit the ground after 4.26 seconds.
    (f) Shooting down means the initial velocity is negative so the height functions becomes $s(t) = -4.9t^2 - 15t + 25$.

50. Make the vertex of the parabola $(0, 10)$ so that the point on the top of the left-hand tower where the cable connects is $(-200, 100)$ and the point on the top of the right-hand tower is $(200, 100)$. Then the parabola is given by $p(x) = \frac{9}{4000}x^2 + 10$. Standing 50 feet to the right of the left-hand tower means you're standing at $x = -150$ and $p(-150) = 60.625$. So the cable is 60.625 feet above the bridge deck there.

51. (a) The line for the Thursday data is $y = -.12x + 237.69$. We have $r = -.9568$ and $r^2 = .9155$ so this is a really good fit.

(b) The line for the Saturday data is $y = -0.000693x + 235.94$. We have $r = -0.008986$ and $r^2 = 0.0000807$ which is horrible. This data is not even close to linear.

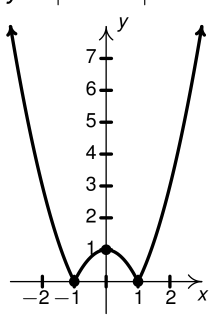
(c) The parabola for the Saturday data is $y = 0.003x^2 - 0.21x + 238.30$. We have $R^2 = .47497$ which isn't good. Thus the data isn't modeled well by a quadratic function, either.

(d) The Thursday linear model had my weight on January 1, 2010 at 193.77 pounds. The Saturday models give 235.69 and 563.31 pounds, respectively. The Thursday line has my weight going below 0 pounds in about five and a half years, so that's no good. The quadratic has a positive leading coefficient which would mean unbounded weight gain for the rest of my life. The Saturday line, which mathematically does not fit the data at all, yields a plausible weight prediction in the end. I think this is why grown-ups talk about "Lies, Damned Lies and Statistics."

52. (a) The quadratic model for the cats in Portage county is $y = 1917803.54x^2 - 16036408.29x + 24094857.7$. Although $R^2 = .70888$ this is not a good model because it's so far off for small values of x. The model gives us 24,094,858 cats when $x = 0$ but we know $N(0) = 2$.

(b) The quadratic model for the hours of daylight in Fairbanks, Alaska is $y = .51x^2 + 6.23x - .36$. Even with $R^2 = .92295$ we should be wary of making predictions beyond the data. Case in point, the model gives -4.84 hours of daylight when $x = 13$. So January 21, 2010 will be "extra dark"? Obviously a parabola pointing down isn't telling us the whole story.

54. $y = |1 - x^2|$



55. $\left(\dfrac{3-\sqrt{7}}{2}, \dfrac{-1+\sqrt{7}}{2}\right), \left(\dfrac{3+\sqrt{7}}{2}, \dfrac{-1-\sqrt{7}}{2}\right)$

56. $D(x) = x^2 + (2x+1)^2 = 5x^2 + 4x + 1$ is minimized when $x = -\frac{2}{5}$. Hence to find the point on $y = 2x + 1$ closest to $(0, 0)$ we substitute $x = -\frac{2}{5}$ into $y = 2x + 1$ to get $\left(-\frac{2}{5}, \frac{1}{5}\right)$.

59. $x = \pm y\sqrt{10}$

60. $x = \pm(y - 2)$

61. $x = \dfrac{m \pm \sqrt{m^2 + 4}}{2}$

62. $y = \dfrac{3 \pm \sqrt{16x + 9}}{2}$

63. $y = 2 \pm x$

64. $t = \dfrac{v_0 \pm \sqrt{v_0^2 + 4gs_0}}{2g}$

65. (a)

i. $L(x) = x^2$ ii. $L(x) = 2x^2 + x$ iii. $L(x) = -x^2 + 5x + 1$

(c) The three points lie on the same line and we get $L(x) = -x + 5$.

(d) To obtain a quadratic function, we require that the points are not collinear (i.e., they do not all lie on the same line.)

Chapter 2

Polynomial Functions

2.1 Graphs of Polynomial Functions

In Chapter 1, we studied functions of the form $f(x) = b$ (constant functions), $f(x) = mx + b$, $m \neq 0$ (linear functions), and $f(x) = ax^2 + bx + c$, $a \neq 0$ (quadratic functions). In each case, we learned how to construct graphs, find zeros, describe behavior, and use the functions in each family to model real-world phenomena. One might wonder about functions of the form $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, or functions containing even higher powers of x. These are the **polynomial functions** and are the subject of study in this chapter.[1] As you may recall, **poly**nomials are the result of adding **mon**omials, so we begin our study of polynomial functions with monomial functions.

2.1.1 Monomial Functions

Definition 2.1. A **monomial function** is a function of the form

$$f(x) = b \qquad \text{or} \qquad f(x) = ax^n,$$

where a and b are real numbers, $a \neq 0$ and $n \in \mathbb{N}$. The domain of a monomial function is $(-\infty, \infty)$.

Monomial functions, by definition, contain the constant functions along with a two parameter family of functions, $f(x) = ax^n$. We use x as the default independent variable here with a and n as parameters. From Section A.1.2, we recall that the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers, so examples of monomial functions include $f(x) = 2x = 2x^1$, $g(t) = -0.1t^2$, and $H(s) = \sqrt{2}\,s^{117}$. Note that the function $f(x) = x^0$ is **not** a monomial function. Even though $x^0 = 1$ for all **nonzero** values of x, 0^0 is undefined,[2] and hence $f(x) = x^0$ does **not** have a domain of $(-\infty, \infty)$.[3]

We begin our study of the graphs of polynomial functions by studying graphs of monomial functions. Starting with $f(x) = x^n$ where n is even, we investigate the cases $n = 2$, 4 and 6 at the top of the next page.

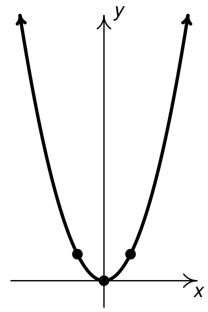
[1] You've seen polynomials before - see Section A.8, for instance. Here, we restrict our attention to polynomial **functions** which for us means **one** independent variable instead of expressions with more than one variable.

[2] More specifically, 0^0 is an **indeterminate form.** These are studied extensively in Calculus.

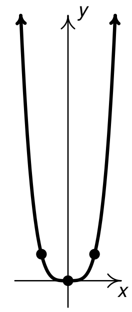
[3] This is why we do not describe monomial functions as having the form $f(x) = ax^n$ for any **whole** number n. See Section A.1.2

Numerically, we see that if $-1 < x < 1$, x^n becomes much smaller as n increases whereas if $x < -1$ or $x > 1$, x^n becomes much larger as n increases. These trends manifest themselves geometrically as the graph 'flattening' for $|x| < 1$ and 'narrowing' for $|x| > 1$ as n increases.[4]

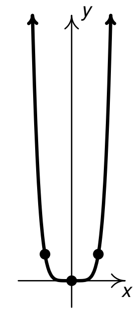
x	x^2	x^4	x^6
-2	4	16	64
-1	1	1	1
-0.5	0.25	0.0625	0.015625
0	0	0	0
0.5	0.25	0.0625	0.015625
1	1	1	1
2	4	16	64



$y = x^2$



$y = x^4$



$y = x^6$

From the graphs, it appears as if the range of each of these functions is $[0, \infty)$. When n is even, $x^n \geq 0$ for all x so the range of $f(x) = x^n$ is contained in $[0, \infty)$. To show that the range of f is all of $[0, \infty)$, we note that the equation $x^n = c$ for $c \geq 0$ has (at least) one solution for every even integer n, namely $x = \sqrt[n]{c}$. (See Section A.13 for a review of this notation.) Hence, $f(\sqrt[n]{c}) = (\sqrt[n]{c})^n = c$ which shows that every non-negative real number is in the range of f.[5]

Another item worthy of note is the symmetry about the line $x = 0$ a.k.a the y-axis. (See Definition A.10 for a review of this concept.) With n being even, $f(-x) = (-x)^n = x^n = f(x)$. At the level of points, we have that for all x, $(-x, f(-x)) = (-x, f(x))$. Hence for every point $(x, f(x))$ on the graph of f, the point symmetric about the y-axis, $(-x, f(x))$ is on the graph, too. We give this sort of symmetry a name honoring its roots here with even-powered monomial functions:

Definition 2.2. A function f is said to be **even** if $f(-x) = f(x)$ for all x in the domain of f.
NOTE: A function f is even if and only if the graph of $y = f(x)$ is symmetric about the y-axis.

An investigation of the odd powered monomial functions ($n \geq 3$) yields similar results with the major difference being that when a negative number is raised to an odd natural number power the result is still negative. Numerically we see that for $|x| > 1$ the values of $|x^n|$ increase as n increases and the values of $|x^n|$ get closer to 0 as n increases. This translates graphically into a flattening behavior on the interval $(-1, 1)$ and a narrowing elsewhere. The graphs are shown on the top of the next page.

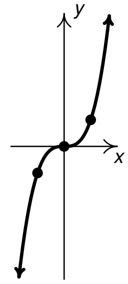
The range of these functions appear to be all real numbers, $(-\infty, \infty)$ which is algebraically sound as the equation $x^n = c$ has a solution for every real number,[6] namely $x = \sqrt[n]{c}$. Hence, for every real number c, choose $x = \sqrt[n]{c}$ so that $f(x) = f(\sqrt[n]{c}) = (\sqrt[n]{c})^n = c$. This shows that every real number is in the range of f.

[4]Recall that $|x| < 1$ is equivalent to $-1 < x < 1$ and $|x| > 1$ is equivalent to $x < -1$ or $x > 1$. Using absolute values allow us to describe these sets of real numbers more succinctly.

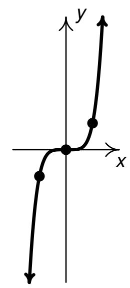
[5]This argument should sound familiar - see the comments regarding the range of $f(x) = x^2$ in Section 1.4.

[6]Do you see the importance of n being odd here?

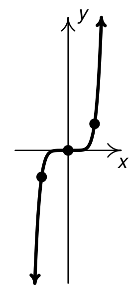
x	x^3	x^5	x^7
-2	-8	-32	-128
-1	-1	-1	-1
-0.5	0.125	-0.03125	-0.0078125
0	0	0	0
0.5	0.125	0.03125	0.0078125
1	1	1	1
2	8	32	128



$y = x^3$



$y = x^5$



$y = x^7$

Here, since n is odd, $f(-x) = (-x)^n = -x^n = -f(x)$. This means that whenever $(x, f(x))$ is on the graph, so is the point symmetric about the origin, $(-x, -f(x))$. (Again, see Definition A.10.) We generalize this property below. Not surprisingly, we name it in honor of its odd powered heritage:

Definition 2.3. A function f is said to be **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
NOTE: A function f is odd if and only if the graph of $y = f(x)$ is symmetric about the origin.

The most important thing to take from the discussion above is the basic shape and common points on the graphs of $y = x^n$ for each of the families when n even and n is odd. While symmetry is nice and should be noted when present, even and odd symmetry are comparatively rare. The point of Definitions 2.2 and 2.3 is to give us the vocabulary to point out the symmetry when appropriate.

Moving on, we take a cue from Theorem 1.2 and prove the following.

Theorem 2.1. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a(x-h)^n + k$ can be obtained from the graph of $f(x) = x^n$ by performing the following operations, in sequence:

1. add h to the x-coordinates of each of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = x^n$ to $y = (x-h)^n$.

2. multiply the y-coordinates of each of the points on the graph obtained in Step 1 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = (x-h)^n$ to $y = a(x-h)^n$.

3. add k to the y-coordinates of each of the points on the graph obtained in Step 2. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a(x-h)^n$ to $y = a(x-h)^n + k$

Proof. Our goal is to start with the graph of $f(x) = x^n$ and build it up to the graph of $F(x) = a(x-h)^n + k$. We begin by examining $F_1(x) = (x-h)^n$. The graph of $f(x) = x^n$ can be described as the set of points $\{(c, c^n) \mid c \in \mathbb{R}\}$.[7] Likewise, the graph of F_1 can be described as the set of points $\{(x, (x-h)^n) \mid x \in \mathbb{R}\}$.

[7] We are using the dummy variable c here instead of x for reasons that will become apparent shortly.

If we relabel $c = x - h$ so that $x = c + h$, then as x varies through all real numbers so does c.[8] Hence, we can describe the graph of F_1 as $\{(c + h, c^n) \mid c \in \mathbb{R}\}$. This means that we can obtain the graph of F_1 from the graph of f by adding h to each of the x-coordinates of the points on the graph of f and that establishes the first step of the theorem.

Next, we consider the graph of $F_2(x) = a(x-h)^n$ as compared to the graph of $F_1(x) = (x-h)^n$. The graph of F_1 is the set of points $\{(x, (x-h)^n \mid x \in \mathbb{R}\}$ while the graph of F_2 is the set of points $\{(x, a(x-h)^n) \mid x \in \mathbb{R}\}$. The only difference between the points $(x, (x-h)^n)$ and $(x, a(x-h)^n)$ is that the y-coordinate in the latter is a times the y-coordinate of the former.

In other words, to produce the graph of F_2 from the graph of F_1, we take the y-coordinate of each point on the graph of F_1 and multiply it by a to get the corresponding point on the graph of F_2. If $a > 0$, all we are doing is scaling the y-axis by a. If $a < 0$, then, in addition to scaling the y-axis, we are also reflecting each point across the x-axis. In either case, we have established the second step of the theorem.

Last, we compare the graph of $F(x) = a(x-h)^n + k$ to that of $F_2(x) = a(x-h)^n$. Once again, we view the graphs as sets of points in the plane. The graph of F_2 is $\{(x, a(x-h)^n) \mid x \in \mathbb{R}\}$ and the graph of F is$\{(x, a(x-h)^n + k) \mid x \in \mathbb{R}\}$. Looking at the corresponding points, $(x, a(x-h)^n)$ and $(x, a(x-h)^n + k)$, we see that we can obtain all of the points on the graph of F by adding k to each of the y-coordinates to points on the graph of F_2. This is equivalent to shifting every point vertically by k units which establishes the third and final step in the theorem. $\square$

This argument should sound familiar. The proof we presented above is more-or-less the same argument we presented after the proof of Theorem 1.2 in Section 1.3 but with '$|\cdot|$' replaced by '$(\cdot)^n$.' Also note that using $n = 2$ in Theorem 2.1 establishes Theorem 1.3 in Section 1.4.

We now use Theorem 2.1 to graph two different "transformed" monomial functions. To provide the reader an opportunity to compare and contrast the graphical behaviors exhibited in the case when n is even versus when n is odd, we graph one of each case.

Example 2.1.1. Use Theorem 2.1 to graph the following functions. Label at least three points on each graph. State the domain and range using interval notation.

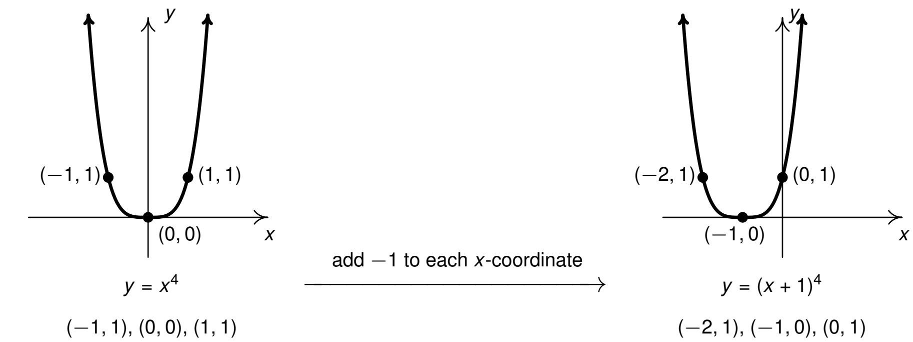
1. $f(x) = -2(x+1)^4 + 3$
2. $g(t) = \dfrac{(2t-1)^3}{5}$

Solution.

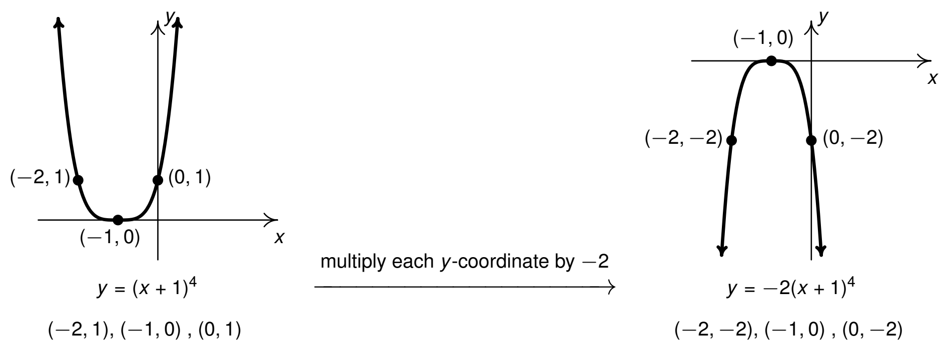
1. For $f(x) = -2(x+1)^4 + 3 = -2(x-(-1))^4 + 3$, we identify $n = 4$, $a = -2$, $h = -1$, and $k = 3$. Thus to graph f, we start with $y = x^4$ and perform the following steps, in sequence, tracking the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$ through each step:

[8]That is, for a fixed number h every real number c can be written as $x - h$ for some real number x, and every real number x can be written as $c + h$ for some real number c.

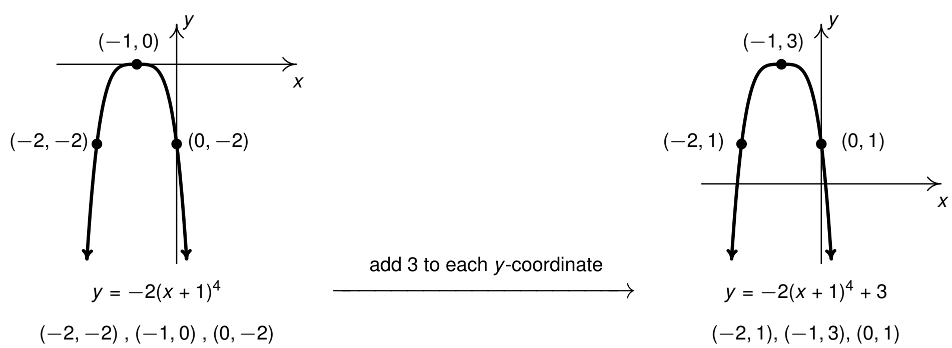
Step 1: add -1 to the x-coordinates of each of the points on the graph of $y = x^4$:



Step 2: multiply the y-coordinates of each of the points on the graph of $y = (x+1)^4$ by -2:



Step 3: add 3 to the y-coordinates of each of the points on the graph of $y = -2(x+1)^4$:



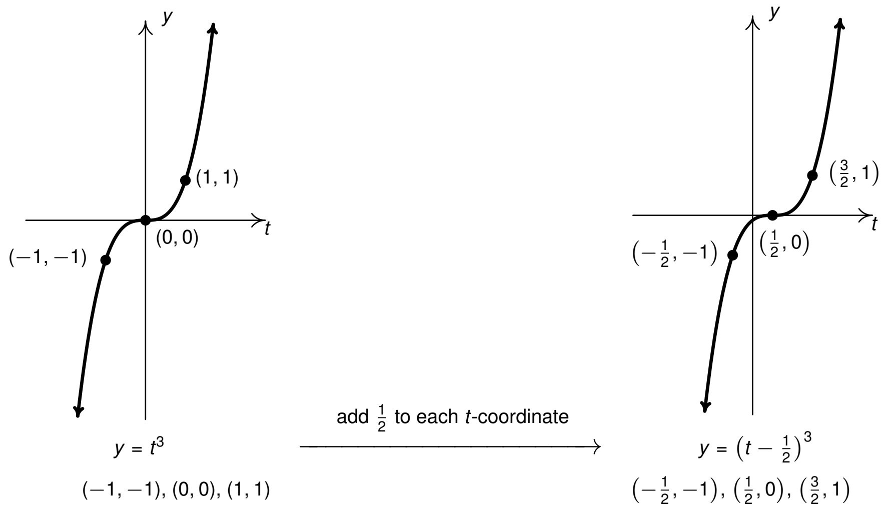
The domain here is $(-\infty, \infty)$ while the range is $(-\infty, 3]$.

2. To use Theorem 2.1 to graph $g(t) = \dfrac{(2t-1)^3}{5}$, we must rewrite the expression for $g(t)$:

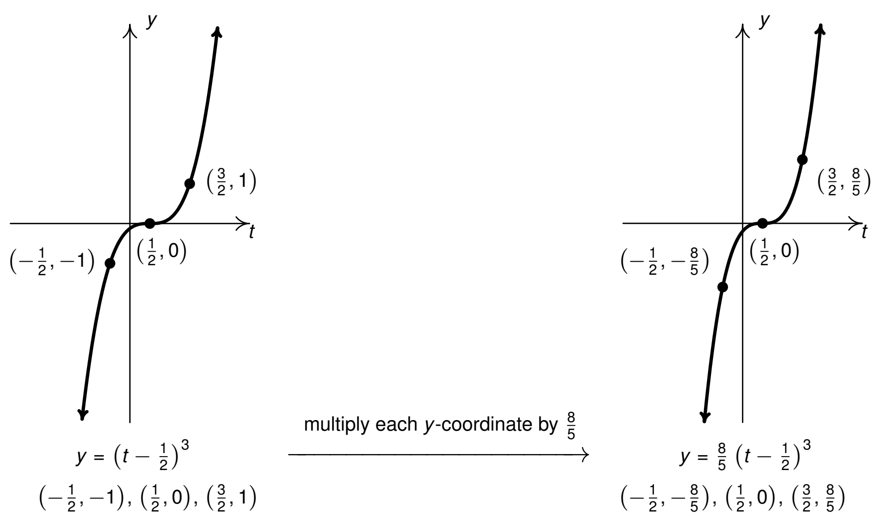
$$g(t) = \frac{(2t-1)^3}{5} = \frac{1}{5}\left(2\left(t - \frac{1}{2}\right)\right)^3 = \frac{1}{5}(2)^3\left(t - \frac{1}{2}\right)^3 = \frac{8}{5}\left(t - \frac{1}{2}\right)^3$$

We identify $n = 3$, $h = \frac{1}{2}$ and $a = \frac{8}{5}$. Hence, we start with the graph of $y = t^3$ and perform the following steps, in sequence, tracking the points $(-1,-1)$, $(0,0)$ and $(1,1)$ through each step:

Step 1: add $\frac{1}{2}$ to each of the t-coordinates of each of the points on the graph of $y = t^3$:



Step 2: multiply each of the y-coordinates of the graph of $y = \left(t - \frac{1}{2}\right)^3$ by $\frac{8}{5}$.



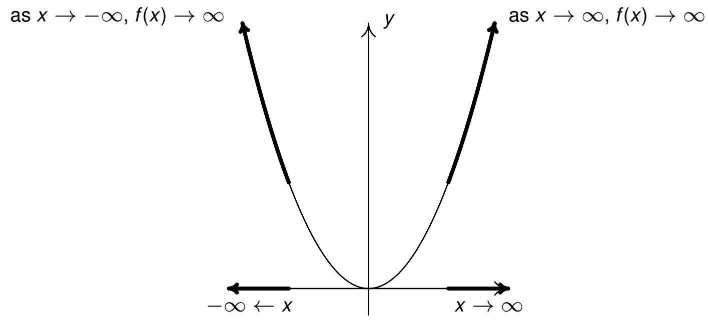
Both the domain and range of g is $(-\infty, \infty)$. □

Example 2.1.1 demonstrates two big ideas in mathematics: first, resolving a complex problem into smaller, simpler steps, and, second, the value of changing form.[9]

[9]We've seen the importance of changing form several times already, but it never hurts to point it out.

Next we wish to focus on the so-called **end behavior** presented in each case.[10] The end behavior of a function is a way to describe what is happening to the outputs from a function as the inputs approach the 'ends' of the domain. Since domain of monomial functions is $(-\infty, \infty)$, we are looking to see what these functions do as their inputs 'approach' ∞ and $-\infty$. The best we can do is sample inputs and outputs and infer general behavior from these observations. The good news is we've wrestled with this concept before. Indeed, every time we add 'arrows' to the graph of a function, we've indicated its end behavior.[11] Let's revisit the graph of $f(x) = x^2$ using the table below.

x	$f(x) = x^2$
-1000	1000000
-100	10000
-10	100
0	0
10	100
100	10000
1000	1000000



$f(x) = x^2$

As x takes on negative values that are larger in absolute value,[12] we see $f(x)$ takes on larger and larger positive values, seemingly without bound.[13] It should be stressed that since '$-\infty$' and '∞' aren't real numbers, we can't write '$f(-\infty) = \infty$,' so in order to communicate this behavior, we write as $x \to -\infty$, $f(x) \to \infty$, or, more succinctly, $\lim\limits_{x \to -\infty} f(x) = \infty$. Note that this latter notation is borrowed from Calculus and is read 'the limit as x approaches $-\infty$ of $f(x)$ is ∞'.

Graphically, the farther to the left we select inputs on the x-axis, the farther up the y-axis the output (function) values are. We indicate this by attaching an 'arrow' on the graph in Quadrant II indicating the graph continues to head upward to the left.

Similarly, observing the behavior of f as $x \to \infty$, we get that $\lim\limits_{x \to \infty} f(x) = \infty$ since as the x values increase without bound, so do the $f(x)$ values. Graphically we indicate this by an arrow on the graph in Quadrant I heading upwards to the right. This behavior holds for all functions $f(x) = x^n$ where $n \geq 2$ is even.[14]

Repeating this investigation for for $f(x) = x^3$, we find as $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to \infty} f(x) = \infty$. This trend holds for all functions $f(x) = x^n$ where n is odd.

[10] Sometimes called the 'long run' behavior.

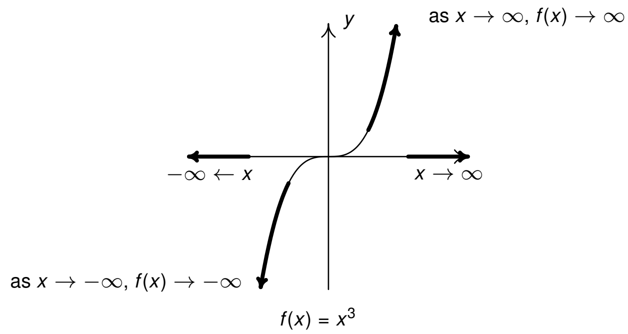
[11] We'll do a much more exhaustive look at what it means for variables to 'approach' infinity in Section 6.1. Hopefully this informal discussion suffices for now.

[12] these are technically 'smaller and smaller' values because of how the real number line is ordered. For example, $x = -1000$ is smaller than $x = -100$ since $x = -1000$ is farther to the left than $x = -100$ on the number line. This being said, $|-1000| = 1000$ is larger than $|-100| = 100$ so in an imprecise, informal, and technically incorrect way, the number -1000 is a 'larger' negative number than the number -100. I would worry about pedants here but hardly anyone reads these footnotes.

[13] That is, the $f(x)$ values grow larger than any positive number. They are '**unbounded**.' The algebra that proves this fact is the same that shows the range of f is $[0, \infty)$.

[14] Can you reason why?

x	$f(x) = x^3$
-1000	-1000000000
-100	-1000000
-10	-1000
0	0
10	1000
100	1000000
1000	1000000000

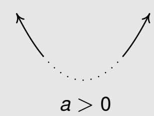


Theorem 2.2 summarizes the end behavior of monomial functions. The results are a consequence of Theorem 2.1 in that the end behavior of a function of the form $y = ax^n$ only differs from that of $y = x^n$ if there is a reflection, that is, if $a < 0$.

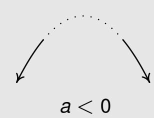
Theorem 2.2. End Behavior of Monomial Functions:
Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \in \mathbb{N}$.

- If n is even:

  if $a > 0$, $\displaystyle\lim_{x \to -\infty} f(x) = \infty$ and $\displaystyle\lim_{x \to \infty} f(x) = \infty$:

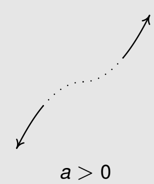


  for $a < 0$, $\displaystyle\lim_{x \to -\infty} f(x) = -\infty$ and $\displaystyle\lim_{x \to \infty} f(x) = -\infty$:

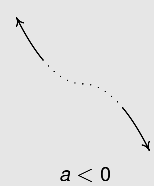


- If n is odd:

  for $a > 0$, $\displaystyle\lim_{x \to -\infty} f(x) = -\infty$ and $\displaystyle\lim_{x \to \infty} f(x) = \infty$:



  for $a < 0$, $\displaystyle\lim_{x \to -\infty} f(x) = \infty$ and $\displaystyle\lim_{x \to \infty} f(x) = -\infty$:



2.1.2 Polynomial Functions

We are now in the position to discuss **polynomial** functions. Simply stated, **poly**nomial functions are sums of **mon**omial functions. The challenge becomes how to describe one of these beasts in general. Up until now, we have used distinct letters to indicate different parameters in our definitions of function families. In other words, we define constant functions as $f(x) = b$, linear functions as $f(x) = mx + b$, and quadratic functions as $f(x) = ax^2 + bx + c$. We even hinted at a function of the form $f(x) = ax^3 + bx^2 + cx + d$. What happens if we wanted to describe a generic polynomial that required, say, 117 different parameters? Our work around is to use subscripted parameters, a_k, that denote the coefficient of x^k. For example, instead of writing a quadratic as $f(x) = ax^2 + bx + c$, we describe it as $f(x) = a_2x^2 + a_1x + a_0$, where a_2, a_1, and a_0 are real numbers and $a_2 \neq 0$. As an added example, consider $f(x) = 4x^5 - 3x^2 + 2x - 5$. We can re-write the formula for f as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. and identify $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$ and $a_0 = -5$. This is the notation we use in the following definition.

Definition 2.4. A **polynomial function** is a function of the form

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are real numbers and $n \in \mathbb{N}$. The domain of a polynomial function is $(-\infty, \infty)$.

As usual, x is used in Definition 2.4 as the independent variable with the a_k each being a parameter. Even though we specify $n \in \mathbb{N}$ so $n \geq 1$, the value of the a_k are unrestricted. Hence, any constant function $f(x) = b$ can be written as $f(x) = 0x + a_0$, and so they are polynomials. Polynomials have an associated vocabulary,[15] and hence, so do polynomial functions.

Definition 2.5.

- Given $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with $n \in \mathbb{N}$ and $a_n \neq 0$, we say
  - The natural number n is called the **degree** of the polynomial f.
  - The term a_nx^n is called the **leading term** of the polynomial f.
  - The real number a_n is called the **leading coefficient** of the polynomial f.
  - The real number a_0 is called the **constant term** of the polynomial f.
- If $f(x) = a_0$, and $a_0 \neq 0$, we say f has degree 0.
- If $f(x) = 0$, we say f has no degree.[a]

[a]Some authors say $f(x) = 0$ has degree $-\infty$ for reasons not even we will go into.

Again, constant functions are split off in their own separate case Definition 2.5 because of the ambiguity of 0^0. (See the remarks following Definition 2.1.) A consequence of Definition 2.5 is that we can now think of nonzero constant functions as 'zeroth' degree polynomial functions, linear functions as 'first' degree polynomial functions, and quadratic functions as 'second' degree polynomial functions.

Example 2.1.2.
Find the degree, leading term, leading coefficient and constant term of the following polynomial functions.

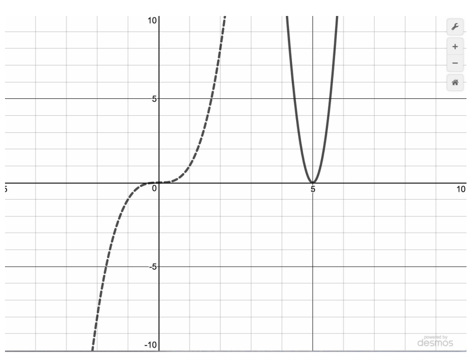
[15]See Section A.8.

1. $f(x) = 4x^5 - 3x^2 + 2x - 5$
2. $g(t) = 12t - t^3$
3. $H(w) = \dfrac{4-w}{5}$
4. $p(z) = (2z-1)^3(z-2)(3z+2)$

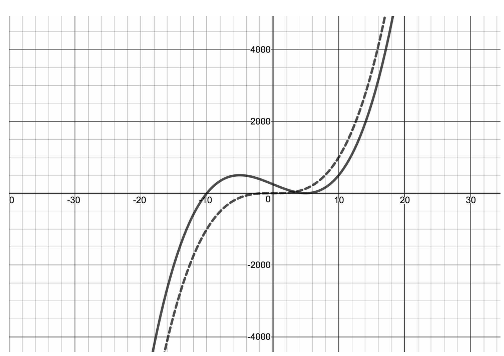
Solution.

1. There are no surprises with $f(x) = 4x^5 - 3x^2 + 2x - 5$. It is written in the form of Definition 2.5, and we see that the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is -5.
2. Two changes here: first, the independent variable is t, not x. Second, the form given in Definition 2.5 specifies the function be written in descending order of the powers of x, or in this case, t. To that end, we re-write $g(t) = 12t - t^3 = -t^3 + 12t$, and see that the degree of g is 3, the leading term is $-t^3$, the leading coefficient is -1 and the constant term is 0.
3. We need to rewrite the formula for $H(w)$ so that it resembles the form given in Definition 2.5: $H(w) = \frac{4-w}{5} = \frac{4}{5} - \frac{w}{5} = -\frac{1}{5}w + \frac{4}{5}$. We see the degree of H is 1, the leading term is $-\frac{1}{5}w$, the leading coefficient is $-\frac{1}{5}$ and the constant term is $\frac{4}{5}$.
4. It may seem that we have some work ahead of us to get p in the form of Definition 2.5. However, it is possible to glean the information requested about p without multiplying out the entire expression $(2z-1)^3(z-2)(3z+2)$. The leading term of p will be the term which has the highest power of z. The way to get this term is to multiply the terms with the highest power of z from each factor together - in other words, the leading term of $p(z)$ is the product of the leading terms of the **factors** of $p(z)$. Hence, the leading term of p is $(2z)^3(z)(3z) = 24z^5$. This means that the degree of p is 5 and the leading coefficient is 24. As for the constant term, we can perform a similar operation. The constant term of p is obtained by multiplying the constant terms from each of the **factors**: $(-1)^3(-2)(2) = 4$. $\Box$

We now turn our attention to graphs of polynomial functions. Since polynomial functions are sums of monomial functions, it stands to reason that some of of the properties of those graphs carry over to more general polynomials. We first discuss end behavior. Consider $f(x) = x^3 - 75x + 250$. Below is the graph of $f(x)$ (solid line) along with the graph of its leading term, $y = x^3$ (dashed line.) Below on the left is a view 'near' the origin while below on the right is a 'zoomed out' view. Near the origin, the graphs have little in common, but as we look farther out, it becomes that the functions begin to look quite similar.



$y = f(x)$ and $y = x^3$ 'near' $(0,0)$



a 'zoomed out' view

This observation is borne out numerically as well. Based on the table below, as $x \to \infty$ or $x \to -\infty$, it certainly appears as if $f(x) \approx g(x)$. One way to think about what is happening numerically is that as $x \to \infty$ or $x \to -\infty$, the leading term x^3 **dominates** the lower order terms $-75x$ and 250. In other words, x^3 grows so much faster than $-75x$ and 250 that these 'lower order terms' don't contribute anything of significance to the x^3 so $f(x) \approx x^3$. To see this, we rewrite $f(x)$ as[16]

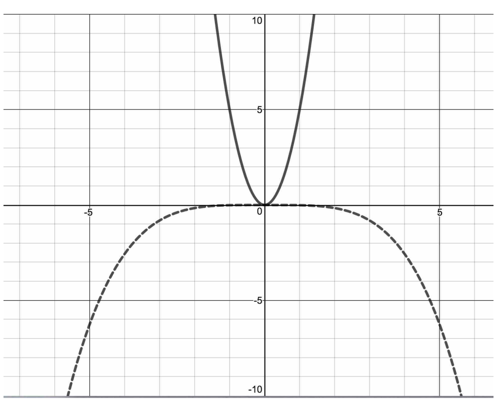
$$f(x) = x^3 - 75x + 250 = x^3\left(1 - \frac{75}{x^2} + \frac{250}{x^3}\right).$$

As $x \to \infty$ or $x \to -\infty$, both $\frac{75}{x^2}$ and $\frac{250}{x^3}$ have constant numerators but denominators that are becoming unbounded. As such, both $\frac{75}{x^2}$ and $\frac{250}{x^3} \to 0$. Therefore, as $x \to \infty$ or $x \to -\infty$,

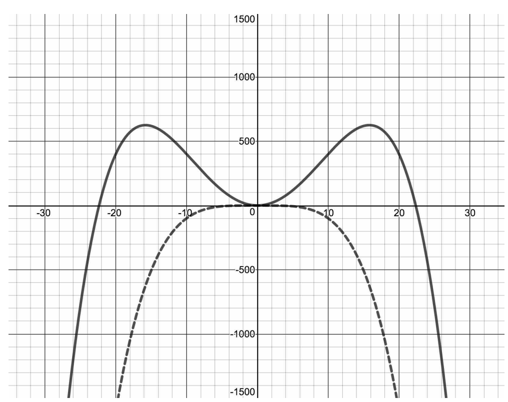
$$f(x) = x^3 - 75x + 250 = x^3\left(1 - \frac{75}{x^2} + \frac{250}{x^3}\right) \approx x^3(1 + 0 + 0) = x^3.$$

x	$f(x) = x^3 - 75x + 250$	x^3	$-75x$	250	$\frac{75}{x^2}$	$\frac{250}{x^3}$
-1000	$\approx -1 \times 10^9$	-1×10^9	75000	250	7.5×10^{-5}	-2.5×10^{-7}
-100	$\approx -9.9 \times 10^5$	-1×10^6	7500	250	0.0075	-2.5×10^{-4}
-10	0	-1000	750	250	0.75	-0.25
10	500	1000	-750	250	0.75	0.25
100	$\approx 9.9 \times 10^5$	1×10^6	-7500	250	0.0075	2.5×10^{-4}
1000	$\approx 1 \times 10^9$	1×10^9	-75000	250	7.5×10^{-5}	2.5×10^{-7}

Next, consider $g(x) = -0.01x^4 + 5x^2$. Following the logic of the above example, we would expect the end behavior of $y = g(x)$ to mimic that of $y = -0.01x^4$. When we graph $y = g(x)$ (solid line) on the same set of axes as $y = -0.01x^4$ (dashed line), a view near the origin seems to suggest the exact opposite. However, zooming out reveals that the two graphs do share the same end behavior.[17]



$y = g(x)$ and $y = x^4$ 'near' $(0, 0)$



a 'zoomed out' view

[16]Since we are considering $x \to \infty$ or $x \to -\infty$, we are not concerned with x even being close to 0, so these fractions will all be defined.

[17]Or at least they appear to within the limits of the technology.

Algebraically, for $x \to \infty$ or $x \to -\infty$, even with the small coefficient of -0.01, $-0.01x^4$ dominates the $5x^2$ term so $g(x) \approx -0.01x^4$. More precisely,

$$g(x) = -0.01x^4 + 5x^2 = x^4\left(-0.01 + \frac{5}{x^2}\right) \approx x^4(-0.01 + 0) = -0.01x^4.$$

The results of these last two examples generalize below in Theorem 2.3.

Theorem 2.3. End Behavior for Polynomial Functions:
The end behavior of polynomial function $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with $a_n \neq 0$ matches the end behavior of $y = a_nx^n$.
That is, the end behavior of a polynomial function is determined by its leading term.

We argue Theorem 2.3 using an argument similar to ones used above. As $x \to \infty$ or $x \to -\infty$,

$$f(x) = x^n\left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \approx x^n(a_n + 0 + \ldots 0) = a_nx^n$$

If this argument looks a little fuzzy, it should. As with all things involving infinity, the precision of Calculus is required here[18] For now, we'll rely on number sense and algebraic intuition.

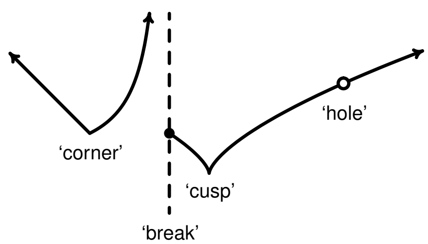
Now that we know how to determine the end behavior of polynomial functions, it's time to investigate what happens 'in between' the ends. First and foremost, polynomial functions are **continuous**. Recall from Section 1.4 that, informally, graphs of continuous functions have no 'breaks' or 'holes' in them.[19] Since monomial functions are continuous (as far as we can tell) and polynomials are sums of monomial functions, it turns out that polynomial functions are continuous as well.

Moreover, the graphs of monomial functions, hence polynomial functions, are **smooth**. Once again, 'smoothness' is a concept defined precisely in Calculus, but for us, functions have no 'corners' or 'sharp turns'. Below we find the graph of a function which is neither smooth nor continuous, and to its right we have a graph of a polynomial, for comparison.

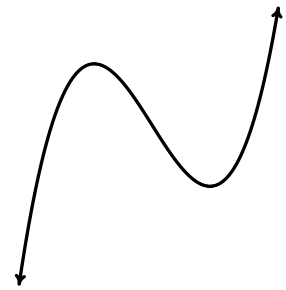
The function whose graph appears on the left fails to be continuous where it has a 'break' or 'hole' in the graph; everywhere else, the function is continuous. The function is continuous at the 'corner' and the 'cusp', but we consider these 'sharp turns', so these are places where the function fails to be smooth. Apart from these four places, the function is smooth and continuous. Polynomial functions are smooth and continuous everywhere, as exhibited in the graph on the right.

The notion of smoothness is what tells us graphically that, for example, $f(x) = |x|$, whose graph is the characteristic '$\vee$' shape, cannot be a polynomial function, even though it is a piecewise-defined function comprised of polynomial functions. Knowing polynomial functions are continuous and smooth gives us an idea of how to 'connect the dots' when sketching the graph from points that we're able to find analytically such as intercepts.

[18] Which we'll touch on in Chapters 3 and 6.
[19] We'll revisit this concept in more formally in Section 6.1.2.



Pathologies not found on graphs of polynomials functions.



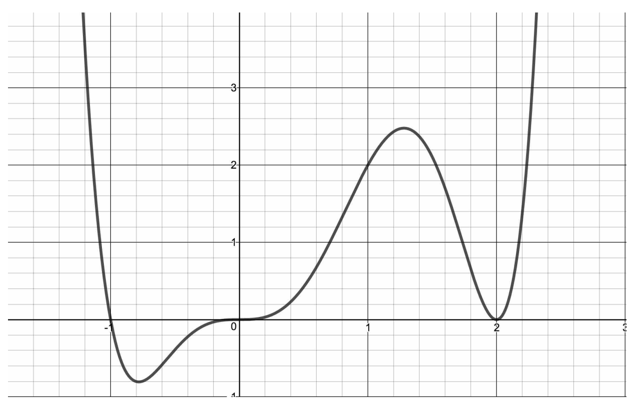
The graph of a polynomial function.

Speaking of intercepts, we next focus our attention on the behavior of the graphs of polynomial functions near their zeros. Recall a zero c of a function f is a solution to $f(x) = 0$. Geometrically, the zeros of a function are the x-coordinates of the x-intercepts of the graph of $y = f(x)$.

Consider the polynomial function $f(x) = x^3(x-2)^2(x+1)$. To find the zeros of f, we set $f(x) = x^3(x-2)^2(x+1) = 0$. Since the expression $f(x)$ is already factored, we set each factor equal to zero.[20]

Solving $x^3 = 0$ gives $x = 0$, $(x-2)^2 = 0$ gives $x = 2$, and $x + 1 = 0$ gives $x = -1$. Hence, our zeros are $x = -1$, $x = 0$, and $x = 2$.

Below, we graph $y = f(x)$ and focus our attention near the x-intercepts $(-1, 0)$, $(0, 0)$ and $(2, 0)$.



$y = f(x) = x^3(x-2)^2(x+1)$

We first note that the graph **crosses** through the x-axis at $(-1, 0)$ and $(0, 0)$, but the graph **touches** and **rebounds** at $(2, 0)$. Moreover, at $(-1, 0)$, the graph crosses through the axis is a fairly 'linear' fashion whereas there is a substantial amount of 'flattening' going on near $(0, 0)$. Or aim is to explain these observations and generalize them.

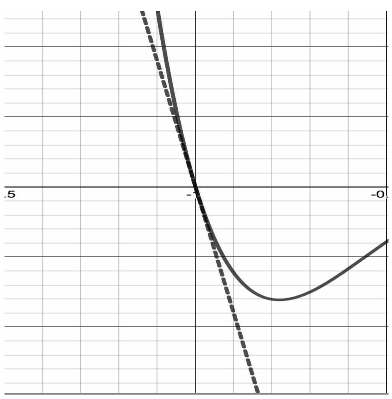
First, let's look at what's happening with the formula $f(x) = x^3(x-2)^2(x+1)$ when $x \approx -1$. We know the x-intercept at $(-1, 0)$ is due to the presence of the $(x+1)$ factor in the expression for $f(x)$. So, in this sense, the factor $(x+1)$ is determining a major piece of the behavior of the graph near $x = -1$. For that reason, we focus instead on the other two factors to see what contribution they make.

[20] in accordance with the Zero Product Property of the Real Numbers - see Section A.2.

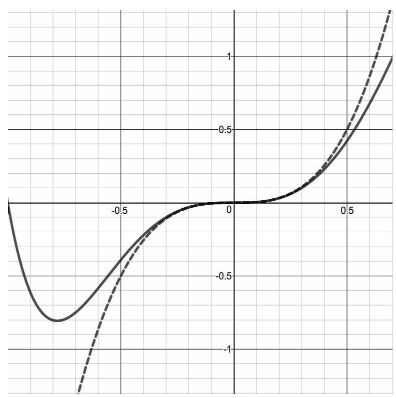
We find when $x \approx -1$, $x^3 \approx (-1)^3 = -1$ and $(x-2)^2 \approx (-1-2)^2 = 9$. Hence, $f(x) = x^3(x-3)^2(x+1) \approx (-1)^3(-1-2)^2(x+1) = -9(x+1)$. Below on the left is a graph of $y = f(x)$ (the solid line) and the graph of $y = -9(x+1)$ (the dashed line.) Sure enough, these graphs approximate one another near $x = -1$.

Likewise, let's look near $x = 0$. The x-intercept $(0,0)$ is due to the x^3 term. For $x \approx 0$, $(x-2)^2 \approx (0-2)^2 = 4$ and $(x+1) \approx (0+1) = 1$, so $f(x) = x^3(x-3)^2(x+1) \approx x^3(-2)^2(1) = 4x^3$. Below in the center picture, we have the graph of $y = f(x)$ (again, the solid line) and $y = 4x^3$ (the dashed line) near $x = 0$. Once again, the graphs verify our analysis.

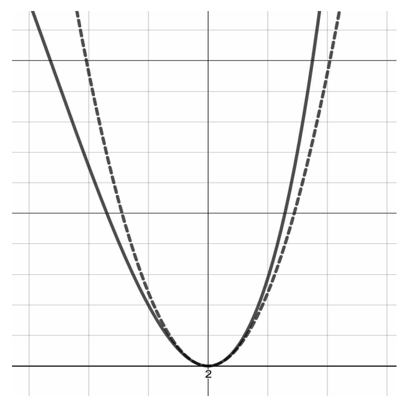
Last, but not least, we analyze f near $x = 2$. Here, the intercept $(2,0)$ is due to the $(x-2)^2$ factor, so we look at the x^3 and $(x+1)$ factors. If $x \approx 2$, $x^3 \approx (2)^3 = 8$ and $(x+1) \approx (2+1) = 3$. Hence, $f(x) = x^3(x-3)^2(x+1) \approx (2)^3(x-2)^2(2+1) = 24(x-2)^2$. Sure enough, as evidenced below on the right, the graphs of $y = f(x)$ and $y = 24(x-2)^2$.



$y = f(x)$ and $y = -9(x+1)$



$y = f(x)$ and $y = 4x^3$



$y = f(x)$ and $y = 24(x-2)^2$

We generalize our observations in Theorem 2.4 below. Like many things we've seen in this text, a more precise statement and proof can be found in a course on Calculus.

Theorem 2.4. Suppose f is a polynomial function and $f(x) = (x-c)^m q(x)$ where $m \in \mathbb{N}$ and $q(c) \neq 0$. Then the the graph of $y = f(x)$ near $(c, 0)$ resembles that of $y = q(c)(x-c)^m$.

Let's see how Theorem 2.4 applies to our findings regarding $f(x) = x^3(x-2)^2(x+1)$. For $c = -1$, $(x-c) = (x-(-1)) = (x+1)$. We rewrite $f(x) = x^3(x-2)^2(x+1) = (x-(-1))^1\left[x^3(x-2)^2\right]$ and identify $m = 1$ and $q(x) = x^3(x-2)^2$. We find $q(c) = q(-1) = (-1)^3(-1-2)^2 = -9$ so Theorem 2.4 says that near $(-1,0)$, the graph of $y = f(x)$ resembles $y = q(-1)(x-(-1))^1 = -9(x+1)$.

For $c = 0$, $(x-c) = (x-0) = x$ and we can rewrite $f(x) = x^3(x-2)^2(x+1) = (x-0)^3\left[(x-2)^2(x+1)\right]$. We identify $m = 3$ and $q(x) = (x-2)^2(x+1)$. In this case $q(c) = q(0) = (0-2)^2(0+1) = 4$, so Theorem 2.4 guarantees the graph of $y = f(x)$ near $x = 0$ resembles $y = q(0)(x-0)^3 = 4x^3$.

Lastly, for $c = 2$, we see $f(x) = (x-2)^2\left[x^3(x+1)\right]$ and identify $m = 2$ and $q(x) = x^3(x+1)$. We find $q(2) = 2^3(2+1) = 24$, so per Theorem 2.4, the graph of $y = f(x)$ resembles $y = 24(x-2)^2$ near $x = 2$.

As we already mentioned, the formal statement and proof of Theorem 2.4 require Calculus. For now, we can understand the theorem as follows.

If we factor a polynomial function as $f(x) = (x - c)^m q(x)$ where $m \geq 1$, then $x = c$ is a zero of f, since $f(c) = (c - c)^m q(c) = 0 \cdot q(c) = 0$. The stipulation that $q(c) \neq 0$ means that we have essentially factored the expression $f(x) = (x - c)^m q(x) = (\text{going to } 0) \cdot (\text{not going to } 0)$.

Thinking back to Theorem 2.1, the graph $y = q(c)(x - c)^m$ has an x-intercept at $(c, 0)$, a basic overall shape determined by the exponent m, and end behavior determined by the sign of $q(c)$.

The fact that **if** $x = c$ is a zero **then** we are guaranteed we can factor $f(x) = (x - c)^m q(x)$ were $q(c) \neq 0$ and, moreover, such a factorization is unique (so that there's only one value of m possible for each zero) is a consequence of two theorems, Theorem 2.6 and The Factor Theorem, Theorem 2.8 which we'll review in Section 2.2. For now, we assume such a factorization is unique in order to define the following.

Definition 2.6. Suppose f is a polynomial function and $m \in \mathbb{N}$. If $f(x) = (x - c)^m q(x)$ where $q(c) \neq 0$, we say $x = c$ is a zero of **multiplicity** m.

So, for $f(x) = x^3(x - 2)^2(x + 1) = (x - 0)^3(x - 2)^2(x - (-1))^1$, $x = 0$ is a zero of multiplicity 3, $x = 2$ is a zero of multiplicity 2, and $x = -1$ is a zero of multiplicity 1. Theorems 2.3 and 2.4 give us the following:

Theorem 2.5. The Role of Multiplicity: Suppose f is a polynomial function and $x = c$ is a zero of multiplicity m.

- If m is even, the graph of $y = f(x)$ touches and rebounds from the x-axis at $(c, 0)$.
- If m is odd, the graph of $y = f(x)$ crosses through the x-axis at $(c, 0)$.

Our next example showcases how all of the above theory can assist in sketching relatively good graphs of polynomial functions without the assistance of technology.

Example 2.1.3. Let $p(x) = (2x - 1)(x + 1)(1 - x^4)$.

1. Find all real zeros of p and state their multiplicities.
2. Describe the behavior of the graph of $y = p(x)$ near each of the x-intercepts.
3. Determine the end behavior and y-intercept of the graph of $y = p(x)$.
4. Sketch $y = p(x)$ and check your answer using a graphing utility.

Solution.

1. To find the zeros of p, we set $p(x) = (2x - 1)(x + 1)(1 - x^4) = 0$. Since the expression $p(x)$ is already (partially) factored, we set each factor equal to 0 and solve. From $(2x - 1) = 0$, we get $x = \frac{1}{2}$; from $(x + 1) = 0$ we get $x = -1$; and from solving $1 - x^4 = 0$ we get $x = \pm 1$. Hence, the zeros are $x = -1$, $x = \frac{1}{2}$, and $x = 1$.

In order to determine the multiplicities, we need to factor $p(x)$ as so we can identify the m and $q(x)$ as described in Definition 2.6. The zero $x = -1$ corresponds to the factor $(x+1)$. Notice, however, that writing $p(x) = (x+1)^1\left[(2x-1)(1-x^4)\right]$ with $m = 1$ and $q(x) = (2x-1)(1-x^4)$ does **not** satisfy Definition 2.6 since here, $q(-1) = (2(-1)-1)(1-(-1)^4) = 0$. Indeed, we can factor $(1-x^4) = (1-x^2)(1+x^2) = (1-x)(1+x)(x^2+1)$ so that

$$p(x) = (2x-1)(x+1)(1-x^4) = (2x-1)(x+1)(1-x)(1+x)(x^2+1) = (x+1)^2\left[(2x-1)(1-x)(x^2+1)\right].$$

Identifying $q(x) = (2x-1)(1-x)(x^2+1)$, we find $q(-1) = (2(-1)-1)(1-(-1))((-1)^2+1) = -12 \neq 0$, which means the multiplicity of $x = -1$ is $m = 2$.

The zero $x = \frac{1}{2}$ came from the factor $(2x-1) = 2(x-\frac{1}{2})$, so we have

$$p(x) = (2x-1)(x+1)^2(1-x)(x^2+1) = (x-\frac{1}{2})^1\left[2(x+1)^2(1-x)(x^2+1)\right].$$

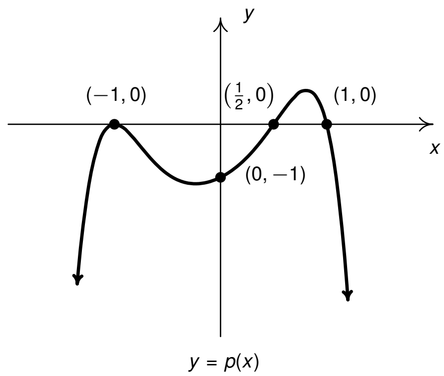
If we identify $q(x) = 2(x+1)^2(1-x)(x^2+1)$, we find $q(\frac{1}{2}) = \frac{45}{16} \neq 0$ so multiplicity here is $m = 1$.

Last but not least, we turn our attention to our last zero, $x = 1$, which we obtained from solving $1-x^4 = 0$. However, from $p(x) = (2x-1)(x+1)^2(1-x)(x^2+1)$, we see the zero $x = 1$ corresponds to the factor $(1-x) = -(x-1)$. We have $p(x) = (x-1)^1\left[-(2x-1)(x+1)^2(x^2+1)\right]$. Identifying $q(x) = -(2x-1)(x+1)^2(x^2+1)$, we see $q(1) = -8$, so the multiplicity $m = 1$ here as well.

2. From Theorem 2.5, since the multiplicities of $x = \frac{1}{2}$ and $x = 1$ are both **odd**, we know the graph of $y = p(x)$ **crosses** through the x-axis at $(\frac{1}{2}, 0)$ and $(1, 0)$. More specifically, since the multiplicity for both of these zeros is 1, the graph will look locally linear at these points. More specifically, based on our calculations above, near $x = \frac{1}{2}$, the graph will resemble the increasing line $y = \frac{45}{16}(x-\frac{1}{2})$, and near $x = 1$, the graph will resemble the decreasing line $y = -8(x-1)$. Since the multiplicity of $x = -1$ is **even**, we know the graph of $y = p(x)$ **touches** and **rebounds** at $(-1, 0)$. Since the multiplicity of $x = -1$ is 2, it will look locally like a parabola. More specifically, the graph near $x = -1$ will resemble $y = -12(x+1)^2$.

3. Per Theorem 2.3, the end behavior of $y = p(x)$, matches the end behavior of its leading term. As in Example 2.1.2, we multiply the leading terms from each factor together to obtain the leading term for $p(x)$: $p(x) = (2x-1)(x+1)(1-x^4) = (2x)(x)(-x^4) + ... = -2x^6 + ...$ Since the degree here, 6, is even and the leading coefficient $-2 < 0$, we know that $\lim_{x\to-\infty} p(x) = -\infty$ and $\lim_{x\to\infty} p(x) = -\infty$. To find the y-intercept, we find $p(0) = (2(0)-1)(0+1)(1-0^4) = -1$, hence, the y-intercept is $(0, -1)$.

4. Since $\lim_{x\to-\infty} p(x) = -\infty$, we start the graph in Quadrant III and head towards $(-1, 0)$. At $(-1, 0)$, we 'bounce' off of the x-axis and head towards the y-intercept, $(0, -1)$. We then head towards $\left(\frac{1}{2}, 0\right)$ and cross through the x-axis there. Finally, we head back to the x-axis and cross through at $(1, 0)$. Since $\lim_{x\to\infty} p(x) = -\infty$, we exit the picture in Quadrant IV. Since polynomial functions are continuous and smooth, we have no holes or gaps in the graph, and all the 'turns' are rounded (no abrupt turns or corners.) We produce something resembling the graph below.



□

A couple of remarks about Example 2.1.3 are in order. First, notice that the factor $(x^2 + 1)$ was more of a spectator in our discussion of the zeros of p. Indeed, if we set $x^2 + 1 = 0$, we have $x^2 = -1$ which provides no **real** solutions.[21] That being said, the factor $x^2 + 1$ **does** affect the shape of the graph. [22]

Next, when connecting up the graph from $(-1, 0)$ to $(0, -1)$ to $\left(\frac{1}{2}, 0\right)$, there really is no way for us to know how low the graph goes, or where the lowest point is between $x = -1$ and $x = \frac{1}{2}$ unless we plot more points. Likewise, we have no idea how high the graph gets between $x = \frac{1}{2}$ and $x = 1$. While there are ways to determine these points analytically, more often than not, finding them requires concepts from Calculus which we'll investigate later. Since these points do play an important role in many applications, we'll need to discuss them in this course and, when required, we'll use technology to find them. For that reason, we have the following definition:

Definition 2.7. Suppose f is a function with $f(a) = b$.

- We say f has a **local minimum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \leq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local minimum value of f.'

  That is, b is the minimum $f(x)$ value over an **open interval** containing a.

  Graphically, no points 'near' a local minimum are lower than (a, b).

- We say f has a **local maximum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \geq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local maximum value of f.'

  That is, b is the maximum $f(x)$ value over an **open interval** containing a.

  Graphically, no points 'near' a local maximum are higher than (a, b).

Taken together, the local maximums and local minimums of a function, if they exist, are called the **local extrema** of the function.

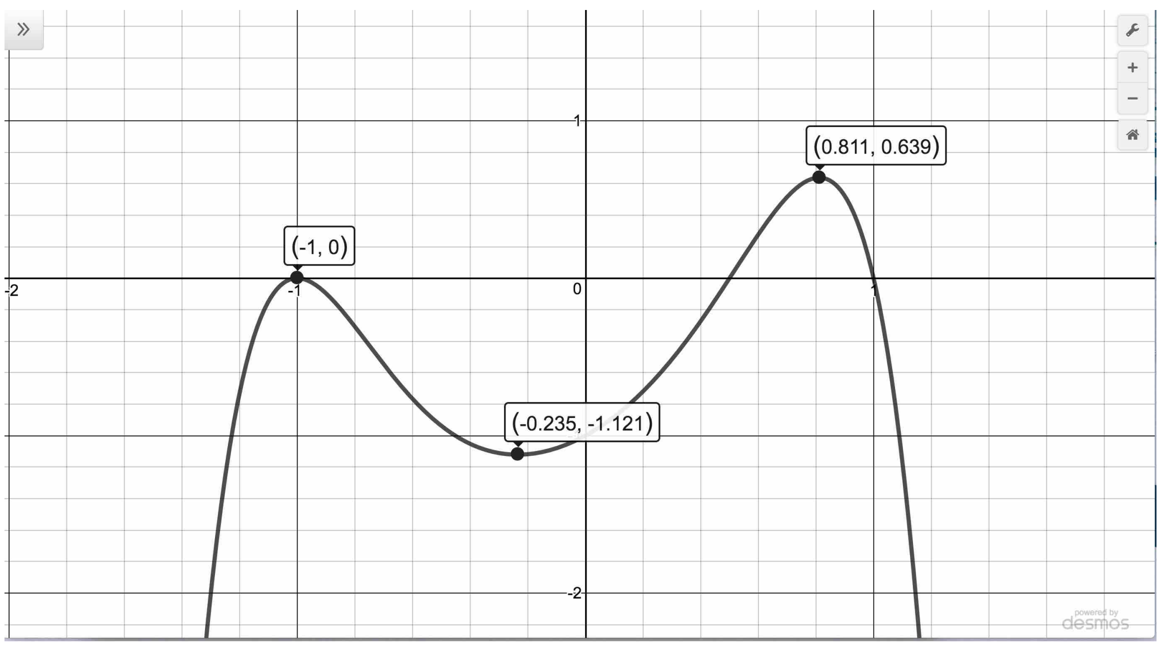
[21] The solutions are $x = \pm i$ - see Section A.11.

[22] See Exercise 60.

Once again, the terminology used in Definition 2.7 blurs the line between the function f and its outputs, $f(x)$. Also, some textbooks use the terms 'relative' minimum and 'relative' maximum instead of the adjective 'local.' Lastly, note the definition of local extrema requires an **open** interval exist in the domain containing a in order for $(a, f(a))$ to be a candidate for a local maximum or local minimum. We'll have more to say about this in later chapters. If our open interval happens to be $(-\infty, \infty)$, then our local extrema are the extrema of f - we'll see an example of this momentarily.

Below we use a graphing utility to graph $y = p(x) = (2x - 1)(x + 1)(1 - x^4)$. We first consider the point $(-1, 0)$. Even though there are points on the graph of $y = p(x)$ that are higher than $(-1, 0)$, locally, $(-1, 0)$ is the top of a hill. To satisfy Definition 2.7, we need to provide an open interval on which $p(-1) = 0$ is the largest, or maximum function value. Note the definition requires us to provide **just one** open interval. One that works is the interval $(-1.5, -0.5)$. We could use any smaller interval or go as large as $\left(-\infty, \frac{1}{2}\right)$.

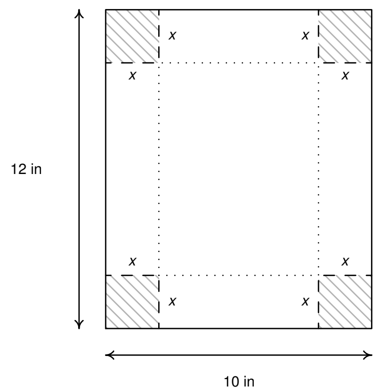
Next we encounter a 'low' point at approximately $(-0.2353, -1.1211)$. More specifically, for all x in the interval, say, $(-0.5, 0)$, $p(x) \geq 1.1211$, Hence, we have a local minimum at $(-0.2353, -1.1211)$. Lastly, at $(0.811, 0.639)$, we are back to a high point. In fact, 0.639 isn't just a local maximum value, based on the graph, it is **the** maximum of p. Here, we may choose the open interval $(-\infty, \infty)$ as the open interval required by Definition 2.7, since for all x, $p(x) \leq 0.639$. It is important to note that there is no minimum value of p despite there being a local minimum value.[23]

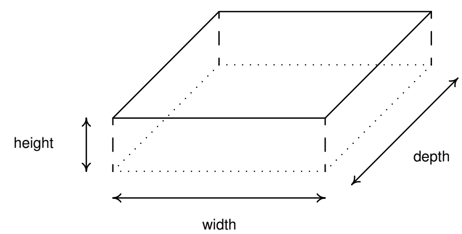


[23]Some books use the adjectives 'global' or 'absolute' when describing the extreme values of a function to distinguish them from their local counterparts.

We close this section with a classic application of a third degree polynomial function.

Example 2.1.4. A box with no top is to be fashioned from a 10 inch $\times$ 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. Let x denote the length of the side of the square which is removed from each corner.





1. Find an expression for $V(x)$, the volume of the box produced by removing squares of edge length x. Include an appropriate domain.

2. Use a graphing utility to help you determine the value of x which produces the box with the largest volume. What is the largest volume? Round your answers to two decimal places.

Solution.

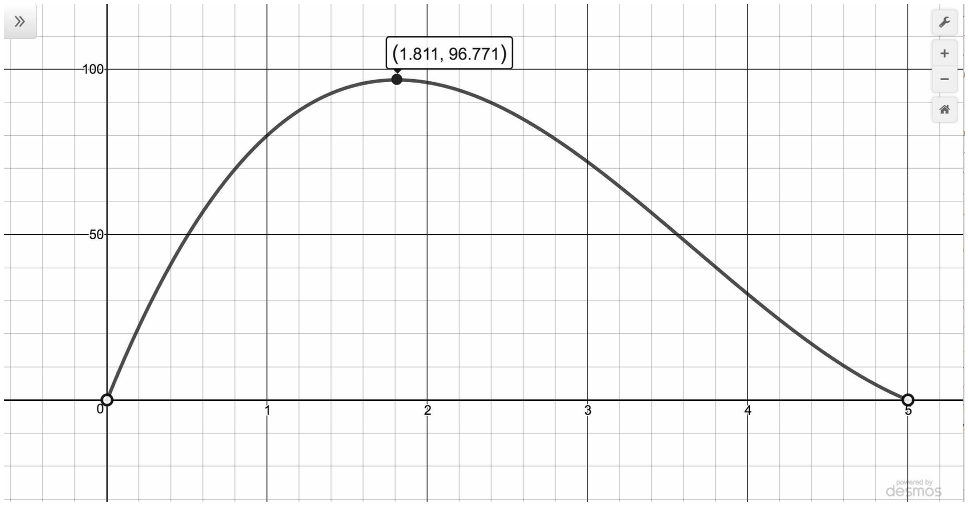
1. From Geometry, we know that Volume = width $\times$ height $\times$ depth. The key is to find each of these quantities in terms of x.

   From the figure, we see that the height of the box is x itself. The cardboard piece is initially 10 inches wide. Removing squares with a side length of x inches from each corner leaves $10 - 2x$ inches for the width.[24] As for the depth, the cardboard is initially 12 inches long, so after cutting out x inches from each side, we would have $12-2x$ inches remaining. Hence, we get $V(x) = x(10-2x)(12-2x)$.

   To find a suitable applied domain, we note that to make a box at all we need $x > 0$. Also the shorter of the two dimensions of the cardboard is 10 inches, and since we are removing $2x$ inches from this dimension, we also require $10 - 2x > 0$ or $x < 5$. Hence, our applied domain is $0 < x < 5$.

2. Using a graphing utility, we find a local maximum at approximately $(1.811, 96.771)$. Because the domain of V is restricted to the interval $(0, 5)$, the maximum of V is here as well.

[24]There's no harm in taking an extra step here and making sure this makes sense. If we chopped out a 1 inch square from each side, then the width would be 8 inches, so chopping out x inches would leave $10 - 2x$ inches.



This means the maximum volume attainable is approximately 96.77 cubic inches when we remove squares approximately 1.81 inches per side. □

Notice that there is a very slight, but important, difference between the function $V(x) = x(10-2x)(12-2x)$, $0 < x < 5$ from Example 2.1.4 and the function $p(x) = x(10-2x)(12-2x)$: their domains. The domain of V is restricted to the interval $(0, 5)$ while the domain of p is $(-\infty, \infty)$. Indeed, the function V has a maximum of (approximately) 96.771 at (approximately) $x = 1.811$ whereas for the function p, 96.771 is a local maximum value only. We leave it to the reader to verify that V has neither a minimum nor a local minimum.

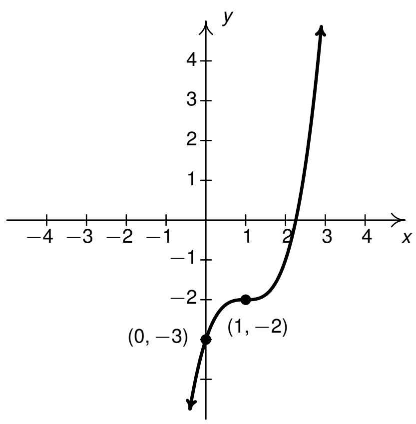
2.1.3 Exercises

In Exercises 1 - 6, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem 2.1. Track at least three points of your choice through the transformations. State the domain and range of g.

1. $f(x) = x^3$, $F(x) = (x + 2)^3 + 1$
2. $f(x) = x^4$, $F(x) = (x + 2)^4 + 1$
3. $f(x) = x^4$, $F(x) = 2 - 3(x - 1)^4$
4. $f(x) = x^5$, $F(x) = -x^5 - 3$
5. $f(x) = x^5$, $F(x) = (x + 1)^5 + 10$
6. $f(x) = x^6$, $F(x) = 8 - x^6$

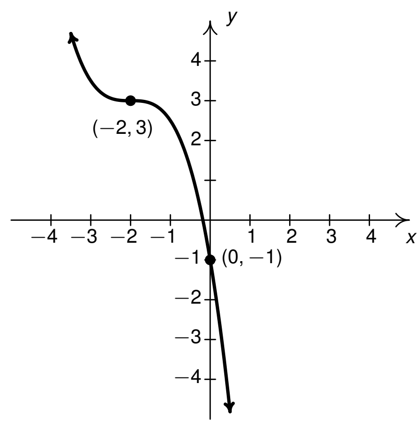
In Exercises 7 - 8, find a formula for each function below in the form $F(x) = a(x - h)^3 + k$.

7.



$y = F(x)$

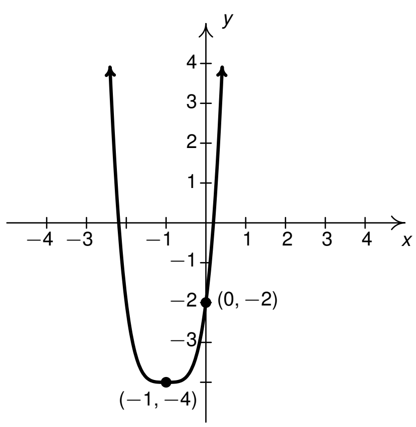
8.



$y = F(x)$

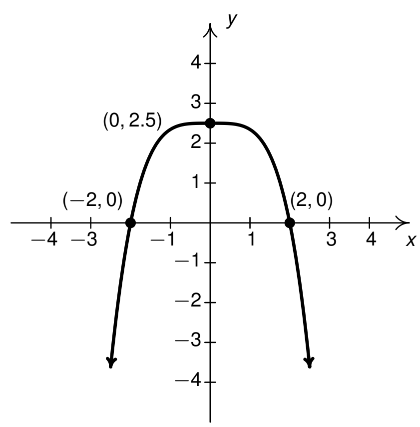
In Exercises 9 - 10, find a formula for each function below in the form $F(x) = a(x - h)^4 + k$.

9.



$y = F(x)$

10.



$y = F(x)$

In Exercises 11 - 20, find the degree, the leading term, the leading coefficient, the constant term and the end behavior of the given polynomial function.

11. $f(x) = 4 - x - 3x^2$
12. $g(x) = 3x^5 - 2x^2 + x + 1$
13. $q(r) = 1 - 16r^4$
14. $Z(b) = 42b - b^3$
15. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
16. $s(t) = -4.9t^2 + v_0 t + s_0$
17. $P(x) = (x - 1)(x - 2)(x - 3)(x - 4)$
18. $p(t) = -t^2(3 - 5t)(t^2 + t + 4)$
19. $f(x) = -2x^3(x + 1)(x + 2)^2$
20. $G(t) = 4(t - 2)^2 \left(t + \frac{1}{2}\right)$

In Exercises 21 - 30, find the real zeros of the given polynomial and their corresponding multiplicities. Use this information along with end behavior to provide a rough sketch of the graph of the polynomial function. Compare your answer with the result from a graphing utility.

21. $a(x) = x(x + 2)^2$
22. $g(t) = t(t + 2)^3$
23. $f(z) = -2(z - 2)^2(z + 1)$
24. $g(x) = (2x + 1)^2(x - 3)$
25. $F(t) = t^3(t + 2)^2$
26. $P(z) = (z - 1)(z - 2)(z - 3)(z - 4)$
27. $Q(x) = (x + 5)^2(x - 3)^4$
28. $h(t) = t^2(t - 2)^2(t + 2)^2$
29. $H(z) = (3 - z)(z^2 + 1)$
30. $Z(x) = x(42 - x^2)$

In Exercises 31 - 45, determine analytically if the following functions are even, odd or neither. Confirm your answer using a graphing utility.

31. $f(x) = 7x$
32. $g(t) = 7t + 2$
33. $p(z) = 7$
34. $F(s) = 3s^2 - 4$
35. $h(t) = 4 - t^2$
36. $g(x) = x^2 - x - 6$
37. $f(x) = 2x^3 - x$
38. $p(z) = -z^5 + 2z^3 - z$
39. $G(t) = t^6 - t^4 + t^2 + 9$
40. $G(s) = s(s^2 - 1)$
41. $f(x) = (x^2 + 1)(x - 1)$
42. $H(t) = (t^2 - 1)(t^4 + t^2 + 3)$
43. $g(t) = t(t - 2)(t + 2)$
44. $P(z) = (2z^5 - 3z)(5z^3 + z)$
45. $f(x) = 0$

46. Suppose $p(x)$ is a polynomial function written in the form of Definition 2.4.
    (a) If the nonzero terms of $p(x)$ consist of even powers of x (or a constant), explain why p is even.
    (b) If the nonzero terms of $p(x)$ consist of odd powers of x, explain why p is odd.
    (c) If $p(x)$ the nonzero terms of $p(x)$ contain at least one odd power of x and one even power of x (or a constant term), then p is neither even nor odd.

47. Use the results of Exercise 46 to determine whether the following functions are even, odd, or neither.

(a) $p(x) = 3x^4 + x^2 - 1$ (b) $F(s) = s^3 - 14s$ (c) $f(t) = 2t^5 - t^2 + 1$ (d) $g(x) = x^3(x^2 + 1)$

48. Show $f(x) = |x|$ is an even function.

49. Rework Example 2.1.4 assuming the box is to be made from an 8.5 inch by 11 inch sheet of paper. Using scissors and tape, construct the box. Are you surprised?[25]

50. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[26] What trends do you observe? How do your answers manifest themselves graphically?

$f(x)$	$[-0.1, 0]$	$[0, 0.1]$	$[0.9, 1]$	$[1, 1.1]$	$[1.9, 2]$	$[2, 2.1]$
1						
x						
x^2						
x^3						
x^4						
x^5						

51. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[27] What trends do you observe? How do your answers manifest themselves graphically?

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
1				
x				
x^2				
x^3				
x^4				
x^5				

In Exercises 52 - 54, suppose the revenue R, in *thousands* of dollars, from producing and selling x *hundred* LCD TVs is given by $R(x) = -5x^3 + 35x^2 + 155x$ for $0 \leq x \leq 10.07$.

52. Use a graphing utility to graph $y = R(x)$ and determine the number of TVs which should be sold to maximize revenue. What is the maximum revenue?

[25] Consider decorating the box and presenting it to your instructor. If done well enough, maybe your instructor will issue you some bonus points. Or maybe not.

[26] See Definition 1.8 in Section 1.2.4 for a review of this concept, as needed.

[27] See Definition 1.8 in Section 1.2.4 for a review of this concept, as needed.

53. Assume the cost, in *thousands* of dollars, to produce x *hundred* LCD TVs is given by the function $C(x) = 200x + 25$ for $x \geq 0$. Find and simplify an expression for the profit function $P(x)$.

    (Remember: Profit = Revenue - Cost.)

54. Use a graphing utility to graph $y = P(x)$ and determine the number of TVs which should be sold to maximize profit. What is the maximum profit?

55. While developing their newest game, Sasquatch Attack!, the makers of the PortaBoy (from Example 1.2.3) revised their cost function and now use $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$. As before, $C(x)$ is the cost to make x PortaBoy Game Systems. Market research indicates that the demand function $p(x) = -1.5x + 250$ remains unchanged. Use a graphing utility to find the production level x that maximizes the *profit* made by producing and selling x PortaBoy game systems.

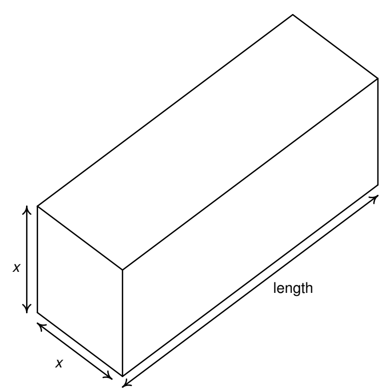
56. According to US Postal regulations, a rectangular shipping box must satisfy the following inequality: "Length + Girth $\leq$ 130 inches" for Parcel Post and "Length + Girth $\leq$ 108 inches" for other services.

    Let's assume we have a closed rectangular box with a square face of side length x as drawn below. The length is the longest side and is clearly labeled. The girth is the distance around the box in the other two dimensions so in our case it is the sum of the four sides of the square, $4x$.

    (a) Assuming that we'll be mailing a box via Parcel Post where Length + Girth = 130 inches, express the length of the box in terms of x and then express the volume V of the box in terms of x.

    (b) Find the dimensions of the box of maximum volume that can be shipped via Parcel Post.

    (c) Repeat parts 56a and 56b if the box is shipped using "other services".



57. This exercise revisits the data set from Exercise 52b in Section 1.4. In that exercise, you were given a chart of the number of hours of daylight they get on the 21$^{\text{st}}$ of each month in Fairbanks, Alaska based on the 2009 sunrise and sunset data found on the U.S. Naval Observatory website. Here $x = 1$ represents January 21, 2009, $x = 2$ represents February 21, 2009, and so on.

Month Number	1	2	3	4	5	6	7	8	9	10	11	12
Hours of Daylight	5.8	9.3	12.4	15.9	19.4	21.8	19.4	15.6	12.4	9.1	5.6	3.3

- Find cubic (third degree) and quartic (fourth degree) polynomials which model this data and comment on the goodness of fit for each. What can we say about using either model to make predictions about the year 2020? (Hint: Think about the end behavior of polynomials.)
- Use the models to see how many hours of daylight they got on your birthday and then check the website to see how accurate the models are.
- Sasquatch are largely nocturnal, so what days of the year according to your models allow for at least 14 hours of darkness for field research on the elusive creatures?

58. An electric circuit is built with a variable resistor installed. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, mW) is given in the table below. [28]

Resistance: ($k\Omega$)	1.012	2.199	3.275	4.676	6.805	9.975
Power: (mW)	1.063	1.496	1.610	1.613	1.505	1.314

(a) Make a scatter diagram of the data using the Resistance as the independent variable and Power as the dependent variable.

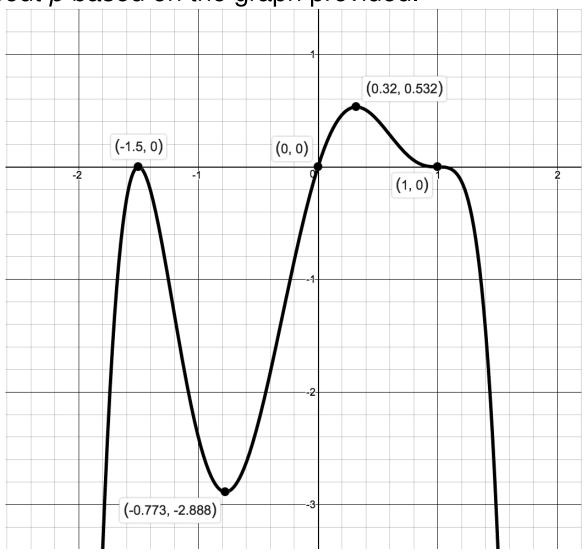
(b) Use your calculator to find quadratic (2nd degree), cubic (3rd degree) and quartic (4th degree) regression models for the data and judge the reasonableness of each.

(c) For each of the models found above, find the predicted maximum power that can be delivered to the load. What is the corresponding resistance value?

(d) Discuss with your classmates the limitations of these models - in particular, discuss the end behavior of each.

[28] The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

59. Below is a graph of a polynomial function $y = p(x)$ as generated by a graphing utility. Answer the following questions about p based on the graph provided.



(a) Describe the end behavior of $y = p(x)$.

(b) List the real zeros of p along with their respective multiplicities.

(c) List the local minimums and local maximums of the graph of $y = p(x)$.

(d) What can be said about the degree of and leading coefficient $p(x)$?

(e) It turns out that $p(x)$ is a seventh degree polynomial.[29] How can this be?

60. (This Exercise is a follow up to Example 2.1.3.) Use a graphing utility to compare and contrast the graphs of $f(x) = (2x - 1)(x + 1)^2(1 - x)(x^2 + 1)$ and $g(x) = (2x - 1)(x + 1)^2(1 - x)$.

61. Use the graph of $y = p(x) = (2x - 1)(x + 1)(1 - x^4)$ on page 150 to estimate the largest open interval containing $x = -0.235$ which satisfies the the criteria for 'local minimum' in Definition 2.7.

62. In light of Definition 2.7, explain why *every* point on the graph of a constant function is both a local maximum and a local minimum.

63. This exercise involves the greatest integer function, $f(x) = \lfloor x \rfloor$, introduced in Example 1.2.2. Explain why the points (k, k) for integers k are local maximums but not local minimums.

64. Use Theorems 2.3 and 2.4 prove Theorem 2.5.

[29]to be exact, $p(x) = -0.1\,(x + 1.5)^2\,(3x)\,(x - 1)^3\,(x + 5)$.

65. Here are a few other questions for you to discuss with your classmates.

    (a) How many and how few local extrema could a polynomial of degree n have?

    (b) Could a polynomial have two local maxima but no local minima?

    (c) If a polynomial has two local maxima and two local minima, can it be of odd degree? Can it be of even degree?

    (d) Can a polynomial have local extrema without having any real zeros?

    (e) Why must every polynomial of odd degree have at least one real zero?

    (f) Can a polynomial have two distinct real zeros and no local extrema?

    (g) Can an x-intercept yield a local extrema? Can it yield an absolute extrema?

    (h) If the y-intercept yields an absolute minimum, what can we say about the degree of the polynomial and the sign of the leading coefficient?

66. (This is a follow-up to Exercises 60 in Section 1.2 and 65 in Section 1.4.) The Lagrange Interpolate function L for four points: (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) where x_0, x_1, x_2, and x_3 are four distinct real numbers is given by the formula:

$$\begin{aligned} L(x) &= y_0\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ &\quad + y_2\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \end{aligned}$$

    (a) Choose four points with different x-values and construct the Lagrange Interpolate for those points. Verify each of the points lies on the polynomial.

    (b) Verify that, in general, $L(x_0) = y_0$, $L(x_1) = y_1$, $L(x_2) = y_2$, and $L(x_3) = y_3$.

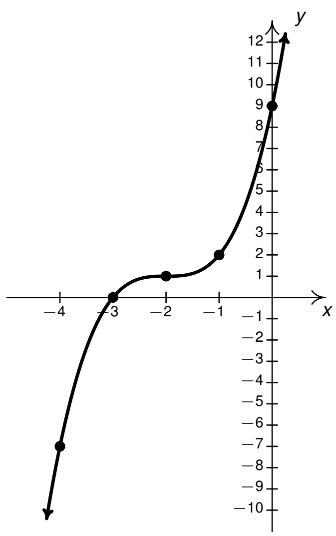
    (c) Find $L(x)$ for the points $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 4)$. What happens?

    (d) Find $L(x)$ for the points $(-1, 0)$, $(0, 1)$, $(1, 2)$ and $(2, 3)$. What happens?

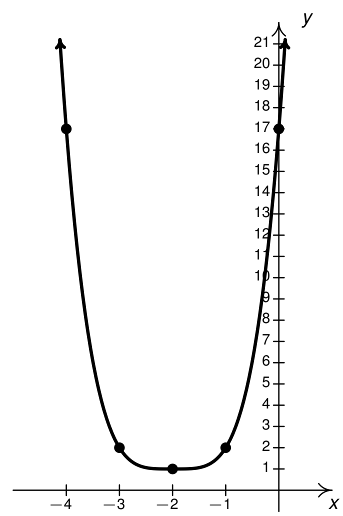
    (e) Generalize the formula for $L(x)$ to five points. What's the pattern?

2.1.4 Answers

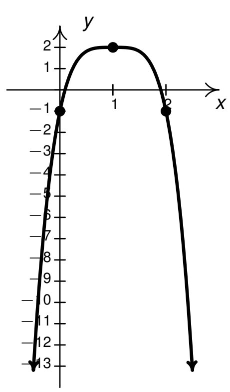
1. $F(x) = (x+2)^3 + 1$
domain: $(-\infty, \infty)$
range: $(-\infty, \infty)$



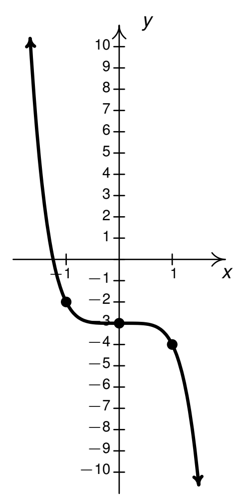
2. $F(x) = (x+2)^4 + 1$
domain: $(-\infty, \infty)$
range: $[1, \infty)$



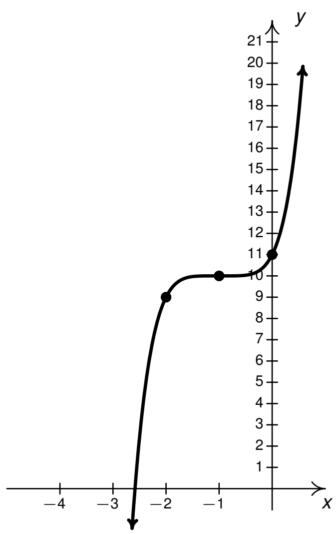
3. $F(x) = 2 - 3(x-1)^4$
domain: $(-\infty, \infty)$
range: $(-\infty, 2]$



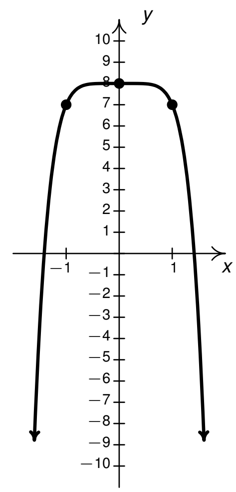
4. $F(x) = -x^5 - 3$
domain: $(-\infty, \infty)$
range: $(-\infty, \infty)$



5. $F(x) = (x+1)^5 + 10$
   domain: $(-\infty, \infty)$
   range: $(-\infty, \infty)$



6. $F(x) = 8 - x^6$
   domain: $(-\infty, \infty)$
   range: $(-\infty, 8]$



7. $F(x) = (x-1)^3 - 2$

8. $F(x) = -\frac{1}{2}(x+2)^3 + 3$

9. $F(x) = 2(x+1)^4 - 4$

10. $F(x) = -0.15625x^4 + 2.5$

11. $f(x) = 4 - x - 3x^2$
    Degree 2
    Leading term $-3x^2$
    Leading coefficient -3
    Constant term 4
    $\lim\limits_{x\to-\infty} f(x) = -\infty$
    $\lim\limits_{x\to\infty} f(x) = -\infty$

12. $g(x) = 3x^5 - 2x^2 + x + 1$
    Degree 5
    Leading term $3x^5$
    Leading coefficient 3
    Constant term 1
    $\lim\limits_{x\to-\infty} g(x) = -\infty$
    $\lim\limits_{x\to\infty} g(x) = \infty$

13. $q(r) = 1 - 16r^4$
    Degree 4
    Leading term $-16r^4$
    Leading coefficient -16
    Constant term 1
    $\lim\limits_{r\to-\infty} q(r) = -\infty$
    $\lim\limits_{r\to\infty} q(r) = -\infty$

14. $Z(b) = 42b - b^3$
    Degree 3
    Leading term $-b^3$
    Leading coefficient -1
    Constant term 0
    $\lim\limits_{b\to-\infty} Z(b) = \infty$
    $\lim\limits_{b\to\infty} Z(b) = -\infty$

15. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
Degree 17
Leading term $\sqrt{3}x^{17}$
Leading coefficient $\sqrt{3}$
Constant term $\frac{1}{3}$
$\lim\limits_{x\to-\infty} f(x) = -\infty$
$\lim\limits_{x\to\infty} f(x) = \infty$

16. $s(t) = -4.9t^2 + v_0 t + s_0$
Degree 2
Leading term $-4.9t^2$
Leading coefficient -4.9
Constant term s_0
$\lim\limits_{t\to-\infty} s(t) = -\infty$
$\lim\limits_{t\to\infty} s(t) = -\infty$

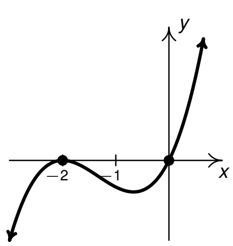
17. $P(x) = (x-1)(x-2)(x-3)(x-4)$
Degree 4
Leading term x^4
Leading coefficient 1
Constant term 24
$\lim\limits_{x\to-\infty} P(x) = \infty$
$\lim\limits_{x\to\infty} P(x) = -\infty$

18. $p(t) = -t^2(3-5t)(t^2+t+4)$
Degree 5
Leading term $5t^5$
Leading coefficient 5
Constant term 0
$\lim\limits_{t\to-\infty} p(t) = -\infty$
$\lim\limits_{t\to\infty} p(t) = \infty$

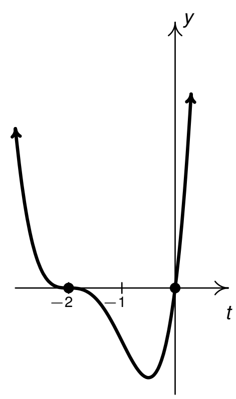
19. $f(x) = -2x^3(x+1)(x+2)^2$
Degree 6
Leading term $-2x^6$
Leading coefficient -2
Constant term 0
$\lim\limits_{x\to-\infty} f(x) = -\infty$
$\lim\limits_{x\to\infty} f(x) = -\infty$

20. $G(t) = 4(t-2)^2\left(t+\frac{1}{2}\right)$
Degree 3
Leading term $4t^3$
Leading coefficient 4
Constant term 8
$\lim\limits_{t\to-\infty} G(t) = -\infty$
$\lim\limits_{t\to\infty} G(t) = \infty$

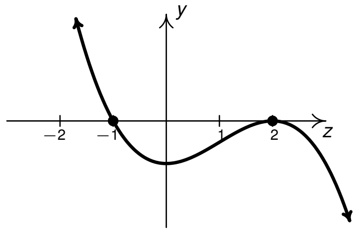
21. $a(x) = x(x+2)^2$
$x = 0$ multiplicity 1
$x = -2$ multiplicity 2



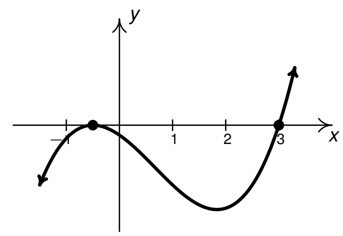
22. $g(t) = t(t+2)^3$
$t = 0$ multiplicity 1
$t = -2$ multiplicity 3



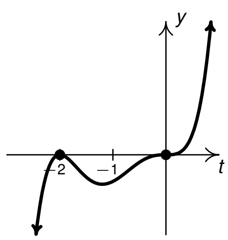
23. $f(z) = -2(z-2)^2(z+1)$
$z = 2$ multiplicity 2
$z = -1$ multiplicity 1



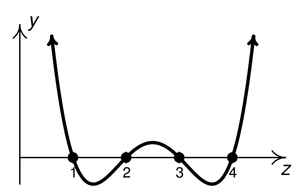
24. $g(x) = (2x+1)^2(x-3)$
$x = -\frac{1}{2}$ multiplicity 2
$x = 3$ multiplicity 1



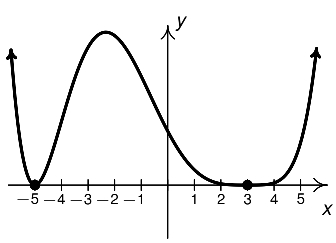
25. $F(t) = t^3(t+2)^2$
$t = 0$ multiplicity 3
$t = -2$ multiplicity 2



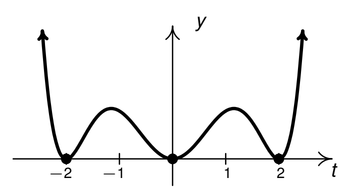
26. $P(z) = (z-1)(z-2)(z-3)(z-4)$
$z = 1$ multiplicity 1
$z = 2$ multiplicity 1
$z = 3$ multiplicity 1
$z = 4$ multiplicity 1



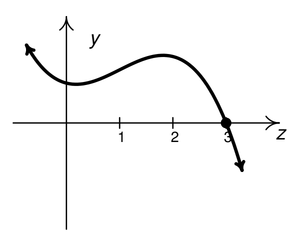
27. $Q(x) = (x+5)^2(x-3)^4$
$x = -5$ multiplicity 2
$x = 3$ multiplicity 4



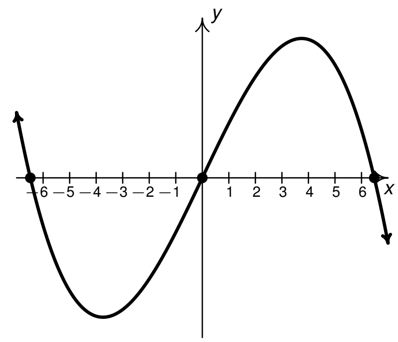
28. $f(t) = t^2(t-2)^2(t+2)^2$
$t = -2$ multiplicity 2
$t = 0$ multiplicity 2
$t = 2$ multiplicity 2



29. $H(z) = (3-z)\left(z^2+1\right)$
$z = 3$ multiplicity 1



30. $Z(x) = x(42 - x^2)$
$x = -\sqrt{42}$ multiplicity 1
$x = 0$ multiplicity 1
$x = \sqrt{42}$ multiplicity 1



31. odd
32. neither
33. even
34. even
35. even
36. neither
37. odd
38. odd
39. even
40. odd
41. neither
42. even
43. odd
44. even
45. even **and** odd

47. (a) even (b) odd (c) neither (d) odd[30]

48. For $f(x) = |x|$, $f(-x) = |-x| = |(-1)x| = |-1||x| = (1)|x| = |x|$. Hence, $f(-x) = f(x)$.

49. $V(x) = x(8.5 - 2x)(11 - 2x) = 4x^3 - 39x^2 + 93.5x$, $0 < x < 4.25$. Volume is maximized when $x \approx 1.58$, so we get the dimensions of the box with maximum volume are: height ≈ 1.58 inches, width ≈ 5.34 inches, and depth ≈ 7.84 inches. The maximum volume is ≈ 66.15 cubic inches.

50. Each of these average rates of change indicate slope of the curve over the given interval. Smaller slopes correspond to 'flatter' curves and higher slopes correspond to 'steeper' curves.

$f(x)$	$[-0.1, 0]$	$[0, 0.1]$	$[0.9, 1]$	$[1, 1.1]$	$[1.9, 2]$	$[2, 2.1]$
1	0	0	0	0	0	0
x	1	1	1	1	1	1
x^2	-0.1	0.1	1.9	2.1	3.9	4.1
x^3	0.01	0.01	2.71	3.31	11.41	12.61
x^4	-0.001	0.001	3.439	4.641	29.679	34.481
x^5	0.0001	0.0001	4.0951	6.1051	72.3901	88.4101

[30] You need to first multiply out the expression for $g(x)$ so it is in the form prescribed by Definition 2.4.

51. As we sample points closer to $x = 1$, the slope of the curve approaches the exponent on x.

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
1	0	0	0	0
x	1	1	1	1
x^2	2	2	2	2
x^3	3.01	3.0001	≈ 3	≈ 3
x^4	4.04	4.0004	≈ 4	≈ 4
x^5	5.1001	≈ 5.001	≈ 5	≈ 5

52. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 6.305$ and $y \approx 1115.417$. Since x represents the number of TVs sold in hundreds, $x = 6.305$ corresponds to 630.5 TVs. Since we can't sell half of a TV, we compare $R(6.30) \approx 1115.415$ and $R(6.31) \approx 1115.416$, so selling 631 TVs results in a (slightly) higher revenue. Since y represents the revenue in *thousands* of dollars, the maximum revenue is \$1,115,416.

53. $P(x) = R(x) - C(x) = -5x^3 + 35x^2 - 45x - 25$, $0 \leq x \leq 10.07$.

54. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 3.897$ and $y \approx 35.255$. Since x represents the number of TVs sold in hundreds, $x = 3.897$ corresponds to 389.7 TVs. Since we can't sell 0.7 of a TV, we compare $P(3.89) \approx 35.254$ and $P(3.90) \approx 35.255$, so selling 390 TVs results in a (slightly) higher revenue. Since y represents the revenue in *thousands* of dollars, the maximum revenue is \$35,255.

55. Making and selling 71 PortaBoys yields a maximized profit of \$5910.67.

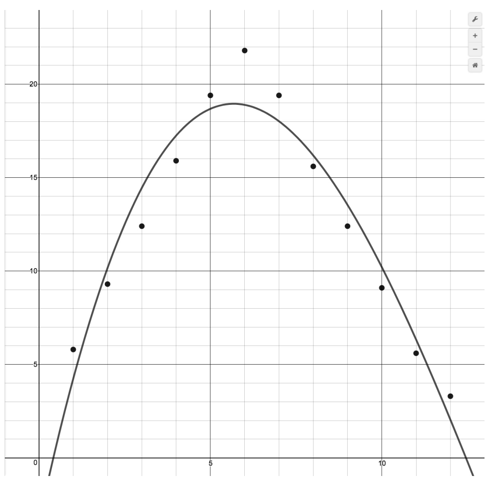
56. (a) To maximize the volume, we assume we start with the maximum Length + Girth of 130, so the length is $130 - 4x$. The volume of a rectangular box is 'length $\times$ width $\times$ height' so we get $V(x) = x^2(130 - 4x) = -4x^3 + 130x^2$.

    (b) Using a graphing utility, we get a (local) maximum of $y = V(x)$ at $(21.67, 20342.59)$. Hence, the maximum volume is 20342.59in.^{3} using a box with dimensions 21.67in. $\times$ 21.67in. $\times$ 43.32in..

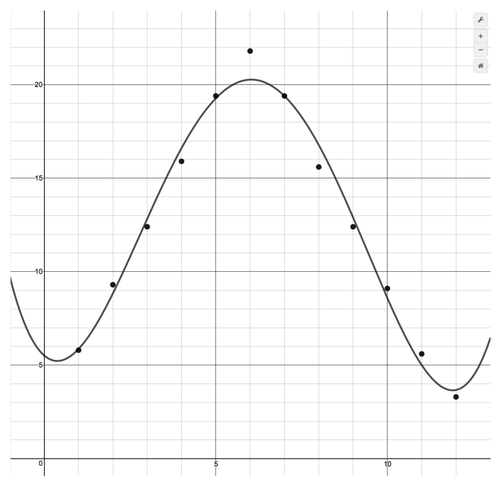
    (c) If we start with Length + Girth = 108 then the length is $108 - 4x$ so $V(x) = -4x^3 + 108x^2$. Graphing $y = V(x)$ shows a (local) maximum at $(18.00, 11664.00)$ so the dimensions of the box with maximum volume are 18.00in.$\times$18.00in.$\times$36in. for a volume of 11664.00in.^{3}. (Calculus will confirm that the measurements which maximize the volume are <u>exactly</u> 18in. by 18in. by 36in., however, as I'm sure you are aware by now, we treat all numerical results as approximations and list them as such.)

57.
- The cubic regression model is $p_3(x) = 0.0226x^3 - 0.9508x^2 + 8.615x - 3.446$. It has $R^2 = 0.9377$ which isn't bad. The graph of $y = p_3(x)$ along with the data is shown below on the left. Note p_3 hits the x-axis at about $x = 12.45$ making this a bad model for future predictions.

- To use the model to approximate the number of hours of sunlight on your birthday, you'll have to figure out what decimal value of x is close enough to your birthday and then plug it into the model. Jeff's birthday is July 31 which is 10 days after July 21 ($x = 7$). Assuming 30 days in a month, I think $x = 7.33$ should work for my birthday and $p_3(7.33) \approx 17.5$. The website says there will be about 18.25 hours of daylight that day.
- To have 14 hours of darkness we need 10 hours of daylight. We see that $p_3(1.96) \approx 10$ and $p_3(10.05) \approx 10$ so it seems reasonable to say that we'll have at least 14 hours of darkness from December 21, 2008 ($x = 0$) to February 21, 2009 ($x = 2$) and then again from October 21,2009 ($x = 10$) to December 21, 2009 ($x = 12$).
- The quartic regression model is $p_4(x) = 0.0144x^4 - 0.3507x^3 + 2.259x^2 - 1.571x + 5.513$. It has $R^2 = 0.9859$ which is good. The graph of $y = p_4(x)$ along with data is shown below on the right. Note $p_4(15)$ is above 24 making this a bad model as well for future predictions.
- Here, $p_4(7.33) \approx 18.71$ so this model more accurately predicts the number of hours of daylight on Jeff's birthday.
- This model says we'll have at least 14 hours of darkness from December 21, 2008 ($x = 0$) to about March 1, 2009 ($x = 2.30$) and then again from October 10, 2009 ($x = 9.667$) to December 21, 2009 ($x = 12$).



$y = p_3(x)$

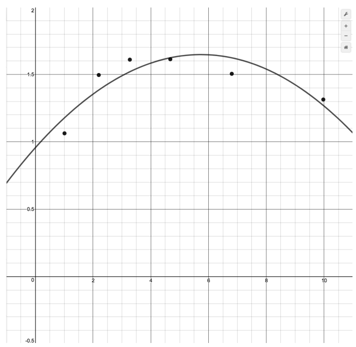


$y = p_4(x)$

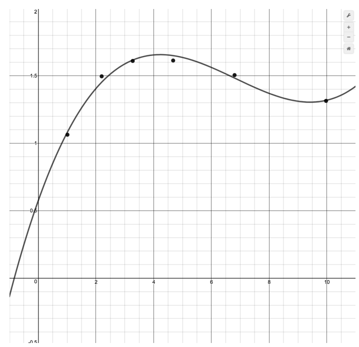
58. (a) The scatter plot is shown below with each of the three regression models.

(b) The quadratic model is $P_2(x) = -0.021x^2 + 0.241x + 0.956$, $R^2 = 0.7771$.
The cubic model is $P_3(x) = 0.005x^3 - 0.103x^2 + 0.602x + 0.573$, $R^2 = 0.9815$.
The quartic model is $P_4(x) = -0.000969x^4 + 0.0253x^3 - 0.240x^2 + 0.944x + 0.330$, $R^2 = 0.9993$.

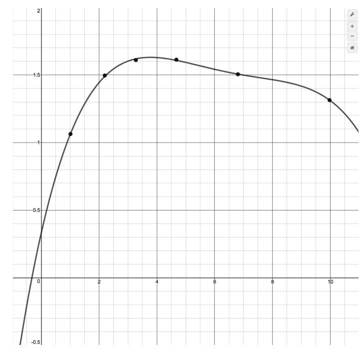
(c) The models give maximums: $P_2(5.737) \approx 1.648$, $P_3(4.232) \approx 1.657$ and $P_4(3.784) \approx 1.630$.



$y = P_2(x)$



$y = P_3(x)$



$y = P_4(x)$

59. (a) $\displaystyle\lim_{x\to-\infty} p(x) = -\infty$ and $\displaystyle\lim_{x\to\infty} p(x) = -\infty$

(b) The zeros appear to be: $x = -1.5$, even multiplicity - probably 2 since it doesn't 'look like' the graph is very flat near $x = 2$; $x = 0$, odd multiplicity - probably 1 since the graph seems fairly linear as it passes through the origin; $x = 1$ odd multiplicity - probably 3 or higher since the graph seems fairly 'flat' near $x = 1$.

(c) local minimum: approximately $(-0.773, -2.888)$; local maximums: approximately $(-1.5, 0)$, and $(0.32, 0.532)$

(d) Based on the graph, even degree (at least 6 based on multiplicities) with a negative leading coefficient based on the end behavior.

(e) We only have a *portion* of the graph represented here.

61. We are looking for the largest open interval containing $x = -0.235$ for which the graph of $y = p(x)$ is at or above $y = -1.121$. Since each of the gridlines on the x-axis correspond to 0.2 units, we approximate this interval as $(-1.25\,\text{ish}, 1.1\,\text{ish})$.

66. (c) $L(x) = x^2$ (d) $L(x) = x + 1$

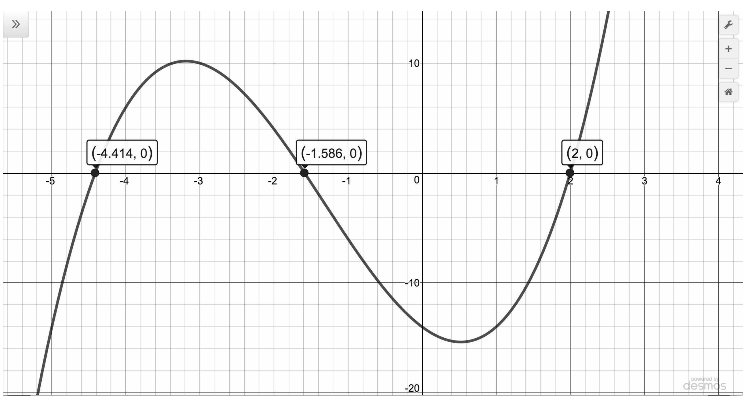
2.2 The Remainder and Factor Theorems

In Section 2.1 we saw how much of the 'local' behavior of the graph of a polynomial function is determined by the zeros of the polynomial function. In that section, the polynomial functions we were given were mostly, if not completely, factored which greatly simplified the process for determining zeros. In this section, we revisit the relationship between zeros and factors with the ultimate aim of taking a polynomial function given to us in the form stated in Definition 2.4 and determining its zeros.

We start by way of example: suppose we wish to determine the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all of the factoring techniques we learned (and forgot!) in Intermediate Algebra, this equation foils[1] us at every turn. Knowing that the zeros of f correspond to x-intercepts on the graph of $y = f(x)$, we use a graphing utility to produce the graph below on the left.

The graph suggests that the function has three zeros, one of which appears to be $x = 2$ and two others for whom we are provided what we assume to be decimal approximations: $x \approx -4.414$ and $x \approx -1.586$. We can verify if these are zeros easily enough. We find $f(2) = (2)^2 + 4(2)^2 - 5(2) - 14 = 0$, but $f(-4.414) \approx 0.0039$ and $f(-1.586) \approx 0.0022$, While these last two values are probably by some measures, 'close' to 0, they are not **exactly** equal to 0. The question becomes: is there a way to use the fact that $x = 2$ is a zero to obtain the other two zeros?

Based on our experience, if $x = 2$ is a zero, it seems that there should be a factor of $(x - 2)$ lurking around in the factorization of $f(x)$. In other words, we should expect that $x^3 + 4x^2 - 5x - 14 = (x - 2)\,q(x)$, where $q(x)$ is some other polynomial. How could we find such a $q(x)$, if it even exists? The answer comes from our old friend, polynomial division. (See Section A.8.2.) Below on the right, we perform the long division: $(x^3 + 4x^2 - 5x - 14) \div (x - 2)$ and obtain $x^2 + 6x + 7$.



$$
\begin{array}{r|rrrr}
 & x^2 & +\ 6x & +\ 7 & \\
\cline{2-5}
x-2 & x^3 & +\ 4x^2 & -\ 5x & -\ 14 \\
 & -\left(x^3\right. & \left.-\ 2x^2\right) & & \\
\cline{2-3}
 & & 6x^2 & -\ 5x & \\
 & & -\left(6x^2\right. & \left.-\ 12x\right) & \\
\cline{3-4}
 & & & 7x & -\ 14 \\
 & & & -\ (7x & -\ 14) \\
\cline{4-5}
 & & & & 0
\end{array}
$$

Said differently, $f(x) = x^3 + 4x^2 - 5x - 14 = (x - 2)\left(x^2 + 6x + 7\right)$. Using this form of $f(x)$, we find the zeros by solving $(x - 2)\left(x^2 + 6x + 7\right) = 0$. Setting each factor equal to 0, we get $x - 2 = 0$ (which gives us our known zero, $x = 2$) as well as $x^2 + 6x + 7 = 0$. The latter doesn't factor nicely, so we apply the Quadratic Formula to get $x = -3 \pm \sqrt{2}$. Sure enough, $-3 - \sqrt{2} \approx -4.414$ and $-3 + \sqrt{2} \approx -1.586$. We leave it to the reader to show $f(-3 - \sqrt{2}) = 0$ and $f(-3 + \sqrt{2}) = 0$. (See Exercise 36.)

[1] pun intended

The point of this section is to generalize the technique applied here. First up is a friendly reminder of what we can expect when we divide polynomials.

Theorem 2.6. Polynomial Division:
Suppose $d(x)$ and $p(x)$ are nonzero polynomial functions where the degree of p is greater than or equal to the degree of d. There exist two unique polynomial functions, $q(x)$ and $r(x)$, such that $p(x) = d(x)\,q(x) + r(x)$, where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

As you may recall, all of the polynomials in Theorem 2.6 have special names. The polynomial p is called the **dividend**; d is the **divisor**; q is the **quotient**; r is the **remainder**. If $r(x) = 0$ then d is called a **factor** of p. The word 'unique' here is critical in that it guarantees there is only **one** quotient and remainder for each division problem.[2] The proof of Theorem 2.6 is usually relegated to a course in Abstract Algebra, but we can still use the result to establish two important facts which are the basis of the rest of the chapter.

Theorem 2.7. The Remainder Theorem: Suppose p is a polynomial function of degree at least 1 and c is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$. Said differently, there is a polynomial function $q(x)$ such that:

$$p(x) = (x - c)q(x) + p(c)$$

The proof of Theorem 2.7 is a direct consequence of Theorem 2.6. Since $x - c$ has degree 1, when a polynomial function is divided by $x - c$, the remainder is either 0 or degree 0 (i.e., a nonzero constant.) In either case, $p(x) = (x - c)\,q(x) + r$, where r, the remainder, is a real number, possibly 0. It follows that $p(c) = (c - c)\,q(c) + r = 0 \cdot q(c) + r = r$, so we get $r = p(c)$ as required. There is one last 'low hanging fruit'[3] to collect which we present below.

Theorem 2.8. The Factor Theorem:
Suppose p is a nonzero polynomial function. The real number c is a zero of p if and only if $(x - c)$ is a factor of $p(x)$.

Once again, we see the phrase 'if and only if' which means there are really two things being said in The Factor Theorem: if $(x - c)$ is a factor of $p(x)$, then c is a zero of p and the **only** way c is a zero of p is if $(x - c)$ is a factor of $p(x)$.

We argue the Factor Theorem as follows: if $(x - c)$ is a factor of $p(x)$, then $p(x) = (x - c)\,q(x)$ for some polynomial q. Hence, $p(c) = (c - c)\,q(c) = 0$, so c is a zero of p. Conversely, suppose c is a zero of p, so $p(c) = 0$. The Remainder Theorem tells us $p(x) = (x - c)q(x) + p(c) = (x - c)q(x) + 0 = (x - c)q(x)$. Hence, $(x - c)$ is a factor of $p(x)$.

We have enough theory to explain why the concept of multiplicity (Definition 2.6) is well-defined. If c is a zero of p, then The Factor Theorem tells us there is a polynomial function q_1 so that $p(x) = (x - c)q_1(x)$. If $q_1(c) = 0$, then we apply the Factor Theorem to q_1 and find a polynomial q_2 so that $q_1(x) = (x - c)q_2(x)$. Hence, we have

$$p(x) = (x - c)q_1(x) = (x - c)(x - c)q_2(x) = (x - c)^2 q_2(x).$$

[2] Hence the use of the definite article 'the' when speaking of **the** quotient and **the** remainder.

[3] Jeff hates this expression and Carl included it just to annoy him.

We now ‘rinse and repeat’ this process. Since the degree of p is a finite number, this process has to end at some point. That is we arrive at a factorization $p(x) = (x - c)^m q(x)$ where $q(c) \neq 0$. Suppose we arrive at a different factorization of p using other methods. That is, we find $p(x) = (x - c)^k Q(x)$, where Q is a polynomial function with $Q(c) \neq 0$. Then we have $(x - c)^m q(x) = (x - c)^k Q(x)$.

If $m \neq k$, then either $m < k$ or $m > k$. If $m < k$, then we may divide both sides by $(x - c)^m$ to get: $q(x) = (x - c)^{k-m} Q(x)$. Since $k > m$, $k - m > 0$ and we would have $q(c) = (c - c)^{k-m} Q(c) = 0$, a contradiction since we are assuming $q(c) \neq 0$. The assumption that $m > k$ likewise ends in a contradiction. Therefore, we have $m = k$, so $p(x) = (x - c)^m q(x) = (x - c)^m Q(x)$. By the uniqueness guaranteed in Theorem 2.6, we must have that $q(x) = Q(x)$. Hence, the number m and quotient polynomial $q(x)$ are unique.

The process outlined above, in which we coax out factors of $p(x)$ one at a time until we have all of them serves as a template for our work to come. Of the things The Factor Theorem tells us, the most pragmatic is that we had better find a more efficient way to divide polynomial functions by quantities of the form $x - c$. Fortunately, people like Ruffini and Horner have already blazed this trail. Let’s take a closer look at the long division we performed at the beginning of the section and try to streamline it. First off, let’s change all of the subtractions into additions by distributing through the -1s.

$$\begin{array}{r|rrrr}
 & & x^2 & +6x & +7 \\
\cline{2-5}
x-2 & x^3 & +4x^2 & -5x & -14 \\
 & -x^3 & +2x^2 & & \\
\cline{2-3}
 & & 6x^2 & -5x & \\
 & & -6x^2 & +12x & \\
\cline{3-4}
 & & & 7x & -14 \\
 & & & -7x & +14 \\
\cline{4-5}
 & & & & 0
\end{array}$$

Next, observe that the terms $-x^3$, $-6x^2$ and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information.

Also note that the terms we ‘bring down’ (namely the $-5x$ and -14) aren’t really necessary to recopy, so we omit them, too.

$$\begin{array}{r|rrrr}
 & & x^2 & +6x & +7 \\
\cline{2-5}
x-2 & x^3 & +4x^2 & -5x & -14 \\
 & & 2x^2 & & \\
\cline{2-3}
 & & 6x^2 & & \\
 & & & 12x & \\
\cline{3-4}
 & & & 7x & \\
 & & & & 14 \\
\cline{4-5}
 & & & & 0
\end{array}$$

Let's move terms up a bit and copy the x^3 into the last row.

$$\begin{array}{r|rrrr} & & x^2 & +\,6x & +\,7 \\ \cline{2-5} x-2 & x^3 & +4x^2 & -\,5x & -14 \\ & & 2x^2 & 12x & 14 \\ \hline & x^3 & 6x^2 & 7x & 0 \end{array}$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results.

If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial. This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$\begin{array}{r|rrrr} -2 & x^3 & +4x^2 & -\,5x & -14 \\ & & 2x^2 & 12x & 14 \\ \hline & x^3 & 6x^2 & 7x & 0 \end{array}$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, by the -2 in $x-2$, then by -1 when we changed the subtraction to addition.

Multiplying by -2 then by -1 is the same as multiplying by 2, so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & 0 \end{array}$$

We have constructed a **synthetic division tableau** for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process.

To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in for the dividend. Then 'bring down' the first coefficient of the dividend.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was 'brought down' to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient is a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$.

The number in the box is the remainder. Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x - c$. It is important to note that it works **only** for these kinds of divisors.[4]

Also take note that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division. While the authors have done their best to indicate where the algorithm comes from, there is no substitute for working through it yourself.

Example 2.2.1. Use synthetic division to perform the following polynomial divisions. Identify the quotient and remainder. Write the dividend, quotient and remainder in the form given in Theorem 2.6.

1. $\left(5x^3 - 2x^2 + 1\right) \div (x - 3)$
2. $\left(t^3 + 8\right) \div (t + 2)$
3. $\dfrac{4 - 8z - 12z^2}{2z - 3}$

Solution.

1. When setting up the synthetic division tableau, the coefficients of even 'missing' terms need to be accounted for, so we enter 0 for the coefficient of x in the dividend.

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & \downarrow & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \boxed{118} \end{array}$$

[4] You'll need to use good old-fashioned polynomial long division for divisors of degree larger than 1.

Since the dividend was a third degree polynomial function, the quotient is a second degree (quadratic) polynomial function with coefficients 5, 13 and 39: $q(x) = 5x^2+13x+39$. The remainder is $r(x) = 118$. According to Theorem 2.6, we have $5x^3 - 2x^2 + 1 = (x-3)\left(5x^2+13x+39\right) + 118$, which we leave to the reader to check.

2. To use synthetic division here, we rewrite $t+2$ as $t-(-2)$ and proceed as before

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

We get the quotient $q(t) = t^2 - 2t + 4$ and the remainder $r(t) = 0$. Relating the dividend, quotient and remainder gives: $t^3 + 8 = (t+2)\left(t^2 - 2t + 4\right)$, which is a specific instance of the 'sum of cubes' formula some of you may recall from Intermediate Algebra.

3. To divide $4 - 8z - 12z^2$ by $2z - 3$, two things must be done. First, we write the dividend in descending powers of z as $-12z^2 - 8z + 4$. Second, since synthetic division works only for factors of the form $z - c$, we factor $2z - 3$ as $2\left(z - \frac{3}{2}\right)$. Hence, we are dividing $-12z^2 - 8z + 4$ by two factors: 2 and $\left(z - \frac{3}{2}\right)$. Dividing first by 2, we obtain $-6z^2 - 4z + 2$. Next, we divide $-6z^2 - 4z + 2$ by $\left(z - \frac{3}{2}\right)$:

$$\begin{array}{r|rrr} \frac{3}{2} & -6 & -4 & 2 \\ & \downarrow & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & \boxed{-\frac{35}{2}} \end{array}$$

Hence, $-6z^2 - 4z + 2 = \left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2}$. However when it comes to writing the dividend, quotient and remainder in the form given in Theorem 2.6, we need to find $q(z)$ and $r(z)$ so that $-12z^2 - 8z + 4 = (2z-3)q(z) + r(z)$. Hence, starting with $-6z^2 - 4z + 2 = \left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2}$, we multiply 2 back on both sides:

$$\begin{array}{rcl} -6z^2 - 4z + 2 & = & \left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2} \\ 2\left(-6z^2 - 4z + 2\right) & = & 2\left[\left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2}\right] \\ -12z^2 - 8z + 4 & = & 2\left(z - \frac{3}{2}\right)(-6z - 13) - 2\left(\frac{35}{2}\right) \\ -12z^2 - 8z + 4 & = & (2z-3)(-6z-13) - 35 \end{array}$$

At this stage, we have written $-12z^2 - 8z + 4$ in the **form** $(2z-3)q(z) + r(z)$, so we identify the quotient as $q(z) = -6z - 13$ and the remainder is $r(z) = -35$.

But how can we be sure these are the same quotient and remainder polynomial functions we would have obtained if we had taken the time to do the long division in the first place? Because of the word 'unique' in Theorem 2.6. The theorem states that there is only **one** way to decompose $-12z^2 - 8z + 4$ as $(2z-3)q(z) + r(z)$. Since we have found such a way, we can be sure it is the only way.[5] □

[5]But it wouldn't hurt to check, just this once.

The next example pulls together all of the concepts discussed in this section.

Example 2.2.2. Let $p(x) = 2x^3 - 5x + 3$.

1. Find $p(-2)$ using The Remainder Theorem. Check your answer by substitution.

2. Verify $x = 1$ is a zero of p and use this information to all the real zeros of p.

Solution.

1. The Remainder Theorem states $p(-2)$ is the remainder when $p(x)$ is divided by $x - (-2)$. We set up our synthetic division tableau below. We are careful to record the coefficient of x^2 as 0:

$$\begin{array}{r|rrrr} -2 & 2 & 0 & -5 & 3 \\ & \downarrow & -4 & 8 & -6 \\ \hline & 2 & -4 & 3 & \boxed{-3} \end{array}$$

   According to the Remainder Theorem, $p(-2) = -3$. We can check this by direct substitution into the formula for $p(x)$: $p(-2) = 2(-2)^3 - 5(-2) + 3 = -16 + 10 + 3 = -3$.

2. We verify $x = 1$ is a zero of p by evaluating $p(1) = 2(1)^3 - 5(1) + 3 = 0$. To see if there are any more real zeros, we need to solve $p(x) = 2x^3 - 5x + 3 = 0$. From the Factor Theorem, we know since $p(1) = 0$, we can factor $p(x)$ as $(x - 1)q(x)$. To find $q(x)$, we use synthetic division:

$$\begin{array}{r|rrrr} 1 & 2 & 0 & -5 & 3 \\ & \downarrow & 2 & 2 & -3 \\ \hline & 2 & 2 & -3 & \boxed{0} \end{array}$$

   As promised, our remainder is 0, and we get $p(x) = (x - 1)\left(2x^2 + 2x - 3\right)$. Setting this form of $p(x)$ equal to 0 we get $(x - 1)\left(2x^2 + 2x - 3\right) = 0$. We recover $x = 1$ from setting $x - 1 = 0$ but we also obtain $x = \frac{-1\pm\sqrt{7}}{2}$ from $2x^2 + 2x - 3 = 0$, courtesy of the Quadratic Formula. $\square$

Our next example demonstrates how we can extend the synthetic division tableau to accommodate zeros of multiplicity greater than 1.

Example 2.2.3. Let $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$. Show $x = \frac{1}{2}$ is a zero of multiplicity 2 and find all of the remaining real zeros of p.

Solution. While computing $p\left(\frac{1}{2}\right) = 0$ shows $x = \frac{1}{2}$ is a zero of p, to prove it has multiplicity 2, we need to factor $p(x) = \left(x - \frac{1}{2}\right)^2 q(x)$ with $q\left(\frac{1}{2}\right) \neq 0$,. We set up for synthetic division, but instead of stopping after the first division, we continue the tableau downwards and divide $\left(x - \frac{1}{2}\right)$ directly into the quotient we obtained from the first division as follows:

$$\begin{array}{r|rrrrr}
\frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\
 & \downarrow & 2 & -1 & -6 & 3 \\
\hline
\frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\
 & \downarrow & 2 & 0 & -6 & \\
\hline
 & 4 & 0 & -12 & \boxed{0} &
\end{array}$$

We get:[6] $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)^2 \left(4x^2 - 12\right)$. Note if we let $q(x) = 4x^2 - 12$, then $q\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 - 12 = -11 \neq 0$ which proves $x = \frac{1}{2}$ is a zero of p of multiplicity 2. To find the remaining zeros of p, we set the quotient $4x^2 - 12 = 0$, so $x^2 = 3$ and extract square roots to get $x = \pm\sqrt{3}$. $\square$

A couple of things about the last example are worth mentioning. First, the extension of the synthetic division tableau for repeated divisions will be a common site in the sections to come. Typically, we will start with a higher order polynomial and peel off one zero at a time until we are left with a quadratic, whose roots can always be found using the Quadratic Formula.

Secondly, we found $x = \pm\sqrt{3}$ are zeros of p. The Factor Theorem guarantees $\left(x - \sqrt{3}\right)$ and $\left(x - \left(-\sqrt{3}\right)\right)$ are both factors of p. We can certainly put the Factor Theorem to the test and continue the synthetic division tableau from above to see what happens.

$$\begin{array}{r|rrrrr}
\frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\
 & \downarrow & 2 & -1 & -6 & 3 \\
\hline
\frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\
 & \downarrow & 2 & 0 & -6 & \\
\hline
\sqrt{3} & 4 & 0 & -12 & \boxed{0} & \\
 & \downarrow & 4\sqrt{3} & 12 & & \\
\hline
-\sqrt{3} & 4 & 4\sqrt{3} & \boxed{0} & & \\
 & \downarrow & -4\sqrt{3} & & & \\
\hline
 & 4 & \boxed{0} & & &
\end{array}$$

This gives us

$$\begin{aligned}
p(x) &= 4x^4 - 4x^3 - 11x^2 + 12x - 3 \\
&= \left(x - \frac{1}{2}\right)^2 \left(x - \sqrt{3}\right)\left(x - \left(-\sqrt{3}\right)\right)(4) \\
&= 4\left(x - \frac{1}{2}\right)^2 \left(x - \sqrt{3}\right)\left(x - \left(-\sqrt{3}\right)\right)
\end{aligned}$$

We have shown that p is a product of its leading coefficient times linear factors of the form $(x - c)$ where c are zeros of p. It may surprise and delight the reader that, in theory, all polynomials can be reduced to this kind of factorization. We leave that discussion to Section 2.4, because the zeros may not be real numbers. Our final theorem in the section gives us an upper bound on the number of real zeros.

Theorem 2.9. Suppose f is a polynomial of degree $n \geq 1$. Then f has at most n real zeros, counting multiplicities.

[6] For those wanting more detail: the first division gives: $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(4x^3 - 2x^2 - 12x + 6\right)$. The second division gives: $4x^3 - 2x^2 - 12x + 6 = \left(x - \frac{1}{2}\right)\left(4x^2 - 12\right)$.

Theorem 2.9 is a consequence of the Factor Theorem and polynomial multiplication. Every zero c of f gives us a factor of the form $(x - c)$ for $f(x)$. Since f has degree n, there can be at most n of these factors. The next section provides us some tools which not only help us determine where the real zeros are to be found, but which real numbers they may be.

We close this section with a summary of several concepts previously presented. You should take the time to look back through the text to see where each concept was first introduced and where each connection to the other concepts was made.

Connections Between Zeros, Factors and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n \geq 1$. The following statements are equivalent:

- The real number c is a zero of p
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x - c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an x-intercept of the graph of $y = p(x)$

2.2.1 Exercises

In Exercises 1 - 14, use synthetic division to perform the following polynomial divisions. Identify the quotient and remainder. Write the divisor, quotient and remainder in the form given in Theorem 2.6.

1. $\left(3x^2 - 2x + 1\right) \div (x - 1)$
2. $\left(x^2 - 5\right) \div (x - 5)$
3. $\left(3 - 4t - 2t^2\right) \div (t + 1)$
4. $\left(4t^2 - 5t + 3\right) \div (t + 3)$
5. $\left(z^3 + 8\right) \div (z + 2)$
6. $\left(4z^3 + 2z - 3\right) \div (z - 3)$
7. $\left(18x^2 - 15x - 25\right) \div \left(x - \frac{5}{3}\right)$
8. $\left(4x^2 - 1\right) \div \left(x - \frac{1}{2}\right)$
9. $\left(2t^3 + t^2 + 2t + 1\right) \div \left(t + \frac{1}{2}\right)$
10. $\left(3t^3 - t + 4\right) \div \left(t - \frac{2}{3}\right)$
11. $\left(2z^3 - 3z + 1\right) \div \left(z - \frac{1}{2}\right)$
12. $\left(4z^4 - 12z^3 + 13z^2 - 12z + 9\right) \div \left(z - \frac{3}{2}\right)$
13. $\left(x^4 - 6x^2 + 9\right) \div \left(x - \sqrt{3}\right)$
14. $\left(x^6 - 6x^4 + 12x^2 - 8\right) \div \left(x + \sqrt{2}\right)$

In Exercises 15 - 24, find $p(c)$ using the Remainder Theorem. If $p(c) = 0$, use the Factor Theorem to partially factor the polynomial function.

15. $p(x) = 2x^2 - x + 1$, $c = 4$
16. $p(x) = 4x^2 - 33x - 180$, $c = 12$
17. $p(t) = 2t^3 - t + 6$, $c = -3$
18. $p(t) = t^3 + 2t^2 + 3t + 4$, $c = -1$
19. $p(z) = 3z^3 - 6z^2 + 4z - 8$, $c = 2$
20. $p(z) = 8z^3 + 12z^2 + 6z + 1$, $c = -\frac{1}{2}$
21. $p(x) = x^4 - 2x^2 + 4$, $c = \frac{3}{2}$
22. $p(x) = 6x^4 - x^2 + 2$, $c = -\frac{2}{3}$
23. $p(t) = t^4 + t^3 - 6t^2 - 7t - 7$, $c = -\sqrt{7}$
24. $p(t) = t^2 - 4t + 1$, $c = 2 - \sqrt{3}$

In Exercises 25 - 34, you are given a polynomial function and one of its zeros. Find the remaining real zeros and factor the polynomial.

25. $x^3 - 6x^2 + 11x - 6, \quad c = 1$
26. $x^3 - 24x^2 + 192x - 512, \quad c = 8$
27. $3t^3 + 4t^2 - t - 2, \quad c = \frac{2}{3}$
28. $2t^3 - 3t^2 - 11t + 6, \quad c = \frac{1}{2}$
29. $z^3 + 2z^2 - 3z - 6, \quad c = -2$
30. $2z^3 - z^2 - 10z + 5, \quad c = \frac{1}{2}$
31. $4x^4 - 28x^3 + 61x^2 - 42x + 9$, $c = \frac{1}{2}$ is a zero of multiplicity 2
32. $t^5 + 2t^4 - 12t^3 - 38t^2 - 37t - 12$, $c = -1$ is a zero of multiplicity 3
33. $125z^5 - 275z^4 - 2265z^3 - 3213z^2 - 1728z - 324$, $c = -\frac{3}{5}$ is a zero of multiplicity 3
34. $x^2 - 2x - 2, \quad c = 1 - \sqrt{3}$

35. Find a quadratic polynomial with integer coefficients which has $x = \frac{3}{5} \pm \frac{\sqrt{29}}{5}$ as its real zeros.

36. For $f(x) = x^3 + 4x^2 - 5x - 14$, show $f(-3 - \sqrt{2}) = 0$ and $f(-3 + \sqrt{2}) = 0$ two ways:

    (a) By direct substitution.

    (b) Using synthetic division and the Factor Theorem

37. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ be a polynomial function with the property that $a_n + a_{n-1} + \ldots + a_1 + a_0 = 0$. (That is, the sum of the coefficients and the constant term is 0.)

    Prove that $(x - 1)$ is a factor of $f(x)$.

    HINT: Show $f(1) = 0$ and invoke the Factor Theorem

38. Verify the result in number 37 with the functions: $f(x) = x^3 - 2x + 1$ and $f(x) = 3x^4 - x - 2$.

39. Suppose a is a nonzero real number. Find the quotients below, using synthetic division as required.

    - $\dfrac{x - a}{x - a}$
    - $\dfrac{x^2 - a^2}{x - a}$
    - $\dfrac{x^3 - a^3}{x - a}$
    - $\dfrac{x^4 - a^4}{x - a}$
    - $\dfrac{x^5 - a^5}{x - a}$

    Based on the pattern that evolves, find the quotient: $\dfrac{x^{10} - a^{10}}{x - a}$. What about $\dfrac{x^n - a^n}{x - a}$?

40. Use your result from number 39 to rewrite the sum: $1 + r + r^2 + \cdots + r^{n-2} + r^{n-1}$ as a quotient. What assumptions need to be made about r?

2.2.2 Answers

1. $\left(3x^2 - 2x + 1\right) = (x - 1)(3x + 1) + 2$
2. $\left(x^2 - 5\right) = (x - 5)(x + 5) + 20$
3. $\left(3 - 4t - 2t^2\right) = (t + 1)(-2t - 2) + 5$
4. $\left(4t^2 - 5t + 3\right) = (t + 3)(4t - 17) + 54$
5. $\left(z^3 + 8\right) = (z + 2)\left(z^2 - 2z + 4\right) + 0$
6. $\left(4z^3 + 2z - 3\right) = (z - 3)\left(4z^2 + 12z + 38\right) + 111$
7. $\left(18x^2 - 15x - 25\right) = \left(x - \frac{5}{3}\right)(18x + 15) + 0$
8. $\left(4x^2 - 1\right) = \left(x - \frac{1}{2}\right)(4x + 2) + 0$
9. $\left(2t^3 + t^2 + 2t + 1\right) = \left(t + \frac{1}{2}\right)\left(2t^2 + 2\right) + 0$
10. $\left(3t^3 - t + 4\right) = \left(t - \frac{2}{3}\right)\left(3t^2 + 2t + \frac{1}{3}\right) + \frac{38}{9}$
11. $\left(2z^3 - 3z + 1\right) = \left(z - \frac{1}{2}\right)\left(2z^2 + z - \frac{5}{2}\right) - \frac{1}{4}$
12. $\left(4z^4 - 12z^3 + 13z^2 - 12z + 9\right) = \left(z - \frac{3}{2}\right)\left(4z^3 - 6z^2 + 4z - 6\right) + 0$
13. $\left(x^4 - 6x^2 + 9\right) = \left(x - \sqrt{3}\right)\left(x^3 + \sqrt{3}\,x^2 - 3x - 3\sqrt{3}\right) + 0$
14. $\left(x^6 - 6x^4 + 12x^2 - 8\right) = \left(x + \sqrt{2}\right)\left(x^5 - \sqrt{2}\,x^4 - 4x^3 + 4\sqrt{2}\,x^2 + 4x - 4\sqrt{2}\right) + 0$
15. $p(4) = 29$
16. $p(12) = 0$, $p(x) = (x - 12)(4x + 15)$
17. $p(-3) = -45$
18. $p(-1) = 2$
19. $p(2) = 0$, $p(z) = (z - 2)\left(3z^2 + 4\right)$
20. $p\left(-\frac{1}{2}\right) = 0$, $p(z) = \left(z + \frac{1}{2}\right)\left(8z^2 + 8z + 2\right)$
21. $p\left(\frac{3}{2}\right) = \frac{73}{16}$
22. $p\left(-\frac{2}{3}\right) = \frac{74}{27}$
23. $p(-\sqrt{7}) = 0$, $p(t) = (t + \sqrt{7})\left(t^3 + (1 - \sqrt{7})t^2 + (1 - \sqrt{7})t - \sqrt{7}\right)$
24. $p(2 - \sqrt{3}) = 0$, $p(t) = (t - (2 - \sqrt{3}))(t - (2 + \sqrt{3}))$
25. $x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$
26. $x^3 - 24x^2 + 192x - 512 = (x - 8)^3$
27. $3t^3 + 4t^2 - t - 2 = 3\left(t - \frac{2}{3}\right)(t + 1)^2$
28. $2t^3 - 3t^2 - 11t + 6 = 2\left(t - \frac{1}{2}\right)(t + 2)(t - 3)$

29. $z^3 + 2z^2 - 3z - 6 = (z+2)(z+\sqrt{3})(z-\sqrt{3})$

30. $2z^3 - z^2 - 10z + 5 = 2\left(z - \frac{1}{2}\right)(z+\sqrt{5})(z-\sqrt{5})$

31. $4x^4 - 28x^3 + 61x^2 - 42x + 9 = 4\left(x - \frac{1}{2}\right)^2 (x-3)^2$

32. $t^5 + 2t^4 - 12t^3 - 38t^2 - 37t - 12 = (t+1)^3(t+3)(t-4)$

33. $125z^5 - 275z^4 - 2265z^3 - 3213z^2 - 1728z - 324 = 125\left(z + \frac{3}{5}\right)^3 (z+2)(z-6)$

34. $x^2 - 2x - 2 = (x - (1-\sqrt{3}))(x - (1+\sqrt{3}))$

35. $p(x) = 5x^2 - 6x - 4$

38.
- For $f(x) = x^3 - 2x + 1$, the coefficients $1 + (-2) + 1 = 0$ and $f(x) = (x-1)(x^2 + x - 1)$.
- For $f(x) = 3x^4 - x - 2$ the coefficients $3 + (-1) + (-2) = 0$ and $f(x) = (x-1)(3x^3 + 3x^2 + 3x + 2)$.

39.
- $\dfrac{x-a}{x-a} = 1$
- $\dfrac{x^2-a^2}{x-a} = x + a$
- $\dfrac{x^3-a^3}{x-a} = x^2 + ax + a^2$
- $\dfrac{x^4-a^4}{x-a} = x^3 + ax^2 + a^2x + a^3$
- $\dfrac{x^5-a^5}{x-a} = x^4 + ax^3 + a^2x^2 + a^3x + a^4$

Following the pattern:

- $\dfrac{x^{10}-a^{10}}{x-a} = x^9 + ax^8 + a^2x^7 + a^3x^6 + a^4x^5 + a^5x^4 + a^6x^3 + a^7x^2 + a^8x + a^9$
- $\dfrac{x^n-a^n}{x-a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + ... + a^{n-2}x + a^{n-1}$

40. Put $x = 1$ and $a = r$ so that $1 + r + r^2 + \cdots + r^{n-2} + r^{n-1} = \dfrac{1-r^n}{1-r}$. Here, $r \neq 1$ as otherwise we'd be dividing by 0.

2.3 Real Zeros of Polynomials

In Section 2.2, we found that we can use synthetic division to determine if a given real number is a zero of a polynomial function. This section presents results which will help us determine good candidates to test using synthetic division. There are two approaches to the topic of finding the real zeros of a polynomial. The first approach is to use a little bit of Mathematics followed by a good use of technology like graphing utilities. The second approach makes good use of mathematical machinery (theorems) only. For completeness, we include the two approaches but in separate subsections. Both approaches benefit from the following two theorems, the first of which is due to the famous mathematician Augustin Cauchy. It gives us an interval on which **all** of the real zeros of a polynomial can be found.

Theorem 2.10. Cauchy's Bound: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$. Let M be the largest of the numbers: $\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}$. Then all the real zeros of f lie in in the interval $[-(M+1), M+1]$.

There's a lot going on in the statement of Cauchy's Bound, so we'll get right to an example and show how it is used. For those wanting a proof of Cauchy's Bound, see Exercise 49 in Section 10.2.

Example 2.3.1. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Determine an interval which contains all of the real zeros of f.

Solution. To find the M stated in Cauchy's Bound, we take the absolute value of the leading coefficient, in this case $|2| = 2$ and divide it into the largest (in absolute value) of the remaining coefficients, in this case $|-6| = 6$. This yields $M = 3$ so it is guaranteed that all of the real zeros of f lie in the interval $[-4, 4]$. $\square$

Whereas the previous result tells us **where** we can find the real zeros of a polynomial, the next theorem gives us a list of **possible** real zeros.

Theorem 2.11. Rational Zeros Theorem: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and a_0, a_1, $\ldots a_n$ are integers. If r is a rational zero of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

The Rational Zeros Theorem gives us a list of numbers to try in our synthetic division and that is a lot nicer than simply guessing. If none of the numbers in the list are zeros, then either the polynomial has no real zeros at all, or all of the real zeros are irrational numbers. To see why the Rational Zeros Theorem works, suppose c is a zero of f and $c = \frac{p}{q}$ in lowest terms. This means p and q have no common factors. Since $f(c) = 0$, we have

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_1 \left(\frac{p}{q}\right) + a_0 = 0.$$

Multiplying both sides of this equation by q^n, we clear the denominators to get

$$a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_1 pq^{n-1} + a_0 q^n = 0$$

Rearranging this equation, we get

$$a_n p^n = -a_{n-1}p^{n-1}q - \ldots - a_1 pq^{n-1} - a_0 q^n$$

Now, the left hand side is an integer multiple of p, and the right hand side is an integer multiple of q. (Can you see why?) This means $a_n p^n$ is both a multiple of p and a multiple of q. Since p and q have no common factors, a_n must be a multiple of q. If we rearrange the equation

$$a_n p^n + a_{n-1} p^{n-1} q + \ldots + a_1 p q^{n-1} + a_0 q^n = 0$$

as

$$a_0 q^n = -a_n p^n - a_{n-1} p^{n-1} q - \ldots - a_1 p q^{n-1}$$

we can play the same game and conclude a_0 is a multiple of p, and we have the result.

Example 2.3.2. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Zeros Theorem to list all of the possible rational zeros of f.

Solution. To generate a complete list of rational zeros, we need to take each of the factors of constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$. The factors of -3 are ± 1 and ± 3. Since the Rational Zeros Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Zeros Theorem gives the list $\left\{\pm \frac{1}{1}, \pm \frac{1}{2}, \pm \frac{3}{1}, \pm \frac{3}{2}\right\}$ or $\left\{\pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 3\right\}$. □

Our discussion now diverges between those who wish to use technology and those who do not.

2.3.1 For Those Wishing to use a Graphing Utility

At this stage, we know not only the interval in which all of the zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ are located, but we also know some potential candidates. We can now use our calculator to help us determine all of the real zeros of f, as illustrated in the next example.

Example 2.3.3. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

1. Graph $y = f(x)$ using a graphing utility over the interval obtained in Example 2.3.1.
2. Use the graph to shorten the list of possible rational zeros obtained in Example 2.3.2.
3. Use synthetic division to find the real zeros of f, and state their multiplicities.

Solution.

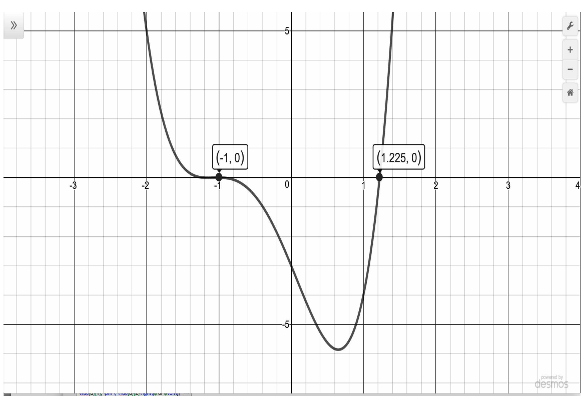
1. In Example 2.3.1, we determined all of the real zeros of f lie in the interval $[-4, 4]$, so we restrict our attention to that portion of the x-axis.
2. In Example 2.3.2, we learned that any rational zero of f must be in the list $\left\{\pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 3\right\}$. From the graph, it looks as if we can rule out any of the positive rational zeros, since the graph seems to cross the x-axis at $x \approx 1.225$. On the negative side, $x = -1$ looks good. Indeed, the shape of the graph near $(-1, 0)$ suggests that if $x = -1$ is a zero, it is of multiplicity at least three. We set about synthetically dividing:

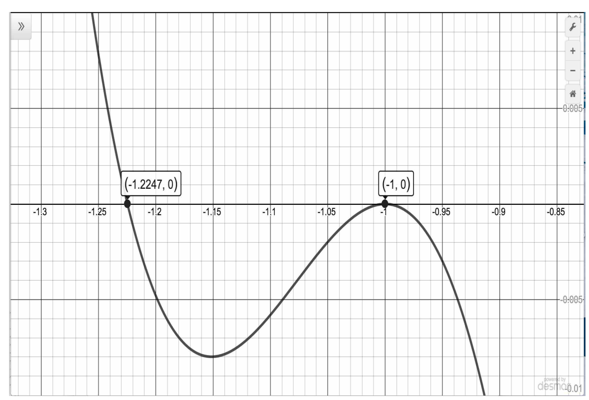
$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline & 2 & 2 & -3 & -3 & \boxed{0} \end{array}$$

Since f is a fourth degree polynomial, we know that our quotient is a third degree polynomial. If we can do one more successful division, we will have reduced the quotient to a quadratic, and we can use the quadratic formula, if needed, to find the two remaining zeros. Continuing with $x = -1$:

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline -1 & 2 & 2 & -3 & -3 & \boxed{0} \\ & \downarrow & -2 & 0 & 3 & \\ \hline & 2 & 0 & -3 & \boxed{0} & \end{array}$$

Our quotient polynomial is now $2x^2 - 3$. Setting this to zero gives $2x^2 - 3 = 0$, or $x^2 = \frac{3}{2}$, which gives us $x = \pm\frac{\sqrt{6}}{2}$. Based on our division work, we know that -1 has a multiplicity of **at least** 2. The Factor Theorem tells us our remaining zeros, $\pm\frac{\sqrt{6}}{2}$, each have multiplicity at least 1. However, Theorem 2.9 tells us f can have at most 4 real zeros, counting multiplicity, and so we conclude that -1 is of multiplicity **exactly** 2 and $\pm\frac{\sqrt{6}}{2} \approx \pm 1.225$ each has multiplicity 1. Thus, we were incorrect in thinking -1 was a zero of multiplicity 3. Sure enough, if we adjust zoom in near $(-1, 0)$ using graphing utility, we find the graph of $y = f(x)$ touches and rebounds from the x-axis at $(-1, 0)$, typical behavior near a zero of even multiplicity. The lesson here is, once again, technology may **suggest** a result, but it is only the mathematics which can **prove** (or in this case, **disprove**) it.





$\square$

Our next example shows how even a mild-mannered polynomial can cause problems.

Example 2.3.4. Let $f(x) = x^4 + x^2 - 12$.

1. Use Cauchy's Bound to determine an interval in which all of the real zeros of f lie.

2. Use the Rational Zeros Theorem to determine a list of possible rational zeros of f.

3. Graph $y = f(x)$ using a graphing utility.

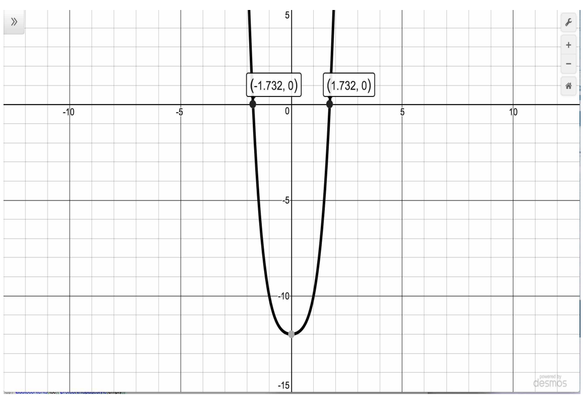
4. Find all of the real zeros of f and their multiplicities.

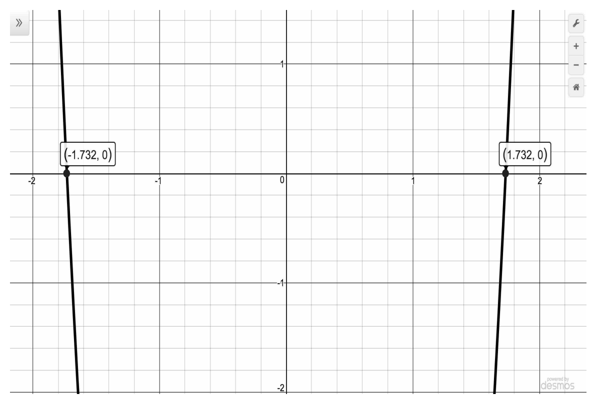
Solution.

1. Applying Cauchy's Bound, we find $M = 12$, so all of the real zeros lie in the interval $[-13, 13]$.

2. Applying the Rational Zeros Theorem with constant term $a_0 = -12$ and leading coefficient $a_4 = 1$, we get the list $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$.

3. Graphing $y = f(x)$ on the interval $[-13, 13]$ produces the graph below on the left. Zooming in a bit gives the graph below on the right. Based on the graph, none of our rational zeros will work. (Do you see why not?)





4. From the graph, we know f has two real zeros, one positive, and one negative. Our only hope at this point is to try and find the zeros of f by setting $f(x) = x^4 + x^2 - 12 = 0$ and solving. If we stare at this equation long enough, we may recognize it as a 'quadratic in disguise' or 'quadratic in form'. (See Section A.10.) In other words, we have three terms: x^4, x^2 and 12, and the exponent on the first term, x^4, is exactly twice that of the second term, x^2. We may rewrite this as $\left(x^2\right)^2 + \left(x^2\right) - 12 = 0$. To better see the forest for the trees, we momentarily replace x^2 with the variable u. In terms of u, our equation becomes $u^2 + u - 12 = 0$, which we can readily factor as $(u + 4)(u - 3) = 0$. In terms of x, this means $x^4 + x^2 - 12 = \left(x^2 - 3\right)\left(x^2 + 4\right) = 0$. We get $x^2 = 3$, which gives us $x = \pm\sqrt{3}$, or $x^2 = -4$, which admits no real solutions. Since $\sqrt{3} \approx 1.73$, the two zeros match what we expected from the graph. Turning our attention now to multiplicities, the Factor Theorem guarantees that since $x = \pm\sqrt{3}$ are zeros, $(x - \sqrt{3})$ and $(x + \sqrt{3})$ are factors of $f(x)$. We've already partially factored $f(x)$ as $f(x) = \left(x^2 - 3\right)\left(x^2 + 4\right)$. Since $x^2 + 4$ as no real zeros, we know both $(x - \sqrt{3})$ and $(x + \sqrt{3})$ must divide $x^2 - 3$. By Theorem 2.9, $x^2 - 3$ can only have a total of two zeros, including multiplicities, so we are forced to conclude $x = \pm\sqrt{3}$ are each zeros of multiplicity 1 of $x^2 - 3$, and hence, $f(x)$.[1] $\square$

[1] Alternatively, we could recognize $x^2 - 3 = x^2 - (\sqrt{3})^2 = (x - \sqrt{3})(x + \sqrt{3})$, but the above argument works for all quadratic functions, even those which aren't as easy to factor.

A couple of remarks are in order. First, the graph of $f(x) = x^4 + x^2 - 12$ appears to be symmetric about the y-axis. Sure enough, we find $f(-x) = (-x)^4 + (-x)^2 - 12 = x^4 + x^2 = 12 = f(x)$ proving f is, indeed, an even function, thus **proving** the symmetry **suggested** by the graph. Second, the technique used to factor $f(x)$ in Example 2.3.4 is called ***u*-substitution**. We shall this technique now and then in the sections to come, so it is worth taking the time to let this idea sink in. In general, substitution can help us identify a 'quadratic in disguise' - in essence, it helps us 'see the forest for the trees.' Last, but not least, it is entirely possible that a polynomial has no real roots at all, or worse, it has real roots but none of the techniques discussed in this section can help us find them exactly. In the latter case, we are forced to approximate using technology.

2.3.2 For Those Wishing NOT to use a Graphing Calculator

Suppose we wish to find the zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ **without** using the calculator. In this subsection, we present some more advanced mathematical tools (theorems) to help us. Our first result is due to René Descartes.

Theorem 2.12. Descartes' Rule of Signs: Suppose $f(x)$ is the formula for a polynomial function written with descending powers of x.

- If P denotes the number of variations of sign in the formula for $f(x)$, then the number of positive real zeros (counting multiplicity) is one of the numbers $\{P, P-2, P-4, \ldots\}$.
- If N denotes the number of variations of sign in the formula for $f(-x)$, then the number of negative real zeros (counting multiplicity) is one of the numbers $\{N, N-2, N-4, \ldots\}$.

A few remarks are in order. First, to use Descartes' Rule of Signs, we need to understand what is meant by a '**variation in sign**' of a polynomial function. Consider $f(x) = 2x^4+4x^3-x^2-6x-3$. If we focus on only the **signs** of the coefficients, we start with a (+), followed by another (+), then switch to (−), and stay (−) for the remaining two coefficients. Since the signs of the coefficients switched **once** as we read from left to right, we say that $f(x)$ has **one** variation in sign. When we speak of the variations in sign of a polynomial function f we assume the formula for $f(x)$ is written with descending powers of x, as in Definition 2.4, and concern ourselves only with the nonzero coefficients. Second, unlike the Rational Zeros Theorem, Descartes' Rule of Signs gives us an estimate to the **number** of positive and negative real zeros, not the actual **value** of the zeros. Lastly, Descartes' Rule of Signs counts multiplicities. This means that, for example, if one of the zeros has multiplicity 2, Descsartes' Rule of Signs would count this as **two** zeros. Lastly, note that the number of positive or negative real zeros always starts with the number of sign changes and decreases by an even number. For example, if $f(x)$ has 7 sign changes, then, counting multplicities, f has either 7, 5, 3 or 1 positive real zero. This implies that the graph of $y = f(x)$ crosses the positive x-axis at least once. If $f(-x)$ results in 4 sign changes, then, counting multiplicities, f has 4, 2 or 0 negative real zeros; hence, the graph of $y = f(x)$ may not cross the negative x-axis at all. The proof of Descartes' Rule of Signs is a bit technical, and can be found here.

Example 2.3.5. Let $f(x) = 2x^4+4x^3-x^2-6x-3$. Use Descartes' Rule of Signs to determine the possible number and location of the real zeros of f.

Solution. As noted above, the variations of sign of $f(x)$ is 1. This means, counting multiplicities, f has exactly 1 positive real zero. Since $f(-x) = 2(-x)^4 + 4(-x)^3 - (-x)^2 - 6(-x) - 3 = 2x^4 - 4x^3 - x^2 + 6x - 3$ has 3 variations in sign, f has either 3 negative real zeros or 1 negative real zero, counting multiplicities. □

Cauchy's Bound gives us a general bound on the zeros of a polynomial function. Our next result helps us determine bounds on the real zeros of a polynomial as we synthetically divide which are often sharper[2] bounds than Cauchy's Bound.

Theorem 2.13. Upper and Lower Bounds: Suppose f is a polynomial of degree $n \geq 1$.

- If $c > 0$ is synthetically divided into f and all of the numbers in the final line of the division tableau have the same signs, then c is an upper bound for the real zeros of f. That is, there are no real zeros greater than c.
- If $c < 0$ is synthetically divided into f and the numbers in the final line of the division tableau alternate signs, then c is a lower bound for the real zeros of f. That is, there are no real zeros less than c.

  NOTE: If the number 0 occurs in the final line of the division tableau in either of the above cases, it can be treated as $(+)$ or $(-)$ as needed.

The Upper and Lower Bounds Theorem works because of Theorem 2.6. For the upper bound part of the theorem, suppose $c > 0$ is divided into f and the resulting line in the division tableau contains, for example, all nonnegative numbers. This means $f(x) = (x-c)q(x) + r$, where the coefficients of the quotient polynomial and the remainder are nonnegative. (Note that the leading coefficient of q is the same as f so $q(x)$ is not the zero polynomial.) If $b > c$, then $f(b) = (b-c)q(b) + r$, where $(b-c)$ and $q(b)$ are both positive and $r \geq 0$. Hence $f(b) > 0$ which shows b cannot be a zero of f. Thus no real number $b > c$ can be a zero of f, as required. A similar argument proves $f(b) < 0$ if all of the numbers in the final line of the synthetic division tableau are non-positive. To prove the lower bound part of the theorem, we note that a lower bound for the negative real zeros of $f(x)$ is an upper bound for the positive real zeros of $f(-x)$, since all we are doing is reflecting the numbers across the $x = 0$. Applying the upper bound portion to $f(-x)$ gives the result. (Do you see where the alternating signs come in?) With the additional mathematical machinery of Descartes' Rule of Signs and the Upper and Lower Bounds Theorem, we can find the real zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ without the use of a graphing utility.

Example 2.3.6. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

1. Find all of the real zeros of f and their multiplicities.
2. Sketch the graph of $y = f(x)$.

Solution.

1. We know from Cauchy's Bound that all of the real zeros lie in the interval $[-4, 4]$ and that our possible rational zeros are $\pm\frac{1}{2}$, ± 1, $\pm\frac{3}{2}$ and ± 3. Descartes' Rule of Signs guarantees us at least one

[2]That is, better, or more accurate.

negative real zero and exactly one positive real zero, counting multiplicity. We try our positive rational zeros, starting with the smallest, $\frac{1}{2}$. Since the remainder isn't zero, we know $\frac{1}{2}$ isn't a zero. Sadly, the final line in the division tableau has both positive and negative numbers, so $\frac{1}{2}$ is not an upper bound. The only information we get from this division is courtesy of the Remainder Theorem which tells us $f\left(\frac{1}{2}\right) = -\frac{45}{8}$ so the point $\left(\frac{1}{2}, -\frac{45}{8}\right)$ is on the graph of f. We continue to our next possible zero, 1. As before, the only information we can glean from this is that $(1, -4)$ is on the graph of f. When we try our next possible zero, $\frac{3}{2}$, we get that it is not a zero, and we also see that it is an upper bound on the zeros of f, since all of the numbers in the final line of the division tableau are positive. This means there is no point trying our last possible rational zero, 3. Descartes' Rule of Signs guaranteed us a positive real zero, and at this point we have shown this zero is irrational.[3]

$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 1 & \frac{5}{2} & \frac{3}{4} & -\frac{21}{8} \\ \hline & 2 & 5 & \frac{3}{2} & -\frac{21}{4} & \boxed{-\frac{45}{8}} \end{array} \qquad \begin{array}{r|rrrrr} 1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 2 & 6 & 5 & -1 \\ \hline & 2 & 6 & 5 & -1 & \boxed{-4} \end{array} \qquad \begin{array}{r|rrrrr} \frac{3}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 3 & \frac{21}{2} & \frac{57}{4} & \frac{99}{8} \\ \hline & 2 & 7 & \frac{19}{2} & \frac{33}{4} & \boxed{\frac{75}{8}} \end{array}$$

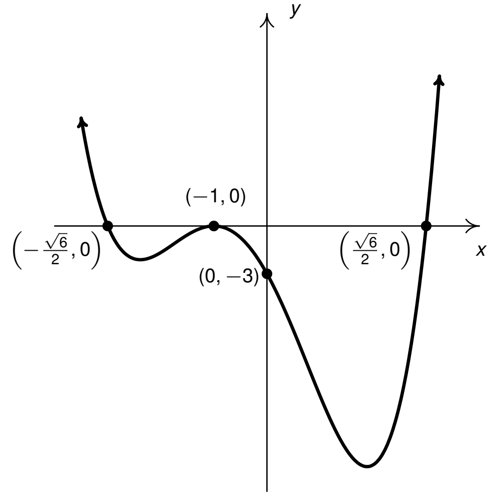
We now turn our attention to negative real zeros. We try the largest possible zero, $-\frac{1}{2}$. Synthetic division shows us it is not a zero, nor is it a lower bound (since the numbers in the final line of the division tableau do not alternate), so we proceed to -1. This division shows -1 is a zero. Descartes' Rule of Signs told us that we may have up to three negative real zeros, counting multiplicity, so we try -1 again, and it works once more. At this point, we have taken f, a fourth degree polynomial, and performed two successful divisions. Our quotient polynomial is quadratic, so we look at it to find the remaining zeros.

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -1 & -\frac{3}{2} & \frac{5}{4} & \frac{19}{8} \\ \hline & 2 & 3 & -\frac{5}{2} & -\frac{19}{4} & \boxed{-\frac{5}{8}} \end{array} \qquad \begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline -1 & 2 & 2 & -3 & -3 & \boxed{0} \\ & \downarrow & -2 & 0 & 3 & \\ \hline & 2 & 0 & -3 & \boxed{0} & \end{array}$$

Setting the quotient polynomial equal to zero yields $2x^2 - 3 = 0$, so that $x^2 = \frac{3}{2}$, or $x = \pm\frac{\sqrt{6}}{2}$. Descartes' Rule of Signs tells us that the positive real zero we found, $\frac{\sqrt{6}}{2}$, has multiplicity 1. Descartes also tells us the total multiplicity of negative real zeros is 3, which forces -1 to be a zero of multiplicity 2 and $-\frac{\sqrt{6}}{2}$ to have multiplicity 1.

2. We know the end behavior of $y = f(x)$ resembles that of its leading term $y = 2x^4$. This means that the graph enters the scene in Quadrant II and exits in Quadrant I. Since $\pm\frac{\sqrt{6}}{2}$ are zeros of multiplicity 1, we have that the graph crosses through the x-axis at the points $\left(-\frac{\sqrt{6}}{2}, 0\right)$ and $\left(\frac{\sqrt{6}}{2}, 0\right)$ in a fairly linear fashion. Since -1 is a zero of multiplicity 2, the graph of $y = f(x)$ touches and rebounds off the x-axis at $(-1, 0)$ in a parabolic manner. Last, but not least, since $f(0) = -3$, we get the y-intercept is $(0, -3)$. Putting all of this together results in the graph below.

[3]Since polynomials are continuous, we know the zero lies between 1 and $\frac{3}{2}$, since $f(1) < 0$ and $f\left(\frac{3}{2}\right) > 0$.



$y = f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$

□

2.3.3 The Intermediate Value Theorem and Inequalities

As we mentioned in Section 2.1, polynomial functions are continuous. An important property of continuous functions is that they cannot change sign between two values unless there is a zero in between. We used this property of quadratic functions when constructing sign diagrams to help us solve inequalities (see Section 1.4.2.) This property is a version of the celebrated **Intermediate Value Theorem**.

Theorem 2.14. The Intermediate Value Theorem (Zero Version): If f is continuous over an interval containing a and b and $f(a)$ and $f(b)$ have different signs, then f has a zero between a and b. That is, for at least one value c between a and b, $f(c) = 0$.

The Intermediate Value Theorem is discussed in greater detail in Calculus, and its proof is usually delayed until a formal analysis course. It is an example of an 'existence' theorem - it tells us that, under suitable conditions, a zero exists - but offers us no algorithm to find it.[4] Its use to us in this section is that it provides the justification needed to create sign diagrams for general polynomial functions in the same manner in which we constructed them for quadratic functions.

Steps for Constructing a Sign Diagram for a Polynomial Function

Suppose f is a polynomial function.

1. Find the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine and record the sign of $f(x)$ for each test value in step 2.

[4]See the notes on the 'Bisection Method' at the end of this section.

The Intermediate Value Theorem justifies the use of just one 'test' value in the algorithm above, since a continuous function cannot change signs on an interval without there being a zero on that interval. Since we have found the zeros in Step 1 of the algorithm and used these to create the intervals for Step 2, there cannot be any sign changes on any of the intervals in Step 2.

Not surprisingly, we use sign diagrams to solve inequalities involving higher order polynomial functions in the same way we used them to solve inequalities involving quadratic functions. We reproduce our algorithm from section 1.4.2 for reference.

Solving Inequalities using Sign Diagrams

To solve an inequality using a sign diagram:

1. Rewrite the inequality so a function $f(x)$ is being compared to '0.'
2. Make a sign diagram for f.
3. Record the solution.

Example 2.3.7.

1. Find all of the real solutions to the equation $2x^5 + 6x^3 + 3 = 3x^4 + 8x^2$.
2. Solve the inequality $2x^5 + 6x^3 + 3 \leq 3x^4 + 8x^2$.
3. Interpret your answer to part 2 graphically, and verify using a graphing utility.

Solution.

1. Finding the real solutions to $2x^5 + 6x^3 + 3 = 3x^4 + 8x^2$ is the same as finding the real solutions to $2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 = 0$. In other words, we are looking for the real zeros of $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$. Using the techniques developed in this section, we get

$$\begin{array}{r|rrrrrr}
1 & 2 & -3 & 6 & -8 & 0 & 3 \\
 & \downarrow & 2 & -1 & 5 & -3 & -3 \\
\hline
1 & 2 & -1 & 5 & -3 & -3 & \boxed{0} \\
 & \downarrow & 2 & 1 & 6 & 3 & \\
\hline
-\frac{1}{2} & 2 & 1 & 6 & 3 & \boxed{0} & \\
 & \downarrow & -1 & 0 & -3 & & \\
\hline
 & 2 & 0 & 6 & \boxed{0} & &
\end{array}$$

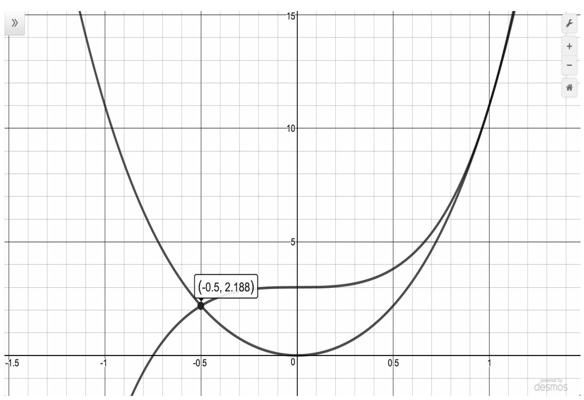
   The quotient polynomial is $2x^2 + 6$ which has no real zeros so we get $x = -\frac{1}{2}$ and $x = 1$.

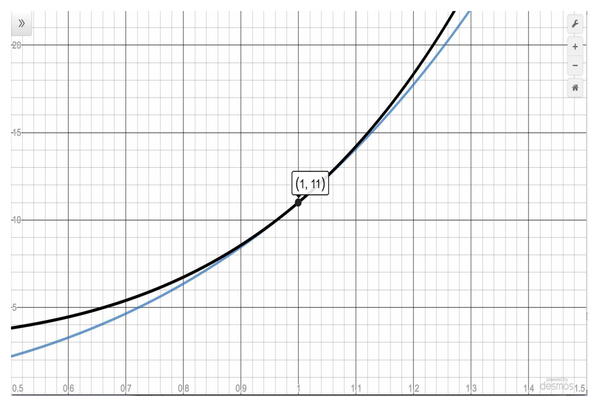
2. Our first step is to rewrite this inequality so as to compare a function $f(x)$ to 0. We have two options, but choose $2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 \leq 0$, since we found the zeros of $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$ to be $x = -\frac{1}{2}$ and $x = 1$. We construct our sign diagram below using the test values -1, 0, and 2.

$$\begin{array}{ccccccc} (-) & 0 & (+) & 0 & (+) \\ \hline \uparrow & -\frac{1}{2} & \uparrow & 1 & \uparrow \\ -1 & & 0 & & 2 \end{array}$$

The solution to $p(x) < 0$ is $\left(-\infty, -\frac{1}{2}\right)$, and we know $p(x) = 0$ at $x = -\frac{1}{2}$ and $x = 1$. Hence, the solution to $p(x) \leq 0$ is $\left(-\infty, -\frac{1}{2}\right] \cup \{1\}$.

3. To interpret this solution graphically, we set $f(x) = 2x^5 + 6x^3 + 3$ and $g(x) = 3x^4 + 8x^2$. Recall from Section 1.3 the solution to $f(x) \leq g(x)$ is the set of x values for which the graph of f is below the graph of g (where $f(x) < g(x)$) along with the x values where the two graphs intersect ($f(x) = g(x)$). Graphing f and g using a graphing utility produces the graph below on the left. (The end behavior should tell you which is which.) We see that the graph of f is below the graph of g on $\left(-\infty, -\frac{1}{2}\right)$. However, it is difficult to see what is happening near $x = 1$. Zooming in (and making the graph of g lighter), we see that the graphs of f and g do intersect at $x = 1$, but the graph of g remains below the graph of f on either side of $x = 1$.





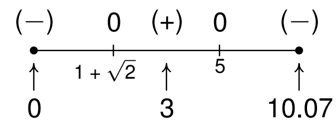
□

Note that we could have used end behavior and the concept of multiplicity to create the sign diagram used in Example 2.3.7 as follows. We know the end behavior of $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$ matches that of $y = 2x^5$ which means $\displaystyle\lim_{x \to -\infty} p(x) = -\infty$. This means for the interval $\left(-\infty, -\frac{1}{2}\right)$, $p(x) < 0$ or $(-)$. From our work finding the zeros of p, we can deduce the multiplicity of the zero $x = -\frac{1}{2}$ is 1 which means the graph of $y = p(x)$ crosses through the x-axis at $\left(-\frac{1}{2}, 0\right)$, hence, changing sign from $(-)$ to $(+)$. Finally, we can deduce the multiplicity of the zero $x = 1$ is 2 which means the graph of $y = p(x)$ rebounds here, meaning the sign of $p(x)$ for $x > 1$ is $(+)$. This matches the end behavior, since $\displaystyle\lim_{x \to \infty} p(x) = \infty$. The reader is encouraged to tackle any given problem using whatever tools are comfortable and convenient, but it also never hurts to think outside the box and revisit a problem from a variety of perspectives.

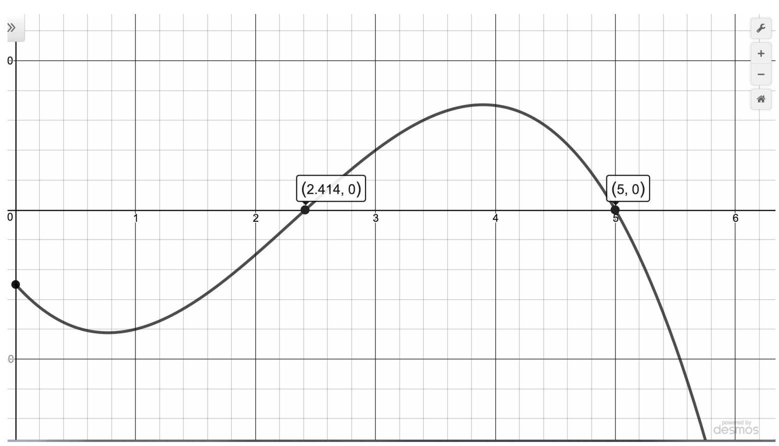
Next up is an application problem torn from page 155 in the Exercises of Section 2.1.

Example 2.3.8. Suppose the profit P, in **thousands** of dollars, from producing and selling x **hundred** LCD TVs is given by $P(x) = -5x^3 + 35x^2 - 45x - 25$, $0 \leq x \leq 10.07$. How many TVs should be produced to make a profit? Check your answer using a graphing utility.

Solution. To 'make a profit' means to solve $P(x) = -5x^3 + 35x^2 - 45x - 25 > 0$, which we do analytically using a sign diagram. To simplify things, we first factor out the -5 common to all the coefficients to get $-5\left(x^3 - 7x^2 + 9x - 5\right) > 0$, so we can just focus on finding the zeros of $f(x) = x^3 - 7x^2 + 9x + 5$. The possible rational zeros of f are ± 1 and ± 5, and going through the usual computations, we find $x = 5$ is the only rational zero. Using this, we factor $f(x) = x^3 - 7x^2 + 9x + 5 = (x - 5)\left(x^2 - 2x - 1\right)$, and we find the remaining zeros by applying the Quadratic Formula to $x^2 - 2x - 1 = 0$. We find three real zeros, $x = 1 - \sqrt{2} = -0.414\ldots$, $x = 1 + \sqrt{2} = 2.414\ldots$, and $x = 5$, of which only the last two fall in the applied domain of $[0, 10.07]$. We choose $x = 0$, $x = 3$ and $x = 10.07$ as our test values and plug them into the function $P(x) = -5x^3 + 35x^2 - 45x - 25$ (not $f(x) = x^3 - 7x^2 + 9x - 5$) to get the sign diagram below.



We see immediately that $P(x) > 0$ on $(1 + \sqrt{2}, 5)$. Since x measures the number of TVs in **hundreds**, $x = 1 + \sqrt{2}$ corresponds to $241.4\ldots$ TVs. Since we can't produce a fractional part of a TV, we need to choose between producing 241 and 242 TVs. From the sign diagram, we see that $P(2.41) < 0$ but $P(2.42) > 0$ so, in this case we take the next **larger** integer value and set the minimum production to 242 TVs. At the other end of the interval, we have $x = 5$ which corresponds to 500 TVs. Here, we take the next **smaller** integer value, 499 TVs to ensure that we make a profit. Hence, in order to make a profit, at least 242, but no more than 499 TVs need to be produced. We graph $y = P(x)$ below using a graphing utility and see $P(x) > 0$ between $x \approx 2.414$ and $x = 5$, as predicted.



□

It would be a sin of omission if the authors left the reader with the impression that the theory in this section is compete in that given **any** polynomial function, provided here are the tools to find all of its real zeros exactly.

The reality is this couldn't be further from the truth. In general, no matter how many theorems you throw at a polynomial, it may well be impossible to express its zeros exactly. The polynomial $f(x) = x^5 - x - 1$ is one such beast.[5] According to Descartes' Rule of Signs, f has exactly one positive real zero, and it could have two negative real zeros, or none at all. The Rational Zeros Test gives us ± 1 as rational zeros to try but neither of these work since $f(1) = f(-1) = -1$. If we try the substitution technique we used in Example 2.3.4, we find $f(x)$ has three terms, but the exponent on the x^5 isn't exactly twice the exponent on x. How could we go about approximating the positive zero? We use the **Bisection Method**.

The first step in the Bisection Method is to find an interval on which f changes sign. We know $f(1) = -1$ and we find $f(2) = 29$. By the Intermediate Value Theorem, we know that the zero of f lies in the interval $[1, 2]$. Next, we 'bisect' this interval by finding the midpoint, 1.5. We compute $f(1.5) \approx 5.09$. Once again, the Intermediate Value Theorem guarantees our zero is between 1 and 1.5, since f changes sign on this interval. Now, we 'bisect' the interval $[1, 1.5]$ and find $f(1.25) \approx 0.80$, so now we have the zero between 1 and 1.25. Bisecting $[1, 1.25]$, we find $f(1.125) \approx -0.32$, which means the zero of f is between 1.125 and 1.25. We continue in this fashion until we have 'sandwiched' the zero between two numbers whose digits agree to a desired amount.[6] You can think of the Bisection Method as reversing the sign diagram process: instead of finding the zeros and checking the sign of f using test values, we are using test values to determine where the signs switch to find the zeros. It is a slow and tedious, yet fool-proof, method for **approximating** a real zero when the other analytical methods fail us.

[5]See this page.

[6]We ask you to approximate this zero to three decimal places using the Bisection Method in Exercise 64.

2.3.4 Exercises

In Exercises 1 - 10, for the given polynomial:

- Use Cauchy's Bound to find an interval containing all of the real zeros.
- Use the Rational Zeros Theorem to make a list of possible rational zeros.
- Use Descartes' Rule of Signs to list the possible number of positive and negative real zeros, counting multiplicities.

1. $f(x) = x^3 - 2x^2 - 5x + 6$
2. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
3. $p(z) = z^4 - 9z^2 - 4z + 12$
4. $p(z) = z^3 + 4z^2 - 11z + 6$
5. $g(t) = t^3 - 7t^2 + t - 7$
6. $g(t) = -2t^3 + 19t^2 - 49t + 20$
7. $f(x) = -17x^3 + 5x^2 + 34x - 10$
8. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
9. $p(z) = 3z^3 + 3z^2 - 11z - 10$
10. $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$

In Exercises 11 - 30, find the real zeros of the polynomial using the techniques specified by your instructor. State the multiplicity of each real zero.

11. $f(x) = x^3 - 2x^2 - 5x + 6$
12. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
13. $p(z) = z^4 - 9z^2 - 4z + 12$
14. $p(z) = z^3 + 4z^2 - 11z + 6$
15. $g(t) = t^3 - 7t^2 + t - 7$
16. $g(t) = -2t^3 + 19t^2 - 49t + 20$
17. $f(x) = -17x^3 + 5x^2 + 34x - 10$
18. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
19. $p(z) = 3z^3 + 3z^2 - 11z - 10$
20. $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$
21. $g(t) = 9t^3 - 5t^2 - t$
22. $g(t) = 6t^4 - 5t^3 - 9t^2$
23. $f(x) = x^4 + 2x^2 - 15$
24. $f(x) = x^4 - 9x^2 + 14$
25. $p(z) = 3z^4 - 14z^2 - 5$
26. $p(z) = 2z^4 - 7z^2 + 6$
27. $g(t) = t^6 - 3t^3 - 10$
28. $g(t) = 2t^6 - 9t^3 + 10$
29. $f(x) = x^5 - 2x^4 - 4x + 8$
30. $f(x) = 2x^5 + 3x^4 - 18x - 27$

In Exercises 31 - 33, use your calculator,[7] to help you find the real zeros of the polynomial. State the multiplicity of each real zero.

31. $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$

32. $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$

33. $f(x) = 90x^4 - 399x^3 + 622x^2 - 399x + 90$

34. Find the real zeros of $f(x) = x^3 - \frac{1}{12}x^2 - \frac{7}{72}x + \frac{1}{72}$ by first finding a polynomial $q(x)$ with integer coefficients such that $q(x) = N \cdot f(x)$ for some integer N. (Recall that the Rational Zeros Theorem required the polynomial in question to have integer coefficients.) Show that f and q have the same real zeros.

In Exercises 35 - 44, find the real solutions of the polynomial equation. (See Example 2.3.7.)

35. $9x^3 = 5x^2 + x$

36. $9x^2 + 5x^3 = 6x^4$

37. $z^3 + 6 = 2z^2 + 5z$

38. $z^4 + 2z^3 = 12z^2 + 40z + 32$

39. $t^3 - 7t^2 = 7 - t$

40. $2t^3 = 19t^2 - 49t + 20$

41. $x^3 + x^2 = \dfrac{11x + 10}{3}$

42. $x^4 + 2x^2 = 15$

43. $14z^2 + 5 = 3z^4$

44. $2z^5 + 3z^4 = 18z + 27$

In Exercises 45 - 54, solve the polynomial inequality and state your answer using interval notation.

45. $-2x^3 + 19x^2 - 49x + 20 > 0$

46. $x^4 - 9x^2 \leq 4x - 12$

47. $(z - 1)^2 \geq 4$

48. $4z^3 \geq 3z + 1$

49. $t^4 \leq 16 + 4t - t^3$

50. $3t^2 + 2t < t^4$

51. $\dfrac{x^3 + 2x^2}{2} < x + 2$

52. $\dfrac{x^3 + 20x}{8} \geq x^2 + 2$

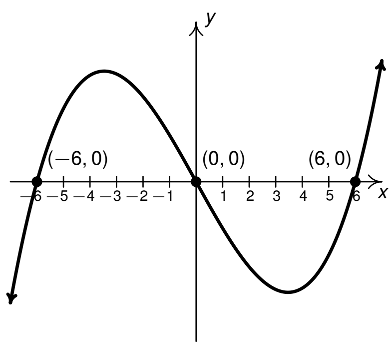
53. $2z^4 > 5z^2 + 3$

54. $z^6 + z^3 \geq 6$

[7] You *can* do these without your calculator, but it may test your mettle!

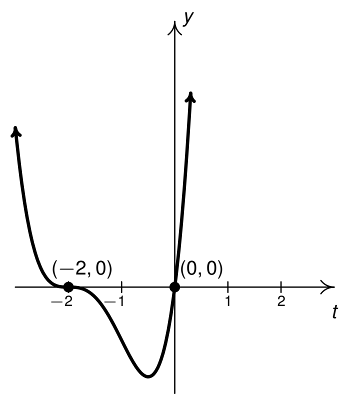
In Exercises 55 - 60, use the the graph of the given polynomial function to solve the stated inequality.

55. Solve $f(x) < 0$.



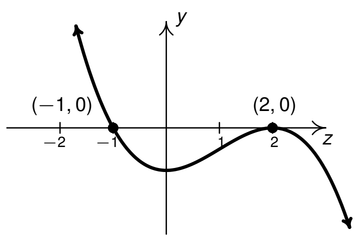
$y = f(x)$

56. Solve $g(t) > 0$.



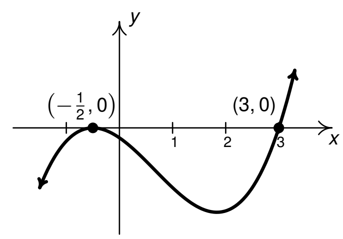
$y = g(t)$

57. Solve $p(z) \geq 0$



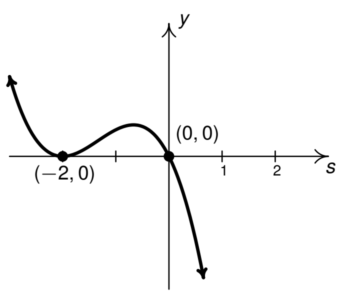
$y = p(z)$

58. Solve $f(x) < 0$.



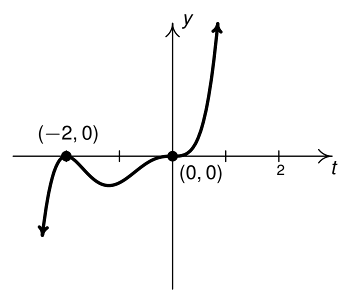
$y = f(x)$

59. Solve $F(s) \leq 0$.



$y = F(s)$

60. Solve $G(t) \geq 0$.



$y = G(t)$

61. Use the Intermediate Value Theorem, Theorem 2.14 to prove that $f(x) = x^3 - 9x + 5$ has a real zero in each of the following intervals: $[-4, -3]$, $[0, 1]$ and $[2, 3]$.

62. Use the concepts of End Behavior and the Intermediate Value Theorem to prove any odd-degree polynomial function with real number coefficients has at least one real zero.

63. Find an even-degree polynomial function with real number coefficients which has no real zeros.

64. Continue the Bisection Method as introduced on 192 to approximate the real zero of $f(x) = x^5 - x - 1$ to three decimal places.

65. In this exercise, we prove $\sqrt{2}$ is an irrational number and approximate its value. Let $f(x) = x^2 - 2$.

    (a) Use Decartes' Rule of Signs to prove f has exactly one positive real zero.

    (b) Use the Intermediate Value Theorem to prove f has a zero in $[1, 2]$.

    (c) Use the Rational Zeros Theorem to prove f has no rational zeros.

    (d) Use the Bisection Method (see 192) to approximate the zero of f on $[1, 2]$ to three decimal places.

66. Generalize the argument given in Exercise 65c to prove:

    (a) If N is not the perfect square of an integer, then $\sqrt{N}$ is irrational. (HINT: Consider $f(x) = x^2 - N$.)

    (b) For natural numbers $n \geq 2$, if N is not the perfect n^{th} power of an integer, then $\sqrt[n]{N}$ is irrational. (HINT: Consider $f(x) = x^n - N$.)

67. In Example 2.1.4 in Section 2.1, a box with no top is constructed from a 10 inch $\times$ 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. We determined the volume of that box (in cubic inches) is given by the function $V(x) = 4x^3 - 44x^2 + 120x$, where x denotes the length of the side of the square which is removed from each corner (in inches), $0 < x < 5$. Solve the inequality $V(x) \geq 80$ analytically and interpret your answer in the context of that example.

68. From Exercise 55 in Section 2.1, $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$ models the cost, in dollars, to produce x PortaBoy game systems. If the production budget is \$5000, find the number of game systems which can be produced and still remain under budget.

69. Let $f(x) = 5x^7 - 33x^6 + 3x^5 - 71x^4 - 597x^3 + 2097x^2 - 1971x + 567$. With the help of your classmates, find the x- and y- intercepts of the graph of f. Find the intervals on which the function is increasing, the intervals on which it is decreasing and the local extrema. Sketch the graph of f, using more than one picture if necessary to show all of the important features of the graph.

70. With the help of your classmates, create a list of five polynomials with different degrees whose real zeros cannot be found using any of the techniques in this section.

2.3.5 Answers

1. For $f(x) = x^3 - 2x^2 - 5x + 6$
   - All of the real zeros lie in the interval $[-7, 7]$
   - Possible rational zeros are ± 1, ± 2, ± 3, ± 6
   - There are 2 or 0 positive real zeros; there is 1 negative real zero
2. For $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
   - All of the real zeros lie in the interval $[-41, 41]$
   - Possible rational zeros are ± 1, ± 2, ± 4, ± 8, ± 16, ± 32
   - There is 1 positive real zero; there are 3 or 1 negative real zeros
3. For $p(z) = z^4 - 9z^2 - 4z + 12$
   - All of the real zeros lie in the interval $[-13, 13]$
   - Possible rational zeros are ± 1, ± 2, ± 3, ± 4, ± 6, ± 12
   - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros
4. For $p(z) = z^3 + 4z^2 - 11z + 6$
   - All of the real zeros lie in the interval $[-12, 12]$
   - Possible rational zeros are ± 1, ± 2, ± 3, ± 6
   - There are 2 or 0 positive real zeros; there is 1 negative real zero
5. For $g(t) = t^3 - 7t^2 + t - 7$
   - All of the real zeros lie in the interval $[-8, 8]$
   - Possible rational zeros are ± 1, ± 7
   - There are 3 or 1 positive real zeros; there are no negative real zeros
6. For $g(t) = -2t^3 + 19t^2 - 49t + 20$
   - All of the real zeros lie in the interval $\left[-\frac{51}{2}, \frac{51}{2}\right]$
   - Possible rational zeros are $\pm\frac{1}{2}$, ± 1, ± 2, $\pm\frac{5}{2}$, ± 4, ± 5, ± 10, ± 20
   - There are 3 or 1 positive real zeros; there are no negative real zeros
7. For $f(x) = -17x^3 + 5x^2 + 34x - 10$
   - All of the real zeros lie in the interval $[-3, 3]$
   - Possible rational zeros are $\pm\frac{1}{17}$, $\pm\frac{2}{17}$, $\pm\frac{5}{17}$, $\pm\frac{10}{17}$, ± 1, ± 2, ± 5, ± 10
   - There are 2 or 0 positive real zeros; there is 1 negative real zero

8. For $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
   - All of the real zeros lie in the interval $\left[-\frac{4}{3}, \frac{4}{3}\right]$
   - Possible rational zeros are $\pm\frac{1}{36}, \pm\frac{1}{18}, \pm\frac{1}{12}, \pm\frac{1}{9}, \pm\frac{1}{6}, \pm\frac{1}{4}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1$
   - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros
9. For $p(z) = 3z^3 + 3z^2 - 11z - 10$
   - All of the real zeros lie in the interval $\left[-\frac{14}{3}, \frac{14}{3}\right]$
   - Possible rational zeros are $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}, \pm 1, \pm 2, \pm 5, \pm 10$
   - There is 1 positive real zero; there are 2 or 0 negative real zeros
10. For $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$
    - All of the real zeros lie in the interval $\left[-\frac{9}{2}, \frac{9}{2}\right]$
    - Possible rational zeros are $\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3$
    - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros
11. $f(x) = x^3 - 2x^2 - 5x + 6$
    $x = -2$, $x = 1$, $x = 3$ (each has mult. 1)
12. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
    $x = -2$ (mult. 3), $x = 4$ (mult. 1)
13. $p(z) = z^4 - 9z^2 - 4z + 12$
    $z = -2$ (mult. 2), $z = 1$ (mult. 1), $z = 3$ (mult. 1)
14. $p(z) = z^3 + 4z^2 - 11z + 6$
    $z = -6$ (mult. 1), $z = 1$ (mult. 2)
15. $g(t) = t^3 - 7t^2 + t - 7$
    $t = 7$ (mult. 1)
16. $g(t) = -2t^3 + 19t^2 - 49t + 20$
    $t = \frac{1}{2}$, $t = 4$, $t = 5$ (each has mult. 1)
17. $f(x) = -17x^3 + 5x^2 + 34x - 10$
    $x = \frac{5}{17}$, $x = \pm\sqrt{2}$ (each has mult. 1)
18. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
    $x = \frac{1}{2}$ (mult. 2), $x = -\frac{1}{3}$ (mult. 2)
19. $p(z) = 3z^3 + 3z^2 - 11z - 10$
    $z = -2$, $z = \frac{3\pm\sqrt{69}}{6}$ (each has mult. 1)

20. $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$
$z = -1$, $z = \frac{1}{2}$, $z = \pm\sqrt{3}$ (each mult. 1)

21. $g(t) = 9t^3 - 5t^2 - t$
$t = 0$, $t = \frac{5\pm\sqrt{61}}{18}$ (each has mult. 1)

22. $g(t) = 6t^4 - 5t^3 - 9t^2$
$t = 0$ (mult. 2), $t = \frac{5\pm\sqrt{241}}{12}$ (each has mult. 1)

23. $f(x) = x^4 + 2x^2 - 15$
$x = \pm\sqrt{3}$ (each has mult. 1)

24. $f(x) = x^4 - 9x^2 + 14$
$x = \pm\sqrt{2}$, $x = \pm\sqrt{7}$ (each has mult. 1)

25. $p(z) = 3z^4 - 14z^2 - 5$
$z = \pm\sqrt{5}$ (each has mult. 1)

26. $p(z) = 2z^4 - 7z^2 + 6$
$z = \pm\frac{\sqrt{6}}{2}$, $z = \pm\sqrt{2}$ (each has mult. 1)

27. $g(t) = t^6 - 3t^3 - 10$
$t = \sqrt[3]{-2} = -\sqrt[3]{2}$, $t = \sqrt[3]{5}$ (each has mult. 1)

28. $g(t) = 2t^6 - 9t^3 + 10$
$t = \frac{\sqrt[3]{20}}{2}$, $t = \sqrt[3]{2}$ (each has mult. 1)

29. $f(x) = x^5 - 2x^4 - 4x + 8$
$x = 2$, $x = \pm\sqrt{2}$ (each has mult. 1)

30. $f(x) = 2x^5 + 3x^4 - 18x - 27$
$x = -\frac{3}{2}$, $x = \pm\sqrt{3}$ (each has mult. 1)

31. $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$
$x = -4$ (mult. 3), $x = 6$ (mult. 2)

32. $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$
$x = \frac{3}{5}$ (mult. 2), $x = 1$ (mult. 3)

33. $f(x) = 90x^4 - 399x^3 + 622x^2 - 399x + 90$
$x = \frac{2}{3}$, $x = \frac{3}{2}$, $x = \frac{5}{3}$, $x = \frac{3}{5}$ (each has mult. 1)

34. We choose $q(x) = 72x^3 - 6x^2 - 7x + 1 = 72 \cdot f(x)$. Clearly $f(x) = 0$ if and only if $q(x) = 0$ so they have the same real zeros. In this case, $x = -\frac{1}{3}$, $x = \frac{1}{6}$ and $x = \frac{1}{4}$ are the real zeros of both f and q.

35. $x=0, \frac{5\pm\sqrt{61}}{18}$

36. $x=0, \frac{5\pm\sqrt{241}}{12}$

37. $z=-2,1,3$

38. $z=-2,4$

39. $t=7$

40. $t=\frac{1}{2},4,5$

41. $x=-2, \frac{3\pm\sqrt{69}}{6}$

42. $x=\pm\sqrt{3}$

43. $z=\pm\sqrt{5}$

44. $z=-\frac{3}{2},\pm\sqrt{3}$

45. $(-\infty,\frac{1}{2})\cup(4,5)$

46. $\{-2\}\cup[1,3]$

47. $(-\infty,-1]\cup[3,\infty)$

48. $\left\{-\dfrac{1}{2}\right\}\cup[1,\infty)$

49. $[-2,2]$

50. $(-\infty,-1)\cup(-1,0)\cup(2,\infty)$

51. $(-\infty,-2)\cup\left(-\sqrt{2},\sqrt{2}\right)$

52. $\{2\}\cup[4,\infty)$

53. $(-\infty,-\sqrt{3})\cup(\sqrt{3},\infty)$

54. $(-\infty,-\sqrt[3]{3})\cup(\sqrt[3]{2},\infty)$

55. $f(x)<0$ on $(-\infty,-6)\cup(0,6)$

56. $g(t)>0$ on $(-\infty,-2)\cup(0,\infty)$

57. $p(z)\geq 0$ on $(-\infty,-1]\cup\{2\}$

58. $f(x)<0$ on $\left(-\infty,-\frac{1}{2}\right)\cup\left(-\frac{1}{2},3\right)$

59. $F(s)\leq 0$ on $\{-2\}\cup[0,\infty)$

60. $G(t)\geq 0$ on $\{-2\}\cup[0,\infty)$

61. Since $f(-4)=-23$, $f(-3)=5$, $f(0)=5$, $f(1)=-3$, $f(2)=-5$ and $f(3)=5$ the Intermediate Value Theorem gives that $f(x)=x^3-9x+5$ has real zeros in the intervals $[-4,-3]$, $[0,1]$ and $[2,3]$.

62. An odd degree polynomial function f has 'mismatched' end behavior. That is, the end behavior of $f(x)$ is either: $\lim\limits_{x\to-\infty} f(x)=-\infty$ and $\lim\limits_{x\to\infty} f(x)=\infty$ or $\lim\limits_{x\to-\infty} f(x)=\infty$ and $\lim\limits_{x\to\infty} f(x)=-\infty$. This means at some point, $f(x)>0$ and at some other point $f(x)<0$. The Intermediate Value Theorem guarantees at least one place where $f(x)=0$.

63. The function $f(x)=x^2+1$ has no real zeros.

64. $x\approx 1.167$.

65. (a) $f(x)$ has only one variation in sign, so the result follows from Descartes' Rule of Signs.
    (b) $f(1)=-1<0$ and $f(2)=2>0$ so the Intermediate Value Theorem promises a zero in $[1,2]$.
    (c) The Rational Zeros Theorem gives the only possible rational zeros of f are ± 1 and ± 2. Since $f(\pm 1)=-1$ and $f(\pm 2)=2$, f has no rational zeros.
    (d) The zero of f is $\sqrt{2}\approx 1.414$.

66. $V(x) \geq 80$ on $[1, 5-\sqrt{5}] \cup [5+\sqrt{5}, \infty)$. Only the portion $[1, 5-\sqrt{5}]$ lies in the applied domain, however. In the context of the problem, this says for the volume of the box to be at least 80 cubic inches, the square removed from each corner needs to have a side length of at least 1 inch, but no more than $5-\sqrt{5} \approx 2.76$ inches.

67. $C(x) \leq 5000$ on (approximately) $(-\infty, 82.18]$. The portion of this which lies in the applied domain is $(0, 82.18]$. Since x represents the number of game systems, we check $C(82) = 4983.04$ and $C(83) = 5078.11$, so to remain within the production budget, anywhere between 1 and 82 game systems can be produced.

2.4 Complex Zeros and the Fundamental Theorem of Algebra

In Section 2.3, we were focused on finding the real zeros of a polynomial function. In this section, we expand our horizons and look for the non-real zeros as well. By 'non-real' here we mean we will be discussing 'imaginary,' and, more generally, 'complex' numbers. Even though the monikers 'non-real' and 'imaginary' suggests these numbers play no role in 'real' world applications, we assure you that electrical engineers live a 'complex' life and these numbers are invaluable to them.[1] That being said, our main use of complex numbers in this section is to present some powerful structure theorems for polynomial functions (this is, after all, a math book!) For a detailed review of the Complex Number system, we refer the reader to Section A.11. For us, it suffices to review the basic vocabulary.

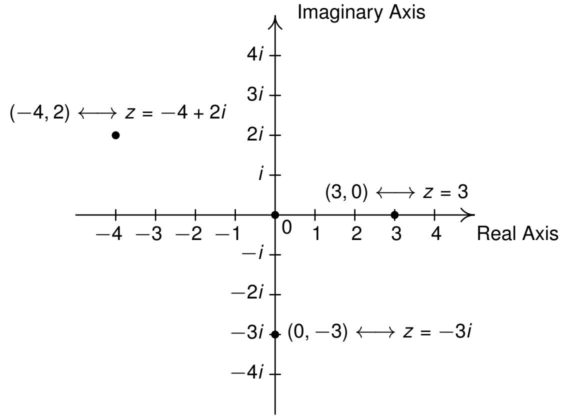
- The imaginary unit $i = \sqrt{-1}$ satisfies the two following properties
  1. $i^2 = -1$
  2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$
- The **complex numbers** are the set of numbers $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$
- Given a complex number $z = a + bi$, the **complex conjugate** of z, $\overline{z} = \overline{a + bi} = a - bi$.

Note that every real number is a complex number, that is $\mathbb{R} \subseteq \mathbb{C}$. To see this, take your favorite real number, say 117. We may write $117 = 117 + 0i$ which puts in the form $a + bi$. Hence, we we speak of the 'complex zeros' of a polynomial function, we are talking about not just the non-real, but also the real zeros.

Complex numbers, by their very definition, are two dimensional creatures. To see this, we may identify a complex number $z = a+bi$ with the point in the Cartesian plane (a, b). The horizontal axis is called the 'real' axis since points here have the form $(a, 0)$ which corresponds to numbers of the form $z = a + 0i = a$ which are the real numbers. The vertical axis is called the 'imaginary' axis since points here are of the form $(0, b)$ which correspond to numbers of the form $z = 0 + bi = bi$, the so-called 'purely imaginary' numbers. Below we plot some complex numbers on this so-called 'Complex Plane.' Plotting a set of complex numbers this way is called an Argand Diagram, and opens up a wealth of opportunities to explore many algebraic properties of complex numbers geometrically. For example, complex conjugation amounts to a reflection about the real axis, and multiplication by i amounts to a 90° rotation.[2] While we won't have much use for the Complex Plane in this section, it is worth introducing this concept now, if, for no other reason, it gives the reader a sense of the vastness of the complex number system and the role of the real numbers in it.

[1] Even a cursory web search for 'use of imaginary numbers in electrical engineering' provides a wealth of source material - enough to convince anyone of their importance to the field (pun intended.) Most of it, however, requires more electrical background than the authors feel comfortable including in the text. Be aware, however, that in electrical applications, the letter j is used to represent $\sqrt{-1}$ since the letter i is reserved for current.

[2] See Exercises 43 - 46.



The Complex Plane

Returning to zeros of polynomials, suppose we wish to find the zeros of $f(x) = x^2 - 2x + 5$. To solve the equation $x^2 - 2x + 5 = 0$, we note that the quadratic doesn't factor nicely, so we resort to the Quadratic Formula, Equation 1.3 and obtain

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Two things are important to note. First, the zeros $1 + 2i$ and $1 - 2i$ are complex conjugates. If ever we obtain non-real zeros to a quadratic function with **real number** coefficients, the zeros will be a complex conjugate pair. (Do you see why?)

We could ask if all of the theory from Section2.2 holds for non-real zeros, in particular the division algorithm and the Remainder and Factor Theorems. The answer is 'yes.'

$$\begin{array}{r|rrr} 1+2i & 1 & -2 & 5 \\ & \downarrow & 1+2i & -5 \\ \hline & 1 & -1+2i & \boxed{0} \end{array}$$

Indeed, the above shows $x^2 - 2x + 5 = (x - [1 + 2i])(x - 1 + 2i) = (x - [1 + 2i])(x - [1 - 2i])$ which demonstrates both $(x - [1 + 2i])$ and $(x - [1 - 2i])$ are factors of $x^2 - 2x + 5$.[3]

But how do we know if a general polynomial has any complex zeros at all? We have many examples of polynomials with no real zeros. Can there be polynomials with no zeros whatsoever? The answer to that last question is "No." and the theorem which provides that answer is The Fundamental Theorem of Algebra.

Theorem 2.15. The Fundamental Theorem of Algebra: Suppose f is a polynomial function with complex number coefficients of degree $n \geq 1$, then f has at least one complex zero.

The Fundamental Theorem of Algebra is an example of an 'existence' theorem in Mathematics. Like the Intermediate Value Theorem, Theorem 2.14, the Fundamental Theorem of Algebra guarantees the existence of at least one zero, but gives us no algorithm to use in finding it. In fact, as we mentioned in Section 2.3, there are polynomials whose real zeros, though they exist, cannot be expressed using

[3]It is a good review of the algebra of complex numbers to start with $(x - [1+2i])(x - [1-2i])$, perform the indicated operations, and simplify the result to $x^2 - 2x + 5$. See part 6 of Example A.11.1 in Section A.11.

the 'usual' combinations of arithmetic symbols, and must be approximated. It took mathematicians literally hundreds of years to prove the theorem in its full generality,[4] and some of that history is recorded here. Note that the Fundamental Theorem of Algebra applies to not only polynomial functions with real coefficients, but to those with complex number coefficients as well.

Suppose f is a polynomial function of degree $n \geq 1$. The Fundamental Theorem of Algebra guarantees us at least one complex zero, z_1. The Factor Theorem guarantees that $f(x)$ factors as $f(x) = (x - z_1)\, q_1(x)$ for a polynomial function q_1, which has degree $n - 1$. If $n - 1 \geq 1$, then the Fundamental Theorem of Algebra guarantees a complex zero of q_1 as well, say z_2, so then the Factor Theorem gives us $q_1(x) = (x - z_2)\, q_2(x)$, and hence $f(x) = (x - z_1)(x - z_2)\, q_2(x)$. We can continue this process exactly n times, at which point our quotient polynomial q_n has degree 0 so it's a constant. This constant is none-other than the leading coefficient of f which is carried down line by line each time we divide by factors of the form $x - c$.

Theorem 2.16. Complex Factorization Theorem: Suppose f is a polynomial function with complex number coefficients. If the degree of f is n and $n \geq 1$, then f has exactly n complex zeros, counting multiplicity. If $z_1, z_2, \ldots, z_k$ are the distinct zeros of f, with multiplicities $m_1, m_2, \ldots, m_k$, respectively, then $f(x) = a\,(x - z_1)^{m_1}\,(x - z_2)^{m_2} \cdots (x - z_k)^{m_k}$.

Theorem 2.16 says two important things: first, every polynomial is a product of linear factors; second, every polynomial function is completely determined by its zeros, their multiplicities, and its leading coefficient. We put this theorem to good use in the next example.

Example 2.4.1. Let $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$.

1. Find all of the complex zeros of f and state their multiplicities.
2. Factor $f(x)$ using Theorem 2.16

Solution.

1. Since f is a fifth degree polynomial, we know that we need to perform at least three successful divisions to get the quotient down to a quadratic function. At that point, we can find the remaining zeros using the Quadratic Formula, if necessary. Using the techniques developed in Section 2.3:

$$\begin{array}{r|rrrrrr}
\frac{1}{2} & 12 & -20 & 19 & -6 & -2 & 1 \\
 & \downarrow & 6 & -7 & 6 & 0 & -1 \\
\hline
\frac{1}{2} & 12 & -14 & 12 & 0 & -2 & \boxed{0} \\
 & \downarrow & 6 & -4 & 4 & 2 & \\
\hline
-\frac{1}{3} & 12 & -8 & 8 & 4 & \boxed{0} & \\
 & \downarrow & -4 & 4 & -4 & & \\
\hline
 & 12 & -12 & 12 & \boxed{0} & &
\end{array}$$

   Our quotient is $12x^2 - 12x + 12$, whose zeros we find to be $\frac{1 \pm i\sqrt{3}}{2}$. From Theorem 2.16, we know f has exactly 5 zeros, counting multiplicities, and as such we have the zero $\frac{1}{2}$ with multiplicity 2, and the zeros $-\frac{1}{3}$, $\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$, each of multiplicity 1.

[4]So if its profound nature and beautiful subtlety escape you, no worries!

2. Applying Theorem 2.16, we are guaranteed that f factors as

$$f(x) = 12\left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right) \left(x - \left[\frac{1 + i\sqrt{3}}{2}\right]\right) \left(x - \left[\frac{1 - i\sqrt{3}}{2}\right]\right) \qquad \Box$$

A true test of Theorem 2.16 would be to take the factored form of $f(x)$ in the previous example and multiply it out[5] to see that it really does reduce to $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$. When factoring a polynomial using Theorem 2.16, we say that it is **factored completely over the complex numbers**, meaning that it is impossible to factor the polynomial any further using complex numbers. If we wanted to completely factor $f(x)$ over the **real numbers** then we would have stopped short of finding the nonreal zeros of f and factored f using our work from the synthetic division to write $f(x) = \left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right) \left(12x^2 - 12x + 12\right)$, or $f(x) = 12\left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right) \left(x^2 - x + 1\right)$. Since the zeros of $x^2 - x + 1$ are nonreal, we call $x^2 - x + 1$ an **irreducible quadratic** meaning it is impossible to break it down any further using **real** numbers.

The last two results of the section show us that, theoretically, the non-real zeros of polynomial functions with real number coefficients come exclusively from irreducible quadratics.

Theorem 2.17. Conjugate Pairs Theorem: If f is a polynomial function with real number coefficients and z is a complex zero of f, then so is $\overline{z}$.

To prove the theorem, let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ be a polynomial function with real number coefficients. If z is a zero of f, then $f(z) = 0$, which means $a_n z^n + a_{n-1}z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0 = 0$. Next, we consider $f\left(\overline{z}\right)$ and apply Theorem A.13 below.

$$\begin{array}{rcll}
f\left(\overline{z}\right) & = & a_n\left(\overline{z}\right)^n + a_{n-1}\left(\overline{z}\right)^{n-1} + \ldots + a_2\left(\overline{z}\right)^2 + a_1\overline{z} + a_0 & \\
& = & a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \ldots + a_2\overline{z^2} + a_1\overline{z} + a_0 & \text{since } \left(\overline{z}\right)^n = \overline{z^n} \\
& = & \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \ldots + \overline{a_2}\,\overline{z^2} + \overline{a_1}\,\overline{z} + \overline{a_0} & \text{since the coefficients are real} \\
& = & \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \ldots + \overline{a_2 z^2} + \overline{a_1 z} + \overline{a_0} & \text{since } \overline{z}\,\overline{w} = \overline{zw} \\
& = & \overline{a_n z^n + a_{n-1}z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0} & \text{since } \overline{z} + \overline{w} = \overline{z + w} \\
& = & \overline{f(z)} & \\
& = & \overline{0} & \\
& = & 0 &
\end{array}$$

This shows that $\overline{z}$ is a zero of f. So, if f is a polynomial function with real number coefficients, Theorem 2.17 tells us that if $a + bi$ is a nonreal zero of f, then so is $a - bi$. In other words, nonreal zeros of f come in conjugate pairs. The Factor Theorem kicks in to give us both $(x - [a + bi])$ and $(x - [a - bi])$ as factors of $f(x)$ which means $(x - [a + bi])(x - [a - bi]) = x^2 + 2ax + \left(a^2 + b^2\right)$ is an irreducible quadratic factor of f. As a result, we have our last theorem of the section.

Theorem 2.18. Real Factorization Theorem: Suppose f is a polynomial function with real number coefficients. Then $f(x)$ can be factored into a product of linear factors corresponding to the real zeros of f and irreducible quadratic factors which give the nonreal zeros of f.

We now present an example which pulls together all of the major ideas of this section.

[5]This is a good chance to test your algebraic mettle and see that all of this does actually work.

Example 2.4.2. Let $f(x) = x^4 + 64$.

1. Use synthetic division to show that $x = 2 + 2i$ is a zero of f.
2. Find the remaining complex zeros of f.
3. Completely factor $f(x)$ over the complex numbers.
4. Completely factor $f(x)$ over the real numbers.

Solution.

1. Remembering to insert the 0's in the synthetic division tableau we have

$$\begin{array}{r|rrrrr} 2+2i & 1 & 0 & 0 & 0 & 64 \\ & \downarrow & 2+2i & 8i & -16+16i & -64 \\ \hline & 1 & 2+2i & 8i & -16+16i & \boxed{0} \end{array}$$

2. Since f is a fourth degree polynomial, we need to make two successful divisions to get a quadratic quotient. Since $2 + 2i$ is a zero, we know from Theorem 2.17 that $2 - 2i$ is also a zero. We continue our synthetic division tableau.

$$\begin{array}{r|rrrrr} 2+2i & 1 & 0 & 0 & 0 & 64 \\ & \downarrow & 2+2i & 8i & -16+16i & -64 \\ \hline 2-2i & 1 & 2+2i & 8i & -16+16i & \boxed{0} \\ & \downarrow & 2-2i & 8-8i & 16-16i & \\ \hline & 1 & 4 & 8 & \boxed{0} & \end{array}$$

   Our quotient polynomial is $x^2 + 4x + 8$. Using the quadratic formula, we solve $x^2 + 4x + 8 = 0$ and find the remaining zeros are $-2 + 2i$ and $-2 - 2i$.

3. Using Theorem 2.16, we get $f(x) = (x - [2 - 2i])(x - [2 + 2i])(x - [-2 + 2i])(x - [-2 - 2i])$.

4. To find the irreducible quadratic factors of $f(x)$, we multiply the factors together which correspond to the conjugate pairs. We find $(x - [2 - 2i])(x - [2 + 2i]) = x^2 - 4x + 8$, and $(x - [-2 + 2i])(x - [-2 - 2i]) = x^2 + 4x + 8$, so $f(x) = \left(x^2 - 4x + 8\right)\left(x^2 + 4x + 8\right)$. □

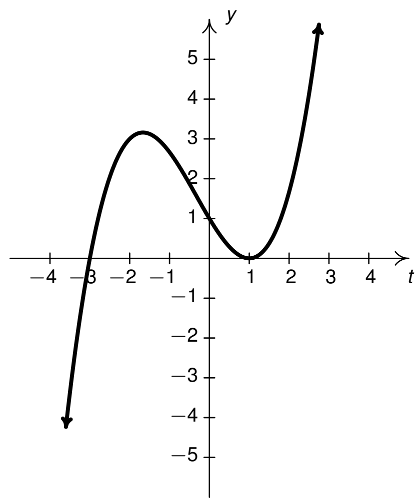
We close this section with an example where we are asked to manufacture a polynomial function with certain characteristics.

Example 2.4.3. 1. Find a polynomial function p of lowest degree that has integer coefficients and satisfies all of the following criteria:

- the graph of $y = p(x)$ touches and rebounds from the x-axis at $\left(\frac{1}{3}, 0\right)$
- $x = 3i$ is a zero of p.

- $\lim\limits_{x\to-\infty} p(x) = -\infty$
- $\lim\limits_{x\to\infty} p(x) = -\infty$

2. Find a possible formula for the polynomial function p graphed below. You may leave your answer in factored form.



Solution.

1. To solve this problem, we will need a good understanding of the relationship between the x-intercepts of the graph of a function and the zeros of a function, the Factor Theorem, the role of multiplicity, complex conjugates, the Complex Factorization Theorem, and end behavior of polynomial functions. (In short, you'll need most of the major concepts of this chapter.) Since the graph of p touches the x-axis at $\left(\frac{1}{3}, 0\right)$, we know $x = \frac{1}{3}$ is a zero of even multiplicity. Since we are after a polynomial of lowest degree, we need $x = \frac{1}{3}$ to have multiplicity exactly 2. The Factor Theorem now tells us $\left(x - \frac{1}{3}\right)^2$ is a factor of $p(x)$. Since $x = 3i$ is a zero and our final answer is to have integer (hence, real) coefficients, $x = -3i$ is also a zero. The Factor Theorem kicks in again to give us $(x-3i)$ and $(x+3i)$ as factors of $p(x)$. We are given no further information about zeros or intercepts so we conclude, by the Complex Factorization Theorem that $p(x) = a\left(x - \frac{1}{3}\right)^2 (x - 3i)(x + 3i)$ for some real number a. Expanding this, we get $p(x) = ax^4 - \frac{2a}{3}x^3 + \frac{82a}{9}x^2 - 6ax + a$. In order to obtain integer coefficients, we know a must be an integer multiple of 9. Our last concern is end behavior. Since the leading term of $p(x)$ is ax^4, we need $a < 0$ to get $\lim\limits_{x\to-\infty} p(x) = -\infty$ and $\lim\limits_{x\to\infty} p(x) = -\infty$. Hence, if we choose $x = -9$, we get $p(x) = -9x^4 + 6x^3 - 82x^2 + 54x - 9$. We can verify our handiwork using the techniques developed in this chapter.

2. The first thing to note is the independent variable here is t, not x as evidenced by the labeling on the horizontal axis. Next, the graph appears to cross through the t-axis at $(-3, 0)$ in a fairly linear fashion, so $t = -3$ is likely a zero of multiplicity 1. Also, the graph touches and rebounds at $(1, 0)$,

indicating $t = 1$ is a zero of even multiplicity. Since the graph doesn't appear too 'flat,' we'll go with multiplicity 2 (though there is really no way of telling.) Using the Complex Factorization Theorem and assuming we have no non-real zeros, we now have $p(t) = a(t-(-3))(t-1)^2 = a(t+3)(t-1)^2$. To determine the leading coefficient, a, we note the graph appears to have a y-intercept at $(0,1)$. Solving $p(0) = 1$ gives $a(3)(-1)^2 = 1$ or $3a = 1$. Hence, $a = \frac{1}{3}$ so $p(t) = \frac{1}{3}(t+3)(t-1)^2$. Since we may leave our answer in factored form, we are done. $\square$

This example concludes our study of polynomial functions.[6] The last few sections have contained what is considered by many to be 'heavy' Mathematics. Like a heavy meal, heavy Mathematics takes time to digest. Don't be overly concerned if it doesn't seem to sink in all at once, and pace yourself in the Exercises or you're liable to get mental cramps. But before we get to the Exercises, we'd like to offer a bit of an epilogue.

Our main goal in presenting the material on the complex zeros of a polynomial was to give the chapter a sense of completeness. Given that it can be shown that some polynomials have real zeros which cannot be expressed using the usual algebraic operations, and still others have no real zeros at all, it was nice to discover that every polynomial of degree $n \geq 1$ has n complex zeros. So like we said, it gives us a sense of closure.[7] As mentioned at the top of the section, complex numbers are very useful in many applied fields such as electrical engineering, but most of the applications require science and mathematics well beyond precalculus material to fully understand them. That does not mean you'll never be be able to understand them; in fact, it is the authors' sincere hope that all of you will reach a point in your studies when the glory, awe and splendor of complex numbers are revealed to you. For now, however, the really good stuff is beyond the scope of this text. We invite you and your classmates to find a few examples of complex number applications and see what you can make of them.

For the remainder of the text, with the exception of Section 14.3 and a few exploratory exercises scattered about, we will restrict our attention to real numbers. We do this primarily because the first Calculus sequence you will take, ostensibly the one that this text is preparing you for, studies only functions of real variables. Also, lots of really cool scientific things don't require any deep understanding of complex numbers to study them, but they do need more Mathematics like exponential, logarithmic and trigonometric functions. We believe it makes more sense pedagogically for you to learn about those functions now then take a course in Complex Function Theory in your junior or senior year once you've completed the Calculus sequence. It is in that course that the true power of the complex numbers is released. But for now, in order to fully prepare you for life immediately after Precalculus, we will say that functions like $f(x) = \frac{1}{x^2+1}$, which we'll study in the very next chapter, have a domain of all real numbers, even though we know $x^2 + 1 = 0$ has two complex solutions, namely $x = \pm i$ which produce a '0' in the denominator. Since $x^2 + 1 > 0$ for all **real** numbers x, the fraction $\frac{1}{x^2+1}$ is never undefined in the real variable setting.

[6] With the exception of the Exercises on the next page, of course.
[7] This is a very deep math pun.

2.4.1 Exercises

In Exercises 1 - 22, find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.

1. $f(x) = x^2 - 4x + 13$
2. $f(x) = x^2 - 2x + 5$
3. $p(z) = 3z^2 + 2z + 10$
4. $p(z) = z^3 - 2z^2 + 9z - 18$
5. $g(t) = t^3 + 6t^2 + 6t + 5$
6. $g(t) = 3t^3 - 13t^2 + 43t - 13$
7. $f(x) = x^3 + 3x^2 + 4x + 12$
8. $f(x) = 4x^3 - 6x^2 - 8x + 15$
9. $p(z) = z^3 + 7z^2 + 9z - 2$
10. $p(z) = 9z^3 + 2z + 1$
11. $g(t) = 4t^4 - 4t^3 + 13t^2 - 12t + 3$
12. $g(t) = 2t^4 - 7t^3 + 14t^2 - 15t + 6$
13. $f(x) = x^4 + x^3 + 7x^2 + 9x - 18$
14. $f(x) = 6x^4 + 17x^3 - 55x^2 + 16x + 12$
15. $p(z) = -3z^4 - 8z^3 - 12z^2 - 12z - 5$
16. $p(z) = 8z^4 + 50z^3 + 43z^2 + 2z - 4$
17. $g(t) = t^4 + 9t^2 + 20$
18. $g(t) = t^4 + 5t^2 - 24$
19. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$
20. $f(x) = x^6 - 64$
21. $f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26$ (Hint: $x = i$ is one of the zeros.)
22. $p(z) = 2z^4 + 5z^3 + 13z^2 + 7z + 5$ (Hint: $z = -1 + 2i$ is a zero.)

In Exercises 23 - 32, use Theorem 2.16 to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.

23.
- The zeros of f are $c = \pm 2$ and $c = \pm 1$.
- The leading term of $f(x)$ is $117x^4$.

24.
- The zeros of p are $c = 1$ and $c = 3$.
- $c = 3$ is a zero of multiplicity 2.
- The leading term of $p(z)$ is $-5z^3$.

25.
- The solutions to $g(t) = 0$ are $t = \pm 3$ and $t = 6$.
- The leading term of $g(t)$ is $7t^4$.
- The point $(-3, 0)$ is a local minimum on the graph of $y = g(t)$.

26.
- The solutions to $f(x) = 0$ are $x = \pm 3$, $x = -2$, and $x = 4$.
- The leading term of $f(x)$ is $-x^5$.
- The point $(-2, 0)$ is a local maximum on the graph of $y = f(x)$.

27.
- p is degree 4.
- $\lim_{z \to \infty} p(z) = -\infty$.
- p has exactly three z-intercepts: $(-6, 0)$, $(1, 0)$ and $(117, 0)$.
- The graph of $y = p(z)$ crosses through the z-axis at $(1, 0)$.

28.
- The zeros of g are $c = \pm 1$ and $c = \pm i$.
- The leading term of $g(t)$ is $42t^4$.

29.
- $c = 2i$ is a zero.
- the point $(-1, 0)$ is a local minimum on the graph of $y = f(x)$.
- the leading term of $f(x)$ is $117x^4$.

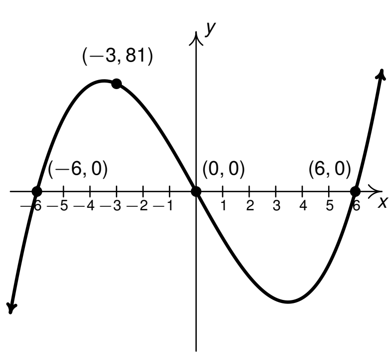
30.
- The solutions to $p(z) = 0$ are $z = \pm 2$ and $z = \pm 7i$.
- The leading term of $p(z)$ is $-3z^5$.
- The point $(2, 0)$ is a local maximum on the graph of $y = p(z)$.

31.
- g is degree 5.
- $t = 6$, $t = i$ and $t = 1 - 3i$ are zeros of g.
- $\lim_{t \to -\infty} g(t) = \infty$

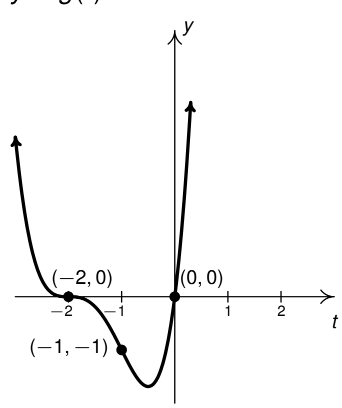
32.
- The leading term of $f(x)$ is $-2x^3$.
- $c = 2i$ is a zero.
- $f(0) = -16$.

In Exercises 33 - 42, find a possible formula for the polynomial function given its graph. You may leave the polynomial in factored form.

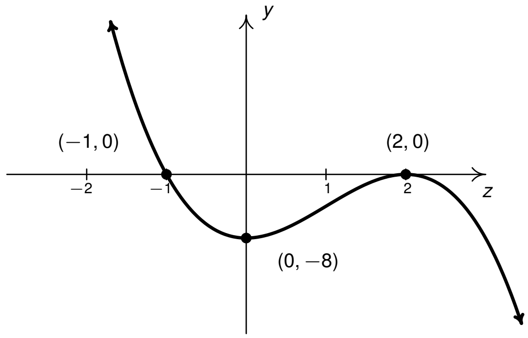
33. $y = f(x)$.



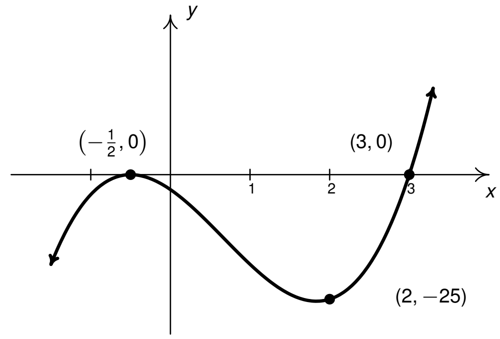
34. $y = g(t)$



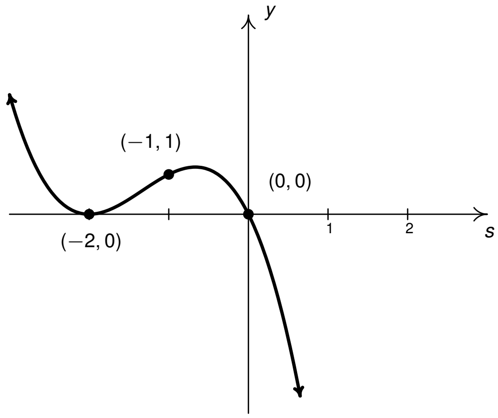
35. $y = p(z)$



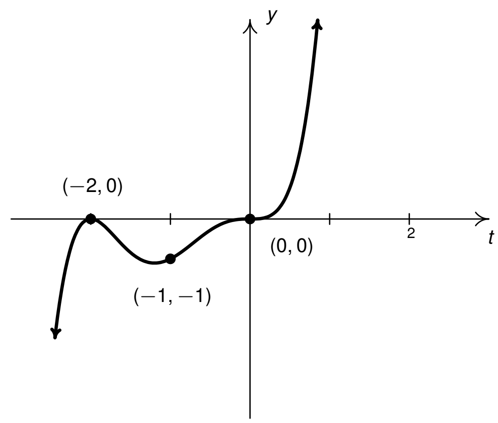
36. $y = f(x)$



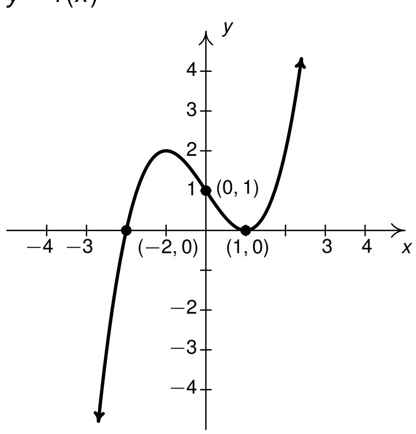
37. $y = F(s)$



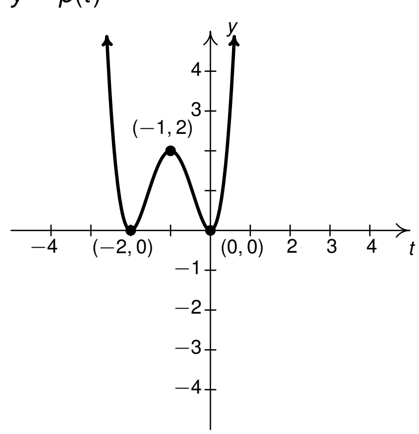
38. $y = G(t)$



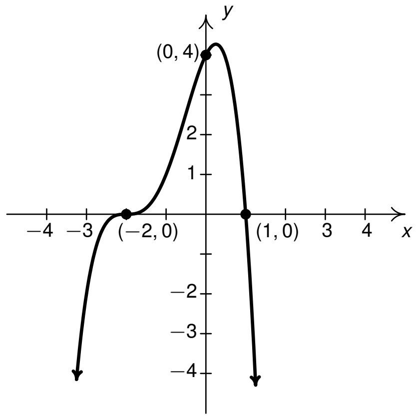
39. $y = f(x)$



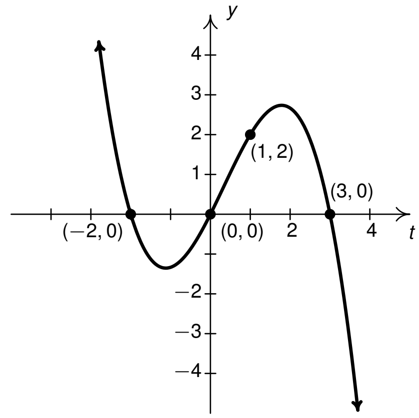
40. $y = p(t)$



41. $y = f(x)$



42. $y = p(t)$



43. With help from your classmates, choose several nonzero complex numbers z, find their complex conjugates $\overline{z}$. Plot each pair z and $\overline{z}$ in the Complex Plane. What appears to be the relationship between these numbers geometrically? State and prove a general result.

44. With help from your classmates, choose several nonzero complex numbers z and find $-z$. Plot each pair z and $-z$ in the Complex Plane. What appears to be the relationship between these numbers geometrically? State and prove a general result.

45. With help from your classmates, choose several different complex numbers z and find the product of i and z, iz. Plot each pair of z and iz in the Complex Plane. In each case, show the line containing the origin and the point corresponding to z is perpendicular[8] to the line containing the origin and the point corresponding to iz. Show this result holds in general for every nonzero complex number.

46. Given a complex number $z = a + bi$, we define the **modulus** of z, $|z|$, by $|z| = \sqrt{a^2 + b^2}$. With help from your classmates, calculate $|z|$ for several different complex numbers, z. What does z measure geometrically? Show that if x is a real number, then the modulus of x is the same as the absolute value of x, and comment how all this relates to Definition A.14 in Section A.7.

47. Let z and w be arbitrary complex numbers. Show that $\overline{z}\,\overline{w} = \overline{zw}$ and $\overline{\overline{z}} = z$.

[8]See Theorem A.3 in Section A.5 of you need a refresher on how to do this.

2.4.2 Answers

1. $f(x) = x^2 - 4x + 13 = (x - (2 + 3i))(x - (2 - 3i))$
   Zeros: $x = 2 \pm 3i$

2. $f(x) = x^2 - 2x + 5 = (x - (1 + 2i))(x - (1 - 2i))$
   Zeros: $x = 1 \pm 2i$

3. $p(z) = 3z^2 + 2z + 10 = 3\left(z - \left(-\frac{1}{3} + \frac{\sqrt{29}}{3}i\right)\right)\left(z - \left(-\frac{1}{3} - \frac{\sqrt{29}}{3}i\right)\right)$ Zeros: $z = -\frac{1}{3} \pm \frac{\sqrt{29}}{3}i$

4. $p(z) = z^3 - 2z^2 + 9z - 18 = (z - 2)\left(z^2 + 9\right) = (z - 2)(z - 3i)(z + 3i)$
   Zeros: $z = 2, \pm 3i$

5. $g(t) = t^3 + 6t^2 + 6t + 5 = (t + 5)(t^2 + t + 1) = (t + 5)\left(t - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(t - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$
   Zeros: $t = -5,\ t = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

6. $g(t) = 3t^3 - 13t^2 + 43t - 13 = (3t - 1)(t^2 - 4t + 13) = (3t - 1)(t - (2 + 3i))(t - (2 - 3i))$
   Zeros: $t = \frac{1}{3},\ t = 2 \pm 3i$

7. $f(x) = x^3 + 3x^2 + 4x + 12 = (x + 3)\left(x^2 + 4\right) = (x + 3)(x + 2i)(x - 2i)$
   Zeros: $x = -3,\ \pm 2i$

8. $f(x) = 4x^3 - 6x^2 - 8x + 15 = \left(x + \frac{3}{2}\right)\left(4x^2 - 12x + 10\right)$
   $= 4\left(x + \frac{3}{2}\right)\left(x - \left(\frac{3}{2} + \frac{1}{2}i\right)\right)\left(x - \left(\frac{3}{2} - \frac{1}{2}i\right)\right)$
   Zeros: $x = -\frac{3}{2},\ x = \frac{3}{2} \pm \frac{1}{2}i$

9. $p(z) = z^3 + 7z^2 + 9z - 2 = (z + 2)\left(z - \left(-\frac{5}{2} + \frac{\sqrt{29}}{2}\right)\right)\left(z - \left(-\frac{5}{2} - \frac{\sqrt{29}}{2}\right)\right)$
   Zeros: $z = -2,\ z = -\frac{5}{2} \pm \frac{\sqrt{29}}{2}$

10. $p(z) = 9z^3 + 2z + 1 = \left(z + \frac{1}{3}\right)\left(9z^2 - 3z + 3\right)$
    $= 9\left(z + \frac{1}{3}\right)\left(z - \left(\frac{1}{6} + \frac{\sqrt{11}}{6}i\right)\right)\left(z - \left(\frac{1}{6} - \frac{\sqrt{11}}{6}i\right)\right)$
    Zeros: $z = -\frac{1}{3},\ z = \frac{1}{6} \pm \frac{\sqrt{11}}{6}i$

11. $g(t) = 4t^4 - 4t^3 + 13t^2 - 12t + 3 = \left(t - \frac{1}{2}\right)^2\left(4t^2 + 12\right) = 4\left(t - \frac{1}{2}\right)^2(t + i\sqrt{3})(t - i\sqrt{3})$
    Zeros: $t = \frac{1}{2},\ t = \pm\sqrt{3}i$

12. $g(t) = 2t^4 - 7t^3 + 14t^2 - 15t + 6 = (t - 1)^2\left(2t^2 - 3t + 6\right)$
    $= 2(t - 1)^2\left(t - \left(\frac{3}{4} + \frac{\sqrt{39}}{4}i\right)\right)\left(t - \left(\frac{3}{4} - \frac{\sqrt{39}}{4}i\right)\right)$
    Zeros: $t = 1,\ t = \frac{3}{4} \pm \frac{\sqrt{39}}{4}i$

13. $f(x) = x^4 + x^3 + 7x^2 + 9x - 18 = (x + 2)(x - 1)\left(x^2 + 9\right) = (x + 2)(x - 1)(x + 3i)(x - 3i)$
    Zeros: $x = -2,\ 1,\ \pm 3i$

14. $f(x) = 6x^4 + 17x^3 - 55x^2 + 16x + 12 = 6\left(x + \frac{1}{3}\right)\left(x - \frac{3}{2}\right)\left(x - \left(-2 + 2\sqrt{2}\right)\right)\left(x - \left(-2 - 2\sqrt{2}\right)\right)$
    Zeros: $x = -\frac{1}{3},\ x = \frac{3}{2},\ x = -2 \pm 2\sqrt{2}$

15. $p(z) = -3z^4 - 8z^3 - 12z^2 - 12z - 5 = (z+1)^2\left(-3z^2 - 2z - 5\right)$
 $= -3(z+1)^2\left(z - \left(-\frac{1}{3} + \frac{\sqrt{14}}{3}i\right)\right)\left(z - \left(-\frac{1}{3} - \frac{\sqrt{14}}{3}i\right)\right)$
 Zeros: $z = -1,\ z = -\frac{1}{3} \pm \frac{\sqrt{14}}{3}i$

16. $p(z) = 8z^4 + 50z^3 + 43z^2 + 2z - 4 = 8\left(z + \frac{1}{2}\right)\left(z - \frac{1}{4}\right)(z - (-3 + \sqrt{5}))(z - (-3 - \sqrt{5}))$
 Zeros: $z = -\frac{1}{2},\ \frac{1}{4},\ z = -3 \pm \sqrt{5}$

17. $g(t) = t^4 + 9t^2 + 20 = \left(t^2 + 4\right)\left(t^2 + 5\right) = (t - 2i)(t + 2i)\left(t - i\sqrt{5}\right)\left(t + i\sqrt{5}\right)$
 Zeros: $t = \pm 2i, \pm i\sqrt{5}$

18. $g(t) = t^4 + 5t^2 - 24 = \left(t^2 - 3\right)\left(t^2 + 8\right) = (t - \sqrt{3})(t + \sqrt{3})\left(t - 2i\sqrt{2}\right)\left(t + 2i\sqrt{2}\right)$
 Zeros: $t = \pm\sqrt{3}, \pm 2i\sqrt{2}$

19. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12 = (x - 1)\left(x^2 + 3\right)\left(x^2 + 4\right)$
 $= (x - 1)(x - i\sqrt{3})(x + i\sqrt{3})(x - 2i)(x + 2i)$
 Zeros: $x = 1,\ \pm\sqrt{3}i,\ \pm 2i$

20. $f(x) = x^6 - 64 = (x - 2)(x + 2)\left(x^2 + 2x + 4\right)\left(x^2 - 2x + 4\right)$
 $= (x - 2)(x + 2)\left(x - \left(-1 + i\sqrt{3}\right)\right)\left(x - \left(-1 - i\sqrt{3}\right)\right)\left(x - \left(1 + i\sqrt{3}\right)\right)\left(x - \left(1 - i\sqrt{3}\right)\right)$
 Zeros: $x = \pm 2,\ x = -1 \pm i\sqrt{3},\ x = 1 \pm i\sqrt{3}$

21. $f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26 = (x^2 - 2x + 26)(x^2 + 1) = (x - (1 + 5i))(x - (1 - 5i))(x + i)(x - i)$
 Zeros: $x = 1 \pm 5i,\ x = \pm i$

22. $p(z) = 2z^4 + 5z^3 + 13z^2 + 7z + 5 = \left(z^2 + 2z + 5\right)\left(2z^2 + z + 1\right)$
 $= 2(z - (-1 + 2i))(z - (-1 - 2i))\left(z - \left(-\frac{1}{4} + i\frac{\sqrt{7}}{4}\right)\right)\left(z - \left(-\frac{1}{4} - i\frac{\sqrt{7}}{4}\right)\right)$
 Zeros: $z = -1 \pm 2i, -\frac{1}{4} \pm i\frac{\sqrt{7}}{4}$

23. $f(x) = 117(x + 2)(x - 2)(x + 1)(x - 1)$

24. $p(z) = -5(z - 1)(z - 3)^2$

25. $g(t) = 7(t + 3)^2(t - 3)(t - 6)$

26. $f(x) = -(x + 2)^2(x - 3)(x + 3)(x - 4)$

27. $p(z) = a(z + 6)^2(z - 1)(z - 117)$ where a can be any real number as long as $a < 0$

28. $g(t) = 42(t - 1)(t + 1)(t - i)(t + i)$

29. $f(x) = 117(x + 1)^2(x - 2i)(x + 2i)$

30. $p(z) = -3(z - 2)^2(z + 2)(z - 7i)(z + 7i)$

31. $g(t) = a(t - 6)(t - i)(t + i)(t - (1 - 3i))(t - (1 + 3i))$ where a is any real number, $a < 0$

32. $f(x) = -2(x - 2i)(x + 2i)(x + 2)$

33. $f(x) = x(x+6)(x-6)$

34. $g(t) = t(t+2)^3$

35. $p(z) = -2(z+1)(z-2)^2$

36. $f(x) = 4\left(x+\frac{1}{2}\right)^2(x-3)$

37. $F(s) = -s(s+2)^2$

38. $G(t) = t^3(t+2)^2$

39. $f(x) = \frac{1}{2}(x-1)^2(x+2)$

40. $p(t) = 2t^2(t+2)^2$

41. $f(x) = -\frac{1}{2}(x-1)(x+2)^3$

42. $p(t) = -\frac{1}{3}t(t+2)(t-3)$

43. If $z = a+bi$, then z corresponds to the point (a, b) in the xy-plane. Hence, $\overline{z} = \overline{a+bi} = a-bi$ corresponds to the point $(a, -b)$. Hence, the points corresponding to z and $\overline{z}$ are reflections about the x-axis.

44. If $z = a+bi$, then z corresponds to the point (a, b) in the xy-plane. Hence, $-z = -(a+bi) = -a-bi$ corresponds to the point $(-a, -b)$. Hence, the points corresponding to z and $-z$ are reflections through the origin.

45. If $z = a+bi$, then z corresponds to the point (a, b) in the xy-plane. Writing out the product iz, we get: $iz = i(a+bi) = ia+bi^2 = ia-b = -b+ia$. Hence, iz corresponds to the point $(-b, a)$. If $z \neq 0$, then neither a nor b is 0 (do you see why?) Hence, the slope of the line containing $(0, 0)$ and (a, b) is $\frac{b}{a}$ and the slope of the line containing $(0, 0)$ and $(-b, a)$ is $-\frac{a}{b}$. Per Theorem A.3, since the slopes of these lines are negative reciprocals, the lines themselves are perpendicular.[9]

46. $|z| = \sqrt{a^2+b^2}$ measures the distance from the origin to the point (a, b). Hence, $|z|$ measures the distance from z to 0 in the Complex Plane. This is exactly how $|x|$ is defined in Definition A.14 in Section A.7. In that section, however, the only part of the Complex Plane under discussion is the real number line.

[9]We'll be able to show in Section 14.3 that, more precisely, multiplication by i rotates the complex number counter-clockwise by 90°.

Chapter 3

Rational Functions

3.1 Introduction to Rational Functions

If we add, subtract, or multiply polynomial functions, the result is another polynomial function. When we divide polynomial functions, however, we may not get a polynomial function. The result of dividing two polynomials is a **rational function**, so named because rational functions are **ratios** of polynomials.

Definition 3.1. A **rational function** is a function which is the ratio of polynomial functions. Said differently, r is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions.[a]

[a] According to this definition, all polynomial functions are also rational functions. (Take $q(x) = 1$).

3.1.1 Laurent Monomial Functions

As with polynomial functions, we begin our study of rational functions with what are, in some sense, the building blocks of rational functions, **Laurent monomial functions**.

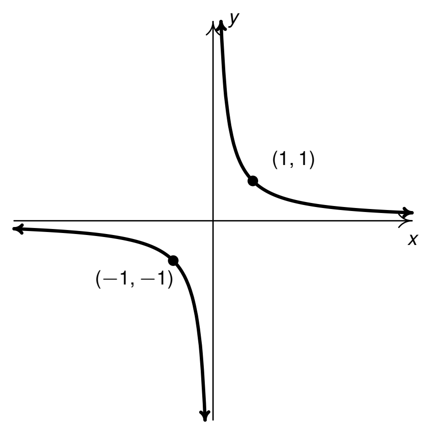
Definition 3.2. A **Laurent monomial function** is either a monomial function (see Definition 2.1) or a function of the form $f(x) = \dfrac{a}{x^n} = ax^{-n}$ for $n \in \mathbb{N}$.

Laurent monomial functions are named in honor of Pierre Alphonse Laurent and generalize the notion of 'monomial function' from Chapter 2 to terms with negative exponents. Our study of these functions begins with an analysis of $r(x) = \frac{1}{x} = x^{-1}$, the reciprocal function. The first item worth noting is that $r(0)$ is not defined owing to the presence of x in the denominator. That is, the domain of r is $\{x \in \mathbb{R} \mid x \neq 0\}$ or, using interval notation, $(-\infty, 0) \cup (0, \infty)$. Of course excluding 0 from the domain of r serves only to pique our curiosity about the behavior of $r(x)$ when $x \approx 0$. Thinking from a number sense perspective, the closer the denominator of $\frac{1}{x}$ is to 0, the larger the value of the fraction (in absolute value.)[1] So it stands to reason that as x gets closer and closer to 0, the values for $r(x) = \frac{1}{x}$ should grow larger and larger (in absolute value.) This is borne out in the table below on the left where it is apparent that for $x \approx 0$, $r(x)$ is becoming unbounded.

As we investigate the end behavior of r, we find that as $x \to -\infty$ and as $x \to \infty$, $r(x) \approx 0$. Again, number sense agrees here with the data, since as the denominator of $\frac{1}{x}$ becomes unbounded, the value of the fraction should diminish.[2] That being said, we could ask if the graph ever reaches the x-axis. If we attempt to solve $y = r(x) = \frac{1}{x} = 0$. we arrive at the contradiction $1 = 0$ hence, 0 is not in the range of r. Every other real number besides 0 is in the range of r, however. To see this, let $c \neq 0$ be a real number. Then $\frac{1}{c}$ is defined and, moreover, $r\left(\frac{1}{c}\right) = \frac{1}{(1/c)} = c$. This shows c is in the range of r. Hence, the range of r is $\{y \in \mathbb{R} \mid y \neq 0\}$ or, using interval notation, $(-\infty, 0) \cup (0, \infty)$.

x	$r(x) = \frac{1}{x}$
-0.01	-100
-0.001	-1000
-0.0001	-10000
-0.00001	-100000
0	undefined
0.00001	100000
0.0001	10000
0.001	1000
0.01	100

x	$r(x) = \frac{1}{x}$
-100000	-0.00001
-10000	-0.0001
-1000	-0.001
-100	-0.01
0	undefined
100	0.01
1000	0.001
10000	0.0001
100000	0.00001



$y = r(x) = \frac{1}{x}$

Like we did in Section 2.1, we'll borrow some notation from Calculus in order for us to codify the behavior as $x \to 0$. First off, note that the behavior of r differs depending on which direction we approach 0. We describe the values $x < 0$ but $x \to 0$ (such as $x = -0.01$, -0.001, etc.) as 'x approaching 0 **from the left**,' written as $x \to 0^-$. If we think of these numbers as all being x-values where $x =$ '$0-$ a little bit', the the '$-$' in the notation '$x \to 0^-$' makes better sense. For these values, the function values $r(x) \to -\infty$. Using the limit notation introduced in Section 2.1, we'd write: $\displaystyle\lim_{x \to 0^-} r(x) = -\infty$.

Similarly, we say 'as x approaches 0 **from the right**,' that is as $x \to 0^+$, $r(x) \to \infty$, or, more succinctly, $\displaystyle\lim_{x \to 0^+} r(x) = \infty$. As before, we understand 'from the right' means we are using x values slightly to the **right** of 0 on the number line: numbers such as $x = 0.001$. These numbers could described as '$0 +$ a little bit,' which justifies the '$+$' in the notation '$x \to 0^+$.'

We can also use this notation to describe the end behavior, but here the numerical roles are reversed. We see as $x \to -\infty$, $r(x) \to 0^-$ and as $x \to \infty$, $r(x) \to 0^+$. When it comes to codifying these results using Calculus, we write $\displaystyle\lim_{x \to -\infty} r(x) = 0$ and $\displaystyle\lim_{x \to \infty} r(x) = 0$. Note that, unfortunately, we lose the directionality

[1] Technically speaking, -1×10^{117} is a 'small' number (since it is very far to the left on the number line.) However, it's absolute value, 1×10^{117} is very large. If you read footnotes, you've seen this clarification before . . . (see Section 2.1.)

[2] We'll talk more about this end behavior shortly - stay tuned!

here on the limiting value - that is, we do not write $\lim_{x\to-\infty} r(x) = 0^-$ or $\lim_{x\to\infty} r(x) = 0^+$. Without getting too much into formal definitions, the reason is that if limiting values are finite, we express them as real numbers.[3] Period.

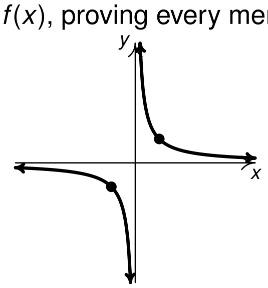
The way we describe what is happening graphically is to say the line $x = 0$ is a **vertical asymptote** to the graph of $y = r(x)$ and the line $y = 0$ is a **horizontal asymptote** to the graph of $y = r(x)$. Roughly speaking, asymptotes are lines which approximate functions as either the inputs or outputs become unbounded.

Definition 3.3. The line $x = c$ is called a **vertical asymptote** of the graph of a function $y = f(x)$ if either of the limits $\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$ (or both) result in ∞ or $-\infty$.

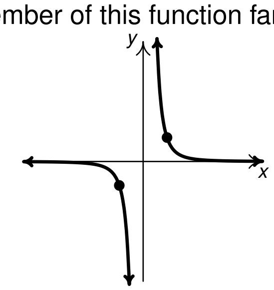
Definition 3.4. The line $y = c$ is called a **horizontal asymptote** of the graph of a function $y = f(x)$ either $\lim_{x\to-\infty} f(x) = c$ or $\lim_{x\to\infty} f(x) = c$ (or both).

The behaviors illustrated in the graph $r(x) = \frac{1}{x}$ are typical of functions of the form $f(x) = \frac{1}{x^n} = x^{-n}$ for natural numbers, n. As with the monomial functions discussed in Section 2.1, the patterns that develop primarily depend on whether n is odd or even. Having thoroughly discussed the graph of $y = \frac{1}{x} = x^{-1}$, we graph it along with $y = \frac{1}{x^3} = x^{-3}$ and $y = \frac{1}{x^5} = x^{-5}$ below. Note the points $(-1, -1)$ and $(1, 1)$ are common to all three graphs as are the asymptotes $x = 0$ and $y = 0$. As the n increases, the graphs become steeper for $|x| < 1$ and flatten out more quickly for $|x| > 1$. Both the domain and range in each case appears to be $(-\infty, 0) \cup (0, \infty)$. Indeed, owing to the x in the denominator of $f(x) = \frac{1}{x^n}$, $f(0)$, and only $f(0)$, is undefined. Hence the domain is $(-\infty, 0) \cup (0, \infty)$. When thinking about the range, note the equation $f(x) = \frac{1}{x^n} = c$ has the solution $x = \sqrt[n]{\frac{1}{c}}$ as long as $c \neq 0$. Thus means $f\left(\sqrt[n]{\frac{1}{c}}\right) = c$ for every nonzero real number c. If $c = 0$, we are in the same situation as before: $\frac{1}{x^n} = 0$ has no real solution. This establishes the range is $(-\infty, 0) \cup (0, \infty)$. Finally, each of the graphs appear to be symmetric about the origin. Indeed, since n is odd, $f(-x) = (-x)^{-n} = (-1)^{-n}x^{-n} = -x^{-n} = -f(x)$, proving every member of this function family is odd.

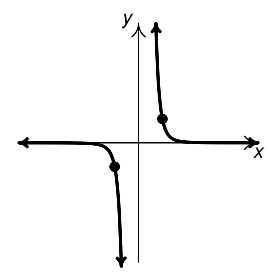
x	$\frac{1}{x} = x^{-1}$	$\frac{1}{x^3} = x^{-3}$	$\frac{1}{x^5} = x^{-5}$
-10	-0.1	-0.001	-0.00001
-1	-1	-1	-1
-0.1	-10	-1000	-100000
0	undefined	undefined	undefined
0.1	10	1000	100000
1	1	1	1
10	0.1	0.001	0.00001



$y = \frac{1}{x} = x^{-1}$



$y = \frac{1}{x^3} = x^{-3}$



$y = \frac{1}{x^5} = x^{-5}$

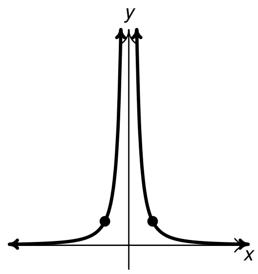
We repeat the same experiment with functions of the form $f(x) = \frac{1}{x^n} = x^{-n}$ where n is even. $y = \frac{1}{x^2} = x^{-2}$, $y = \frac{1}{x^4} = x^{-4}$ and $y = \frac{1}{x^6} = x^{-6}$. These graphs all share the points $(-1, 1)$ and $(1, 1)$, and asymptotes $x = 0$ and $y = 0$. Note here that both $\lim_{x\to 0^-} f(x) = \infty$ and $\lim_{x\to 0^+} f(x) = \infty$, so we may simply write $\lim_{x\to 0} f(x) = \infty$.

The same remarks about the steepness for $|x| < 1$ and the flattening for $|x| > 1$ also apply. For the same reasons as given above, the domain of each of these functions is $(-\infty, 0) \cup (0, \infty)$. When it comes to the

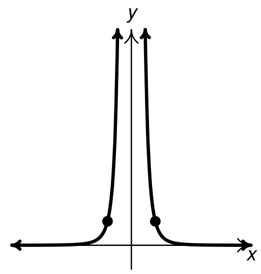
[3]We will, of course, offer more detail if the situation presents itself.

range, the fact n is even tells us there are solutions to $\frac{1}{x^n} = c$ only if $c > 0$. It follows that the range is $(0, \infty)$ for each of these functions. Concerning symmetry, as n is even, $f(-x) = (-x)^{-n} = (-1)^{-n}x^{-n} = x^{-n} = f(x)$, proving each member of this function family is even. Hence, the graphs of these functions are symmetric about the y-axis.

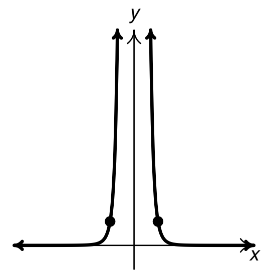
x	$\frac{1}{x^2} = x^{-2}$	$\frac{1}{x^4} = x^{-4}$	$\frac{1}{x^6} = x^{-6}$
-10	0.01	0.0001	1×10^{-6}
-1	1	1	1
-0.1	100	10000	1×10^{6}
0	undefined	undefined	undefined
0.1	100	10000	1×10^{6}
1	1	1	1
10	0.01	0.0001	1×10^{-6}



$y = \frac{1}{x^2} = x^{-2}$



$y = \frac{1}{x^3} = x^{-3}$



$y = \frac{1}{x^6} = x^{-6}$

Not surprisingly, we have an analog to Theorem 2.1 for this family of Laurent monomial functions.

Theorem 3.1. For real numbers a, h, and k with $a \neq 0$, the graph of $F(x) = \frac{a}{(x-h)^n} + k = a(x-h)^{-n} + k$ can be obtained from the graph of $f(x) = \frac{1}{x^n} = x^{-n}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = x^{-n}$ to $y = (x-h)^{-n}$.

   The vertical asymptote moves from $x = 0$ to $x = h$.

2. multiply the y-coordinates of the points on the graph obtained in Step 1 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = (x-h)^{-n}$ to $y = a(x-h)^{-n}$.

3. add k to each of the y-coordinates of the points on the graph obtained in Step 2. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a(x-h)^{-n}$ to $y = a(x-h)^{-n} + k$.

   The horizontal asymptote moves from $y = 0$ to $y = k$.

The proof of Theorem 3.1 is **identical** to the proof of Theorem 2.1 - just replace x^n with x^{-n}. We nevertheless encourage the reader to work through the details[4] and compare the results of this theorem with Theorems 1.2, 1.3, and 2.1.

We put Theorem 3.1 to good use in the following example.

Example 3.1.1. Use Theorem 3.1 to graph the following. Label at least two points and the asymptotes. State the domain and range using interval notation.

[4] We are, in fact, building to Theorem 5.11 in Section 5.4, so the more you see the forest for the trees, the better off you'll be when the time comes to generalize these moves to all functions.

1. $f(x) = (2x-3)^{-2}$

2. $g(t) = \dfrac{2t-1}{t+1}$

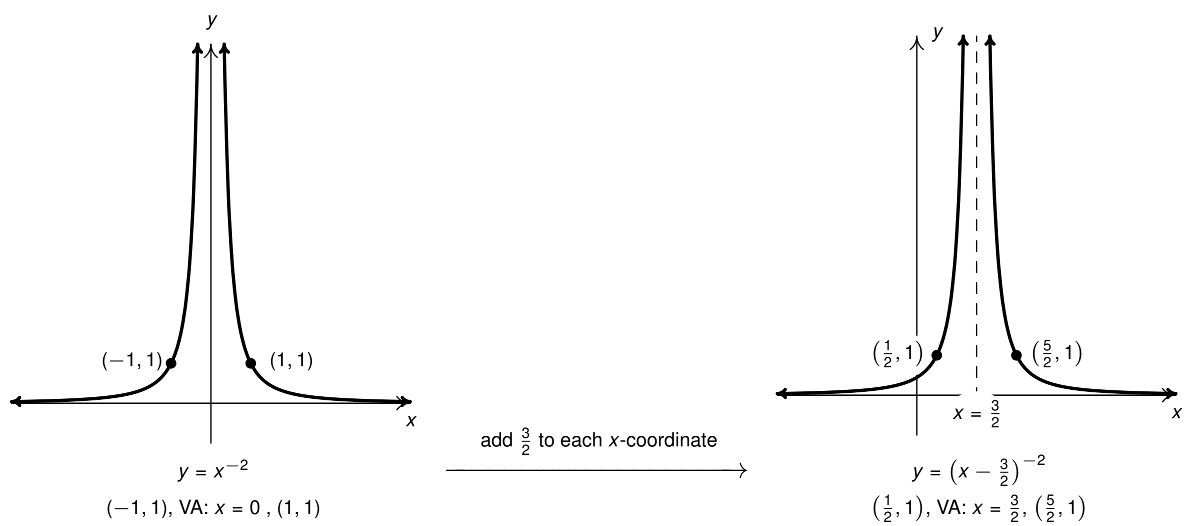
Solution.

1. In order to use Theorem 3.1, we first must put $f(x) = (2x-3)^{-2}$ into the form prescribed by the theorem. To that end, we factor:

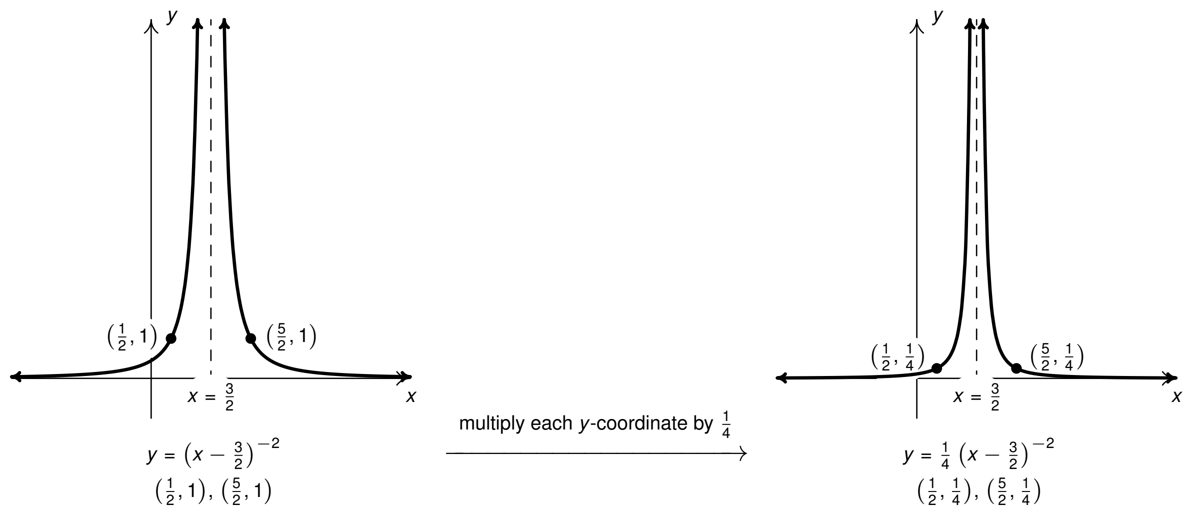
$$f(x) = \left(2\left[x - \frac{3}{2}\right]\right)^{-2} = 2^{-2}\left(x - \frac{3}{2}\right)^{-2} = \frac{1}{4}\left(x - \frac{3}{2}\right)^{-2}$$

We identify $n = 2$, $a = \frac{1}{4}$ and $h = \frac{3}{2}$ (and $k = 0$.) Per the theorem, we begin with the graph of $y = x^{-2}$ and track the two points $(-1, 1)$ and $(1, 1)$ along with the vertical and horizontal asymptotes $x = 0$ and $y = 0$, respectively through each step.

Step 1: add $\frac{3}{2}$ to each of the x-coordinates of each of the points on the graph of $y = x^{-2}$. This moves the vertical asymptote from $x = 0$ to $x = \frac{3}{2}$ (which we represent by a dashed line.)



Step 2: multiply each of the y-coordinates of each of the points on the graph of $y = \left(x - \frac{3}{2}\right)^{-2}$ by $\frac{1}{4}$.



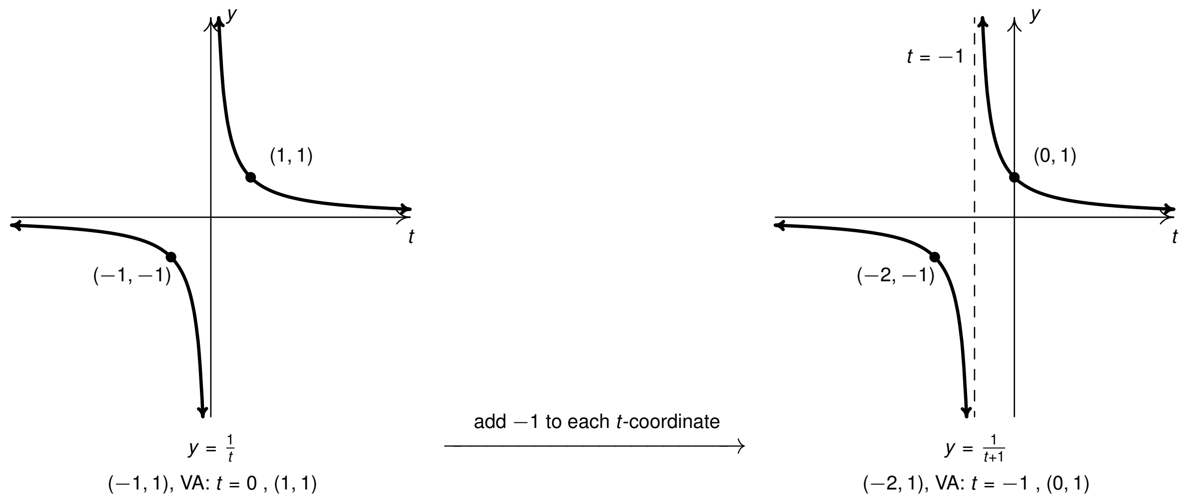
Since we did not shift the graph vertically, the horizontal asymptote remains $y = 0$. We can determine the domain and range of f by tracking the changes to the domain and range of our progenitor function, $y = x^{-2}$. We get the domain and range of f is $\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$ and the range of f is $(-\infty, 0) \cup (0, \infty)$.

2. Using either long or synthetic division, we get

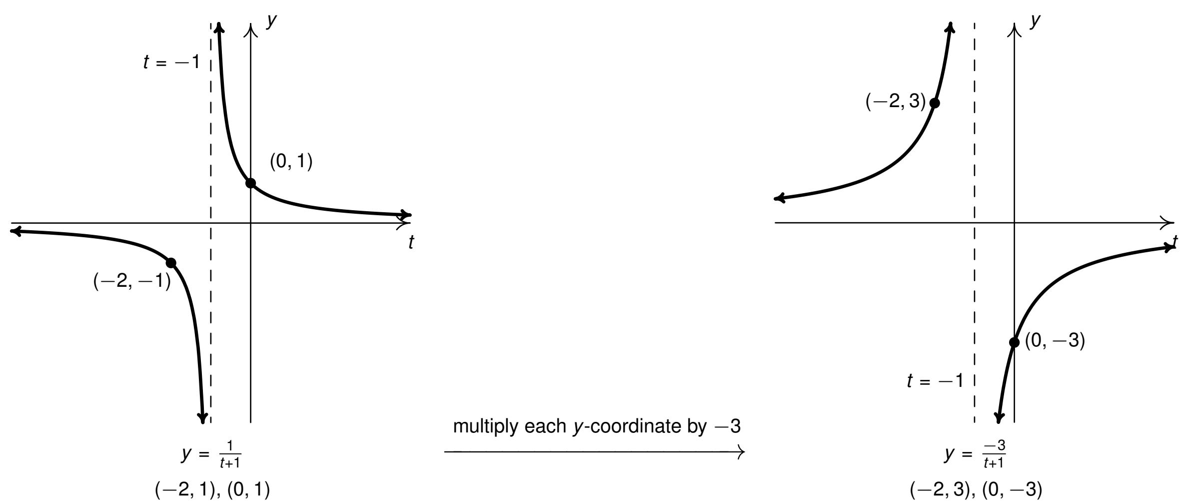
$$g(t) = \frac{2t - 1}{t + 1} = -\frac{3}{t + 1} + 2 = \frac{-3}{(t - (-1))^1} + 2$$

so we identify $n = 1$, $a = -3$, $h = -1$, and $k = 2$. We start with the graph of $y = \frac{1}{t}$ with points $(-1, -1)$, $(1, 1)$ and asymptotes $t = 0$ and $y = 0$ and track these through each of the steps.

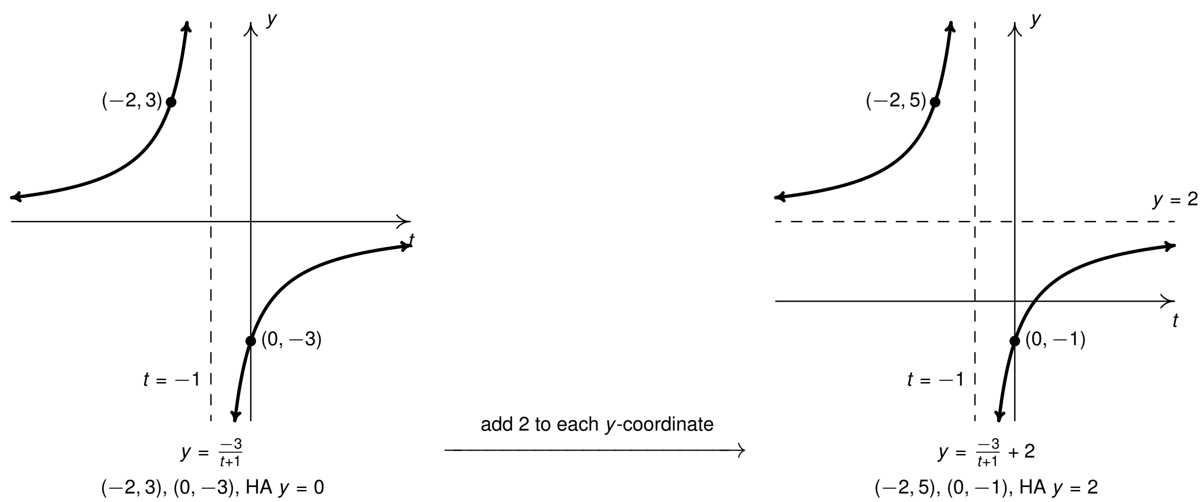
Step 1: Add -1 to each of the t-coordinates of each of the points on the graph of $y = \frac{1}{t}$. This moves the vertical asymptote from $t = 0$ to $t = -1$.



Step 2: multiply each of the y-coordinates of each of the points on the graph of $y = \frac{1}{t+1}$ by -3.



Step 3: add 2 to each of the y-coordinates of each of the points on the graph of $y = \frac{-3}{t+1}$. This moves the horizontal asymptote from $y = 0$ to $y = 2$.



As above, we determine the domain and range of g by tracking the changes in the domain and range of $y = \frac{1}{t}$. We find the domain of g is $(-\infty, -1) \cup (-1, \infty)$ and the range is $(-\infty, 2) \cup (2, \infty)$. $\Box$

In Example 3.1.1, we once again see the benefit of changing the form of a function to make use of an important result. A natural question to ask is to what extent general rational functions can be rewritten to use Theorem 3.1. In the same way polynomial functions are sums of monomial functions, it turns out, allowing for non-real number coefficients, that every rational function can be written as a sum of (possibly shifted) Laurent monomial functions.[5]

3.1.2 Local Behavior near Excluded Values

We take time now to focus on behaviors of the graphs of rational functions near excluded values. We've already seen examples of one type of behavior: vertical asymptotes. Our next example gives us a physical interpretation of a vertical asymptote. This type of model arises from a family of equations cheerily named 'doomsday' equations.[6]

Example 3.1.2. A mathematical model for the population $P(t)$, in thousands, of a certain species of bacteria, t days after it is introduced to an environment is given by $P(t) = \frac{100}{(5-t)^2}$, $0 \leq t < 5$.

1. Find and interpret $P(0)$.
2. When will the population reach 100,000?
3. Graph $y = P(t)$.
4. Find and interpret $\lim\limits_{t \to 5^-} P(t)$.

[5]i.e., Laurent 'Polynomials.' This result is a combination of Theorems 2.16 in Section 2.4 and Theorem 9.10 in Section 9.6.
[6]These functions arise in Differential Equations. The unfortunate name will make sense shortly.

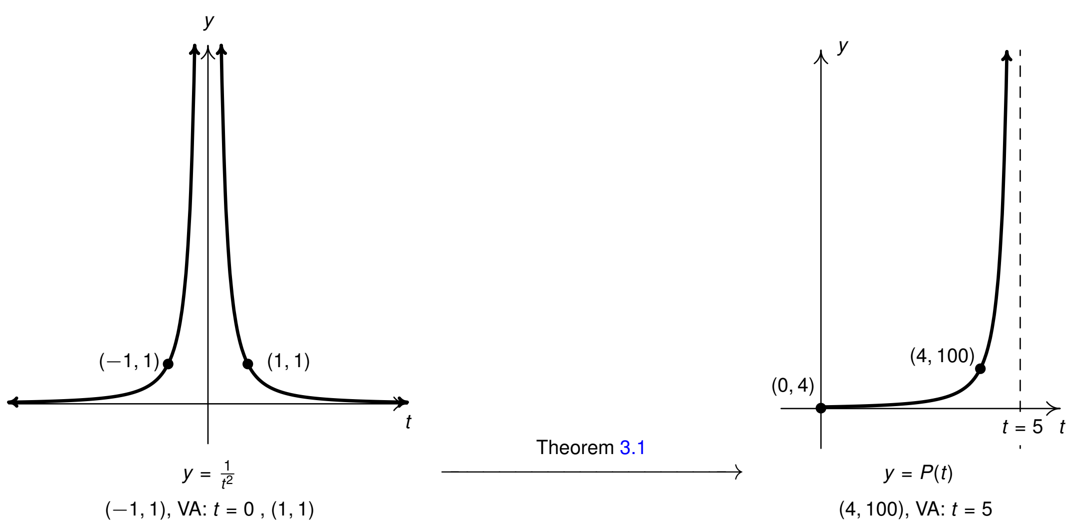
Solution.

1. Substituting $t = 0$ gives $P(0) = \frac{100}{(5-0)^2} = 4$. Since t represents the number of days **after** the bacteria are introduced into the environment, $t = 0$ corresponds to the day the bacteria are introduced. Since $P(t)$ is measured in **thousands**, $P(t) = 4$ means 4000 bacteria are initially introduced into the environment.

2. To find when the population reaches 100,000, we first need to remember that $P(t)$ is measured in **thousands**. In other words, 100,000 bacteria corresponds to $P(t) = 100$. Hence, we need to solve $P(t) = \frac{100}{(5-t)^2} = 100$. Clearing denominators and dividing by 100 gives $(5-t)^2 = 1$, which, after extracting square roots, produces $t = 4$ or $t = 6$. Of these two solutions, only $t = 4$ in our domain, so this is the solution we keep. Hence, it takes 4 days for the population of bacteria to reach 100,000.

3. After a slight re-write, we have $P(t) = \frac{100}{(5-t)^2} = \frac{100}{[(-1)(t-5)]^2} = \frac{100}{(t-5)^2}$. Using Theorem 3.1, we start with the graph of $y = \frac{1}{t^2}$ below on the left. After shifting the graph to the right 5 units and stretching it vertically by a factor of 100 (note, the graphs are not to scale!), we restrict the domain to $0 \leq t < 5$ to arrive at the graph of $y = P(t)$ below on the right.

4. From the graph, we see as $t \to 5^-$, $P(t) \to \infty$, so $\displaystyle\lim_{t \to 5^-} P(t) = \infty$. This means that the population of bacteria is increasing without bound as we near 5 days, which cannot actually happen. For this reason, $t = 5$ is called the 'doomsday' for this population. There is no way any environment can support infinitely many bacteria, so shortly before $t = 5$ the environment would collapse.



□

Will all values excluded from the domain of a rational function produce vertical asymptotes in the graph? The short answer is 'no.' There are milder interruptions that can occur - holes in the graph - which we explore in our next example.

To this end, we formalize the notion of **average velocity** - a concept we first encountered in Example 1.2.8 in Section 1.2. In that example, the function $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. The function s is an example of a **position function** since it provides information about **where** the rocket is at time t. In that example, we interpreted the average rate of change of s over an interval as the average velocity of the rocket over that interval. The average velocity provides two pieces of information: the average speed of the rocket along with the rocket's direction.

Suppose we have a position function s defined over an interval containing some fixed time t_0. We can define the average velocity as a function of any time t other than t_0:

> **Definition 3.5.** Let $s(t)$ be the position of an object at time t and t_0 be a fixed time in the domain of s. The **average velocity** between time t and time t_0 for $t \neq t_0$ is given by
>
> $$\overline{v}(t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t) - s(t_0)}{t - t_0}.$$

We must exclude $t = t_0$ from the domain of $\overline{v}$ in Definition 3.5 since, otherwise, we would have a 0 in the denominator. What is interesting in this case however, is that substituting $t = t_0$ also produces 0 in the **numerator**. (Do you see why?) While '$\frac{0}{0}$' is undefined, it is more precisely called an 'indeterminate form' and is studied extensively in Calculus. We explore this phenomenon in the next example.

Example 3.1.3. Let $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ give the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff.

1. Find and simplify an expression for the average velocity of the rocket between times t and 15, $\overline{v}(t)$.
2. Find and interpret $\overline{v}(14)$.
3. Find and interpret $\displaystyle\lim_{t \to 15} \overline{v}(t)$.
4. Graph $y = \overline{v}(t)$. Interpret your answer to part 3 graphically.

Solution.

1. Using Definition 3.5 with $t_0 = 15$, we get:

$$\begin{aligned}
\overline{v}(t) &= \frac{s(t) - s(15)}{t - 15}, \qquad t \neq 15 \\
&= \frac{(-5t^2 + 100t) - 375}{t - 15} \\
&= \frac{-5(t^2 - 20t + 75)}{t - 15} \\
&= \frac{-5(t - 15)(t - 5)}{t - 15} \\
&= \frac{-5\cancel{(t - 15)}(t - 5)}{\cancel{(t - 15)}} \\
&= -5(t - 5) = -5t + 25, \quad t \neq 15
\end{aligned}$$

Since the domain of s is $0 \leq t \leq 20$, our final answer is $\overline{v}(t) = -5t + 25$, for $t \in [0, 15) \cup (15, 20]$.

2. We find $\overline{v}(14) = -5(14) + 25 = -45$. This means that between 14 and 15 seconds after launch, the rocket was traveling, on average, a speed 45 feet per second **downwards**, or falling back to the Moon's surface.

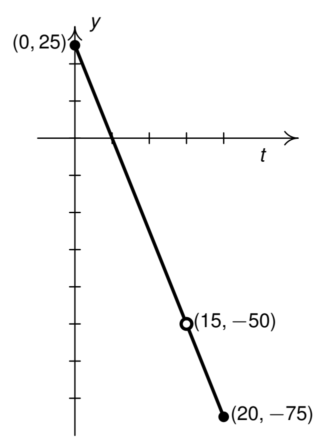
3. To find $\lim\limits_{t \to 15} \overline{v}(t)$, we need to analyze the outputs from $\overline{v}(t)$ as $t \to 15$. Using the table below on the left, it certainly **appears** to be the case that $\lim\limits_{t \to 15} \overline{v}(t) = -50$. Of course, we could be choosing the number -50 because of a bias towards nice, integer answers.[7]

   However, we can argue a stronger case algebraically using the simplified formula $\overline{v}(t) = -5t+25$. As $t \to 15$, $-5t \to -5(15) = -75$ so $-5t + 25 \to -50$. It stands to reason, then, that $\lim\limits_{t \to 15} \overline{v}(t) = -50$.

   This means our average velocity approaches -50 feet per second as we sample times closer and closer to $t = 15$ seconds after liftoff. Since we're pushing the t-values to $t = 15$, instead of viewing the limit -50 as an **average** velocity calculated **between** two time values, we view -50 feet per second as the **instantaneous velocity** of the rocket **at** $t = 15$. That is, **at** $t = 15$ seconds after liftoff, the rocket **is traveling** downwards at a rate of 50 feet per second.[8]

4. From part 1, we know $\overline{v}(t) = -5t + 25$, for $t \in [0, 15) \cup (15, 20]$ Hence the graph of $\overline{v}(t)$ is a portion of the line $y = -5t + 25$. Since $\overline{v}(t)$ is not defined when $t = 15$, our graph is the line segment starting at $(0, 25)$ and ending at $(20, 75)$ which skips over the point $(15, -50)$, creating a hole in the graph.[9]

t	$\overline{v}(t)$
14.9	-49.5
14.99	-49.95
14.999	-49.995
15	undefined
15.001	-50.005
15.01	-50.05
15.1	-50.5



$y = \overline{v}(t)$

□

Some notes about Example 3.1.3 are in order. First, excluded values from the domain of a rational function don't necessarily cause vertical asymptotes in the graph. Even though $\overline{v}(15)$ doesn't exist, the fact that

[7] Or maybe the limit is actually -49.999999865 which, owing to calculation limitations of the graphing utility rounds to -50.

[8] It is worth noting that Exercise 56 from Section 1.2 has you arrive at this limit via a table.

[9] We've seen and discussed such holes in the graph as far back as Example 1.1.4 in Section 1.1 . . .

$\lim_{t\to 15} \overline{v}(t) = -50$ means that we **expect** $\overline{v}(15)$ to be -50.This sentiment is exactly what a hole in the graph at $(15, -50)$ communicates.

Second, in finding $\lim_{t\to 15} \overline{v}(t) = -50$, we've taken some (more) steps into Calculus. Specifically, we used properties of the limit process that we've not yet formalized, let alone justified. The main idea is that the '$\frac{0}{0}$' indeterminate form which occurs when we attempt to evaluate $\overline{v}(15)$ using the formula $\overline{v}(t) = \frac{-5t^2+100t-375}{t-15}$ is resolved when the factor $(t-15)$ cancels from the denominator. We can algebraically reason what to expect out of the expression $\overline{v}(t) = -5t + 25$ as $t \to 15$ because there is no longer any division by 0.

We will revisit these sorts of machinations later in the text in a bit more generality.[10] For now, we'll work to build some intuition with some classic hand-waving which we hope will do more good than harm.

Our next theorem generalizes our reasoning from this last example.

Theorem 3.2. Location of Vertical Asymptotes and Holes:[a] Suppose r is a rational function and c is not in the domain of r.

- If $\lim_{x\to c} r(x) = L$ where L is a real number, then the graph of r has a hole at (c, L).
- Otherwise, the graph of $y = r(x)$ has a vertical asymptote $x = c$.

[a] Or, 'How to tell your asymptote from a hole in the graph.'

Of course the first question to ask is how do we know if $\lim_{x\to c} r(x)$ results in a real number, L, and if so, how do we find L? It turns out that if the limit exists, then the same sort of algebraic cancellation which occurred Example 3.1.3 is guaranteed to happen.

Let's consider a generic rational function $r(x) = \frac{p(x)}{q(x)}$ where p and q are polynomial functions. The values 'c' excluded from the domain of r are the zeros of q: $q(c) = 0$. We now have two cases to consider.

If $p(c) \neq 0$, then as $x \to c$, $r(x) = \frac{p(x)}{q(x)} \to \frac{p(c), \text{ a nonzero number}}{0}$ which results in unbounded behavior (graphically, a vertical asymptote.)

If $p(c) = 0$, then as $x \to c$, $r(x) = \frac{p(x)}{q(x)} \to \frac{0}{0}$, an indeterminate form. The Factor Theorem,[11] guarantees both $p(x)$ and $q(x)$ contain factors of $(x-c)$. This means we can simplify the expression $r(x)$ by cancelling common factors of $(x-c)$. If all of the factors of $(x-c)$ in the denominator, $q(x)$, cancel with factors in the numerator, $p(x)$, then the division by 0 is eliminated and we can proceed as in Example 3.1.3 to determine the limit.[12] If some factors of $(x-c)$ remain in the denominator, then we're back to the first scenario and the graph will have a vertical asymptote.

We practice this methodology in the following example.

Example 3.1.4. For each function below:

- determine the values excluded from the domain.
- determine whether each excluded value corresponds to a vertical asymptote or hole in the graph.

[10] See Section 6.1.

[11] Theorem 2.8 in Section 2.2

[12] Using the vocabulary from Section 2.1, the limit will exist if the multiplicity of c as a zero for p is greater than or equal to the multiplicity of c as a zero for q.

- verify your answers using a graphing utility.
- describe the behavior of the graph near each excluded value using proper notation.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $f(x) = \dfrac{2x}{x^2 - 3}$
2. $g(t) = \dfrac{t^2 - t - 6}{t^2 - 9}$
3. $h(t) = \dfrac{t^2 - t - 6}{t^2 + 9}$
4. $r(t) = \dfrac{t^2 - t - 6}{t^2 + 4t + 4}$

Solution.

1. We begin by finding the values excluded from the domain by setting the denominator equal to 0. Solving $x^2 - 3 = 0$, we get $x = \pm\sqrt{3}$ which factors the denominator as $(x - \sqrt{3})(x + \sqrt{3})$. Since $f(x)$ is in lowest terms (can you see why?), no cancellation occurs, hence, the lines $x = -\sqrt{3}$ and $x = \sqrt{3}$ are vertical asymptotes to the graph of $y = f(x)$.

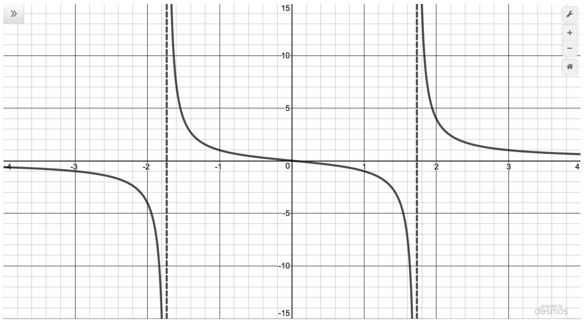
   A graphing utility verifies this claim, and from the graph, we see that $\displaystyle\lim_{x \to -\sqrt{3}^-} f(x) = -\infty$, $\displaystyle\lim_{x \to -\sqrt{3}^+} f(x) = \infty$, $\displaystyle\lim_{x \to \sqrt{3}^-} f(x) = -\infty$, and $\displaystyle\lim_{x \to \sqrt{3}^+} f(x) = \infty$.

   As a side note, the graph of f appears to be symmetric about the origin. Sure enough, we find: $f(-x) = \frac{2(-x)}{(-x)^2-3} = -\frac{2x}{x^2-3} = -f(x)$, proving f is odd.

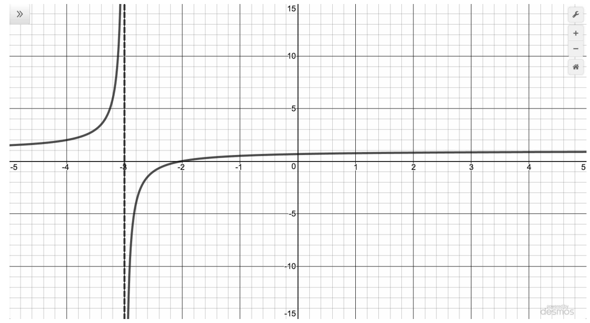
2. As above, we find the values excluded from the domain of g finding the zeros of the denominator. Solving $t^2 - 9 = 0$ gives $t = \pm 3$. In this case, we can simplify the formula for $g(t)$: $\frac{t^2-t-6}{t^2-9} = \frac{(t-3)(t+2)}{(t-3)(t+3)} = \frac{\cancel{(t-3)}(t+2)}{\cancel{(t-3)}(t+3)} = \frac{t+2}{t+3}$. Hence, $g(t) = \frac{t+2}{t+3}$ provided $t \neq 3$.

   Since the factor $(t+3)$, which corresponds to the zero $t = -3$, did not cancel from the denominator of $g(t)$, we expect a vertical asymptote to the graph at $t = -3$. On the other hand, as $t \to 3$, $t + 2 \to 5$ and $t + 3 \to 6$ so $\frac{t+2}{t+3} \to \frac{3+2}{3+3} = \frac{5}{6}$. This gives $\displaystyle\lim_{t \to 3} g(t) = \frac{5}{6}$. Hence we have a hole in the graph of $y = g(t)$ at $\left(3, \frac{5}{6}\right)$.

   Graphing g we can definitely see the vertical asymptote $t = -3$: as $\displaystyle\lim_{t \to -3^-} g(t) = \infty$ and $\displaystyle\lim_{t \to -3^+} g(t) = -\infty$. Near $t = 3$, the graph seems to have no interruptions, but we know g is undefined at $t = 3$. Depending on the graphing utility, we may or may not convince the display to show us the hole at $\left(3, \frac{5}{6}\right)$.



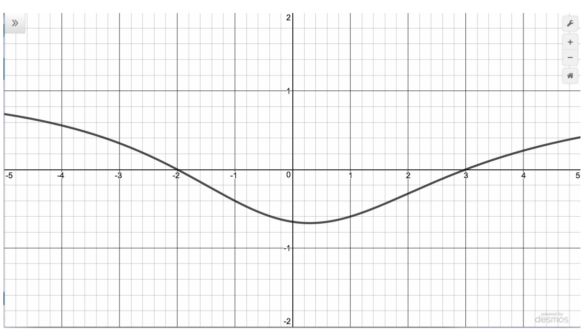
The graph of $y = f(x)$



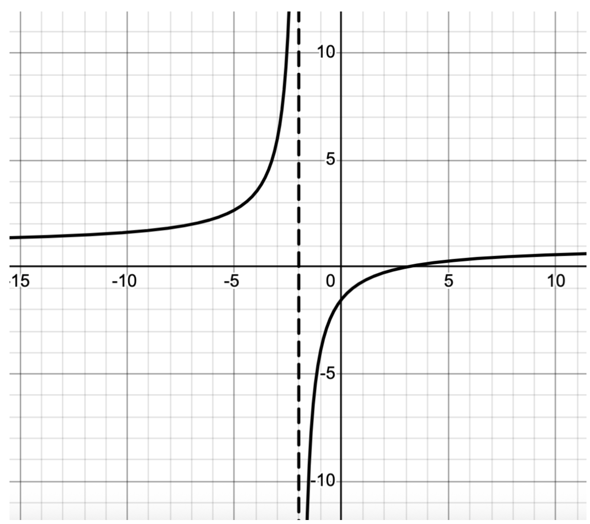
The graph of $y = g(t)$

3. Setting the denominator of the expression for $h(t)$ to 0 gives $t^2 + 9 = 0$, which has no real solutions. Accordingly, the graph of $y = h(t)$ (at least as much as we can discern from the technology) is devoid of both vertical asymptotes and holes. Using terms defined in Section 2.1, the function h is both continuous and smooth.[13]

4. Setting the denominator of $r(t)$ to zero gives the equation $t^2 + 4t + 4 = 0$. We get the (repeated!) solution $t = -2$. Simplifying, we get $\frac{t^2-t-6}{t^2+4t+4} = \frac{(t-3)(t+2)}{(t+2)(t+2)} = \frac{(t-3)\cancel{(t+2)}}{(t+2)\cancel{(t+2)}} = \frac{t-3}{t+2}$. Since not all factors of $(t+2)$ cancelled from the denominator, $t = -2$ continues to produce a 0 in the denominator. Hence $t = -2$ is a vertical asymptote to the graph. A graphing utility bears this out. Specifically, $\displaystyle\lim_{t \to -2^-} r(t) = \infty$ and $\displaystyle\lim_{t \to -2^+} r(t) = -\infty$.



The graph of $y = h(t)$



The graph of $y = r(t)$

□

[13] We'll remind you more about continuous functions in Section 3.2 . . .

3.1.3 End Behavior

Now that we've discussed behavior near values excluded from the domains of rational functions, let's focus our attention on end behavior. We have already seen one example of this in the form of horizontal asymptotes. Our next example of the section gives us a real-world application of a horizontal asymptote.[14]

Example 3.1.5. The number of students $N(t)$ at local college who have had the flu t months after the semester begins can be modeled by:

$$N(t) = \frac{1500t + 50}{3t + 1}, \quad t \geq 0.$$

1. Find and interpret $N(0)$.
2. How long will it take until 300 students will have had the flu?
3. Use Theorem 3.1 to graph $y = N(t)$.
4. Find and interpret $\displaystyle\lim_{t\to\infty} N(t)$.

Solution.

1. Substituting $t = 0$ gives $N(0) = \frac{1500(0)+50}{1+3(0)} = 50$. Since t represents the number of months since the beginning of the semester, $t = 0$ describes the state of the flu outbreak at the beginning of the semester. Hence, at the beginning of the semester, 50 students have had the flu.

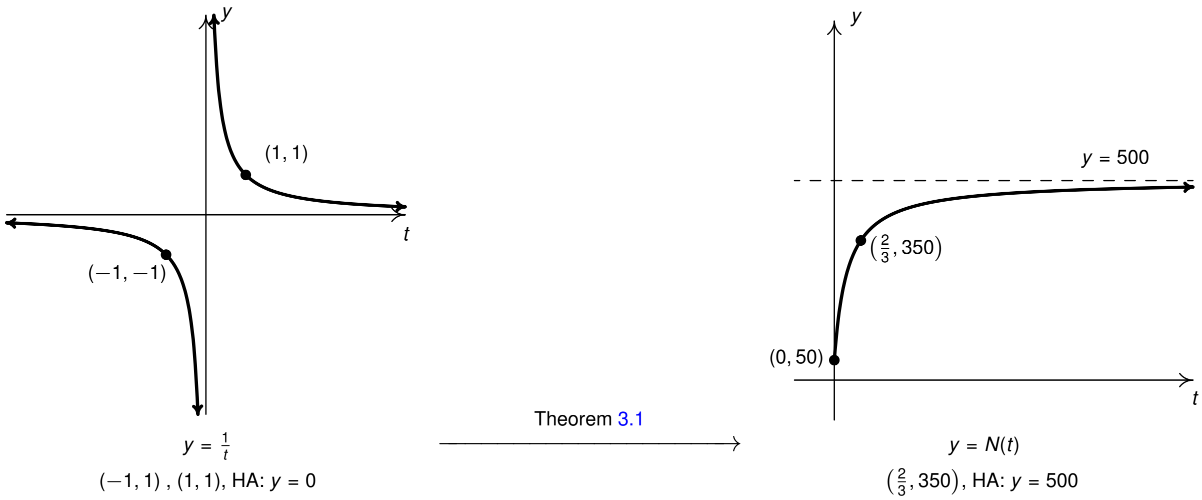
2. We set $N(t) = \frac{1500t+50}{3t+1} = 300$ and solve. Clearing denominators gives $1500t + 50 = 300(3t + 1)$ from which we get $t = \frac{5}{12}$. This means it will take $\frac{5}{12}$ months, or about 13 days, for 300 students to have had the flu.

3. To graph $y = N(t)$, we first use long division to rewrite $N(t) = \frac{-450}{3t+1} + 500$. From there, we get

$$N(t) = -\frac{450}{3t + 1} + 500 = \frac{-450}{3\left(t + \frac{1}{3}\right)} + 500 = \frac{-150}{t + \frac{1}{3}} + 500$$

Using Theorem 3.1, we start with the graph of $y = \frac{1}{t}$ below on the left and perform the following steps: shift the graph to the left by $\frac{1}{3}$ units, stretch the graph vertically by a factor of 150, reflect the graph across the t-axis, and finally, shift the graph up 500 units. As the domain of N is $t \geq 0$, we obtain the graph below on the right.

[14]Though the population below is more accurately modeled with the functions in Chapter 7, we approximate it (using Calculus, of course!) using a rational function.



4. Owing to the horizontal asymptote, $y = 500$, we have $\lim\limits_{t \to \infty} N(t) = 500$. (More specifically, as $t \to \infty$, $N(t) \to 500^{-}$.) This means as time goes by, only a total of 500 students will have ever had the flu. $\square$

We determined the horizontal asymptote to the graph of $y = N(t)$ in Example 3.1.5 by rewriting $N(t)$ into a form compatible with Theorem 3.1, and while there is nothing wrong with this approach, it will simply not work for general rational functions which cannot be rewritten this way. To that end, we revisit this problem using Theorem 2.3 from Section 2.1. The end behavior of the numerator of $N(t) = \frac{1500t+50}{3t+1}$ is determined by its leading term, $1500t$, and the end behavior of the denominator is likewise determined by its leading term, $3t$. Hence, as $t \to \infty$:

$$N(t) = \frac{1500t + 50}{3t + 1} \approx \frac{1500t}{3t} = 500.$$

Hence $\lim\limits_{t \to \infty} N(t) = 500$ so $y = 500$ is the horizontal asymptote. This same reasoning can be used in general to argue the following theorem.

Theorem 3.3. Location of Horizontal Asymptotes: Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions with leading coefficients a and b, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $\lim\limits_{x \to \infty} r(x) = \frac{a}{b}$ so $y = \frac{a}{b}$ is the[a] horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $\lim\limits_{x \to \infty} r(x) = 0$ so $y = 0$ (the x-axis) is the horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y = r(x)$ has no horizontal asymptotes.

[a]The use of the definite article will be justified momentarily.

So see why Theorem 3.3 works, suppose $r(x) = \frac{p(x)}{q(x)}$ where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$. As $x \to -\infty$ or $x \to \infty$, Theorem 2.3 gives $r(x) \approx \frac{ax^n}{bx^m}$, where n and m are the degrees of $p(x)$ and $q(x)$, respectively.

If the degree of $p(x)$ and the degree of $q(x)$ are the same, then $n = m$ so that $r(x) \approx \frac{ax^n}{bx^n} = \frac{a}{b}$. Hence $\lim\limits_{x \to -\infty} r(x) = \frac{a}{b}$ and $\lim\limits_{x \to \infty} r(x) = \frac{a}{b}$ which means $y = \frac{a}{b}$ is the horizontal asymptote in this case.

If the degree of $p(x)$ is less than the degree of $q(x)$, then $n < m$, so $m - n$ is a positive number, and hence, $r(x) \approx \frac{ax^n}{bx^m} = \frac{a}{bx^{m-n}} \to 0$. As $x \to -\infty$ or $x \to \infty$, $r(x)$ is more or less a fraction with a constant numerator, a, but a denominator which is unbounded. Hence, $\lim\limits_{x \to -\infty} r(x) = 0$ and $\lim\limits_{x \to \infty} r(x) = 0$ producing the horizontal asymptote $y = 0$.

If the degree of $p(x)$ is greater than the degree of $q(x)$, then $n > m$, and hence $n - m$ is a positive number and $r(x) \approx \frac{ax^n}{bx^m} = \frac{ax^{n-m}}{b}$, which is a monomial function from Section 2.1. As such, r becomes unbounded as $x \to -\infty$ or $x \to \infty$.

Note that in the two cases which produce horizontal asymptotes, the behavior of r is identical as $x \to -\infty$ and $x \to \infty$. Hence, if the graph of a rational function has a horizontal asymptote, there is only one.[15]

We put Theorem 3.3 to good use in the following example.

Example 3.1.6. For each function below:

- use Theorem 2.3 to analytically determine the horizontal asymptotes to the graph, if any.
- check your answers using Theorem 3.3 and a graphing utility.
- describe the end behavior of the graph using proper notation.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $F(s) = \dfrac{5s}{s^2 + 1}$
2. $g(x) = \dfrac{x^2 - 4}{x + 1}$

[15]We will (first) encounter functions with more than one horizontal asymptote in Chapter 4.1.

3. $h(t) = \dfrac{6t^3 - 3t + 1}{5 - 2t^3}$

4. $r(x) = 2 - \dfrac{3x^2}{1 - x^2}$

Solution.

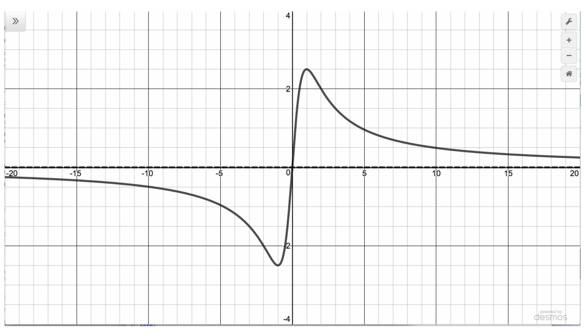
1. Using Theorem 2.3, we get as $s \to -\infty$ or $s \to \infty$, $F(s) = \frac{5s}{s^2+1} \approx \frac{5s}{s^2} = \frac{5}{s}$. Hence, $\displaystyle\lim_{s \to -\infty} F(s) = 0$ and $\displaystyle\lim_{s \to \infty} F(s) = 0$ so $y = 0$ is a horizontal asymptote to the graph.

   Alternatively, to use Theorem 3.3 note the degree of the numerator of $F(s)$, 1, is less than the degree of the denominator, 2, so $y = 0$ as the horizontal asymptote using this approach as well.

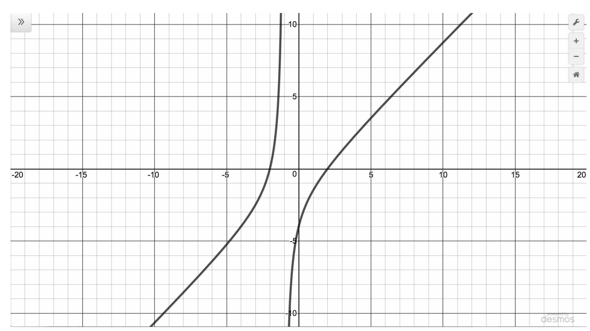
   Graphically, as $s \to -\infty$ or $s \to \infty$, the graph $y = F(s)$ approaches the s-axis ($y = 0$). More specifically, as $s \to -\infty$, $F(s) \to 0^-$ and as $s \to \infty$, $F(s) \to 0^+$.

   As a side note, the graph of F appears to be symmetric about the origin. Indeed, $F(-s) = \frac{5(-s)}{(-s)^2+1} = -\frac{5s}{s^2+1}$ proving F is odd.

2. As $x \to -\infty$ or $x \to \infty$, $g(x) = \frac{x^2-4}{x+1} \approx \frac{x^2}{x} = x$, and while $y = x$ is a line, it is not a horizontal line. Hence, we conclude the graph of $y = g(x)$ has no horizontal asymptotes. Sure enough, Theorem 3.3 supports this since the degree of the numerator of $g(x)$ is 2 which is greater than the degree of the denominator, 1. From the graph, we see that the graph of $y = g(x)$ doesn't appear to level off to a constant value, confirming there is no horizontal asymptote.[16]



The graph of $y = F(s)$



The graph of $y = g(x)$

3. As $t \to -\infty$ or $t \to \infty$, $h(t) = \frac{6t^3-3t+1}{5-2t^3} \approx \frac{6t^3}{-2t^3} = -3$. Hence, $\displaystyle\lim_{t \to -\infty} h(t) = -3$ and $\displaystyle\lim_{t \to \infty} h(t) = -3$, indicating a horizontal asymptote $y = -3$. Sure enough, since the degrees of the numerator and denominator of $h(t)$ are both three, Theorem 3.3 tells us $y = \frac{6}{-2} = -3$ is the horizontal asymptote. We see from the graph of $y = h(t)$ that as $t \to -\infty$, $h(t) \to -3^+$, and as $t \to \infty$, $h(t) \to -3^-$.

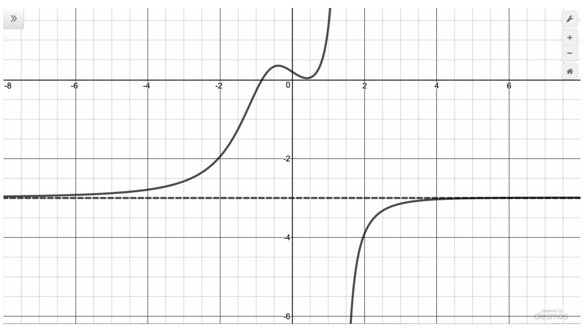
4. If we apply Theorem 2.3 to the term $\frac{3x^2}{1-x^2}$ in the expression for $r(x)$, we find $\frac{3x^2}{1-x^2} \approx \frac{3x^2}{-x^2} = -3$ as $x \to -\infty$ or $x \to \infty$. It seems reasonable to conclude, then, that $\displaystyle\lim_{x \to -\infty} r(x) = 2 - (-3) = 5$ and likewise $\displaystyle\lim_{x \to \infty} r(x) = 2 - (-3) = 5$ so $y = 5$ is our horizontal asymptote.

[16] Sit tight! We'll revisit this function and its end behavior shortly.

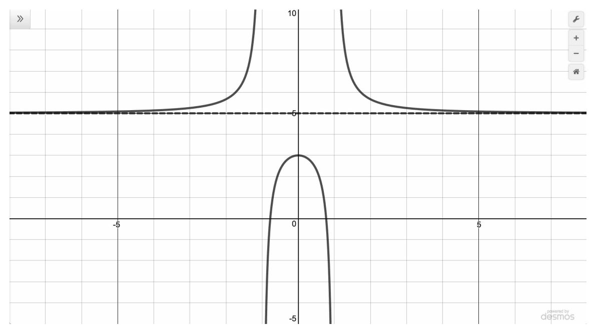
In order to double check this calculation using Theorem 3.3, however, we need to rewrite the expression $r(x)$ with a single denominator: $r(x) = 2 - \frac{3x^2}{1-x^2} = \frac{2(1-x^2)-3x^2}{1-x^2} = \frac{2-5x^2}{1-x^2}$. Now we apply Theorem 3.3 and note since the numerator and denominator have the same degree, we are guaranteed the horizontal asymptote is $y = \frac{-5}{-1} = 5$.

Both calculations are borne out graphically below where it appears as if as $x \to -\infty$ or $x \to \infty$, $r(x) \to 5^+$.

As a final note, the graph of r appears to be symmetric about the y axis. We find $r(-x) = 2 - \frac{3(-x)^2}{1-(-x)^2} = 2 - \frac{3x^2}{1-x^2} = r(x)$, proving r is even.



The graph of $y = h(t)$



The graph of $y = r(x)$

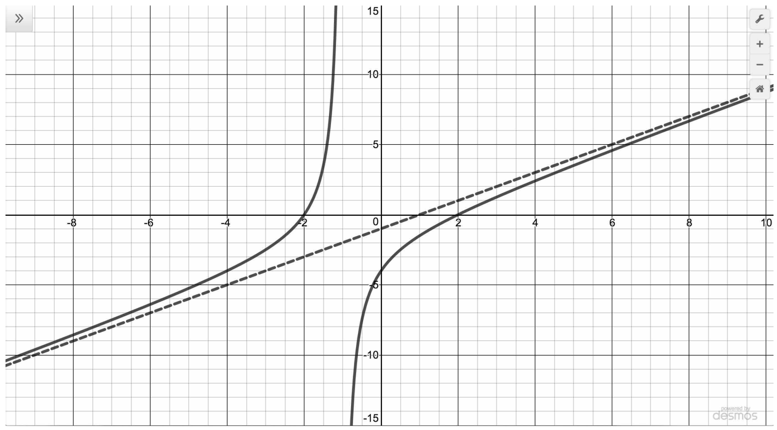
□

We close this section with a discussion of the **third** (and final!) kind of asymptote which can be associated with the graphs of rational functions. Let us return to the function $g(x) = \frac{x^2-4}{x+1}$ in Example 3.1.6. Performing long division,[17] we get $g(x) = \frac{x^2-4}{x+1} = x - 1 - \frac{3}{x+1}$. Since the term $\frac{3}{x+1} \to 0$ as $x \to -\infty$ and as $x \to \infty$, it stands to reason that as x becomes unbounded, the function values $g(x) = x - 1 - \frac{3}{x+1} \approx x - 1$. Geometrically, this means that the graph of $y = g(x)$ should resemble the line $y = x - 1$ as $x \to -\infty$ and $x \to \infty$. We see this play out both numerically and graphically below. (As usual, the asymptote $y = x - 1$ is denoted by a dashed line.)

x	$g(x)$	$x-1$
-10	≈ -10.6667	-11
-100	≈ -100.9697	-101
-1000	≈ -1000.9970	-1001
-10000	≈ -10000.9997	-10001

x	$g(x)$	$x-1$
10	≈ 8.7273	9
100	≈ 98.9703	99
1000	≈ 998.9970	999
10000	≈ 9998.9997	9999

[17] See the remarks following Theorem 3.3.



The way we symbolize the relationship between the end behavior of $y = g(x)$ with that of the line $y = x - 1$ is to write 'as $x \to -\infty$ and $x \to \infty$, $g(x) \to x - 1$' in order to have some notational consistency with what we have done earlier in this section when it comes to end behavior.[18] In this case, we say the line $y = x - 1$ is a **slant asymptote**[19] to the graph of $y = g(x)$. Informally, the graph of a rational function has a slant asymptote if, as $x \to -\infty$ or as $x \to \infty$, the graph resembles a non-horizontal, or 'slanted' line. More formally, we define a slant asymptote as follows.

Definition 3.6. The line $y = mx + b$ where $m \neq 0$ is called a **slant asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to mx + b$.

A few remarks are in order. First, note that the stipulation $m \neq 0$ in Definition 3.6 is what makes the 'slant' asymptote 'slanted' as opposed to the case when $m = 0$ in which case we'd have a horizontal asymptote. Secondly, while we have motivated what me mean intuitively by the notation '$f(x) \to mx + b$,' like so many ideas in this section, the formal definition requires Calculus. Another way to express this sentiment, however, is to rephrase '$f(x) \to mx + b$' as '$[f(x) - (mx + b)] \to 0$.' In other words, the graph of $y = f(x)$ has the **slant** asymptote $y = mx + b$ if and only if the graph of $y = f(x) - (mx + b)$ has a **horizontal** asymptote $y = 0$. This last sentiment can be encoded using limit notation as follows.

Definition 3.7. The line $y = mx + b$ where $m \neq 0$ is called a **slant asymptote** of the graph of a function $y = f(x)$ if either $\displaystyle\lim_{x \to -\infty} [f(x) - (mx + b)] = 0$ or $\displaystyle\lim_{x \to \infty} [f(x) - (mx + b)] = 0$ (or both!).

Our next task is to determine the conditions under which the graph of a rational function has a slant asymptote, and if it does, how to find it. In the case of $g(x) = \frac{x^2-4}{x+1}$, the degree of the numerator $x^2 - 4$ is 2, which is **exactly one more** than the degree if its denominator $x + 1$ which is 1. This results in a **linear** quotient polynomial, and it is this quotient polynomial which is the slant asymptote. Generalizing this situation gives us the following theorem.[20]

[18] Other notations include $g(x) \asymp x - 1$ or $g(x) \sim x - 1$.

[19] Also called an 'oblique' asymptote in some, ostensibly higher class (and more expensive), texts.

[20] Once again, this theorem is brought to you courtesy of Theorem 2.2 and Calculus.

Theorem 3.4. Determination of Slant Asymptotes: Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where the degree of p is **exactly** one more than the degree of q. Then the graph of $y = r(x)$ has the slant asymptote $y = L(x)$ where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$.

In the same way that Theorem 3.3 gives us an easy way to see if the graph of a rational function $r(x) = \frac{p(x)}{q(x)}$ has a horizontal asymptote by comparing the degrees of the numerator and denominator, Theorem 3.4 gives us an easy way to check for slant asymptotes. Unlike Theorem 3.3, which gives us a quick way to **find** the horizontal asymptotes (if any exist), Theorem 3.4 gives us no such 'short-cut'. If a slant asymptote exists, we have no recourse but to use long division to find it.[21]

Example 3.1.7. For each of the following functions:

- find the slant asymptote, if it exists.
- verify your answer using a graphing utility.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$
2. $g(t) = \dfrac{t^2 - 4}{t - 2}$
3. $h(x) = \dfrac{x^3 + 1}{x^2 - 4}$
4. $r(t) = 2t - 1 + \dfrac{4t^3}{1 - t^2}$

Solution.

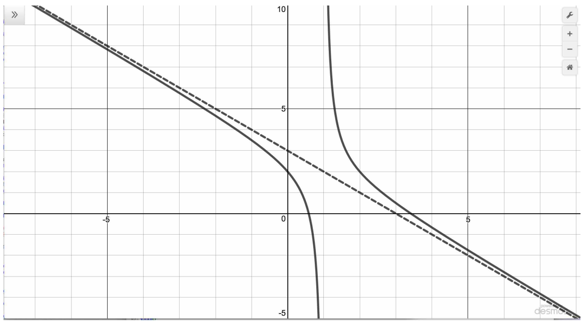
1. The degree of the numerator is 2 and the degree of the denominator is 1, so Theorem 3.4 guarantees us a slant asymptote. To find it, we divide $1 - x = -x + 1$ into $x^2 - 4x + 2$ and get a quotient of $-x + 3$, so our slant asymptote is $y = -x + 3$. We confirm this graphically below.

2. As with the previous example, the degree of the numerator $g(t) = \frac{t^2-4}{t-2}$ is 2 and the degree of the denominator is 1, so Theorem 3.4 applies. In this case,

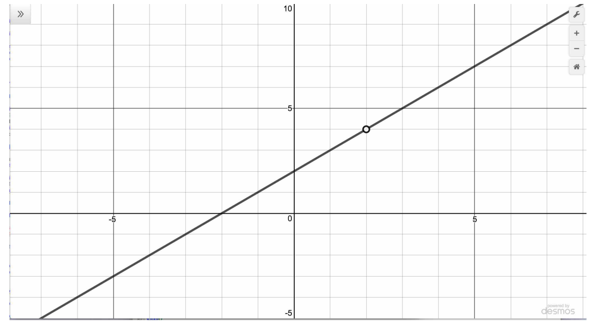
$$g(t) = \frac{t^2 - 4}{t - 2} = \frac{(t+2)(t-2)}{(t-2)} = \frac{(t+2)\cancel{(t-2)}}{\cancel{(t-2)}^{\,1}} = t + 2, \quad t \neq 2$$

so we have that the slant asymptote $y = t+2$ is identical to the graph of $y = g(t)$ except at $t = 2$ (where the latter has a 'hole' at $(2, 4)$.) While the word 'asymptote' has the connotation of 'approaching but not equaling,' Definitions 3.4 and 3.6 allow for these extreme cases.

[21] That's OK, though. In the next section, we'll use long division to analyze end behavior and it's worth the effort!



The graph of $y = f(x)$



The graph of $y = g(t)$

3. For $h(x) = \frac{x^3+1}{x^2-4}$, the degree of the numerator is 3 and the degree of the denominator is 2 so again, we are guaranteed the existence of a slant asymptote. The long division $\left(x^3+1\right) \div \left(x^2-4\right)$ gives a quotient of just x, so our slant asymptote is the line $y = x$. The graphing utility confirms this.

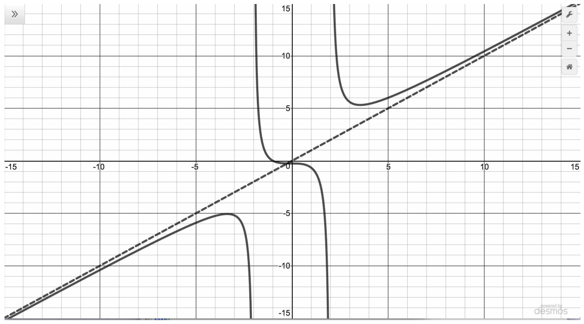
   Note the graph of h appears to be symmetric about the origin. We check $h(-x) = \frac{(-x)^3+1}{(-x)^2-4} = \frac{-x^3+1}{x^2-4} = -\frac{x^3-1}{x^2-4}$. However, $-h(x) = -\frac{x^3+1}{x^2-4}$, so it appears as if $h(-x) \neq -h(x)$ for all x. Checking $x = 1$, we find $h(1) = -\frac{2}{3}$ but $h(-1) = 0$ which shows the graph of h, is in fact, **not** symmetric about the origin.

4. For our last example, $r(t) = 2t - 1 + \frac{4t^3}{1-t^2}$, the expression $r(t)$ is not in the form to apply Theorem 3.4 directly. We can, nevertheless, appeal to the spirit of the theorem and use long division to rewrite the term $\frac{4t^3}{1-t^2} = -4t + \frac{4t}{1-t^2}$. We then get:

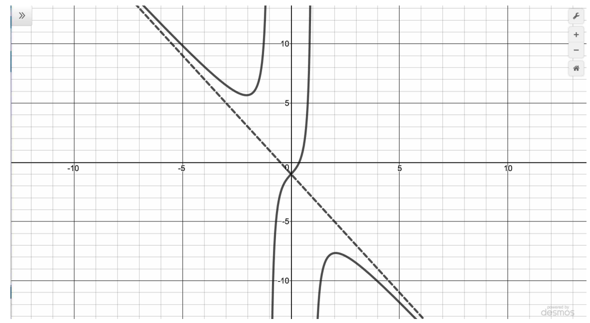
$$
\begin{aligned}
r(t) &= 2t - 1 + \frac{4t^3}{1-t^2} \\
&= 2t - 1 - 4t + \frac{4t}{1-t^2} \\
&= -2t - 1 + \frac{4t}{1-t^2}
\end{aligned}
$$

   As $t \to -\infty$ or $x \to \infty$, Theorem 2.3 gives $\frac{4t}{1-t^2} \approx \frac{4t}{-t^2} = -\frac{4}{t} \to 0$. Hence, as $t \to -\infty$ or $t \to \infty$, $r(t) \to -2t - 1$, so $y = -2t - 1$ is the slant asymptote to the graph as confirmed by the graphing utility below. From a distance, the graph of r appears to be symmetric about the origin. However, if we look carefully, we see the y-intercept is $(0, -1)$, as borne out by the computation $r(0) = -1$. Hence r cannot be odd.[22]

[22]Do you see why?



The graph of $y = h(x)$



The graph of $y = r(t)$

□

Our last example gives a real-world application of a slant asymptote. The problem features the concept of **average profit**. The average profit, denoted $\overline{P}(x)$, is the total profit, $P(x)$, divided by the number of items sold, x. In English, the average profit tells us the profit made per item sold. It, along with average cost, is defined below.

Definition 3.8. Let $C(x)$ and $P(x)$ represent the cost and profit to make and sell x items, respectively.

- The **average cost**, $\overline{C}(x) = \dfrac{C(x)}{x}$, $x > 0$.

  NOTE: The average cost is the cost per item produced.

- The **average profit**, $\overline{P}(x) = \dfrac{P(x)}{x}$, $x > 0$.

  NOTE: The average profit is the profit per item sold.

You'll explore average cost (and its relation to variable cost) in Exercise 37. For now, we refer the reader to to Example 1.4.3 in Section 1.4.

Example 3.1.8. Recall the profit (in dollars) when x PortaBoy game systems are produced and sold is given by $P(x) = -1.5x^2 + 170x - 150$, $0 \leq x \leq 166$.

1. Find and simplify an expression for the average profit, $\overline{P}(x)$. What is the domain of $\overline{P}$?

2. Find and interpret $\overline{P}(50)$.

3. Determine the slant asymptote to the graph of $y = \overline{P}(x)$. Check your answer using a graphing utility.

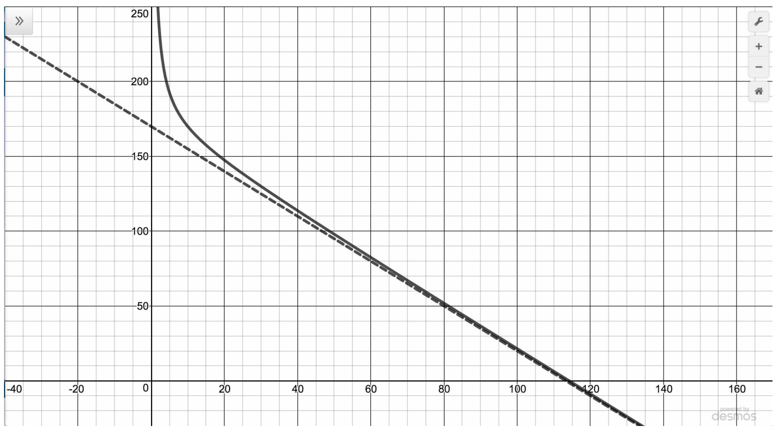
4. Interpret the slope of the slant asymptote.

Solution.

1. We find $\overline{P}(x) = \frac{P(x)}{x} = \frac{-1.5x^2+170x-150}{x} = -1.5x + 170 - \frac{150}{x}$. Since the domain of P is $[0, 166]$ but $x \neq 0$, the domain of $\overline{P}$ is $(0, 166]$.

2. We find $\overline{P}(50) = -1.5(50) + 170 - \frac{150}{50} = 92$. This means that when 50 PortaBoy systems are sold, the average profit is \$92 per system.

3. Technically, the graph of $y = \overline{P}(x)$ has no slant asymptote since the domain of the function is restricted to $(0, 166]$. That being said, if we were to let $x \to \infty$, the term $\frac{150}{x} \to 0$, so we'd have $\overline{P}(x) \to -1.5x + 170$. This means the slant asymptote would be $y = -1.5x + 170$. We graph $y = \overline{P}(x)$ and $y = -1.5x + 170$.



4. The slope of the slant asymptote $y = -1.5x + 170$ is -1.5. Allowing for $x \to \infty$, $\overline{P}(x) \approx -1.5x + 170$ which means as we sell more systems, the average profit is **decreasing** at about a rate of \$1.50 per system. If the number 1.5 sounds familiar to this problem situation, it should. In Example 1.2.4 in Section 1.2, we determined the slope of the demand function to be -1.5. In that situation, the -1.5 meant that in order to sell an additional system, the price had to drop by \$1.50. The fact the average profit is decreasing at more or less this same rate means the loss in profit per system can be attributed to the reduction in price needed to sell each additional system.[23] $\square$

[23] We generalize this result in Exercise 38.

3.1.4 Exercises

(Review of Long Division):[24] In Exercises 1 - 6, use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.

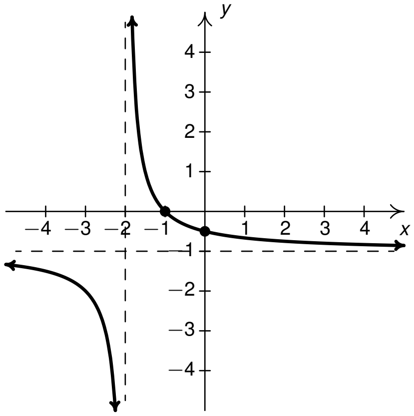
1. $\left(4x^2 + 3x - 1\right) \div (x - 3)$
2. $\left(2x^3 - x + 1\right) \div \left(x^2 + x + 1\right)$
3. $\left(5x^4 - 3x^3 + 2x^2 - 1\right) \div \left(x^2 + 4\right)$
4. $\left(-x^5 + 7x^3 - x\right) \div \left(x^3 - x^2 + 1\right)$
5. $\left(9x^3 + 5\right) \div (2x - 3)$
6. $\left(4x^2 - x - 23\right) \div \left(x^2 - 1\right)$

In Exercises 7 - 10, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem 3.1. Track at least two points and the asymptotes. State the domain and range using interval notation.

7. $f(x) = \dfrac{1}{x}$, $F(x) = \dfrac{1}{x-2} + 1$
8. $f(x) = \dfrac{1}{x}$, $F(x) = \dfrac{2x}{x+1}$
9. $f(x) = x^{-1}$, $F(x) = 4x(2x+1)^{-1}$
10. $f(x) = x^{-2}$, $F(x) = -(x-1)^{-2} + 3$

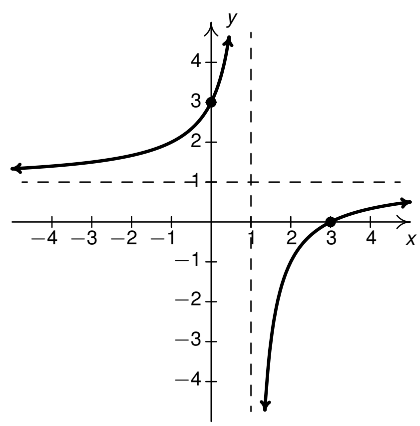
In Exercises 11 - 12, find a formula for each function below in the form $F(x) = \dfrac{a}{x-h} + k$.

11. $y = F(x)$



x-intercept $(-1, 0)$, y-intercept $\left(0, -\frac{1}{2}\right)$

12. $y = F(x)$



x-intercept $(3, 0)$, y-intercept $(0, 3)$

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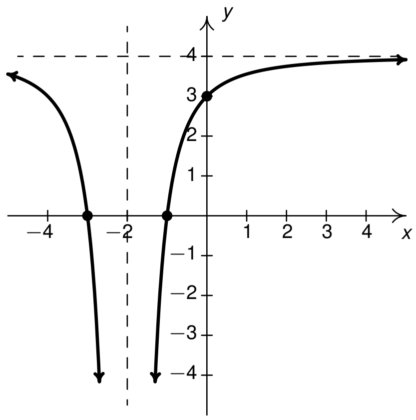
[24] For more review, see Section A.8.2.

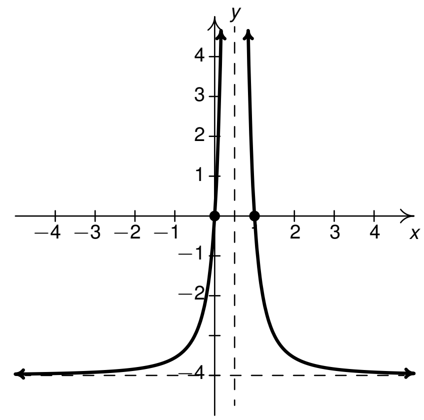
In Exercises 13 - 14, find a formula for each function below in the form $F(x) = \dfrac{a}{(x-h)^2} + k$.

13. $y = F(x)$



x-intercepts $(-3, 0)$, $(-1, 0)$, y-intercept $(0, 3)$

14. $y = F(x)$



x-intercepts $(0, 0)$, $(1, 0)$, Vertical Asymptote: $x = \frac{1}{2}$

In Exercises 15 - 32, for the given rational function:

- State the domain.
- Identify any vertical asymptotes of the graph.
- Identify any holes in the graph.
- Find the horizontal asymptote, if it exists.
- Find the slant asymptote, if it exists.
- Graph the function using a graphing utility and describe the behavior near the asymptotes.

15. $f(x) = \dfrac{x}{3x-6}$

16. $f(x) = \dfrac{3+7x}{5-2x}$

17. $f(x) = \dfrac{x}{x^2+x-12}$

18. $g(t) = \dfrac{t}{t^2+1}$

19. $g(t) = \dfrac{t+7}{(t+3)^2}$

20. $g(t) = \dfrac{t^3+1}{t^2-1}$

21. $r(z) = \dfrac{4z}{z^2+4}$

22. $r(z) = \dfrac{4z}{z^2-4}$

23. $r(z) = \dfrac{z^2-z-12}{z^2+z-6}$

24. $f(x) = \dfrac{3x^2-5x-2}{x^2-9}$

25. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2}$

26. $f(x) = \dfrac{x^3-3x+1}{x^2+1}$

27. $g(t) = \dfrac{2t^2+5t-3}{3t+2}$

28. $g(t) = \dfrac{-t^3+4t}{t^2-9}$

29. $g(t) = \dfrac{-5t^4-3t^3+t^2-10}{t^3-3t^2+3t-1}$

30. $r(z) = \dfrac{z^3}{1-z}$

31. $r(z) = \dfrac{18-2z^2}{z^2-9}$

32. $r(z) = \dfrac{z^3-4z^2-4z-5}{z^2+z+1}$

33. The cost $C(p)$ in dollars to remove p% of the invasive Ippizuti fish species from Sasquatch Pond is:

$$C(p) = \frac{1770p}{100 - p}, \quad 0 \leq p < 100$$

(a) Find and interpret $C(25)$ and $C(95)$.

(b) What does the vertical asymptote at $x = 100$ mean within the context of the problem?

(c) What percentage of the Ippizuti fish can you remove for \$40000?

34. In the scenario of Example 3.1.3, $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. For each of the times t_0 listed below, find and simplify a the formula for the average velocity $\overline{v}(t)$ between t and t_0 (see Definition 3.5) and use $\overline{v}(t)$ to find and interpret the instantaneous velocity of the rocket at $t = t_0$ (See Example 3.1.3).

(a) $t_0 = 5$ (b) $t_0 = 9$ (c) $t_0 = 10$ (d) $t_0 = 11$

35. The population of Sasquatch in Portage County t years after the year 1803 is modeled by the function

$$P(t) = \frac{150t}{t + 15}.$$

Find and interpret the horizontal asymptote of the graph of $y = P(t)$ and explain what it means.

36. The cost in dollars, $C(x)$ to make x dOpi media players is $C(x) = 100x + 2000$, $x \geq 0$. You may wish to review the concepts of fixed and variable costs introduced in Example 1.2.3 in Section 1.2.2.

(a) Find a formula for the average cost $\overline{C}(x)$.

(b) Find and interpret $\overline{C}(1)$ and $\overline{C}(100)$.

(c) How many dOpis need to be produced so that the average cost per dOpi is \$200?

(d) Find and interpret $\displaystyle\lim_{x \to 0^+} \overline{C}(x)$.

(e) Interpret the behavior of $\overline{C}(x)$ as $x \to \infty$.

37. This exercise explores the relationships between fixed cost, variable cost, and average cost. The reader is encouraged to revisit Example 1.2.3 in Section 1.2.2 as needed. Suppose the cost in dollars $C(x)$ to make x items is given by $C(x) = mx + b$ where m and b are positive real numbers.

(a) Show the fixed cost (the money spent even if no items are made) is b.

(b) Show the variable cost (the increase in cost per item made) is m.

(c) Find a formula for the average cost when making x items, $\overline{C}(x)$.

(d) Show $\overline{C}(x) > m$ for all $x > 0$ and, moreover, $\overline{C}(x) \to m^+$ as $x \to \infty$.

(e) Interpret $\overline{C}(x) \to m^+$ both geometrically and in terms of fixed, variable, and average costs.

38. Suppose the price-demand function for a particular product is given by $p(x) = mx + b$ where x is the number of items made and sold for $p(x)$ dollars. Here, $m < 0$ and $b > 0$. If the cost (in dollars) to make x of these products is also a linear function $C(x)$, show that the graph of the average profit function $\overline{P}(x)$ has a slant asymptote with slope m and interpret.

39. In Exercise 58 in Section 2.1, we fit a few polynomial models to the following electric circuit data. The circuit was built with a variable resistor. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, mW) is given below.[25]

Resistance: ($k\Omega$)	1.012	2.199	3.275	4.676	6.805	9.975
Power: (mW)	1.063	1.496	1.610	1.613	1.505	1.314

Using some fundamental laws of circuit analysis mixed with a healthy dose of algebra, we can derive the actual formula relating power $P(x)$ to resistance x:

$$P(x) = \frac{25x}{(x+3.9)^2}, \quad x \geq 0.$$

(a) Graph the data along with the function $y = P(x)$ using a graphing utility.

(b) Use a graphing utility to approximate the maximum power that can be delivered to the load. What is the corresponding resistance value?

(c) Find and interpret the end behavior of $P(x)$ as $x \to \infty$.

40. Let $f(x) = \dfrac{ax^2 - c}{x + 3}$. Find values for a and c so the graph of f has a hole at $(-3, 12)$.

41. Let $f(x) = \dfrac{ax^n - 4}{2x^2 + 1}$.

(a) Find values for a and n so the graph of $y = f(x)$ has the horizontal asymptote $y = 3$.

(b) Find values for a and n so the graph of $y = f(x)$ has the slant asymptote $y = 5x$.

42. Suppose p is a polynomial function and a is a real number. Define $r(x) = \dfrac{p(x) - p(a)}{x - a}$. Use the Factor Theorem, Theorem 2.8, to prove the graph of $y = r(x)$ has a hole at $x = a$.

43. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[26] What trends do you observe? How do your answers manifest themselves graphically? How do you results compare with those of Exercise 51 in Section 2.1?

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
x^{-1}				
x^{-2}				
x^{-3}				
x^{-4}				

[25]The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

[26]See Definition 1.8 in Section 1.2.4 for a review of this concept, as needed.

44. In his now famous 1919 dissertation The Learning Curve Equation, Louis Leon Thurstone presents a rational function which models the number of words a person can type in four minutes as a function of the number of pages of practice one has completed.[27] Using his original notation and original language, we have $Y = \frac{L(X+P)}{(X+P)+R}$ where L is the predicted practice limit in terms of speed units, X is pages written, Y is writing speed in terms of words in four minutes, P is equivalent previous practice in terms of pages and R is the rate of learning. In Figure 5 of the paper, he graphs a scatter plot and the curve $Y = \frac{216(X+19)}{X+148}$. Discuss this equation with your classmates. How would you update the notation? Explain what the horizontal asymptote of the graph means. You should take some time to look at the original paper. Skip over the computations you don't understand yet and try to get a sense of the time and place in which the study was conducted.

[27] This paper, which is now in the public domain and can be found here, is from a bygone era when students at business schools took typing classes on manual typewriters.

3.1.5 Answers

1. $4x^2 + 3x - 1 = (x - 3)(4x + 15) + 44$

2. $2x^3 - x + 1 = \left(x^2 + x + 1\right)(2x - 2) + (-x + 3)$

3. $5x^4 - 3x^3 + 2x^2 - 1 = \left(x^2 + 4\right)\left(5x^2 - 3x - 18\right) + (12x + 71)$

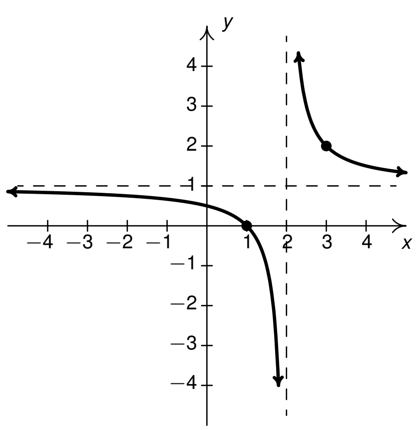
4. $-x^5 + 7x^3 - x = \left(x^3 - x^2 + 1\right)\left(-x^2 - x + 6\right) + \left(7x^2 - 6\right)$

5. $9x^3 + 5 = (2x - 3)\left(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}\right) + \frac{283}{8}$

6. $4x^2 - x - 23 = \left(x^2 - 1\right)(4) + (-x - 19)$

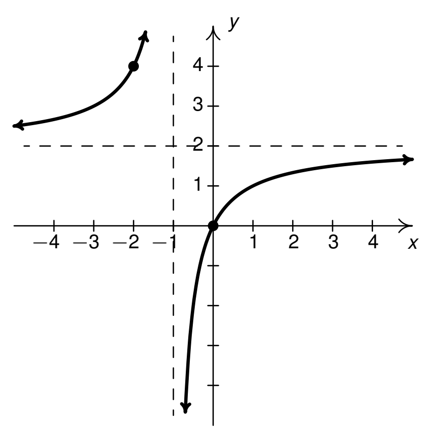
7. $F(x) = \dfrac{1}{x - 2} + 1$

   Domain: $(-\infty, 2) \cup (2, \infty)$
   Range: $(-\infty, 1) \cup (1, \infty)$
   Vertical asymptote: $x = 2$
   Horizontal asymptote: $y = 1$



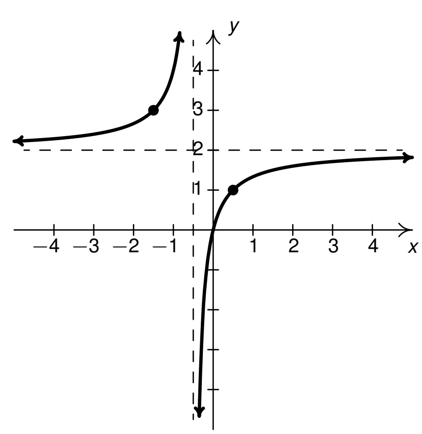
8. $F(x) = \dfrac{2x}{x + 1} = \dfrac{-2}{x + 1} + 2$

   Domain: $(-\infty, -1) \cup (-1, \infty)$
   Range: $(-\infty, 2) \cup (2, \infty)$
   Vertical asymptote: $x = -1$
   Horizontal asymptote: $y = 2$



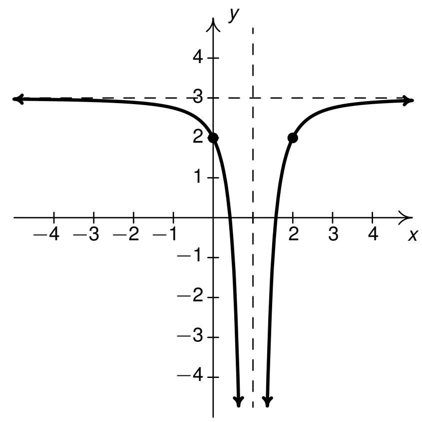
9. $F(x) = 4x(2x + 1)^{-1} = \dfrac{4x}{2x + 1} = \dfrac{-1}{x + \frac{1}{2}} + 2$

   Domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$
   Range: $(-\infty, 2) \cup (2, \infty)$
   Vertical asymptote: $x = -\frac{1}{2}$
   Horizontal asymptote: $y = 2$



10. $F(x) = -(x-1)^{-2} + 3 = \dfrac{-1}{(x-1)^2} + 3$

Domain: $(-\infty, 1) \cup (1, \infty)$
Range: $(-\infty, 3) \cup (3, \infty)$
Vertical asymptote: $x = 1$
Horizontal asymptote: $y = 3$



11. $F(x) = \dfrac{1}{x+2} - 1$

12. $F(x) = \dfrac{-2}{x-1} + 1$

13. $F(x) = \dfrac{-4}{(x+2)^2} + 4$

14. $F(x) = \dfrac{1}{\left(x - \frac{1}{2}\right)^2} - 4$

15. $f(x) = \dfrac{x}{3x-6}$
Domain: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
$\lim\limits_{x \to 2^-} f(x) = -\infty$, $\lim\limits_{x \to 2^+} f(x) = \infty$
No holes in the graph
Horizontal asymptote: $y = \frac{1}{3}$
$\lim\limits_{x \to -\infty} f(x) = \frac{1}{3}$
More specifically: as $x \to -\infty$, $f(x) \to \frac{1}{3}^-$
$\lim\limits_{x \to \infty} f(x) = \frac{1}{3}$
More specifically: as $x \to \infty$, $f(x) \to \frac{1}{3}^+$

16. $f(x) = \dfrac{3+7x}{5-2x}$
Domain: $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$
Vertical asymptote: $x = \frac{5}{2}$
$\lim\limits_{x \to \frac{5}{2}^-} f(x) = \infty$, $\lim\limits_{x \to \frac{5}{2}^+} f(x) = -\infty$
No holes in the graph
Horizontal asymptote: $y = -\frac{7}{2}$
$\lim\limits_{x \to -\infty} f(x) = -\frac{7}{2}$
More specifically: as $x \to -\infty$, $f(x) \to -\frac{7}{2}^+$
$\lim\limits_{x \to \infty} f(x) = -\frac{7}{2}$
More specifically: as $x \to \infty$, $f(x) \to -\frac{7}{2}^-$

17. $f(x) = \dfrac{x}{x^2+x-12} = \dfrac{x}{(x+4)(x-3)}$
Domain: $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$
Vertical asymptotes: $x=-4$, $x=3$
$\lim\limits_{x\to -4^-} f(x) = -\infty$, $\lim\limits_{x\to -4^+} f(x) = \infty$
$\lim\limits_{x\to 3^-} f(x) = -\infty$, $\lim\limits_{x\to 3^+} f(x) = \infty$
No holes in the graph
Horizontal asymptote: $y=0$
$\lim\limits_{x\to -\infty} f(x) = 0$
More specifically, as $x\to -\infty$, $f(x)\to 0^-$
$\lim\limits_{x\to \infty} f(x) = 0$
More specifically, as $x\to \infty$, $f(x)\to 0^+$

18. $g(t) = \dfrac{t}{t^2+1}$
Domain: $(-\infty,\infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y=0$
$\lim\limits_{t\to -\infty} g(t) = 0$
More specifically, as $t\to -\infty$, $g(t)\to 0^-$
$\lim\limits_{t\to \infty} g(t) = 0$
More specifically, as $t\to \infty$, $g(t)\to 0^+$

19. $g(t) = \dfrac{t+7}{(t+3)^2}$
Domain: $(-\infty,-3)\cup(-3,\infty)$
Vertical asymptote: $t=-3$
$\lim\limits_{t\to -3} g(t) = \infty$
No holes in the graph
Horizontal asymptote: $y=0$
$\lim\limits_{t\to -\infty} g(t) = 0$
[28]More specifically, as $t\to -\infty$, $g(t)\to 0^-$
$\lim\limits_{t\to \infty} g(t) = 0$
More specifically, as $t\to \infty$, $g(t)\to 0^+$

20. $g(t) = \dfrac{t^3+1}{t^2-1} = \dfrac{t^2-t+1}{t-1}$
Domain: $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$
Vertical asymptote: $t=1$
$\lim\limits_{t\to 1^-} g(t) = -\infty$, $\lim\limits_{t\to 1^+} g(t) = \infty$
Hole at $(-1,-\frac{3}{2})$
Slant asymptote: $y=t$
$\lim\limits_{t\to -\infty} g(t) = -\infty$
As $t\to -\infty$, the graph is below $y=t$
$\lim\limits_{t\to \infty} g(t) = \infty$
As $t\to \infty$, the graph is above $y=t$

21. $r(z) = \dfrac{4z}{z^2+4}$
Domain: $(-\infty,\infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y=0$
$\lim\limits_{z\to -\infty} r(z) = 0$
More specifically, as $z\to -\infty$, $r(z)\to 0^-$
$\lim\limits_{z\to \infty} r(z) = 0$
More specifically, as $z\to \infty$, $r(z)\to 0^+$

22. $r(z) = \dfrac{4z}{z^2-4} = \dfrac{4z}{(z+2)(z-2)}$
Domain: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$
Vertical asymptotes: $z=-2$, $z=2$
$\lim\limits_{z\to -2^-} r(z) = -\infty$, $\lim\limits_{z\to -2^+} r(z) = \infty$
$\lim\limits_{z\to 2^-} r(z) = -\infty$, $\lim\limits_{z\to 2^+} r(z) = \infty$
No holes in the graph
Horizontal asymptote: $y=0$
$\lim\limits_{z\to -\infty} r(z) = 0$
More specifically, as $z\to -\infty$, $r(z)\to 0^-$
$\lim\limits_{z\to \infty} r(z) = 0$
More specifically, as $z\to \infty$, $r(z)\to 0^+$

[28]This is hard to see on the calculator, but trust me, the graph is below the t-axis to the left of $t=-7$.

23. $r(z) = \dfrac{z^2 - z - 12}{z^2 + z - 6} = \dfrac{z-4}{z-2}$
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
Vertical asymptote: $z = 2$
$\displaystyle\lim_{z\to 2^-} r(z) = \infty$, $\displaystyle\lim_{z\to 2^+} r(z) = -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y = 1$
$\displaystyle\lim_{z\to -\infty} r(z) = 1$
More specifically, as $z \to -\infty$, $r(z) \to 1^+$
$\displaystyle\lim_{z\to \infty} r(z) = 1$
More specifically, as $z \to \infty$, $r(z) \to 1^-$

24. $f(x) = \dfrac{3x^2 - 5x - 2}{x^2 - 9} = \dfrac{(3x+1)(x-2)}{(x+3)(x-3)}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -3$, $x = 3$
$\displaystyle\lim_{x\to -3^-} f(x) = \infty$, $\displaystyle\lim_{x\to -3^+} f(x) = -\infty$
$\displaystyle\lim_{x\to 3^-} f(x) = -\infty$, $\displaystyle\lim_{x\to 3^+} f(x) = \infty$
No holes in the graph
Horizontal asymptote: $y = 3$
$\displaystyle\lim_{x\to -\infty} f(x) = 3$
More specifically, as $x \to -\infty$, $f(x) \to 3^+$
$\displaystyle\lim_{x\to \infty} f(x) = 3$
More specifically, as $x \to \infty$, $f(x) \to 3^-$

25. $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2} = \dfrac{x(x+1)}{x-2}$
Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
$\displaystyle\lim_{x\to 2^-} f(x) = -\infty$, $\displaystyle\lim_{x\to 2^+} f(x) = \infty$
Hole at $(-1, 0)$
Slant asymptote: $y = x + 3$
$\displaystyle\lim_{x\to -\infty} f(x) = -\infty$
As $x \to -\infty$, the graph is below $y = x + 3$
$\displaystyle\lim_{x\to \infty} f(x) = \infty$
As $x \to \infty$, the graph is above $y = x + 3$

26. $f(x) = \dfrac{x^3 - 3x + 1}{x^2 + 1}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Slant asymptote: $y = x$
$\displaystyle\lim_{x\to -\infty} f(x) = -\infty$
As $x \to -\infty$, the graph is above $y = x$
$\displaystyle\lim_{x\to \infty} f(x) = \infty$
As $x \to \infty$, the graph is below $y = x$

27. $g(t) = \dfrac{2t^2 + 5t - 3}{3t + 2}$
Domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
Vertical asymptote: $t = -\frac{2}{3}$
$\displaystyle\lim_{t\to -\frac{2}{3}^-} g(t) = \infty$, $\displaystyle\lim_{t\to -\frac{2}{3}^+} g(t) = -\infty$
No holes in the graph
Slant asymptote: $y = \frac{2}{3}t + \frac{11}{9}$
$\displaystyle\lim_{t\to -\infty} g(t) = -\infty$
As $t \to -\infty$, the graph is above $y = \frac{2}{3}t + \frac{11}{9}$
$\displaystyle\lim_{t\to \infty} g(t) = \infty$
As $t \to \infty$, the graph is below $y = \frac{2}{3}t + \frac{11}{9}$

28. $g(t) = \dfrac{-t^3 + 4t}{t^2 - 9} = \dfrac{-t^3 + 4t}{(t-3)(t+3)}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Vertical asymptotes: $t = -3$, $t = 3$
$\displaystyle\lim_{t\to -3^-} g(t) = \infty$, $\displaystyle\lim_{t\to -3^+} g(t) = -\infty$
$\displaystyle\lim_{t\to 3^-} g(t) = \infty$, $\displaystyle\lim_{t\to 3^+} g(t) = -\infty$
No holes in the graph
Slant asymptote: $y = -t$
$\displaystyle\lim_{t\to -\infty} g(t) = \infty$
As $t \to -\infty$, the graph is above $y = -t$
$\displaystyle\lim_{t\to \infty} g(t) = -\infty$
As $t \to \infty$, the graph is below $y = -t$

29. $g(t) = \dfrac{-5t^4 - 3t^3 + t^2 - 10}{t^3 - 3t^2 + 3t - 1} = \dfrac{-5t^4 - 3t^3 + t^2 - 10}{(t-1)^3}$
Domain: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptotes: $t = 1$
$\lim\limits_{t \to 1^-} g(t) = \infty$, $\lim\limits_{t \to 1^+} g(t) = -\infty$
No holes in the graph
Slant asymptote: $y = -5t - 18$
$\lim\limits_{t \to -\infty} g(t) = \infty$
As $t \to -\infty$, the graph is above $y = -5t - 18$
$\lim\limits_{t \to \infty} g(t) = -\infty$
As $t \to \infty$, the graph is below $y = -5t - 18$

30. $r(z) = \dfrac{z^3}{1 - z}$
Domain: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptote: $z = 1$
$\lim\limits_{z \to 1^-} r(z) = \infty$
$\lim\limits_{z \to 1^+} r(z) = -\infty$
No holes in the graph
No horizontal or slant asymptote
$\lim\limits_{z \to -\infty} r(z) = -\infty$
$\lim\limits_{z \to \infty} r(z) = -\infty$

31. $r(z) = \dfrac{18 - 2z^2}{z^2 - 9} = -2$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
No vertical asymptotes
Holes in the graph at $(-3, -2)$ and $(3, -2)$
Horizontal asymptote $y = -2$
$\lim\limits_{z \to -\infty} r(z) = -2$
$\lim\limits_{z \to \infty} r(z) = -2$

32. $r(z) = \dfrac{z^3 - 4z^2 - 4z - 5}{z^2 + z + 1} = z - 5$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Slant asymptote: $y = z - 5$
$\lim\limits_{z \to -\infty} r(z) = -\infty$
$\lim\limits_{z \to \infty} r(z) = \infty$
$r(z) = z - 5$ everywhere.

33. (a) $C(25) = 590$ means it costs \$590 to remove 25% of the fish and and $C(95) = 33630$ means it would cost \$33630 to remove 95% of the fish from the pond.

(b) The vertical asymptote at $x = 100$ means that as we try to remove 100% of the fish from the pond, the cost increases without bound; i.e., it's impossible to remove all of the fish.

(c) For \$40000 you could remove about 95.76% of the fish.

34. (a) $\overline{v}(t) = \frac{s(t)-s(5)}{t-5} = \frac{-5t^2+100t-375}{t-5} = -5t + 75$, $t \neq 5$. The instantaneous velocity of the rocket when $t_0 = 5$ is $-5(5) + 75 = 50$ meaning it is traveling 50 feet per second upwards.

(b) $\overline{v}(t) = \frac{s(t)-s(9)}{t-9} = \frac{-5t^2+100t-495}{t-9} = -5t + 55$, $t \neq 9$. The instantaneous velocity of the rocket when $t_0 = 9$ is $-5(9) + 55 = 10$, so the rocket has slowed to 10 feet per second (but still heading up.)

(c) $\overline{v}(t) = \frac{s(t)-s(10)}{t-10} = \frac{-5t^2+100t-495}{t-10} = -5t + 50$, $t \neq 10$. The instantaneous velocity of the rocket when $t_0 = 10$ is $-5(10) + 50 = 0$, so the rocket has momentarily stopped! In Example 1.2.8, we learned the rocket reaches its maximum height when $t = 10$ seconds, which means the rocket must change direction from heading up to coming back down, so it makes sense that for this instant, its velocity is 0.

(d) $\overline{v}(t) = \frac{s(t)-s(11)}{t-11} = \frac{-5t^2+100t-495}{t-11} = -5t+45,\ t \neq 11$. The instantaneous velocity of the rocket when $t_0 = 11$ is $-5(11)+45 = -10$ meaning the rocket has, indeed, changed direction and is heading downwards at a rate of 10 feet per second. (Note the symmetry here between this answer and our answer when $t = 9$.)

35. The horizontal asymptote of the graph of $P(t) = \frac{150t}{t+15}$ is $y = 150$ and it means that the model predicts the population of Sasquatch in Portage County will never exceed 150.

36. (a) $\overline{C}(x) = \frac{100x+2000}{x} = 100 + \frac{2000}{x},\ x > 0$.

(b) $\overline{C}(1) = 2100$ and $\overline{C}(100) = 120$. When just 1 dOpi is produced, the cost per dOpi is \$2100, but when 100 dOpis are produced, the cost per dOpi is \$120.

(c) $\overline{C}(x) = 200$ when $x = 20$. So to get the cost per dOpi to \$200, 20 dOpis need to be produced.

(d) We find $\lim\limits_{x \to 0^+} \overline{C}(x) = \infty$. This means that as fewer and fewer dOpis are produced, the cost per dOpi becomes unbounded. In this situation, there is a fixed cost of \$2000 ($C(0) = 2000$), we are trying to spread that \$2000 over fewer and fewer dOpis.

(e) As $x \to \infty$, $\overline{C}(x) \to 100^+$. This means that as more and more dOpis are produced, the cost per dOpi approaches \$100, but is always a little more than \$100. Since \$100 is the variable cost per dOpi ($C(x) = \underline{100}x + 2000$), it means that no matter how many dOpis are produced, the average cost per dOpi will always be a bit higher than the variable cost to produce a dOpi. As before, we can attribute this to the \$2000 fixed cost, which factors into the average cost per dOpi no matter how many dOpis are produced.

37. (a) The cost to make 0 items is $C(0) = m(0) + b = b$. Hence, so the fixed costs are b.

(b) $C(x) = mx + b$ is a linear function with slope $m > 0$. Hence, the cost increases at a rate of m dollars per item made. Hence, the variable cost is m.

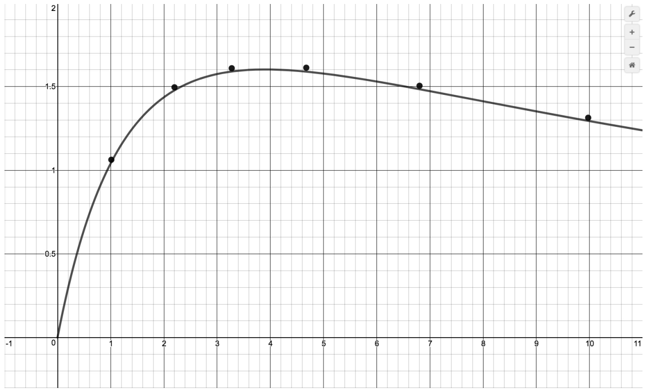
(c) $\overline{C}(x) = \frac{C(x)}{x} = \frac{mx+b}{x} = m + \frac{b}{x}$ for $x > 0$.

(d) Since $b > 0$, $\overline{C}(x) = m + \frac{b}{x} > m$ for $x > 0$. As $x \to \infty$, $\frac{b}{x} \to 0$ so $\overline{C}(x) = m + \frac{b}{x} \to m$.

(e) Geometrically, the graph of $y = \overline{C}(x)$ has a horizontal asymptote $y = m$, the variable cost. In terms of costs, as more items are produced, the affect of the fixed cost on the average cost, $\frac{b}{x}$ falls away so that the average cost per item approaches the variable cost to make each item.

38. If $p(x) = mx + b$ and $C(x)$ is linear, say $C(x) = rx + s$, then we can compute the the profit function (in general) as: $P(x) = xp(x) - C(x) = x(mx+b) - (rx+s)$ which simplifies to $P(x) = mx^2 + (b-r)x - s$. Hence, the average profit $\overline{P}(x) = \frac{P(x)}{x} = \frac{mx^2+(b-r)x-s}{x} = mx + (b-r) - \frac{s}{x}$. We see that as $x \to \infty$, $\frac{s}{x} \to 0$ so $\overline{P}(x) \approx mx + (b-r)$. Hence, $y = mx + (b-r)$ is the slant asymptote to $y = \overline{P}(x)$. This means that as more items are sold, the average profit is decreasing at approximately the same rate as the price function is decreasing, m dollars per item. That is, to sell one additional item, we drop the price $p(x)$ by m dollars which results in a drop in the average profit by approximately m dollars.

39. (a)



(b) The maximum power is approximately 1.603 mW which corresponds to 3.9 $k\Omega$.

(c) As $x \to \infty$, $P(x) \to 0^+$ which means as the resistance increases without bound, the power diminishes to zero.

40. $a = -2$ and $c = -18$ so $f(x) = \dfrac{-2x^2 + 18}{x + 3}$.

41. (a) $a = 6$ and $n = 2$ so $f(x) = \dfrac{6x^2 - 4}{2x^2 + 1}$ (b) $a = 10$ and $n = 3$ so $f(x) = \dfrac{10x^3 - 4}{2x^2 + 1}$.

42. If we define $f(x) = p(x) - p(a)$ then f is a polynomial function with $f(a) = p(a) - p(a) = 0$. The Factor Theorem guarantees $(x - a)$ is a factor of $f(x)$, that is, $f(x) = p(x) - p(a) = (x - a)q(x)$ for some polynomial $q(x)$. Hence, $r(x) = \frac{p(x)-p(a)}{x-a} = \frac{(x-a)q(x)}{x-a} = q(x)$ so the graph of $y = r(x)$ is the same as the graph of the polynomial $y = q(x)$ except for a hole when $x = a$.

43. The slope of the curves near $x = 1$ matches the exponent on x. This exactly what we saw in Exercise 51 in Section 2.1.

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
x^{-1}	-1.0101	-1.0001	≈ -1	≈ -1
x^{-2}	-2.0406	-2.0004	≈ -2	≈ -2
x^{-3}	-3.1021	-3.0010	≈ -3	≈ -3
x^{-4}	-4.2057	-4.0020	≈ -4	≈ -4

3.2 Graphs of Rational Functions

In Section 3.1, we learned about the types of behaviors to expect from graphs of rational functions: vertical asymptotes, holes in graph, horizontal and slant asymptotes. Moreover, Theorems 3.2, 3.3 and 3.4 tell us exactly when and where these behaviors will occur. We used graphing technology extensively in the last section to help us verify results. In this section, we delve more deeply into graphing rational functions with the goal of sketching relatively accurate graphs without the aid of a graphing utility. Your instructor will ultimately communicate the level of detail expected out of you when it comes to producing graphs of rational functions; what we provide here is an attempt to glean as much information about the graph as possible given the analytical tools at our disposal.

One of the standard tools we will use is the sign diagram which was first introduced in Section 1.4, and then revisited in Section 2.3. In these sections, to construct a sign diagram for a function f, we first found the zeros of f. The zeros broke the domain of f into a series of intervals. We determined the sign of $f(x)$ over the *entire* interval by finding the sign of $f(x)$ for just *one* test value per interval. The theorem that justified this approach was the Intermediate Value Theorem, Theorem 2.14, which says that *continuous* functions cannot change their sign between two values unless there is a zero between those two values.

This strategy fails in general with rational functions. Indeed, the very first function we studied in Section 3.1, $r(x) = \frac{1}{x}$ changes sign between $x = -1$ and $x = 1$, but there is no zero between these two values - instead, the graph changes sign across a vertical asymptote. We could also well imagine the graph of a rational function having a hole where an x-intercept should be.[1] With Calculus we can show rational functions are **continuous on their domains** which means when constructing sign diagrams, we need to choose test values on either side of values excluded from the domain in addition to checking around zeros.[2]

Steps for Constructing a Sign Diagram for a Rational Function

Suppose f is a rational function.

1. Place any values excluded from the domain of f on the number line with an '‽' above them.[a]
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine and record the sign of $f(x)$ for each test value in step 3.

[a] '‽' is a nonstandard symbol called the interrobang. We use this symbol to convey a sense of surprise, caution and wonderment - an appropriate attitude to take when approaching these points.

We now present our procedure for graphing rational functions and apply it to a few exhaustive examples. Please note that we decrease the amount of detail given in the explanations as we move through the examples. The reader should be able to fill in any details in those steps which we have abbreviated.

[1] Take $f(x) = \frac{x^2}{x}$, for instance.
[2] Since here excluded values are zeros of the denominator, we can think of this as really just generalizing what we already do.

Steps for Graphing Rational Functions

Suppose r is a rational function.

1. Find the domain of r.
2. Reduce $r(x)$ to lowest terms, if applicable.[a]
3. Determine the location of any vertical asymptotes or holes in the graph, if they exist.
4. Find the axis intercepts, if they exist.
5. Analyze the end behavior of r. Find the horizontal or slant asymptote, if one exists.
6. Use a sign diagram and plot additional points, as needed, to sketch the graph.[b]

[a]This helps us determine limits for the next step.
[b]It doesn't hurt to check for symmetry at this point, if convenient.

Example 3.2.1. Sketch a detailed graph of $f(x) = \dfrac{3x}{x^2-4}$.

Solution. We follow the six step procedure outlined above.

1. To find the domain, we first find the excluded values. To that end, we solve $x^2 - 4 = 0$ and find $x = \pm 2$. Our domain is $\{x \in \mathbb{R} \mid x \neq \pm 2\}$, or, using interval notation, $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

2. We check if $f(x)$ is in lowest terms by factoring: $f(x) = \frac{3x}{(x-2)(x+2)}$. There are no common factors which means $f(x)$ is already in lowest terms.

3. Per Theorem 3.2, vertical asymptotes and holes in the graph come from values excluded from the domain of f. The two numbers excluded from the domain of f are $x = -2$ and $x = 2$ and since $f(x)$ didn't reduce, we know f will be unbounded near $x = -2$ and $x = 2$, so we have are vertical asymptotes there. We can actually go a step further at this point and determine exactly how the graph approaches the asymptote near each of these values. Though not absolutely necessary,[3] it is good practice for those heading off to Calculus. For the discussion that follows, we use the factored form of $f(x) = \frac{3x}{(x-2)(x+2)}$.

    - *The behavior of $y = f(x)$ as $x \to -2$:* Suppose $x \to -2^-$. If we were to build a table of values, we'd use x-values a little less than -2, say -2.1, -2.01 and -2.001. While there is no harm in actually building a table like we did in Section 3.1, we want to develop a 'number sense' here. Let's think about each factor in the formula of $f(x)$ as we imagine substituting a number like $x = -2.000001$ into $f(x)$. The quantity $3x$ would be very close to -6, the quantity $(x-2)$ would be very close to -4, and the factor $(x+2)$ would be very close to 0. More specifically, $(x+2)$

[3]The sign diagram in step 6 will also determine the behavior near the vertical asymptotes.

would be a little less than 0, in this case, -0.000001. We will call such a number a 'very small $(-)$', 'very small' meaning close to zero in absolute value.[4] So, mentally, as $x \to -2^-$,

$$f(x) = \frac{3x}{(x-2)(x+2)} \approx \frac{-6}{(-4)\,(\text{very small }(-))} = \frac{3}{2\,(\text{very small }(-))}$$

Now, the closer x gets to -2, the smaller $(x+2)$ will become, so even though we are multiplying our 'very small $(-)$' by 2, the denominator will continue to get smaller and smaller, and remain negative. The result is a fraction whose numerator is positive, but whose denominator is very small and negative. Mentally,

$$f(x) \approx \frac{3}{2\,(\text{very small }(-))} \approx \frac{3}{\text{very small }(-)} \approx \text{very big }(-)$$

The term 'very big $(-)$' means a number with a large absolute value which is negative.[5] What all of this means is that $\displaystyle\lim_{x\to -2^-} f(x) = -\infty$.

Now we turn our attention to $x \to -2^+$. If we imagine substituting something a little larger than -2 in for x, say -1.999999, we mentally estimate

$$f(x) \approx \frac{-6}{(-4)\,(\text{very small }(+))} = \frac{3}{2\,(\text{very small }(+))} \approx \frac{3}{\text{very small }(+)} \approx \text{very big }(+)$$

We conclude that $\displaystyle\lim_{x\to -2^+} f(x) = \infty$.

- *The behavior of $y = f(x)$ as $x \to 2$:* Consider $x \to 2^-$. We imagine substituting $x = 1.999999$. Approximating $f(x)$ as we did above, we get

$$f(x) \approx \frac{6}{(\text{very small }(-))\,(4)} = \frac{3}{2\,(\text{very small }(-))} \approx \frac{3}{\text{very small }(-)} \approx \text{very big }(-)$$

We conclude that $\displaystyle\lim_{x\to 2^-} f(x) = -\infty$. Similarly, as $x \to 2^+$, we imagine substituting $x = 2.000001$ to get $f(x) \approx \frac{3}{\text{very small }(+)} \approx \text{very big }(+)$. So $\displaystyle\lim_{x\to 2^+} f(x) = \infty$.

We interpret this graphically below on the left.

4. To find the x-intercepts of the graph, we set $y = f(x) = 0$. Solving $\frac{3x}{(x-2)(x+2)} = 0$ results in $x = 0$. Since $x = 0$ is in our domain, $(0, 0)$ is the x-intercept. This is also the y-intercept,[6] as we can quickly verify since $f(0) = \frac{3(0)}{0^2-4} = 0$.

5. Next, we determine the end behavior of the graph of $y = f(x)$. Since the degree of the numerator is 1, and the degree of the denominator is 2, Theorem 3.3 tells us that $y = 0$ is the horizontal asymptote. As with the vertical asymptotes, we can glean more detailed information using 'number sense'. For the discussion below, we use the formula $f(x) = \frac{3x}{x^2-4}$.

[4] This is the third or fourth time we've used this convention (if you've read the footnotes) so we hope we can just roll with it now.

[5] The actual retail value of $f(-2.000001)$ is approximately $-1{,}500{,}000$.

[6] Per Exercise 36, functions can have at most one y-intercept. Since $(0, 0)$ is on the graph, it is *the* y-intercept.

- *The behavior of $y = f(x)$ as $x \to -\infty$:* If we were to make a table of values to discuss the behavior of f as $x \to -\infty$, we would substitute very 'large' negative numbers in for x, say for example, $x = -1$ billion. The numerator $3x$ would then be -3 billion, whereas the denominator $x^2 - 4$ would be $(-1 \text{ billion})^2 - 4$, which is pretty much the same as $1(\text{billion})^2$. Hence,

$$f(-1 \text{ billion}) \approx \frac{-3 \text{ billion}}{1(\text{billion})^2} \approx -\frac{3}{\text{billion}} \approx \text{very small } (-)$$

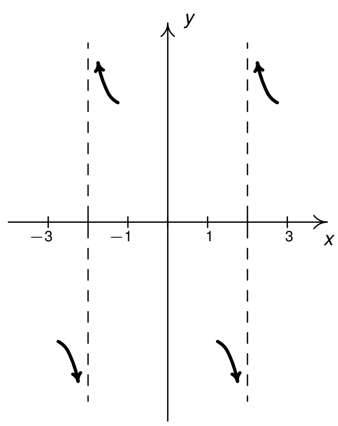
  Notice that if we substituted in $x = -1$ trillion, essentially the same kind of cancellation would happen, and we would be left with an even 'smaller' negative number. This not only confirms the fact that as $x \to -\infty$, $f(x) \to 0$, it tells us that $f(x) \to 0^-$. In other words, the graph of $y = f(x)$ is a little bit *below* the x-axis as we move to the far left.

- *The behavior of $y = f(x)$ as $x \to \infty$:* On the flip side, we can imagine substituting very large positive numbers in for x and looking at the behavior of $f(x)$. For example, let $x = 1$ billion. Proceeding as before, we get

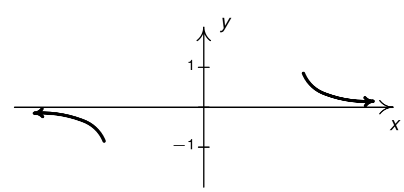
$$f(1 \text{ billion}) \approx \frac{3 \text{ billion}}{1(\text{billion})^2} \approx \frac{3}{\text{billion}} \approx \text{very small } (+)$$

  The larger the number we put in, the smaller the positive number we would get out. In other words, as $x \to \infty$, $f(x) \to 0^+$, so the graph of $y = f(x)$ is a little bit *above* the x-axis as we look toward the far right.

We interpret these findings graphically below on the right.



behavior near $x = \pm 2$

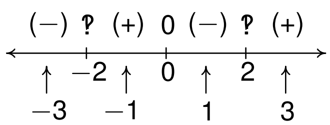


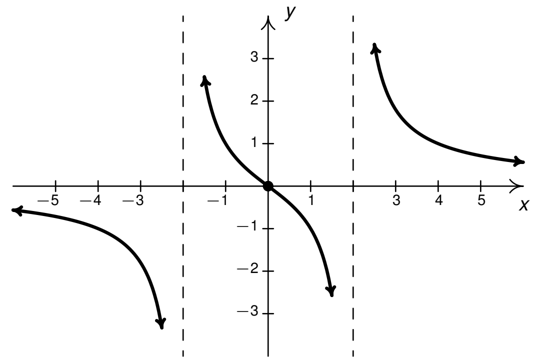
end behavior

6. Lastly, we construct a sign diagram for $f(x)$. The x-values excluded from the domain of f are $x = \pm 2$, and the only zero of f is $x = 0$. Displaying these appropriately on the number line gives us four test intervals, and we choose the test values[7] $x = -3$, $x = -1$, $x = 1$ and $x = 3$. We find $f(-3)$

[7] In this particular case, we don't need test values since our analysis of the behavior of f near the vertical asymptotes and our end behavior analysis have given us the signs on each of the test intervals. In general, however, this won't always be the case, so for demonstration purposes, we continue with our usual construction.

is $(-)$, $f(-1)$ is $(+)$, $f(1)$ is $(-)$ and $f(3)$ is $(+)$. As we begin our sketch, it certainly appears as if the graph could be symmetric about the origin. Taking a moment to check for symmetry, we find $f(-x) = \frac{3(-x)}{(-x)^2-4} = -\frac{3x}{x^2-4} = -f(x)$. Hence, f is odd and the graph of $y = f(x)$ is symmetric about the origin. Putting all of our work together, we get the graph below.





$\square$

Something important to note about the above example is that while $y = 0$ is the horizontal asymptote, the graph of f actually crosses the x-axis at $(0, 0)$. The myth that graphs of rational functions can't cross their horizontal asymptotes is completely false,[8] as we shall see again in our next example.

Example 3.2.2. Sketch a detailed graph of $g(t) = \dfrac{2t^2 - 3t - 5}{t^2 - t - 6}$.

Solution.

1. To find the values excluded from the domain of g, we solve $t^2 - t - 6 = 0$ and find $t = -2$ and $t = 3$. Hence, our domain is $\{t \in \mathbb{R} \mid t \neq -2, 3\}$, or using interval notation: $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.

2. To check if $g(t)$ is in lowest terms, we factor: $g(t) = \frac{(2t-5)(t+1)}{(t-3)(t+2)}$. There is no cancellation, so $g(t)$ is in lowest terms.

3. Since $g(t)$ was given to us in lowest terms, we know the graph has vertical asymptotes $t = -2$ and $t = 3$. Keeping in mind $g(t) = \frac{(2t-5)(t+1)}{(t-3)(t+2)}$, we proceed to our analysis near each of these values.

   - *The behavior of $y = g(t)$ as $t \to -2$:* As $t \to -2^-$, we imagine substituting a number a little bit less than -2. We have
   $$g(t) \approx \frac{(-9)(-1)}{(-5)(\text{very small }(-))} \approx \frac{9}{\text{very small }(+)} \approx \text{very big }(+)$$
   so $\displaystyle\lim_{t \to -2^-} g(t) = \infty$. On the flip side, as $t \to -2^+$, we get
   $$g(t) \approx \frac{9}{\text{very small }(-)} \approx \text{very big }(-)$$
   so $\displaystyle\lim_{t \to -2^+} g(t) = -\infty$.

[8]That's why we called it a MYTH!

- *The behavior of $y = g(t)$ as $t \to 3$:* As $t \to 3^-$, we imagine substituting a number just shy of 3. We have

$$g(t) \approx \frac{(1)(4)}{(\text{ very small } (-))(5)} \approx \frac{4}{\text{very small } (-)} \approx \text{very big } (-)$$

Hence, $\displaystyle\lim_{t \to 3^-} g(t) = -\infty$. As $t \to 3^+$, we get

$$g(t) \approx \frac{4}{\text{very small } (+)} \approx \text{very big } (+)$$

so $\displaystyle\lim_{t \to 3^+} g(t) = \infty$.

We interpret this analysis graphically below on the left.

4. To find the t-intercepts we set $y = g(t) = 0$. Using the factored form of $g(t)$ above, we find the zeros to be the solutions of $(2t - 5)(t + 1) = 0$. We obtain $t = \frac{5}{2}$ and $t = -1$. Since both of these numbers are in the domain of g, we have two t-intercepts, $\left(\frac{5}{2}, 0\right)$ and $(-1, 0)$. To find the y-intercept, we find $y = g(0) = \frac{5}{6}$, so our y-intercept is $\left(0, \frac{5}{6}\right)$.

5. Since the degrees of the numerator and denominator of $g(t)$ are the same, we know from Theorem 3.3 that we can find the horizontal asymptote of the graph of g by taking the ratio of the leading terms coefficients, $y = \frac{2}{1} = 2$. However, if we take the time to do a more detailed analysis, we will be able to reveal some 'hidden' behavior which would be lost otherwise. Using long division, we may rewrite $g(t)$ as $g(t) = 2 - \frac{t-7}{t^2-t-6}$. We focus our attention on the term $\frac{t-7}{t^2-t-6}$.

- *The behavior of $y = g(t)$ as $t \to -\infty$:* If imagine substituting $t = -1$ billion into $\frac{t-7}{t^2-t-6}$, we estimate $\frac{t-7}{t^2-t-6} \approx \frac{-1 \text{ billion}}{1\text{billion}^2} = \frac{-1}{\text{billion}} \approx \text{very small } (-)$.[9] Hence,

$$g(t) = 2 - \frac{t-7}{t^2 - t - 6} \approx 2 - \text{very small } (-) = 2 + \text{very small } (+)$$

Hence, as $t \to -\infty$, the graph is a little bit *above* the line $y = 2$.

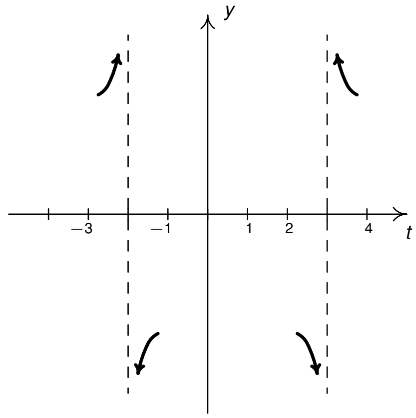
- *The behavior of $y = g(t)$ as $t \to \infty$:* To consider $\frac{t-7}{t^2-t-6}$ as $t \to \infty$, we imagine substituting $t = 1$ billion and, going through the usual mental routine, find

$$\frac{t-7}{t^2 - t - 6} \approx \text{very small } (+)$$

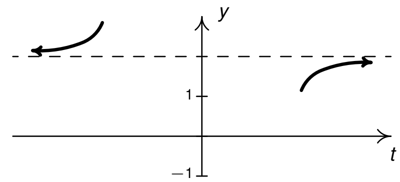
Hence, $g(t) \approx 2 -$ very small $(+)$, so the graph is just *below* the line $y = 2$ as $t \to \infty$.

We sketch the end behavior below on the right.

[9] We are once again using the fact that for polynomials, end behavior is determined by the leading term, so in the denominator, the t^2 term dominates the t and constant terms.

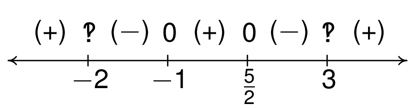


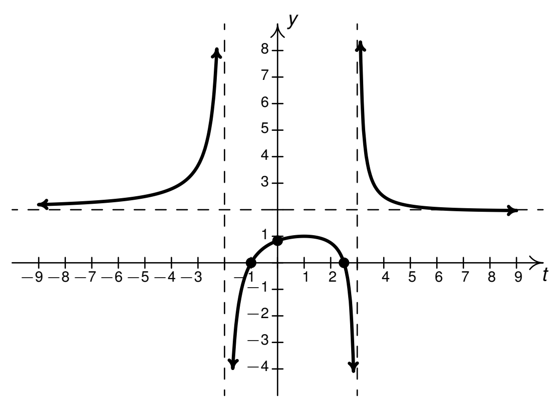
behavior near $t = -2$ and $t = 3$



end behavior

6. Finally we construct our sign diagram. We place an '?' above $t = -2$ and $t = 3$, and a '0' above $t = \frac{5}{2}$ and $t = -1$. Choosing test values in the test intervals gives us $g(t)$ is (+) on the intervals $(-\infty, -2)$, $\left(-1, \frac{5}{2}\right)$ and $(3, \infty)$, and $(-)$ on the intervals $(-2, -1)$ and $\left(\frac{5}{2}, 3\right)$. As we piece together all of the information, it stands to reason the graph must cross the horizontal asymptote at some point after $t = 3$ in order for it to approach $y = 2$ from underneath.[10] To find where $y = g(t)$ intersects $y = 2$, we solve $g(t) = 2 - \frac{t-7}{t^2-t-6} = 2$ and get $t - 7 = 0$, or $t = 7$. Note that $t - 7$ is the remainder when $2t^2 - 3t - 5$ is divided by $t^2 - t - 6$, so it makes sense that for $g(t)$ to equal the quotient 2, the remainder from the division must be 0. Sure enough, we find $g(7) = 2$. The location of the t-intercepts alone dashes all hope of the function being even or odd (do you see why?) so we skip the symmetry check in this case.





□

More can be said about the graph of $y = g(t)$ above. It stands to reason that g must attain a local minimum at some point past $t = 7$ since the graph of g crosses through $y = 2$ at $(2, 7)$ but approaches $y = 2$ from below as $t \to \infty$. Calculus verifies a local minimum at $(13, 1.96)$. We invite the reader to verify this claim using a graphing utility.

[10]This subtlety would have been missed had we skipped the long division and subsequent end behavior analysis.

Example 3.2.3. Sketch a detailed graph of $h(x) = \dfrac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2}$.

Solution.

1. Solving $x^2 + 3x + 2 = 0$ gives $x = -2$ and $x = -1$ as our excluded values. Hence, the domain is $\{x \in \mathbb{R} \mid x \neq -1, -2\}$ or, using interval notation, $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

2. To reduce $h(x)$, we need to factor the numerator and denominator. To factor the numerator, we use the techniques[11] set forth in Section 2.3 and get

$$h(x) = \frac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2} = \frac{(2x+1)(x+1)^2}{(x+2)(x+1)} = \frac{(2x+1)(x+1)^{\cancel{2}\,1}}{(x+2)\cancel{(x+1)}} = \frac{(2x+1)(x+1)}{x+2}$$

   Note we can use this formula for $h(x)$ in our analysis of the graph of h as long as we are not substituting $x = -1$. To make this exclusion specific, we write $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$.

3. Since the factor $(x + 2)$ remains in the denominator of $h(x)$ in lowest terms, we expect the graph to have the vertical asymptote $x = -2$. As for $x = -1$, the factor $(x + 1)$ was canceled from the denominator when we reduced $h(x)$, so there will be a hole when $x = -1$.

   - *The behavior of $y = h(x)$ as $x \to -2$:* As $x \to -2^-$, we imagine substituting a number a little bit less than -2. We have $h(x) \approx \frac{(-3)(-1)}{(\text{very small }(-))} \approx \frac{3}{(\text{very small }(-))} \approx$ very big $(-)$. Hence, $\lim\limits_{x \to -2^-} h(x) = -\infty$. On the other side of -2, as $x \to -2^+$, we find that $h(x) \approx \frac{3}{\text{very small }(+)} \approx$ very big $(+)$, so $\lim\limits_{x \to -2^+} h(x) = \infty$.
   - *The behavior of $y = h(x)$ as $x \to -1$.* As $x \to -1$, we have $2x + 1 \to -1$, $x + 1 \to 0$, and $x + 2 \to 1$. Hence, $\frac{(2x+1)(x+1)}{x+2} \to \frac{(-1)(0)}{1} = 0$ so $\lim\limits_{x \to -1} h(x) = 0$. This means we have a hole at $(-1, 0)$. More specifically,[12] we note that as $x \to -1^-$, $h(x) > 0$ whereas as $x \to -1^+$, $h(x) < 0$. This helps us sketch the graph of h near $(-1, 0)$.

4. To find the x-intercepts, as usual, we set $h(x) = 0$ and solve. Solving $\frac{(2x+1)(x+1)}{x+2} = 0$ yields $x = -\frac{1}{2}$ and $x = -1$. The latter isn't in the domain of h, in fact, we know there is a hole at $(-1, 0)$, so we exclude it. Our only x-intercept is $\left(-\frac{1}{2}, 0\right)$. To find the y-intercept, we set $x = 0$. Since $0 \neq -1$, we can use the reduced formula for $h(x)$ and we get $h(0) = \frac{1}{2}$ for a y-intercept of $\left(0, \frac{1}{2}\right)$.

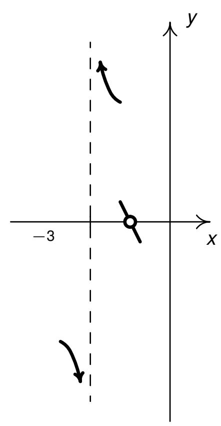
5. For end behavior, we note that the degree of the numerator of $h(x)$, $2x^3 + 5x^2 + 4x + 1$, is 3 and the degree of the denominator, $x^2 + 3x + 2$, is 2 so by Theorem 3.4, the graph of $y = h(x)$ has a slant asymptote. For $x \to -\infty$ or $x \to \infty$, we are far enough away from $x = -1$ to use the reduced formula, $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$. To perform long division, we multiply out the numerator and get $h(x) = \frac{2x^2+3x+1}{x+2}$, $x \neq -1$, and rewrite $h(x) = 2x - 1 + \frac{3}{x+2}$, $x \neq -1$. By Theorem 3.4, the slant asymptote is $y = 2x - 1$, and to better see *how* the graph approaches the asymptote, we focus our attention on the term generated from the remainder, $\frac{3}{x+2}$.

[11] Bet you never thought you'd never see *that* stuff again before the Final Exam!

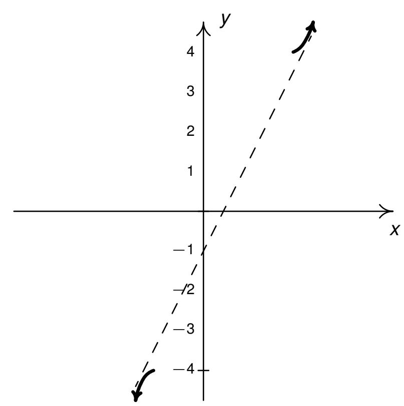
[12] We'll also see this when we make the sign diagram later . . .

- *The behavior of $y = h(x)$ as $x \to -\infty$:* Substituting $x = -1$ billion into $\frac{3}{x+2}$, we get the estimate $\frac{3}{-1 \text{ billion}} \approx$ very small $(-)$. Hence, $h(x) = 2x - 1 + \frac{3}{x+2} \approx 2x - 1 +$ very small $(-)$. This means the graph of $y = h(x)$ is a little bit *below* the line $y = 2x - 1$ as $x \to -\infty$.
- *The behavior of $y = h(x)$ as $x \to \infty$:* If $x \to \infty$, then $\frac{3}{x+2} \approx$ very small $(+)$. This means $h(x) \approx 2x - 1 +$ very small $(+)$, or that the graph of $y = h(x)$ is a little bit *above* the line $y = 2x - 1$ as $x \to \infty$.

We sketch the end behavior below on the right.

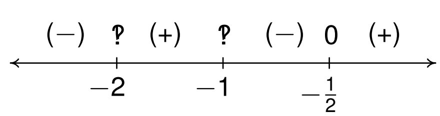


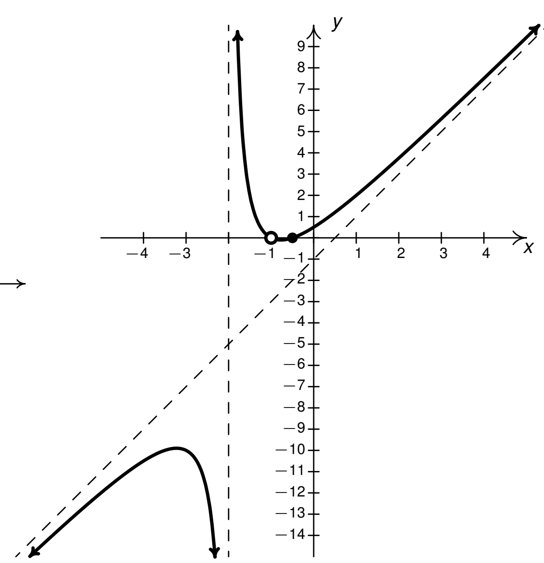
behavior near $x = -2$ and $x = -1$



end behavior

6. To make our sign diagram, we place an '‽' above $x = -2$ and $x = -1$ and a '0' above $x = -\frac{1}{2}$. On our four test intervals, we find $h(x)$ is $(+)$ on $(-2, -1)$ and $\left(-\frac{1}{2}, \infty\right)$ and $h(x)$ is $(-)$ on $(-\infty, -2)$ and $\left(-1, -\frac{1}{2}\right)$. Putting all of our work together yields the graph below.





To find if the graph of h ever crosses the slant asymptote, we solve $h(x) = 2x - 1 + \frac{3}{x+2} = 2x - 1$. This results in $\frac{3}{x+2} = 0$, which has no solution.[13] Hence, the graph of h never crosses its slant asymptote.[14] $\square$

Our last graphing example is challenging in that our six step process provides us little information to work with.

Example 3.2.4. Sketch the graph of $r(x) = \dfrac{x^4+1}{x^2+1}$.

Solution.

1. The denominator $x^2 + 1$ is never zero which means there are no excluded values. The domain is $\mathbb{R}$, or using interval notation, $(-\infty, \infty)$.

2. With no real zeros in the denominator, $x^2 + 1$ is an irreducible quadratic. Our only hope of reducing $r(x)$ is if $x^2 + 1$ is a factor of $x^4 + 1$. Performing long division gives us

$$\frac{x^4+1}{x^2+1} = x^2 - 1 + \frac{2}{x^2+1}$$

   The remainder is not zero so $r(x)$ is already reduced.

3. Since there are no numbers excluded from the domain of r, there are no vertical asymptotes or holes in the graph of r.

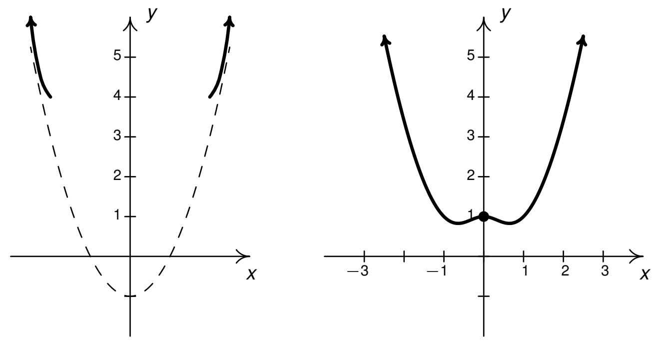
4. To find the x-intercept, we'd set $r(x) = 0$. Since there are no real solutions to $x^4 + 1 = 0$, we have no x-intercepts. Since $r(0) = 1$, we do get $(0, 1)$ as the y-intercept.

5. For end behavior, we note that since the degree of the numerator is exactly *two* more than the degree of the denominator, neither Theorems 3.3 nor 3.4 apply.[15] We know from our attempt to reduce $r(x)$ that we can rewrite $r(x) = x^2 - 1 + \frac{2}{x^2+1}$, so we focus our attention on the term corresponding to the remainder, $\frac{2}{x^2+1}$ It should be clear that as $x \to -\infty$ or $x \to \infty$, $\frac{2}{x^2+1} \approx$ very small $(+)$, which means $r(x) \approx x^2 - 1 +$ very small $(+)$. So the graph $y = r(x)$ is a little bit *above* the graph of the parabola $y = x^2 - 1$ as $x \to -\infty$ and $x \to \infty$.

6. There isn't much work to do for a sign diagram for $r(x)$, since its domain is all real numbers and it has no zeros. Our sole test interval is $(-\infty, \infty)$, and since we know $r(0) = 1$, we conclude $r(x)$ is $(+)$ for all real numbers. We check for symmetry, and find $r(-x) = \frac{(-x)^4+1}{(-x)^2+1} = \frac{x^4+1}{x^2+1} = r(x)$, so r is even and, hence, the graph is symmetric about the y-axis. It may be tempting at this point to call it quits, reach for a graphing utility, or ask someone who knows Calculus.[16] It turns out, we can do a little bit better. Recall from Section 2.1.1, that when $|x| < 1$ but $x \neq 0$, $x^4 < x^2$, hence $x^4 + 1 < x^2 + 1$. This means for $-1 < x < 0$ and $0 < x < 1$, $r(x) = \frac{x^4+1}{x^2+1} < 1$. Since we know $r(0) = 1$, this means

[13] Alternatively, the remainder after the long division was $r = 3$ which is never 0.
[14] But rest assured, some graphs do!
[15] This won't stop us from giving it the old community college try, however!
[16] This is exactly what the authors did in the Third Edition. Special thanks go to Erik Boczko from Ohio University for showing us that, in fact, we could do more with this example algeraically.

the graph of $y = r(x)$ must fall to either side before heading off to ∞. This means $(0,1)$ is a local maximum and, moreover, there are at least two local minimums, at least one on either side of $(0,1)$. We invite the reader to confirm this using a graphing utility.

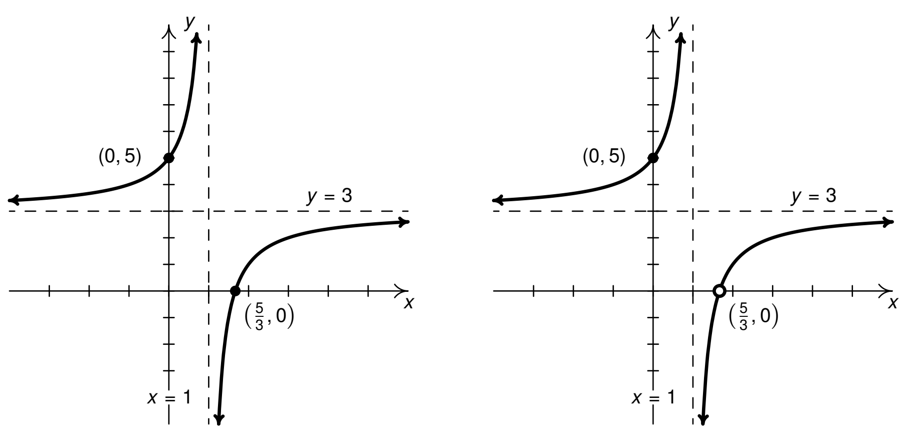


end behavior   sketch of $y = r(x)$

□

Our last example turns the tables and invites us to write formulas for rational functions given their graphs.

Example 3.2.5. Write formulas for rational functions $r(x)$ and $F(x)$ given their graphs below:



Solution. The good news is the graph of r closely resembles the graph of F, so once we know an expression for $r(x)$, we should be able to modify it to obtain $F(x)$. We are told r is a rational function, so we know there are polynomial functions p and q so that $r(x) = \frac{p(x)}{q(x)}$. We know from Theorem 2.16 that we can factor $p(x)$ and $q(x)$ completely in terms of their leading coefficients and their zeros. For simplicity's sake, we assume neither p nor q has any non-real zeros.

We focus our attention first on finding an expression for $p(x)$. When finding the x-intercepts, we look for the zeros of r by solving $r(x) = \frac{p(x)}{q(x)} = 0$. This equation quickly reduces to solving $p(x) = 0$. Since $\left(\frac{5}{3}, 0\right)$ is an x-intercept of the graph, we know $x = \frac{5}{3}$ is a zero of r, and, hence, a zero of p. Since we are shown no other x-intercepts, we assume r, hence p have no other real zeros (and no non-real zeros by our assumption.)

Theorem 2.16 gives $p(x) = a\left(x - \frac{5}{3}\right)^m$ where a is the leading coefficient of $p(x)$ and m is the multiplicity of the zero $x = \frac{5}{3}$. Since the graph of $y = r(x)$ crosses through the x-axis in what appears to be a fairly linear fashion at $\left(\frac{5}{3}, 0\right)$, it seems reasonable to set $m = 1$. Hence, $p(x) = a\left(x - \frac{5}{3}\right)$.
Next, we focus our attention on finding $q(x)$. Per Theorem 3.2, the vertical asymptote $x = 1$ comes from a factor of $(x-1)$ in the denominator of $r(x)$. This means $(x-1)$ is a factor of $q(x)$. Since there are no other vertical asymptotes or holes in the graph, $x = 1$ is the only real zero, hence (per our assumption) only zero of q. At this point, we have $q(x) = b(x-1)^m$ where b is the leading coefficient of $q(x)$ and m is the multiplicity of the zero $x = 1$. Since the graph of r has the *horizontal* asymptote $y = 3$,Theorem 3.4 tells us two things: first, degree of q must match the degree of p; second, the ratio $\frac{a}{b} = 3$. Hence, the degree of q is 1 so that:

$$\begin{aligned} r(x) &= \frac{a\left(x - \frac{5}{3}\right)}{b(x-1)} \\ &= \frac{a}{b}\left(\frac{x - \frac{5}{3}}{x-1}\right) \\ &= 3\left(\frac{x - \frac{5}{3}}{x-1}\right) \\ &= \frac{3x-5}{x-1}. \end{aligned}$$

We have yet to use the y-intercept, $(0, 5)$. In this case, we use it as a partial check: $r(0) = \frac{3(0)-5}{0-1} = 5$, as required. We can sketch $y = r(x)$ by hand, or with a graphing utility, to give a better check of our work.
Now it is time to find a formula for $F(x)$. The graphs of r and F look identical except the graph has a hole in the graph at $\left(\frac{5}{3}, 0\right)$ instead of an x-intercept. Theorem 3.2 tells us this happens because a factor of $\left(x - \frac{5}{3}\right)$ cancels from the denominator when the formula for $F(x)$ is reduced. Hence, we reverse this process and multiply the numerator and denominator of our expression for $r(x)$ by $\left(x - \frac{5}{3}\right)$:

$$\begin{aligned} F(x) &= r(x) \cdot \frac{\left(x - \frac{5}{3}\right)}{\left(x - \frac{5}{3}\right)} \\ &= \frac{3x-5}{x-1} \cdot \frac{\left(x - \frac{5}{3}\right)}{\left(x - \frac{5}{3}\right)} \\ &= \frac{3x^2 - 10x + \frac{25}{3}}{x^2 - \frac{8}{3}x + \frac{5}{3}} && \text{expand} \\ &= \frac{9x^2 - 30x + 25}{3x^2 - 8x + 5} && \text{multiply by } 1 = \tfrac{3}{3} \text{ to reduce the complex fraction.} \end{aligned}$$

Again, we can check our answer by applying the six step method to this function or, for a quick verification, we can use a graphing utility.[17] □

Another way to approach Example 3.2.5 is to take a cue from Theorem 3.1. The graph of $y = r(x)$ certainly *appears* to be the result of moving around the graph of $f(x) = \frac{1}{x}$. To that end, suppose $r(x) = \frac{a}{x-k} + k$. Since the vertical asymptote is $x = 1$ and the horizontal asymptote is $y = 3$, we get $h = 1$ and $k = 3$. At

[17] Be warned, however, a graphing utility may not show the hole at $\left(\frac{5}{3}, 0\right)$.

this point, we have $r(x) = \frac{a}{x-1} + 3$. We can determine a by using the y-intercept, $(0, 5)$: $r(0) = 5$ gives us $-a + 3 = 5$ so $a = -2$. Hence, $r(x) = \frac{-2}{x-1} + 3$. At this point we could check the x-intercept $\left(\frac{5}{3}, 0\right)$ is on the graph, check our answer using a graphing utility, or even better, get common denominators and write $r(x)$ as a single rational expression to compare with our answer in the above example.

As usual, the authors offer no apologies for what may be construed as 'pedantry' in this section. We feel that the detail presented in this section is necessary to obtain a firm grasp of the concepts presented here and it also serves as an introduction to the methods employed in Calculus. In the end, your instructor will decide how much, if any, of the kinds of details presented here are 'mission critical' to your understanding of Precalculus. Without further delay, we present you with this section's Exercises.

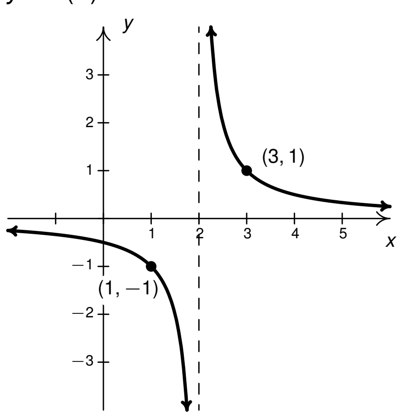
3.2.1 Exercises

In Exercises 1 - 16, use the six-step procedure to graph the rational function. Be sure to draw any asymptotes as dashed lines.

1. $f(x) = \dfrac{4}{x+2}$
2. $f(x) = 5x(6-2x)^{-1}$
3. $g(t) = t^{-2}$
4. $g(t) = \dfrac{1}{t^2+t-12}$
5. $r(z) = \dfrac{2z-1}{-2z^2-5z+3}$
6. $r(z) = \dfrac{z}{z^2+z-12}$
7. $f(x) = 4x(x^2+4)^{-1}$
8. $f(x) = 4x(x^2-4)^{-1}$
9. $g(t) = \dfrac{t^2-t-12}{t^2+t-6}$
10. $g(t) = 3 - \dfrac{5t-25}{t^2-9}$
11. $r(z) = \dfrac{z^2-z-6}{z+1}$
12. $r(z) = -z-2+\dfrac{6}{3-z}$
13. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2}$
14. $f(x) = \dfrac{5x}{9-x^2} - x$
15. $g(t) = \dfrac{1}{2}t - 1 + \dfrac{t+1}{t^2+1}$
16.[18] $g(t) = \dfrac{t^2-2t+1}{t^3+t^2-2t}$

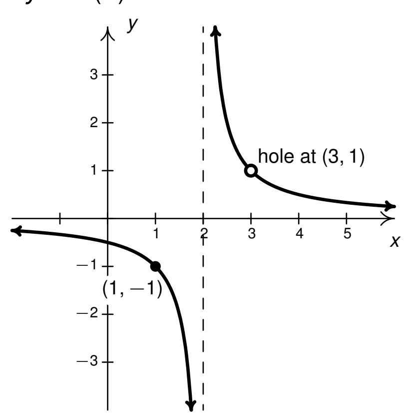
In Exercises 17 - 20, find a possible formula for the function whose graph is given.

17. $y = f(x)$



Vertical asymptote: $x = 2$

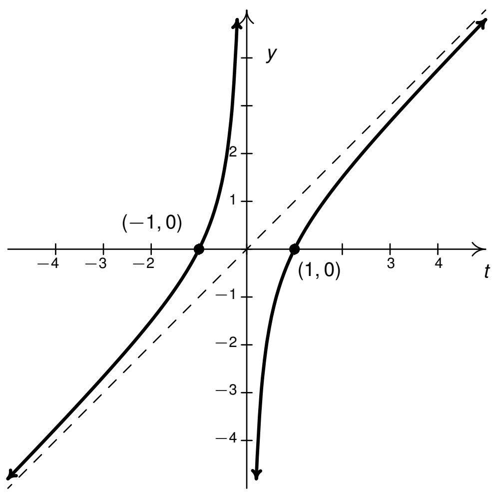
18. $y = F(x)$



Vertical asymptote: $x = 2$

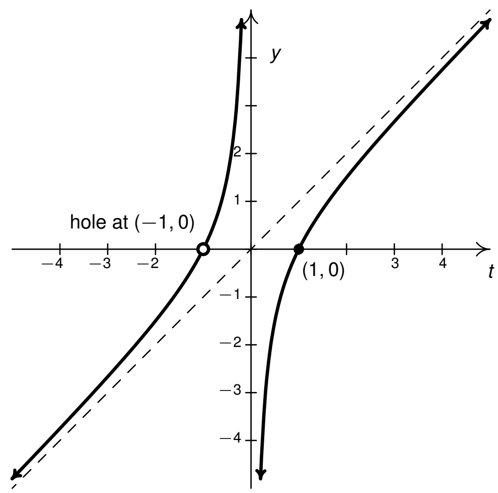
[18] Once you've done the six-step procedure, use a graphing utility to graph this function on the window $[0, 12] \times [0, 0.25]$ . . .

19. $y = g(t)$



Slant asymptote: $y = t$

20. $y = G(t)$



Slant asymptote: $y = t$

21. Let $g(x) = \dfrac{x^4 - 8x^3 + 24x^2 - 72x + 135}{x^3 - 9x^2 + 15x - 7}$. With the help of your classmates:

- find the x- and y- intercepts of the graph of g.
- find all of the asymptotes of the graph of g and any holes in the graph, if they exist.
- find the intervals on which the function is increasing, the intervals on which it is decreasing and the local maximums and minimums, if any exist.
- sketch the graph of g, using more than one picture if necessary to show all of the important features of the graph.

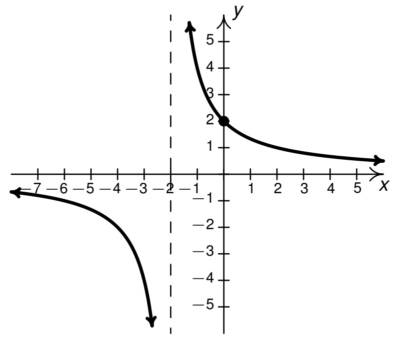
Example 3.2.4 showed us that the six-step procedure cannot tell us everything of importance about the graph of a rational function and that sometimes there are things that are easy to miss. Without Calculus, we may need to use graphing utilities to reveal the hidden behavior of rational functions. Working with your classmates, use a graphing utility to examine the graphs of the rational functions given in Exercises 22 - 25. Compare and contrast their features. Which features can the six-step process reveal and which features cannot be detected by it?

22. $f(x) = \dfrac{1}{x^2 + 1}$ 23. $f(x) = \dfrac{x}{x^2 + 1}$ 24. $f(x) = \dfrac{x^2}{x^2 + 1}$ 25. $f(x) = \dfrac{x^3}{x^2 + 1}$

3.2.2 Answers

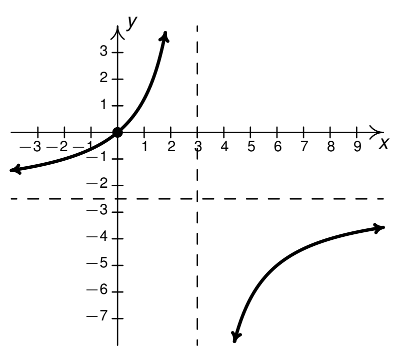
1. $f(x) = \dfrac{4}{x+2}$

   Domain: $(-\infty, -2) \cup (-2, \infty)$
   No x-intercepts
   y-intercept: $(0, 2)$
   Vertical asymptote: $x = -2$
   $\displaystyle\lim_{x \to -2^-} f(x) = -\infty, \ \lim_{x \to -2^+} f(x) = \infty$
   Horizontal asymptote: $y = 0$
   $\displaystyle\lim_{x \to -\infty} f(x) = 0$
   More specifically, as $x \to -\infty, \ f(x) \to 0^-$
   $\displaystyle\lim_{x \to \infty} f(x) = 0$
   More specifically, as $x \to \infty, \ f(x) \to 0^+$



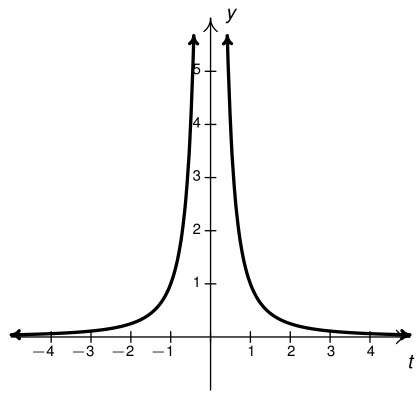
2. $f(x) = 5x(6-2x)^{-1} = \dfrac{5x}{6-2x}$

   Domain: $(-\infty, 3) \cup (3, \infty)$
   x-intercept: $(0, 0)$
   y-intercept: $(0, 0)$
   Vertical asymptote: $x = 3$
   $\displaystyle\lim_{x \to 3^-} f(x) = \infty, \ \lim_{x \to 3^+} f(x) = -\infty$
   Horizontal asymptote: $y = -\frac{5}{2}$
   $\displaystyle\lim_{x \to -\infty} f(x) = -\frac{5}{2}$
   More specifically, as $x \to -\infty, \ f(x) \to -\frac{5}{2}^+$
   $\displaystyle\lim_{x \to -\infty} f(x) = -\frac{5}{2}$
   More specifically, as $x \to \infty, \ f(x) \to -\frac{5}{2}^-$



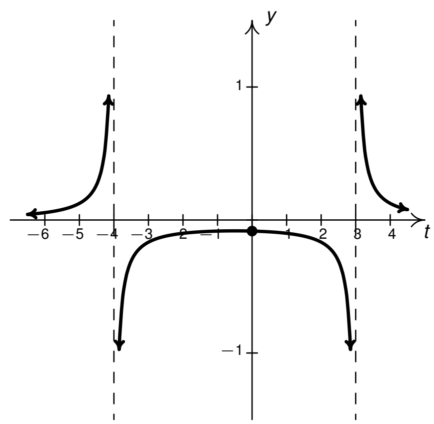
3. $g(t) = t^{-2} = \dfrac{1}{t^2}$

Domain: $(-\infty, 0) \cup (0, \infty)$
No t-intercepts
No y-intercepts
Vertical asymptote: $t = 0$
$\displaystyle\lim_{t\to 0} g(t) = \infty$
Horizontal asymptote: $y = 0$
$\displaystyle\lim_{t\to -\infty} g(t) = 0$
More specifically, as $t \to -\infty$, $g(t) \to 0^+$
$\displaystyle\lim_{t\to \infty} g(t) = 0$
More specifically, as $t \to \infty$, $g(t) \to 0^+$

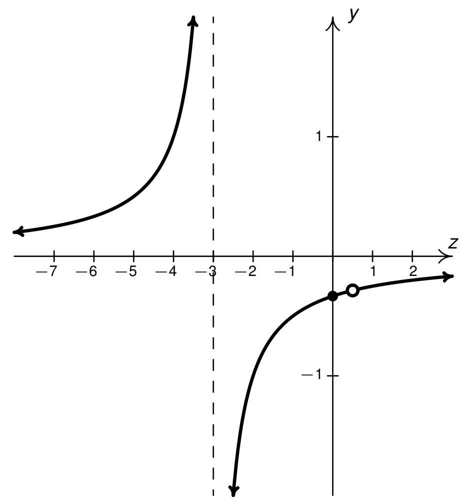


4. $g(t) = \dfrac{1}{t^2 + t - 12} = \dfrac{1}{(t-3)(t+4)}$

Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
No t-intercepts
y-intercept: $(0, -\frac{1}{12})$
Vertical asymptotes: $t = -4$ and $t = 3$
$\displaystyle\lim_{t\to -4^-} g(t) = \infty$, $\displaystyle\lim_{t\to -4^+} g(t) = -\infty$
$\displaystyle\lim_{t\to 3^-} g(t) = -\infty$, $\displaystyle\lim_{t\to 3^+} g(t) = \infty$
Horizontal asymptote: $y = 0$
$\displaystyle\lim_{t\to -\infty} g(t) = 0$
More specifically, as $t \to -\infty$, $g(t) \to 0^+$
$\displaystyle\lim_{t\to \infty} g(t) = 0$
More specifically, as $t \to \infty$, $g(t) \to 0^+$

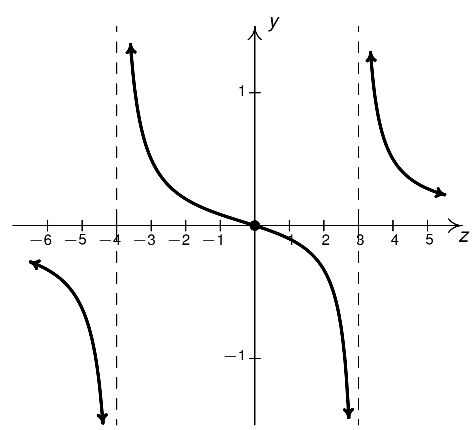


5. $r(z) = \dfrac{2z-1}{-2z^2-5z+3} = -\dfrac{2z-1}{(2z-1)(z+3)}$



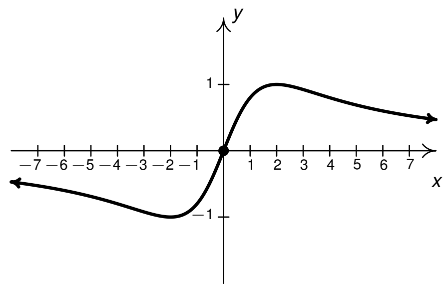
Domain: $(-\infty,-3)\cup(-3,\frac{1}{2})\cup(\frac{1}{2},\infty)$
No z-intercepts
y-intercept: $(0,-\frac{1}{3})$
$r(z) = \dfrac{-1}{z+3},\ z \neq \frac{1}{2}$
Hole in the graph at $(\frac{1}{2},-\frac{2}{7})$
Vertical asymptote: $z=-3$
$\displaystyle\lim_{z\to-3^-} r(z) = \infty,\ \lim_{z\to-3^+} r(z) = -\infty$
Horizontal asymptote: $y=0$
$\displaystyle\lim_{z\to-\infty} r(z) = 0$
More specifically, as $z \to -\infty,\ r(z) \to 0^+$
$\displaystyle\lim_{z\to\infty} r(z) = 0$
More specifically, as $z \to \infty,\ r(z) \to 0^-$

6. $r(z) = \dfrac{z}{z^2+z-12} = \dfrac{z}{(z-3)(z+4)}$



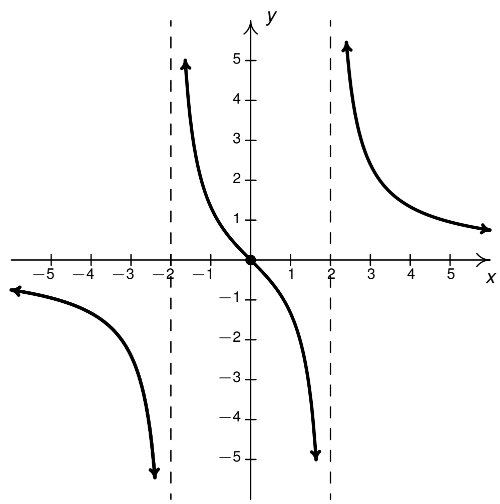
Domain: $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$
z-intercept: $(0,0)$
y-intercept: $(0,0)$
Vertical asymptotes: $z=-4$ and $z=3$
$\displaystyle\lim_{z\to-4^-} r(z) = -\infty,\ \lim_{z\to-4^+} r(z) = \infty$
$\displaystyle\lim_{z\to3^-} r(z) = -\infty,\ \lim_{z\to3^+} r(z) = \infty$
Horizontal asymptote: $y=0$
$\displaystyle\lim_{z\to-\infty} r(z) = 0$
More specifically, as $z \to -\infty,\ r(z) \to 0^-$
$\displaystyle\lim_{z\to\infty} r(z) = 0$
More specifically, as $z \to \infty,\ r(z) \to 0^+$

7. $f(x) = 4x(x^2+4)^{-1} = \dfrac{4x}{x^2+4}$



Domain: $(-\infty, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
$\displaystyle\lim_{x \to -\infty} f(x) = 0$
More specifically, as $x \to -\infty$, $f(x) \to 0^-$
$\displaystyle\lim_{x \to \infty} f(x) = 0$
More specifically, as $x \to \infty$, $f(x) \to 0^+$

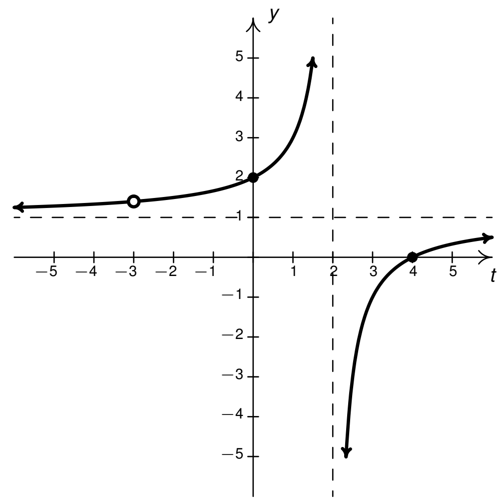
8. $f(x) = 4x(x^2-4)^{-1} = \dfrac{4x}{x^2-4} = \dfrac{4x}{(x+2)(x-2)}$



Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Vertical asymptotes: $x = -2$, $x = 2$
$\displaystyle\lim_{x \to -2^-} f(x) = -\infty$, $\displaystyle\lim_{x \to -2^+} f(x) = \infty$
$\displaystyle\lim_{x \to 2^-} f(x) = -\infty$, $\displaystyle\lim_{x \to 2^+} f(x) = \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
$\displaystyle\lim_{x \to -\infty} f(x) = 0$
More specifically, as $x \to -\infty$, $f(x) \to 0^-$
$\displaystyle\lim_{x \to \infty} f(x) = 0$
More specifically, as $x \to \infty$, $f(x) \to 0^+$

9. $g(t) = \dfrac{t^2 - t - 12}{t^2 + t - 6} = \dfrac{t - 4}{t - 2},\ t \neq -3$

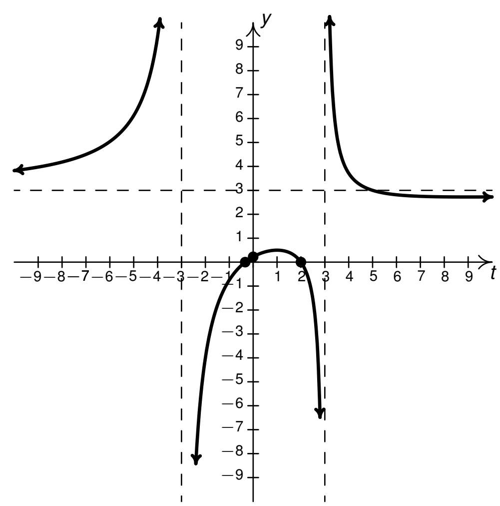
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
t-intercept: $(4, 0)$
y-intercept: $(0, 2)$
Vertical asymptote: $t = 2$
$\lim_{t \to 2^-} g(t) = \infty$, $\lim_{t \to 2^+} g(t) = -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y = 1$
$\lim_{t \to -\infty} g(t) = 1$
More specifically, as $t \to -\infty$, $g(t) \to 1^+$
$\lim_{t \to \infty} g(t) = 1$
More specifically, as $t \to \infty$, $g(t) \to 1^-$



10. $g(t) = 3 - \dfrac{5t - 25}{t^2 - 9} = \dfrac{3t^2 - 5t - 2}{t^2 - 9}$

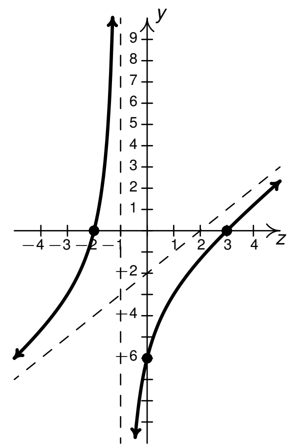
$= \dfrac{(3t + 1)(t - 2)}{(t + 3)(t - 3)}$

Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
t-intercepts: $\left(-\frac{1}{3}, 0\right)$, $(2, 0)$
y-intercept: $\left(0, \frac{2}{9}\right)$
Vertical asymptotes: $t = -3$, $t = 3$
$\lim_{t \to -3^-} g(t) = \infty$, $\lim_{t \to -3^+} g(t) = -\infty$
$\lim_{t \to 3^-} g(t) = -\infty$, $\lim_{t \to 3^+} g(t) = \infty$
Horizontal asymptote: $y = 3$
$\lim_{t \to -\infty} g(t) = 3$
More specifically, as $t \to -\infty$, $g(t) \to 3^+$
$\lim_{t \to \infty} g(t) = 3$
More specifically, as $t \to \infty$, $g(t) \to 3^-$



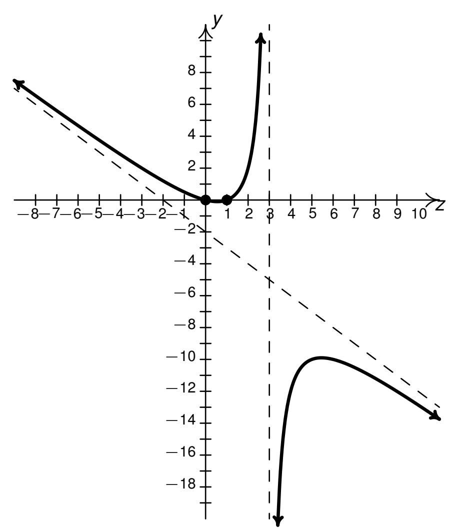
11. $r(z) = \dfrac{z^2 - z - 6}{z + 1} = \dfrac{(z-3)(z+2)}{z+1}$

Domain: $(-\infty, -1) \cup (-1, \infty)$
z-intercepts: $(-2, 0)$, $(3, 0)$
y-intercept: $(0, -6)$
Vertical asymptote: $z = -1$
$\displaystyle\lim_{z \to -1^-} r(z) = \infty$, $\displaystyle\lim_{z \to -1^+} r(z) = -\infty$
Slant asymptote: $y = z - 2$
$\displaystyle\lim_{z \to -\infty} r(z) = -\infty$
As $z \to -\infty$, the graph is above $y = z - 2$
$\displaystyle\lim_{z \to \infty} r(z) = \infty$
As $z \to \infty$, the graph is below $y = z - 2$



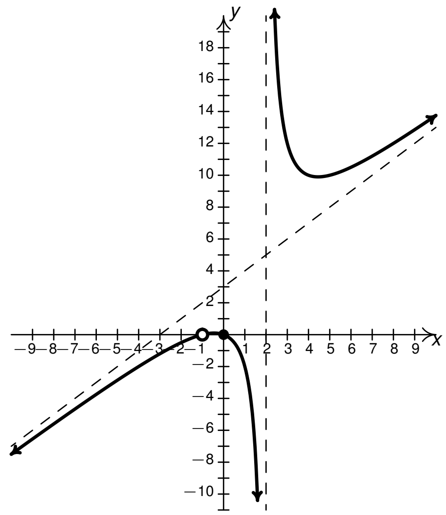
12. $r(z) = -z - 2 + \dfrac{6}{3 - z} = \dfrac{z^2 - z}{3 - z}$

Domain: $(-\infty, 3) \cup (3, \infty)$
z-intercepts: $(0, 0), (1, 0)$
y-intercept: $(0, 0)$
Vertical asymptote: $z = 3$
$\displaystyle\lim_{z \to 3^-} r(z) = \infty$, $\displaystyle\lim_{z \to 3^+} r(z) = -\infty$
Slant asymptote: $y = -z - 2$
$\displaystyle\lim_{z \to -\infty} r(z) = \infty$
As $z \to -\infty$, the graph is above $y = -z - 2$
$\displaystyle\lim_{z \to \infty} r(z) = -\infty$
As $z \to \infty$, the graph is below $y = -z - 2$



13. $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2} = \dfrac{x(x+1)}{x-2},\ x \neq -1$

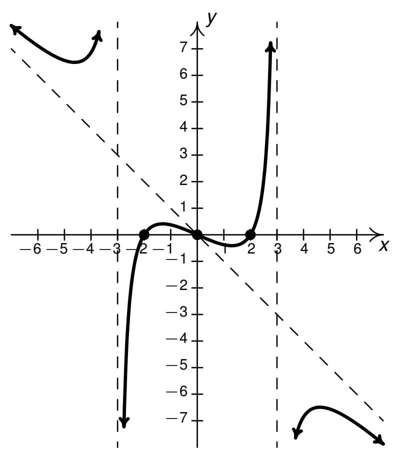
Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Vertical asymptote: $x = 2$
$\lim\limits_{x \to 2^-} f(x) = -\infty,\ \lim\limits_{x \to 2^+} f(x) = \infty$
Hole at $(-1, 0)$
Slant asymptote: $y = x + 3$
$\lim\limits_{x \to -\infty} f(x) = -\infty$
As $x \to -\infty$, the graph is below $y = x + 3$
$\lim\limits_{x \to \infty} f(x) = \infty$
As $x \to \infty$, the graph is above $y = x + 3$



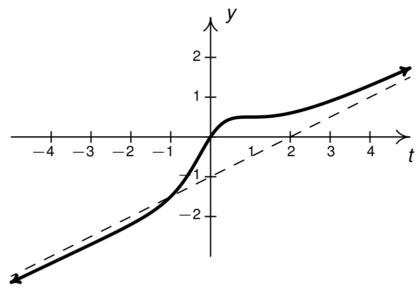
14. $f(x) = \dfrac{5x}{9 - x^2} - x = \dfrac{x^3 - 4x}{9 - x^2}$

$= \dfrac{x(x-2)(x+2)}{-(x-3)(x+3)}$

Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
x-intercepts: $(-2, 0), (0, 0), (2, 0)$
y-intercept: $(0, 0)$
Vertical asymptotes: $x = -3, x = 3$
$\lim\limits_{x \to -3^-} f(x) = \infty,\ \lim\limits_{x \to -3^+} f(x) = -\infty$
$\lim\limits_{x \to 3^-} f(x) = \infty,\ \lim\limits_{x \to 3^+} f(x) = -\infty$
Slant asymptote: $y = -x$
$\lim\limits_{x \to -\infty} f(x) = \infty$
As $x \to -\infty$, the graph is above $y = -x$
$\lim\limits_{x \to \infty} f(x) = -\infty$
As $x \to \infty$, the graph is below $y = -x$

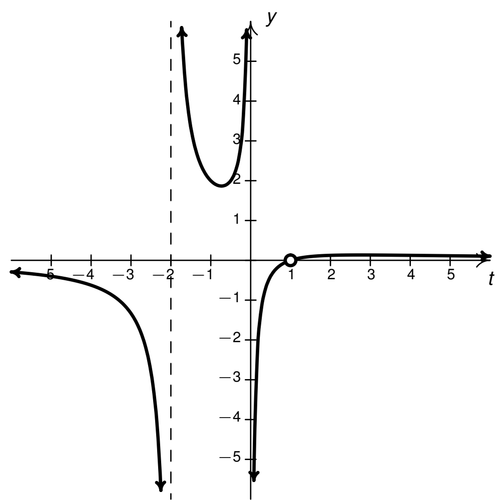


15. $g(t) = \frac{1}{2}t - 1 + \frac{t+1}{t^2+1} = \frac{t(t^2-2t+3)}{2t^2+2}$



Domain: $(-\infty, \infty)$
t-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Slant asymptote: $y = \frac{1}{2}t - 1$
$\lim\limits_{t \to -\infty} g(t) = -\infty$
As $t \to -\infty$, the graph is below $y = \frac{1}{2}t - 1$
$\lim\limits_{t \to \infty} g(t) = \infty$
As $t \to \infty$, the graph is above $y = \frac{1}{2}t - 1$

16. $g(t) = \frac{t^2-2t+1}{t^3+t^2-2t} = \frac{t-1}{t(t+2)}, \ t \neq 1$



Domain: $(-\infty, -2) \cup (-2, 0) \cup (0, 1) \cup (1, \infty)$
No t-intercepts
No y-intercepts
Vertical asymptotes: $t = -2$ and $t = 0$
$\lim\limits_{t \to -2^-} g(t) = -\infty$, $\lim\limits_{t \to -2^+} g(t) = \infty$
$\lim\limits_{t \to 0^-} g(t) = \infty$, $\lim\limits_{t \to 0^+} g(t) = -\infty$
Hole in the graph at $(1, 0)$
Horizontal asymptote: $y = 0$
$\lim\limits_{t \to -\infty} g(t) = 0$
More specifically, as $t \to -\infty$, $g(t) \to 0^-$
$\lim\limits_{t \to \infty} g(t) = 0$
More specifically, as $t \to \infty$, $g(t) \to 0^+$

17. $f(x) = \frac{1}{x-2}$

18. $F(x) = \frac{x-3}{(x-2)(x-3)} = \frac{x-3}{x^2-5x+6}$

19. $g(t) = \frac{t^2-1}{t}$

20. $G(t) = \frac{(t^2-1)(t+1)}{t(t+1)} = \frac{t^3+t^2-t-1}{t^2+t}$

3.3 Inequalities involving Rational Functions and Applications

In this section, we solve equations and inequalities involving rational functions and explore associated application problems. Our first example showcases the critical difference in procedure between solving equations and inequalities.

Example 3.3.1.

1. Solve $\dfrac{x^3-2x+1}{x-1} = \dfrac{1}{2}x-1$.
2. Solve $\dfrac{x^3-2x+1}{x-1} \geq \dfrac{1}{2}x-1$.
3. Verify your solutions using a graphing utility.

Solution.

1. To solve the equation, we clear denominators

$$\begin{array}{rcll}
\dfrac{x^3-2x+1}{x-1} & = & \dfrac{1}{2}x-1 & \\
\left(\dfrac{x^3-2x+1}{x-1}\right)\cdot 2(x-1) & = & \left(\dfrac{1}{2}x-1\right)\cdot 2(x-1) & \\
2x^3-4x+2 & = & x^2-3x+2 & \text{expand} \\
2x^3-x^2-x & = & 0 & \\
x(2x+1)(x-1) & = & 0 & \text{factor} \\
x & = & -\frac{1}{2},\, 0,\, 1 &
\end{array}$$

Since we cleared denominators, we need to check for extraneous solutions. Sure enough, we see that $x=1$ does not satisfy the original equation, so our only solutions are $x=-\frac{1}{2}$ and $x=0$.

2. To solve the inequality, it may be tempting to begin as we did with the equation − namely by multiplying both sides by the quantity $(x-1)$. The problem is that, depending on x, $(x-1)$ may be positive (which doesn't affect the inequality) or $(x-1)$ could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and make a sign diagram using the technique given on page 253 in Section 3.2.

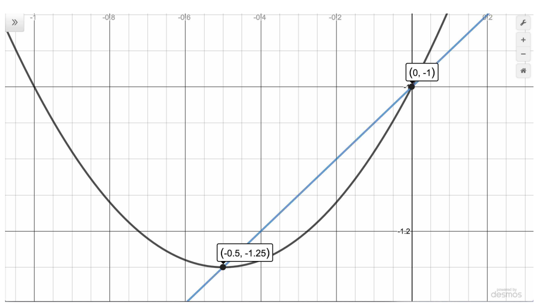
$$\begin{array}{rcll}
\dfrac{x^3-2x+1}{x-1} & \geq & \dfrac{1}{2}x-1 & \\
\dfrac{x^3-2x+1}{x-1} - \dfrac{1}{2}x+1 & \geq & 0 & \\
\dfrac{2\left(x^3-2x+1\right)}{2(x-1)} - \dfrac{x(x-1)}{2(x-1)} + \dfrac{2(x-1)}{2(x-1)} & \geq & 0 & \text{get a common denominator} \\
\dfrac{2\left(x^3-2x+1\right)-x(x-1)+2(x-1)}{2(x-1)} & \geq & 0 & \\
\dfrac{2x^3-x^2-x}{2x-2} & \geq & 0 & \text{expand}
\end{array}$$

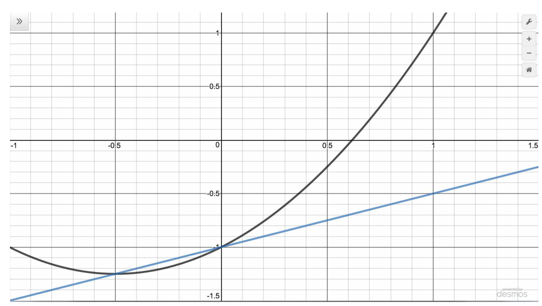
Viewing the left hand side as a rational function $r(x)$ we make a sign diagram. The only value excluded from the domain of r is $x = 1$ which is the solution to $2x - 2 = 0$. The zeros of r are the solutions to $2x^3 - x^2 - x = 0$, which we have already found to be $x = 0$, $x = -\frac{1}{2}$ and $x = 1$, the latter was discounted as a zero because it is not in the domain. Choosing test values in each test interval, we construct the sign diagram below.

$$\begin{array}{ccccccc} (+) & 0 & (-) & 0 & (+) & ? & (+) \\ & -\frac{1}{2} & & 0 & & 1 & \end{array}$$

We are interested in where $r(x) \geq 0$. We see $r(x) > 0$, or (+), on the intervals $\left(-\infty, -\frac{1}{2}\right)$, $(0, 1)$ and $(1, \infty)$. We know $r(x) = 0$ when $x = -\frac{1}{2}$ and $x = 0$. Hence, $r(x) \geq 0$ on $\left(-\infty, -\frac{1}{2}\right] \cup [0, 1) \cup (1, \infty)$.

3. To check our answers graphically, let $f(x) = \frac{x^3 - 2x + 1}{x - 1}$ and $g(x) = \frac{1}{2}x - 1$. The solutions to $f(x) = g(x)$ are the x-coordinates of the points where the graphs of $y = f(x)$ and $y = g(x)$ intersect. We graph both f and g below (the graph of g is the line and is slightly lighter in color.) We find only two intersection points, $(-0.5, -1.25)$ and $(0, -1)$ which correspond to our solutions $x = -\frac{1}{2}$ and $x = 0$. The solution to $f(x) \geq g(x)$ represents not only where the graphs meet, but the intervals over which the graph of $y = f(x)$ is above ($>$) the graph of $g(x)$. From the graph, this *appears* to happen on on $\left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ which *almost* matches the answer we found analytically. We have to remember that f is not defined at $x = 1$, so it cannot be included in our solution.[1]





□

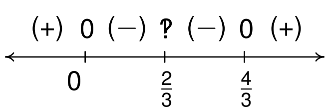
The important take-away from Example 3.3.1 is not to clear fractions when working with an inequality unless you know for certain the sign of the denominators. We offer another example.

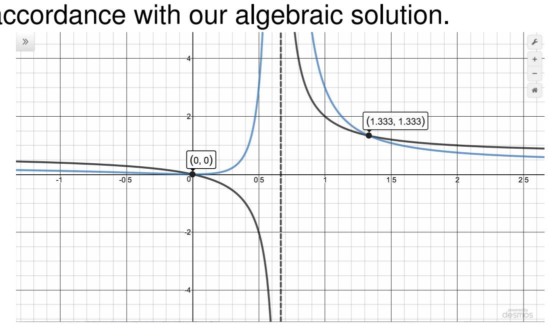
Example 3.3.2. Solve: $2t(3t - 2)^{-1} \leq 3t^2(3t - 2)^{-2}$. Check your answer using a graphing utility.
Solution. We begin by rewriting the terns with negative exponents as fractions and gathering all nonzero terms to one side of the inequality:

[1] We invite the reader to show there is a hole in the graph of $y = f(x)$ at $(1, 1)$.

$$\begin{array}{rcll}
2t(3t-2)^{-1} & \leq & 3t^2(3t-2)^{-2} & \\
\dfrac{2t}{3t-2} & \leq & \dfrac{3t^2}{(3t-2)^2} & \\
\dfrac{2t}{3t-2} - \dfrac{3t^2}{(3t-2)^2} & \leq & 0 & \\
\dfrac{2t(3t-2)}{(3t-2)^2} - \dfrac{3t^2}{(3t-2)^2} & \leq & 0 & \text{get a common denominator} \\
\dfrac{2t(3t-2)-3t^2}{(3t-2)^2} & \leq & 0 & \\
\dfrac{3t^2-4t}{(3t-2)^2} & \leq & 0 & \text{expand}
\end{array}$$

We define $r(t) = \frac{3t^2-4t}{(3t-2)^2}$ and set about constructing a sign diagram for r. Solving $(3t-2)^2 = 0$ gives $t = \frac{2}{3}$, our sole excluded value. To find the zeros of r, we set $r(t) = \frac{3t^2-4t}{(3t-2)^2} = 0$ and solve $3t^2 - 4t = 0$. Factoring gives $t(3t-4) = 0$ so our solutions are $t = 0$ and $t = \frac{4}{3}$. After choosing test values, we get the sign diagram below on the left. Since we are looking for where $r(t) \leq 0$, we are looking for where $r(t)$ is $(-)$ or $r(t) = 0$. Hence, our final answer is $\left[0, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \frac{4}{3}\right]$. Below on the right, we graph $f(t) = 2t(3t-1)^{-1}$ (the darker curve), $g(t) = 3t^2(3t-2)^{-2}$, and vertical asymptote $x = \frac{2}{3}$, the dashed line. Sure enough, the graph of f intersects the graph of g when $t = 0$ and $t = \frac{4}{3}$. Moreover, the graph of f is below the graph of g everywhere they are defined between these values, in accordance with our algebraic solution.





□

One thing to note about Example 3.3.2 is that the quantity $(3t-2)^2 \geq 0$ for all values of t. Hence, as long as we remember $t = \frac{2}{3}$ is excluded from consideration, we could actually multiply both sides of the inequality in Example 3.3.2 by $(3t-2)^2$ to obtain $2t(3t-2) \leq 3t^2$. We could then solve this (slightly easier) inequality using the methods of Section 1.4 as long as we remember to exclude $t = \frac{2}{3}$ from our solution. Once again, the more you *understand*, the less you have to *memorize*. If you know the '*why*' behind an algorithm instead of just the '*how*,' you will know when you can short-cut it.

Our next example is an application of average cost. Recall from Definition 3.8 if $C(x)$ represents the cost to make x items then the average cost per item is given by $\overline{C}(x) = \frac{C(x)}{x}$, for $x > 0$.

Example 3.3.3. Recall from Example 1.2.3 that the cost, $C(x)$, in dollars, to produce x PortaBoy game systems for a local retailer is $C(x) = 80x + 150$, $x \geq 0$.

1. Find an expression for the average cost function, $\overline{C}(x)$.
2. Solve $\overline{C}(x) < 100$ and interpret.
3. Find and interpret $\lim_{x\to\infty} \overline{C}(x)$.

Solution.

1. From $\overline{C}(x) = \frac{C(x)}{x}$, we obtain $\overline{C}(x) = \frac{80x+150}{x}$. The domain of C is $x \geq 0$, but since $x = 0$ causes problems for $\overline{C}(x)$, we get our domain to be $x > 0$, or $(0, \infty)$.

2. Solving $\overline{C}(x) < 100$ means we solve $\frac{80x+150}{x} < 100$. We proceed as in the previous example.

$$\begin{aligned} \frac{80x+150}{x} &< 100 \\ \frac{80x+150}{x} - 100 &< 0 \\ \frac{80x+150-100x}{x} &< 0 \qquad \text{common denominator} \\ \frac{150-20x}{x} &< 0 \end{aligned}$$

If we take the left hand side to be a rational function $r(x)$, we need to keep in mind that the applied domain of the problem is $x > 0$. This means we consider only the positive half of the number line for our sign diagram. On $(0, \infty)$, r is defined everywhere so we need only look for zeros of r. Setting $r(x) = 0$ gives $150 - 20x = 0$, so that $x = \frac{15}{2} = 7.5$. The test intervals on our domain are $(0, 7.5)$ and $(7.5, \infty)$. We find $r(x) < 0$ on $(7.5, \infty)$.

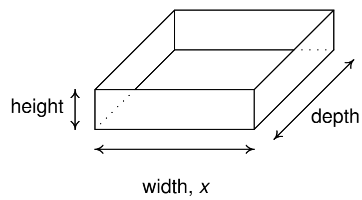
$$\begin{array}{ccccc} \text{?} & (+) & 0 & (-) & \\ 0 & & 7.5 & & \end{array}$$

In the context of the problem, x represents the number of PortaBoy games systems produced and $\overline{C}(x)$ is the average cost to produce each system. Solving $\overline{C}(x) < 100$ means we are trying to find how many systems we need to produce so that the average cost is less than \$100 per system. Our solution, $(7.5, \infty)$ tells us that we need to produce more than 7.5 systems to achieve this. Since it doesn't make sense to produce half a system, our final answer is $[8, \infty)$.

3. To find $\lim_{x\to\infty} \overline{C}(x)$, we note that we can rewrite $\overline{C}(x) = \frac{80x+150}{x} = 80 + \frac{150}{x}$. As $x \to \infty$, note that $\frac{150}{x} \to 0$ so $\lim_{x\to\infty} \overline{C}(x) = 80 + 0 = 80$. Thus the average cost per system is getting closer to \$80 per system. Since $\frac{150}{x} > 0$ for all $x > 0$, we have that $\overline{C}(x) > 80$ for all $x > 0$. This means that the average cost per system is always greater than \$80 per system, but the average cost is approaching this amount as more and more systems are produced. Looking back at Example 1.2.3, we realize \$80 is the variable cost per system $-$ the cost per system above and beyond the fixed initial cost of \$150. Another way to interpret our answer is that 'infinitely' many systems would need to be produced to effectively 'zero out' the fixed cost. □

Note that number 2 in Example 3.3.3 is another opportunity to short-cut the standard algorithm and obtain the solution more quickly if we take stock of the situation. Since the applied domain is $x > 0$, we can multiply through the inequality $\frac{80x+150}{x} < 100$ by x without worrying about changing the sense of the inequality. This reduces the problem to $80x + 150 < 100x$, a basic linear inequality whose solution is readily seen to be $x > 7.5$. It is absolutely critical here that $x > 0$. Indeed, any time you decide to multiply an inequality by a variable expression, it is necessary to justify why the inequality is preserved. Our next example is another classic 'box with no top' problem. The reader is encouraged to compare and contrast this problem with Example 2.1.4 in Section 2.1.

Example 3.3.4. A box with a square base and no top is to be constructed so that it has a volume of 1000 cubic centimeters. Let x denote the width of the box, in centimeters as seen below.



1. Explain why the height of the box (in centimeters) is a function of the width x. Call this function h and find an expression for $h(x)$, complete with an appropriate applied domain.

2. Solve $h(x) \geq x$ and interpret.

3. Find and interpret $\displaystyle\lim_{x\to 0^+} h(x)$ and $\displaystyle\lim_{x\to\infty} h(x)$.

4. Express the surface area of the box as a function of x, $S(x)$ and state the applied domain.

5. Use a graphing utility to approximate (to two decimal places) the dimensions of the box which minimize the surface area.

Solution.

1. We are told that the volume of the box is 1000 cubic centimeters and that x represents the width, in centimeters. Since x represents a physical dimension of a box, we have that $x > 0$. From geometry, we know volume = width $\times$ height $\times$ depth. Since the base of the box is a square, the width and the depth are both x centimeters. Hence, $1000 = x^2(\text{height})$. Solving for the height, we get height $= \frac{1000}{x^2}$. In other words, for each width $x > 0$, we are able to compute *the*[2] corresponding height using the formula $\frac{1000}{x^2}$. Hence, the height is a function of x. Using function notation, we write $h(x) = \frac{1000}{x^2}$. As mentioned before, our only restriction is $x > 0$ so the domain of h is $(0, \infty)$.

2. To solve $h(x) \geq x$, we proceed as before and collect all nonzero terms on one side of the inequality in order to use a sign diagram.

[2]that is, the one and only one

$$\begin{aligned} h(x) &\geq x \\ \frac{1000}{x^2} &\geq x \\ \frac{1000}{x^2} - x &\geq 0 \\ \frac{1000 - x^3}{x^2} &\geq 0 \quad \text{common denominator} \end{aligned}$$

We consider the left hand side of the inequality as our rational function $r(x)$. We see immediately the only value excluded from the domain of r is 0, but since our applied domain is $x > 0$, we restrict our attention to the interval $(0, \infty)$. The sole zero of r comes when $1000 - x^3 = 0$, or when $x = 10$. Choosing test values in the intervals $(0, 10)$ and $(10, \infty)$ gives the following:

$$\begin{array}{ccccc} \text{?} & (+) & 0 & (-) & \\ \hline 0 & & 10 & & \rightarrow \end{array}$$

We see $r(x) > 0$ on $(0, 10)$, and since $r(x) = 0$ at $x = 10$, our solution is $(0, 10]$. In the context of the problem, $h(x)$ represents the height of the box while x represents the width (and depth) of the box. Solving $h(x) \geq x$ is tantamount to finding the values of x which result in a box where the height is at least as big as the width (and, in this case, depth.) Our answer tells us the width of the box can be at most 10 centimeters for this to happen.[3]

3. Since $h(x) = \frac{1000}{x^2}$, $\lim\limits_{x\to 0^+} h(x) = \infty$. This means that the smaller the width x (and, in this case, depth), the larger the height h has to be in order to maintain a volume of 1000 cubic centimeters. On the other hand, $\lim\limits_{x\to\infty} h(x) = 0$, which means that in order to maintain a volume of 1000 cubic centimeters, the width and depth must get bigger as the height becomes smaller.

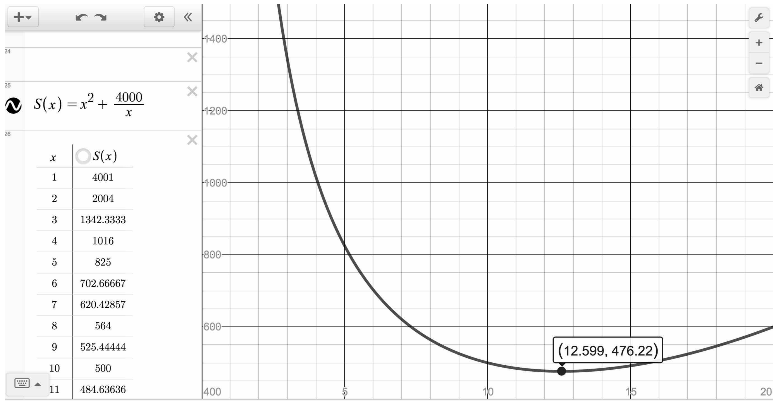
4. Since the box has no top, the surface area can be found by adding the area of each of the sides to the area of the base. The base is a square of dimensions x by x, and each side has dimensions x by $h(x)$. We get the surface area, $S(x) = x^2 + 4xh(x)$. Since $h(x) = \frac{1000}{x^2}$, we have $S(x) = x^2 + 4x\left(\frac{1000}{x^2}\right) = x^2 + \frac{4000}{x}$. The domain of S is the same as h, namely $(0, \infty)$, for the same reasons as above.

5. To graph $y = S(x)$, we create a table of values to help us define a good viewing window. Doing so, we find a local minim when $x \approx 12.60$. As far as we can tell,[4] this is the only local extremum, so it is the (absolute) minimum as well. This means that the width and depth of the box should each measure approximately 12.60 centimeters. To determine the height, we find $h(12.60) \approx 6.30$, so the height of the box should be approximately 6.30 centimeters.[5]

[3]As with the previous example, knowing $x > 0$ means $x^2 > 0$ so we can clear denominators right away and solve $x^3 \leq 1000$, or $x \leq 10$. Coupled with our applied domain, $x > 0$, we would arrive at the same solution, $(0, 10]$.

[4]without Calculus, that is...

[5]The y-coordinate here, 476.22 means the minimum surface area possible is 476.22 square centimeters. Minimizing the surface area minimizes the material required to make the box, therein helping to reduce the cost of the box.



□

Our last example uses regression to verify a very famous scientific law.

Example 3.3.5. Boyle's Law states that when temperature is held constant, the pressure of a gas is inversely proportional to the volume of the gas.[6] According to this website the actual data relating the volume V of a gas and its pressure P used by Boyle and his assistant in 1662 to formulate this law is given below. (NOTE: both pressure and volume here are given in 'arbitrary units.')

V	48	46	44	42	40	38	36	34	32	30	28	26	24
P	29.13	30.56	31.94	33.5	35.31	37	39.31	41.63	44.19	47.06	50.31	54.31	58.81

V	23	22	21	20	19	18	17	16	15	14	13	12
P	61.31	64.06	67.06	70.69	74.13	77.88	82.75	87.88	93.06	100.44	107.81	117.56

1. Assuming P and V are inversely proportional, estimate the constant of proportionality, k.

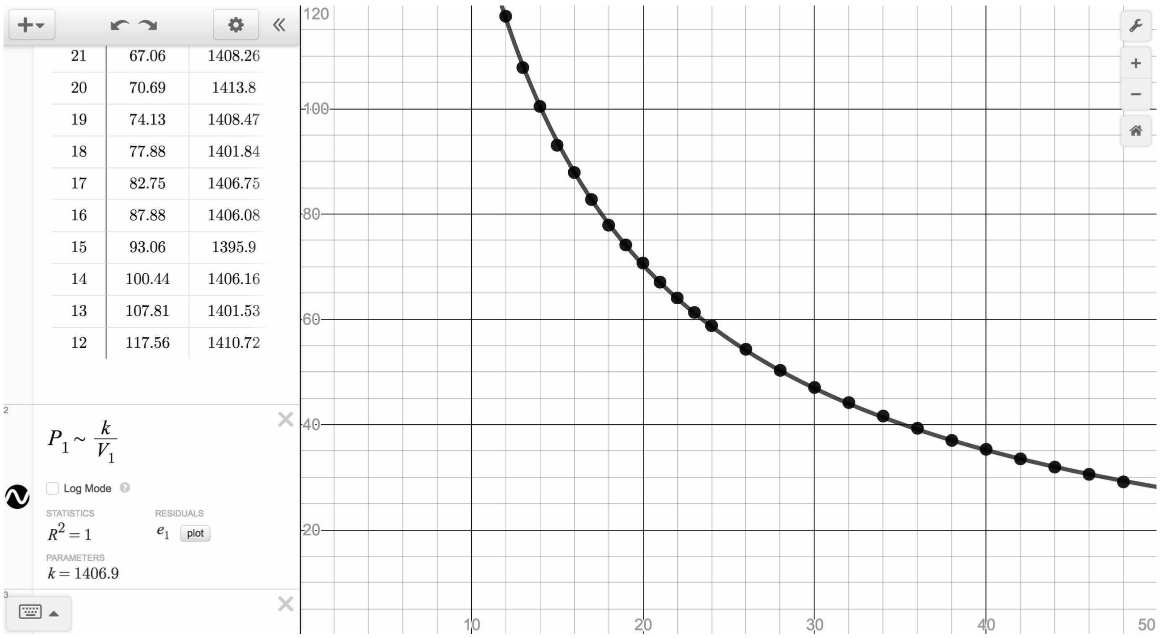
2. Use a graphing utility to fit a curve of the form $P = \dfrac{k}{V}$ to these data.

Solution.

1. Recall if P and V are inversely proportional, there is a real number k so $PV = k$ for all values of P and V. Multiplying the corresponding P and V values from the data together result in numbers which are consistently approximately 1400. This gives us confidence in the claim P and V are inversely proportional and suggests $k \approx 1400$.

2. We plot the pairs (V, P) and run a regression, the results of which are below. To our amazement, the graphing utility reports $k \approx 1406.9$ with $R^2 \approx 1$. This means the data are a very good fit to the model $P = \frac{k}{V}$, or $PV = k$, hence verifying Boyle's Law for this set of data.

[6] For a review of what this means, see Section A.14.



21 67.06 1408.26
20 70.69 1413.8
19 74.13 1408.47
18 77.88 1401.84
17 82.75 1406.75
16 87.88 1406.08
15 93.06 1395.9
14 100.44 1406.16
13 107.81 1401.53
12 117.56 1410.72
$P_1 \sim \frac{k}{V_1}$
Log Mode
STATISTICS
$R^2 = 1$
RESIDUALS
e_1 plot
PARAMETERS
$k = 1406.9$
120
100
80
60
40
20
10
20
30
40
50

$\square$

3.3.1 Exercises

(Review of Solving Equations):[7] In Exercises 1 - 6, solve the rational equation. Be sure to check for extraneous solutions.

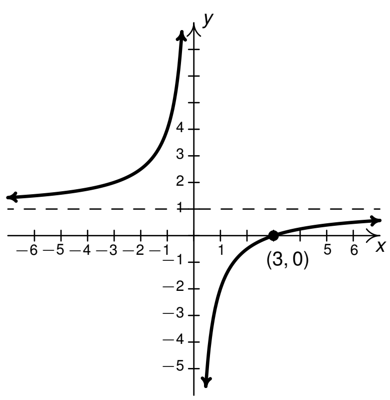
1. $\dfrac{x}{5x+4} = 3$
2. $\dfrac{3x-1}{x^2+1} = 1$
3. $\dfrac{1}{t+3} + \dfrac{1}{t-3} = \dfrac{t^2-3}{t^2-9}$
4. $\dfrac{2t+17}{t+1} = t+5$
5. $\dfrac{z^2-2z+1}{z^3+z^2-2z} = 1$
6. $\dfrac{4z-z^3}{z^2-9} = 4z$

In Exercises 7 - 22, solve the rational inequality. Express your answer using interval notation.

7. $\dfrac{1}{x+2} \geq 0$
8. $\dfrac{5}{x+2} \geq 1$
9. $\dfrac{x}{x^2-1} < 0$
10. $\dfrac{4t}{t^2+4} \geq 0$
11. $\dfrac{2t+6}{t^2+t-6} < 1$
12. $\dfrac{5}{t-3} + 9 < \dfrac{20}{t+3}$
13. $\dfrac{6z+6}{2+z-z^2} \leq z+3$
14. $\dfrac{6}{z-1} + 1 > \dfrac{1}{z+1}$
15. $\dfrac{3z-1}{z^2+1} \leq 1$
16. $(2x+17)(x+1)^{-1} > x+5$
17. $(4x-x^3)(x^2-9)^{-1} \geq 4x$
18. $(x^2+1)^{-1} < 0$
19. $(2t-8)(t+1)^{-1} \leq (t^2-8t)(t+1)^{-2}$
20. $(t-3)(2t+7)(t^2+7t+6)^{-2} \geq (t^2+7t+6)^{-1}$
21. $60z^{-2} + 23z^{-1} \geq 7(z-4)^{-1}$
22. $2z + 6(z-1)^{-1} \geq 11 - 8(z+1)^{-1}$

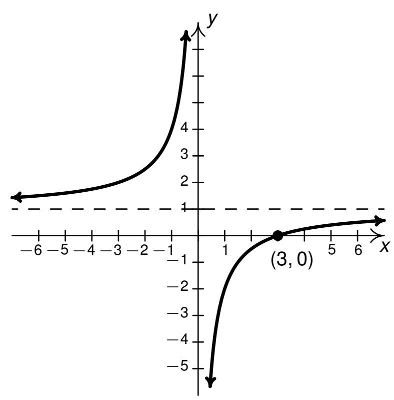
In Exercises 23 - 28, use the the graph of the given rational function to solve the stated inequality.

23. Solve $f(x) \geq 0$.



$y = f(x)$, asymptotes: $x = 0$, $y = 1$.

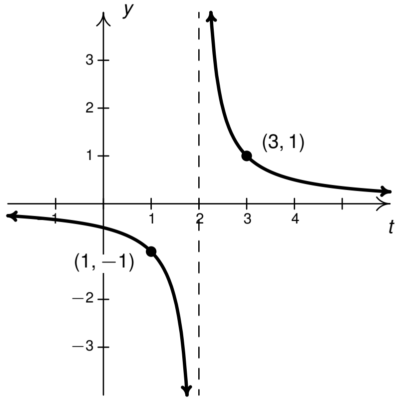
24. Solve $f(x) < 1$.



$y = f(x)$, asymptotes: $x = 0$, $y = 1$.

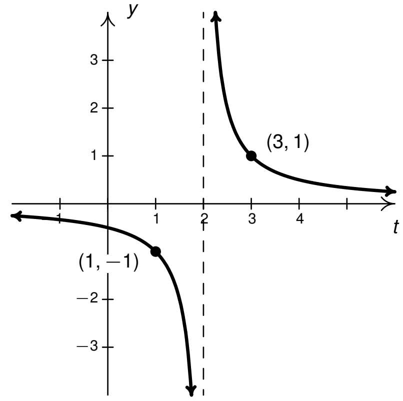
[7] For more review, see Section A.12.

25. Solve $g(t) \geq -1$.



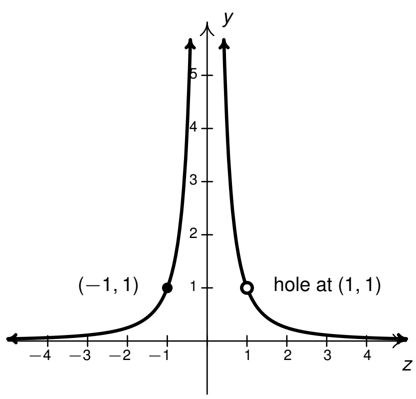
$y = g(t)$, asymptotes: $t = 2$, $y = 0$.

26. Solve $-1 \leq g(t) < 1$.



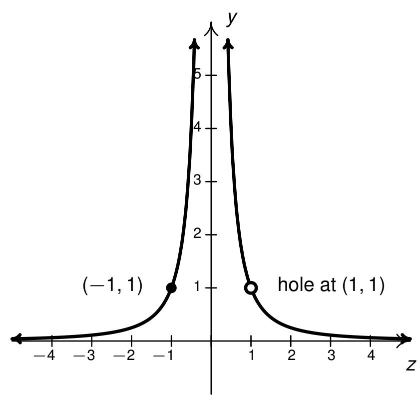
$y = g(t)$, asymptotes: $t = 2$, $y = 0$.

27. Solve $r(z) \leq 1$



$y = r(z)$, asymptotes: $z = 0$, $y = 0$.

28. Solve $r(z) > 0$.



$y = r(z)$, asymptotes: $z = 0$, $y = 0$.

29. In Exercise 55 in Section 2.1, the function $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$ was used to model the cost (in dollars) to produce x PortaBoy game systems. Using this cost function, find the number of PortaBoys which should be produced to minimize the average cost $\overline{C}$. Round your answer to the nearest number of systems.

30. Suppose we are in the same situation as Example 3.3.4. If the volume of the box is to be 500 cubic centimeters, use a graphing utility to find the dimensions of the box which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

31. The box for the new Sasquatch-themed cereal, 'Crypt-Os', is to have a volume of 140 cubic inches. For aesthetic reasons, the height of the box needs to be 1.62 times the width of the base of the box.[8] Find the dimensions of the box which will minimize the surface area of the box. What is the minimum surface area? Round your answers to two decimal places.

[8] 1.62 is a crude approximation of the so-called 'Golden Ratio' $\phi = \frac{1+\sqrt{5}}{2}$.

32. Sally is Skippy's neighbor from Exercise 43 in Section 1.4. Sally also wants to plant a vegetable garden along the side of her home. She doesn't have any fencing, but wants to keep the size of the garden to 100 square feet. What are the dimensions of the garden which will minimize the amount of fencing she needs to buy? What is the minimum amount of fencing she needs to buy? Round your answers to the nearest foot. (Note: Since one side of the garden will border the house, Sally doesn't need fencing along that side.)

33. Another Classic Problem: A can is made in the shape of a right circular cylinder and is to hold one pint. (For dry goods, one pint is equal to 33.6 cubic inches.)[9]

    (a) Find an expression for the volume V of the can in terms of the height h and the base radius r.
    (b) Find an expression for the surface area S of the can in terms of the height h and the base radius r. (Hint: The top and bottom of the can are circles of radius r and the side of the can is really just a rectangle that has been bent into a cylinder.)
    (c) Using the fact that $V = 33.6$, write S as a function of r and state its applied domain.
    (d) Use your graphing calculator to find the dimensions of the can which has minimal surface area.

34. A right cylindrical drum is to hold 7.35 cubic feet of liquid. Find the dimensions (radius of the base and height) of the drum which would minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

35. In Exercise 35 in Section 3.1, the population of Sasquatch in Portage County is modeled by

$$P(t) = \frac{150t}{t+15}, \quad t \geq 0,$$

    where $t = 0$ corresponds to the year 1803. According to this model, when were there fewer than 100 Sasquatch in Portage County?

[9] According to www.dictionary.com, there are different values given for this conversion. We use 33.6in^3 for this problem.

3.3.2 Answers

1. $x = -\frac{6}{7}$
2. $x = 1,\ x = 2$
3. $t = -1$
4. $t = -6,\ x = 2$
5. No solution
6. $z = 0,\ z = \pm 2\sqrt{2}$
7. $(-2, \infty)$
8. $(-2, 3]$
9. $(-\infty, -1) \cup (0, 1)$
10. $[0, \infty)$
11. $(-\infty, -3) \cup (-3, 2) \cup (4, \infty)$
12. $\left(-3, -\frac{1}{3}\right) \cup (2, 3)$
13. $(-1, 0] \cup (2, \infty)$
14. $(-\infty, -3) \cup (-2, -1) \cup (1, \infty)$
15. $(-\infty, 1] \cup [2, \infty)$
16. $(-\infty, -6) \cup (-1, 2)$
17. $(-\infty, -3) \cup \left[-2\sqrt{2}, 0\right] \cup \left[2\sqrt{2}, 3\right)$
18. No solution
19. $[-4, -1) \cup (-1, 2]$
20. $(-\infty, -6) \cup (-6, -3] \cup [9, \infty)$
21. $[-3, 0) \cup (0, 4) \cup [5, \infty)$
22. $\left(-1, -\frac{1}{2}\right] \cup (1, \infty)$
23. $f(x) \geq 0$ on $(-\infty, 0) \cup [3, \infty)$.
24. $f(x) < 1$ on $(0, \infty)$.
25. $g(t) \geq -1$ on $(-\infty, 1] \cup (2, \infty)$.
26. $-1 \leq g(t) < 1$ on $(-\infty, 1] \cup (3, \infty)$.
27. $r(z) \leq 1$ on $(-\infty, -1] \cup (1, \infty)$.
28. $r(z) > 0$ on $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.
29. The absolute minimum of $y = \overline{C}(x)$ occurs at $\approx (75.73, 59.57)$. Since x represents the number of game systems, we check $\overline{C}(75) \approx 59.58$ and $\overline{C}(76) \approx 59.57$. Hence, to minimize the average cost, 76 systems should be produced at an average cost of \$59.57 per system.
30. The width (and depth) should be 10.00 centimeters, the height should be 5.00 centimeters. The minimum surface area is 300.00 square centimeters.
31. The width of the base of the box should be approximately 4.12 inches, the height of the box should be approximately 6.67 inches, and the depth of the base of the box should be approximately 5.09 inches. The minimum surface area is approximately 164.91 square inches.

32. The dimensions are approximately 7 feet by 14 feet. Hence, the minimum amount of fencing required is approximately 28 feet.

33. (a) $V = \pi r^2 h$

    (b) $S = 2\pi r^2 + 2\pi rh$

    (c) $S(r) = 2\pi r^2 + \frac{67.2}{r}$, Domain $r > 0$

    (d) $r \approx 1.749$ in. and $h \approx 3.498$ in.

34. The radius of the drum should be approximately 1.05 feet and the height of the drum should be approximately 2.12 feet. The minimum surface area of the drum is approximately 20.93 cubic feet.

35. $P(t) < 100$ on $(-15, 30)$, and the portion of this which lies in the applied domain is $[0, 30)$. Since $t = 0$ corresponds to the year 1803, from 1803 through the end of 1832, there were fewer than 100 Sasquatch in Portage County.

Chapter 4

Root, Radical and Power Functions

4.1 Root and Radical Functions

In Sections 1.2, 1.3 and 1.4, we studied constant, linear, absolute value,[1] and quadratic functions. Constant, linear and quadratic functions were specific examples of polynomial functions, which we studied in generality in Chapter 2. Chapter 2 culminated with the Real Factorization Theorem, Theorem 2.18, which says that all polynomial functions with real coefficients can be thought of as products of linear and quadratic functions. Our next step was to enlarge our field[2] of study to rational functions in Chapter 3. Being quotients of polynomials, we can ultimately view this family of functions as being built up of linear and quadratic functions as well. So in some sense, Sections 1.2, 1.3 and 1.4 along with Chapters 2 and 3 can be thought of as an exhaustive study of linear and quadratic[3] functions. We now turn our attention to functions involving radicals which cannot be written in terms of linear functions. For a more detailed review of the basics of roots and radicals, we refer the reader to Sections A.2 and A.13.

4.1.1 Root Functions

As with polynomial functions and rational functions, we begin our study of functions involving radical with a special family of functions: the (principal) root functions.

Definition 4.1. Let $n \in \mathbb{N}$ with $n \geq 2$. The **nth (principal) root function** is the function $f(x) = \sqrt[n]{x}$.
NOTE: If n is even, the domain of f is $[0, \infty)$; if n is odd, the domain of f is $(-\infty, \infty)$.

The domain restriction for even indexed roots means that, once again, we are restricting our attention to *real* numbers.[4] We graph a few members of the root function family below, and quickly notice that, as with the monomial, and, more generally, the Laurent monomial functions, the behavior of the root functions depends primarily on whether the root is even or odd.

[1] These were introduced, as you may recall, as piecewise-defined linear functions.

[2] This is a really bad math pun.

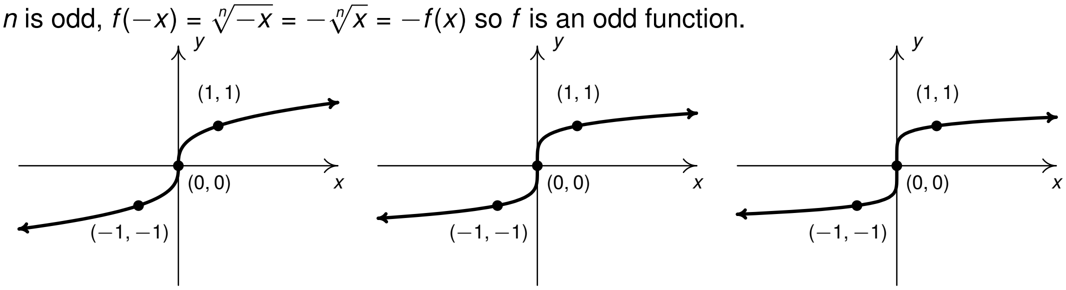
[3] If we broaden our concept of functions to allow for complex valued coefficients, the Complex Factorization Theorem, Theorem 2.16, tells us every function we have studied thus far is a combination of linear functions.

[4] Although we discussed imaginary numbers in Section 2.4, we restrict our attention to real numbers in this section. See the epilogue on page 208 for more details.

In addition to having the common domain of $[0, \infty)$, the graphs of $f(x) = \sqrt[n]{x}$ for even indices n all share the points $(0, 0)$ and $(1, 1)$. As n increases, the functions become 'steeper' near the y-axis and 'flatter' as $x \to \infty$. To show $f(x) \to \infty$ as $x \to \infty$, we show, more generally, the range of f is $[0, \infty)$. Indeed, if $c \geq 0$ is a real number, then $f(c^n) = \sqrt[n]{c^n} = c$ so c is in the range of f. Note that f is increasing: that is, if $a < b$, then $f(a) = \sqrt[n]{a} < \sqrt[n]{b} = f(b)$. This property is useful in solving certain types of polynomial inequalities.[5]

$y = \sqrt{x}$ $y = \sqrt[4]{x}$ $y = \sqrt[6]{x}$

The functions $f(x) = \sqrt[n]{x}$ for odd natural numbers $n \geq 3$ also follow a predictable trend - steepening near $x = 0$ and flattening as $x \to -\infty$ and $x \to -\infty$. The range for these functions is $(-\infty, \infty)$ since if c is any real number, $f(c^n) = \sqrt[n]{c^n} = c$, so c is in the range of f. Like the even indexed roots, the odd indexed roots are also increasing. Moreover, these graphs appear to be symmetric about the origin. Sure enough, when n is odd, $f(-x) = \sqrt[n]{-x} = -\sqrt[n]{x} = -f(x)$ so f is an odd function.



$y = \sqrt[3]{x}$ $y = \sqrt[5]{x}$ $y = \sqrt[7]{x}$

At this point, you're probably expecting a theorem like Theorems 1.2, 1.3, 2.1, 3.1 - that is, a theorem which tells us how to obtain the graph of $F(x) = a\sqrt[n]{x - h} + k$ from the graph of $f(x) = \sqrt[n]{x}$ - and you would not be wrong. Here, however, we need to add an extra parameter 'b' to the recipe and discuss functions of the form $F(x) = a\sqrt[n]{bx - h} + k$. The reason is that, with all of the previous function families, we were always able to factor out the coefficient of x. We list some examples of this below, and invite the reader to revisit other examples in the text:

- $F(x) = |6 - 2x| = |-2x + 6| = |-2(x + 3)| = |-2||x + 3| = 2|x + 3|$.
- $F(x) = (2x - 1)^2 + 1 = \left[2\left(x - \frac{1}{2}\right)\right]^2 + 1 = (2)^2\left(x - \frac{1}{2}\right)^2 + 1 = 4\left(x - \frac{1}{2}\right)^2 + 1$
- $F(x) = \frac{2}{(1-x)^3} - 5 = \frac{2}{[(-1)(x-1)]^3} - 5 = \frac{2}{(-1)^3(x-1)^3} - 5 = \frac{2}{-(x-1)^3} - 5 = \frac{-2}{(x-1)^3} - 5$.

For a function like $F(x) = \sqrt{4x - 12} + 1 = \sqrt{4(x - 6)} + 1 = \sqrt{4}\sqrt{x - 3} + 1 = 2\sqrt{x - 3} + 1$, this approach works fine. However, if the coefficient of x is *negative*, for example, $F(x) = \sqrt{1 - x} = \sqrt{(-1)(x - 1)}$ we get stuck the product rule for radicals doesn't extend to negative quantities when the index is even.[6] Hence we add an extra parameter which means we have an extra step. We state Theorem 4.1 below.

[5] See Exercise 13.

[6] Since, otherwise, $-1 = i^2 = i \cdot i = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$, a contradition.

Theorem 4.1. For real numbers a, b, h, and k with $a, b \neq 0$, the graph of $F(x) = a\sqrt[n]{bx-h}+k$ can be obtained from the graph of $f(x) = \sqrt[n]{x}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{x}$ to $y = \sqrt[n]{x-h}$.

2. divide the x-coordinates of the points on the graph obtained in Step 1 by b. This results in a horizontal scaling, but may also include a reflection about the y-axis if $b < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{x-h}$ to $y = \sqrt[n]{bx-h}$.

3. multiply the y-coordinates of the points on the graph obtained in Step 2 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{bx-h}$ to $y = a\sqrt[n]{bx-h}$.

4. add k to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a\sqrt[n]{bx-h}$ to $y = a\sqrt[n]{bx-h}+k$.

Proof. As usual, we 'build' the graph of $F(x) = a\sqrt[n]{bx-h}+k$ starting with the graph of $f(x) = \sqrt[n]{x}$ one step at a time. First, we consider the graph of $F_1(x) = \sqrt{x-h}$. A generic point on the graph of F_1 looks like $(x, \sqrt[n]{x-h})$. Note that if n is odd, x can be any real number whereas if n is even $x - h \geq 0$ so $x \geq h$. If we let $c = x - h$, then $x = c + h$ and we can change (dummy) variables[7] and obtain a new representation of the point: $(c + h, \sqrt[n]{c})$. Note that if n is odd, x and c vary through all real numbers; if n is even, $x \geq h$ and, hence, $c \geq 0$. Since a generic point on the graph of $f(x) = \sqrt[n]{x}$ can be represented as $(c, \sqrt[n]{c})$ for applicable values of c, we see that we can obtain every point on the graph of F_1 by adding h to each x-coordinate of the graph of f, establishing step 1 of the theorem.
Proceeding to (the new!) step 2, a point on the graph of $F_2(x) = \sqrt[n]{bx-h}$ has the form $(x, \sqrt[n]{bx-h})$. If n is odd, as usual, x can vary through all real numbers. If n is even, we require $bx - h \geq 0$ or $bx \geq h$. If $b > 0$, this gives $x \geq \frac{h}{b}$. If, on the other hand, $b < 0$, the we have $x \leq \frac{h}{b}$. Let $c = bx$ and since by assumption $b \neq 0$, we have $x = \frac{c}{b}$. Once again, we change dummy variables from x to c and describe a generic point on the graph of F_2 as $\left(\frac{c}{b}, \sqrt[n]{c-h}\right)$. If n is odd, x and c can vary through all real numbers. If n is even and $b > 0$, then $x \geq \frac{h}{b}$ and, hence, $c = bx \geq h$; if $b < 0$, then $x \leq \frac{h}{b}$ also gives $c = bx \geq h$. Since a generic point on the graph of F_1 can be represented as $(c, \sqrt{c-h})$ for applicable values of c, we see we can obtain every point on the graph of F_2 by dividing every x-coordinate on the graph of F_1 by b, as per step 2 of the theorem.
The proof of steps 3 and 4 of Theorem 4.1 are identical to the proof of Theorem 2.1 (just with $\sqrt[n]{\cdot}$ instead of $(\cdot)^n$) so we invite the reader to work through the details on their own. $\square$
We demonstrate Theorem 4.1 in the following example.

Example 4.1.1. Theorem 4.1 to graph the following functions. Label at least three points on the graph. State the domain and range using interval notation.

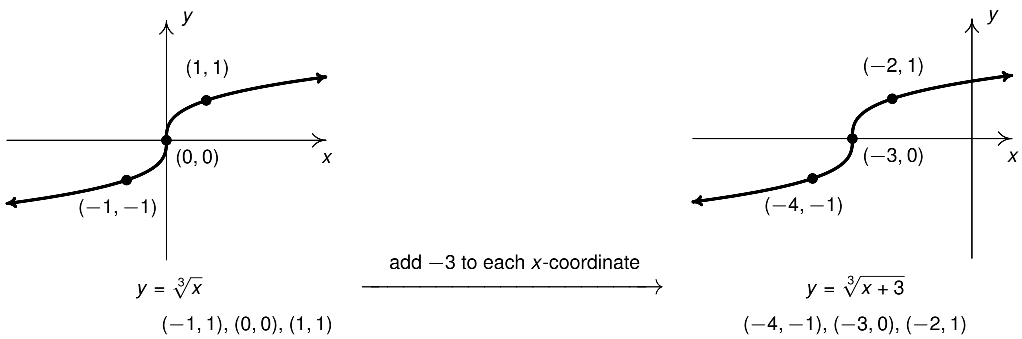
[7] again this is because every real number can be represented as both $x - h$ for some value x and as $c + h$ for some value c.

1. $f(x) = 1 - 2\sqrt[3]{x+3}$
2. $g(t) = \dfrac{\sqrt{1-2t}}{4}$

Solution.

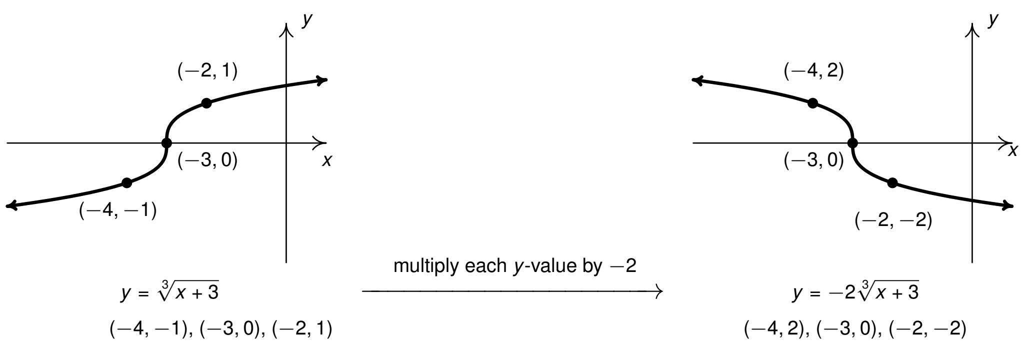
1. We begin by rewriting the expression for $f(x)$ in the form prescribed Theorem 4.1: $f(x) = -2\sqrt[3]{x+3} + 1$. We identify $n = 3$, $a = -2$, $b = 1$, $h = -3$ and $k = 1$.

   Step 1: add -3 to each of the x-coordinates of each of the points on the graph of $y = \sqrt[3]{x}$:

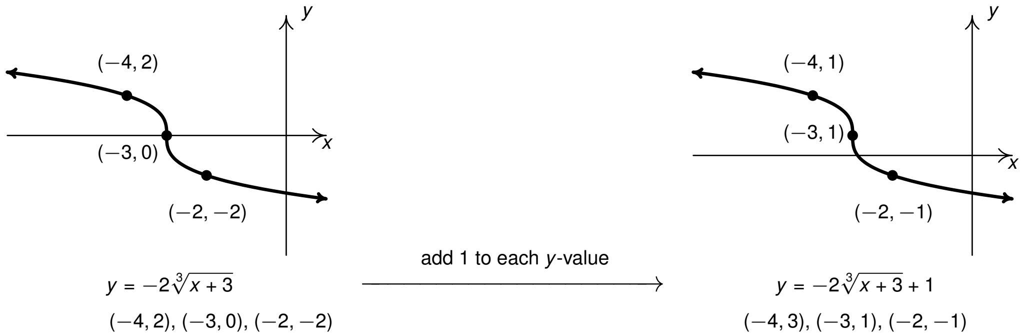


   Since $b = 1$, we can proceed to Step 3 (since dividing a real by 1 just results in the same real number.)

   Step 3: multiply each of the y-coordinates of each point on the graph of $y = \sqrt[3]{x+3}$ by -2:



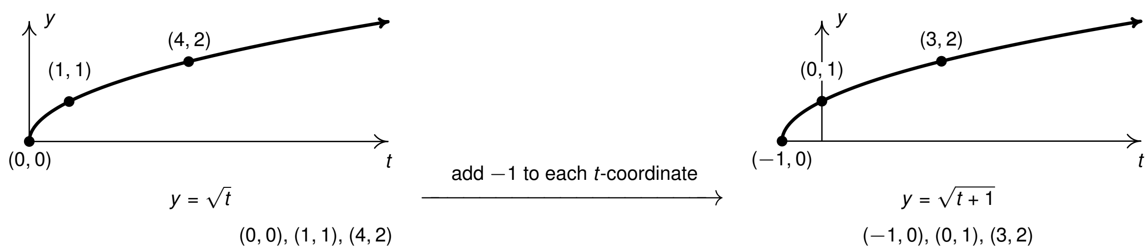
   Step 4: add 1 to y-coordinates of each point on the graph of $y = -2\sqrt[3]{x+3}$:



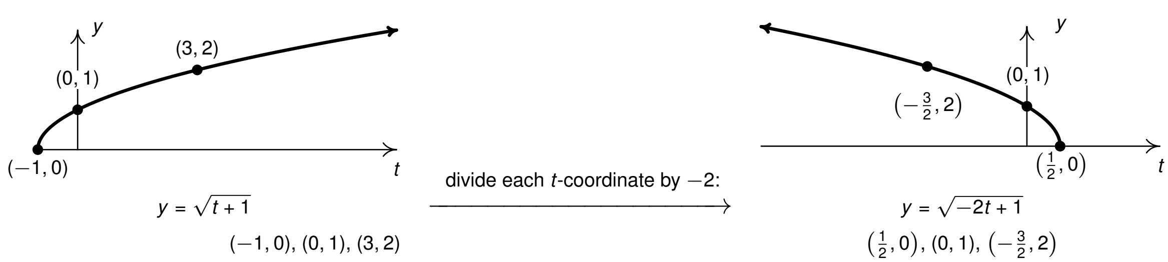
We get the domain and range of f are $(-\infty, \infty)$.

2. For $g(t) = \dfrac{\sqrt{1-2t}}{4} = \frac{1}{4}\sqrt{-2t+1}$, we identify $n = 2$, $a = \frac{1}{4}$, $b = -2$, $h = -1$ and $k = 0$. Since we are asked to label *three* points on the graph, we track $(4, 2)$ along with $(0, 0)$ and $(1, 1)$.[8]

   Step 1: add -1 to each of the t-coordinates of each of the points on the graph of $y = \sqrt{t}$:

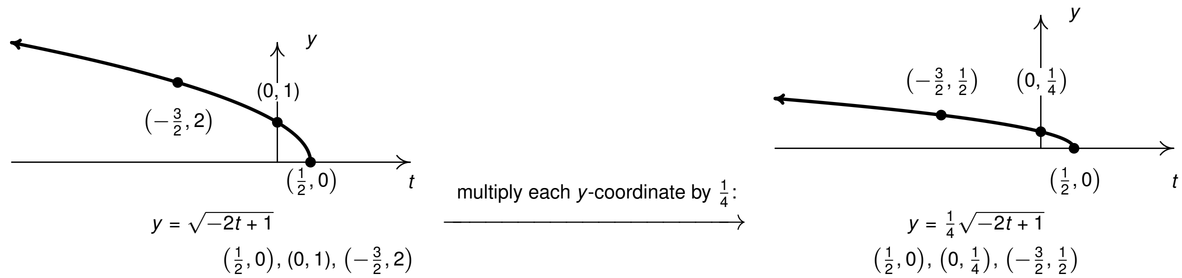


   Step 2: divide each of the t-coordinates of each of the points on the graph of $y = \sqrt{t+1}$ by -2:



   Step 3: multiply each of the y-coordinates of each of the points on the graph of $y = \sqrt{-2t+1}$ by $\frac{1}{4}$:

[8] As $\sqrt{4} = 2$, we know $(4, 2)$ is on the graph of $y = \sqrt{t}$.



We get the domain is $\left(-\infty, \frac{1}{2}\right]$ and the range is $[0, \infty)$. $\square$

4.1.2 Other Functions involving Radicals

Now that we have some practice with basic root functions, we turn our attention to more general functions involving radicals. In general, Calculus is the best tool with which to study these functions. Nevertheless, we will use what algebra we know in combination with a graphing utility to help us visualize these functions and preview concepts which are studied in greater depth in later courses. In the table below, we summarize some of the properties of radicals from elsewhere in this text (and Intermediate Algebra) we will be using in the coming examples.

Theorem 4.2. Some Useful Properties of Radicals: Suppose $\sqrt[n]{x}$, $\sqrt[n]{a}$, and $\sqrt[n]{b}$ are real numbers.[a]

Simplifying n th powers and n th roots:[b]

- $\left(\sqrt[n]{x}\right)^n = x$.
- if n is odd, then $\sqrt[n]{x^n} = x$
- if n is even, then $\sqrt[n]{x^n} = |x|$.

Root Functions Preserve Inequality:[c] if $a \leq b$, then $\sqrt[n]{a} \leq \sqrt[n]{b}$.

[a] i.e., if n is odd, x, a, and b can be any real numbers; if, on the other hand n is even, $x \geq 0$, $a \geq 0$, and $b \geq 0$.
[b] a.k.a., 'Inverse Properties.' See Section 5.6.
[c] i.e., root functions are increasing.

Example 4.1.2. For the following functions:

- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.

- Graph the function with help from a graphing utility and determine:
  - the range.
  - the local extrema, if they exist.
  - intervals of increase.
  - intervals of decrease.
- Construct a sign diagram for each function using the intercepts and graph.[9]

1. $f(x) = 3x\sqrt[3]{2-x}$
2. $g(t) = \sqrt[3]{\dfrac{8t}{t+1}}$
3. $h(x) = \dfrac{3x}{\sqrt{x^2+1}}$
4. $r(t) = t^{-1}\sqrt{16t^4-1}$

Solution.

1. When looking for the domain, we have two thing to watch out for: denominators (which we must make sure aren't 0) and even indexed radicals (whose radicands we must ensure are nonnegative.) Looking at the expression for $f(x)$, we have no denominators nor do we have an even indexed radical, so we are confident the domain is all real numbers, $(-\infty, \infty)$.

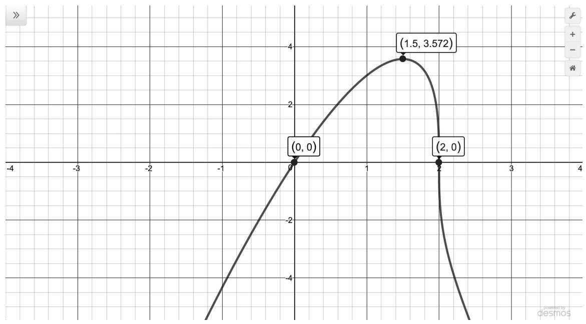
   To find the x-intercepts, we find the zeros of f by solving $f(x) = 3x\sqrt[3]{2-x} = 0$. Using the zero product property, we get $3x = 0$ or $\sqrt[3]{2-x} = 0$. The former gives $x = 0$ and to solve the latter, we cube both sides and get $2 - x = 0$ or $x = 2$. Hence, the x-intercepts are $(0,0)$ and $(2,0)$. Since $(0,0)$ is also on the y-axis and functions can can have at most one y-intercept, we know $(0,0)$ is the only y-intercept.[10] That being said, we can quickly verify $f(0) = 3(0)\sqrt[3]{2-0} = 0$.

   To determine the end behavior, we first consider $f(x)$ as $x \to \infty$. Using 'number sense,' [11] we have $f(x) = 3x\sqrt[3]{2-x} = 3x\sqrt[3]{-x+2} \approx (\text{big } (+))\sqrt[3]{\text{big } (-)} = (\text{big } (+))(\text{big } (-)) = \text{big } (-)$, so $\displaystyle\lim_{x\to\infty} f(x) = -\infty$. As $x \to -\infty$ we get $f(x) = 3x\sqrt[3]{-x+2} \approx (\text{big } (-))\sqrt[3]{\text{big } (+)} = (\text{big } (-))(\text{big } (+)) = \text{big } (-)$, so $\displaystyle\lim_{x\to-\infty} f(x) = -\infty$ as well.

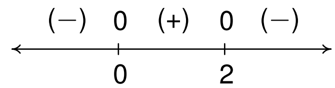
   We graph f below on the left. From the graph, the range appears to be $(-\infty, 3.572]$ with a local maximum (which also happens to be *the* maximum) at $(1.5, 3.572)$. We also see f appears to be increasing on $(-\infty, 1.5)$ and decreasing on $(1.5, \infty)$. It is also worth noting that there appears to be 'unusual steepness' near the x-intercept $(2,0)$. We invite the reader to zoom in on the graph near $(2,0)$ to see that the function is 'locally vertical.'[12]

   To create a sign diagram for $f(x)$, we note that the function has zeros $x = 0$ and $x = 2$. For $x < 0$, $f(x) < 0$ or $(-)$, for $0 < x < 2$, $f(x) > 0$ or $(+)$, and for $x > 2$, $f(x) < 0$ or $(-)$. The sign diagram for $f(x)$ is below on the right.

[9]We'll revisit sign diagrams for these functions in Section 4.3 where we will use them to solve inequalities (surprised?)
[10]Why is this, again?
[11]remember this means we use the adjective 'big' here to mean large in *absolute value*
[12]Of course, the Vertical Line Test prohibits the graph from actually *being* a vertical line. This behavior is more precisely defined and more closely studied in Calculus.



The graph of $y = f(x)$



Sign Diagram for $f(x)$

2. The index of the radical in the expression for $g(t)$ is odd, so our only concern is the denominator. Setting $t + 1 = 0$ gives $t = -1$, which we exclude, so our domain is $\{t \in \mathbb{R} \mid t \neq -1\}$ or using interval notation, $(-\infty, -1) \cup (-1, \infty)$.

   If we take the time to analyze the behavior of g near $t = -1$, we find that as $t \to -1^-$, $g(t) = \sqrt[3]{\frac{8t}{t+1}} \approx \sqrt[3]{\frac{-8}{\text{small }(-)}} \approx \sqrt[3]{\text{big}(+)} = \text{big }(+)$. That is, $\displaystyle\lim_{t\to -1^-} g(t) = \infty$. Likewise, as $t \to -1^+$, $g(t) \approx \sqrt[3]{\frac{-8}{\text{small }(+)}} \approx \sqrt[3]{\text{big }(-)} = \text{big }(-)$. This suggests $\displaystyle\lim_{t\to -1^+} g(t) = -\infty$. This behavior points to a vertical asymptote, $t = -1$.

   To find the t-intercepts of the graph of g, we find the zeros of g by setting $g(t) = \sqrt[3]{\frac{8t}{t+1}} = 0$. Cubing both sides and clearing denominators gives $8t = 0$ or $t = 0$. Hence our t-, and in this case, y-intercept is $(0, 0)$.

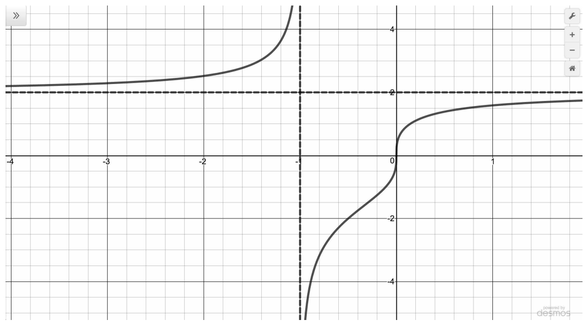
   To determine the end behavior, we note that as $t \to -\infty$ or $t \to \infty$, $\frac{8t}{t+1} \approx \frac{8t}{t} = 8$. Since $g(t) = \sqrt[3]{\frac{8t}{t+1}}$ it stands to reason that $\displaystyle\lim_{t\to -\infty} g(t) = \sqrt[3]{8} = 2$ and, likewise, $\displaystyle\lim_{t\to\infty} g(t) = 2$ This suggests the graph of $y = g(t)$ has a horizontal asymptote at $y = 2$.

   We graph $y = g(t)$ below on the left. The graph confirms our suspicions about the asymptotes $t = -1$ and $y = 2$. Moreover, the range appears to be $(-\infty, 2) \cup (2, \infty)$.

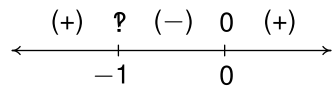
   We could check if the graph ever crosses its horizontal asymptote by attempting to solve $g(t) = \sqrt[3]{\frac{8t}{t+1}} = 2$. Cubing both sides and clearing denominators gives $8t = 8(t + 1)$ which gives $0 = 8$, a contradiction. This proves 2 is not in the range, as we had suspected.

   Scanning the graph, there appears to be no local extrema, and, moreover, the graph suggests g is increasing on $(-\infty, -1)$ and again on $(-1, \infty)$. As with the previous example, the graph appears locally vertical near its intercept $(0, 0)$.

   To create a sign diagram for $g(t)$, we note that the function is undefined when $t = -1$ (so we place a '?' above it) and has a zero $t = 0$. When $t < -1$, $g(t) > 0$ or (+), for $-1 < t < 0$, $g(t) < 0$ or $(-)$, and for $t > 0$, $g(t) > 0$ or (+). Below on the right is a sign diagram for $g(t)$.



The graph of $y = g(t)$



Sign Diagram for $g(t)$

3. The expression for $h(x) = \frac{3x}{\sqrt{x^2+1}}$ has both a denominator and an even-indexed radical, so we have to be extra cautious here. Fortunately for us, the quantity $x^2 + 1 > 0$ for all real numbers x. Not only does this mean $\sqrt{x^2+1}$ is always defined, it also tells us $\sqrt{x^2+1} > 0$ for all x, too. This means the domain of h is all real numbers, $(-\infty, \infty)$.

   Solving for the zeros of h gives only $x = 0$, and we find, once again, $(0, 0)$ is both our lone x- and y-intercept.

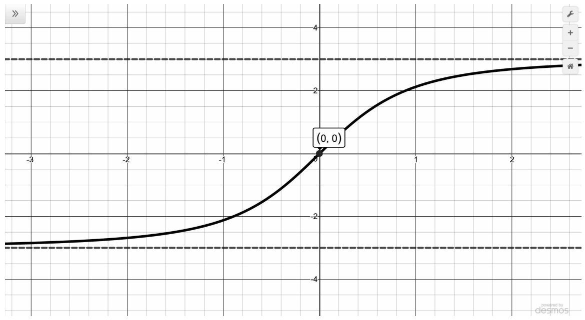
   Moving on to end behavior, as $x \rightarrow -\infty$ or $x \rightarrow \infty$, the term x^2 is the dominant term in the radicand in the denominator. As such, $h(x) = \frac{3x}{\sqrt{x^2+1}} \approx \frac{3x}{\sqrt{x^2}} = \frac{3x}{|x|}$. As $x \rightarrow -\infty$, $|x| = -x$ (since $x < 0$) and hence, $h(x) \approx \frac{3x}{-x} = -3$, so $\displaystyle\lim_{x \rightarrow -\infty} h(x) = -3$. As $x \rightarrow \infty$, $|x| = x$ (since $x > 0$), so $h(x) \approx \frac{3x}{x} = 3$, so $\displaystyle\lim_{x \rightarrow \infty} h(x) = 3$.

   This analysis suggests the graph of $y = h(x)$ has not one, but *two* horizontal asymptotes.[13] The graph of h below on the left bears this out.

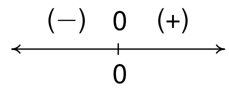
   From the graph, we see the range of h appears to be $(-3, 3)$. Attempting to solve $h(x) = \frac{3x}{\sqrt{x^2+1}} = -3$ or $h(x) = \frac{3x}{\sqrt{x^2+1}} = 3$ gives, in either case, $9x^2 = 9(x^2 + 1)$ which reduces to $0 = 9$, a contradiction. Hence, the graph of $y = h(x)$ never reaches its horizontal asymptotes. Moreover, h appears to be always increasing, with no local extrema or 'unusual' steepness. One last remark: it appears as if the graph of h is symmetric about the origin. We check $h(-x) = \frac{3(-x)}{\sqrt{(-x)^2+1}} = -\frac{3x}{\sqrt{x^2+1}} = -h(x)$ which verifies h is odd.

   Since the domain of h is all real number and the only zero of h is $x = 0$, the sign diagram for $h(x)$ is fairly straight forward. For $x < 0$, $h(x) < 0$ or $(-)$ and for $x > 0$, $h(x) > 0$ or $(+)$. The sign diagram for $h(x)$ is below on the right.

[13] We warned you this was coming . . . see the discussion following Theorem 3.3 in Section 3.1.



The graph of $y = h(x)$



Sign Diagram for $h(x)$

4. The first thing to note about the expression $r(t) = t^{-1}\sqrt{16t^4 - 1}$ is that $t^{-1} = \frac{1}{t}$. Hence, we must exclude $t = 0$ from the domain straight away. Next, we have an even-indexed radical expression: $\sqrt{16t^4 - 1}$. In order for this to return a real number, we require $16t^4 - 1 \geq 0$. Instead of using a sign diagram to solve this,[14] we opt instead to *carefully* use properties of radicals. Isolating t^4, we have $t^4 \geq \frac{1}{16}$. Since the root functions are increasing, we can apply the fourth root to both sides and preserve the inequality: $\sqrt[4]{t^4} \geq \sqrt[4]{\frac{1}{16}}$ which gives[15] $|t| \geq \frac{1}{2}$. Note that since $t = 0$ does *not* satisfy this inequality, restricting t in this manner takes care of *both* domain issues, so the domain is $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$.

   Next, we look for zeros. Setting $r(t) = t^{-1}\sqrt{16t^4 - 1} = \frac{\sqrt{16t^4-1}}{t} = 0$ gives $\sqrt{16t^4 - 1} = 0$. After squaring both sides, we get $16t^4 - 1 = 0$ or $t^4 = \frac{1}{16}$. Extracting fourth roots, we get $t = \pm\frac{1}{2}$. Both of these are (barely!) in the domain of r, so our t intercepts are $\left(-\frac{1}{2}, 0\right)$ and $\left(\frac{1}{2}, 0\right)$. Note, the graph of r has no y-intercept, since $r(0)$ is undefined ($t = 0$ is not in the domain of r).

   Concerning end behavior, we note the term $16t^4$ dominates the radicand $\sqrt{16t^4 - 1}$ as $t \rightarrow -\infty$ or $t \rightarrow \infty$, hence, $r(t) = \frac{\sqrt{16t^4-1}}{t} \approx \frac{\sqrt{16t^4}}{t} = \frac{4t^2}{t} = 4t$. This suggests the graph of $y = r(t)$ has a slant asymptote with slope 4.[16] At this point, we can at least write $\lim\limits_{t\rightarrow-\infty} r(t) = -\infty$. and $\lim\limits_{t\rightarrow\infty} r(t) = \infty$.

   We graph $y = r(t)$ below on the left. We see the range appears to be all real numbers, $(-\infty, \infty)$. It appears as if r is increasing on $\left(-\infty, -\frac{1}{2}\right]$ and again on $\left[\frac{1}{2}, \infty\right)$. The graph does appear to be asymptotic to $y = 4t$, and it also appears to be symmetric about the origin. Sure enough, we find $r(-t) = \frac{\sqrt{16(-t)^4-1}}{-t} = -\frac{\sqrt{16t^4-1}}{t} = -r(t)$, proving r is an odd function.

   To construct the sign diagram for $r(t)$ we note r has two zeros, $t = \pm\frac{1}{2}$. For $t < \frac{1}{2}$, $r(t) < 0$ or $(-)$ and when $t > \frac{1}{2}$, $r(t) > 0$ or $(+)$. When $-\frac{1}{2} < t < \frac{1}{2}$, r is undefined so we have removed that segment from the diagram, as seen below on the right.

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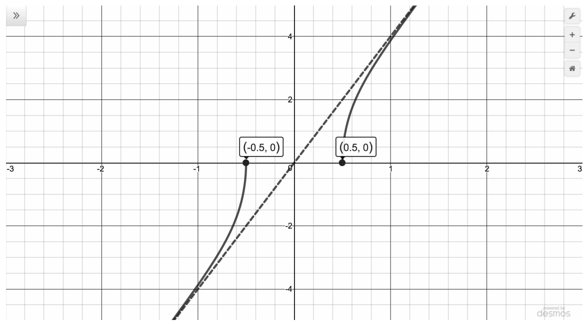
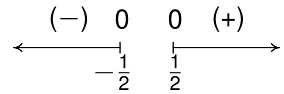
[14] See Section 2.3

[15] Recall: $\sqrt[n]{x^n} = |x|$, not x, if n is even.

[16] Note: this analysis suggests the slant asymptote is $y = 4t + b$, but from this analysis, we cannot determine the value of b. As with slant asymptotes in Section 3.1, we'd need to perform a more detailed analysis which we omit in this case owing to the complexity of the function. (You'll have the tools in Calculus, however!)



The graph of $y = r(t)$



Sign Diagram for $r(t)$.

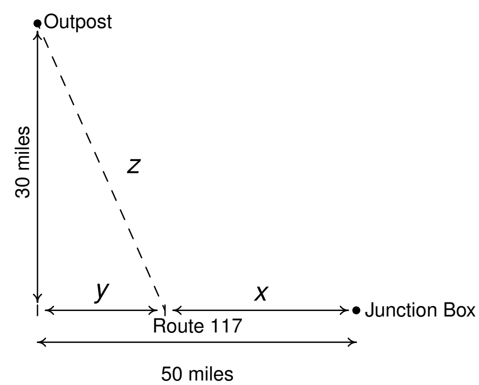
□

We end this section with a classic application of root functions.

Example 4.1.3. Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles down the road from the post, as indicated in the diagram below. Suppose it costs \$15 per mile to run cable along the road and \$20 per mile to run cable off of the road.

1. Find an expression $C(x)$ which computes the cost of connecting the Junction Box to the Outpost as a function of x, the number of miles the cable is run along Route 117 before heading off road directly towards the Outpost. Determine a reasonable applied domain for the problem.

2. Use your calculator to graph $y = C(x)$ on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off of the road?



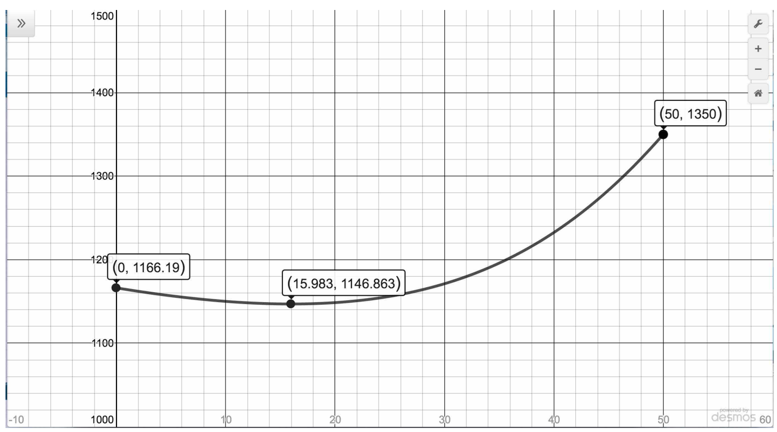
Solution.

1. The cost is broken into two parts: the cost to run cable along Route 117 at \$15 per mile, and the cost to run it off road at \$20 per mile. Since x represents the miles of cable run along Route 117, the cost for that portion is $15x$. From the diagram, we see that the number of miles the cable is run off road is z, so the cost of that portion is $20z$. Hence, the total cost is $15x + 20z$.

   Our next goal is to determine z in terms of x. The diagram suggests we can use the Pythagorean Theorem to get $y^2 + 30^2 = z^2$. But we also see $x + y = 50$ so that $y = 50 - x$. Substituting $(50 - x)$ in for y we obtain $z^2 = (50 - x)^2 + 900$. Solving for z, we obtain $z = \pm\sqrt{(50 - x)^2 + 900}$. Since z represents a distance, we choose $z = \sqrt{(50 - x)^2 + 900}$.

   Hence, the cost as a function of x is given by $C(x) = 15x + 20\sqrt{(50 - x)^2 + 900}$. From the context of the problem, we have $0 \leq x \leq 50$.

2. We graph $y = C(x)$ below and find our (local) minimum to be at the point $(15.98, 1146.86)$. Here the x-coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route 117 and then turn off of the road and head towards the outpost. The y-coordinate tells us that the minimum cost, in dollars, to do so is \$1146.86. The ability to stream live SasquatchCasts? Priceless.



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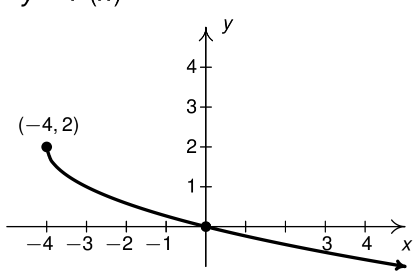
4.1.3 Exercises

In Exercises 1 - 8, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem 4.1. Track at least two points and state the domain and range using interval notation.

1. $f(x) = \sqrt{x}$, $F(x) = \sqrt{x+3} - 2$
2. $f(x) = \sqrt{x}$, $F(x) = \sqrt{4-x} - 1$
3. $f(x) = \sqrt[3]{x}$, $F(x) = \sqrt[3]{x-1} - 2$
4. $f(x) = \sqrt[3]{x}$, $F(x) = -\sqrt[3]{8x+8} + 4$
5. $f(x) = \sqrt[4]{x}$, $F(x) = \sqrt[4]{x-1} - 2$
6. $f(x) = \sqrt[4]{x}$, $F(x) = -3\sqrt[4]{x-7} + 1$
7. $f(x) = \sqrt[5]{x}$, $F(x) = \sqrt[5]{x+2} + 3$
8. $f(x) = \sqrt[8]{x}$, $F(x) = \sqrt[8]{-x} - 2$

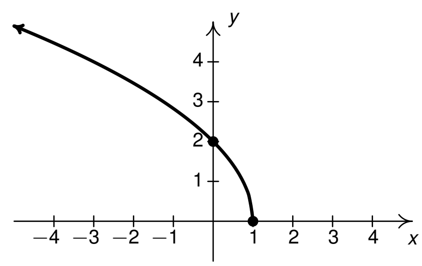
In Exercises 9 - 10, find a formula for each function below in the form $F(x) = a\sqrt{bx-h} + k$.
NOTE: There may be more than one solution!

9. $y = F(x)$



x,y-intercept $(0, 0)$

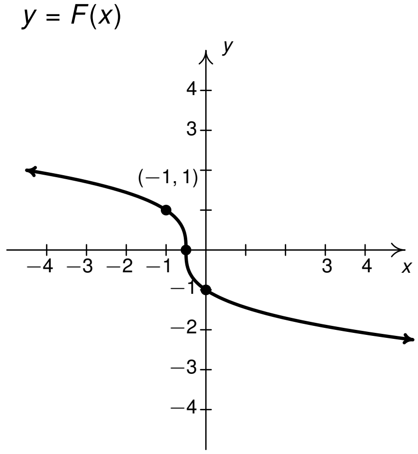
10. $y = F(x)$



x-intercept $(1, 0)$, y-intercept $(0, 2)$

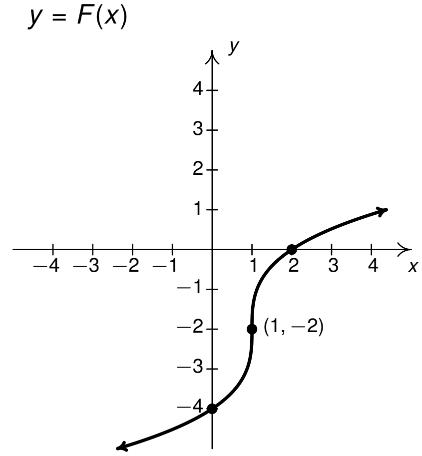
In Exercises 11 - 12, find a formula for each function below in the form $F(x) = a\sqrt[3]{bx-h} + k$.
NOTE: There may be more than one solution!

11. $y = F(x)$



x-intercept $\left(-\frac{1}{2}, 0\right)$, y-intercept $(0, -1)$

12. $y = F(x)$



x-intercept $(2, 0)$, y-intercept $(0, -4)$

13. Use the fact that the nth root functions are increasing to solve the following polynomial inequalities:

(a) $x^3 \leq 64$ (b) $2 - t^5 < 34$ (c) $\dfrac{(2z+1)^3}{4} \geq 2$

For the following inequalities, remember $\sqrt[n]{x^n} = |x|$ if n is even:

(d) $x^4 \leq 16$ (e) $6 - t^6 < -58$ (f) $\dfrac{(2z+1)^4}{3} \geq 27$

For each function in Exercises 14 - 21 below

- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.
- Graph the function with help from a graphing utility and determine:
  - the range.
  - the local extrema, if they exist.
  - intervals of increase/decrease.
  - any 'unusual steepness' or 'local' verticality.
  - vertical asymptotes.
  - horizontal / slant asymptotes.
- Construct a sign diagram for each function using the intercepts and graph.
- Comment on any observed symmetry.

14. $f(x) = \sqrt{1 - x^2}$
15. $f(x) = \sqrt{x^2 - 1}$
16. $g(t) = t\sqrt{1 - t^2}$
17. $g(t) = t\sqrt{t^2 - 1}$
18. $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$
19. $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$
20. $g(t) = \sqrt{t(t+5)(t-4)}$
21. $g(t) = \sqrt[3]{t^3 + 3t^2 - 6t - 8}$
22. Rework Example 4.1.3 so that the outpost is 10 miles from Route 117 and the nearest junction box is 30 miles down the road for the post.
23. The volume V of a right cylindrical cone depends on the radius of its base r and its height h and is given by the formula $V = \frac{1}{3}\pi r^2 h$. The surface area S of a right cylindrical cone also depends on r and h according to the formula $S = \pi r\sqrt{r^2 + h^2}$. In the following problems, suppose a cone is to have a volume of 100 cubic centimeters.

(a) Use the formula for volume to find the height as a function of r, $h(r)$.

(b) Use the formula for surface area along with your answer to 23a to find the surface area as a function of r, $S(r)$.

(c) Use your calculator to find the values of r and h which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

24. The period of a pendulum in seconds is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

(for small displacements) where L is the length of the pendulum in meters and $g = 9.8$ meters per second per second is the acceleration due to gravity. My Seth-Thomas antique schoolhouse clock needs $T = \frac{1}{2}$ second and I can adjust the length of the pendulum via a small dial on the bottom of the bob. At what length should I set the pendulum?

25. According to Einstein's Theory of Special Relativity, the observed mass of an object is a function of how fast the object is traveling. Specifically, if m_r is the mass of the object at rest, v is the speed of the object and c is the speed of light, then the observed mass of the object $m(v)$ is given by:

$$m(v) = \frac{m_r}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(a) Find the applied domain of the function.

(b) Compute $m(.1c)$, $m(.5c)$, $m(.9c)$ and $m(.999c)$.

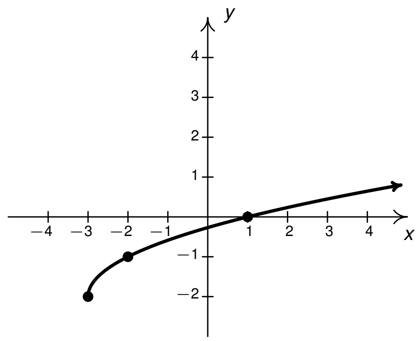
(c) Find $\displaystyle\lim_{v \to c^-} m(v)$.

(d) How slowly must the object be traveling so that the observed mass is no greater than 100 times its mass at rest?

26. Find the inverse of $k(x) = \dfrac{2x}{\sqrt{x^2 - 1}}$.

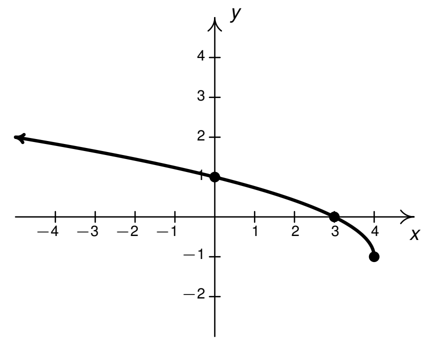
4.1.4 Answers

1. $F(x) = \sqrt{x+3} - 2$



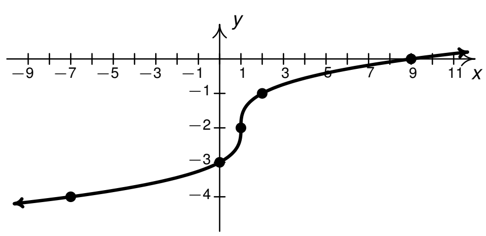
Domain: $[-3, \infty)$, Range: $[-2, \infty)$

2. $F(x) = \sqrt{4-x} - 1 = \sqrt{-x+4} - 1$



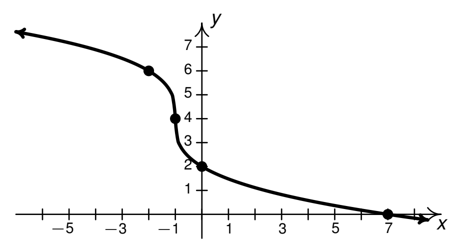
Domain: $(-\infty, 4]$, Range: $[-1, \infty)$

3. $F(x) = \sqrt[3]{x-1} - 2$



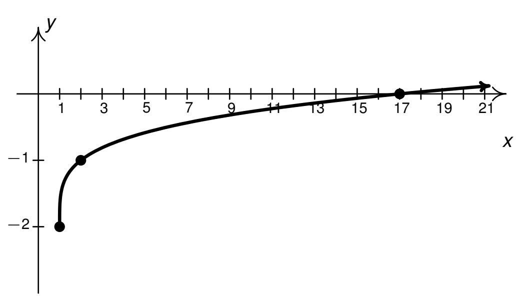
Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$

4. $F(x) = -\sqrt[3]{8x+8} + 4$



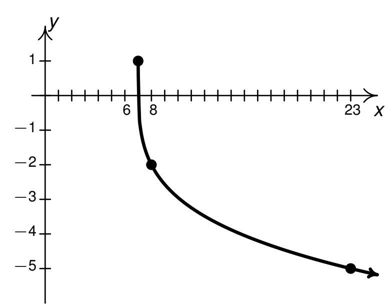
Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$

5. $F(x) = \sqrt[4]{x-1} - 2$



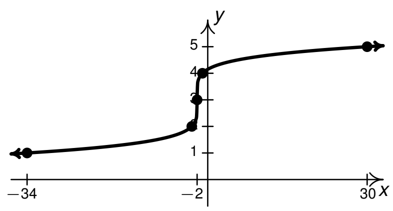
Domain: $[1, \infty)$, Range: $[-2, \infty)$

6. $F(x) = -3\sqrt[4]{x-7} + 1$



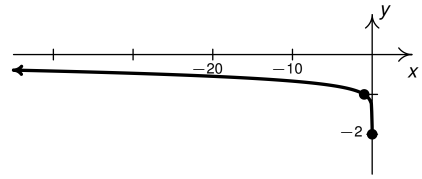
Domain: $[7, \infty)$, Range: $(-\infty, 1]$

7. $F(x) = \sqrt[5]{x+2} + 3$



Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$

8. $F(x) = \sqrt[8]{-x} - 2$



Domain: $(-\infty, 0]$, Range: $[-2, \infty)$

9. One solution is: $F(x) = -\sqrt{x+4} + 2$

10. One solution is: $F(x) = 2\sqrt{-x+1}$

11. One solution is: $F(x) = -\sqrt[3]{2x+1}$

12. One solution is: $F(x) = 2\sqrt[3]{x-1} - 2$

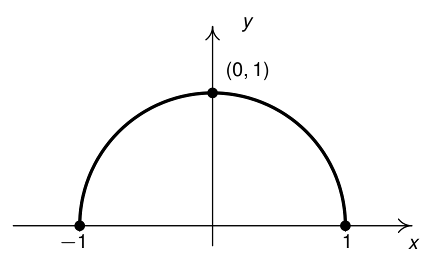
13. (a) $(-\infty, 4]$ (b) $(-2, \infty)$ (c) $\left[\frac{1}{2}, \infty\right)$

(d) $[-2, 2]$ (e) $(-\infty, -2) \cup (2, \infty)$ (f) $(-\infty, -2] \cup [1, \infty)$

14. $f(x) = \sqrt{1-x^2}$
Domain: $[-1, 1]$
Intercepts: $(-1, 0)$, $(1, 0)$
Range: $[0, 1]$
Local maximum: $(0, 1)$
Increasing: $[-1, 0]$, Decreasing: $[0, 1]$
Unusual steepness[17] at $x = -1$ and $x = 1$
Sign Diagram:

0 (+) 0
-1 1

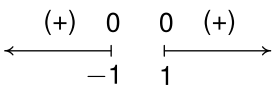
Graph:



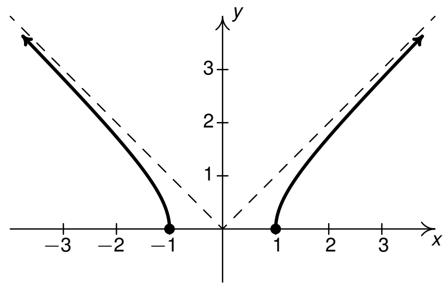
Note: f is even.

[17] You may need to zoom in to see this.

15. $f(x) = \sqrt{x^2 - 1}$
Domain: $(-\infty, -1] \cup [1, \infty)$
Intercepts: $(-1, 0)$, $(1, 0)$
$\displaystyle\lim_{x \to -\infty} f(x) = \infty$
[18] $\displaystyle\lim_{x \to \infty} f(x) = \infty$
Range: $[0, \infty)$
Increasing: $[1, \infty)$, Decreasing: $(-\infty, -1]$
Unusual steepness[19] at $x = -1$ and $x = 1$
Sign Diagram:

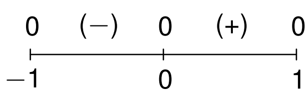


Graph:

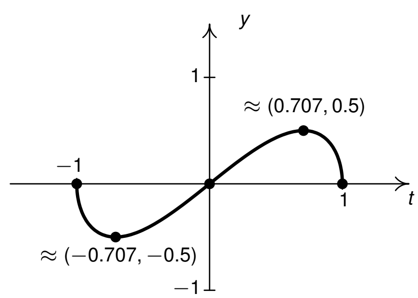


Note: f is even.

16. $g(t) = t\sqrt{1 - t^2}$
Domain: $[-1, 1]$
Intercepts: $(-1, 0)$, $(0, 0)$, $(1, 0)$
Range: $\approx [-0.5, 0.5]$
Local minimum $\approx (-0.707, -0.5)$
Local maximum: $\approx (0.707, 0.5)$
Increasing: $\approx [-0.707, 0.707]$
Decreasing: $\approx [-1, -0.707]$, $[0.707, 1]$
Unusual steepness at $t = -1$ and $t = 1$
Sign Diagram:

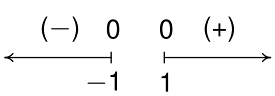


Graph:

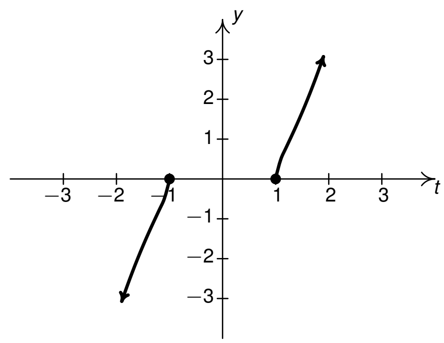


Note: g is odd.

17. $g(t) = t\sqrt{t^2 - 1}$
Domain: $(-\infty, -1] \cup [1, \infty)$
Intercepts: $(-1, 0)$, $(1, 0)$
$\displaystyle\lim_{t \to -\infty} g(t) = -\infty$
$\displaystyle\lim_{t \to \infty} g(t) = \infty$
Range: $(-\infty, \infty)$
Increasing: $(-\infty, -1]$, $[1, \infty)$
Unusual steepness at $t = -1$ and $t = 1$
Sign Diagram:



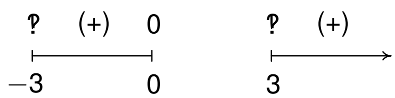
Graph:



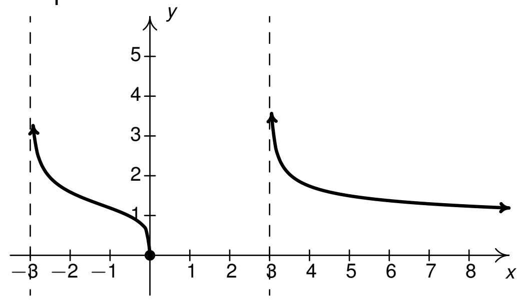
Note: g is odd.

[18] Using Calculus, one can show $y = -x$ and $y = -x$ are slant asymptotes to the graph.
[19] You may need to zoom in to see this.

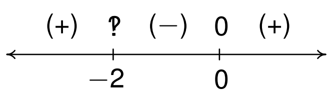
18. $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$
    Domain: $(-3, 0] \cup (3, \infty)$
    Intercept: $(0, 0)$
    Range: $[0, \infty)$
    Decreasing: $(-3, 0]$, $(3, \infty)$
    Unusual steepness at $x = 0$
    Vertical asymptotes: $x = -3$ and $x = 3$
    Horizontal asymptote: $y = 0$
    Sign Diagram:



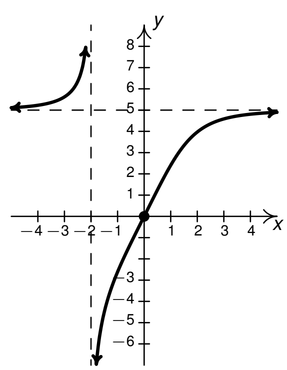
Graph:



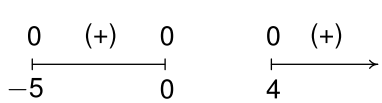
19. $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$
    Domain: $(-\infty, -2) \cup (-2, \infty)$
    Intercept: $(0, 0)$
    Range: $(-\infty, 5) \cup (5, \infty)$
    Increasing: $(-\infty, -2)$, $(-2, \infty)$
    Vertical asymptote $x = -2$
    Horizontal asymptote $y = 5$
    Sign Diagram:



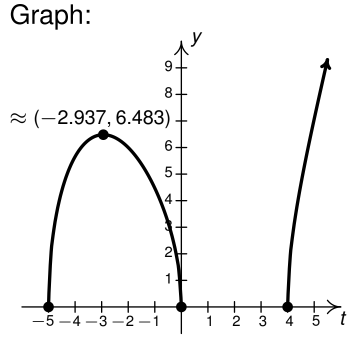
Graph:



20. $g(t) = \sqrt{t(t + 5)(t - 4)}$
    Domain: $[-5, 0] \cup [4, \infty)$
    Intercepts $(-5, 0)$, $(0, 0)$, $(4, 0)$
    $\displaystyle\lim_{t\to\infty} g(t) = \infty$
    Range: $[0, \infty)$
    Local maximum $\approx (-2.937, 6.483)$
    Increasing: $\approx [-5, -2.937]$, $[4, \infty)$
    Decreasing: $\approx [-2.937, 0]$
    Unusual steepness at $t = -5$, $t = 0$ and $t = 4$
    Sign Diagram:



Graph:

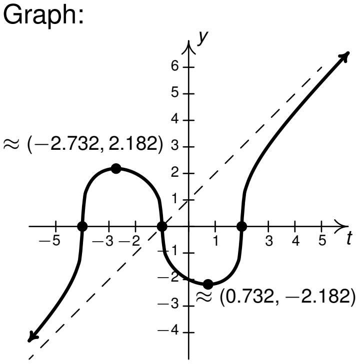


21. $g(t) = \sqrt[3]{t^3 + 3t^2 - 6t - 8}$
Domain: $(-\infty, \infty)$
Intercepts: $(-4, 0)$, $(-1, 0)$, $(0, -2)$, $(2, 0)$
[20] $\displaystyle\lim_{t \to -\infty} g(t) = -\infty$
$\displaystyle\lim_{t \to \infty} g(t) = \infty$
Range: $(-\infty, \infty)$
Local maximum: $\approx (-2.732, 2.182)$
Local minimum: $\approx (0.732, -2.182)$
Increasing: $\approx (-\infty, -2.732]$, $[0.732, \infty)$
Decreasing: $\approx [-2.732, 0.732]$
Unusual steepness at $t = -4$, $t = -1$ and $t = 2$

Sign Diagram:

(−) 0 (+) 0 (−) 0 (+)
−4 −1 2

Graph:



22. $C(x) = 15x + 20\sqrt{100 + (30 - x)^2}$, $0 \leq x \leq 30$. The calculator gives the absolute minimum at approximately $(18.66, 582.29)$. This means to minimize the cost, approximately 18.66 miles of cable should be run along Route 117 before turning off the road and heading towards the outpost. The minimum cost to run the cable is approximately \$582.29.

23. (a) $h(r) = \frac{300}{\pi r^2}$, $r > 0$.

(b) $S(r) = \pi r \sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2} = \frac{\sqrt{\pi^2 r^6 + 90000}}{r}$, $r > 0$

(c) The calculator gives the absolute minimum at the point $\approx (4.07, 90.23)$. This means the radius should be (approximately) 4.07 centimeters and the height should be 5.76 centimeters to give a minimum surface area of 90.23 square centimeters.

24. $9.8 \left(\dfrac{1}{4\pi}\right)^2 \approx 0.062$ meters or 6.2 centimeters

25. (a) $[0, c)$

(b) $m(.1c) = \dfrac{m_r}{\sqrt{.99}} \approx 1.005 m_r$, $m(.5c) = \dfrac{m_r}{\sqrt{.75}} \approx 1.155 m_r$, $m(.9c) = \dfrac{m_r}{\sqrt{.19}} \approx 2.294 m_r$, $m(.999c) = \dfrac{m_r}{\sqrt{.0.001999}} \approx 22.366 m_r$.

(c) $\displaystyle\lim_{v \to c^-} m(x) \to \infty$; as the object's velocity approaches the speed of light, mass becomes infinite.

(d) If the object is traveling no faster than approximately 0.99995 times the speed of light, then its observed mass will be no greater than $100m_r$.

26. $k^{-1}(x) = \dfrac{x}{\sqrt{x^2 - 4}}$

[20]Using Calculus it can be shown that $y = t + 1$ is a slant asymptote of this graph.

4.2 Power Functions

Monomial, and, more generally, Laurent monomial functions are specific examples of a much larger class of functions called **power functions**, as defined below.

Definition 4.2. Let a and p be nonzero real numbers. A **power function** is either a constant function or a function of the form $f(x) = ax^p$.

Definition 4.2 broadens our scope of functions to include non-integer exponents such as $f(x) = 2x^{4/3}$, $g(t) = t^{0.4}$ and $h(w) = w^{\sqrt{2}}$. Our primary aim in this section is to ascribe meaning to these quantities.

4.2.1 Rational Number Exponents

The road to real number exponents starts by defining rational number exponents.

Definition 4.3. Let r be a rational number where in lowest terms $r = \frac{m}{n}$ where m is an integer and n is a natural number.[a] If $n = 1$, then $x^r = x^m$. If $n > 1$, then

$$x^r = x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m},$$

whenever $\left(\sqrt[n]{x}\right)^m$ is defined.[b]

[a]Recall 'lowest terms' means m and n have no common factors other than 1.

[b]That is, if n is even, $x \geq 0$ and if $m < 0$, $x \neq 0$.

There are quite a few items worthy of note which are consequences of Definition 4.3. First off, if m is an integer, then $x^{\frac{m}{1}} = x^m$ so expressions like $x^{\frac{3}{1}}$ are synonymous with x^3, as we would expect.[1] Second, the definition of $x^{\frac{m}{n}}$ can be taken as just $\left(\sqrt[n]{x}\right)^m$ and shown to be equal to $\sqrt[n]{x^m}$ (or vice-versa) courtesy of properties of radicals. We state both in Definition 4.3 to allow for the reader to choose whichever form is more convenient in a given situation. The critical point to remember is no matter which representation you choose, keep in mind the restrictions if n is even, $x \geq 0$ and if $m < 0$, $x \neq 0$.

Moreover, per this definition, $x^{\frac{1}{n}} = \sqrt[n]{x^1} = \sqrt[n]{x}$, so we may rewrite principal roots as exponents: $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[5]{x} = x^{\frac{1}{5}}$. This makes sense from an algebraic standpoint since per Theorem 4.2, $\left(\sqrt[n]{x}\right)^n = x$. Hence if we were to assign an exponent notation to $\sqrt[n]{x}$, say $\sqrt[n]{x} = x^r$, then $\left(\sqrt[n]{x}\right)^n = (x^r)^n = x$. If the properties of exponents are to hold, then, necessarily, $(x^r)^n = x^{rn} = x = x^1$, so $rn = 1$ or $r = \frac{1}{n}$. While this argument helps *motivate* the notation, as we shall see shortly, great care must be exercised in applying exponent properties in these cases. The long and short of this is that root functions as defined in Section 4.1 are all members of the 'power functions' family.

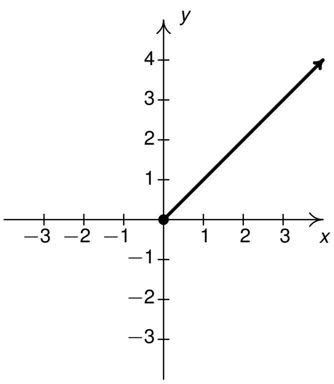
Another important item worthy of note in Definition 4.3 is that it is absolutely essential we express the rational number r in *lowest terms* before applying the root-power definition. For example, consider $x^{0.4}$. Expressing r in lowest terms, we get: $r = 0.4 = \frac{4}{10} = \frac{2}{5}$. Hence, $x^{0.4} = x^{2/5} = (\sqrt[5]{x})^2$ or $\sqrt[5]{x^2}$, either of which is defined for all real numbers x. In contrast, consider the equivalence $r = 0.4 = \frac{4}{10}$. Here, the expression $(\sqrt[10]{x})^4$ is defined only for $x \geq 0$ owing to the presence of the even indexed root, $\sqrt[10]{x}$. Hence, $(\sqrt[10]{x})^4 \neq x^{\frac{4}{10}} = x^{\frac{2}{5}}$ unless $x \geq 0$. On the other hand, the expression $\sqrt[10]{x^4}$ *is* defined for all numbers,

[1]Either $n = 1$ is a special case in Definition 4.3 or we need to define what is meant by $\sqrt[1]{x}$. The authors chose the former.

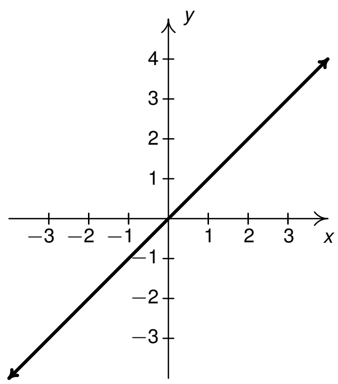
x, since $x^4 \geq 0$ for all x. In fact, it can be shown that $\sqrt[10]{x^4} = \sqrt[5]{x^2}$ for all real numbers. This means $\sqrt[10]{x^4} = \sqrt[5]{x^2} = x^{\frac{2}{5}} = x^{\frac{4}{10}}$. So, to review, in general we have: $x^{\frac{4}{10}} = \sqrt[10]{x^4}$, but $x^{\frac{4}{10}} \neq \left(\sqrt[10]{x}\right)^4$ unless $x \geq 0$. Once again the easiest way to avoid confusion here is to *reduce* the exponent to *lowest terms* before converting it to root-power notation.

Likewise, we have to be careful about the properties of exponents when it comes to rational exponents. Consider, for instance, the product rule for integer exponents: $x^m x^n = x^{m+n}$. Consider $f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}}$ and $g(x) = x^{\frac{1}{2}+\frac{1}{2}}$. In the first case, $f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}} = \sqrt{x}\sqrt{x} = (\sqrt{x})^2 = x$ only for $x \geq 0$. In the second case, $g(x) = x^{\frac{1}{2}+\frac{1}{2}} = x^{\frac{2}{2}} = x^1 = x$ for all real numbers x. Even though $f(x) = g(x)$ for $x \geq 0$, f and g are *different functions* since they have *different domains*.

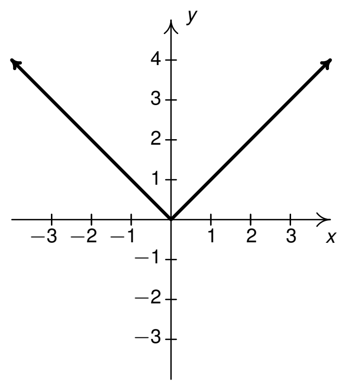
Similarly, the power rule for integer exponents: $(x^n)^m = x^{nm}$ does not hold in general for rational exponents. To see this, consider the three functions: $f(x) = (x^{\frac{1}{2}})^2$, $g(x) = x^{\frac{2}{2}}$, and $h(x) = (x^2)^{\frac{1}{2}}$. In the first case, $f(x) = (x^{\frac{1}{2}})^2 = (\sqrt{x})^2 = x$ for $x \geq 0$ only (this is the same function f above.) In the second case, the rational number $r = \frac{2}{2} = 1$, so $g(x) = x^{\frac{2}{2}} = x^{\frac{1}{1}} = x^1 = x$ for *all* real numbers, x (this is the same function g from above.) In the last case, $h(x) = (x^2)^{\frac{1}{2}} = \sqrt{x^2} = |x|$ for all real numbers, x. Once again, despite $f(x) = g(x) = h(x)$ for all $x \geq 0$, f, g and h and are *three different functions*. We graph f, g, and h below.



$f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}} = (x^{\frac{1}{2}})^2 = x,\ x \geq 0$



$g(x) = x^{\frac{1}{2}+\frac{1}{2}} = x^{\frac{2}{2}} = x$



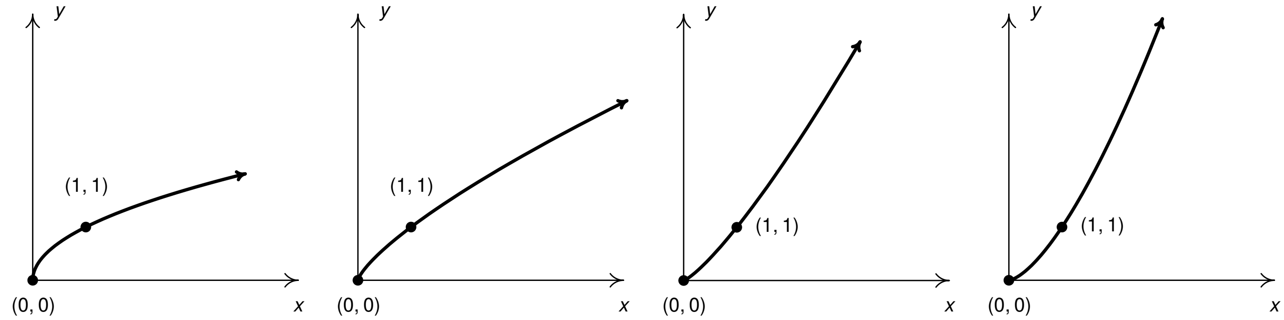
$h(x) = (x^2)^{\frac{1}{2}} = |x|$

In general, the properties of integer exponents *do not extend* to rational exponents *unless* the bases involved represent non-negative real numbers *or* the roots involved are *odd*. We have the following:

Theorem 4.3. Let r and s are rational numbers. The following properties hold provided none of the computations results in division by 0 and **either** r and s have odd denominators **or** $x \geq 0$ **and** $y \geq 0$:

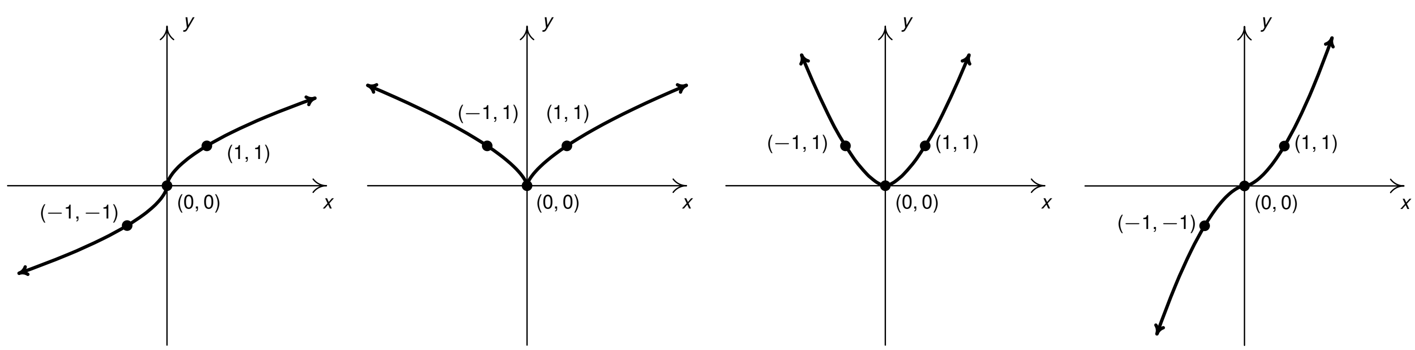
- **Product Rules:** $x^r x^s = x^{r+s}$ and $(xy)^r = x^r y^r$.
- **Quotient Rules:** $\dfrac{x^r}{x^s} = x^{r-s}$ and $\left(\dfrac{x}{y}\right)^r = \dfrac{x^r}{y^r}$
- **Power Rule:** $(x^r)^s = x^{rs}$

Next, we turn our attention to the graphs of $f(x) = x^r = x^{\frac{m}{n}}$ for varying values of m and n. When n is even, the domain is restricted owing to the presence of the even indexed root to $[0, \infty)$. The range is likewise $[0, \infty)$, a fact leave to the reader. All of the functions below are increasing on their domains, and it turns out this is always the case provided $r > 0$. There is, however, is a difference in *how* the functions are increasing - and this is the concept of *concavity*. As with many concepts we've encountered so far in the text, concavity is most precisely defined using Calculus terminology, but we can nevertheless get a sense of concavity geometrically. For us, a curve is **concave up** over an interval if it resembles a portion of a '$\smile$' shape. Similarly, a curve is called **concave down** over an interval if resembles part of a '$\frown$' shape. When $0 < r < 1$, the graphs of $f(x) = x^r$ resemble the left half of $\frown$ and so are concave down; when $r > 1$, the graphs resemble the right half of a '$\smile$' and are hence described as 'concave up.'



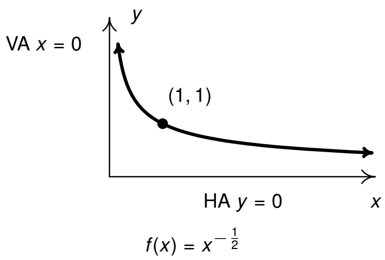
$f(x) = x^{\frac{1}{2}}$ $f(x) = x^{\frac{3}{4}}$ $f(x) = x^{\frac{5}{4}}$ $f(x) = x^{\frac{3}{2}}$

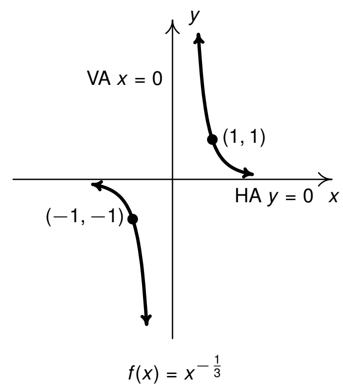
Below we graph several examples of $f(x) = x^r = x^{\frac{m}{n}}$ where n is odd. Here, the domain is $(-\infty, \infty)$ since the index on the root here is odd. Note that when m is even, the graphs appear to be symmetric about the y-axis and the range looks to be $[0, \infty)$. When m is odd, the graphs appear to be symmetric about the origin with range $(-\infty, \infty)$. We leave verification of these facts to the reader. Note here also that for $x \geq 0$, the graphs are down for $0 < r < 1$ and concave up for $r > 1$.



$f(x) = x^{\frac{3}{5}}$ $f(x) = x^{\frac{2}{3}}$ $f(x) = x^{\frac{8}{5}}$ $f(x) = x^{\frac{5}{3}}$

When $r < 0$, we have variables appear in the denominator which open the opportunities for vertical and horizontal asymptotes. Below are graphed two examples





Unsurprisingly, Theorem 4.1, which, as stated, applied to root functions, generalizes to all rational powers.

Theorem 4.4. For real numbers a, b, h, and k and rational number r with $a, b, r \neq 0$, the graph of $F(x) = a(bx - h)^r + k$ can be obtained from the graph of $f(x) = x^r$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = x^r$ to $y = (x - h)^r$.

2. divide the x-coordinates of the points on the graph obtained in Step 1 by b. This results in a horizontal scaling, but may also include a reflection about the y-axis if $b < 0$.

   NOTE: This transforms the graph of $y = (x - h)^r$ to $y = (bx - h)^r$.

3. multiply the y-coordinates of the points on the graph obtained in Step 2 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = (bx - h)^r$ to $y = a(bx - h)^r$.

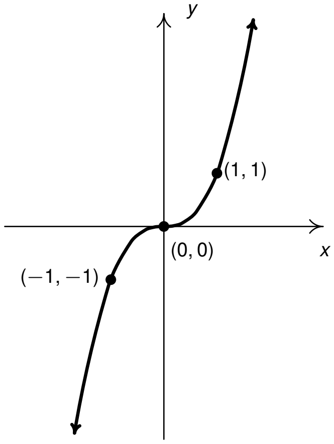
4. add k to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a(bx - h)^r$ to $y = a(bx - h)^r + k$.

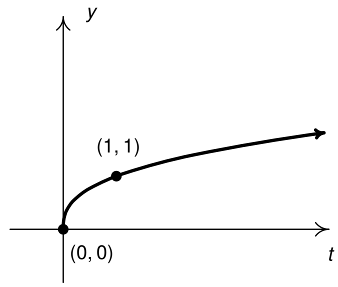
The proof of Theorem 4.4 is identical to that of Theorem 4.1, and we suggest the reader work through the details. We give Theorem 4.4 a test run in the following example.

Example 4.2.1. Use the given graphs of f and g below long with Theorem 4.4 to graph F and G. State the domain and range of F and G using interval notation.

1. Graph $F(x) = (2x - 1)^{2.6}$.
2. Graph $G(t) = 1 - 2(t + 3)^{\frac{3}{8}}$.



$f(x) = x^{2.6}$

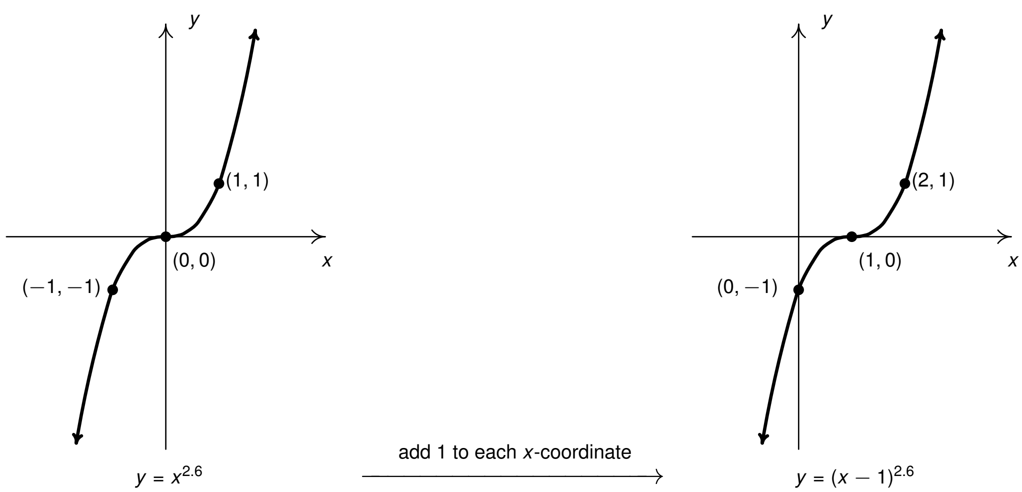


$g(t) = t^{\frac{3}{8}}$

Solution.

1. The expression $F(x) = (2x - 1)^{2.6}$ is given to us in the form prescribed by Theorem 4.4, and we identify $r = 2.6$, $a = 1$, $b = 2$, $h = 1$, and $k = 0$. Even though the graph of $f(x) = x^{2.6}$ is given to us, it's worth taking a moment to reinforce some concepts. Since, in lowest terms, $2.6 = \frac{26}{10} = \frac{13}{5}$, it makes sense the domain and range of $f(x) = x^{2.6}$ are both all real numbers and the graph is symmetric about the origin.[2] Moreover, since $2.6 > 1$, the concavity matches what we would expect, too. We proceed as we have several times in the past, beginning with the horizontal shift.

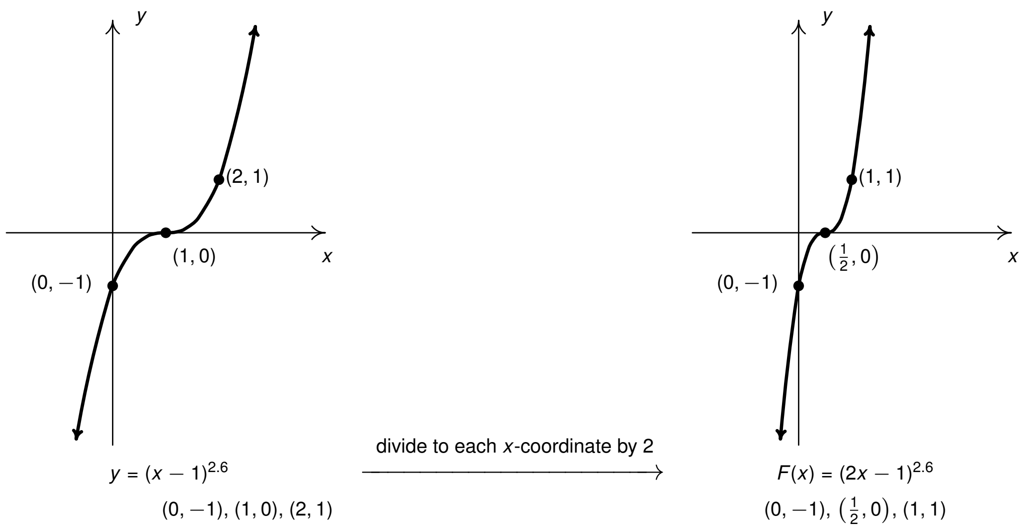
   Step 1: add 1 to each of the x-coordinates of each of the points on the graph of $y = x^{2.6}$:



$y = x^{2.6}$ $\quad$ $(-1, 1), (0, 0), (1, 1)$ $\qquad$ $y = (x - 1)^{2.6}$ $\quad$ $(0, -1), (1, 0), (2, 1)$

[2]The domain is all real numbers since the denominator (root) 5 is odd; the range is all real numbers since the numerator (power) 13 is odd. Since both power and root are odd, the function itself is an odd function, hence the symmetry about the origin.

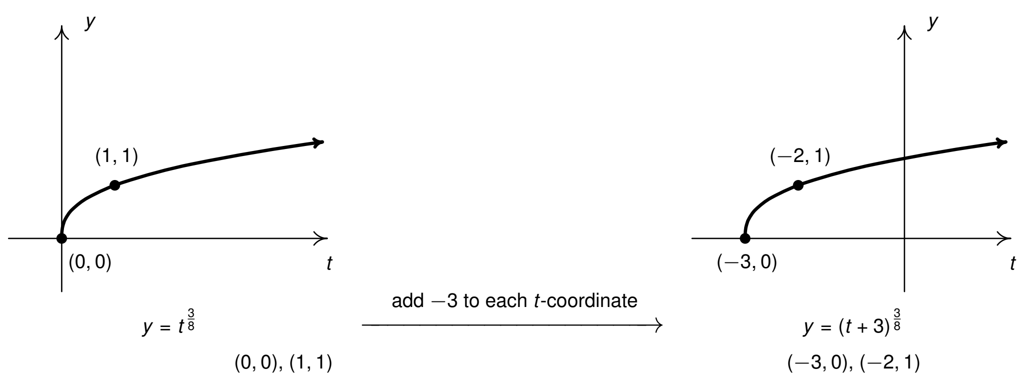
Step 2: divide each of the x-coordinates of each of the points on the graph of $y = (x-1)^{2.6}$ by 2:



We get the domain and range here are both $(-\infty, \infty)$.

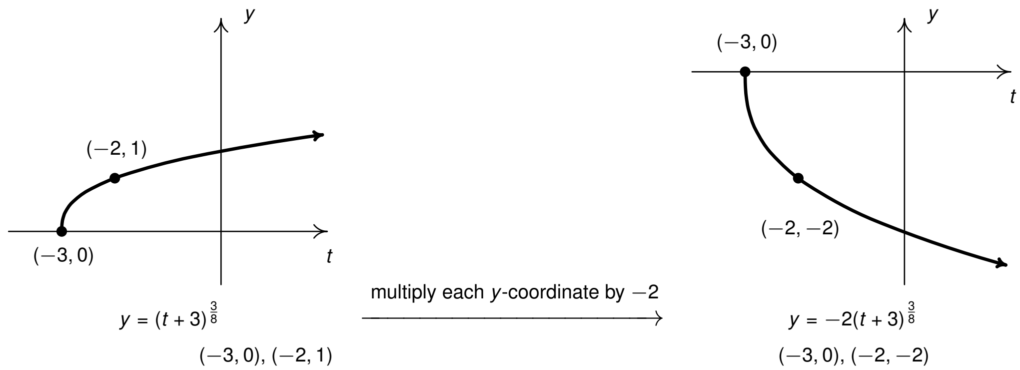
2. We first need to rewrite $G(t) = 1-2(t+3)^{\frac{3}{8}}$ in the form required by Theorem 4.4: $G(t) = -2(t+3)^{\frac{3}{8}}+1$. We identify $r = \frac{3}{8}$, $a = -2$, $b = 1$, $h = -3$, and $k = 1$. Since $\frac{3}{8}$ is in lowest terms and has an even denominator, it makes sense the domain and range of $g(t) = t^{\frac{3}{8}}$ is $[0, \infty)$, since the root here, 8 is even. Also, since $0 < \frac{3}{8} < 1$, the graph of $y = t^{\frac{3}{8}}$ is concave down, as we would expect. As usual, we start with the horizontal shift.

   Step 1: add -3 to each of the t-coordinates of each of the points on the graph of $y = t^{\frac{3}{8}}$:

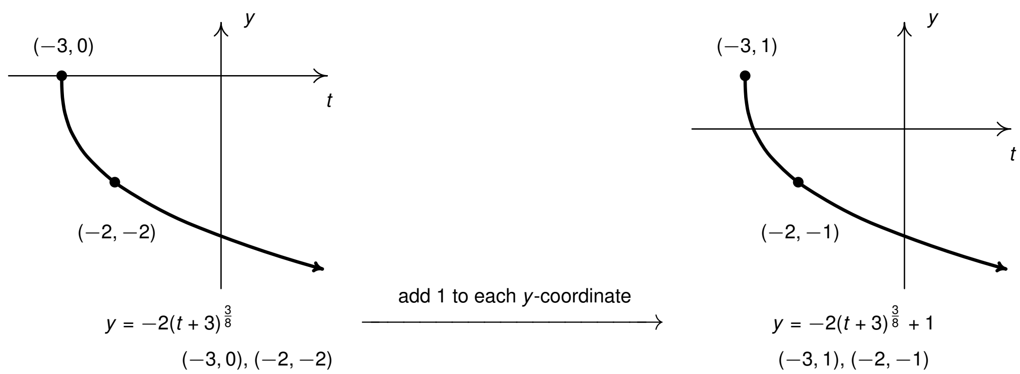


   Step 2: Since $b = 1$, we proceed directly to Step 3.

   Step 3: multiply each of the y-coordinates of each of the points on the graph of $y = (t+3)^{\frac{3}{8}}$ by -2:



Step 4: add 1 to each of the y-coordinates of each of the points on the graph of $y = -2(t+3)^{\frac{3}{8}}$:



From the graph, we get the domain is $[-3, \infty)$ and the range is $(-\infty, 1]$.

We now turn our attention to more complicated functions involving rational exponents.

Example 4.2.2. For the following functions:

- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.
- Graph the function with help from a graphing utility and determine:
  - the range.
  - the local extrema, if they exist.
  - intervals of increase.
  - intervals of decrease.
- Construct a sign diagram for each function using the intercepts and graph.

1. $f(x) = 3x^2(x^3 - 8)^{-\frac{2}{3}}$

2. $g(t) = \dfrac{(t^2 - 4)^{\frac{3}{2}}}{t^2 - 36}$

Solution.

1. We first note that, owing to the negative exponent, the quantity $(x^3 - 8)^{\frac{2}{3}}$ is in the denominator, alerting us to a potential domain issue. Rewriting $(x^3 - 8)^{\frac{2}{3}}$ we set about solving $\sqrt[3]{(x^3 - 8)^2} = 0$. Cubing both sides and extracting square roots gives $x^3 - 8 = 0$ or $x = 2$. Hence, $x = 2$ is excluded from the domain.[3] Since the root involved here is odd (3), the only issue we have is with the denominator, hence our domain is $\{x \in \mathbb{R} \mid x \neq 2\}$ or $(-\infty, 2) \cup (2, \infty)$.

   While not required to do so, we analyze the behavior of f near $x = 2$. As $x \to 2^-$, $3x^2 \approx 12$ and $x^3 - 8 \approx \text{small } (-)$. Hence, $(x^3 - 8)^{\frac{2}{3}} = \sqrt[3]{(x^3 - 8)^2} \approx \sqrt[3]{(\text{small } (-))^2} \approx \sqrt[3]{\text{small } (+)} \approx \text{small}(+)$. As such, $f(x) \approx \frac{12}{\text{small}(+)} \approx \text{big } (+)$. We conclude $\displaystyle\lim_{x \to 2^-} f(x) = \infty$. As $x \to 2^+$, $3x^2 \approx 12$ and $x^3 - 8 \approx \text{small } (+)$, and we likewise get $\displaystyle\lim_{x \to 2^+} f(x) = \infty$. Since the unbounded behavior agrees from both directions as $x \to 2$, we write $\displaystyle\lim_{x \to 2} f(x) = \infty$. Our analysis suggests $x = 2$ is a vertical asymptote to the graph.

   To find the x-intercepts, we set $f(x) = 3x^2(x^3 - 8)^{-\frac{2}{3}} = 0$, so that $3x^2 = 0$ or $x = 0$. We get $(0, 0)$ is our only x- (and y-)intercept.

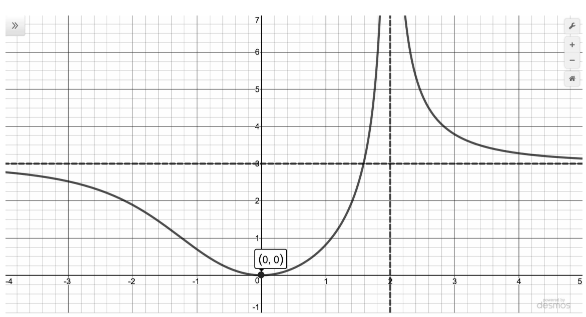
   For end behavior, note that in the denominator the x^3 term dominates so as $x \to -\infty$ and $x \to \infty$,

   $$f(x) = 3x^2(x^3 - 8)^{-\frac{2}{3}} = \frac{3x^2}{(x^3 - 8)^{\frac{2}{3}}} \approx \frac{3x^2}{(x^3)^{\frac{2}{3}}} = \frac{3x^2}{\sqrt[3]{(x^3)^2}} = \frac{3x^2}{\sqrt[3]{x^6}} = \frac{3x^2}{x^2} = 3.$$

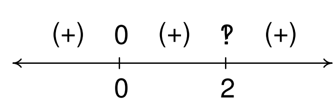
   This suggests $\displaystyle\lim_{x \to -\infty} f(x) = 3$ and $\displaystyle\lim_{x \to \infty} f(x) = 3$ so $y = 3$ is a horizontal asymptote to the graph.

   Graphing $y = f(x)$ below on the right bears out our analysis regarding zeros and asymptotes. The range appears to be $[0, \infty)$, with the graph of $y = f(x)$ crossing its horizontal asymptote between $x = 1$ and $x = 2$. We see we have a single local minimum at $(0, 0)$ with f is decreasing on $(-\infty, 0]$ and $(2, \infty)$ and increasing on $[0, 2)$.

   For the sign diagram, we note that f has only one zero, $x = 0$ and is undefined at $x = 2$. For all x values between these two numbers, $f(x) > 0$ or $(+)$. Our sign diagram for $f(x)$ is below on the right.



The graph of $y = f(x)$



Sign Diagram for $f(x)$

[3]In general if $u^p = 0$ where $p > 0$, then $u = 0$.

2. To find the domain of $g(t) = \frac{(t^2-4)^{\frac{3}{2}}}{t^2-36}$, we have two issues to address: the denominator and an even (square) root. Solving $t^2 - 36 = 0$ gives two excluded values, $t = \pm 6$. For the numerator, we may rewrite $(t^2-4)^{\frac{3}{2}} = (\sqrt{t^2-4})^3$, so we require $t^2 - 4 \geq 0$, or $t^2 \geq 4$. Extracting square roots, we have $\sqrt{t^2} \geq \sqrt{4}$ or $|t| \geq 2$ which means $t \leq -2$ or $t \geq 2$. Taking into account our excluded values $t = \pm 6$, we get the domain of g is $(-\infty, -6) \cup (-6, -2] \cup [2, 6) \cup (6, \infty)$.

   Looking near $t = -6$, we note that as $t \to -6$, $(t^2-4)^{\frac{3}{2}} \approx 32^{\frac{3}{2}} = 32^{1.5}$, a positive number. As $t \to -6^-$, $t^2 - 36 \approx$ small (+), so $g(t) \approx \frac{32^{1.5}}{\text{small}(+)} \approx$ big (+). This suggests $\displaystyle\lim_{t \to -6^-} g(t) = \infty$. On the other hand, as $t \to -6^+$, $t^2 - 36 \approx$ small $(-)$, so $g(t) \approx \frac{32^{1.5}}{\text{small}(-)} \approx$ big $(-)$, suggesting $\displaystyle\lim_{t \to -6^+} g(t) = -\infty$. Similarly, we find as $\displaystyle\lim_{t \to 6^-} g(t) = -\infty$ and as $\displaystyle\lim_{t \to 6^+} g(t) = \infty$. This suggests we have two vertical asymptotes to the graph of $y = g(t)$: $t = -6$ and $t = 6$.

   To find the t-intercepts, we set $g(t) = 0$ and solve $(t^2-4)^{\frac{3}{2}} = 0$. This reduces to $t^2 - 4 = 0$ or $t = \pm 2$. As these are (just barely!) in the domain of g, we have two t-intercepts, $(-2, 0)$ and $(2, 0)$. The graph of g has no y-intercepts, since 0 is not in the domain of g, so $g(0)$ is undefined.

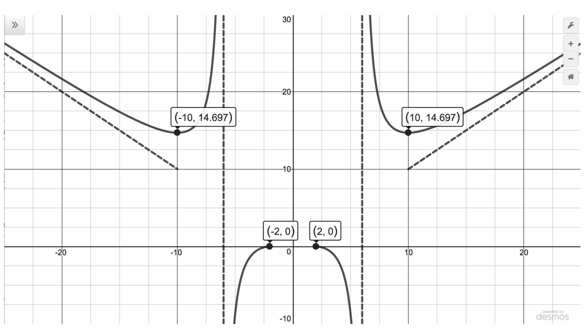
   Regarding end behavior, as $t \to -\infty$ and $t \to \infty$, the t^2 in both numerator and denominator dominate the constant terms, so we have

$$g(t) = \frac{(t^2-4)^{\frac{3}{2}}}{t^2-36} \approx \frac{\left(t^2\right)^{\frac{3}{2}}}{t^2} = \frac{\left(\sqrt{t^2}\right)^3}{t^2} = \frac{|t|^3}{t^2} = \frac{|t||t|^2}{t^2} = \frac{|t|t^2}{t^2} = |t|.$$

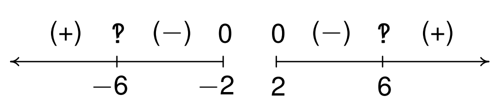
   This suggests that as $t \to -\infty$ and $t \to \infty$, the graph of $y = g(t)$ resembles $y = |t|$. Hence, $\displaystyle\lim_{t \to -\infty} g(t) = \infty$ and $\displaystyle\lim_{t \to \infty} g(t) = \infty$. Using the piecewise definition of $|t|$, we have that as $t \to -\infty$, $g(t) \approx -t$ and as $t \to \infty$, $g(t) \approx t$. In other words, the graph of $y = g(t)$ has *two* slant asymptotes with slopes ± 1.

   Graphing $y = g(t)$ below on the left verifies our analysis. From the graph, the range appears to be $(-\infty, 0] \cup [14.697, \infty)$. The points $(-10, 14.697)$ and $(10, 14.697)$ are local minimums. g appears to be decreasing on $(-\infty, -10]$, $[2, 6)$, and $(6, 10]$. Likewise, g is increasing on $[-10, -6)$, $(-6, -2]$ and $[10, \infty)$. The graph of $y = g(t)$ certainly appears to be symmetric about the y-axis. We leave it to the reader to show g is, indeed, an even function.

   For the sign diagram for $g(t)$, we note that g has zeros $t = \pm 2$ and is undefined at $t = \pm 6$. Moreover, there is a gap in the domain for all values in the interval $(-2, 2)$, so we excise that portion of the real number line for our discussion. We find $g(t) > 0$ or (+) on the intervals $(-\infty - 6)$ and $(6, \infty)$ while $g(t) < 0$ or $(-)$ on $(-6, -2)$ and $(2, 6)$. Our sign diagram for $g(t)$ is below on the right.



The graph of $y = g(t)$



Sign Diagram for $g(t)$

□

4.2.2 Real Number Exponents

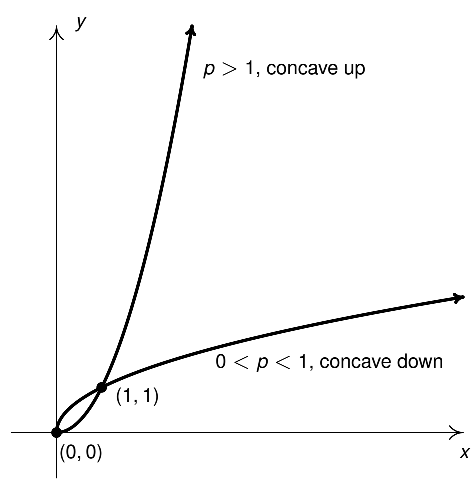
We wish now to extend the concept of 'exponent' from rational to all real numbers which means we need to discuss how to interpret an irrational exponent. Once again, the notions presented here are best discussed using the language of Calculus or Analysis, but we nevertheless do what we can with the notions we have. Consider the wildly famous irrational number π. The number π is defined geometrically as the ratio of the circumference of a circle to that circle's diameter.[4] The reason we use the *symbol* 'π' instead of any numerical expression is that π is an irrational number, and, as such, its decimal representation neither terminates nor repeats. Hence we *approximate* π as $\pi \approx 3.14$ or $\pi \approx 3.14159265$. No matter how many digits we write, however, what we have is a *rational number* approximation of π.

The good news is we can approximate π to any desired accuracy using rational numbers by taking enough digits, so while we'll never 'reach' the *exact* value of π with rational numbers, we can get *as close as we like* to π using rational numbers. That being said, we assume π exists on the real number line, despite the fact the list of digits to pinpoint its location is, in some sense, infinite.

We take this tack when defining the value of a number raised to an irrational exponent. Consider, for instance, 2^{π}. We can compute $2^3 = 8$, $2^{3.1} = 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} \approx 8.574$, $2^{3.14} = 2^{\frac{314}{100}} = 2^{\frac{157}{50}} = \sqrt[50]{2^{157}} \approx 8.8512$, and so on, so one way to *define* 2^{π} as the unique real number we obtain as the exponents 'approach' π.

It is with this understanding that we present the notion of a 'power function,' as described in Definition 4.2: $f(x) = ax^p$ where a and p are nonzero real number parameters. Here the exponent p is open to any (nonzero) real number. Because of how we define real number exponents, if p is irrational, then $x \geq 0$ to avoid having negatives under even-indexed roots as we go through the approximation process.[5]

In general, real number exponents inherit their properties from rational number exponents. For instance, Theorem 4.3 also holds for all real number exponents and the graphs of power functions inherit their behavior from graphs of rational exponent functions. More specifically, the graphs of functions of the form $f(x) = x^p$ where $p > 0$ all contain the points $(0, 0)$ and $(1, 1)$. Moreover, these functions are increasing and their graphs are concave down if $0 < p < 1$ and concave up if $p > 1$.



$f(x) = x^p$ for varying values of p.

[4] This works for each and every circle, by the way, regardless of how large or small the circle is!
[5] or $x > 0$ if p is negative.

Theorem 4.4 generalizes to real number power functions, so, for instance to graph $F(x) = (x - 2)^{\pi}$, one need only start with $y = x^{\pi}$ and shift horizontally two units to the right. (See the Exercises.)
We close this section with an application to economics. According to the US Census, Table 2, the share of money income (2014-2015) is given in the table below on the left. From these data, we can create a cumulative distribution, $y = L(x)$ called the **Lorenz Curve**.
The number $L(x)$ gives the percentage of the total national income earned by the bottom x percent of wage earners, ranked from lowest income to highest income. Since the population here is separated into 'quintiles,' each data point corresponds to 20% of the population. So, for example, $L(20)$ is the percentage of money income earned by the lowest 20% of wage earners. In this case, we see $L(20) = 3.1$. The number $L(40)$ is the percentage of the money income earned by the bottom 40% of wage earners - so this includes not only the money from the Second Quintile, but also the Lowest Quintile: $L(40) = 8.2 + L(20) = 8.2 + 3.1 = 11.3$. Likewise, $L(60)$ is the total income share of the bottom 60% of wage earners which includes the income from the Middle, Second, and Lowest Quintiles: $L(60) = 14.3 + L(40) = 14.3 + (8.2 + 3.1) = 25.6$. Continuing in this manner, we get $L(80) = 48.8$ and $L(100) = 100$, which is what we would expect: 100% of the income is earned by 100% of the population. We summarize these findings below on the right.

Portion of Population	Percent of Money Income
Lowest Quintile	3.1
Second Quintile	8.2
Middle Quintile	14.3
Fourth Quintile	23.2
Highest Quintile	51.2

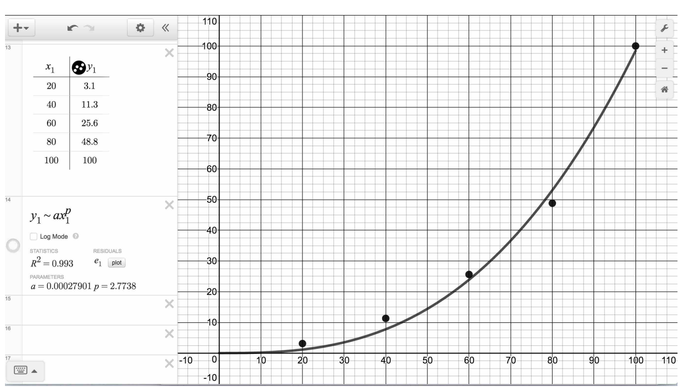
percent wage earners, x	percent income, $L(x)$
20	3.1
40	11.3
60	25.6
80	48.8
100	100

Example 4.2.3.

1. Find power function to model the Lorenz Curve: $L(x) = ax^p$. Comment on the goodness of fit.
2. Find and interpret $L(90)$.

Solution.

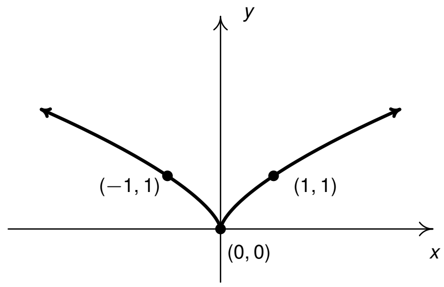
1. Using desmos, we get $L(x) = 0.00027901x^{2.7738}$ with $R^2 = 0.993$, indicating a pretty good fit.



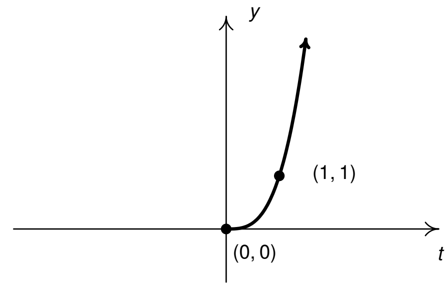
2. We compute $L(90) = 0.00027901(90)^{2.7738} \approx 73.5$ meaning that the bottom 90% of the wage earners brought home 73.5% of the total income. Said differently, the top 10% of wage earners made over 25% of the total national income. $\square$

4.2.3 Exercises

In Exercises 1 - 6, use the given graphs along with Theorem 4.4 to graph the given function. Track at least two points and state the domain and range using interval notation.



$f(x) = x^{\frac{2}{3}}$

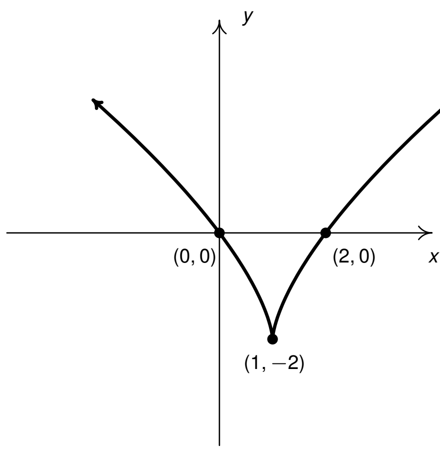


$g(t) = t^{\pi}$

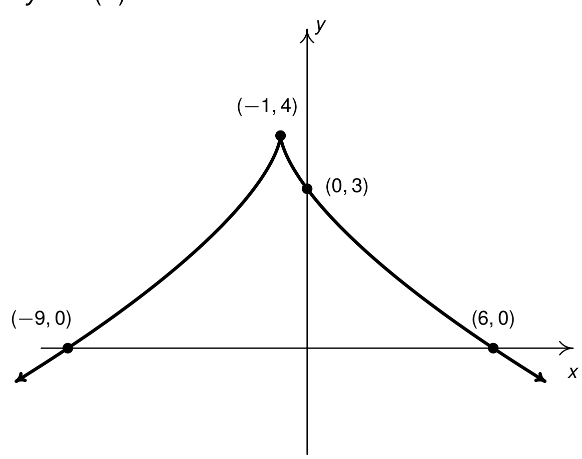
1. $F(x) = (x-2)^{\frac{2}{3}} - 1$
2. $G(t) = (t+3)^{\pi} + 1$
3. $F(x) = 3 - x^{\frac{2}{3}}$
4. $G(t) = (1-t)^{\pi} - 2$
5. $F(x) = (2x+5)^{\frac{2}{3}} + 1$
6. $G(t) = \left(\dfrac{t+3}{2}\right)^{\pi} - 1$

In Exercises 7 - 8, find a formula for each function below in the form $F(x) = a(bx-h)^{\frac{2}{3}} + k$.
NOTE: There may be more than one solution!

7. $y = F(x)$



8. $y = F(x)$



For each function in Exercises 9 - 16 below

- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.
- Graph the function with help from a graphing utility and determine:
  - the range.
  - the local extrema, if they exist.
  - intervals of increase/decrease.
  - any 'unusual steepness' or 'local' verticality.
  - vertical asymptotes.
  - horizontal / slant asymptotes.
- Construct a sign diagram for each function using the intercepts and graph.
- Comment on any observed symmetry.

9. $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$
10. $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$
11. $g(t) = 2t(t+3)^{-\frac{1}{3}}$
12. $g(t) = t^{\frac{3}{2}}(t-2)^{-\frac{1}{2}}$
13. $f(x) = x^{0.4}(3-x)^{0.6}$
14. $f(x) = x^{0.5}(3-x)^{0.5}$
15. $g(t) = 4t(9-t^2)^{-\sqrt{2}}$
16. $g(t) = 3(t^2+1)^{-\pi}$
17. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[6] What trends do you observe? How do your answers manifest themselves graphically? Compare the results of this exercise with those of Exercise 51 in Section 2.1 and Exercise 43 in Section 3.1

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
$x^{\frac{1}{2}}$				
$x^{\frac{2}{3}}$				
$x^{-0.23}$				
x^{π}				

18. The National Weather Service uses the following formula to calculate the wind chill:

$$W = 35.74 + 0.6215\, T_a - 35.75\, V^{0.16} + 0.4275\, T_a\, V^{0.16}$$

where W is the wind chill temperature in °F, T_a is the air temperature in °F, and V is the wind speed in miles per hour. Note that W is defined only for air temperatures at or lower than 50°F and wind speeds above 3 miles per hour.

[6] See Definition 1.8 in Section 1.2.4 for a review of this concept, as needed.

(a) Suppose the air temperature is 42° and the wind speed is 7 miles per hour. Find the wind chill temperature. Round your answer to two decimal places.

(b) Suppose the air temperature is 37°F and the wind chill temperature is 30°F. Find the wind speed. Round your answer to two decimal places.

19. As a follow-up to Exercise 18, suppose the air temperature is 28°F.

    (a) Use the formula from Exercise 18 to find an expression for the wind chill temperature as a function of the wind speed, $W(V)$.

    (b) Solve $W(V) = 0$, round your answer to two decimal places, and interpret.

    (c) Graph the function W using a graphing utility and check your answer to part 19b.

20. Suppose Fritzy the Fox, positioned at a point (x, y) in the first quadrant, spots Chewbacca the Bunny at $(0, 0)$. Chewbacca begins to run along a fence (the positive y-axis) towards his warren. Fritzy, of course, takes chase and constantly adjusts his direction so that he is always running directly at Chewbacca. If Chewbacca's speed is v_1 and Fritzy's speed is v_2, the path Fritzy will take to intercept Chewbacca, provided v_2 is directly proportional to, but not equal to, v_1 is modeled by

$$y = \frac{1}{2}\left(\frac{x^{1+v_1/v_2}}{1+v_1/v_2} - \frac{x^{1-v_1/v_2}}{1-v_1/v_2}\right) + \frac{v_1 v_2}{v_2^2 - v_1^2}$$

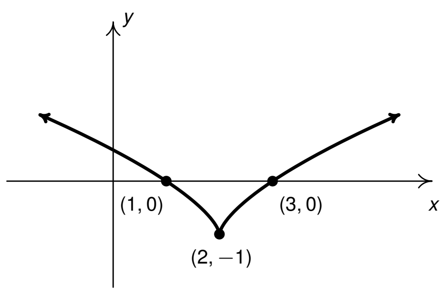
    (a) Determine the path that Fritzy will take if he runs exactly twice as fast as Chewbacca; that is, $v_2 = 2v_1$. Use your calculator to graph this path for $x \geq 0$. What is the significance of the y-intercept of the graph?

    (b) Determine the path Fritzy will take if Chewbacca runs exactly twice as fast as he does; that is, $v_1 = 2v_2$. Use a graphing utility to graph this path for $x > 0$. Describe the behavior of y as $x \to 0^+$ and interpret this physically.

    (c) With the help of your classmates, generalize parts (a) and (b) to two cases: $v_2 > v_1$ and $v_2 < v_1$. We will discuss the case of $v_1 = v_2$ in Exercise 34 in Section 7.6.

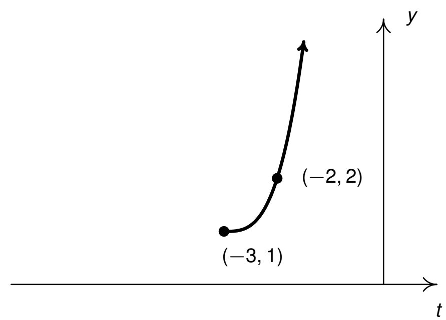
4.2.4 Answers

1. $F(x) = (x-2)^{\frac{2}{3}} - 1$



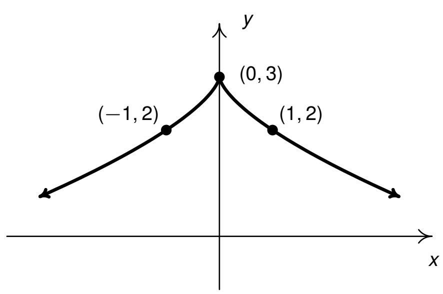
Domain: $(-\infty, \infty)$, Range: $[-1, \infty)$

2. $G(t) = (t+3)^{\pi} + 1$



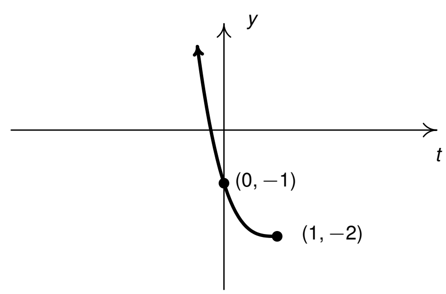
Domain: $[-3, \infty)$, Range: $[1, \infty)$

3. $F(x) = 3 - x^{\frac{2}{3}} = (-1)x^{\frac{2}{3}} + 3$



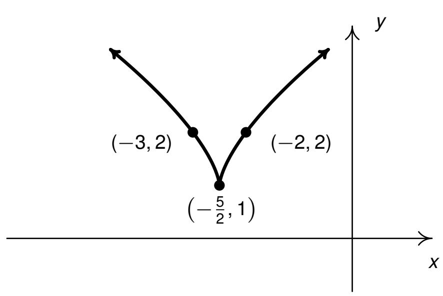
Domain: $(-\infty, \infty)$, Range: $(-\infty, 3]$

4. $G(t) = (1-t)^{\pi} - 2 = ((-1)t+1)^{\pi} - 2$



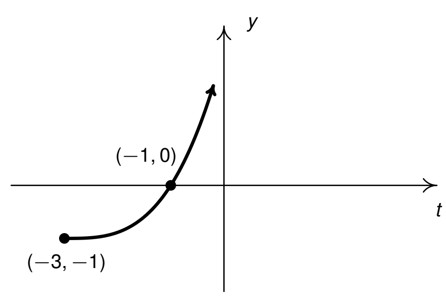
Domain: $(-\infty, 1]$, Range: $[-2, \infty)$

5. $F(x) = (2x+5)^{\frac{2}{3}} + 1$



Domain: $(-\infty, \infty)$, Range: $[1, \infty)$

6. $G(t) = \left(\dfrac{t+3}{2}\right)^{\pi} - 1 = \left(\frac{1}{2}t + \frac{3}{2}\right)^{\pi} - 1$

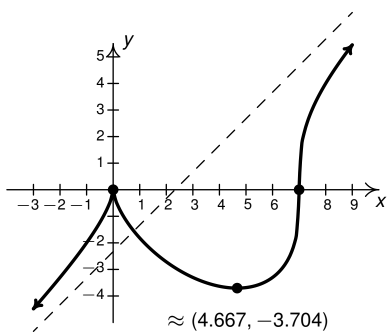


Domain: $[-3, \infty)$, Range: $[-1, \infty)$

7. One solution is: $F(x) = 2(x-1)^{\frac{2}{3}} - 2$

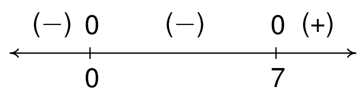
8. One solution is: $F(x) = -(x+1)^{\frac{2}{3}} + 4$

9. $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$
Domain: $(-\infty, \infty)$
Intercepts: $(0,0)$, $(7,0)$
Graph:

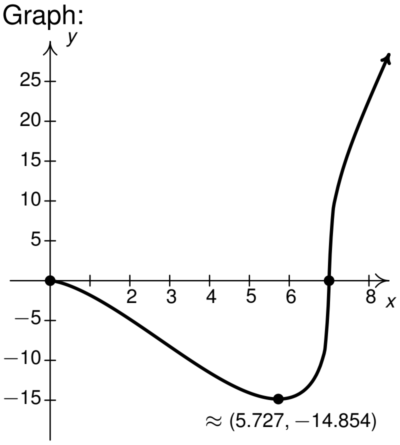


$\displaystyle\lim_{x\to-\infty} f(x) = -\infty$, $\displaystyle\lim_{x\to\infty} f(x) = \infty$[7]
Range: $(-\infty, \infty)$
Local minimum: $\approx (4.667, -3.704)$
Local maximum: $(0,0)$ (this is a cusp)
Increasing: $(-\infty, 0]$, $\approx [4.667, \infty)$
Decreasing: $[0, 4.667]$
Unusual steepness at $x = 7$

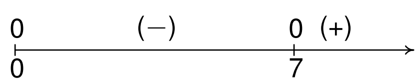
Sign Diagram:



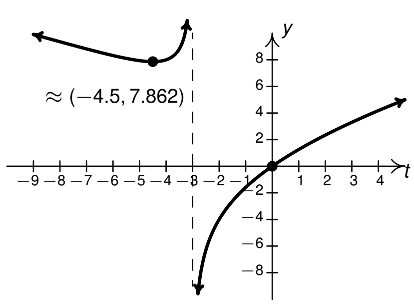
10. $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$
Graph:



Domain: $[0, \infty)$
Intercepts: $(0,0)$, $(7,0)$
$\displaystyle\lim_{x\to\infty} f(x) = \infty$
Range: $\approx [-14.854, \infty)$
Local minimum: $\approx (5.727, -14.854)$
Increasing: $\approx [5.727, \infty)$
Decreasing: $\approx [0, 5.727]$
Unusual steepness at $x = 7$
Sign Diagram:

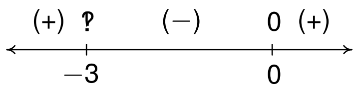


11. $g(t) = 2t(t+3)^{-\frac{1}{3}}$
Graph:

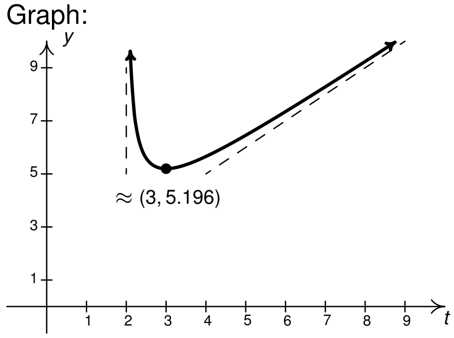


Domain: $(-\infty, -3) \cup (-3, \infty)$
Intercept: $(0,0)$
$\displaystyle\lim_{t\to-\infty} g(t) = \infty$
$\displaystyle\lim_{t\to\infty} g(t) = \infty$
Range: $(-\infty, \infty)$
Local minimum: $\approx (-4.5, 7.862)$
Increasing: $\approx [-4.5, -3)$, $(-3, \infty)$
Decreasing: $\approx (-\infty, -4.5]$
Vertical Asymptote: $t = -3$
Sign Diagram:

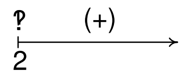
[7]Using Calculus it can be shown that $y = x - \frac{7}{3}$ is a slant asymptote of this graph.



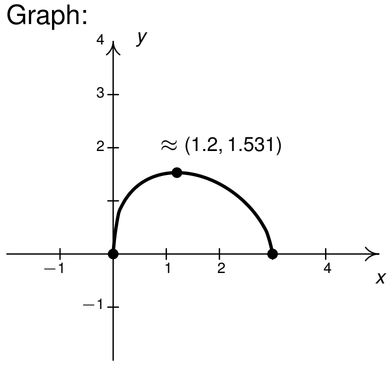
12. $g(t) = t^{\frac{3}{2}}(t-2)^{-\frac{1}{2}}$
Domain: $(2, \infty)$
$\lim_{t \to \infty} g(t) = \infty$
Graph:



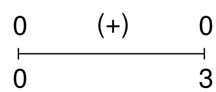
$\lim_{t \to \infty} g(t) = \infty$[8]
Range: $\approx [5.196, \infty)$
Local minimum: $\approx (3, 5.196)$
Increasing: $\approx [3, \infty)$
Decreasing: $\approx (2, 3]$
Vertical asymptote: $t = 2$
Sign Diagram:



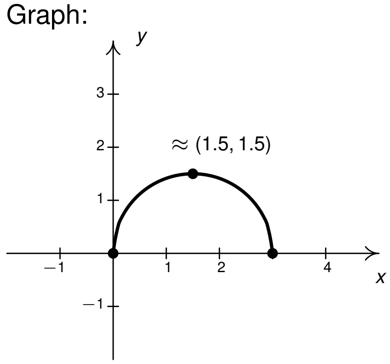
13. $f(x) = x^{0.4}(3-x)^{0.6}$
Graph:



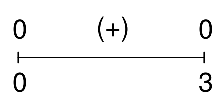
Domain: $[0, 3]$
Intercepts: $(0, 0)$, $(3, 0)$
Range: $\approx [0, 1.5]$
Increasing: $\approx [0, 1.2]$
Decreasing: $\approx [1.2, 3]$
Unusual Steepness:[9] $x = 0$, $x = 3$
Sign Diagram:



14. $f(x) = x^{0.5}(3-x)^{0.5}$
Graph:

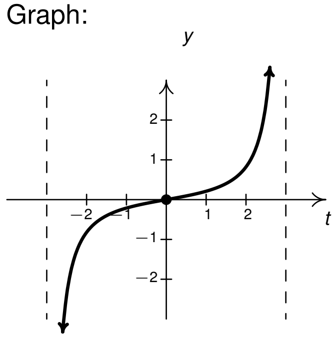


Domain: $[0, 3]$
Intercepts: $(0, 0)$, $(3, 0)$
Range: $\approx [0, 1.5]$
Increasing: $\approx [0, 1.5]$
Decreasing: $\approx [1.5, 3]$
Unusual Steepness:[10] $x = 0$, $x = 3$
Sign Diagram:



[8] Using Calculus it can be shown that $y = t + 1$ is a slant asymptote of this graph.
[9] Note you may need to zoom in to see this.
[10] Note you may need to zoom in to see this.

15. $g(t) = 4t(9 - t^2)^{-\sqrt{2}}$
Graph:

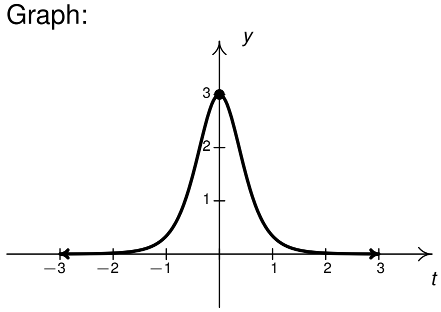


Domain: $(-3, 3)$
Intercepts: $(0, 0)$
Range: $(-\infty, \infty)$
$\lim_{t \to -3^+} g(t) = -\infty$, $\lim_{t \to 3^-} g(t) = \infty$
Vertical asymptotes: $t = -3$ and $t = 3$
Increasing: $(-3, 3)$
Sign Diagram:

‽	$(-)$	0	$(+)$	‽
-3		0		3

Note: g is odd

16. $g(t) = 3(t^2 + 1)^{-\pi}$
Domain: $(-\infty, \infty)$
Graph:



Intercept: $(0, 3)$
$\lim_{t \to -\infty} g(t) = 0$
$\lim_{t \to \infty} g(t) = 0$
Range: $(0, 3]$
Increasing: $(-\infty, 0]$
Decreasing: $[0, \infty)$
Horizontal asymptote: $y = 0$
Sign Diagram:
$(+)$

Note: g is even

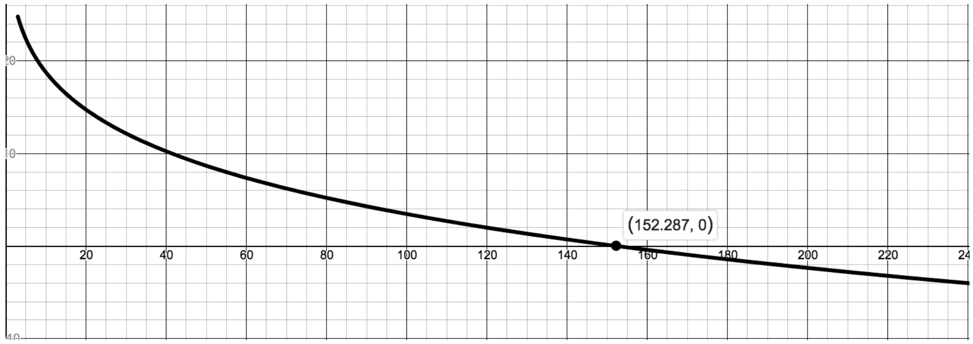
17. As in Exercise 51 in Section 2.1 and Exercise 43 in Section 3.1, the slopes of these curves near $x = 1$ approach the value of the exponent on x.

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
$x^{\frac{1}{2}}$	0.5006	$\approx \frac{1}{2}$	$\approx \frac{1}{2}$	$\approx \frac{1}{2}$
$x^{\frac{2}{3}}$	0.6672	0.6667	$\approx \frac{2}{3}$	$\approx \frac{2}{3}$
$x^{-0.23}$	-0.2310	≈ -0.23	≈ -0.23	≈ -0.23
x^{π}	3.1544	3.1417	$\approx \pi$	$\approx \pi$

18. (a) $W \approx 37.55^\circ$F.
(b) $V \approx 9.84$ miles per hour.

19. (a) $W(V) = 53.142 - 23.78V^{0.16}$. Since we are told in Exercise 18 that wind chill is only effect for wind speeds of more than 3 miles per hour, we restrict the domain to $V > 3$.
(b) $W(V) = 0$ when $V \approx 152.29$. This means, according to the model, for the wind chill temperature to be 0°F, the wind speed needs to be 152.29 miles per hour.

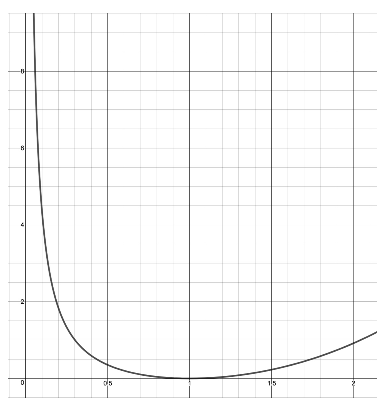
(c) The graph of $y = W(V)$ is below.



20. (a) $y = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}} + \frac{2}{3}$. The point $\left(0, \frac{2}{3}\right)$ is when Fritzy's path crosses Chewbacca's path - in other words, where Fritzy catches Chewbacca.

$y = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}} + \frac{2}{3}$

(b) $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1} - \frac{2}{3}$. We find as $x \to 0^+$, $y \to \infty$ which means, in this case, Fritzy's pursuit never ends; he never catches Chewbacca. This makes sense since Chewbacca has a head start and is running faster than Fritzy.



$y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1} - \frac{2}{3}$

4.3 Equations and Inequalities involving Power Functions

In this section, we set about solving equations and inequalities involving power functions. Our first example demonstrates the usual sorts of strategies to employ when solving equations.

Example 4.3.1. Solve the following equations analytically and verify your answers using a graphing utility.

1. $(7-x)^{\frac{3}{2}} = 8$
2. $(2t-1)^{\frac{2}{3}} - 4 = 0$
3. $(x+3)^{0.5} = 2(7-x)^{0.5} + 1$
4. $2t^{\frac{2}{3}} + 5t^{\frac{1}{3}} = 3$
5. $2(3x-1)^{-0.5} = 3x(3x-1)^{-1.5}$
6. $6(9-t^2)^{\frac{1}{3}} = 4t^2(9-t^2)^{-\frac{2}{3}}$

Solution.

1. One way to proceed to solve $(7-x)^{\frac{3}{2}} = 8$ is to use Definition 4.3 to rewrite $(7-x)^{\frac{3}{2}}$ as either $(\sqrt{7-x})^3$ or $\sqrt{(7-x)^3}$. We opt for the former since, thinking ahead, 8 is a perfect cube:

$$\begin{array}{rclr} (7-x)^{\frac{3}{2}} & = & 8 & \\ (\sqrt{7-x})^3 & = & 8 & \text{rewrite using Definition 4.3} \\ \sqrt[3]{(\sqrt{7-x})^3} & = & \sqrt[3]{8} & \text{extract cube roots} \\ \sqrt{7-x} & = & 2 & \sqrt[3]{u^3} = u \end{array}$$

From $\sqrt{7-x} = 2$, we square both sides and obtain $7-x=4$, so $x=3$. We verify our answer analytically by substituting $x=3$ into the original equation and it checks.

Geometrically, we are looking for where the graph of $f(x) = (7-x)^{\frac{3}{2}}$ intersects the graph of $g(x) = 8$. While we could sketch both curves by hand and gauge the reasonableness of the result,[1] we are instructed to use a graphing utility. Below on the left and see the intersection point of both graphs is $(3,8)$, thereby checking our solution $x=3$.

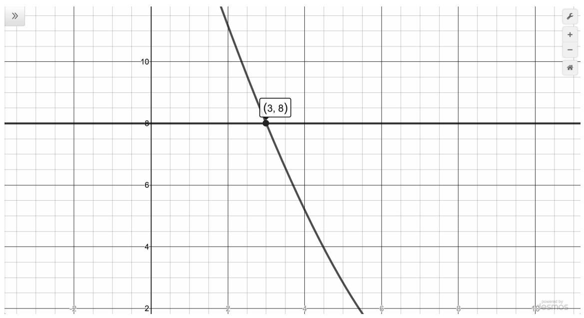
2. Proceeding similarly to the above, to solve $(2t-1)^{\frac{2}{3}} - 4 = 0$, we rewrite $(2t-1)^{\frac{2}{3}}$ as $(\sqrt[3]{2t-1})^2$ and solve:

$$\begin{array}{rclr} (2t-1)^{\frac{2}{3}} - 4 & = & 0 & \\ (\sqrt[3]{2t-1})^2 - 4 & = & 0 & \text{rewrite using Definition 4.3} \\ (\sqrt[3]{2t-1})^2 & = & 4 & \text{isolate the variable term} \\ \sqrt{(\sqrt[3]{2t-1})^2} & = & \sqrt{4} & \text{extract square roots} \\ |\sqrt[3]{2t-1}| & = & 2 & \sqrt{u^2} = |u| \\ \sqrt[3]{2t-1} & = & \pm 2 & \text{for } c > 0,\ |u| = c \text{ is equivalent to } u = \pm c. \end{array}$$

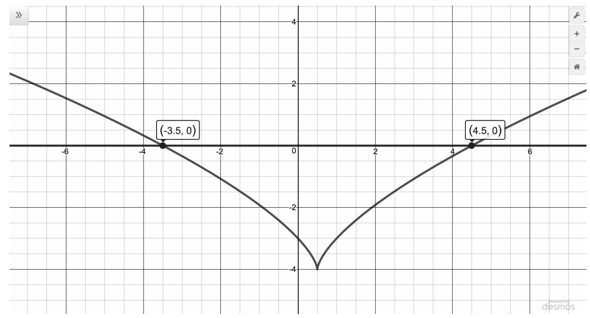
[1] consider this an exercise!

From $\sqrt[3]{2t-1} = 2$ we cube both sides and obtain $2t - 1 = 8$, so $t = \frac{9}{2} = 4.5$. Similarly, from $\sqrt[3]{2t-1} = -2$, we cube both sides and obtain $2t - 1 = -8$, so $t = -\frac{7}{2} = -3.5$. Both of these solutions check in the given equation.

In this case we are looking for where the graph of $f(t) = (2t-1)^{\frac{2}{3}} - 4$ intersects the graph of $g(t) = 0$ - i.e., the t-intercepts of the graph of g. We find these are $(-3.5, 0)$ and $(4.5, 0)$, as predicted.



Checking $(7-x)^{\frac{3}{2}} = 8$



Checking $(2t-1)^{\frac{2}{3}} - 4 = 0$

3. Since $0.5 = \frac{1}{2}$, we may rewrite $(x+3)^{0.5} = 2(7-x)^{0.5} + 1$ as $(x+3)^{\frac{1}{2}} = 2(7-x)^{\frac{1}{2}} + 1$. Using Definition 4.3, we then have $\sqrt{x+3} = 2\sqrt{7-x} + 1$. Since one of the square roots is already isolated, we can rid ourselves of it by squaring both sides.

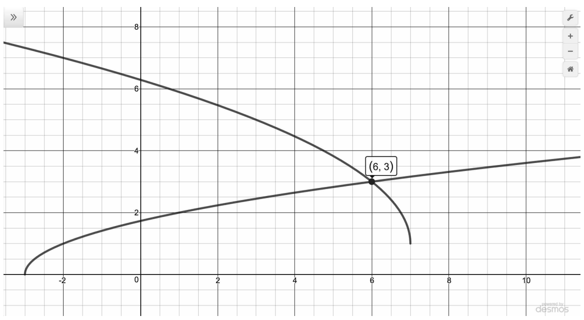
$$\begin{array}{rcll}
\sqrt{x+3} & = & 2\sqrt{7-x}+1 & \\
(\sqrt{x+3})^2 & = & (2\sqrt{7-x}+1)^2 & \text{square both sides} \\
x+3 & = & (2\sqrt{7-x})^2 + 2(2\sqrt{7-x})(1) + 1 & (\sqrt{u})^2 = u \text{ and } (a+b)^2 = a^2 + 2ab + b^2 \\
x+3 & = & 4(7-x) + 4\sqrt{7-x} + 1 & (ab)^2 = a^2b^2 \text{ and, again, } (\sqrt{u})^2 = u \\
x+3 & = & 28 - 4x + 4\sqrt{7-x} + 1 & \\
5x - 26 & = & 4\sqrt{7-x} & \text{isolate } \sqrt{7-x}
\end{array}$$

We square both sides *again* and get $(5x-26)^2 = (4\sqrt{7-x})^2$ which reduces to $25x^2 - 260x + 676 = 16(7-x)$. At last, we have a quadratic equation which we can solve by setting to zero and factoring. We get $25x^2 - 244x + 564 = 0$, so $(x-6)(25x-94) = 0$ so $x = 6$ or $x = \frac{94}{25} = 3.76$. When we go to check these answers, we find $x = 6$ does check, but $x = 3.76$ does not. Hence, $x = 3.76$ is an 'extraneous' solution.[2]

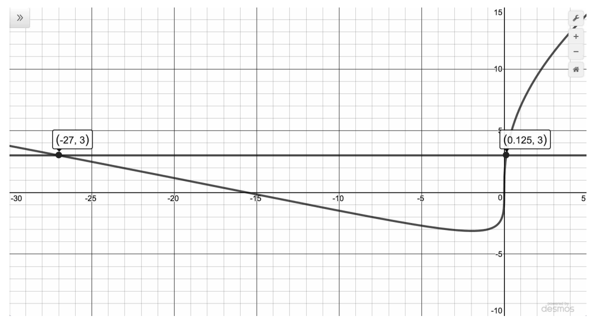
We graph both $f(x) = \sqrt{x+3}$ and $g(x) = 2\sqrt{7-x} + 1$ below (once again, we could graph these by hand!) and confirm there is only one intersection point, $(6, 3)$.

[2] We invite the reader to see at which point in our machinations $x = 3.76$ *does* check. Knowing a solution is extraneous is one thing; understanding *how* it came about is another.

4. While we *could* approach solving $2t^{\frac{2}{3}} + 5t^{\frac{1}{3}} = 3$ as the previous example, we would encounter cubing binomials[3] which we would prefer to avoid. Instead, we take a step back and notice there are three terms here with the exponent on one term, $t^{\frac{2}{3}}$ exactly twice the exponent on another term, $t^{\frac{1}{3}}$. We have ourselves a 'quadratic in disguise.'[4] To help us see the forest for the trees, we let $u = t^{\frac{1}{3}}$ so that $u^2 = t^{\frac{2}{3}}$. (Note that since root here, 3, is odd, we can use the properties of exponents stated in Theorem 4.3.) Hence, in terms of u, the equation $2t^{\frac{2}{3}} + 5t^{\frac{1}{3}} = 3$ becomes the quadratic $2u^2 + 5u = 3$, or $2u^2 + 5u - 3 = 0$. Factoring gives $(2u - 1)(u + 3) = 0$ so $u = t^{\frac{1}{3}} = \frac{1}{2}$ or $u = t^{\frac{1}{3}} = -3$. Since $t^{\frac{1}{3}} = \sqrt[3]{t}$, we solve both equations by cubing both sides to get $t = \frac{1}{8} = 0.125$ and $t = -27$. Both of these solutions check in our original equation. Looking at the graphs of $f(t) = 2t^{\frac{2}{3}} + 5t^{\frac{1}{3}}$ and $g(t) = 3$, we find two intersection points, $(-27, 3)$ and $(0.125, 3)$, as required.



Checking $(x + 3)^{0.5} = 2(7 - x)^{0.5} + 1$



Checking $2t^{\frac{2}{3}} + 5t^{\frac{1}{3}} = 3$

5. Next we are to solve $2(3x - 1)^{-0.5} = 3x(3x - 1)^{-1.5}$ which, when written without negative exponents is: $\frac{2}{(3x-1)^{0.5}} = \frac{3x}{(3x-1)^{1.5}}$. Since the rational exponents here are $0.5 = \frac{1}{2}$ and $1.5 = \frac{3}{2}$, both involve an *even* indexed root (the square root in this case!) which means $3x - 1 \geq 0$. Moreover, since the $3x - 1$ resides in the *denominator* $3x - 1 \neq 0$ so our equation is really valid only for values of x where $3x - 1 > 0$ or $x > \frac{1}{3}$. Hence, we clear denominators and can apply Theorem 4.3:

$$\begin{aligned}
\frac{2}{(3x-1)^{0.5}} &= \frac{3x}{(3x-1)^{1.5}} \\
\left[\frac{2}{(3x-1)^{0.5}}\right] \cdot (3x-1)^{1.5} &= \left[\frac{3x}{(3x-1)^{1.5}}\right] \cdot (3x-1)^{1.5} \\
2 \cdot \frac{(3x-1)^{1.5}}{(3x-1)^{0.5}} &= 3x \\
2(3x-1)^{1.5-0.5} &= 3x && \text{Theorem 4.3 applies since } 3x - 1 > 0. \\
2(3x-1)^{1} &= 3x
\end{aligned}$$

We get $6x - 2 = 3x$, or $x = \frac{2}{3}$. Since $x = \frac{2}{3} > \frac{1}{3}$, we keep it and, sure enough, it checks in our

[3] that is, expanding things like $(a + b)^3$.

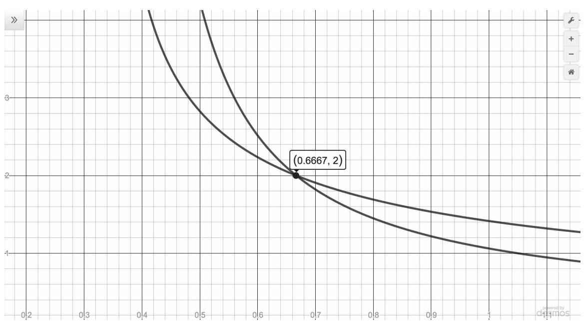
[4] See Section A.10 or, more recently, Example 2.3.4 in Section 2.3.

original equation. Graphically we see $f(x) = 2(3x - 1)^{-0.5}$ intersects $g(x) = 3x(3x - 1)^{-1.5}$ at the point $(0.6667, 2)$ which is the graphing utility's way of representing $\left(\frac{2}{3}, 2\right)$.

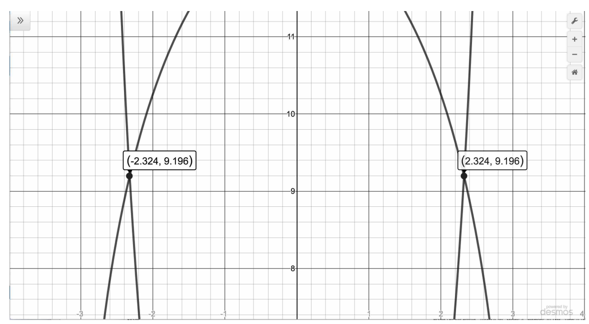
6. Our last equation to solve is $6(9 - t^2)^{\frac{1}{3}} = 4t^2(9 - t^2)^{-\frac{2}{3}}$, which, when rewritten without negative exponents is: $6(9 - t^2)^{\frac{1}{3}} = \frac{4t^2}{(9-t^2)^{\frac{2}{3}}}$. Again, the root here (3) is odd, so we can use the exponent properties listed in Theorem 4.3. We begin by clearing denominators:

$$\begin{array}{rcll} 6(9 - t^2)^{\frac{1}{3}} & = & \frac{4t^2}{(9-t^2)^{\frac{2}{3}}} & \\ 6(9 - t^2)^{\frac{1}{3}} \cdot (9 - t^2)^{\frac{2}{3}} & = & \left[\frac{4t^2}{(9-t^2)^{\frac{2}{3}}}\right] \cdot (9 - t^2)^{\frac{2}{3}} & \\ 6(9 - t^2)^{\frac{1}{3}+\frac{2}{3}} & = & 4t^2 & \text{Theorem 4.3 applies since the root here 3 is odd.} \\ 6(9 - t^2)^1 & = & 4t^2 & \end{array}$$

We get $54 - 6t^2 = 4t^2$ or $10t^2 = 54$. As fraction $t^2 = \frac{54}{10} = \frac{27}{5}$ so $t = \pm\sqrt{\frac{27}{5}} = \pm 3\sqrt{155}$. While not the easiest to check analytically, both of these solutions do work in the original equation. Graphing $f(t) = 6(9 - t^2)^{\frac{1}{3}}$ and $g(t) = 4t^2(9 - t^2)^{-\frac{2}{3}}$ below, we see the graphs intersect when $t \approx \pm 2.324$ which are decimal approximations of our exact answers.



Checking $2(3x - 1)^{-0.5} = 3x(3x - 1)^{-1.5}$



Checking $6(9 - t^2)^{\frac{1}{3}} = 4t^2(9 - x^2)^{\frac{2}{3}}$

□

Note that Example 4.3.1, there are several ways to correctly solve each equation, and we endeavored to demonstrate a variety of methods. For example, for number 1, instead of converting $(7 - x)^{\frac{3}{2}}$ to a radical equation, we could use Theorem 4.3. Since the root here (2) is even, we know $7 - x \geq 0$ or $x \leq 7$. Hence we may apply exponent properties:

$$\begin{array}{rcll} (7 - x)^{\frac{3}{2}} & = & 8 & \\ \left[(7 - x)^{\frac{3}{2}}\right]^{\frac{2}{3}} & = & 8^{\frac{2}{3}} & \text{raise both sides to the } \frac{2}{3} \text{ power} \\ (7 - x)^{\frac{3}{2}\cdot\frac{2}{3}} & = & 4 & \text{Theorem 4.3} \\ (7 - x)^1 & = & 4 & \end{array}$$

from which we get $x = 3$. If we try this same approach to solve number 2, however, we encounter difficulty. From $(2t-1)^{\frac{2}{3}} - 4 = 0$, we get $(2t-1)^{\frac{2}{3}} = 4$.

$$\begin{array}{rcll} (2t-1)^{\frac{2}{3}} & = & 4 & \\ \left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} & = & 4^{\frac{3}{2}} & \text{raise both sides to the } \tfrac{3}{2} \text{ power} \end{array}$$

Since the root here (3) is odd, we have no restriction on $2t-1$ but the exponent $\frac{3}{2}$ has an even denominator. Hence, Theorem 4.3 *does not apply*. That is,

$$\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} \neq (2t-1)^{\frac{2}{3}\cdot\frac{3}{2}} = (2t-1)^1 = (2t-1).$$

Note that if we weren't careful, we'd have $2t - 1 = 4^{\frac{3}{2}} = 8$ which gives $t = \frac{9}{2} = 4.5$ only. We'd have missed the solution $t = -3.5$. Truth be told, you *can* simplify $\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}}$ - just not using Theorem 4.3. We leave it as an exercise to show $\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} = |2t-1|$ and, more generally, $\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = |x|$.

Our next example is an application of the Cobb Douglas production model of an economy. The Cobb-Douglas model states that the yearly total dollar value of the production output in an economy is a function of *two* variables: labor (the total number of hours worked in a year) and capital (the total dollar value of the physical goods required for manufacturing.) The equation relating the production output level P, labor L and capital K takes the form $P = aL^bK^{1-b}$ where $0 < b < 1$; that is, the production level varies jointly with some power of the labor and capital.

Example 4.3.2. In their original paper *A Theory of Production*[5] Cobb and Douglas modeled the output of the US Economy (using 1899 as a baseline) using the formula $P = 1.01L^{0.75}K^{0.25}$ where P, L, and K were percentages of the 1899 figures for total production, labor, and capital, respectively.

1. For 1910, the recorded labor and capital figures for the US Economy are 144% and 208% of the 1899 figures, respectively. Find P using these figures and interpret your answer.

2. The recorded production value figure for 1920 is 231% of the 1899 figure. Use this to write K as a function of L, $K = f(L)$. Find and interpret $f(193)$.

3. Graph $K = f(L)$ and interpret the behavior as $L \to 0^+$ and $L \to \infty$.

Solution.

1. In this case, $P = 1.01L^{0.75}K^{0.25} = 1.01(144)^{0.75}(208)^{0.25} \approx 159$ which means the dollar value of the total US Production in 1920 was approximately 159% of what it was in 1899.[6]

[5] available here.

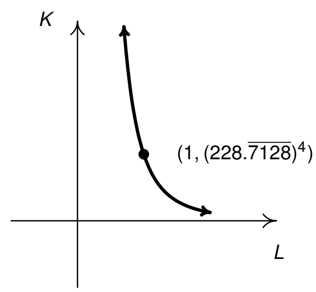
[6] This answer is remarkably accurate. Note: all the dollar values here are recorded in '1880' dollars, per the source article.

2. We are given $P = 231 = 1.01L^{0.75}K^{0.25}$, so to write K as a function of L, we need to solve this equation for K. Since L and K are positive by definition, we can employ properties of exponents:

$$\begin{array}{rcll}
231 & = & 1.01L^{0.75}K^{0.25} & \\
\dfrac{231}{1.01L^{0.75}} & = & \dfrac{1.01L^{0.75}K^{0.25}}{1.01L^{0.75}} & L > 0, \text{ hence } L^{0.75} \neq 0. \\
K^{0.25} & = & 228.\overline{7128}L^{-0.75} & \text{rewrite} \\
\left(K^{0.25}\right)^{\frac{1}{0.25}} & = & \left(228.\overline{7128}L^{-0.75}\right)^{\frac{1}{0.25}} & \\
K^{\frac{0.25}{0.25}} & = & (228.\overline{7128})^{\frac{1}{0.25}} L^{-\frac{0.75}{0.25}} & \text{Theorem 4.3} \\
K & = & (228.\overline{7128})^4 L^{-3} & \text{simplify}
\end{array}$$

Hence, $K = f(L) = (228.\overline{7128})^4 L^{-3}$ where $L > 0$. We find $f(193) = (228.\overline{7128})^4(193)^{-3} \approx 381$ meaning that in order to maintain a production level of 231% of 1889 with a labor level at 193% of 1889, the required capital is 381% that of 1889.[7]

3. The function $f(L)$ is a Laurent Monomial (see Section 3.1) with $n = 3$ and $a = (228.\overline{7128})^4$. As such, as $L \to 0^+$, $f(L) \to \infty$. This means that in order to maintain the given production level, as the available labor diminishes, the capital requirement become unbounded. As $L \to \infty$, we have $f(L) \to 0$ meaning that as the available labor increases, the need for capital diminishes. The graph of f is called an 'isoquant' - meaning 'same quantity.' In this context, the graph displays all combinations of labor and capital, (L, K) which result in the *same* production level, in this case, 231% of what was produced in 1889.



$K = f(L) = (228.\overline{7128})^4 L^{-3}$

□

Next, we move on to solving inequalities with power functions. As we've seen with other types of non-linear inequalities,[8] an invaluable tool for us is the Sign Diagram.

[7] The actual recorded figure is 407.

[8] see Sections 1.4, 2.3, and 3.3

Steps for Constructing a Sign Diagram for an Algebraic Function

Suppose f is an algebraic function.

1. Place any values excluded from the domain of f on the number line with an '?' above them.
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine and record the sign of $f(x)$ for each test value in step 3.

As you may recall, since sign diagrams compare functions to 0, the first step in solving inequalities using a sign diagram is to gather all the nonzero terms one one side of the inequality. We demonstrate this technique in the following example.

Example 4.3.3.
Solve the following inequalities. Check your answers graphically with a calculator.

1. $2 - \sqrt[4]{x+3} \geq 0$
2. $t^{2/3} < t^{4/3} - 6$
3. $3(2-x)^{\frac{1}{3}} \leq x(2-x)^{-\frac{2}{3}}$
4. $(t-4)^{\frac{2}{3}} \geq -\dfrac{2t}{3(t-4)^{\frac{1}{3}}}$

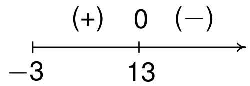
Solution.

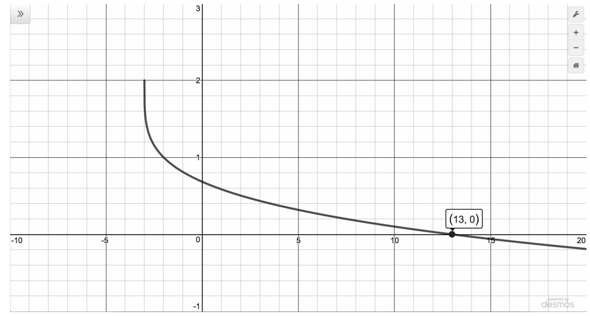
1. To solve $2 - \sqrt[4]{x+3} \geq 0$, it is tempting to rewrite this inequality as $2 \geq \sqrt[4]{x+3}$ and rid ourselves of the fourth root by raising both sides of this inequality to the fourth power. While this technique works *sometimes*, it doesn't work *all* the time since raising both sides of an inequality to the fourth (more generally, to an even) power does not necessarily preserve inequalities .[9] For that reason, we solve this inequality using a sign diagram since this technique will *always* produce a correct solution.

   We already have all the nonzero terms on one side of the inequality, so we let $r(x) = 2 - \sqrt[4]{x+3}$ and proceed to make a sign diagram. Owing to the presence of the fourth root, we know $x + 3 \geq 0$ or $x \geq -3$. Hence, we only concern ourselves with the portion of the number line representing $[3, \infty)$. Next, we find the zeros of r by solving $r(x) = 2 - \sqrt[4]{x+3} = 0$. We get $\sqrt[4]{x+3} = 2$, so $x + 3 = 16$ and we get $x = 13$. We find this solution checks in our original equation,[10] and proceed to construct the sign diagram below on the left. Since we are looking for where $r(x) = 2 - \sqrt[4]{x+3} \geq 0$, we are looking for the zeros of r along with the intervals over which $r(x)$ is $(+)$. We record our answer as $[-3, 13]$. Below on the right is the graph of $y = 2 - \sqrt[4]{x+3}$, and we can see that, indeed, the graph is above the x-axis ($y = 0$) from $[-3, 13)$ and meets the x-axis at $x = 13$, verifying our answer.

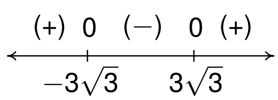
[9] For instance, $-2 \leq 1$ but $(-2)^4 \geq (1)^2$. We invite the reader to see what goes wrong if attempting to solve either of the following inequalities using this method: $-2 \geq \sqrt[4]{x+3}$, which has no solution, or $-2 \leq \sqrt[4]{x+3}$, whose solution is $[-3, \infty)$.

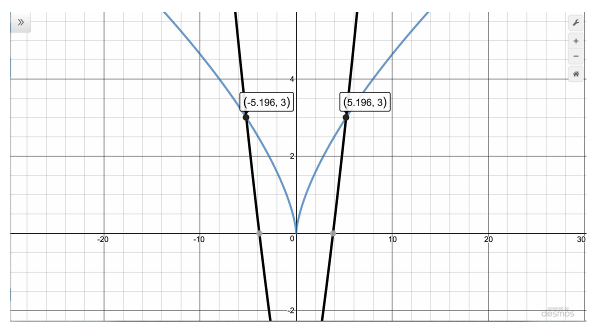
[10] Recall that raising both sides to an even power could produce extraneous solutions, so it is important we check here.





2. To solve $t^{\frac{2}{3}} < t^{\frac{4}{3}} - 6$, we first rewrite as $t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 > 0$. We set $r(t) = t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6$ and note that since the denominators in the exponents are 3, they correspond to cube roots, which means the domain of r is $(-\infty, \infty)$. To find the zeros for the sign diagram, we set $r(t) = 0$ and attempt to solve $t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 = 0$. Since there are three terms, and the exponent on one of the variable terms, $t^{\frac{4}{3}}$, is exactly twice that of the other, $t^{\frac{2}{3}}$, we have ourselves a 'quadratic in disguise.' If we let $u = t^{\frac{2}{3}}$, then $u^2 = t^{\frac{4}{3}}$, so in terms of u, we have $u^2 - u - 6 = 0$. Solving we get $u = -2$ or $u = 3$, hence $t^{\frac{2}{3}} = -2$ or $t^{\frac{2}{3}} = 3$. In root-power notation, these are $\sqrt[3]{t^2} = -2$ or $\sqrt[3]{t^2} = 3$. Cubing both sides of these equations results in $t^2 = -8$, which admits no real solution, or $t^2 = 27$, which gives $t = \pm 3\sqrt{3}$. Using these zeros, we construct the sign diagram below on the left. We find $r(t) = t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 > 0$ on $\left(-\infty, -3\sqrt{3}\right) \cup \left(3\sqrt{3}, \infty\right)$. To check our answer graphically, we set $f(t) = t^{\frac{2}{3}}$ and $g(t) = t^{\frac{4}{3}} - 6$. The solution to $t^{\frac{2}{3}} < t^{\frac{4}{3}} - 6$ corresponds to the inequality $f(t) < g(t)$, which means we are looking for the t values for which the graph of f is *below* the graph of g. On the graph below on the right, we see the graph of f (the lighter colored curve) is below the graph of g (the darker colored curve) for $t < -5.196$ and again for $t > 5.196$, which are the grapher's approximations to $\pm 3\sqrt{3}$.





3. To solve $3(2-x)^{\frac{1}{3}} \leq x(2-x)^{-\frac{2}{3}}$, we first gather all the nonzero terms to one side and obtain $3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \leq 0$. Setting $r(x) = 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}}$, we note since the denominators of the rational exponents are odd, we have no domain concerns owing to even indexed roots. However, the negative exponent on the second term indicates a denominator. Rewriting $r(x)$ with positive exponents, we obtain

$$r(x) = 3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}}$$

Setting the denominator equal to zero we get $(2-x)^{\frac{2}{3}} = 0$, which reduces to $2 - x = 0$, or $x = 2$. Hence, the domain of r is $(-\infty, 2) \cup (2, \infty)$.

To find the zeros of r, we set $r(x) = 0$, so we set about solving

$$3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}} = 0.$$

Clearing denominators, we get $3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}} - x = 0$. Since the denominators of the exponents are odd, we may use Theorem 4.3 to simplify this to $3(2-x)^1 - x = 0$, and obtain $6-4x = 0$ or $x = \frac{3}{2}$. In order for us to be able to more easily determine the sign of $r(x)$ at the test values, we rewrite $r(x)$ as a single term.[11] There are two schools of thought on how to proceed, so we demonstrate both.

- *Factoring Approach.* From $r(x) = 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}}$, we note that the quantity $(2-x)$ is common to both terms. When we factor out common factors, we factor out the quantity with the *smaller* exponent. In this case, since $-\frac{2}{3} < \frac{1}{3}$, we factor $(2-x)^{-\frac{2}{3}}$ from both quantities. While it may seem odd to do so, we need to factor $(2-x)^{-\frac{2}{3}}$ *from* $(2-x)^{\frac{1}{3}}$, which results in subtracting the exponent $-\frac{2}{3}$ from $\frac{1}{3}$. We proceed using the usual properties of exponents.

$$\begin{aligned}
r(x) &= 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \\
&= (2-x)^{-\frac{2}{3}} \left[3(2-x)^{\frac{1}{3}-\left(-\frac{2}{3}\right)} - x\right] \\
&= (2-x)^{-\frac{2}{3}} \left[3(2-x)^{\frac{3}{3}} - x\right] \\
&= (2-x)^{-\frac{2}{3}} \left[3(2-x)^{1} - x\right] \\
&= (2-x)^{-\frac{2}{3}} (6-4x) \\
&= (2-x)^{-\frac{2}{3}} (6-4x)
\end{aligned}$$

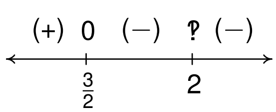
Written without negative exponents, we have $r(x) = \dfrac{6-4x}{(2-x)^{\frac{2}{3}}}$.

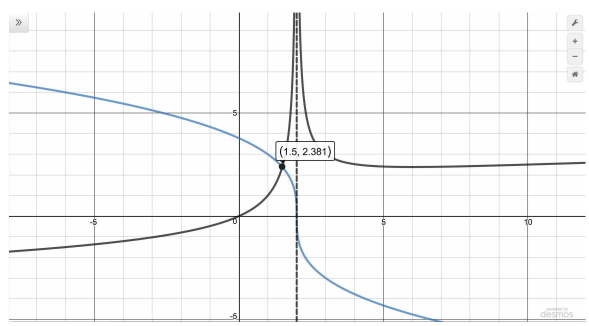
[11] This also gives us a chance to review some good intermediate algebra!

- *Common Denominator Approach.* We rewrite

$$\begin{aligned}
r(x) &= 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \\
&= 3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} && \text{common denominator} \\
&= \frac{3(2-x)^{\frac{1}{3}+\frac{2}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} && \text{Theorem 4.3} \\
&= \frac{3(2-x)^{\frac{3}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{3(2-x)^{1}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{3(2-x)-x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{6-4x}{(2-x)^{\frac{2}{3}}}
\end{aligned}$$

Using either approach, we end up with the same, simpler, expression for $r(x)$ and we use that to create our sign diagram as shown below on the left. We find $r(x) \leq 0$ on $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$. To check this graphically, we set $f(x) = 3(2-x)^{\frac{1}{3}}$ (the lighter curve) and $g(x) = x(2-x)^{-\frac{2}{3}}$ (the darker curve). We confirm that the graphs intersect at $x = \frac{3}{2}$ and the graph of f is below the graph of g for $x > \frac{3}{2}$, with the exception of $x = 2$ where it appears the graph of g has a vertical asymptote.

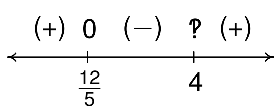


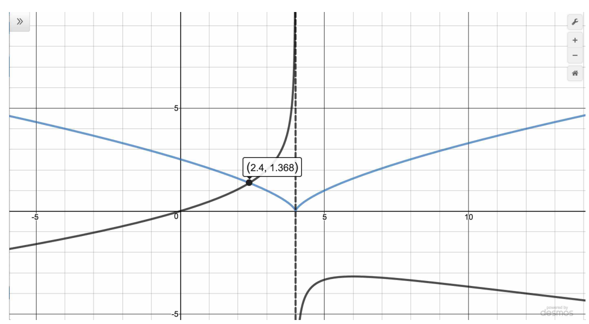


4. While it may be tempting to begin solving our last inequality by clearing denominators, owing to the odd root, the quantity $3(t-4)^{\frac{1}{3}}$ can be both positive and negative for different values of t. This means that if we chose to multiply both sides of our inequality by this quantity, we have no guarantee if the inequality would be preserved. Hence we proceed as usual by gathering all the nonzero terms to one side, and, with the ultimate goal of creating a sign diagram, get common denominators.

$$
\begin{aligned}
(t-4)^{\frac{2}{3}} &\geq -\frac{2t}{3(t-4)^{\frac{1}{3}}} \\
(t-4)^{\frac{2}{3}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} &\geq 0 \\
\frac{(t-4)^{\frac{2}{3}} \cdot 3(t-4)^{\frac{1}{3}}}{3(t-4)^{\frac{1}{3}}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} &\geq 0 \qquad \text{common denominator} \\
\frac{3(t-4)^{\frac{2}{3}+\frac{1}{3}}}{3(t-4)^{\frac{1}{3}}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} &\geq 0 \qquad \text{Theorem 4.3} \\
\frac{3(t-4)^{1}}{3(t-4)^{\frac{1}{3}}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} &\geq 0 \\
\frac{3(t-4)+2t}{3(t-4)^{\frac{1}{3}}} &\geq 0 \\
\frac{5t-12}{3(t-4)^{\frac{1}{3}}} &\geq 0
\end{aligned}
$$

We identify $r(t)$ as the left hand side of the inequality and see right away we must exclude $t = 4$ from the domain owing to the quantity $(t-4)$ in the denominator. As we have already mentioned, the root here (3) is odd, so we have no domain issues stemming from that. To find the zeros of r, we set $r(t) = 0$ which quickly reduces to solving $5t - 12 = 0$. We get $t = \frac{12}{5}$. From the sign diagram, we find $r(t) \geq 0$ on $\left(-\infty, \frac{5}{12}\right] \cup (4, \infty)$. Graphing $f(t) = (t-4)^{\frac{2}{3}}$ (the lighter curve) and $g(t) = -\frac{2t}{3(t-4)^{\frac{1}{3}}}$ (the darker curve), we see the graph of f is above the graph of g for $t < 2.4$ and again for $t > 4$, with an intersection point at $t = 2.4 = \frac{12}{5}$.





□

Note that in Example 4.3.3 number 3, since $(2-x)^{\frac{2}{3}}$ is always positive for $x \neq 2$ (owing to the squared exponent), we *could* have short-cut the sign diagram, choosing to clear denominators instead:

$$\begin{array}{rcl} 3(2-x)^{\frac{1}{3}} & \leq & x(2-x)^{-\frac{2}{3}} \\ 3(2-x)^{\frac{1}{3}} & \leq & \dfrac{x}{(2-x)^{\frac{2}{3}}} \\ \left[3(2-x)^{\frac{1}{3}}\right]\left[(2-x)^{\frac{2}{3}}\right] & \leq & \dfrac{x}{(2-x)^{\frac{2}{3}}}\left[(2-x)^{\frac{2}{3}}\right] \quad \text{provided } x \neq 2 \\ 3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}} & \leq & x \\ 3(2-x)^{\frac{1}{3}+\frac{2}{3}} & \leq & x \\ 3(2-x) & \leq & x \end{array}$$

Hence, we get $6-3x \leq x$ or $x \geq \frac{3}{2}$, provided $x \neq 2$. This matches our solution $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$. If, on the other hand, we tried this same manipulation with number 4, we would clear denominators, assuming $t \neq 4$ to obtain $3(t-4) \geq -2t$ or $t \geq \frac{12}{5}$ which is *not* the correct solution. The moral of the story is the more you understand, the less you need to rely on memorized processes and the more efficient your solution methodologies can become. The sign diagram algorithm is a fail-safe method, but, in some cases, may be far from the most efficient one. It's always best to understand the *why* of a procedure as much as the *how*.

4.3.1 Exercises

In Exercises 1 - 30, solve the equation or inequality.

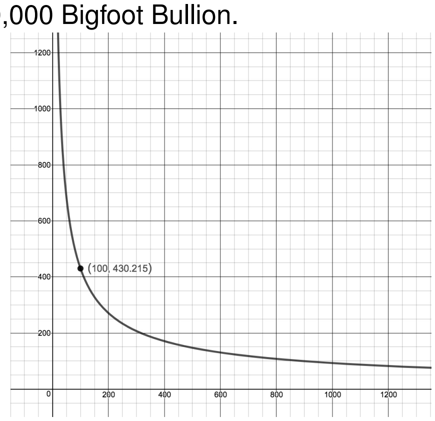
1. $x + 1 = (3x + 7)^{\frac{1}{2}}$
2. $2x + 1 = (3 - 3x)^{\frac{1}{2}}$
3. $t + (3t + 10)^{0.5} = -2$
4. $3t + (6 - 9t)^{0.5} = 2$
5. $x^{-1.5} = 8$
6. $2x - 1 = (x + 1)^{-0.5}$
7. $t^{\frac{2}{3}} = 4$
8. $(t - 2)^{\frac{1}{2}} + (t - 5)^{\frac{1}{2}} = 3$
9. $(2x + 1)^{\frac{1}{2}} = 3 + (4 - x)^{\frac{1}{2}}$
10. $5 - (4 - 2x)^{\frac{2}{3}} = 1$
11. $2t^{\frac{2}{3}} = 6 - t^{\frac{1}{3}}$
12. $2t^{\frac{1}{3}} = 1 - 3t^{\frac{2}{3}}$
13. $2x^{1.5} = 15x^{0.75} + 8$
14. $35x^{-0.75} = x^{-1.5} + 216$
15. $10 - \sqrt{t - 2} \leq 11$
16. $t^{\frac{2}{3}} \leq 4$
17. $\sqrt[3]{x} \leq x$
18. $(2 - 3x)^{\frac{1}{3}} > 3x$
19. $(t^2 - 1)^{-\frac{1}{2}} \geq 2$
20. $(t^2 - 1)^{-\frac{1}{3}} \leq 2$
21. $3(x - 1)^{\frac{1}{3}} + x(x - 1)^{-\frac{2}{3}} \geq 0$
22. $3(x - 1)^{\frac{2}{3}} + 2x(x - 1)^{-\frac{1}{3}} \geq 0$
23. $2(t - 2)^{-\frac{1}{3}} - \frac{2}{3}t(t - 2)^{-\frac{4}{3}} \leq 0$
24. $-\frac{4}{3}(t - 2)^{-\frac{4}{3}} + \frac{8}{9}t(t - 2)^{-\frac{7}{3}} \geq 0$
25. $2x^{-\frac{1}{3}}(x - 3)^{\frac{1}{3}} + x^{\frac{2}{3}}(x - 3)^{-\frac{2}{3}} \geq 0$
26. $\sqrt[3]{x^3 + 3x^2 - 6x - 8} > x + 1$
27. $4(7 - t)^{0.75} - 3t(7 - t)^{-0.25} \leq 0$
28. $4t^{0.75}(t - 3)^{-\frac{2}{3}} + 9t^{-0.25}(t - 3)^{\frac{1}{3}} < 0$
29. $x^{-\frac{1}{3}}(x - 3)^{-\frac{2}{3}} - x^{-\frac{4}{3}}(x - 3)^{-\frac{5}{3}}(x^2 - 3x + 2) \geq 0$
30. $\frac{2}{3}(t + 4)^{\frac{3}{5}}(t - 2)^{-\frac{1}{3}} + \frac{3}{5}(t + 4)^{-\frac{2}{5}}(t - 2)^{\frac{2}{3}} \geq 0$
31. The Cobb-Douglas production model[12] for the country of Sasquatchia is $P = 1.25L^{0.4}K^{0.6}$. Here, P represents the country's production (measured in thousands of Bigfoot Bullion), L represents the total labor (measured in thousands of hours) and K represents the total investment in capital (measured in Bigfoot Bullion.)
    (a) Let $P = 300$ and solve for K as a function of L. If $L = 100$, what is K? Interpret each of the quantities in this case.
    (b) Graph your answer to 31a using a graphing utility. What information does an ordered pair (L, K) on this graph represent?

[12] See Example 4.3.2 for more details on these sorts of models.

4.3.2 Answers

1. $x = 3$
2. $x = \frac{1}{4}$
3. $t = -3$
4. $t = -\frac{1}{3},\ \frac{2}{3}$
5. $x = \frac{1}{4}$
6. $x = \frac{\sqrt{3}}{2}$
7. $t = \pm 8$
8. $t = 6$
9. $x = 4$
10. $x = -2, 6$
11. $t = -8, \frac{27}{8}$
12. $t = -1, \frac{1}{27}$
13. $x = 16$
14. $x = \frac{1}{81}, \frac{1}{16}$
15. $[2, \infty)$
16. $[-8, 8]$
17. $[-1, 0] \cup [1, \infty)$
18. $\left(-\infty, \frac{1}{3}\right)$
19. $\left[-\frac{\sqrt{5}}{2}, -1\right) \cup \left(1, \frac{\sqrt{5}}{2}\right]$
20. $\left(-\infty, -\frac{3\sqrt{2}}{4}\right] \cup (-1, 1) \cup \left[\frac{3\sqrt{2}}{4}, \infty\right)$
21. $\left[\frac{3}{4}, 1\right) \cup (1, \infty)$
22. $\left(-\infty, \frac{3}{5}\right] \cup (1, \infty)$
23. $(-\infty, 2) \cup (2, 3]$
24. $(2, 6]$
25. $(-\infty, 0) \cup [2, 3) \cup (3, \infty)$
26. $(-\infty, -1)$
27. $[4, 7)$
28. $\left(0, \frac{27}{13}\right)$
29. $(-\infty, 0) \cup (0, 3)$
30. $(-\infty, -4) \cup \left(-4, -\frac{22}{19}\right] \cup (2, \infty)$
31. (a) $K = f(L) = (240)^{\frac{5}{3}} L^{-\frac{2}{3}}$. $f(100) \approx 430.2148$. This means in order for the production level of Sasquatchia to reach 300,000 Bigfoot Bullion with a labor investment of 100,000 hours, the country needs to invest approximately 430 Bigfoot Bullion into capital.

    (b) If a point (L, K) is on the graph of this function, it means a combination of L thousand hours of labor with an investment of K Bigfoot Bullion into the Sasquatian Economy will result in a production level of 300,000 Bigfoot Bullion.



Chapter 5

Further Topics on Functions

5.1 Graphs of Functions

Up until this point in the text, we have primarily focused on studying particular *families* of functions. These families and their relationships to one another provide useful *examples* of more abstract function structures and relationships. The notions introduced in this chapter will not only provide us a more formal vocabulary with which to describe the connections between the function families we have already studied, but, more importantly, give us additional lenses through which to view new families of functions that we'll encounter.

In this section, we review of the concepts associated with the graphs of functions. We introduced the notion of the graph of a function in Section 1.1, and the vast majority of the graphs we have encountered in this text were generated from an algebraic representation of a function. In this section, we define the functions geometrically from the outset and review the important concepts associated with the graphs of functions.

Recall the **domain** of a function is the set of inputs to the function and the **range** of a function is the set of outputs from the function. When graphing a function whose domain and range are subsets of real numbers, we plot the ordered pairs (input, output) on the Cartesian plane. Hence, the domain values are found on the horizontal axis while the range values are found on the vertical axis.

Recall from Definition 1.3 that the largest output from the function (if there is one) is called the **maximum** or, when there may be some confusion, the **absolute maximum** of the function. Likewise, the smallest output from the function (again, if there is one) is called the **minimum** or **absolute minimum**.

A concept related to 'absolute' maximum and minimum is the concept of 'local' maximum and minimum as described in Definition 2.7. Here, a point (a, b) on the graph of a function f is a **local maximum** if b is the maximum function value for some open interval in the domain containing a. The notion of 'local' here meaning instead of surveying the entire domain, we instead restrict our attention to inputs 'local' or 'near' the input a. The concept of **local minimum** is defined similarly.

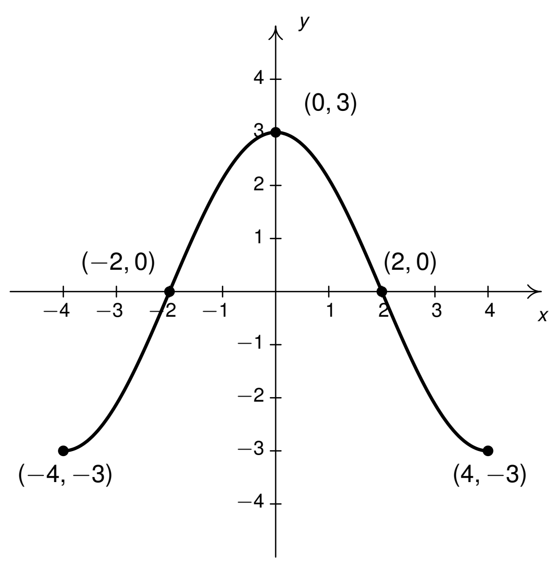
Next, we review the notions of **increasing**, **decreasing**, and **constant** as described in Definition 1.7. Recall a function is increasing over an interval if, as the inputs increase, do the outputs. This means that, geometrically, the graph of the function rises as we move left to right. Similarly, a function is decreasing over an interval if the outputs decrease as the inputs increase. Geometrically, a decreasing function falls as we move left to right. Finally, a function is constant over an interval if the output is the same regardless of the input. If a function is constant over an interval, its graph remains 'flat' - a horizontal line.

Last, and according to some[1] least, we briefly review the notion of symmetry in the graphs of functions. Recall from Definition 2.2 that a function f is called **even** if $f(-x) = f(x)$ for all x in the domain of f. The graphs of even functions are symmetric about the vertical (usually y-) axis. In a similar manner, Definition 2.3 tells us a function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f. Geometrically, the graphs of odd functions are symmetric about the origin.

The next example reviews all of the aforementioned concepts as well as many more.

Example 5.1.1. Given the graph of $y = f(x)$ below, answer all of the following questions.

1. Find the domain of f.
2. Find the range of f.
3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the x-intercepts, if any exist.
6. List the y-intercepts, if any exist.
7. Find the zeros of f.
8. Solve $f(x) < 0$.
9. Determine $f(2)$.
10. Solve $f(x) = -3$.
11. Find the number of solutions to $f(x) = 1$.
12. Does f appear to be even, odd, or neither?
13. List the local maximums, if any exist.
14. List the local minimums, if any exist.
15. List the intervals on which f is increasing.
16. List the intervals on which f is decreasing.



[1] Jeff

Solution.

1. To find the domain of f, we proceed as in Section 1.1. By projecting the graph to the x-axis, we see that the portion of the x-axis which corresponds to a point on the graph is everything from -4 to 4, inclusive. Hence, the domain is $[-4, 4]$.

2. To find the range, we project the graph to the y-axis. We see that the y values from -3 to 3, inclusive, constitute the range of f. Hence, our answer is $[-3, 3]$.

3. The maximum value of f is the largest y-coordinate which is 3.

4. The minimum value of f is the smallest y-coordinate which is -3.

5. The x-intercepts are the points on the graph with y-coordinate 0, namely $(-2, 0)$ and $(2, 0)$.

6. The y-intercept is the point on the graph with x-coordinate 0, namely $(0, 3)$.

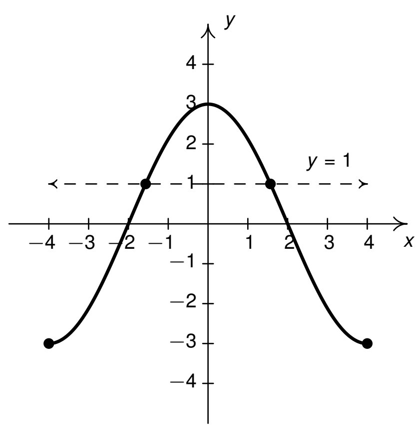
7. The zeros of f are the x-coordinates of the x-intercepts of the graph of $y = f(x)$ which are $x = -2, 2$.

8. To solve $f(x) < 0$, we look for the x values of the points on the graph where the $y = f(x)$ is negative. Graphically, we are looking for where the graph is *below* the x-axis. This happens for the x values from -4 to -2 and again from 2 to 4. So our answer is $[-4, -2) \cup (2, 4]$.

9. Since the graph of f is the graph of the equation $y = f(x)$, $f(2)$ is the y-coordinate of the point which corresponds to $x = 2$. Since the point $(2, 0)$ is on the graph, we have $f(2) = 0$.

10. To solve $f(x) = -3$, we look where $y = f(x) = -3$. We find two points with a y-coordinate of -3, namely $(-4, -3)$ and $(4, -3)$. Hence, the solutions to $f(x) = -3$ are $x = \pm 4$.

11. As in the previous problem, to solve $f(x) = 1$, we look for points on the graph where the y-coordinate is 1. If we imagine the horizontal like $y = 1$ superimposed over the graph of f as sketched below, we get two intersections. Hence, even though these points aren't specified, we know there are *two* points on the graph of f whose y-coordinate is 1. Hence, there are two solutions to $f(x) = 1$.



12. The graph appears to be symmetric about the y-axis. This suggests[2] that f is even.

13. The function has its only local maximum at $(0,3)$.

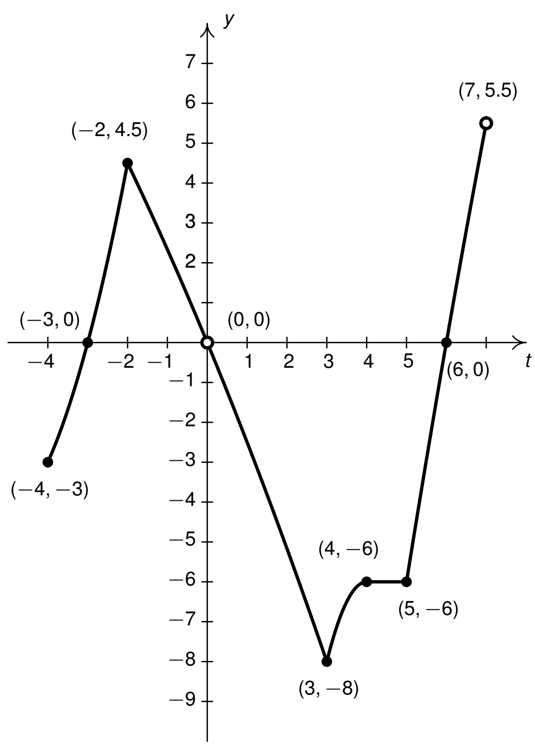
14. There are no local minimums. Why don't $(-4,-3)$ and $(4,-3)$ count? Let's consider the point $(-4,-3)$ for a moment. Recall that, in the definition of local minimum, there needs to be an open interval containing $x = -4$ which is in the domain of f. In this case, there is no open interval containing $x = -4$ which lies entirely in the domain of f, $[-4,4]$. Because we are unable to fulfill the requirements of the definition for a local minimum, we cannot claim that f has one at $(-4,-3)$. The point $(4,-3)$ fails for the same reason – no open interval around $x = 4$ stays within the domain of f.

15. As we move from left to right, the graph rises from $(-4,-3)$ to $(0,3)$. This means f is increasing on the interval $[-4,0]$. (Remember, the answer here is an interval on the x-axis.)

16. As we move from left to right, the graph falls from $(0,3)$ to $(4,-3)$. This means f is decreasing on the interval $[0,4]$. (Again, the answer here is an interval on the x-axis.) $\square$

Our next example involves a more complicated function and asks more complicated questions.

Example 5.1.2. Consider the graph of the function g below.



The graph of $y = g(t)$

[2]but does not prove

1. Find the domain of g.
2. Find the range of g.
3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the local maximums, if any exist.
6. List the local minimums, if any exist.
7. Solve $(t^2 - 25)g(t) = 0$.
8. Solve $\dfrac{g(t)}{t^2 + t - 30} \geq 0$.

Solution.

1. Projecting the graph of g to the t-axis, we see the domain contains values of t from -4 up to, but not including $t = 0$ and values greater than $t = 0$ up to, but not including $t = 7$. Using interval notation, we write the domain as $[-4, 0) \cup (0, 7)$.

2. Projecting the graph of g to the y-axis, we see the range of g contains all real numbers from $y = -8$ up to, but not including, $y = 5.5$. Note that even though there is a hole in the graph at $(0, 0)$, the points $(-3, 0)$ and $(6, 0)$ put $y = 0$ in the range of g. Hence, the range of g is $[-8, 5.5)$.

3. Owing to the hole in the graph at $(7, 5.5)$, g has no maximum.[3]

4. The minimum of g is -8 which occurs at the point $(3, -8)$.

5. The point $(-2, 4.5)$ is clearly a local maximum, but there are actually infinitely many more. Per Definition 2.7, all points of the form $(t, -6)$ for $4 \leq t < 5$ are also local maximums. For each of these points, we can find an open interval on the t axis within which we produce no points on the graph higher than $(t, -6)$. (You may think about 'zooming in' on the point $(4.5, -6)$ to see how this works.)

6. The local minimums of the graph are $(3, -8)$ along with points of the form $(t, -6)$ for $4 < t \leq 5$. Note the point $(-4, -3)$ is not a local minimum since there is no open interval containing $t = -4$ which lies entirely within the domain of g.

7. To solve $(t^2 - 25)g(t) = 0$, we use the zero product property of real numbers[4] to conclude either $t^2 - 25 = 0$ or $g(t) = 0$.

   From $t^2 - 25 = 0$, we get $t = \pm 5$. However, since $t = -5$ isn't in the domain of g, it cannot be regarded as a solution to the equation $(t^2 - 25)g(t) = 0$. (If we substitute $t = -5$ into the equation, we'd get $((-5)^2 - 25)g(-5) = 0 \cdot g(-5)$. Since $g(-5)$ is undefined, so is $0 \cdot g(-5)$.)

   To solve $g(t) = 0$, we look for the zeros of g which are $t = -3$ and $t = 6$. (Again, there is a hole at $(0, 0)$, so $t = 0$ doesn't count as a zero.) Our final answer to $(t^2 - 25)g(t) = 0$ is $t = -3$, 5, or 6.

[3] There is no real number 'right before' 5.5 . . .

[4] see Section A.2, 1333

8. To solve $\frac{g(t)}{t^2+t-30} \geq 0$, we employ a sign diagram as we (most recently) have done in Section 4.3.[5] To that end, we define $F(t) = \frac{g(t)}{t^2+t-30}$ and we set about finding the domain of f.

   First, we note that since F is defined in terms of g, the domain of F is restricted to some subset of the domain of g, namely $[-4, 0) \cup (0, 7)$. Since $t^2 + t - 30$ is in the denominator of $F(t)$, we must also exclude the values where $t^2+t-30 = (t+6)(t-5) = 0$. Hence, we must exclude $t = -6$ (which isn't in the domain of g in the first place) along with $t = 5$. Hence, the domain of F is $[-4, 0) \cup (0, 5) \cup (5, 7)$.

   Next, we find the zeros of F. Setting $F(t) = \frac{g(t)}{t^2+t-30} = 0$ amounts to solving $g(t) = 0$. Graphically, we see this occurs when $t = -3$ and $t = 6$. Hence, we need to select test values in each of the following intervals: $[-4, -3)$, $(-3, 0)$, $(0, 5)$, $(5, 6)$ and $(6, 7)$.

   For the interval $[-4, -3)$, we may choose $t = -4$. $F(-4) = \frac{g(-4)}{(-4)^2+(-4)-30} = \frac{-3}{-18} > 0$ so is $(+)$. For the interval $(-3, 0)$ we choose $t = -2$ and get $F(-2) = \frac{g(-2)}{(-2)^2+(-2)-30} = \frac{4.5}{-28} < 0$ so is $(-)$. For the interval $(0, 5)$, we choose $t = 3$ and find $F(3) = \frac{g(3)}{(3)^2+(3)-30} = \frac{-8}{-18} > 0$ which is $(+)$ again.

   For the last two intervals, $(5, 6)$ and $(6, 7)$, we do not have specific function values for g. However, all we are interested in is the *sign* of the function over these intervals, and we can get that information about g graphically.

   For the interval $(5, 6)$, we choose $t = 5.5$ as our test value. Since the graph of $y = g(t)$ is *below* the t-axis when $t = 5.5$. we know $g(5.5)$ is $(-)$. Hence, $F(5.5) = \frac{g(5.5)}{(5.5)^2+(5.5)-30} = \frac{(-)}{5.75} < 0$ so is $(-)$. Similarly, when $t = 6.5$, the graph of $y = g(t)$ is *above* the t-axis so $F(6.5) = \frac{g(6.5)}{(6.5)^2+(6.5)-30} = \frac{(+)}{18.75} > 0$ so is $(+)$. Putting all of this together, we get the sign diagram for $F(t) = \frac{g(t)}{t^2+t-30}$ below:

$(+)$	0 $(-)$	‽ $(+)$	‽ $(-)$	0 $(+)$	‽
-4	-3	0	5	6	7

   Hence, $F(t) \geq 0$ on $[-4, -3] \cup (0, 5) \cup [6, 7)$. □

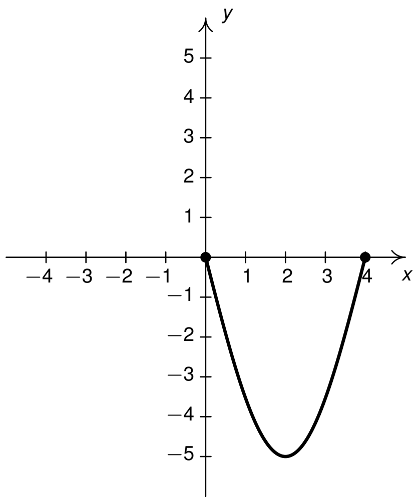
Our last example focuses on symmetry. The reader is encouraged to review the notes about symmetry as summarized on page 1357 in Section A.3.

Example 5.1.3. Below are the partial graphs of functions f and g.

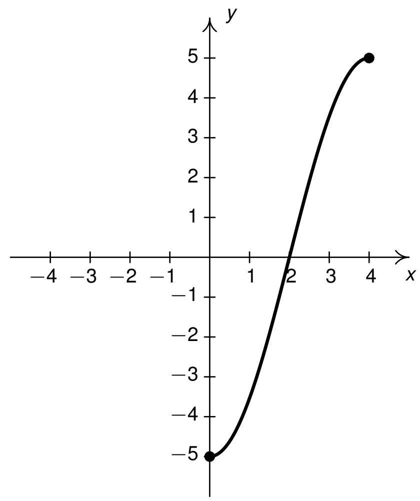
1. If possible, complete the graphs of f and g assuming both functions are even.

2. If possible, complete the graphs of f and g assuming both functions are odd.

[5] Note that g is continuous on its domain, and hence, it follows that $\frac{g(t)}{t^2+t-30}$ is, too. (Thank Calculus!) This means the Intermediate Value Theorem applies so a Sign Diagram approach is valid.



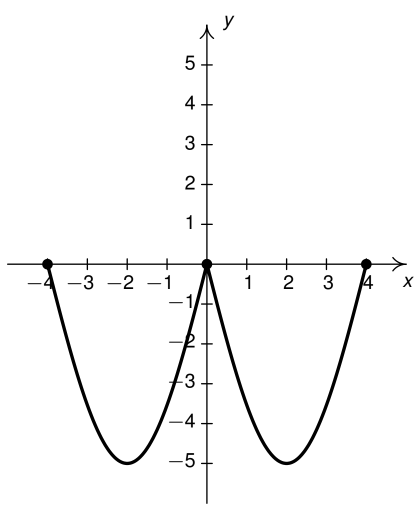
Partial graph of $y = f(x)$



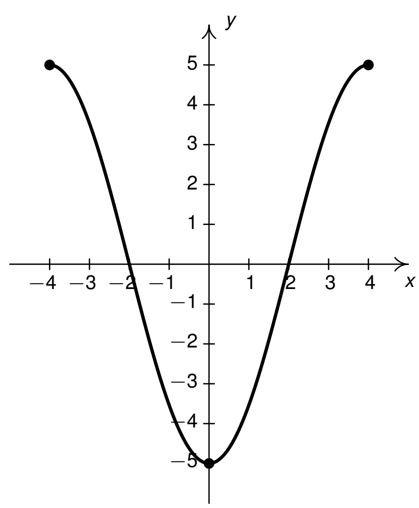
Partial graph of $y = g(x)$

Solution.

1. If f and g are even then their graphs are symmetric about the y-axis. Hence, to complete each graph, we reflect each point on the graphs of f and g about the y-axis.



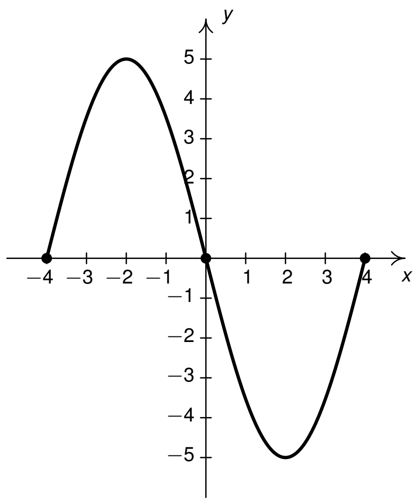
The graph of f assuming f is even.



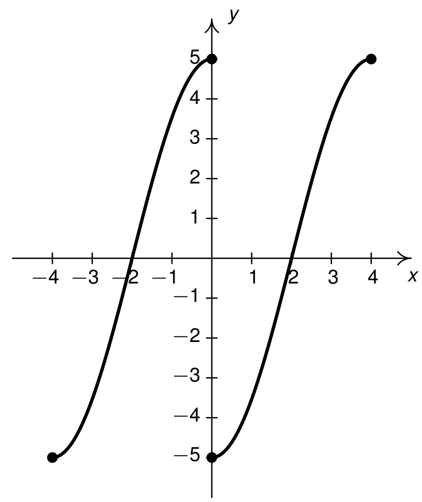
The graph of g assuming g is even.

2. If f and g are odd then their graphs are symmetric about the origin. Hence, to complete each graph, we imagine reflecting each of the points on their graphs through the origin. We complete the process on the graph of f with no issues.

   However, when attempting to do the same with the graph of the function g, we find the point $(0, -5)$ is reflected to the point $(0, 5)$. Hence, this new graph doesn't pass the vertical line test and hence is not a function. Therefore, g cannot be odd.[6]



The graph of f assuming f is odd.



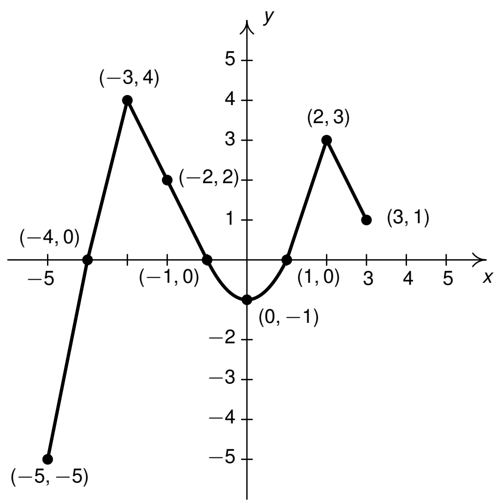
This graph fails the vertical line test.

□

[6]We leave it as an exercise to show that if a function f is odd and 0 is in the domain of f, then, necessarily, $f(0) = 0$.

5.1.1 Exercises

In Exercises 1 - 4, use the graph of $y = f(x)$ given below to answer the question.



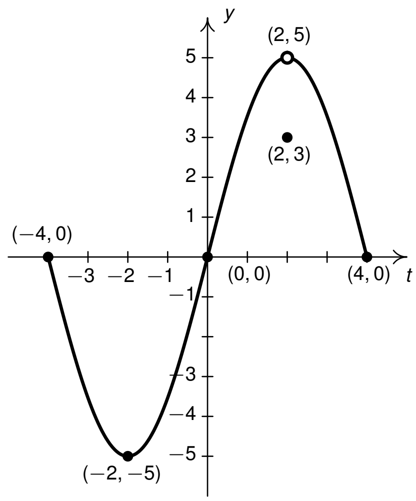
$y = f(x)$

1. Find the domain of f.
2. Find the range of f.
3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the local maximums, if any exist.
6. List the local minimums, if any exist.
7. List the intervals where f is increasing.
8. List the intervals where f is decreasing.
9. Determine $f(-2)$.
10. Solve $f(x) = 4$.
11. List the x-intercepts, if any exist.
12. List the y-intercepts, if any exist.
13. Find the zeros of f.
14. Solve $f(x) \geq 0$.
15. Find the number of solutions to $f(x) = 1$.
16. Find the number of solutions to $|f(x)| = 1$.
17. Solve $(x^2 - x - 2)f(x) = 0$
18. Solve $(x^2 - x - 2)f(x) > 0$

With help from your classmates:

19. Find the domain of $R(x) = \dfrac{1}{f(x)}$
20. Find the range of $R(x) = \dfrac{1}{f(x)}$

In Exercises 21 - 24, use the graph of $y = g(t)$ given below to answer the question.

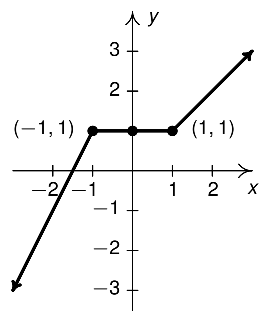


$y = g(t)$

21. Find the domain of g.
22. Find the range of g.
23. Find the maximum, if it exists.
24. Find the minimum, if it exists.
25. List the local maximums, if any exist.
26. List the local minimums, if any exist.
27. List the intervals where g is increasing.
28. List the intervals where g is decreasing.
29. Determine $g(2)$.
30. Solve $g(t) = -5$.
31. List the t-intercepts, if any exist.
32. List the y-intercepts, if any exist.
33. Find the zeros of g.
34. Solve $g(t) \leq 0$.
35. Find the domain of $G(t) = \dfrac{g(t)}{t+2}$.
36. Solve $\dfrac{g(t)}{t+2} \leq 0$.
37. How many solutions are there to $[g(t)]^2 = 9$?
38. Does g appear to be even, odd, or neither?
39. Prove that if f is an odd function and 0 is in the domain of f, then $f(0) = 0$.
40. Let $R(x)$ be the function defined as: $R(x) = 1$ if x is a rational number, $R(x) = 0$ if x is an irrational number. With help from your classmates, try to graph R. What difficulties do you encounter?

    NOTE: Between every pair of real numbers, there is both a rational and an irrational number ...

41. Consider the graph of the function f given below.



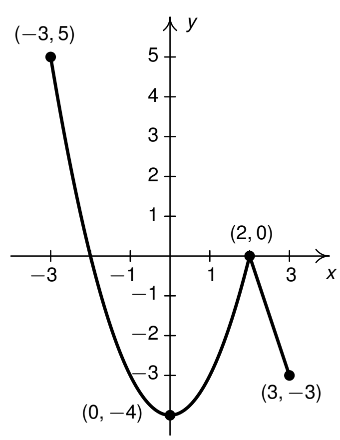
(a) Explain why f has a local maximum but not a local minimum at the point $(-1, 1)$.

(b) Explain why f has a local minimum but not a local maximum at the point $(1, 1)$.

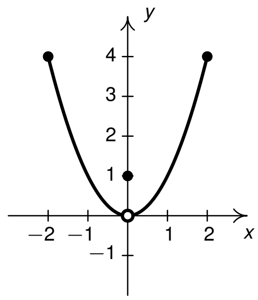
(c) Explain why f has a local maximum AND a local minimum at the point $(0, 1)$.

(d) Explain why f is constant on the interval $[-1, 1]$ and thus has both a local maximum AND a local minimum at every point $(x, f(x))$ where $-1 < x < 1$.

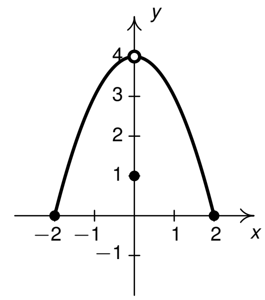
42. Explain why the function g whose graph is given below does not have a local maximum at $(-3, 5)$ nor does it have a local minimum at $(3, -3)$. Find its extrema, both local and absolute and find the intervals on which g is increasing and those on which g is decreasing.



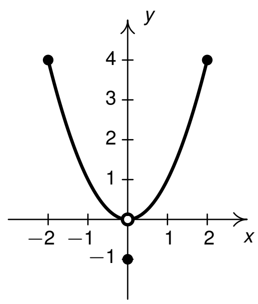
43. For each function below, find the local maximum or local minimum and list the interval over which the function is increasing and the interval over which the function is decreasing.



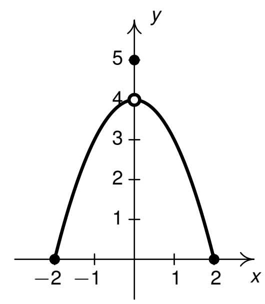
(a) Function I



(b) Function II



(c) Function III



(d) Function IV

5.1.2 Answers

1. $[-5, 3]$
2. $[-5, 4]$
3. $f(-3) = 4$
4. $f(-5) = -5$
5. $(-3, 4)$, $(2, 3)$
6. $(0, -1)$
7. $[-5, -3]$, $[0, 2]$
8. $[-3, 0]$, $[2, 3]$
9. $f(-2) = 2$
10. $x = -3$
11. $(-4, 0)$, $(-1, 0)$, $(1, 0)$
12. $(0, -1)$
13. -4, -1, 1
14. $[-4, -1]$, $[1, 3]$
15. 4
16. 6
17. $x = -4, -1, 1, 2$
18. $(-4, -1) \cup (-1, 1) \cup (2, 3)$
19. To find the domain of $R(x) = \frac{1}{f(x)}$, we start with the domain of f and exclude values where $f(x) = 0$. Hence, the domain of R is $[-5, -4) \cup (-4, -1) \cup (-1, 1) \cup (1, 3]$.
20. To find the range of $R(x) = \frac{1}{f(x)}$, we start with the range of f (excluding 0) and take reciprocals. If $-5 \leq y < 0$, then $\frac{1}{y} \leq -\frac{1}{5}$. If $0 < y \leq 4$, then $\frac{1}{y} \geq \frac{1}{4}$. Hence the range of R is $\left(-\infty, -\frac{1}{5}\right] \cup \left[\frac{1}{4}, \infty\right)$.
21. $[-4, 4]$
22. $[-5, 5)$
23. none
24. $g(-2) = -5$
25. none
26. $(-2, -5)$, $(2, 3)$
27. $[-2, 2)$
28. $[-4, -2]$, $(2, 4]$
29. $g(2) = 3$
30. $t = -2$
31. $(-4, 0)$, $(0, 0)$, $(4, 0)$
32. $(0, 0)$
33. -4, 0, 4
34. $[-4, 0] \cup \{4\}$
35. $[-4, -2) \cup (-2.4]$
36. $\{-4\} \cup (-2, 0] \cup \{4\}$
37. 5
38. Neither.

43. (a) Local maximum: $(0, 1)$, no local minimum. Increasing: $(0, 2]$, decreasing: $[-2, 0)$.
    (b) No local maximum, local minimum: $(0, 1)$. Increasing: $[-2, 0)$, decreasing: $(0, 2]$.
    (c) No local maximum, local minimum: $(0, -1)$. Increasing: $[0, 2]$, decreasing: $[-2, 0]$.
    (d) Local maximum: $(0, 5)$, no local minimum. Increasing: $[-2, 0]$, decreasing: $[0, 2]$.

5.2 Function Arithmetic

As we mentioned in Section 5.1, in this chapter, we are studying functions in a more abstract and general setting. In this section, we begin our study of what can be considered as the *algebra of functions* by defining *function arithmetic.*

Given two real numbers, we have four primary arithmetic operations available to us: addition, subtraction, multiplication, and division (provided we don't divide by 0.) Since the functions we study in this text have ranges which are sets of real numbers, it makes sense we can extend these arithmetic notions to functions.

For example, to add two functions means we add their outputs; to subtract two functions, we subtract their outputs, and so on and so forth. More formally, given two functions f and g, we *define* a new function $f+g$ whose rule is determined by adding the outputs of f and g. That is $(f+g)(x) = f(x)+g(x)$. While this looks suspiciously like some kind of distributive property, it is nothing of the sort. The '+' sign in the expression '$f+g$' is part of the *name* of the function we are defining,[1] whereas the plus sign '+' sign in the expression $f(x)+g(x)$ represents real number addition: we are adding the output from f, $f(x)$ with the output from g, $g(x)$ to determine the output from the sum function, $(f+g)(x)$.

Of course, in order to define $(f+g)(x)$ by the formula $(f+g)(x) = f(x)+g(x)$, both $f(x)$ and $g(x)$ need to be defined in the first place; that is, x must be in the domain of f *and* the domain of g. You'll recall[2] this means x must be in the *intersection* of the domains of f and g. We define the following.

Definition 5.1. Suppose f and g are functions and x is in both the domain of f and the domain of g.

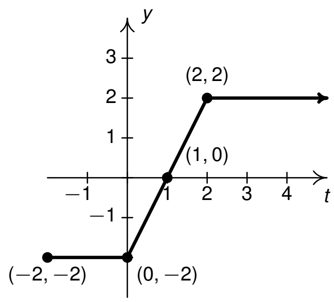
- The **sum** of f and g, denoted $f+g$, is the function defined by the formula
$$(f+g)(x) = f(x)+g(x)$$
- The **difference** of f and g, denoted $f-g$, is the function defined by the formula
$$(f-g)(x) = f(x)-g(x)$$
- The **product** of f and g, denoted fg, is the function defined by the formula
$$(fg)(x) = f(x)g(x)$$
- The **quotient** of f and g, denoted $\dfrac{f}{g}$, is the function defined by the formula
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$
provided $g(x) \neq 0$.

We put these definitions to work for us in the next example.

[1] We could have just as easily called this new function $S(x)$ for 'sum' of f and g and defined S by $S(x) = f(x)+g(x)$.
[2] see Section A.1.

Example 5.2.1. Consider the following functions:

- $f(x) = 6x^2 - 2x$
- $g(t) = 3 - \dfrac{1}{t},\ t > 0$
- $h = \{(-3, 2), (-2, 0.4), (0, \sqrt{2}), (3, -6)\}$
- s whose graph is given below:



$y = s(t)$

1. Find and simplify the following function values:

   (a) $(f + g)(1)$ (b) $(s - f)(-1)$ (c) $(fg)(2)$ (d) $\left(\dfrac{s}{h}\right)(0)$

   (e) $((s + g) + h)(3)$ (f) $(s + (g + h))(3)$ (g) $\left(\dfrac{f + h}{s}\right)(3)$ (h) $(f(g - h))(-2)$

2. Find the domain of each of the following functions:

   (a) hg (b) $\dfrac{f}{s}$

3. Find expressions for the functions below. State the domain for each.

   (a) $(fg)(x)$ (b) $\left(\dfrac{g}{f}\right)(t)$

Solution.

1. (a) By definition, $(f + g)(1) = f(1) + g(1)$. We find $f(1) = 6(1)^2 - 2(1) = 4$ and $g(1) = 3 - \frac{1}{1} = 2$. So we get $(f + g)(1) = 4 + 2 = 6$.

   (b) To find $(s - f)(-1) = s(-1) - f(-1)$, we need both $s(-1)$ and $f(-1)$. To get $s(-1)$, we look to the graph of $y = s(t)$ and look for the y-coordinate of the point on the graph with the t-coordinate of -1. While not labeled directly, we infer the point $(-1, -2)$ is on the graph which means $s(-1) = -2$. For $f(-1)$, we compute: $f(-1) = 6(-1)^2 - 2(-1) = 8$. Putting it all together, we get $(s - f)(-1) = (-2) - (8) = -10$.

   (c) Since $(fg)(2) = f(2)g(2)$, we first compute $f(2)$ and $g(2)$. We find $f(2) = 6(2)^2 - 2(2) = 20$ and $g(2) = 2 + \frac{1}{2} = \frac{5}{2}$, so $(fg)(2) = f(2)g(2) = (20)\left(\frac{5}{2}\right) = 50$.

(d) By definition, $\left(\frac{s}{h}\right)(0) = \frac{s(0)}{h(0)}$. Since $(0,-2)$ is on the graph of $y = s(t)$, so we know $s(0) = -2$. Likewise, the ordered pair $(0, \sqrt{2}) \in h$, so $h(0) = \sqrt{2}$. We get $\left(\frac{s}{h}\right)(0) = \frac{s(0)}{h(0)} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$.

(e) The expression $((s+g)+h)(3)$ involves *three* functions. Fortunately, they are grouped so that we can apply Definition 5.1 by first considering the sum of the two functions $(s+g)$ and h, then to the sum of the two functions s and g: $((s+g)+h)(3) = (s+g)(3) + h(3) = (s(3)+g(3)) + h(3)$. To get $s(3)$, we look to the graph of $y = s(t)$. We infer the point $(3,2)$ is on the graph of s, so $s(3) = 2$. We compute $g(3) = 3 - \frac{1}{3} = \frac{8}{3}$. To find $h(3)$, we note $(3,-6) \in h$, so $h(3) = -6$. Hence, $((s+g)+h)(3) = (s+g)(3) + h(3) = (s(3)+g(3)) + h(3) = \left(2 + \frac{8}{3}\right) + (-6) = -\frac{4}{3}$.

(f) The expression $(s+(g+h))(3)$ is very similar to the previous problem, $((s+g)+h)(3)$ except that the g and h are grouped together here instead of the s and g. We proceed as above applying Definition 5.1 twice and find $(s+(g+h))(3) = s(3) + (g+h)(3) = s(3) + (g(3)+h(3))$. Substituting the values for $s(3)$, $g(3)$ and $h(3)$, we get $(s+(g+h))(3) = 2 + \left(\frac{8}{3} + (-6)\right) = -\frac{4}{3}$, which, not surprisingly, matches our answer to the previous problem.

(g) Once again, we find the expression $\left(\frac{f+h}{s}\right)(3)$ has more than two functions involved. As with all fractions, we treat '$-$' as a grouping symbol and interpret $\left(\frac{f+h}{s}\right)(3) = \frac{(f+h)(3)}{s(3)} = \frac{f(3)+h(3)}{s(3)}$. We compute $f(3) = 6(3)^2 - 2(3) = 48$ and have $h(3) = -6$ and $s(3) = 2$ from above. Hence, $\left(\frac{f+h}{s}\right)(3) = \frac{f(3)+h(3)}{s(3)} = \frac{48+(-6)}{2} = 21$.

(h) We need to need to exercise caution in parsing $(f(g-h))(-2)$. In this context, f, g, and h are all functions, so we interpret $(f(g-h))$ as the function and -2 as the argument. We view the function $f(g-h)$ as the product of f and the function $g-h$. Hence, $(f(g-h))(-2) = f(-2)[(g-h)(-2)] = f(-2)[g(-2) - h(-2)]$. We compute $f(-2) = 6(-2)^2 - 2(-2) = 28$, and $g(-2) = 3 - \frac{1}{-2} = 3 + \frac{1}{2} = \frac{7}{2} = 3.5$. Since $(-2, 0.4) \in h$, $h(-2) = 0.4$. Putting this altogether, we get $(f(g-h))(-2) = f(-2)[(g-h)(-2)] = f(-2)[g(-2) - h(-2)] = 28(3.5-0.4) = 28(3.1) = 86.8$.

2. (a) To find the domain of hg, we need to find the real numbers in both the domain of h and the domain of g. The domain of h is $\{-3,-2,0,3\}$ and the domain of g is $\{t \in \mathbb{R} \mid t > 0\}$ so the only real number in common here is 3. Hence, the domain of hg is $\{3\}$, which may be small, but it's better than nothing.[3]

(b) To find the domain of $\frac{f}{s}$, we first note the domain of f is all real numbers, but that the domain of s, based on the graph, is just $[-2,\infty)$. Moreover, $s(t) = 0$ when $t = 1$, so we must exclude this value from the domain of $\frac{f}{s}$. Hence, we are left with $[-2,1) \cup (1,\infty)$.

3. (a) By definition, $(fg)(x) = f(x)g(x)$. We are given $f(x) = 6x^2 - 2x$ and $g(t) = 3 - \frac{1}{t}$ so $g(x) = 3 - \frac{1}{x}$. Hence,

[3] Since $(hg)(3) = h(3)g(3) = (-6)\left(\frac{8}{3}\right) = -16$, we can write $hg = \{(3,-16)\}$.

$$\begin{aligned}
(fg)(x) &= f(x)g(x) \\
&= \left(6x^2 - 2x\right)\left(3 - \frac{1}{x}\right) \\
&= 18x^2 - 6x^2\left(\frac{1}{x}\right) - 2x(3) + 2x\left(\frac{1}{x}\right) \quad \text{distribute} \\
&= 18x^2 - 6x - 6x + 2 \\
&= 18x^2 - 12x + 2
\end{aligned}$$

To find the domain of fg, we note the domain of f is all real numbers, $(-\infty, \infty)$ whereas the domain of g is restricted to $\{t \in \mathbb{R} \,|\, t > 0\} = (0, \infty)$. Hence, the domain of fg is likewise restricted to $(0, \infty)$. Note if we relied solely on the **simplified formula** for $(fg)(x) = 18x^2 - 12x + 2$, we would have obtained the *incorrect* answer for the domains of fg.

(b) To find an expression for $\left(\frac{g}{f}\right)(t) = \frac{f(t)}{g(t)}$ we first note $f(t) = 6t^2 - 2t$ and $g(t) = 3 - \frac{1}{t}$. Hence:

$$\begin{aligned}
\left(\frac{g}{f}\right)(t) &= \frac{g(t)}{f(t)} \\
&= \frac{3 - \frac{1}{t}}{6t^2 - 2t} = \frac{3 - \frac{1}{t}}{6t^2 - 2t} \cdot \frac{t}{t} \quad \text{simplify compound fractions} \\
&= \frac{\left(3 - \frac{1}{t}\right)t}{\left(6t^2 - 2t\right)t} = \frac{3t - 1}{\left(6t^2 - 2t\right)t} \\
&= \frac{3t - 1}{2t^2(3t - 1)} = \frac{\cancelto{1}{(3t - 1)}}{2t^2\cancel{(3t - 1)}} \quad \text{factor and cancel} \\
&= \frac{1}{2t^2}
\end{aligned}$$

Hence, $\left(\frac{g}{f}\right)(t) = \frac{1}{2t^2} = \frac{1}{2}t^{-2}$. To find the domain of $\frac{g}{f}$, a real number must be both in the domain of g, $(0, \infty)$, and the domain of f, $(-\infty, \infty)$ so we start with the set $(0, \infty)$. Additionally, we require $f(t) \neq 0$. Solving $f(t) = 0$ amounts to solving $6t^2 - 2t = 0$ or $2t(3t - 1) = 0$. We find $t = 0$ or $t = \frac{1}{3}$ which means we need to exclude these values from the domain. Hence, our final answer for the domain of $\frac{g}{f}$ is $\left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$. Note that, once again, using the *simplified formula* for $\left(\frac{g}{f}\right)(t)$ to determine the domain of $\frac{g}{f}$, would have produced erroneous results. □

A few remarks are in order. First, in number 1 parts 1e through 1h, we first encountered combinations of *three* functions despite Definition 5.1 only addressing combinations of *two* functions at a time. It turns out that function arithmetic inherits many of the same properties of real number arithmetic. For example, we showed above that $((s + g) + h)(3) = (s + (g + h))(3)$. In general, given any three functions f, g, and h, $(f + g) + h = f + (g + h)$ that is, function addition is *assocative*. To see this, choose an element x common to the domains of f, g, and h. Then

$$\begin{array}{rcll} ((f+g)+h)(x) & = & (f+g)(x)+h(x) & \text{definiton of } ((f+g)+h)(x) \\ & = & (f(x)+g(x))+h(x) & \text{definition of } (f+g)(x) \\ & = & f(x)+(g(x)+h(x)) & \text{associative property of real number addition} \\ & = & f(x)+(g+h)(x) & \text{definition of } (g+h)(x) \\ & = & (f+(g+h))(x) & \text{definition of } (f+(g+h))(x) \end{array}$$

The key step to the argument is that $(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$ which is true courtesy of the associative property of real number addition. And just like with real number addition, because function addition is associative, we may write $f + g + h$ instead of $(f + g) + h$ or $f + (g + h)$ even though, when it comes down to computations, we can only add two things together at a time.[4]

For completeness, we summarize the properties of function arithmetic in the theorem below. The proofs of the properties all follow along the same lines as the proof of the associative property and are left to the reader. We investigate some additional properties in the exercises.

Theorem 5.1. Suppose f, g and h are functions.

- **Commutative Law of Addition:** $f + g = g + f$
- **Associative Law of Addition:** $(f + g) + h = f + (g + h)$
- **Additive Identity:** The function $Z(x) = 0$ satisfies: $f + Z = Z + f = f$ for all functions f.
- **Additive Inverse:** The function $F(x) = -f(x)$ for all x in the domain of f satisfies:

$$f + F = F + f = Z.$$

- **Commutative Law of Multiplication:** $fg = gf$
- **Associative Law of Multiplication:** $(fg)h = f(gh)$
- **Multiplicative Identity:** The function $I(x) = 1$ satisfies: $fI = If = f$ for all functions f.
- **Multiplicative Inverse:** If $f(x) \neq 0$ for all x in the domain of f, then $F(x) = \dfrac{1}{f(x)}$ satisfies:

$$fF = Ff = I$$

- **Distributive Law of Multiplication over Addition:** $f(g + h) = fg + fh$

In the next example, we decompose given functions into sums, differences, products and/or quotients of other functions. Note that there are infinitely many different ways to do this, including some trivial ones. For example, suppose we were instructed to decompose $f(x) = x + 2$ into a sum or difference of functions. We could write $f = g + h$ where $g(x) = x$ and $h(x) = 2$ or we could choose $g(x) = 2x + 3$ and $h(x) = -x - 1$.

[4] Addition is a 'binary' operation - meaning it is defined only on two objects at once. Even though we write $1+2+3 = 6$, mentally, we add just two of numbers together at any given time to get our answer: for example, $1 + 2 + 3 = (1 + 2) + 3 = 3 + 3 = 6$.

More simply, we could write $f = g + h$ where $g(x) = x + 2$ and $h(x) = 0$. We'll call this last decomposition a 'trivial' decomposition. Likewise, if we ask for a decomposition of $f(x) = 2x$ as a product, a nontrivial solution would be $f = gh$ where $g(x) = 2$ and $h(x) = x$ whereas a trivial solution would be $g(x) = 2x$ and $h(x) = 1$. In general, non-trivial solutions to decomposition problems avoid using the additive identity, 0, for sums and differences and the multiplicative identity, 1, for products and quotients.

Example 5.2.2. 1. For $f(x) = x^2 - 2x$, find functions g, h and k to decompose f nontrivially as:

(a) $f = g - h$ (b) $f = g + h$ (c) $f = gh$ (d) $f = g(h - k)$

2. For $F(t) = \dfrac{2t+1}{\sqrt{t^2-1}}$, find functions G, H and K to decompost F nontrivially as:

(a) $F = \dfrac{G}{H}$ (b) $F = GH$ (c) $F = G + H$ (d) $F = \dfrac{G+H}{K}$

Solution.

1. (a) To decompose $f = g - h$, we need functions g and h so $f(x) = (g - h)(x) = g(x) - h(x)$. Given $f(x) = x^2 - 2x$, one option is to let $g(x) = x^2$ and $h(x) = 2x$. To check, we find $(g - h)(x) = g(x) - h(x) = x^2 - 2x = f(x)$ as required. In addition to checking the formulas match up, we also need to check domains. There isn't much work here since the domains of g and h are all real numbers which combine to give the domain of f which is all real numbers.

   (b) In order to write $f = g + h$, we need $f(x) = (g + h)(x) = g(x) + h(x)$. One way to accomplish this is to write $f(x) = x^2 - 2x = x^2 + (-2x)$ and identify $g(x) = x^2$ and $h(x) = -2x$. To check, $(g + h)(x) = g(x) + h(x) = x^2 - 2x = f(x)$. Again, the domains for both g and h are all real numbers which combine to give f its domain of all real numbers.

   (c) To write $f = gh$, we require $f(x) = (gh)(x) = g(x)h(x)$. In other words, we need to factor $f(x)$. We find $f(x) = x^2 - 2x = x(x - 2)$, so one choice is to select $g(x) = x$ and $h(x) = x - 2$. Then $(gh)(x) = g(x)h(x) = x(x - 2) = x^2 - 2x = f(x)$, as required. As above, the domains of g and h are all real numbers which combine to give f the correct domain of $(-\infty, \infty)$.

   (d) We need to be careful here interpreting the equation $f = g(h-k)$. What we have is an equality of *functions* so the parentheses here *do not* represent function notation here, but, rather function *multiplication*. The way to parse $g(h-k)$, then, is the function g *times* the function $h-k$. Hence, we seek functions g, h, and k so that $f(x) = [g(h-k)](x) = g(x)[(h-k)(x)] = g(x)(h(x) - k(x))$. From the previous example, we know we can rewrite $f(x) = x(x - 2)$, so one option is to set $g(x) = h(x) = x$ and $k(x) = 2$ so that $[g(h - k)](x) = g(x)[(h - k)(x)] = g(x)(h(x) - k(x)) = x(x - 2) = x^2 - 2x = f(x)$, as required. As above, the domain of all constituent functions is $(-\infty, \infty)$ which matches the domain of f.

2. (a) To write $F = \frac{G}{H}$, we need $G(t)$ and $H(t)$ so $F(t) = \left(\frac{G}{H}\right)(t) = \frac{G(t)}{H(t)}$. We choose $G(t) = 2t + 1$ and $H(t) = \sqrt{t^2 - 1}$. Sure enough, $\left(\frac{G}{H}\right)(t) = \frac{G(t)}{H(t)} = \frac{2t+1}{\sqrt{t^2-1}} = F(t)$ as required. When it comes to the domain of F, owing to the square root, we require $t^2 - 1 \geq 0$. Since we have a denominator as well, we require $\sqrt{t^2 - 1} \neq 0$. The former requirement is the same restriction on H, and the

latter requirement comes from Definition 5.1. Starting with the domain of G, all real numbers, and working through the details, we arrive at the correct domain of F, $(-\infty, -1) \cup (1, \infty)$.

(b) Next, we are asked to find functions G and H so $F(t) = (GH)(t) = G(t)H(t)$. This means we need to rewrite the expression for $F(t)$ as a product. One way to do this is to convert radical notation to exponent notation:

$$F(t) = \frac{2t+1}{\sqrt{t^2-1}} = \frac{2t+1}{\left(t^2-1\right)^{\frac{1}{2}}} = (2t+1)\left(t^2-1\right)^{-\frac{1}{2}}.$$

Choosing $G(t) = 2t+1$ and $H(t) = \left(t^2-1\right)^{-\frac{1}{2}}$, we see $(GH)(t) = G(t)H(t) = (2t+1)\left(t^2-1\right)^{-\frac{1}{2}}$ as required. The domain restrictions on F stem from the presence of the square root in the denominator - both are addressed when finding the domain of H. Hence, we obtain the correct domain of F.

(c) To express F as a sum of functions G and H, we could rewrite

$$F(t) = \frac{2t+1}{\sqrt{t^2-1}} = \frac{2t}{\sqrt{t^2-1}} + \frac{1}{\sqrt{t^2-1}},$$

so that $G(t) = \frac{2t}{\sqrt{t^2-1}}$ and $H(t) = \frac{1}{\sqrt{t^2-1}}$. Indeed, $(G+H)(t) = G(t) + H(t) = \frac{2t}{\sqrt{t^2-1}} + \frac{1}{\sqrt{t^2-1}} = \frac{2t+1}{\sqrt{t^2-1}} = F(t)$, as required. Moreover, the domain restrictions for F are the same for both G and H, so we get agreement on the domain, as required.

(d) Last, but not least, to write $F = \frac{G+H}{K}$, we require $F(t) = \left(\frac{G+H}{K}\right)(t) = \frac{(G+H)(t)}{K(t)} = \frac{G(t)+H(t)}{K(t)}$. Identifying $G(t) = 2t$, $H(t) = 1$, and $K(t) = \sqrt{t^2-1}$, we get

$$\left(\frac{G+H}{K}\right)(t) = \frac{(G+H)(t)}{K(t)} = \frac{G(t)+H(t)}{K(t)} = \frac{2t+1}{\sqrt{t^2-1}} = F(t).$$

Concerning domains, the domain of both G and H are all real numbers, but the domain of K is restricted to $t^2 - 1 \geq 0$. Coupled with the restriction stated in Definition 5.1 that $K(t) \neq 0$, we recover the domain of F, $(-\infty, -1) \cup (1, \infty)$. □

5.2.1 The Arithmetic of Change

Recall the **average rate of change** of a function over the interval $[a, b]$ is the slope of the line connecting the two points $(a, f(a))$ and $(b, f(b))$ and is given by

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

For the purposes of this section, consider a function f defined over an interval containing x and $x + \Delta x$ where $\Delta x \neq 0$. The average rate of change of f over the interval $[x, x + \Delta x]$ is thus given by the formula:[5]

[5]assuming $\Delta x > 0$; otherwise, the interval is $[x + \Delta x, x]$. We get the same formula for the difference quotient either way.

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0.$$

Our aim in this section is to develop formulas which relate the rate of change of arithmetic combinations of functions to the rates of change of the component functions. Our first step is to study the **difference operator** 'Δ' and how it works with the standard arithmetic operations.

In general, if u is some quantity which assumes two values in a particular order, say u_1 (the 'first' or 'initial' value) and u_2 (the 'second' or 'final' value), then $\Delta u = u_2 - u_1$. For example, if u represents the temperature of an object before (u_1) heat is applied and after (u_2) heat is applied, $\Delta u = u_2 - u_1$ represents the increase in temperature of the object.

In the context of functions and rates of change, u is the function f defined on the interval $[x, x+\Delta x]$ with $u_1 = f(x)$ and $u_2 = f(x+\Delta x)$. Here, $\Delta u = u_2 - u_1 = f(x+\Delta x) - f(x) = \Delta[f(x)]$.

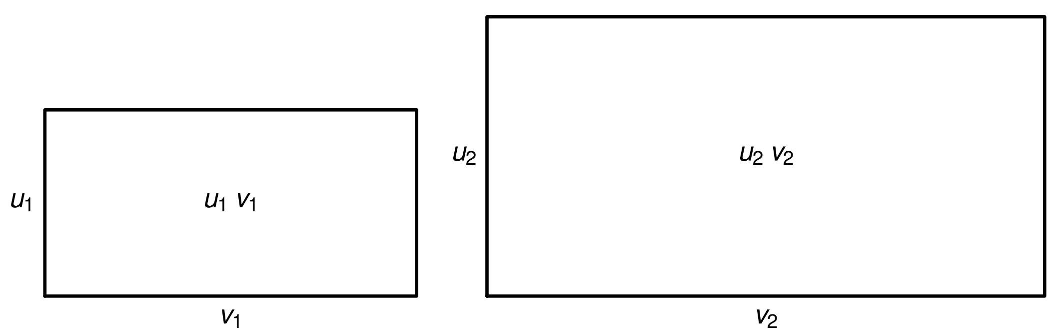
Suppose we have two quantities, u and v with $\Delta u = u_2 - u_1$ and $\Delta v = v_2 - v_1$. What do we mean by $\Delta[u+v]$? The initial value of the sum $u+v$ would be the sum of the initial values $u_1 + v_1$. Likewise, the final value of the sum would be the sum of the final values $u_2 + v_2$. Hence:

$$\begin{aligned} \Delta[u+v] &= (u_2+v_2) - (u_1+v_1) \\ &= u_2 + v_2 - u_1 - v_1 \\ &= (u_2 - u_1) + (v_2 - v_1) \\ &= \Delta u + \Delta v \end{aligned}$$

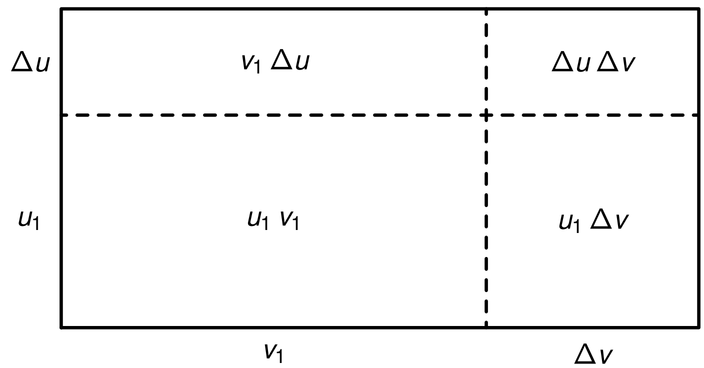
A similar calculation gives $\Delta[u - v] = \Delta u - \Delta v$.

Let's turn our attention to products. We have $\Delta[uv] = u_2 v_2 - u_1 v_1$. We'd like to express $\Delta[uv]$ in terms of Δu and Δv and there seems to be no way obvious way to do that. We take to some geometric reasoning for inspiration. Let's assume the all the quantities we're working with are positive.

We imagine the product $u_1 v_1$ as being the area of a rectangle with width u_1 and length v_1. Likewise, the product $u_2 v_2$ is the area of a (larger) rectangle with width u_2 and length v_2.



From $\Delta u = u_2 - u_1$, we get $u_2 = u_1 + \Delta u$ and, likewise, $v_2 = v_1 + \Delta v$. Doing so allows us to decompose the larger rectangle into four smaller rectangles.



Using this schematic, we see the area $u_2\,v_2$ is the sum of the areas of four smaller rectangles:

$$u_2\,v_2 = u_1\,v_1 + v_1\,\Delta u + u_1\,\Delta v + \Delta u\,\Delta v$$

Hence, $\Delta[uv] = u_2\,v_2 - u_1\,v_1 = v_1\,\Delta u + u_1\,\Delta v + \Delta u\,\Delta v$.

To prove this formula holds in general, we can substitute $u_2 = u_1 + \Delta u$ and $v_2 = v_1 + \Delta v$ into $\Delta[uv] = u_2\,v_2 - u_1\,v_1$ and simplify. We leave the details to the reader.[6]

Next, we turn our attention to quotients. We begin with: $\Delta\left[\frac{u}{v}\right] = \frac{u_2}{v_2} - \frac{u_1}{v_1}$.

Instead of appealing to geometric reasoning here,[7] we take a cue from the previous discussion and substitute $u_2 = u_1 + \Delta u$ and $v_2 = v_1 + \Delta v$ and set about getting a common denominator:

$$\begin{aligned}
\Delta\left[\frac{u}{v}\right] &= \frac{u_2}{v_2} - \frac{u_1}{v_1} \\
&= \frac{u_1 + \Delta u}{v_1 + \Delta v} - \frac{u_1}{v_1} \\
&= \frac{(u_1 + \Delta u)\,v_1}{(v_1 + \Delta v)\,v_1} - \frac{u_1\,(v_1 + \Delta v)}{v_1\,(v_1 + \Delta v)} \\
&= \frac{u_1\,v_1 + v_1\,\Delta u - u_1\,v_1 - u_1\,\Delta v}{v_1\,(v_1 + \Delta v)} \\
&= \frac{v_1\,\Delta u - u_1\,\Delta v}{v_1\,(v_1 + \Delta v_1)}
\end{aligned}$$

We summarize these results in the following theorem.

[6] Why not do this from the start, then? Carl trained as a geometric topologist.

[7] If you come up with or know of a nice geometric argument and don't mind sharing, feel free to contact Carl.

Theorem 5.2. Suppose $\Delta u = u_2 - u_1$ and $\Delta v = v_2 - v_1$:

- **The Sum Rule for Change:** $\Delta[u + v] = \Delta u + \Delta v$
- **The Difference Rule for Change:** $\Delta[u - v] = \Delta u - \Delta v$
- **The Product Rule for Change:** $\Delta[uv] = v_1\, \Delta u + u_1\, \Delta v + \Delta u\, \Delta v$
  - **The Constant Multiple Rule for Change:** If c is a constant, then $\Delta[cu] = c\, \Delta u$.
- **The Quotient Rule for Change:** $\Delta\left[\dfrac{u}{v}\right] = \dfrac{v_1\, \Delta u - u_1\, \Delta v}{v_1\,(v_1 + \Delta v_1)}$

You'll note we've called out a special case for the Product Rule, the Constant Multiple Rule. If one of the factors is a constant, c, then $\Delta c = 0$ (since constants don't change.)

In the following example, we use the Quotient Rule for Change to help approximate the **propagated error** when using measured quantities (with associated uncertainties) in calculations.[8]

Example 5.2.3. The density of a substance, ρ, is calculated by dividing its mass, m, by its volume, V: $\rho = \frac{m}{V}$. A scientist collects 5 milliliters (mL) of a substance and determines its mass to be 68.2 grams (g). She computes the density as: $\rho = \frac{68.2\,\text{g}}{5\,\text{mL}} = 13.64\,\frac{\text{g}}{\text{mL}}$.

Since every measurement in the lab has an associated uncertainty, she notes the pipet she used to measure the volume has an uncertainty of ± 0.125 mL and the balance she used to mass the substance has an uncertainty of ± 0.01 g. This means the actual volume measurement can be anywhere from as low as $5 - 0.125 = 4.875$ mL and as high as $5 + 0.125 = 5.125$ mL. Likewise, the actual mass of the substance can be anywhere from $68.2 - 0.01 = 68.19$ g to $68.2 + 0.01 = 68.21$g. Our goal is to help estimate the associated uncertainty for the density, ρ.

1. Use Theorem 5.2 to find an expression for the uncertainty in the volume $\Delta\rho$ produced as a result in the uncertainties in the measurements of mass, Δm, and volume, ΔV.

2. Calculate $\frac{\Delta\rho}{\rho}$ and interpret your answer.

Solution.

1. In this scenario, we have two quantities, the volume, V and the mass, m. We'll take V_1 and m_1 to be the measured values of volume and mass, respectively, and use the uncertainties in each of the respective measurements as ΔV and Δm. Since $\Delta\rho = \Delta\left[\frac{m}{V}\right]$, using the Quotient Rule from Theorem 5.2 gives:

$$\Delta\rho = \frac{V_1\, \Delta m - m_1\, \Delta V}{V_1(V_1 + \Delta V)}.$$

[8] The adjective 'propagated' here means than when we use measured quantities in calculations, the uncertainties in the measured quantities will produce, or 'propagate' uncertainty in the calculated quantity.

2. Substituting $V_1 = 5$ mL, $m_1 = 68.2$ g, $\Delta V = \pm 0.125$ mL and $\Delta m = \pm 0.01$ g gives:

$$\Delta\rho = \frac{(\pm 0.01\,\text{g})(5\,\text{mL}) - (68.2\,\text{g})(\pm 0.125\,\text{mL})}{(5\,\text{mL})(5\,\text{mL} \pm 0.125\,\text{mL})}.$$

Since we have no idea the exact value of each uncertainty, we need to make a judgement call as to which of the sign values, '$\pm$' , to use. To get the largest (most conservative) answer for $\Delta\rho$, we select the $\pm$ which generate the largest numerator and smallest denominator:

$$\Delta\rho = \frac{(+0.01\,\text{g})(5\,\text{mL}) - (68.2\,\text{g})(-0.125\,\text{mL})}{(5\,\text{mL})(5\,\text{mL} - 0.125\,\text{mL})} = \frac{343\,\text{g mL}}{975\,\text{mL}^2} \approx 0.3517\,\frac{\text{g}}{\text{mL}}$$

Hence, $\frac{\Delta\rho}{\rho} \approx \frac{0.3517\,\frac{\text{g}}{\text{mL}}}{13.64\,\frac{\text{g}}{\text{mL}}} \approx 0.0258 = 2.58\%$.

We may interpret this as the uncertainties in the measurements for mass and volume in this situation could produce up to a 2.58% error in the calculated density, $\square$

In order to establish formulas for the average rate of change for functions, we substitute $f(x)$ for u_1 and $g(x)$ for v_1 and divide each of the expressions in Theorem 5.2 by Δx. For example, if we to find an expression for the average rate of change of fg in terms of f, g, and their respective average rates of change:

$$\frac{\Delta[(fg)(x)]}{\Delta x} = \frac{\Delta[f(x)]g(x) + f(x)\Delta[g(x)] + \Delta[f(x)]\Delta[g(x)]}{\Delta x} = \frac{\Delta[f(x)]}{\Delta x}\,g(x) + f(x)\,\frac{\Delta[g(x)]}{\Delta x} + \frac{\Delta[f(x)]\Delta[g(x)]}{\Delta x}.$$

Note that with the last term, we may associate the 'Δx' with either of the factors in the numerator:

$$\frac{\Delta[f(x)]\Delta[g(x)]}{\Delta x} = \frac{\Delta[f(x)]}{\Delta x}\,\Delta[g(x)] = \Delta[f(x)]\,\frac{\Delta[g(x)]}{\Delta x}.$$

Either way, we've managed to express the average rate of change of the function fg in terms of the changes and rates of change of f and g.

In the result below, we abbreviate the average rate of change as 'ARoC' for convenience.

Theorem 5.3. Suppose f and g are functions defined on an interval containing x and $x + \Delta x$, $\Delta x \neq 0$:

- **The Sum Rule for ARoC:** $\text{ARoC}[(f+g)(x)] = \text{ARoC}[f(x)] + \text{ARoC}[g(x)]$
- **The Difference Rule for ARoC:** $\text{ARoC}[(f-g)(x)] = \text{ARoC}[f(x)] - \text{ARoC}[g(x)]$
- **The Product Rule for ARoC:**

$$\begin{aligned} \text{ARoC}[(fg)(x)] &= \text{ARoC}[f(x)]\, g(x) + f(x)\, \text{ARoC}[g(x)] + \text{ARoC}[f(x)]\, g(x) \\ &= \text{ARoC}[f(x)]\, g(x) + f(x)\, \text{ARoC}[g(x)] + f(x)\, \text{ARoC}[g(x)] \end{aligned}$$

  - **The Constant Multiple Rule for ARoC:** If c is a constant, then $\text{ARoC}[c\, f(x)] = c\, \text{ARoC}[f(x)]$.

- **The Quotient Rule for ARoC:**

$$\text{ARoC}\left[\left(\frac{f}{g}\right)(x)\right] = \frac{\text{ARoC}[f(x)]\, g(x) - f(x)\, \text{ARoC}[g(x)]}{g(x)(g(x) + \Delta[g(x)])}$$

Our final example revisits the scenario in Exercise 55 in Section 2.1.

Example 5.2.4. The cost, in dollars, $C(x)$, to produce x 'PortaBoy' handheld game systems is given by: $C(x) = .03x^3 - 4.5x^2 + 225x + 250$. The revenue generated by selling x of the systems, $R(x)$, also in dollars, is given by $R(x) = -1.5x^2 + 250x$.

1. Find and interpret the average rate of change of C and R over the interval $[70, 71]$.
2. Use Theorem 5.3 to determine the average rate of change of the profit function, P over the interval $[70, 71]$ using your answers to part 1. What does your answer suggest?

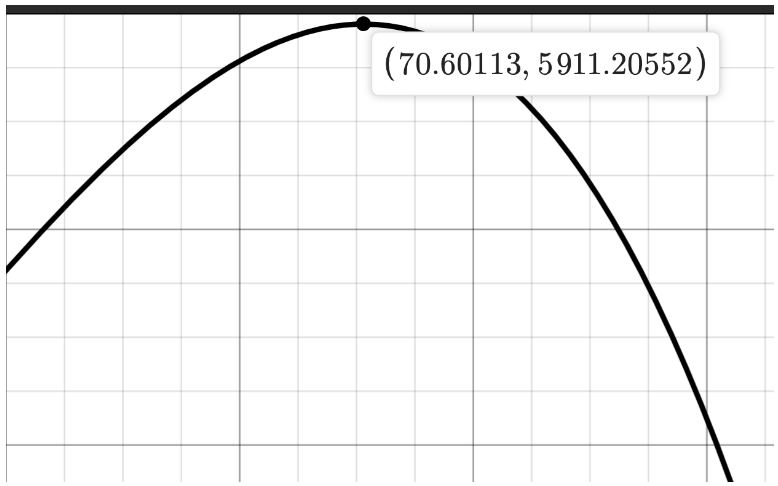
Solution.

1. We find $\text{ARoC}[C(x)] = \frac{C(71)-C(70)}{71-70} = \frac{4277.83-4240}{1} = 37.83$. This means that as we move from producing 70 to 71 PortaBoy systems the cost will increase by \$37.83 per system.

   For revenue, we find $\text{ARoC}[R(x)] = \frac{R(71)-R(70)}{71-70} = \frac{10188.5-10150}{1} = 38.5$. This means that as we move from selling 70 to 71 PortaBoy systems, the revenue generated will increase by \$38.5 per system.

2. Since $P(x) = R(x) - C(x)$, the Difference Rule of Theorem 5.3 gives $\text{ARoC}[P(x)] = \text{ARoC}[R(x)] - \text{ARoC}[C(x)]$. In this case, we'd get $\text{ARoC}[P(x)] = 38.5 - 37.83 = 0.67$. This means as we move from producing and selling 70 to 71 PortaBoy systems, the profit generated will increase by just 67 cents per system. At this point, the increase in revenue is nearly balanced out by the increase in cost. Since costs typically continue to rise as the number of items is produced while the revenue falls as we try to sell more items,[9] we are likely near a maximum point with the profit. A quick check of the graph on Desmos confirms our suspicions.

[9]we've seen this before: to sell more, we lower the price which, in turn, lowers revenue . . .



□

Note that in Example 5.2.4, since $\Delta x = 1$, the average rate of change for the cost moving from producing 70 to 71 systems, $\frac{C(71)-C(70)}{71-70}$ is the same numerical value as the additional cost incurred by producing the 71st system, $C(71) - C(70)$. The same goes for the revenue and profit calculations. This is the concept of **marginal analysis** is studied at length in Economics and Business Calculus classes.[10] For us, it's time for some Exercises.

[10]We'll do some more exploration as well. See, for example, Exercise 53.

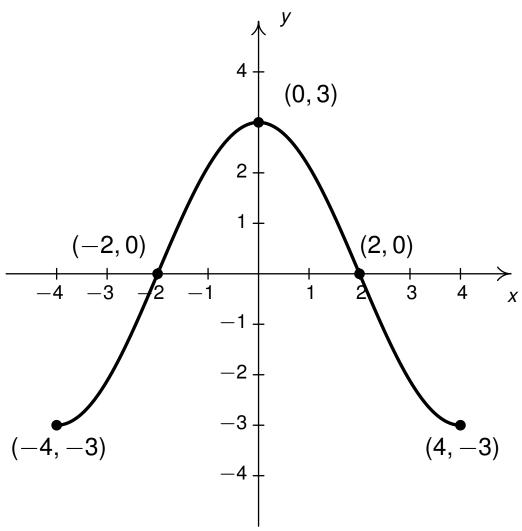
5.2.2 Exercises

In Exercises 1 - 10, use the pair of functions f and g to find the following values if they exist.

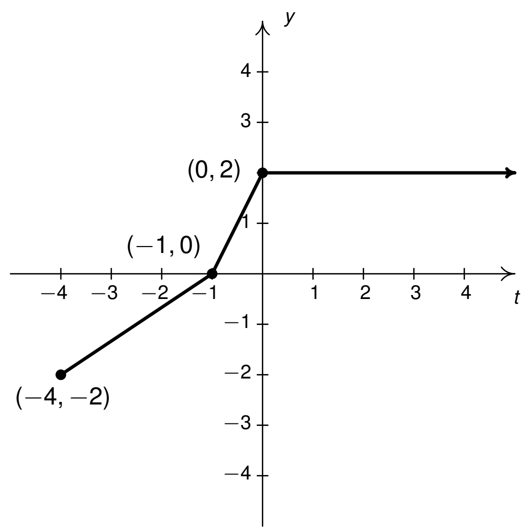
- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x + 1$ and $g(t) = 4 - t$
2. $f(x) = x^2$ and $g(t) = -2t + 1$
3. $f(x) = x^2 - x$ and $g(t) = 12 - t^2$
4. $f(x) = 2x^3$ and $g(t) = -t^2 - 2t - 3$
5. $f(x) = \sqrt{x+3}$ and $g(t) = 2t - 1$
6. $f(x) = \sqrt{4-x}$ and $g(t) = \sqrt{t+2}$
7. $f(x) = 2x$ and $g(t) = \dfrac{1}{2t+1}$
8. $f(x) = x^2$ and $g(t) = \dfrac{3}{2t-3}$
9. $f(x) = x^2$ and $g(t) = \dfrac{1}{t^2}$
10. $f(x) = x^2 + 1$ and $g(t) = \dfrac{1}{t^2+1}$

Exercises 11 - 20 refer to the functions f and g whose graphs are below.



$y = f(x)$



$y = g(t)$

11. $(f+g)(-4)$
12. $(f+g)(0)$
13. $(f-g)(4)$
14. $(fg)(-4)$
15. $(fg)(-2)$
16. $(fg)(4)$
17. $\left(\dfrac{f}{g}\right)(0)$
18. $\left(\dfrac{f}{g}\right)(2)$
19. $\left(\dfrac{g}{f}\right)(-1)$
20. Find the domains of $f+g$, $f-g$, fg, $\dfrac{f}{g}$ and $\dfrac{g}{f}$.

In Exercises 21 - 32, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

and let g be the function defined by

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}$$

Compute the indicated value if it exists.

21. $(f+g)(-3)$
22. $(f-g)(2)$
23. $(fg)(-1)$
24. $(g+f)(1)$
25. $(g-f)(3)$
26. $(gf)(-3)$
27. $\left(\frac{f}{g}\right)(-2)$
28. $\left(\frac{f}{g}\right)(-1)$
29. $\left(\frac{f}{g}\right)(2)$
30. $\left(\frac{g}{f}\right)(-1)$
31. $\left(\frac{g}{f}\right)(3)$
32. $\left(\frac{g}{f}\right)(-3)$

In Exercises 33 - 42, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f+g)(x)$
- $(f-g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

33. $f(x) = 2x+1$ and $g(x) = x-2$
34. $f(x) = 1-4x$ and $g(x) = 2x-1$
35. $f(x) = x^2$ and $g(x) = 3x-1$
36. $f(x) = x^2 - x$ and $g(x) = 7x$
37. $f(x) = x^2 - 4$ and $g(x) = 3x+6$
38. $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$
39. $f(x) = \dfrac{x}{2}$ and $g(x) = \dfrac{2}{x}$
40. $f(x) = x-1$ and $g(x) = \dfrac{1}{x-1}$
41. $f(x) = x$ and $g(x) = \sqrt{x+1}$
42. $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$

In Exercises 43 - 47, write the given function as a nontrivial decomposition of functions as directed.

43. For $p(z) = 4z - z^3$, find functions f and g so that $p = f - g$.
44. For $p(z) = 4z - z^3$, find functions f and g so that $p = f + g$.
45. For $g(t) = 3t|2t-1|$, find functions f and h so that $g = fh$.
46. For $r(x) = \dfrac{3-x}{x+1}$, find functions f and g so $r = \dfrac{f}{g}$.
47. For $r(x) = \dfrac{3-x}{x+1}$, find functions f and g so $r = fg$.

48. Can $f(x) = x$ be decomposed as $f = g - h$ where $g(x) = x + \dfrac{1}{x}$ and $h(x) = \dfrac{1}{x}$?

49. Discuss with your classmates how to phrase the quantities revenue and profit in Definition 1.12 terms of function arithmetic as defined in Definition 5.1.

50. In this exercise, we explore decomposing a function into its positive and negative parts. Given a function f, we define the **positive part** of f, denoted f_+ and **negative part** of f, denoted f_- by:

$$f_+(x) = \frac{f(x) + |f(x)|}{2}, \qquad \text{and} \qquad f_-(x) = \frac{f(x) - |f(x)|}{2}.$$

(a) Using a graphing utility, graph each of the functions f below along with f_+ and f_-.

- $f(x) = x - 3$
- $f(x) = x^2 - x - 6$
- $f(x) = 4x - x^3$

Why is f_+ called the 'positive part' of f and f_- called the 'negative part' of f?

(b) Show that $f = f_+ + f_-$.

(c) Use Definition 1.9 to rewrite the expressions for $f_+(x)$ and $f_-(x)$ as piecewise defined functions.

51. Let U be the unit step function defined in Exercise 11 in Section 1.2. For each function $f(t)$ below:

- Write $(Uf)(t)$ as a piecewise-defined function.
- Graph $y = f(t)$ and $y = (Uf)(t)$.

(a) $f(t) = t - 3$ (b) $f(t) = |t + 2|$ (c) $f(t) = (t - 1)^2$

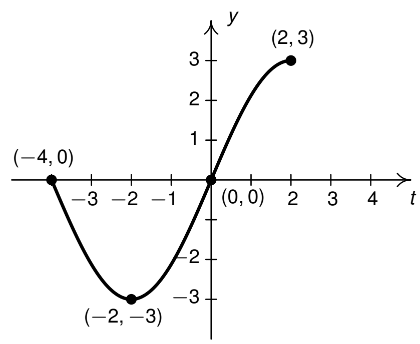
(d) $f(t) = (t + 1)^{-1}$ (e) $f(t) = \sqrt[3]{t - 1}$ (f) $f(t) = (t - 2)^{\frac{2}{3}}$

(g) Write a general formula for $(Uf)(t)$ for a function f. (Assume the domain of f is $(-\infty, \infty)$.)

(h) Explain how to obtain the graph of $y = (Uf)(t)$ from $y = f(t)$.

(i) The function $U(t)$ is used to model a change in state from 'off' to 'on' (like flipping a light switch.) How does this relate to your observations?

(j) Use the graph of $y = f(t)$ below to graph $y = (Uf)(t)$.



$y = f(t)$

52. Use Example 5.2.3 as a guide to help find the following uncertainties.

    (a) A chemist combines the solutions from two graduated cylinders into a beaker. The volume of the first solution, A, an acid, is read as $A_1 = 101 \pm 0.5$ milliliters (mL). The volume of the second solution, a base, B, is measured to be $B_1 = 16 \pm 0.5$ mL. Estimate the percent propagated error in calculating the volume of the combined solution as $V = A_1 + B_1 = 101 + 16 = 117$ mL.

    (b) A student measures the length, ℓ, and width, w, of a piece of paper. They find $\ell_1 = 280 \pm 0.5$ millimeters (mm) $w_1 = 216 \pm 0.5$ mm. Estimate the percent propagated error in calculating the area of the piece of paper as $A = \ell_1 \, w_1 = 280 \times 216 = 60480 \, \text{mm}^2$.

    (c) An airplane passenger observers a car travel a distance $d_1 = 1320 \pm 2$ feet (ft) in time $t_1 = 15 \pm 0.5$ seconds (s). Estimate the percent propagated error in calculating the speed of the car as $v = \frac{d_1}{t_1} = \frac{1320}{15} = 88 \, \frac{\text{ft}}{\text{s}}$.

53. Let us return to Example 5.2.4 where $C(x) = .03x^3 - 4.5x^2 + 225x + 250$ denotes the cost, in dollars, of producing x PortaBoy game systems. Recall the **average cost**[11] is defined as $\overline{C}(x) = \frac{C(x)}{x}$, $x > 0$, is the cost per item.

    (a) Find and interpret $\overline{C}(75)$.

    (b) Define the **marginal cost** $MC(x) = C(x+1) - C(x)$. Find and interpret $MC(75)$.

    (c) How do your answers to parts 53a and 53b compare?

    (d) Graph $y = \overline{C}(x)$ with help from a graphing utility. What is happening graphically near $x = 75$?

    (e) Use Theorem 5.3 to show that, in general, $\text{ARoC}[\overline{C}(x)] = 0$ when $MC(x) = \overline{C}(x)$.

    HINT: Note that, by definition, $MC(x) = C(x+1) - C(x) = \Delta[C(x)]$ when $\Delta x = 1$.

    Hence, $\text{ARoC}[C(x)] = \frac{\Delta[C(x)]}{\Delta x} = \frac{\Delta[C(x)]}{1} = \Delta[C(x)] = MC(x)$ in this case ...

[11] First mentioned in Definition 3.8 in Section 3.1.

5.2.3 Answers

1. For $f(x) = 3x + 1$ and $g(x) = 4 - x$
   - $(f + g)(2) = 9$
   - $(f - g)(-1) = -7$
   - $(g - f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = \frac{35}{4}$
   - $\left(\frac{f}{g}\right)(0) = \frac{1}{4}$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{6}{5}$

2. For $f(x) = x^2$ and $g(x) = -2x + 1$
   - $(f + g)(2) = 1$
   - $(f - g)(-1) = -2$
   - $(g - f)(1) = -2$
   - $(fg)\left(\frac{1}{2}\right) = 0$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{5}{4}$

3. For $f(x) = x^2 - x$ and $g(x) = 12 - x^2$
   - $(f + g)(2) = 10$
   - $(f - g)(-1) = -9$
   - $(g - f)(1) = 11$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{47}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{4}{3}$

4. For $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$
   - $(f + g)(2) = 5$
   - $(f - g)(-1) = 0$
   - $(g - f)(1) = -8$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{17}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{3}{16}$

5. For $f(x) = \sqrt{x + 3}$ and $g(x) = 2x - 1$
   - $(f + g)(2) = 3 + \sqrt{5}$
   - $(f - g)(-1) = 3 + \sqrt{2}$
   - $(g - f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = 0$
   - $\left(\frac{f}{g}\right)(0) = -\sqrt{3}$
   - $\left(\frac{g}{f}\right)(-2) = -5$

6. For $f(x) = \sqrt{4 - x}$ and $g(x) = \sqrt{x + 2}$
   - $(f + g)(2) = 2 + \sqrt{2}$
   - $(f - g)(-1) = -1 + \sqrt{5}$
   - $(g - f)(1) = 0$
   - $(fg)\left(\frac{1}{2}\right) = \frac{\sqrt{35}}{2}$
   - $\left(\frac{f}{g}\right)(0) = \sqrt{2}$
   - $\left(\frac{g}{f}\right)(-2) = 0$

7. For $f(x) = 2x$ and $g(x) = \frac{1}{2x+1}$

- $(f+g)(2) = \frac{21}{5}$
- $(f-g)(-1) = -1$
- $(g-f)(1) = -\frac{5}{3}$
- $(fg)\left(\frac{1}{2}\right) = \frac{1}{2}$
- $\left(\frac{f}{g}\right)(0) = 0$
- $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$

8. For $f(x) = x^2$ and $g(x) = \frac{3}{2x-3}$

- $(f+g)(2) = 7$
- $(f-g)(-1) = \frac{8}{5}$
- $(g-f)(1) = -4$
- $(fg)\left(\frac{1}{2}\right) = -\frac{3}{8}$
- $\left(\frac{f}{g}\right)(0) = 0$
- $\left(\frac{g}{f}\right)(-2) = -\frac{3}{28}$

9. For $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$

- $(f+g)(2) = \frac{17}{4}$
- $(f-g)(-1) = 0$
- $(g-f)(1) = 0$
- $(fg)\left(\frac{1}{2}\right) = 1$
- $\left(\frac{f}{g}\right)(0)$ is undefined.
- $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$

10. For $f(x) = x^2 + 1$ and $g(x) = \frac{1}{x^2+1}$

- $(f+g)(2) = \frac{26}{5}$
- $(f-g)(-1) = \frac{3}{2}$
- $(g-f)(1) = -\frac{3}{2}$
- $(fg)\left(\frac{1}{2}\right) = 1$
- $\left(\frac{f}{g}\right)(0) = 1$
- $\left(\frac{g}{f}\right)(-2) = \frac{1}{25}$

11. $(f+g)(-4) = -5$
12. $(f+g)(0) = 5$
13. $(f-g)(4) = -5$
14. $(fg)(-4) = 6$
15. $(fg)(-2) = 0$
16. $(fg)(4) = -6$
17. $\left(\dfrac{f}{g}\right)(0) = \dfrac{3}{2}$
18. $\left(\dfrac{f}{g}\right)(2) = 0$
19. $\left(\dfrac{g}{f}\right)(-1) = 0$
20. The domains of $f+g$, $f-g$ and fg are all $[-4, 4]$. The domain of $\frac{f}{g}$ is $[-4, -1) \cup (-1, 4]$ and the domain of $\frac{g}{f}$ is $[-4, -2) \cup (-2, 2) \cup (2, 4]$.
21. $(f+g)(-3) = 2$
22. $(f-g)(2) = 3$
23. $(fg)(-1) = 0$
24. $(g+f)(1) = 0$
25. $(g-f)(3) = 3$
26. $(gf)(-3) = -8$
27. $\left(\frac{f}{g}\right)(-2)$ does not exist
28. $\left(\frac{f}{g}\right)(-1) = 0$
29. $\left(\frac{f}{g}\right)(2) = 4$
30. $\left(\frac{g}{f}\right)(-1)$ does not exist
31. $\left(\frac{g}{f}\right)(3) = -2$
32. $\left(\frac{g}{f}\right)(-3) = -\frac{1}{2}$

33. For $f(x) = 2x + 1$ and $g(x) = x - 2$

- $(f + g)(x) = 3x - 1$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x + 3$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 2x^2 - 3x - 2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}$
  Domain: $(-\infty, 2) \cup (2, \infty)$

34. For $f(x) = 1 - 4x$ and $g(x) = 2x - 1$

- $(f + g)(x) = -2x$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = 2 - 6x$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = -8x^2 + 6x - 1$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{1-4x}{2x-1}$
  Domain: $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

35. For $f(x) = x^2$ and $g(x) = 3x - 1$

- $(f + g)(x) = x^2 + 3x - 1$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 3x + 1$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3 - x^2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}$
  Domain: $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

36. For $f(x) = x^2 - x$ and $g(x) = 7x$

- $(f + g)(x) = x^2 + 6x$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 8x$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 7x^3 - 7x^2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-1}{7}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

37. For $f(x) = x^2 - 4$ and $g(x) = 3x + 6$

- $(f + g)(x) = x^2 + 3x + 2$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 3x - 10$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3 + 6x^2 - 12x - 24$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-2}{3}$
  Domain: $(-\infty, -2) \cup (-2, \infty)$

38. For $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$

- $(f + g)(x) = x - 3$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = -2x^2 + x + 15$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = -x^4 + x^3 + 15x^2 - 9x - 54$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = -\frac{x+2}{x+3}$
  Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

39. For $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$

- $(f + g)(x) = \frac{x^2+4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $(f - g)(x) = \frac{x^2-4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $(fg)(x) = 1$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

40. For $f(x) = x - 1$ and $g(x) = \frac{1}{x-1}$

- $(f + g)(x) = \frac{x^2-2x+2}{x-1}$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $(f - g)(x) = \frac{x^2-2x}{x-1}$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $(fg)(x) = 1$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $\left(\frac{f}{g}\right)(x) = x^2 - 2x + 1$
  Domain: $(-\infty, 1) \cup (1, \infty)$

41. For $f(x) = x$ and $g(x) = \sqrt{x + 1}$

- $(f + g)(x) = x + \sqrt{x + 1}$
  Domain: $[-1, \infty)$
- $(f - g)(x) = x - \sqrt{x + 1}$
  Domain: $[-1, \infty)$
- $(fg)(x) = x\sqrt{x + 1}$
  Domain: $[-1, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}}$
  Domain: $(-1, \infty)$

42. For $f(x) = \sqrt{x - 5}$ and $g(x) = f(x) = \sqrt{x - 5}$

- $(f + g)(x) = 2\sqrt{x - 5}$
  Domain: $[5, \infty)$
- $(f - g)(x) = 0$
  Domain: $[5, \infty)$
- $(fg)(x) = x - 5$
  Domain: $[5, \infty)$
- $\left(\frac{f}{g}\right)(x) = 1$
  Domain: $(5, \infty)$

43. One solution is $f(z) = 4z$ and $g(z) = z^3$.

44. One solution is $f(z) = 4z$ and $g(z) = -z^3$.

45. One solution is $f(t) = 3t$ and $h(t) = |2t - 1|$

46. One solution is $f(x) = 3 - x$ and $g(x) = x + 1$.

47. One solution is $f(x) = 3 - x$ and $g(x) = (x + 1)^{-1}$.

48. No. The equivalence does not hold when $x = 0$.

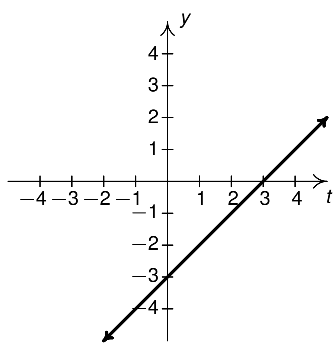
50. (b) $(f_+ + f_-)(x) = f_+(x) + f_-(x) = \dfrac{f(x) + |f(x)|}{2} + \dfrac{f(x) - |f(x)|}{2} = \dfrac{2f(x)}{2} = f(x)$.

(c)

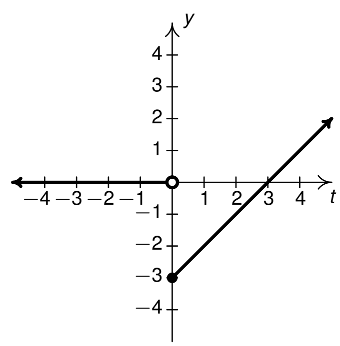
$$f_+(x) = \begin{cases} 0 & \text{if } f(x) < 0 \\ f(x) & \text{if } f(x) \geq 0 \end{cases}, \qquad f_-(x) = \begin{cases} f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geq 0 \end{cases}$$

51.

(a) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ t - 3 & \text{if } t \geq 0. \end{cases}$

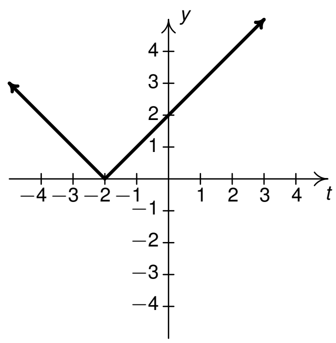


$y = f(t)$

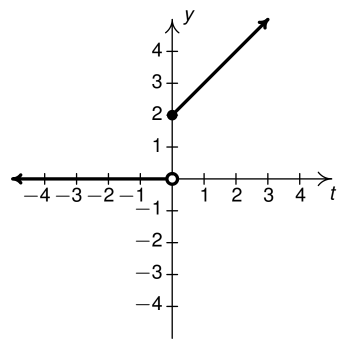


$y = (Uf)(t)$

(b) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ |t+2| = t+2 & \text{if } t \geq 0. \end{cases}$

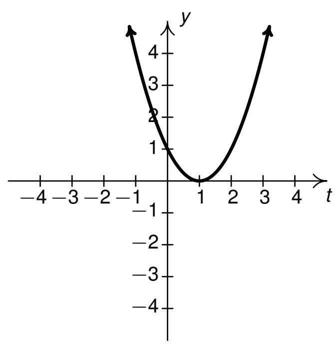


$y = f(t)$

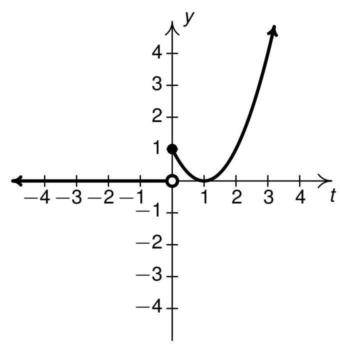


$y = (Uf)(t)$

(c) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ (t-1)^2 & \text{if } t \geq 0. \end{cases}$

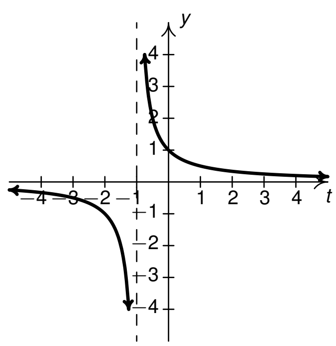


$y = f(t)$

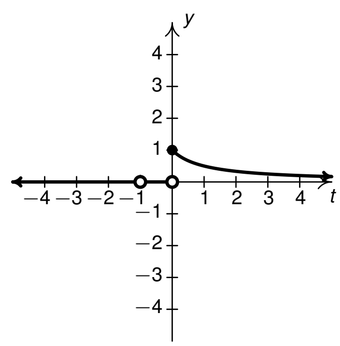


$y = (Uf)(t)$

(d) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0,\ t \neq -1 \\ (t+1)^{-1} & \text{if } t \geq 0. \end{cases}$

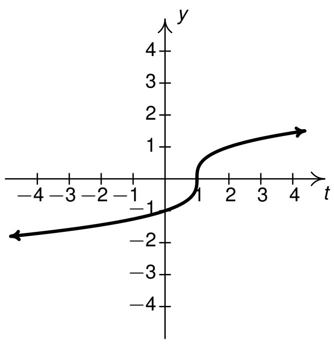


$y = f(t)$

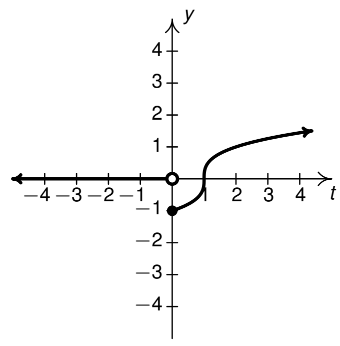


$y = (Uf)(t)$ - note the hole at $(-1, 0)$

(e) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ \sqrt[3]{t-1} & \text{if } t \geq 0. \end{cases}$

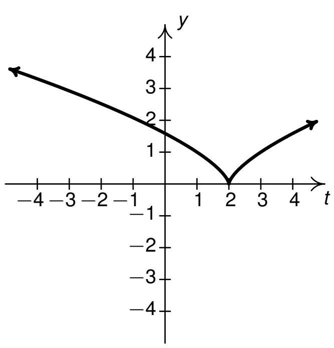


$y = f(t)$

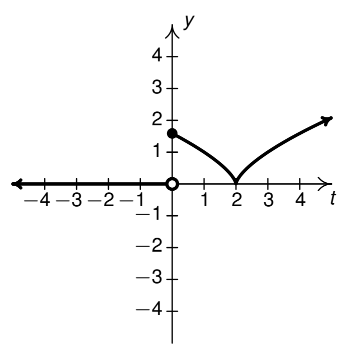


$y = (Uf)(t)$

(f) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ (t-2)^{\frac{2}{3}} & \text{if } t \geq 0. \end{cases}$



$y = f(t)$



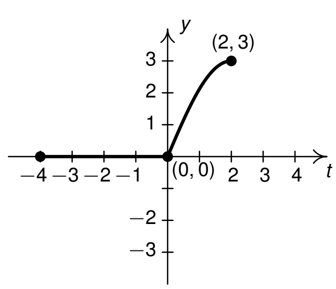
$y = (Uf)(t)$

(g) $(Uf)(t) = \begin{cases} 0 & \text{if } t < 0, \\ f(t) & \text{if } t \geq 0 \end{cases}$

(h) The graph of $(Uf)(t)$ is $y = 0$ for $t < 0$ and $y = f(t)$ for $t \geq 0$.

(i) The unit step function keeps the function 'off' until $t = 0$ then turns the function 'on' for $t \geq 0$.

(j)



$y = (Uf)(t)$

52. (a) $\Delta V = \Delta[A+B] = \Delta A + \Delta B = \pm 0.5\,\text{mL} + \pm 0.5\,\text{mL} = \pm 1\,\text{mL}$.
$\frac{\Delta V}{V} = \pm\frac{1}{117} \approx 0.85\,\%$.
(b) $\Delta A = \Delta[\ell w] = w_1 \Delta\ell + \ell_1 \Delta w + \Delta\ell \Delta w = (216\,\text{mm})(\pm 0.5\,\text{mm}) + (280\,\text{mm})(\pm 0.5\,\text{mm}) + (\pm 0.5\,\text{mm})(\pm 0.5\,\text{mm}) = \pm 248.25\,\text{mm}^2$.
$\frac{\Delta A}{A} = \pm\frac{248.25}{60480} \approx 0.41\,\%$
(c) $\Delta v = \Delta\left[\frac{d}{t}\right] = \frac{t_1\,\Delta d - d_1\,\Delta t}{t_1(t_1+\Delta t)} = \frac{(15\,\text{s})(\pm 2\,\text{ft}) - (1320\,\text{ft})(\pm 0.5\,\text{s})}{(15\,\text{s})(15 \pm 0.5\,\text{s})} = \pm\frac{92}{29}\,\frac{\text{ft}}{\text{s}} \approx 3.17\,\frac{\text{ft}}{\text{s}}$.
$\frac{\Delta v}{v} \approx \pm\frac{3.17}{88} \approx 3.60\,\%$

53. (a) $\overline{C}(75) \approx 59.58$. When making 75 systems, the cost per system is approximately $59.58.
(b) $MC(75) = C(76) - C(75) = 58.53$. It costs an additional $58.53 to make the 76th system.
(c) $\overline{C}(75)$ and $MC(75)$ appear to be 'pretty close.'
(d) The graph $y = \overline{C}(x)$ has a local (absolute) minimum right near $x = 75$.
(e) Per Theorem 5.3, since

$$\begin{aligned} \text{ARoC}[\overline{C}(x)] = \text{ARoC}\left[\frac{C(x)}{x}\right] &= \frac{\text{ARoC}[C(x)]\,x - C(x)\,\text{ARoC}[x]}{x(x+\Delta x)} \\ &= \frac{\text{ARoC}[C(x)]\,x - C(x)(1)}{x(x+\Delta x)} \quad \text{Since ARoC}[x] = \frac{\Delta x}{\Delta x} = 1 \end{aligned}$$

If $\text{ARoC}[\overline{C}(x)] = 0$, then the numerator, $\text{ARoC}[C(x)]\,x - C(x) = 0$. Solving for $\text{ARoC}[C(x)]$, we get $\text{ARoC}[C(x)] = \frac{C(x)}{x} = \overline{C}(x)$. If we are working with a whole number of items, the smallest meaningful value of Δx is 1, in which case $\text{ARoC}[C(x)] = MC(x)$. Hence, $\text{ARoC}[\overline{C}(x)] = 0$ when $MC(x) = \overline{C}(x)$, that is, when the marginal cost and average cost are the same. At this point, the graph of $y = \overline{C}(x)$ levels off (at a minumum.)[12] Can you reason why this creates a minimum?

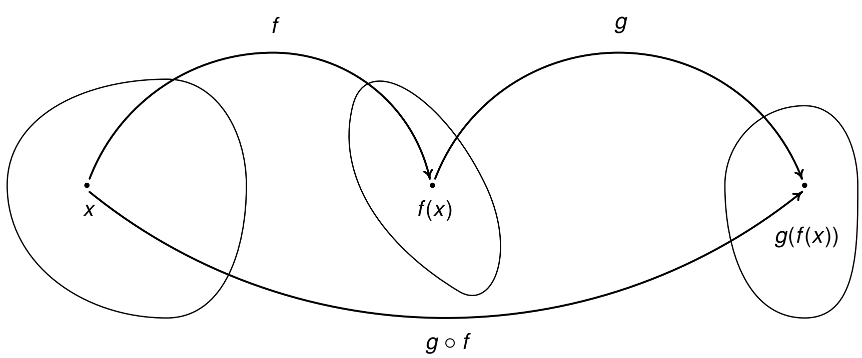
[12] We'll have more to say on this after Section 6.3 in Exercise 8.

5.3 Function Composition

In Section 5.2, we saw how the arithmetic of real numbers carried over into an arithmetic of functions. In this section, we discuss another way to combine functions which is unique to functions and isn't shared with real numbers - function **composition**.

Definition 5.2. Let f and g be functions where the real number x is in the domain of f and the real number $f(x)$ is in the domain of g. The **composite** of g with f, denoted $g \circ f$, and read 'g composed with f' is defined by the formula: $(g \circ f)(x) = g(f(x))$.

To compute $(g \circ f)(x)$, we use the formula given in Defintion 5.2: $(g \circ f)(x) = g(f(x))$. However, from a procedural viewpoint, Defintion 5.2 tells us the output from $g \circ f$ is found by taking the output from f, $f(x)$, and then making that the input to g. From this perspective, we see $g \circ f$ as a two step process taking an input x and first applying the procedure f then applying the procedure g. Abstractly, we have



In the expression $g(f(x))$, the function f is often called the 'inside' function while g is often called the 'outside' function. When evaluating composite function values we present two methods in the example below: the 'inside out' and 'outside in' methods.

Example 5.3.1. Let $f(x) = x^2 - 4x$, $g(t) = 2 - \sqrt{t+3}$, and $h(s) = \dfrac{2s}{s+1}$.
In numbers 1 - 3, find the indicated function value.

1. $(g \circ f)(1)$
2. $(f \circ g)(1)$
3. $(g \circ g)(6)$

In numbers 4 - 10, find and simplify the indicated composite functions. State the domain of each.

4. $(g \circ f)(x)$
5. $(f \circ g)(t)$
6. $(g \circ h)(s)$
7. $(h \circ g)(t)$
8. $(h \circ h)(x)$
9. $(h \circ (g \circ f))(x)$
10. $((h \circ g) \circ f)(x)$

Solution.

1. Using Definition 5.2, $(g \circ f)(1) = g(f(1))$. Since $f(1) = (1)^2 - 4(1) = -3$ and $g(-3) = 2 - \sqrt{(-3)+3} = 2$, we have $(g \circ f)(1) = g(f(1)) = g(-3) = 2$.

2. By definition, $(f \circ g)(1) = f(g(1))$. We find $g(1) = 2 - \sqrt{1+3} = 0$, and $f(0) = (0)^2 - 4(0) = 0$, so $(f \circ g)(1) = f(g(1)) = f(0) = 0$. Comparing this with our answer to the last problem, we see that $(g \circ f)(1) \neq (f \circ g)(1)$ which tells us function composition is not commutative,[1]

3. Since $(g \circ g)(6) = g(g(6))$, we 'iterate' the process g: that is, we apply the process g to 6, then apply the process g again. We find $g(6) = 2 - \sqrt{6+3} = -1$, and $g(-1) = 2 - \sqrt{(-1)+3} = 2 - \sqrt{2}$, so $(g \circ g)(6) = g(g(6)) = g(-1) = 2 - \sqrt{2}$.

4. By definition, $(g \circ f)(x) = g(f(x))$. We now illustrate *two* ways to approach this problem.

   - *inside out*: We substitute $f(x) = x^2 - 4x$ in for t in the expression $g(t)$ and get

     $$(g \circ f)(x) = g(f(x)) = g\left(x^2 - 4x\right) = 2 - \sqrt{\left(x^2 - 4x\right) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

     Hence, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   - *outside in*: We use the formula for g first to get

     $$(g \circ f)(x) = g(f(x)) = 2 - \sqrt{f(x) + 3} = 2 - \sqrt{\left(x^2 - 4x\right) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

     We get the same answer as before, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   To find the domain of $g \circ f$, we need to find the elements in the domain of f whose outputs $f(x)$ are in the domain of g. Since the domain of f is all real numbers, we focus on finding the range elements compatible with g. Owing to the presence of the square root in the formula $g(t) = 2 - \sqrt{t+3}$ we require $t \geq -3$. Hence, we need $f(x) \geq -3$ or $x^2 - 4x \geq -3$. To solve this inequality we rewrite as $x^2 - 4x + 3 \geq 0$ and use a sign diagram. Letting $r(x) = x^2 - 4x + 3$, we find the zeros of r to be $x = 1$ and $x = 3$ and obtain

   $$\begin{array}{ccccc} (+) & 0 & (-) & 0 & (+) \\ \longleftarrow & 1 & \longrightarrow & 3 & \longrightarrow \end{array}$$

   Our solution to $x^2 - 4x + 3 \geq 0$, and hence the domain of $g \circ f$, is $(-\infty, 1] \cup [3, \infty)$.

5. To find $(f \circ g)(t)$, we find $f(g(t))$.

   - *inside out*: We substitute the expression $g(t) = 2 - \sqrt{t+3}$ in for x in the formula $f(x)$ and get

     $$\begin{aligned} (f \circ g)(t) &= f(g(t)) = f\left(2 - \sqrt{t+3}\right) \\ &= \left(2 - \sqrt{t+3}\right)^2 - 4\left(2 - \sqrt{t+3}\right) \\ &= 4 - 4\sqrt{t+3} + \left(\sqrt{t+3}\right)^2 - 8 + 4\sqrt{t+3} \end{aligned}$$

[1] That is, in general, $g \circ f \neq f \circ g$. This shouldn't be too surprising, since, in general, the order of processes matters: adding eggs to a cake batter then baking the cake batter has a much different outcome than baking the cake batter then adding eggs.

$$\begin{aligned} &= 4 + t + 3 - 8 \\ &= t - 1 \end{aligned}$$

- *outside in*: We use the formula for $f(x)$ first to get

$$\begin{aligned} (f \circ g)(t) &= f(g(t)) = (g(t))^2 - 4\,(g(t)) \\ &= \left(2 - \sqrt{t+3}\right)^2 - 4\left(2 - \sqrt{t+3}\right) \\ &= t - 1 && \text{same algebra as before} \end{aligned}$$

Thus we get $(f \circ g)(t) = t - 1$. To find the domain of $f \circ g$, we look for the elements t in the domain of g whose outputs, $g(t)$ are in the domain of f. As mentioned previously, the domain of g is limited by the presence of the square root to $\{t \in \mathbb{R} \,|\, t \geq -3\}$ while the domain of f is all real numbers. Hence, the domain of $f \circ g$ is restricted only by the domain of g and is $\{t \in \mathbb{R} \,|\, t \geq -3\}$ or, using interval notation, $[-3, \infty)$. Note that as with Example 5.2.1 in Section 5.2, had we used the simplified formula for $(f \circ g)(t) = t - 1$ to determine domain, we would have arrived at the incorrect answer.

6. To find $(g \circ h)(s)$, we compute $g(h(s))$.

- *inside out*: We substitute $h(s)$ in for t in the expression $g(t)$ to get

$$\begin{aligned} (g \circ h)(s) &= g(h(s)) = g\left(\frac{2s}{s+1}\right) \\ &= 2 - \sqrt{\left(\frac{2s}{s+1}\right) + 3} \\ &= 2 - \sqrt{\frac{2s}{s+1} + \frac{3(s+1)}{s+1}} && \text{get common denominators} \\ &= 2 - \sqrt{\frac{5s+3}{s+1}} \end{aligned}$$

- *outside in*: We use the formula for $g(t)$ first to get

$$\begin{aligned} (g \circ h)(s) &= g(h(s)) = 2 - \sqrt{h(s) + 3} \\ &= 2 - \sqrt{\left(\frac{2s}{s+1}\right) + 3} \\ &= 2 - \sqrt{\frac{5s+3}{s+1}} && \text{get common denominators as before} \end{aligned}$$

To find the domain of $g \circ h$, we need the elements in the domain of h so that $h(s)$ is in the domain of g. Owing to the $s + 1$ in the denominator of the expression $h(s)$, we require $s \neq -1$. Once again, because of the square root in $g(t) = 2 - \sqrt{t+3}$, we need $t \geq -3$ or, in this case $h(s) \geq -3$. To use a sign diagram to solve, we rearrange this inequality:

$$\begin{aligned}
\frac{2s}{s+1} &\geq -3 \\
\frac{2s}{s+1}+3 &\geq 0 \\
\frac{5s+3}{s+1} &\geq 0 \qquad \text{get common denominators as before}
\end{aligned}$$

Defining $r(s) = \frac{5s+3}{s+1}$, we see r is undefined at $s = -1$ (a carry over from the domain restriction of h) and $r(s) = 0$ at $s = -\frac{3}{5}$. Our sign diagram is

$$\begin{array}{ccccc} (+) & ? & (-) & 0 & (+) \\ & -1 & & -\frac{3}{5} & \end{array}$$

hence our domain is $(-\infty, -1) \cup \left[-\frac{3}{5}, \infty\right)$.

7. We find $(h \circ g)(t)$ by finding $h(g(t))$.

   - *inside out*: We substitute the expression $g(t)$ for s in the formula $h(s)$

$$\begin{aligned}
(h \circ g)(t) &= h(g(t)) = h\left(2-\sqrt{t+3}\right) \\
&= \frac{2\left(2-\sqrt{t+3}\right)}{\left(2-\sqrt{t+3}\right)+1} \\
&= \frac{4-2\sqrt{t+3}}{3-\sqrt{t+3}}
\end{aligned}$$

   - *outside in*: We use the formula for $h(s)$ first to get

$$\begin{aligned}
(h \circ g)(t) &= h(g(t)) = \frac{2\,(g(t))}{(g(t))+1} \\
&= \frac{2\left(2-\sqrt{t+3}\right)}{\left(2-\sqrt{t+3}\right)+1} \\
&= \frac{4-2\sqrt{t+3}}{3-\sqrt{t+3}}
\end{aligned}$$

   To find the domain of $h \circ g$, we need the elements of the domain of g so that $g(t)$ is in the domain of h. As we've seen already, for t to be in the domain of g, $t \geq -3$. For s to be in the domain of h, $s \neq -1$, so we require $g(t) \neq -1$. Hence, we solve $g(t) = 2 - \sqrt{t+3} = -1$ with the intent of excluding the solutions. Isolating the radical expression gives $\sqrt{t+3} = 3$ or $t = 6$. Sure enough, we check $g(6) = -1$ so we exclude $t = 6$ from the domain of $h \circ g$. Our final answer is $[-3, 6) \cup (6, \infty)$.

8. To find $(h \circ h)(s)$ we find $h(h(s))$:

- *inside out*: We substitute the expression $h(s)$ for s in the expression $h(s)$ into h to get

$$\begin{aligned}
(h \circ h)(s) &= h(h(s)) = h\left(\frac{2s}{s+1}\right) \\
&= \frac{2\left(\dfrac{2s}{s+1}\right)}{\left(\dfrac{2s}{s+1}\right)+1} \\
&= \frac{\dfrac{4s}{s+1}}{\dfrac{2s}{s+1}+1} \cdot \frac{(s+1)}{(s+1)} \\
&= \frac{\dfrac{4s}{s+1} \cdot (s+1)}{\left(\dfrac{2s}{s+1}\right) \cdot (s+1) + 1 \cdot (s+1)} \\
&= \frac{\dfrac{4s}{\cancel{(s+1)}} \cdot \cancel{(s+1)}}{\dfrac{2s}{\cancel{(s+1)}} \cdot \cancel{(s+1)} + s + 1} \\
&= \frac{4s}{3s+1}
\end{aligned}$$

- *outside in*: This approach yields

$$\begin{aligned}
(h \circ h)(s) &= h(h(s)) = \frac{2(h(s))}{h(s)+1} \\
&= \frac{2\left(\dfrac{2s}{s+1}\right)}{\left(\dfrac{2s}{s+1}\right)+1} \\
&= \frac{4s}{3s+1} && \text{same algebra as before}
\end{aligned}$$

To find the domain of $h \circ h$, we need to find the elements in the domain of h so that the outputs, $h(s)$ are also in the domain of h. The only domain restriction for h comes from the denominator: $s \neq -1$, so in addition to this, we also need $h(s) \neq -1$. To this end, we solve $h(s) = -1$ and exclude the answers. Solving $\frac{2s}{s+1} = -1$ gives $s = -\frac{1}{3}$. The domain of $h \circ h$ is $(-\infty, -1) \cup \left(-1, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, \infty\right)$.

9. The expression $(h \circ (g \circ f))(x)$ indicates that we first find the composite, $g \circ f$ and compose the function h with the result. We know from number 4 that $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$ with domain $(-\infty, 1] \cup [3, \infty)$. We now proceed as usual.

- *inside out*: We substitute the expression $(g \circ f)(x)$ for s in the expression $h(s)$ first to get

$$\begin{aligned}(h \circ (g \circ f))(x) &= h((g \circ f)(x)) = h\left(2 - \sqrt{x^2 - 4x + 3}\right) \\ &= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

- *outside in*: We use the formula for $h(s)$ first to get

$$\begin{aligned}(h \circ (g \circ f))(x) &= h((g \circ f)(x)) = \frac{2\left((g \circ f)(x)\right)}{((g \circ f)(x)) + 1} \\ &= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

To find the domain of $h \circ (g \circ f)$, we need the domain elements of $g \circ f$, $(-\infty, 1] \cup [3, \infty)$, so that $(g \circ f)(x)$ is in the domain of h. As we've seen several times already, the only domain restriction for h is $s \neq -1$, so we set $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3} = -1$ and exclude the solutions. We get $\sqrt{x^2 - 4x + 3} = 3$, and, after squaring both sides, we have $x^2 - 4x + 3 = 9$. We solve $x^2 - 4x - 6 = 0$ using the quadratic formula and obtain $x = 2 \pm \sqrt{10}$. The reader is encouraged to check that both of these numbers satisfy the original equation, $2 - \sqrt{x^2 - 4x + 3} = -1$ and also belong to the domain of $g \circ f$, $(-\infty, 1] \cup [3, \infty)$, and so must be excluded from our final answer.[2] Our final domain for $h \circ (f \circ g)$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$.

10. The expression $((h \circ g) \circ f)(x)$ indicates that we first find the composite $h \circ g$ and then compose that with f. From number 7, we have

$$(h \circ g)(t) = \frac{4 - 2\sqrt{t + 3}}{3 - \sqrt{t + 3}}$$

with domain $[-3, 6) \cup (6, \infty)$.

- *inside out*: We substitute the expression $f(x)$ for t in the expression $(h \circ g)(t)$ to get

[2] We can approximate $\sqrt{10} \approx 3$ so $2 - \sqrt{10} \approx -1$ and $2 + \sqrt{10} \approx 5$.

$$\begin{aligned}((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = (h \circ g)\left(x^2 - 4x\right) \\ &= \frac{4 - 2\sqrt{\left(x^2 - 4x\right) + 3}}{3 - \sqrt{\left(x^2 - 4x\right) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

- *outside in*: We use the formula for $(h \circ g)(t)$ first to get

$$\begin{aligned}((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = \frac{4 - 2\sqrt{(f(x)) + 3}}{3 - \sqrt{f(x)) + 3}} \\ &= \frac{4 - 2\sqrt{\left(x^2 - 4x\right) + 3}}{3 - \sqrt{\left(x^2 - 4x\right) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

Since the domain of f is all real numbers, the challenge here to find the domain of $(h \circ g) \circ f$ is to determine the values $f(x)$ which are in the domain of $h \circ g$, $[-3, 6) \cup (6, \infty)$. At first glance, it appears as if we have two (or three!) inequalities to solve: $-3 \leq f(x) < 6$ and $f(x) > 6$. Alternatively, we could solve $f(x) = x^2 - 4x \geq -3$ and exclude the solutions to $f(x) = x^2 - 4x = 6$ which is not only easier from a procedural point of view, but also easier since we've already done both calculations. In number 4, we solved $x^2 - 4x \geq -3$ and obtained the solution $(-\infty, 1] \cup [3, \infty)$ and in number 9, we solved $x^2 - 4x - 6 = 0$ and obtained $x = 2 \pm \sqrt{10}$. Hence, the domain of $(h \circ g) \circ f$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$. $\square$

As previously mentioned, it should be clear from Example 5.3.1 that, in general, $g \circ f \neq f \circ g$, in other words, function composition is not *commutative*. However, numbers 9 and 10 demonstrate the **associative** property of function composition. That is, when composing three (or more) functions, as long as we keep the order the same, it doesn't matter which two functions we compose first. We summarize the important properties of function composition in the theorem below.

Theorem 5.4. Properties of Function Composition: Suppose f, g, and h are functions.

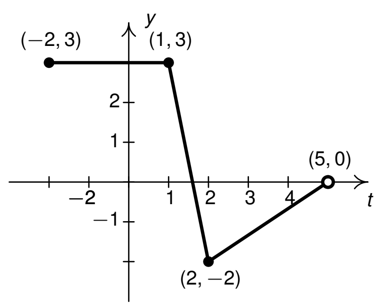
- **Associative Law of Composition:** $h \circ (g \circ f) = (h \circ g) \circ f$, provided the composite functions are defined.
- **Composition Identity:** The function $I(x) = x$ satisfies: $I \circ f = f \circ I = f$ for all functions, f.

By repeated applications of Definition 5.2, we find $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$. Similarly, $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$. This establishes that the formulas for the two functions are the same. We leave it to the reader to think about why the domains of these two functions are identical, too. These two facts establish the equality $h \circ (g \circ f) = (h \circ g) \circ f$. A consequence of the associativity of function composition is that there is no need for parentheses when we write $h \circ g \circ f$. The second property can also be verified using Definition 5.2. Recall that the function $I(x) = x$ is called the *identity function* and was introduced in Exercise 35 in Section 1.2. If we compose the function I with a function f, then we have $(I \circ f)(x) = I(f(x)) = f(x)$, and a similar computation shows $(f \circ I)(x) = f(I(x)) = f(x)$. This establishes that we have an identity for function composition much in the same way the function $I(x) = 1$ is an identity for function multiplication.

As we know, not all functions are described by formulas, and, moreover, not all functions are described by just *one* formula. The next example applies the concept of function composition to functions represented in various and sundry ways.

Example 5.3.2. Consider the following functions:

- $f(x) = 6x - x^2$
- $g(t) \begin{cases} 2t - 1 & \text{if } -1 \leq t < 3, \\ t^2 & \text{if } t \geq 3. \end{cases}$
- $h = \{(-3, 1), (-2, 6), (0, -2), (1, 5), (3, -1)\}$
- s whose graph is given below:



$y = s(t)$

1. Find and simplify the following function values:

   (a) $(g \circ f)(2)$ (b) $(h \circ g)(-1)$ (c) $(h \circ s)(-2)$ (d) $(f \circ s)(0)$

2. Find and simplify a formula for $(g \circ f)(x)$.

3. Write $s \circ h$ as a set of ordered pairs.

Solution.

1. (a) To find $(g \circ f)(2) = g(f(2))$ we first find $f(2) = 6(2) - (2)^2 = 8$. Since $8 \geq 3$, we use the rule $g(t) = t^2$ so $g(8) = (8)^2 = 64$. Hence, $(g \circ f)(3) = g(f(3)) = g(8) = 64$.

   (b) Since $(h \circ g)(-1) = h(g(-1))$ we first need $g(-1)$. Since $-1 \leq -1 < 3$, we use the rule $g(t) = 2t - 1$ and find $g(-1) = 2(-1) - 1 = -3$. Next, we need $h(-3)$. Since $(-3, 1) \in h$, we have that $h(-3) = 1$. Putting it all together, we find $(h \circ g)(-1) = h(g(-1)) = h(-3) = 1$.

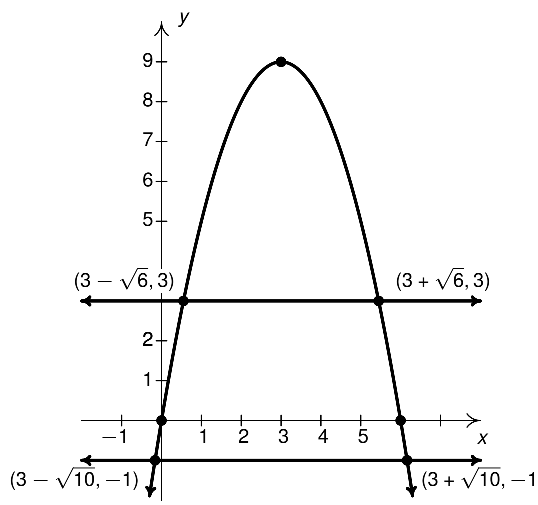
(c) To find $(h \circ s)(-2) = h(s(-2))$, we first need $s(-2)$. We see the point $(-2, 3)$ is on the graph of s, so $s(-2) = 3$. Next, we see $(3, -1) \in h$, so $h(3) = -1$. Hence, $(h \circ s)(-2) = h(s(-2)) = h(3) = -1$.

(d) To find $(f \circ s)(0) = f(s(0))$ we infer from the graph of s that it contains the point $(0, 3)$, so $s(0) = 3$. Sine $f(3) = 6(3) - (3)^2 = 9$, we have $(f \circ s)(0) = f(s(0)) = f(3) = 9$.

2. To find a formula for $(g \circ f)(x) = g(f(x))$, we substitute $f(x) = 6x - x^2$ in for t in the formula for $g(t)$:

$$(g \circ f)(x) = g(f(x)) = g(6x - x^2) = \begin{cases} 2(6x - x^2) - 1 & \text{if } -1 \leq 6x - x^2 < 3, \\ (6x - x^2)^2 & \text{if } 6x - x^2 \geq 3. \end{cases}$$

Simplifying each expression, we get $2(6x - x^2) - 1 = -2x^2 + 12x - 1$ for the first piece and $(6x - x^2)^2 = x^4 - 12x^3 + 36x^2$ for the second piece. The real challenge comes in solving the inequalities $-1 \leq 6x - x^2 < 3$ and $6x - x^2 \geq 3$. While we could solve each individually using a sign diagram, a graphical approach works best here. We graph the parabola $y = 6x - x^2$, finding the vertex is $(3, 9)$ with intercepts $(0, 0)$ and $(6, 0)$ along with the horizontal lines $y = -1$ and $y = 3$ below. We determine the intersection points by solving $6x - x^2 = -1$ and $6x - x^2 = 3$. Using the quadratic formula, we find the solutions to each equation are $x = 3 \pm \sqrt{10}$ and $x = 3 \pm \sqrt{6}$, respectively.

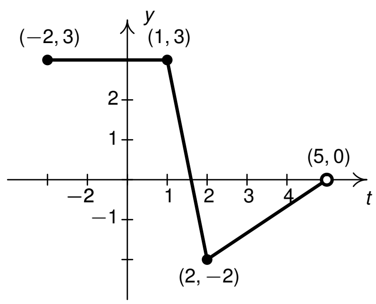


$y = 6x - x^2$, $y = -1$, $y = 3$

From the graph, we see the parabola $y = 6x - x^2$ is between the lines $y = -1$ and $y = 3$ from $x = 3 - \sqrt{10}$ to $x = 3 - \sqrt{6}$ and again from $x = 3 + \sqrt{6}$ to $x = 3 + \sqrt{10}$. Hence the solution to $-1 \leq 6x - x^2 < 3$ is $[3 - \sqrt{10}, 3 - \sqrt{6}) \cup (3 + \sqrt{6}, 3 + \sqrt{10}]$. We also note $y = 6x - x^2$ is above the line $y = 3$ for all x between $x = 3 - \sqrt{6}$ and $3 + \sqrt{6}$. Hence, the solution to $6x - x^2 \geq 3$ is $[3 - \sqrt{6}, 3 + \sqrt{6}]$. Hence,

$$(g \circ f)(x) = \begin{cases} -2x^2 + 12x - 1 & \text{if } x \in [3 - \sqrt{10}, 3 - \sqrt{6}) \cup (3 + \sqrt{6}, 3 + \sqrt{10}], \\ x^4 - 12x^3 + +36x^2 & \text{if } x \in [3 - \sqrt{6}, 3 + \sqrt{6}]. \end{cases}$$

3. Last but not least, we are tasked with representing $s \circ h$ as a set of ordered pairs. Since h is described by the discrete set of points $h = \{(-3,1), (-2,6), (0,-2), (1,5), (3,-1)\}$, we find $s \circ h$ point by point. We keep the graph of s handy and construct the table below to help us organize our work.



$y = s(t)$

x	$h(x)$	$s(h(x))$
-3	1	3
-2	6	undefined
0	-2	3
1	5	undefined
3	-1	3

Since neither 6 nor 5 are in the domain of s, -2 and 1 are not in the domain of $s \circ h$. Hence, we get $s \circ h = \{(-3,3), (0,3), (3,3)\}$. □

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions which our last example illustrates. As with Example 5.2.2, we want to avoid trivial decompositions, which, when it comes to function composition, are those involving the identity function $I(x) = x$ as described in Theorem 5.4.

Example 5.3.3.

1. Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

   (a) $F(x) = |3x - 1|$ (b) $G(t) = \dfrac{2}{t^2+1}$ (c) $H(s) = \dfrac{\sqrt{s}+1}{\sqrt{s}-1}$

2. For $F(x) = \sqrt{\dfrac{2x-1}{x^2+4}}$, find functions f, g, and h to decompose F nontrivially as $F = f \circ \left(\dfrac{g}{h}\right)$.

Solution. There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. (a) Our goal is to express the function F as $F = g \circ f$ for functions g and f. From Definition 5.2, we know $F(x) = g(f(x))$, and we can think of $f(x)$ as being the 'inside' function and g as being the 'outside' function. Looking at $F(x) = |3x - 1|$ from an 'inside versus outside' perspective, we can think of $3x - 1$ being inside the absolute value symbols. Taking this cue, we define $f(x) = 3x - 1$. At this point, we have $F(x) = |f(x)|$. What is the outside function? The function which takes the absolute value of its input, $g(x) = |x|$. Sure enough, this checks: $(g \circ f)(x) = g(f(x)) = |f(x)| = |3x - 1| = F(x)$.

(b) We attack deconstructing G from an operational approach. Given an input t, the first step is to square t, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write G as a composite of *three* functions: f, g and h. Our first function, f, is the function that squares its input, $f(t) = t^2$. The next function is the function that adds 1 to its input, $g(t) = t + 1$. Our last function takes its input and divides it into 2, $h(t) = \frac{2}{t}$. The claim is that $G = h \circ g \circ f$ which checks:

$$(h \circ g \circ f)(t) = h(g(f(t))) = h(g\left(t^2\right)) = h\left(t^2 + 1\right) = \frac{2}{t^2 + 1} = G(x).$$

(c) If we look $H(s) = \frac{\sqrt{s}+1}{\sqrt{s}-1}$ with an eye towards building a complicated function from simpler functions, we see the expression $\sqrt{s}$ is a simple piece of the larger function. If we define $f(s) = \sqrt{s}$, we have $H(s) = \frac{f(s)+1}{f(s)-1}$. If we want to decompose $H = g \circ f$, then we can glean the formula for $g(s)$ by looking at what is being done to $f(s)$. We take $g(s) = \frac{s+1}{s-1}$, and check below:

$$(g \circ f)(s) = g(f(s)) = \frac{f(s) + 1}{f(s) - 1} = \frac{\sqrt{s} + 1}{\sqrt{s} - 1} = H(s).$$

□

2. To write $F = f \circ \left(\frac{g}{h}\right)$ means

$$F(x) = \sqrt{\frac{2x - 1}{x^2 + 4}} = \left(f \circ \left(\frac{g}{h}\right)\right)(x) = f\left(\left(\frac{g}{h}\right)(x)\right) = f\left(\frac{g(x)}{h(x)}\right).$$

Working from the inside out, we have a rational expression with numerator $g(x)$ and denominator $h(x)$. Looking at the formula for $F(x)$, one choice is $g(x) = 2x - 1$ and $h(x) = x^2 + 4$. Making these identifications, we have

$$F(x) = \sqrt{\frac{2x - 1}{x^2 + 4}} = \sqrt{\frac{g(x)}{h(x)}}.$$

Since F takes the square root of $\frac{g(x)}{h(x)}$, the our last function f is the function that takes the square root of its input, i.e., $f(x) = \sqrt{x}$. We leave it to the reader to check that, indeed, $F = f \circ \left(\frac{g}{h}\right)$. □

We close this section of a real-world application of function composition.

Example 5.3.4. The surface area of a sphere is a function of its radius r and is given by the formula $S(r) = 4\pi r^2$. Suppose a spherical balloon is inflated so that the radius of the sphere is increasing according to the formula $r(t) = 2t$, where t is measured in minutes (min), $t \geq 0$, and r is measured in centimeters (cm). Find and interpret $(S \circ r)(t)$.

Solution. The function $S(r)$ gives the surface area of the sphere and $r(t)$ gives the radius of the sphere at a given time. Given a specific time, t, we find the radius at that time, $r(t)$ and feed that into $S(r)$ to find the surface area. Hence, the surface area S is ultimately a function of time t and we find $(S \circ r)(t) = S(r(t)) = 4\pi(r(t))^2 = 4\pi(2t)^2 = 16\pi t^2$. This formula allows us to compute the surface area directly given the time without going through the 'intermediary variable' r. □

5.3.1 Related Rates

In Section 5.2.1, we studied the difference operator, Δ and showed how average rates of change operate with the basic function arithmetic. In this section, we explore how rates of change of composite functions are related to the rates of change of their constituent functions. As in that section, we'll use the formulation:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0,$$

adjusting the names of functions and independent variables as needed.

As a motivational example, we revisit the scenario in Example 5.3.4.

Example 5.3.5. The surface area of a sphere is a function of its radius r and is given by the formula $S(r) = 4\pi r^2$. Suppose a spherical balloon is inflated so that the radius of the sphere is increasing according to the formula $r(t) = 2t$, where t is measured in minutes (min), $t \geq 0$, and r is measured in centimeters (cm).

1. Find and simplify an expression for the average rate of change of S with respect to r. Find and interpret the average rate of change of S with respect to r over the interval $[1, 3]$.
2. Find, simplify, and interpret an expression for the average rate of change of r with respect to t.
3. Find and simplify an expression for the average rate of change of S with respect to t. Find and interpret the average rate of change of S with respect to t over the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$.
4. Multiply your answers to 1 and 2 and compare those to your answer in 3.

Solution. It is important to note that as we work through the expressions below, the variables r and Δr as well as t and Δt are distinct. That is, they do not combine as 'like terms.'

1. We start by simplifying $\dfrac{\Delta[S(r)]}{\Delta r} = \dfrac{S(r+\Delta r) - S(r)}{\Delta r}$:

$$\begin{aligned}
\frac{\Delta[S(r)]}{\Delta r} &= \frac{S(r+\Delta r) - S(r)}{\Delta r} \\
&= \frac{4\pi(r+\Delta r)^2 - 4\pi r^2}{\Delta r} \\
&= \frac{4\pi\left[r^2 + 2r\,\Delta r + (\Delta r)^2\right] - 4\pi r^2}{\Delta r} \\
&= \frac{4\pi r^2 + 8\pi r\,\Delta r + 4\pi(\Delta r)^2 - 4\pi r^2}{\Delta r} \\
&= \frac{8\pi r\,\Delta r + 4\pi(\Delta r)^2}{\Delta r}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(\Delta r)\,(8\pi r + 4\pi\,\Delta r)}{\Delta r} \\
&= \frac{\cancel{(\Delta r)}\,(8\pi r + 4\pi\,\Delta r)}{\cancel{(\Delta r)}^{1}} \\
&= 8\pi r + 4\pi\,\Delta r
\end{aligned}$$

To find the average rate of change of S over the interval $[1, 3]$, we take $r = 1$ and $\Delta r = 3 - 1 = 2$:

$$\frac{\Delta[S(r)]}{\Delta r} = 8\pi(1) + 4\pi(2) = 16\pi.$$

This means as the radius of the balloon increases from 1 centimeter to 3 centimeters, the surface area is increasing at an average rate of $16\pi\,\frac{\text{cm}^2}{\text{cm}}$.

Note that the units here, cm, do cancel and we could write the average rate of change as 16π cm. This somewhat hides the fact this number represents a ratio. Any time area and length are measured in compatible units, the ratio of units $\frac{\text{area}}{\text{length}}$ will simplify to units of length.[3]

2. Next, we simplify $\dfrac{\Delta[r(t)]}{\Delta t} = \dfrac{r(t+\Delta t) - r(t)}{\Delta t}$:

$$\begin{aligned}
\frac{\Delta[r(t)]}{\Delta t} &= \frac{r(t+\Delta t) - r(t)}{\Delta t} \\
&= \frac{2(t+\Delta t) - 2t}{\Delta t} \\
&= \frac{2t + 2\Delta t - 2t}{\Delta t} \\
&= \frac{2\cancel{\Delta t}}{\cancel{(\Delta t)}^{1}} \\
&= 2
\end{aligned}$$

The fact that the average rate of change here is constant shouldn't be too surprising. Note $r(t) = 2t$ is a linear function with slope 2. Hence, '2' is the (constant) rate of change of r.[4] This means that the radius of the balloon is increasing at a constant rate of $2\,\frac{\text{cm}}{\text{min}}$.

3. To find $\dfrac{\Delta[S(t)]}{\Delta t} = \dfrac{S(t+\Delta t) - S(t)}{\Delta t}$, we start with our answer from Example 5.3.4: $S(t) = 16\pi t^2$:

$$\frac{\Delta[S(t)]}{\Delta t} = \frac{S(t+\Delta t) - S(t)}{\Delta t}$$

[3] As always, context is key!

[4] We could probably have lead with that and avoided some tedious computations ...

$$\begin{aligned}
&= \frac{16\pi(t+\Delta t)^2 - 16\pi t^2}{\Delta t} \\
&= \frac{16\pi\left[t^2 + 2t\,\Delta t + (\Delta t)^2\right] - 16\pi t^2}{\Delta t} \\
&= \frac{16\pi t^2 + 32\pi t\,\Delta t + 16\pi(\Delta t)^2 - 16\pi t^2}{\Delta t} \\
&= \frac{32\pi t\,\Delta t + 16\pi(\Delta t)^2}{\Delta t} \\
&= \frac{(\Delta t)(32\pi t + 16\pi\,\Delta t)}{\Delta t} \\
&= \frac{\cancel{(\Delta t)}(32\pi t + 16\pi\,\Delta t)}{\cancel{(\Delta t)}^{1}} \\
&= 32\pi t + 16\pi\,\Delta t
\end{aligned}$$

To find the average rate of change of S over the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$, we take $t = \frac{1}{2}$ and $\Delta t = \frac{3}{2} - \frac{1}{2} = 1$:

$$\frac{\Delta[S(t)]}{\Delta t} = 32\pi\left(\frac{1}{2}\right) + 16\pi(1) = 32\pi.$$

This means the surface area of the balloon is increasing at an average rate of $32\pi\,\frac{\text{cm}^2}{\text{min}}$ over the time span of $\frac{1}{2}$ minute (30 seconds) after the start of inflation to $\frac{3}{2}$ (90 seconds) after the start of inflation.

4. We begin with:

$$\frac{\Delta[S(r)]}{\Delta r} \cdot \frac{\Delta[r(t)]}{\Delta t} \quad = \quad (8\pi r + 4\pi\,\Delta r)(2),$$

which doesn't look like much unless we substitute $r = 2t$ and $\Delta r = 2\Delta t$. We get:

$$\begin{aligned}
\frac{\Delta[S(r)]}{\Delta r} \cdot \frac{\Delta[r(t)]}{\Delta t} &= (8\pi r + 4\pi\,\Delta r)(2) \\
&= (8\pi(2t) + 4\pi\,(2\Delta t)(2) \\
&= (16\pi t + 8\pi\,\Delta t)(2) \\
&= 32\pi t + 16\pi\,\Delta t
\end{aligned}$$

We find in this case,

$$\frac{\Delta[S(r)]}{\Delta r} \cdot \frac{\Delta[r(t)]}{\Delta t} = \frac{\Delta[S(r)]}{\Delta t}.$$

Moreover, we note that the time interval $\frac{1}{2} \leq t \leq \frac{3}{2}$ corresponds to the interval $1 \leq r \leq 3$ so it makes sense to multiply our numerical answers as well:

$$\frac{\Delta[S(r)]}{\Delta r} \cdot \frac{\Delta[r(t)]}{\Delta t} = 16\pi \text{ cm} \cdot 2 \, \frac{\text{cm}}{\text{min}} = 32\pi \, \frac{\text{cm}^2}{\text{min}}.$$

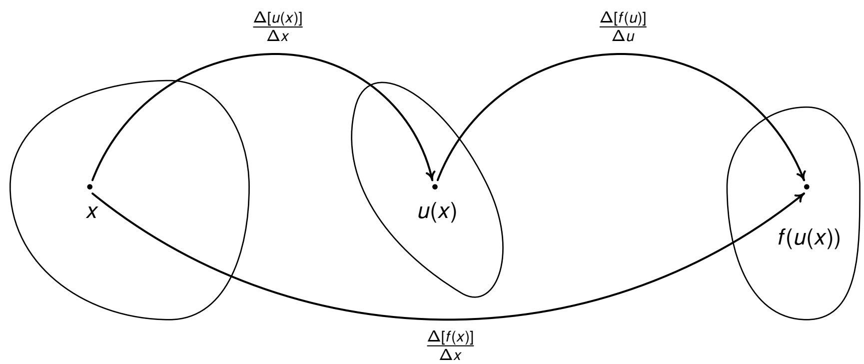
□

Example 5.3.5 verifies a property of rates we formalize below.

Theorem 5.5. Related Rates: Let f and u be functions where u is defined over an interval containing x and $x + \Delta x$ and f is defined over an interval containing $u(x)$ and $u(x + \Delta x)$. Then:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{\Delta[f(u)]}{\Delta u} \cdot \frac{\Delta[u(x)]}{\Delta x}, \quad \Delta x \neq 0, \Delta u \neq 0.$$

If we think of u as being an 'intermediary' variable, Theorem 5.5 allows us to determine the rate of change of f with respect to x by multiplying the rate of change of f with respect to this 'intermediary' u by the rate of change of the 'intermediary' u with respect to x. That is, we are looking for rate information on how f depends on x by decomposing the rate into two rates as visualized below.



We close the section with one last example.

Example 5.3.6. The drag force F, in Newtons (N), of a perfectly fine OER Precalculus Textbook which has been discarded off of a cliff because it didn't have enough Calculus in it is given by: $F(v) = 0.6\,v^2$, where v is the speed of the book as it hurtles towards the Earth. If the speed is increasing at a constant rate of 9.8 meters per second per second, $\frac{\text{m/s}}{\text{s}}$, determine the rate of change of F with respect to time as the speed changes from 5 $\frac{\text{m}}{\text{s}}$ to 6 $\frac{\text{m}}{\text{s}}$

Solution. In this scenario, the drag force, F depends directly on the speed, v and the speed, v depends directly on time. (The longer the book falls, the faster it falls.[5]) By Theorem 5.5, we know

$$\frac{\Delta[F(t)]}{\Delta t} = \frac{\Delta[F(v)]}{\Delta v} \cdot \frac{\Delta v}{\Delta t}.$$

[5]Well until it reaches terminal velocity . . .

We are told that the speed is increasing at a constant rate of 9.8 meters per second per second, so we know $\frac{\Delta v}{\Delta t} = 9.8 \, \frac{\text{m/s}}{\text{s}}$ for all time (and hence, speeds.).

To find $\frac{\Delta[F(v)]}{\Delta v}$ as the speed changes from 5 $\frac{\text{m}}{\text{s}}$ to 6 $\frac{\text{m}}{\text{s}}$, we calculate:

$$\frac{\Delta[F(v)]}{\Delta v} = \frac{F(6) - F(5)}{6 - 5} = \frac{0.6(6)^2 - 0.6(5)^2}{1} = 6.6.$$

The units on $\frac{\Delta[F(v)]}{\Delta v}$ would be the units of F, N, divided by the units of v, $\frac{\text{m}}{\text{s}}$ which works out[6] to $\frac{\text{N s}}{\text{m}}$.

Hence,

$$\frac{\Delta[F(t)]}{\Delta t} = \frac{\Delta[F(v)]}{\Delta v} \cdot \frac{\Delta v}{\Delta t} = \left(6.6 \, \frac{\text{N s}}{\text{m}}\right)\left(9.8 \, \frac{\text{m/s}}{\text{s}}\right) = 64.68 \, \frac{\text{N}}{\text{s}}.$$

The force is increasing at an average rate of 64.68 Newtons per second. □

Note that we never needed to know explicitly how the speed, v, directly depended on time in order to answer the question posed in Example 5.3.6. All we needed was the rate.

[6]Note: we could simplify this a bit farther. A Newton, N, has units $\frac{\text{kg m}}{\text{s}^2}$, so some units could cancel to give $\frac{\text{kg}}{\text{s}}$. We leave things as they are for now for a more simple calculation later.

5.3.2 Exercises

In Exercises 1 - 12, use the given pair of functions to find the following values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

1. $f(x) = x^2$, $g(t) = 2t + 1$
2. $f(x) = 4 - x$, $g(t) = 1 - t^2$
3. $f(x) = 4 - 3x$, $g(t) = |t|$
4. $f(x) = |x - 1|$, $g(t) = t^2 - 5$
5. $f(x) = 4x + 5$, $g(t) = \sqrt{t}$
6. $f(x) = \sqrt{3 - x}$, $g(t) = t^2 + 1$
7. $f(x) = 6 - x - x^2$, $g(t) = t\sqrt{t + 10}$
8. $f(x) = \sqrt[3]{x + 1}$, $g(t) = 4t^2 - t$
9. $f(x) = \dfrac{3}{1 - x}$, $g(t) = \dfrac{4t}{t^2 + 1}$
10. $f(x) = \dfrac{x}{x + 5}$, $g(t) = \dfrac{2}{7 - t^2}$
11. $f(x) = \dfrac{2x}{5 - x^2}$, $g(t) = \sqrt{4t + 1}$
12. $f(x) = \sqrt{2x + 5}$, $g(t) = \dfrac{10t}{t^2 + 1}$

In Exercises 13 - 24, use the given pair of functions to find and simplify expressions for the following functions and state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(t)$
- $(f \circ f)(x)$

13. $f(x) = 2x + 3$, $g(t) = t^2 - 9$
14. $f(x) = x^2 - x + 1$, $g(t) = 3t - 5$
15. $f(x) = x^2 - 4$, $g(t) = |t|$
16. $f(x) = 3x - 5$, $g(t) = \sqrt{t}$
17. $f(x) = |x + 1|$, $g(t) = \sqrt{t}$
18. $f(x) = 3 - x^2$, $g(t) = \sqrt{t + 1}$
19. $f(x) = |x|$, $g(t) = \sqrt{4 - t}$
20. $f(x) = x^2 - x - 1$, $g(t) = \sqrt{t - 5}$
21. $f(x) = 3x - 1$, $g(t) = \dfrac{1}{t + 3}$
22. $f(x) = \dfrac{3x}{x - 1}$, $g(t) = \dfrac{t}{t - 3}$
23. $f(x) = \dfrac{x}{2x + 1}$, $g(t) = \dfrac{2t + 1}{t}$
24. $f(x) = \dfrac{2x}{x^2 - 4}$, $g(t) = \sqrt{1 - t}$

In Exercises 25 - 30, use $f(x) = -2x$, $g(t) = \sqrt{t}$ and $h(s) = |s|$ to find and simplify expressions for the following functions and state the domain of each using interval notation.

25. $(h \circ g \circ f)(x)$
26. $(h \circ f \circ g)(t)$
27. $(g \circ f \circ h)(s)$
28. $(g \circ h \circ f)(x)$
29. $(f \circ h \circ g)(t)$
30. $(f \circ g \circ h)(s)$

In Exercises 31 - 43, let f be the function defined by

$$f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

and let g be the function defined by

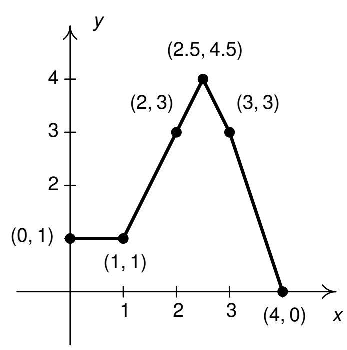
$$g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}.$$

Find the following, if it exists.

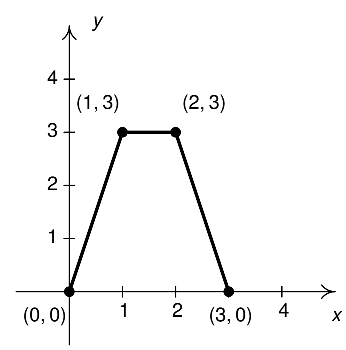
31. $(f \circ g)(3)$
32. $f(g(-1))$
33. $(f \circ f)(0)$
34. $(f \circ g)(-3)$
35. $(g \circ f)(3)$
36. $g(f(-3))$
37. $(g \circ g)(-2)$
38. $(g \circ f)(-2)$
39. $g(f(g(0)))$
40. $f(f(f(-1)))$
41. $f(f(f(f(f(1)))))$
42. $\underbrace{(g \circ g \circ \cdots \circ g)}_{n \text{ times}}(0)$

43. Find the domain and range of $f \circ g$ and $g \circ f$.

In Exercises 44 - 50, use the graphs of $y = f(x)$ and $y = g(x)$ below to find the following if it exists.



$y = f(x)$



$y = g(x)$

44. $(g \circ f)(1)$
45. $(f \circ g)(3)$
46. $(g \circ f)(2)$
47. $(f \circ g)(0)$
48. $(f \circ f)(4)$
49. $(g \circ g)(1)$

50. Find the domain and range of $f \circ g$ and $g \circ f$.

In Exercises 51 - 60, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

51. $p(x) = (2x+3)^3$

52. $P(x) = \left(x^2 - x + 1\right)^5$

53. $h(t) = \sqrt{2t-1}$

54. $H(t) = |7-3t|$

55. $r(s) = \dfrac{2}{5s+1}$

56. $R(s) = \dfrac{7}{s^2-1}$

57. $q(z) = \dfrac{|z|+1}{|z|-1}$

58. $Q(z) = \dfrac{2z^3+1}{z^3-1}$

59. $v(x) = \dfrac{2x+1}{3-4x}$

60. $w(x) = \dfrac{x^2}{x^4+1}$

61. Write the function $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}}$ as a composition of three or more non-identity functions.

62. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?

63. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?

64. Let $f(x) = 3x+1$ and let $g(x) = \begin{cases} 2x-1 & \text{if } x \leq 3 \\ 4-x & \text{if } x > 3 \end{cases}$. Find expressions for $(f \circ g)(x)$ and $(g \circ f)(x)$.

65. The volume V of a cube is a function of its side length x. Let's assume that $x = t+1$ is also a function of time t, where x is measured in inches and t is measured in minutes. Find a formula for V as a function of t.

66. Suppose a local vendor charges \$2 per hot dog and that the number of hot dogs sold per hour x is given by $x(t) = -4t^2 + 20t + 92$, where t is the number of hours since 10 AM, $0 \leq t \leq 4$.

    (a) Find an expression for the revenue per hour R as a function of x.

    (b) Find and simplify $(R \circ x)(t)$. What does this represent?

    (c) What is the revenue per hour at noon?

    (d) Using Example 5.3.5 as a guide, verify $\frac{\Delta[R(x)]}{\Delta x} \cdot \frac{\Delta[x(t)]}{\Delta t} = \frac{\Delta[R(t)]}{\Delta t}$.

67. The book in Example 5.3.6 plunges into a lake and generates a circular wave pattern. If the waves are tracked as traveling at a constant 0.5 meters per second $\left(\frac{\text{m}}{\text{s}}\right)$, use Theorem 5.5 to find the rate at which the area of the disturbance is changing with respect to time as the radius changes from $r = 1$ to $r = 1.1$ meters (m). Be sure to include units on your answer.

    HINT: Recall the area, A, enclosed by a circle of radius r is given by $A = \pi r^2$. Here, $\frac{\Delta r}{\Delta t} = 0.5 \, \frac{\text{m}}{\text{s}}$.

68. Perfectly fine precalculus textbooks which have no Calculus content are being fed into a shredder at a rate of 2 books per minute in order to make room for precalculus textbooks with Calculus content. The shredder creates a pile of debris which is in the shape of a right circular cone whose height is twice its width.

    (a) Assume the volume of the conical pile, V, is given by $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the base of the pile and h is the height of the pile. Given the pile is twice as tall as it is wide, show we can write $V = \frac{4}{3}\pi r^3$.

    (b) Assuming a typical precalculus textbook is 0.10 cubic feet $\left(\text{ft}^3\right)$, use Theorem 5.5 to find the rate of change of the radius of the pile with respect to time as the radius changes from 2 to 2.1 feet. Be sure to include units on your answer.

69. Discuss with your classmates how 'real-world' processes such as filling out federal income tax forms or computing your final course grade could be viewed as a use of function composition. Find a process for which composition with itself (iteration) makes sense.

5.3.3 Answers

1. For $f(x) = x^2$ and $g(t) = 2t + 1$,
   - $(g \circ f)(0) = 1$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = 16$
   - $(g \circ f)(-3) = 19$
   - $(f \circ g)\left(\frac{1}{2}\right) = 4$
   - $(f \circ f)(-2) = 16$

2. For $f(x) = 4 - x$ and $g(t) = 1 - t^2$,
   - $(g \circ f)(0) = -15$
   - $(f \circ g)(-1) = 4$
   - $(f \circ f)(2) = 2$
   - $(g \circ f)(-3) = -48$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{13}{4}$
   - $(f \circ f)(-2) = -2$

3. For $f(x) = 4 - 3x$ and $g(t) = |t|$,
   - $(g \circ f)(0) = 4$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = 10$
   - $(g \circ f)(-3) = 13$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{5}{2}$
   - $(f \circ f)(-2) = -26$

4. For $f(x) = |x - 1|$ and $g(t) = t^2 - 5$,
   - $(g \circ f)(0) = -4$
   - $(f \circ g)(-1) = 5$
   - $(f \circ f)(2) = 0$
   - $(g \circ f)(-3) = 11$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{23}{4}$
   - $(f \circ f)(-2) = 2$

5. For $f(x) = 4x + 5$ and $g(t) = \sqrt{t}$,
   - $(g \circ f)(0) = \sqrt{5}$
   - $(f \circ g)(-1)$ is not real
   - $(f \circ f)(2) = 57$
   - $(g \circ f)(-3)$ is not real
   - $(f \circ g)\left(\frac{1}{2}\right) = 5 + 2\sqrt{2}$
   - $(f \circ f)(-2) = -7$

6. For $f(x) = \sqrt{3 - x}$ and $g(t) = t^2 + 1$,
   - $(g \circ f)(0) = 4$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = \sqrt{2}$
   - $(g \circ f)(-3) = 7$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt{7}}{2}$
   - $(f \circ f)(-2) = \sqrt{3 - \sqrt{5}}$

7. For $f(x) = 6 - x - x^2$ and $g(t) = t\sqrt{t + 10}$,
   - $(g \circ f)(0) = 24$
   - $(f \circ g)(-1) = 0$
   - $(f \circ f)(2) = 6$
   - $(g \circ f)(-3) = 0$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{27 - 2\sqrt{42}}{8}$
   - $(f \circ f)(-2) = -14$

8. For $f(x) = \sqrt[3]{x+1}$ and $g(t) = 4t^2 - t$,

- $(g \circ f)(0) = 3$
- $(f \circ g)(-1) = \sqrt[3]{6}$
- $(f \circ f)(2) = \sqrt[3]{\sqrt[3]{3} + 1}$
- $(g \circ f)(-3) = 4\sqrt[3]{4} + \sqrt[3]{2}$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt[3]{12}}{2}$
- $(f \circ f)(-2) = 0$

9. For $f(x) = \frac{3}{1-x}$ and $g(t) = \frac{4t}{t^2+1}$,

- $(g \circ f)(0) = \frac{6}{5}$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = \frac{3}{4}$
- $(g \circ f)(-3) = \frac{48}{25}$
- $(f \circ g)\left(\frac{1}{2}\right) = -5$
- $(f \circ f)(-2)$ is undefined

10. For $f(x) = \frac{x}{x+5}$ and $g(t) = \frac{2}{7-t^2}$,

- $(g \circ f)(0) = \frac{2}{7}$
- $(f \circ g)(-1) = \frac{1}{16}$
- $(f \circ f)(2) = \frac{2}{37}$
- $(g \circ f)(-3) = \frac{8}{19}$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{8}{143}$
- $(f \circ f)(-2) = -\frac{2}{13}$

11. For $f(x) = \frac{2x}{5-x^2}$ and $g(t) = \sqrt{4t+1}$,

- $(g \circ f)(0) = 1$
- $(f \circ g)(-1)$ is not real
- $(f \circ f)(2) = -\frac{8}{11}$
- $(g \circ f)(-3) = \sqrt{7}$
- $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{3}$
- $(f \circ f)(-2) = \frac{8}{11}$

12. For $f(x) = \sqrt{2x+5}$ and $g(t) = \frac{10t}{t^2+1}$,

- $(g \circ f)(0) = \frac{5\sqrt{5}}{3}$
- $(f \circ g)(-1)$ is not real
- $(f \circ f)(2) = \sqrt{11}$
- $(g \circ f)(-3)$ is not real
- $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{13}$
- $(f \circ f)(-2) = \sqrt{7}$

13. For $f(x) = 2x + 3$ and $g(t) = t^2 - 9$

- $(g \circ f)(x) = 4x^2 + 12x$, domain: $(-\infty, \infty)$
- $(f \circ g)(t) = 2t^2 - 15$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = 4x + 9$, domain: $(-\infty, \infty)$

14. For $f(x) = x^2 - x + 1$ and $g(t) = 3t - 5$

- $(g \circ f)(x) = 3x^2 - 3x - 2$, domain: $(-\infty, \infty)$
- $(f \circ g)(t) = 9t^2 - 33t + 31$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = x^4 - 2x^3 + 2x^2 - x + 1$, domain: $(-\infty, \infty)$

15. For $f(x) = x^2 - 4$ and $g(t) = |t|$

    - $(g \circ f)(x) = |x^2 - 4|$, domain: $(-\infty, \infty)$
    - $(f \circ g)(t) = |t|^2 - 4 = t^2 - 4$, domain: $(-\infty, \infty)$
    - $(f \circ f)(x) = x^4 - 8x^2 + 12$, domain: $(-\infty, \infty)$

16. For $f(x) = 3x - 5$ and $g(t) = \sqrt{t}$

    - $(g \circ f)(x) = \sqrt{3x - 5}$, domain: $\left[\frac{5}{3}, \infty\right)$
    - $(f \circ g)(t) = 3\sqrt{t} - 5$, domain: $[0, \infty)$
    - $(f \circ f)(x) = 9x - 20$, domain: $(-\infty, \infty)$

17. For $f(x) = |x + 1|$ and $g(t) = \sqrt{t}$

    - $(g \circ f)(x) = \sqrt{|x + 1|}$, domain: $(-\infty, \infty)$
    - $(f \circ g)(t) = |\sqrt{t} + 1| = \sqrt{t} + 1$, domain: $[0, \infty)$
    - $(f \circ f)(x) = ||x + 1| + 1| = |x + 1| + 1$, domain: $(-\infty, \infty)$

18. For $f(x) = 3 - x^2$ and $g(t) = \sqrt{t + 1}$

    - $(g \circ f)(x) = \sqrt{4 - x^2}$, domain: $[-2, 2]$
    - $(f \circ g)(t) = 2 - t$, domain: $[-1, \infty)$
    - $(f \circ f)(x) = -x^4 + 6x^2 - 6$, domain: $(-\infty, \infty)$

19. For $f(x) = |x|$ and $g(t) = \sqrt{4 - t}$

    - $(g \circ f)(x) = \sqrt{4 - |x|}$, domain: $[-4, 4]$
    - $(f \circ g)(t) = |\sqrt{4 - t}| = \sqrt{4 - t}$, domain: $(-\infty, 4]$
    - $(f \circ f)(x) = ||x|| = |x|$, domain: $(-\infty, \infty)$

20. For $f(x) = x^2 - x - 1$ and $g(t) = \sqrt{t - 5}$

    - $(g \circ f)(x) = \sqrt{x^2 - x - 6}$, domain: $(-\infty, -2] \cup [3, \infty)$
    - $(f \circ g)(t) = t - 6 - \sqrt{t - 5}$, domain: $[5, \infty)$
    - $(f \circ f)(x) = x^4 - 2x^3 - 2x^2 + 3x + 1$, domain: $(-\infty, \infty)$

21. For $f(x) = 3x - 1$ and $g(t) = \frac{1}{t+3}$

    - $(g \circ f)(x) = \frac{1}{3x+2}$, domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
    - $(f \circ g)(t) = -\frac{t}{t+3}$, domain: $(-\infty, -3) \cup (-3, \infty)$
    - $(f \circ f)(x) = 9x - 4$, domain: $(-\infty, \infty)$

22. For $f(x) = \frac{3x}{x-1}$ and $g(t) = \frac{t}{t-3}$

- $(g \circ f)(x) = x$, domain: $(-\infty, 1) \cup (1, \infty)$
- $(f \circ g)(t) = t$, domain: $(-\infty, 3) \cup (3, \infty)$
- $(f \circ f)(x) = \frac{9x}{2x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 1\right) \cup (1, \infty)$

23. For $f(x) = \frac{x}{2x+1}$ and $g(t) = \frac{2t+1}{t}$

- $(g \circ f)(x) = \frac{4x+1}{x}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right), \cup (0, \infty)$
- $(f \circ g)(t) = \frac{2t+1}{5t+2}$, domain: $\left(-\infty, -\frac{2}{5}\right) \cup \left(-\frac{2}{5}, 0\right) \cup (0, \infty)$
- $(f \circ f)(x) = \frac{x}{4x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$

24. For $f(x) = \frac{2x}{x^2-4}$ and $g(t) = \sqrt{1-t}$

- $(g \circ f)(x) = \sqrt{\frac{x^2-2x-4}{x^2-4}}$, domain: $(-\infty, -2) \cup \left[1-\sqrt{5}, 2\right) \cup \left[1+\sqrt{5}, \infty\right)$
- $(f \circ g)(t) = -\frac{2\sqrt{1-t}}{t+3}$, domain: $(-\infty, -3) \cup (-3, 1]$
- $(f \circ f)(x) = \frac{4x-x^3}{x^4-9x^2+16}$, domain: $\left(-\infty, -\frac{1+\sqrt{17}}{2}\right) \cup \left(-\frac{1+\sqrt{17}}{2}, -2\right) \cup \left(-2, \frac{1-\sqrt{17}}{2}\right) \cup \left(\frac{1-\sqrt{17}}{2}, \frac{-1+\sqrt{17}}{2}\right) \cup \left(\frac{-1+\sqrt{17}}{2}, 2\right) \cup \left(2, \frac{1+\sqrt{17}}{2}\right) \cup \left(\frac{1+\sqrt{17}}{2}, \infty\right)$

25. $(h \circ g \circ f)(x) = |\sqrt{-2x}| = \sqrt{-2x}$, domain: $(-\infty, 0]$

26. $(h \circ f \circ g)(t) = |-2\sqrt{t}| = 2\sqrt{t}$, domain: $[0, \infty)$

27. $(g \circ f \circ h)(s) = \sqrt{-2|s|}$, domain: $\{0\}$

28. $(g \circ h \circ f)(x) = \sqrt{|-2x|} = \sqrt{2|x|}$, domain: $(-\infty, \infty)$

29. $(f \circ h \circ g)(t) = -2|\sqrt{t}| = -2\sqrt{t}$, domain: $[0, \infty)$

30. $(f \circ g \circ h)(s) = -2\sqrt{|s|}$, , domain: $(-\infty, \infty)$

31. $(f \circ g)(3) = f(g(3)) = f(2) = 4$

32. $f(g(-1)) = f(-4)$ which is undefined

33. $(f \circ f)(0) = f(f(0)) = f(1) = 3$

34. $(f \circ g)(-3) = f(g(-3)) = f(-2) = 2$

35. $(g \circ f)(3) = g(f(3)) = g(-1) = -4$

36. $g(f(-3)) = g(4)$ which is undefined

37. $(g \circ g)(-2) = g(g(-2)) = g(0) = 0$

38. $(g \circ f)(-2) = g(f(-2)) = g(2) = 1$

39. $g(f(g(0))) = g(f(0)) = g(1) = -3$

40. $f(f(f(-1))) = f(f(0)) = f(1) = 3$

41. $f(f(f(f(f(1))))) = f(f(f(f(3)))) = f(f(f(-1))) = f(f(0)) = f(1) = 3$

42. $\underbrace{(g \circ g \circ \cdots \circ g)}_{n \text{ times}}(0) = 0$

43.
- The domain of $f \circ g$ is $\{-3, -2, 0, 1, 2, 3\}$ and the range of $f \circ g$ is $\{1, 2, 3, 4\}$.
- The domain of $g \circ f$ is $\{-2, -1, 0, 1, 3\}$ and the range of $g \circ f$ is $\{-4, -3, 0, 1, 2\}$.

44. $(g \circ f)(1) = 3$

45. $(f \circ g)(3) = 1$

46. $(g \circ f)(2) = 0$

47. $(f \circ g)(0) = 1$

48. $(f \circ f)(4) = 1$

49. $(g \circ g)(1) = 0$

50.
- The domain of $f \circ g$ is $[0, 3]$ and the range of $f \circ g$ is $[1, 4.5]$.
- The domain of $g \circ f$ is $[0, 2] \cup [3, 4]$ and the range is $[0, 3]$.

51. Let $f(x) = 2x + 3$ and $g(x) = x^3$, then $p(x) = (g \circ f)(x)$.

52. Let $f(x) = x^2 - x + 1$ and $g(x) = x^5$, $P(x) = (g \circ f)(x)$.

53. Let $f(t) = 2t - 1$ and $g(t) = \sqrt{t}$, then $h(t) = (g \circ f)(t)$.

54. Let $f(t) = 7 - 3t$ and $g(t) = |t|$, then $H(t) = (g \circ f)(t)$.

55. Let $f(s) = 5s + 1$ and $g(s) = \frac{2}{s}$, then $r(s) = (g \circ f)(s)$.

56. Let $f(s) = s^2 - 1$ and $g(s) = \frac{7}{s}$, then $R(s) = (g \circ f)(s)$.

57. Let $f(z) = |z|$ and $g(z) = \frac{z+1}{z-1}$, then $q(z) = (g \circ f)(z)$.

58. Let $f(z) = z^3$ and $g(z) = \frac{2z+1}{z-1}$, then $Q(z) = (g \circ f)(z)$.

59. Let $f(x) = 2x$ and $g(x) = \frac{x+1}{3-2x}$, then $v(x) = (g \circ f)(x)$.

60. Let $f(x) = x^2$ and $g(x) = \frac{x}{x^2+1}$, then $w(x) = (g \circ f)(x)$.

61. $F(x) = \sqrt{\frac{x^3+6}{x^3-9}} = (h(g(f(x)))$ where $f(x) = x^3$, $g(x) = \frac{x+6}{x-9}$ and $h(x) = \sqrt{x}$.

62. $F(x) = 3\sqrt{-x+2} - 4 = k(j(f(h(g(x)))))$

63. One solution is $F(x) = -\frac{1}{2}(2x - 7)^3 + 1 = k(j(f(h(g(x)))))$ where $g(x) = 2x$, $h(x) = x - 7$, $j(x) = -\frac{1}{2}x$ and $k(x) = x + 1$. You could also have $F(x) = H(f(G(x)))$ where $G(x) = 2x - 7$ and $H(x) = -\frac{1}{2}x + 1$.

64. $(f \circ g)(x) = \begin{cases} 6x - 2 & \text{if } x \leq 3 \\ 13 - 3x & \text{if } x > 3 \end{cases}$ and $(g \circ f)(x) = \begin{cases} 6x + 1 & \text{if } x \leq \dfrac{2}{3} \\ 3 - 3x & \text{if } x > \dfrac{2}{3} \end{cases}$

65. $V(x) = x^3$ so $V(x(t)) = (t + 1)^3$

66. (a) $R(x) = 2x$

(b) $(R \circ x)(t) = -8t^2 + 40t + 184$, $0 \leq t \leq 4$. This gives the revenue per hour as a function of time.

(c) Noon corresponds to $t = 2$, so $(R \circ x)(2) = 232$. The hourly revenue at noon is \$232 per hour.

(d) $\frac{\Delta[R(x)]}{\Delta x} = 2$, $\frac{\Delta[x(t)]}{\Delta t} = -8t + 4\Delta t + 20$.
$\frac{\Delta[R(x)]}{\Delta x} \cdot \frac{\Delta[x(t)]}{\Delta t} = (2)(-8t + 4\Delta t + 20) = -16t + 8\Delta t + 40 = \frac{\Delta[R(t)]}{\Delta t}$ ✓

67. $\frac{\Delta A}{\Delta t} = \frac{\Delta A}{\Delta r} \cdot \frac{\Delta r}{\Delta t}$. $\frac{\Delta A}{\Delta r} = \frac{A(1.1)-A(1)}{1.1-1} = \frac{\pi(1.1)^2-\pi(1)^2}{0.1} = 2.1\,\frac{\text{m}^2}{\text{m}}$, $\frac{\Delta r}{\Delta t} = 0.5\,\frac{\text{m}}{\text{s}}$.
Hence, $\frac{\Delta A}{\Delta t} = \left(2.1\,\frac{\text{m}^2}{\text{m}}\right)\left(0.5\,\frac{\text{m}}{\text{s}}\right) = 1.05\,\frac{\text{m}^2}{\text{s}}$

68. (a) The 'width' of the pile is the diameter of the circular base of the pile. Since the diameter of a circle is twice the radius, $h = 2(2r) = 4r$. Hence, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(4r) = \frac{4}{3}\pi r^3$.

(b) We have $\frac{\Delta V}{\Delta t} = \frac{\Delta V}{\Delta r} \cdot \frac{\Delta r}{\Delta t}$. Two textbooks per minute into the shredder amounts to the volume of the cone increasing at a rate of $2(0.1) = 0.2\,\frac{\text{ft}^3}{\text{min}}$. $\frac{\Delta V}{\Delta r} = \frac{V(2.1)-V(2)}{2.1-2} = 16.81\overline{3}\,\pi\,\frac{\text{ft}^3}{\text{ft}}$.
Hence, $0.2\,\frac{\text{ft}^3}{\text{min}} = 16.81\overline{3}\,\pi\,\frac{\text{ft}^3}{\text{ft}}\,\frac{\Delta r}{\Delta t}$ so $\frac{\Delta r}{\Delta t} = \frac{0.2}{16.81\overline{3}\,\pi} \approx 0.0038\,\frac{\text{ft}}{\text{min}}$.

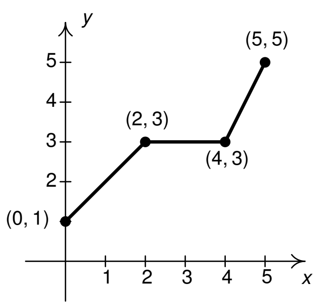
5.4 Transformations of Graphs

Theorems 1.2, 1.3, 2.1, 3.1, 4.1 and 4.4 all describe ways in which the graph of a function can changed, or 'transformed' to obtain the graph of a related function. The results and proofs of each of these theorems are virtually identical, and with the language of function composition, we can see better why.

Consider, for instance, Theorem 4.4, in which we describe how to transform the graph of $f(x) = x^r$ to $F(x) = a(bx - h)^r + k$. We may think of F as being build up from f by composing f with linear functions. Specifically, if we let $i(x) = bx - h$, then $(f \circ i)(x) = f(i(x)) = f(bx - h) = (bx - h)^r$. If, additionally, we let and $j(x) = ax + k$, then $(j \circ (f \circ i))(x) = j((f \circ i)(x)) = j((bx - h)^r) = a(bx - h)^r + k = F(x)$. Hence, we can view $F = j \circ f \circ i$.

In this section, our goal is to generalize the aforementioned theorems to the graphs of *all* functions. Along the way, you'll see some very familiar arguments, but, additionally, we hope this section affords the reader an opportunity to not only see *how* these transformations work they way they do, but *why*.

Our motivational example for the results in this section is the graph of $y = f(x)$ below. While we could formulate an expression for $f(x)$ as a piecewise-defined function consisting of linear and constant parts, we wish to focus more on the geometry here. That being said, we do record some of the function values - the 'key points' if you will - to track through each transformation.



$y = f(x)$

x	$(x, f(x))$	$f(x)$
0	$(0, 1)$	1
2	$(2, 3)$	3
4	$(4, 3)$	3
5	$(5, 5)$	5

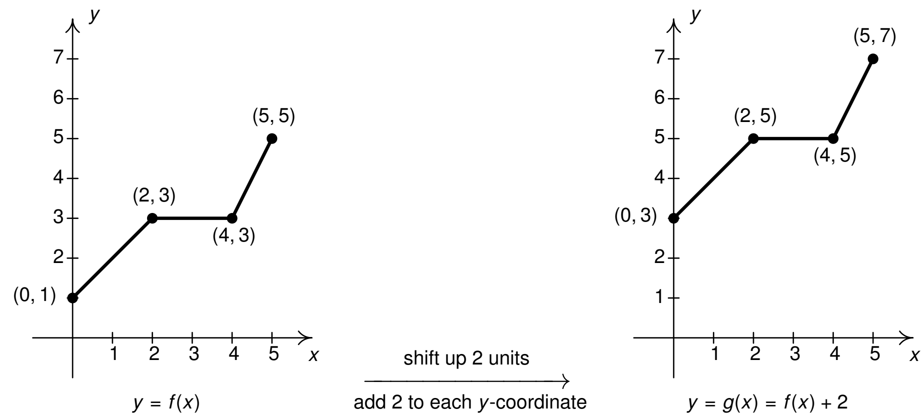
5.4.1 Vertical and Horizontal Shifts

Suppose we wished to graph $g(x) = f(x) + 2$. From a procedural point of view, we start with an input x to the function f and we obtain the output $f(x)$. The function g takes the output $f(x)$ and adds 2 to it. Using the sample values for f from the table above we can create a table of values for g below, hence generating points on the graph of g.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0, 1)$	1	$1 + 2 = 3$	$(0, 3)$
2	$(2, 3)$	3	$3 + 2 = 5$	$(2, 5)$
4	$(4, 3)$	3	$3 + 2 = 5$	$(4, 5)$
5	$(5, 5)$	5	$5 + 2 = 7$	$(5, 7)$

In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Hence, $g(a) = f(a) + 2 = b + 2$, so the point $(a, b+2)$ is on the graph of g. In other words, to obtain the graph of g, we add 2 to the y-coordinate of each point on the graph of f.

Geometrically, adding 2 to the y-coordinate of a point moves the point 2 units above its previous location. Adding 2 to every y-coordinate on a graph *en masse* is moves or 'shifts' the entire graph of f up 2 units. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four 'key points' we moved in the same manner in which they were connected before.



You'll note that the domain of f and the domain of g are the same, namely $[0, 5]$, but that the range of f is $[1, 5]$ while the range of g is $[3, 7]$. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range.

You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of f, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of j is the same as f, but the range of j is $[-1, 3]$. In general, we have:

Theorem 5.6. Vertical Shifts. Suppose f is a function and k is a real number.
To graph $F(x) = f(x) + k$, add k to each of the y-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a vertical shift up k units if $k > 0$ or down k units if $k < 0$.

To prove Theorem 5.6, we first note that f and F have the same domain (why?) Let c be an element in the domain of F and, hence, the domain of f. The fact that f and F are *functions* guarantees there is *exactly one* point on each of their graphs corresponding to $x = c$. On $y = f(x)$, this point is $(c, f(c))$; on $y = F(x)$, this point is $(c, F(c)) = (c, f(c)+k)$. This sets up a nice correspondence between the two graphs and shows that each of the points on the graph of F can be obtained to by adding k to each of the y-coordinates of the corresponding point on the graph of f. This proves Theorem 5.6. In the language of 'inputs' and 'outputs', Theorem 5.6 says adding to the *output* of a function causes the graph to shift *vertically*.

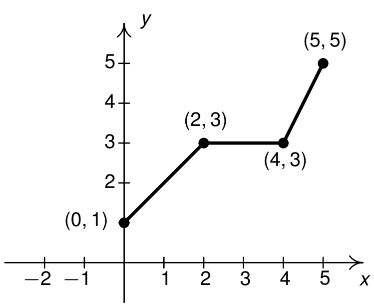
Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x + 2)$. In other words, we are looking to see what happens when we add 2 to the input of the function. Let's try to generate a table of values of g based on those we know for f. We quickly find that we run into some difficulties. For instance, when we substitute $x = 4$ into the formula $g(x) = f(x + 2)$, we are asked to find $f(4 + 2) = f(6)$ which doesn't exist because the domain of f is only $[0, 5]$. The same thing happens when we attempt to find $g(5)$.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x+2)$	$(x, g(x))$
0	$(0, 1)$	1	$g(0) = f(0+2) = f(2) = 3$	$(0, 3)$
2	$(2, 3)$	3	$g(2) = f(2+2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$g(4) = f(4+2) = f(6) = ?$	
5	$(5, 5)$	5	$g(5) = f(5+2) = f(7) = ?$	

What we need here is a new strategy. We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of g, we need to figure out what value of x we must substitute into $g(x) = f(x+2)$ so that the quantity $x + 2$, works out to be 0. Solving $x + 2 = 0$ gives $x = -2$, and $g(-2) = f((-2) + 2) = f(0) = 1$ so $(-2, 1)$ on the graph of g. To use the fact $f(2) = 3$, we set $x + 2 = 2$ to get $x = 0$. Substituting gives $g(0) = f(0 + 2) = f(2) = 3$. Continuing in this fashion, we produce the table below.

x	$x+2$	$g(x) = f(x+2)$	$(x, g(x))$
-2	0	$g(-2) = f(-2+2) = f(0) = 1$	$(-2, 1)$
0	2	$g(0) = f(0+2) = f(2) = 3$	$(0, 3)$
2	4	$g(2) = f(2+2) = f(4) = 3$	$(2, 3)$
3	5	$g(3) = f(3+2) = f(5) = 5$	$(3, 5)$

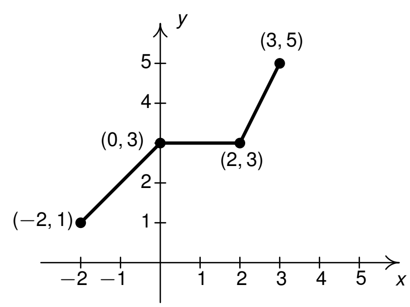
In summary, the points $(0, 1)$, $(2, 3)$, $(4, 3)$ and $(5, 5)$ on the graph of $y = f(x)$ give rise to the points $(-2, 1)$, $(0, 3)$, $(2, 3)$ and $(3, 5)$ on the graph of $y = g(x)$, respectively. In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x + 2 = a$ gives $x = a - 2$ so that $g(a - 2) = f((a - 2) + 2) = f(a) = b$. As such, $(a - 2, b)$ is on the graph of $y = g(x)$. The point $(a - 2, b)$ is exactly 2 units to the *left* of the point (a, b) so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units, as pictured below.



$y = f(x)$

$\xrightarrow{\text{shift left 2 units}}$

subtract 2 from each x-coordinate



$y = g(x) = f(x + 2)$

Note that while the ranges of f and g are the same, the domain of g is $[-2, 3]$ whereas the domain of f is $[0, 5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph $j(x) = f(x - 2)$, we would find ourselves *adding* 2 to all of the x values of the points on the graph of $y = f(x)$ to effect a shift to the *right* 2 units. Generalizing these notions produces the following result.

Theorem 5.7. Horizontal Shifts. Suppose f is a function and h is a real number.
To graph $F(x) = f(x - h)$, add h to each of the x-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a horizontal shift right h units if $h > 0$ or left h units if $h < 0$.

To prove Theorem 5.7, we first note the domains of f and F may be different. If c is in the domain of f, then the only number we know for sure is in the domain of F is $c + h$, since $F(c + h) = f((c + h) - h) = f(c)$. This sets up a nice correspondence between the domain of f and the domain of F which spills over to a correspondence between their graphs, The point $(c, f(c))$ is the one and only point on the graph of $y = f(x)$ corresponding to $x = c$ just as the point $(c + h, F(c + h)) = (c + h, f(c))$ is the one and only point on the graph of $y = F(x)$ corresponding to $x = c + h$. This correspondence shows we may obtain the graph of F by adding h to each x-coordinate of each point on the graph of f, which establishes the theorem. In words, Theorem 5.7 says that subtracting from the *input* to a function amounts to shifting the graph *horizontally*.

Theorems 5.6 and 5.7 present a theme which will run common throughout the section: changes to the *outputs* from a function result in some kind of *vertical change*; changes to the *inputs* to a function result in some kind of *horizontal* change. We demonstrate Theorems 5.6 and 5.7 in the example below.

Example 5.4.1. Use Theorems 5.6 and 5.7 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1, 3)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (a) $y = f(x) + 5$ (b) $y = f(x + 5)$ (c) $f(x - 7) + 4$

2. Find a formula for a function $g(t)$ whose graph is the same as $f(t) = |t| - 2t$ but is shifted:

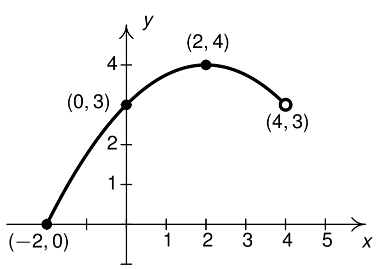
   (a) to the right 4 units. (b) down 2 units.

3. Predict how the graph of $F(x) = \dfrac{(x-2)^{\frac{2}{3}}}{x}$ relates to the graph of $f(x) = \dfrac{x^{\frac{2}{3}}}{x+2}$.

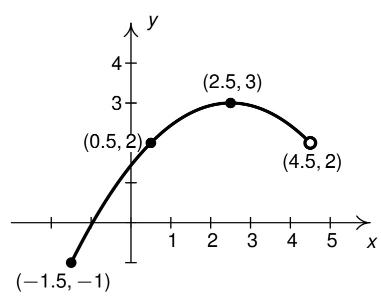
4. Below on the left is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = f(x - 2)$ (b) $F(x) = f(x) + 1$ (c) $F(x) = f(x + 1) - 2$

5. Below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.



$y = f(x)$



$y = g(x)$

Solution.

1. (a) To apply Theorem 5.6, we identify $f(x) + 5 = f(x) + k$ so $k = 5$. Hence, we add 5 to the y-coordinate of $(-1, 3)$ and get $(-1, 3 + 5) = (-1, 8)$. To check our answer note since $(-1, 3)$ is on the graph of f this means $f(-1) = 3$. Substituting $x = -1$ into the formula $y = f(x) + 5$, we get $y = f(-1) + 5 = 3 + 5 = 8$. Hence, $(-1, 8)$ is on the graph of $f(x) + 5$.

   (b) We note that $f(x + 5)$ can be written as $f(x - (-5)) = f(x - h)$ so we apply Theorem 5.7 with $h = -5$. Adding -5 to (subtracting 5 from) the x-coordinate of $(-1, 3)$ gives $(-1 + (-5), 3) = (-6, 3)$. To check our answer, since $(-1, 3)$ is on the graph of f, $f(-1) = 3$. Substituting $x = -6$ into $y = f(x+5)$ gives $y = f(-6+5) = f(-1) = 3$, proving $(-6, 3)$ is on the graph of $y = f(x+5)$.

   (c) Note that the expression $f(x - 7) + 4$ differs from $f(x)$ in *two* ways indicating two different transformations. In situations like this, its best if we handle each transformation in turn, starting with the graph of $y = f(x)$ and 'building up' to the graph of $y = f(x - 7) + 4$.

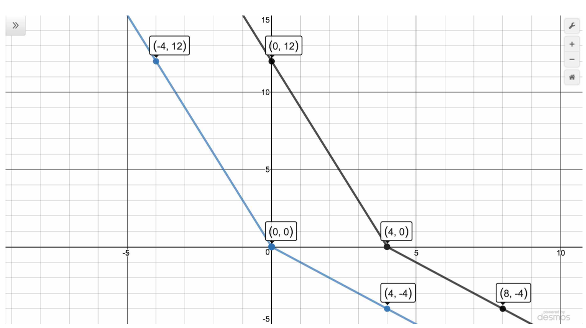
   We choose to work from the 'inside' (argument) out and use Theorem 5.7 to first get a point on the graph of $y = f(x - 7) = f(x - h)$. Identifying $h = 7$, we add 7 to the x-coordinate of $(-1, 3)$ to get $(-1 + 7, 3) = (6, 3)$. Hence, $(6, 3)$ is a point on the graph of $y = f(x - 7)$.

   Next, we apply Theorem 5.6 to graph $y = f(x - 7) + 4$ starting with $y = f(x - 7)$. Viewing $f(x - 7) + 4 = f(x - 7) + k$, we identify $k = 4$ and add 4 to the y-coordinate of $(6, 3)$ to get $(6, 3 + 4) = (6, 7)$. To check, we note that if we substitute $x = 6$ into $y = f(x - 7) + 4$, we get $y = f(6 - 7) + 4 = f(-1) + 4 = 3 + 4 = 7$.

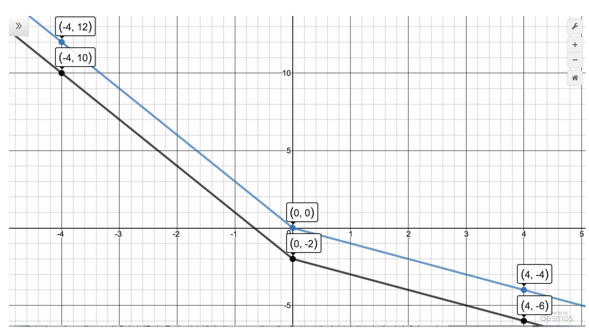
2. Here the independent variable is t instead of x which doesn't affect the geometry in any way since our convention is the independent variable is used to label the horizontal axis and the dependent variable is used to label the vertical axis.

   (a) Per Theorem 5.7, the graph of $g(t) = f(t - 4) = |t - 4| - 2(t - 4) = |t - 4| - 2t + 8$ should be the graph of $f(t) = |t| - 2t$ shifted to the right 4 units. Our check is below on the left.

   (b) Per Theorem 5.6, the graph of $g(t) = f(t) + (-2) = |t| - 2t + (-2) = |t| - 2t - 2$ should be the graph of $f(t) = |t| - 2t$ shifted down 2 units. Our check is below on the right.



$y = |t| - 2t$ (lighter color) and $y = |t - 4| - 2t + 8$

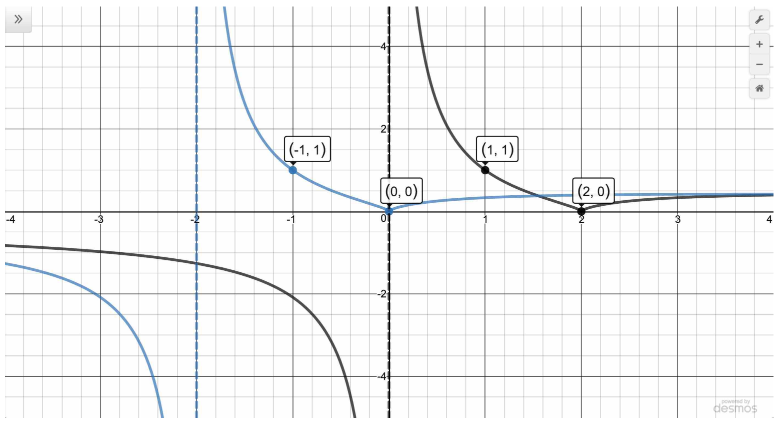


$y = |t| - 2t$ (lighter color) and $y = |t| - 2t - 2$

3. Comparing *formulas*, it appears as if $F(x) = f(x-2)$. We check:

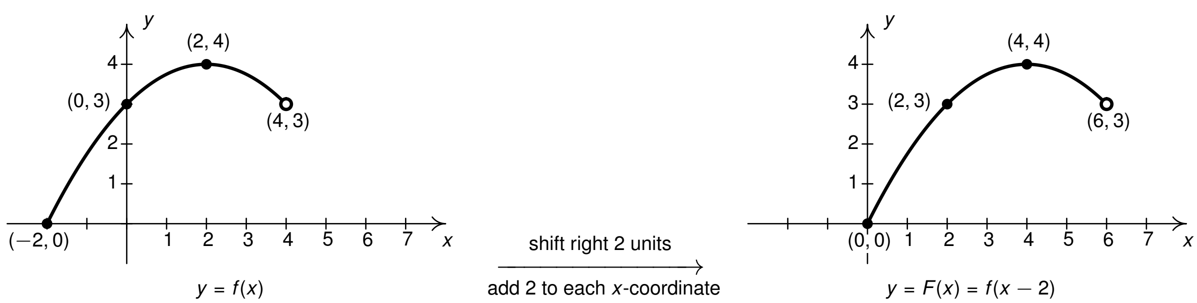
$$f(x-2) = \frac{(x-2)^{\frac{2}{3}}}{(x-2)+2} = \frac{(x-2)^{\frac{2}{3}}}{x} = F(x),$$

so, per Theorem 5.7, the graph of $y = F(x)$ should be the graph of $y = f(x)$ but shifted to the right 2 units. We graph both functions below to confirm our answer.



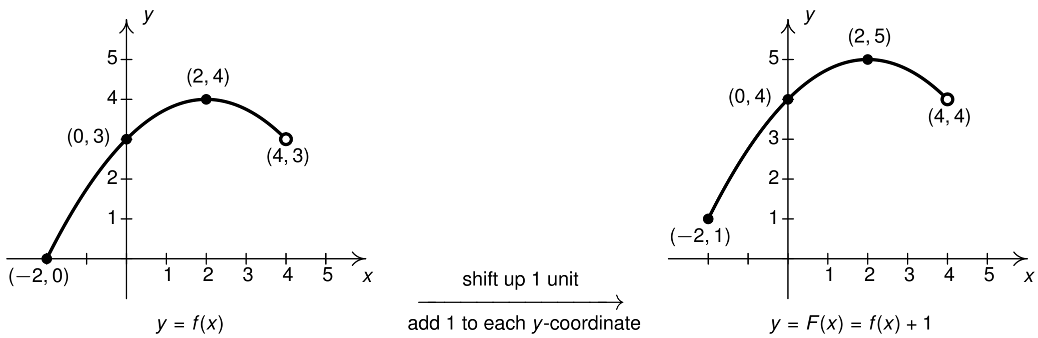
$y = \dfrac{x^{\frac{2}{3}}}{x+2}$ (lighter color) and $y = \dfrac{(x-2)^{\frac{2}{3}}}{x}$

4. (a) We recognize $F(x) = f(x-2) = f(x-h)$. With $h = 2$, Theorem 5.7 tells us to add 2 to each of the x-coordinates of the points on the graph of f, moving the graph of f to the *right* two units.



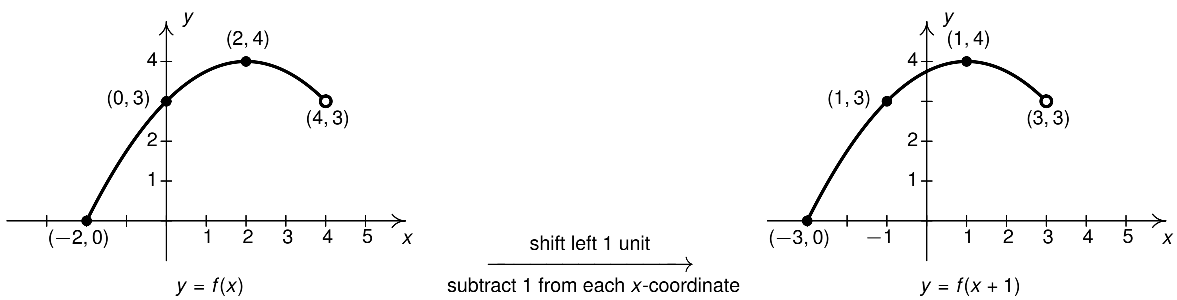
We can check our answer by showing each ordered pair (x, y) listed on our final graph satisfies the equation $y = f(x-2)$. Starting with $(0, 0)$, we substitute $x = 0$ into $y = f(x-2)$ and get $y = f(0-2) = f(-2)$. Since $(-2, 0)$ is on the graph of f, we know $f(-2) = 0$. Hence, $y = f(0-2) = f(-2) = 0$, showing the point $(0, 0)$ is on the graph of $y = f(x-2)$. We invite the reader to check the remaining points.

(b) We have $F(x) = f(x) + 1 = f(x) + k$ where $k = 1$, so Theorem 5.6 tells us to move the graph of f *up* 1 unit by adding 1 to each of the y-coordinates of the points on the graph of f.

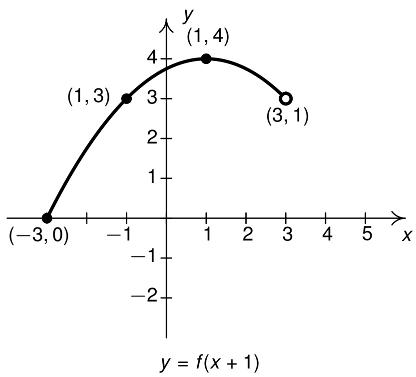


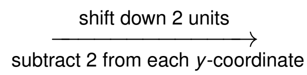
To check our answer, we proceed as above. Starting with the point $(-2, 1)$, we substitute $x = -2$ into $y = f(-2) + 1$ to get $y = f(-2) + 1$. Since $(-2, 0)$ is on the graph of f, we know $f(-2) = 0$. Hence, $y = f(-2) + 1 = 0 + 1 = 1$. This proves $(-2, 1)$ is on the graph of $y = f(x) + 1$. We encourage the reader to check the remaining points in kind.

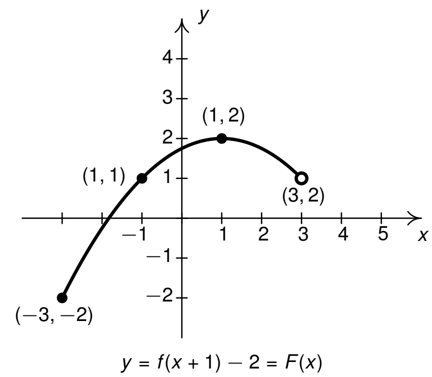
(c) We are asked to graph $F(x) = f(x+1) - 2$. As above, when we have more than one modification to do, we work from the inside out and build up to $F(x) = f(x + 1) - 2$ from $f(x)$ in stages. First, we apply Theorem 5.7 to graph $y = f(x + 1)$ from $y = f(x)$. Rewriting $f(x + 1) = f(x - (-1))$, we identify $h = -1$, so we add -1 to (subtract 1 from) each of the x-coordinates on the graph of f, shifting it to the *left* 1 unit.



Next, we apply Theorem 5.6 to graph $y = f(x + 1) - 2$ starting with the graph of $y = f(x + 1)$. Writing $f(x+1) - 2 = f(x+1) + (-2) = f(x+1) + k$, we identify $k = -2$ so Theorem 5.6 instructs us to add -2 to (subtract 2 from) each of the y-coordinates on the graph of $y = f(x+1)$, thereby shifting the graph *down* two units.







To check, we start with the point $(-3,-2)$. We find when we substitute $x = -3$ into the equation $y = f(x+1) - 2$ we get $y = f(-3+1) - 2 = f(-2) - 2$. Since $(-2, 0)$ is on the graph of f, we know $f(-2) = 0$, so $y = f(-3+1) - 2 = f(-2) - 2 = 0 - 2 = -2$. This proves $(-3,-2)$ is on the graph of $y = f(x+1) - 2$. We leave the checks of the remaining points to the reader.

5. To write $g(x)$ in terms of $f(x)$, we note that based on points which are labeled, it appears as if the graph of g can be obtained from the graph of f by shifting the graph of f to the right 0.5 units and down 1 unit.

   Per Theorems 5.7 and 5.6, $g(x)$ must take the form $g(x) = f(x-h) + k$. Since the horizontal shift is to the *right* 0.5 units, $h = 0.5$ and since the vertical shift is *down* 1 unit, $k = -1$. Hence, we get $g(x) = f(x - 0.5) - 1$.

   We can check our answer by working through both transformations, in sequence, as in the previous example. To write $f(x)$ in terms of $g(x)$, we need to reverse the process - that is, we need to shift the graph of g *left* one half of a unit and *up* one unit. Theorems 5.7 and 5.6 suggest the formula $f(x) = g(x + 0.5) + 1$. We leave it to the reader to check. □

5.4.2 Reflections about the Coordinate Axes

We now turn our attention to reflections. We know from Section A.3 that to reflect a point (x, y) across the x-axis, we replace y with $-y$. If (x, y) is on the graph of f, then $y = f(x)$, so replacing y with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y = -f(x)$ is the graph of f reflected across the x-axis. Similarly, the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis.[1]

> **Theorem 5.8. Reflections.** Suppose f is a function.
> To graph $F(x) = -f(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by -1.
> **NOTE:** This results in a reflection across the x-axis.
> To graph $F(x) = f(-x)$, multiply each of the x-coordinates of the points on the graph of $y = f(x)$ by -1.
> **NOTE:** This results in a reflection across the y-axis.

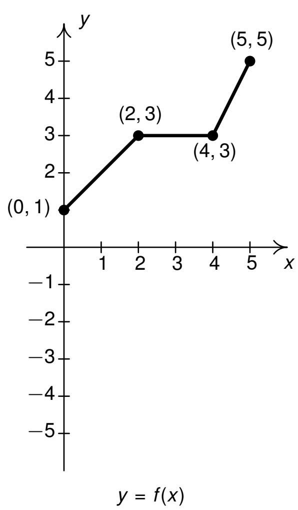
[1] The expressions $-f(x)$ and $f(-x)$ should look familiar - they are the quantities we used in Section 2.1 to determine if a function was even, odd or neither. We explore impact of symmetry on reflections in Exercise 74.

The proof of Theorem 5.8 follows in much the same way as the proofs of Theorems 5.6 and 5.7. If c is an element of the domain of f and $F(x) = -f(x)$, then the point $(c, f(c))$ corresponds to the point $(c, F(c)) = (c, -f(c))$. Comparing the corresponding points $(c, f(c))$ and $(c, -f(c))$, we see they only difference is the y-coordinates are the exact opposite - indicating they are mirror-images across the x-axis. Similarly, if c is an element in the domain of f, then c corresponds to the element $-c$ in the domain of $F(x) = f(-x)$ since $F(-c) = f(-(-c)) = f(c)$. Hence, the corresponding points here are $(c, f(c))$ and $(-c, F(-c)) = (-c, f(c))$. Comparing $(c, f(c))$ with $(-c, f(c))$, we see they are reflections about the y-axis.

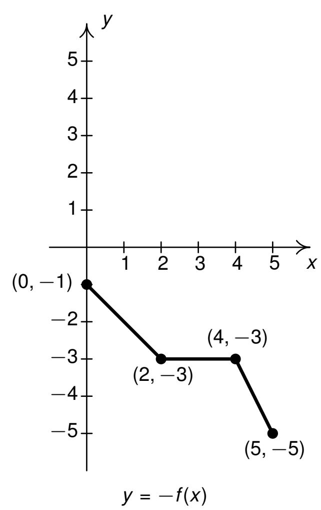
Using the language of inputs and outputs, Theorem 5.8 says that multiplying the *outputs* from a function by -1 reflects its graph across the *horizontal* axis, while multiplying the *inputs* to a function by -1 reflects the graph across the *vertical* axis.

Applying Theorem 5.8 to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of f about the x-axis.

x	$(x, f(x))$	$f(x)$	$g(x) = -f(x)$	$(x, g(x))$
0	$(0, 1)$	1	-1	$(0, -1)$
2	$(2, 3)$	3	-3	$(2, -3)$
4	$(4, 3)$	3	-3	$(4, -3)$
5	$(5, 5)$	5	-5	$(5, -5)$

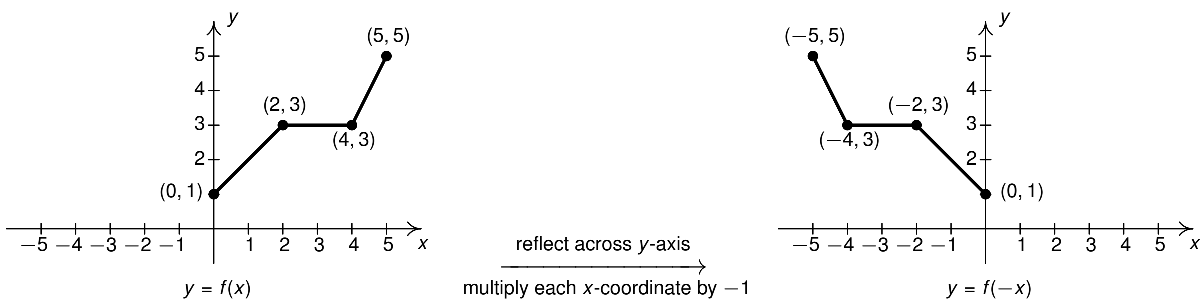


reflect across x-axis $\longrightarrow$ multiply each y-coordinate by -1



By reflecting the graph of f across the y-axis, we obtain the graph of $y = f(-x)$.

x	$-x$	$g(x) = f(-x)$	$(x, g(x))$
0	0	$g(0) = f(-(-0)) = f(0) = 1$	$(0, 1)$
-2	2	$g(-2) = f(-(-2)) = f(2) = 3$	$(-2, 3)$
-4	4	$g(-4) = f(-(-4)) = f(4) = 3$	$(-4, 3)$
-5	5	$g(-5) = f(-(-5)) = f(5) = 5$	$(-5, 5)$



Example 5.4.2. Use Theorems 5.6, 5.7 and 5.8 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(2, -5)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (a) $y = f(-x)$ (b) $y = -f(x + 2)$ (c) $f(8 - x)$

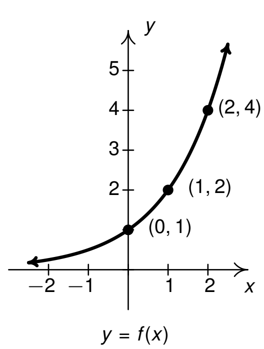
2. Find a formula for a function $H(s)$ whose graph is the same as $t = h(s) = s^3 - s^2$ but is reflected across the t-axis.

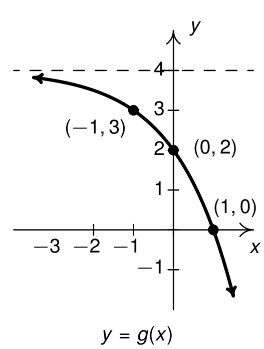
3. Predict how the graph of $G(t) = \dfrac{t + 4}{t - 3}$ relates to the graph of $g(t) = \dfrac{t + 4}{3 - t}$.

4. Below on the left is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = f(-x) + 1$ (b) $F(x) = 1 - f(2 - x)$

5. Below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.





NOTE: The x-axis, $y = 0$, is a horizontal asymptote to the graph of $y = f(x)$ and the line $y = 4$ is a horizontal asymptote to the graph of $y = g(x)$.

Solution.

1. (a) To find a point on the graph of $y = f(-x)$, Theorem 5.8 tells us to multiply the x-coordinate of the point on the graph of $y = f(x)$ by -1: $((-1)2, -5) = (-2, -5)$.

   To check, since $(2, -5)$ is on the graph of f, we know $f(2) = -5$. Hence, when we substitute $x = -2$ into $y = f(-x)$, we get $y = f(-(-2)) = f(2) = -5$, proving $(-2, -5)$ is on the graph of $y = f(-x)$.

   (b) To find a point on the graph of $y = -f(x + 2)$, we first note we have two transformations at work here, so we work our way from the inside out and build $f(x)$ to $-f(x + 2)$.

   First, we find a point on the graph of $y = f(x + 2)$. Writing $f(x + 2) = f(x - (-2))$, we apply Theorem 5.7 with $h = -2$ and add -2 to (or subtract 2 from) the x-coordinate of the point we know is on $y = f(x)$: $(2 - 2, -5) = (0, -5)$.

   Next we apply Theorem 5.8 to the graph of $y = f(x+2)$ to get a point on the graph of $y = -f(x+2)$ by multiplying the y-coordinate of $(0, -5)$ by -1: $(0, (-1)(-5)) = (0, 5)$.

   To check, recall $f(2) = -5$ so that when we substitute $x = 0$ into the equation $y = -f(x + 2)$, we get $y = -f(0 + 2) = -f(2) = -(-5) = 5$, as required.

   (c) Rewriting $f(8 - x) = f(-x + 8)$ we see we have two transformations at play here: a reflection across the y-axis and a horizontal shift. Since both of these transformations affect the x-coordinates of the graph, the question becomes which transformation to address first. To help us with this decision, we attack the problem algebraically.

   Recall that since $(2, -5)$ is on the graph of f, we know $f(2) = -5$. Hence, to get a point on the graph of $y = f(-x + 8)$, we need to match up the arguments of $f(-x + 8)$ and $f(2)$: $-x + 8 = 2$.

   To solve this equation, we first subtract 8 from both sides to get $-x = -6$. Geometrically, subtracting 8 from the x-coordinate of $(2, -5)$, shifts the point $(2, -5)$ left 8 units to get the point $(-6, -5)$.

   Next, we multiply both sides of the equation $-x = -6$ by -1 to get $x = 6$. Geometrically, multiplying the x-coordinate of $(-6, -5)$ by -1 reflects the point $(-6, -5)$ across the y-axis to $(6, -5)$.

   To check we substitute $x = 6$ into $y = f(-x + 8)$, and obtain $y = f(-6 + 8) = f(2) = -5$.

   Even though we have found our answer, we re-examine this process from a 'build' perspective. We began with a point on the graph of $y = f(x)$ and first shifted the graph to the left 8 units. Per Theorem 5.7, this point is on the graph of $y = f(x + 8)$.

   Next we took a point on the graph of $y = f(x+8)$ and reflected it about the y-axis. Per Theorem 5.8, this put the point on the graph of $y = f(-x + 8)$.

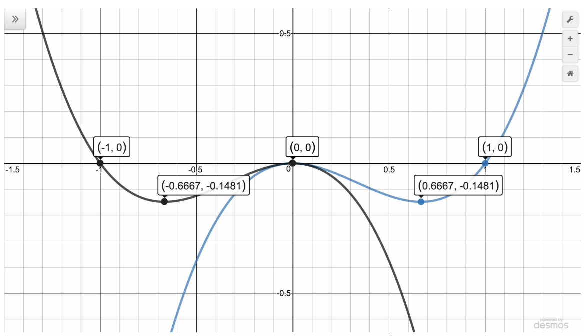
   In general, when faced with graphing functions in which there is both a horizontal shift and a reflection about the y-axis, we'll deal with the shift first.

2. In this example, the independent variable is s and the dependent variable is t. We are asked to reflect the graph of h about the t-axis, which in this case is the *vertical* axis. Hence, $H(s) = h(-s) = (-s)^3 - (-s)^2 = -s^3 - s^2$. Our confirmation is below on the left.

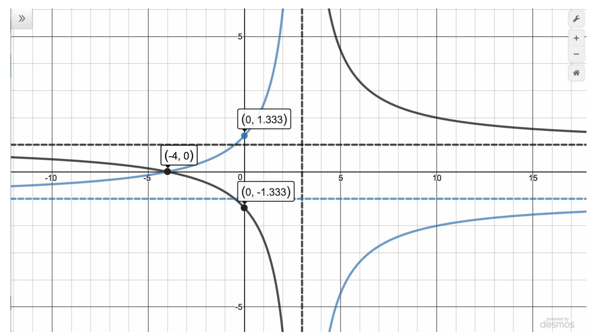
3. Comparing the formulas for $G(t) = \frac{t+4}{t-3}$ and $g(t) = \frac{t+4}{3-t}$, we have the same numerators, but in the denominator, we have $(t-3) = -(3-t)$:

$$G(t) = \frac{t+4}{t-3} = \frac{t+4}{-(3-t)} = -\frac{t+4}{3-t} = -g(t).$$

Hence, the graph of $y = G(t)$ should be the graph of $y = g(t)$ reflected across the t-axis. We check our answer below on the right.

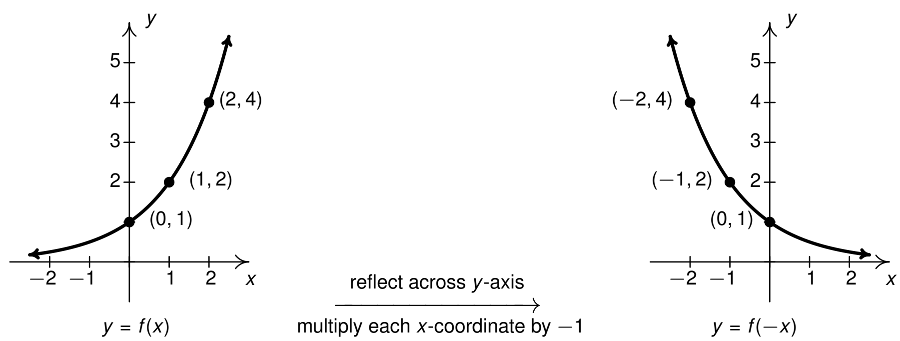


$t = s^3 - s^2$ (lighter color) and $t = -s^3 - s^2$

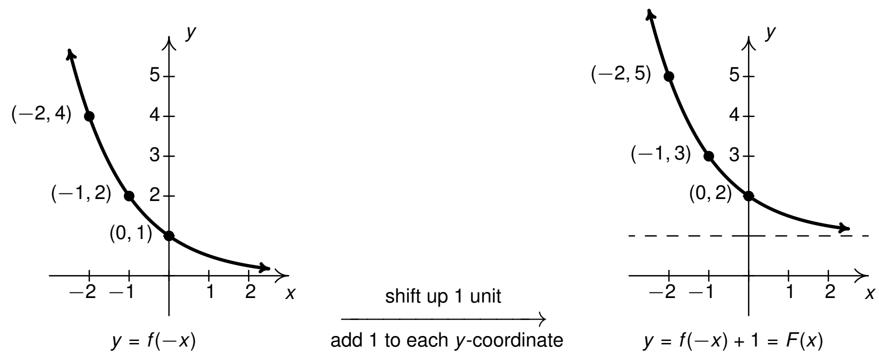


$y = \dfrac{t+4}{3-t}$ (lighter color) and $y = \dfrac{t+4}{t-3}$

4. (a) We have two transformations indicated with the formula $F(x) = f(-x) + 1$: a reflection across the y-axis and a vertical shift. Working from the inside out, we first tackle the reflection. Per Theorem 5.8, to obtain the graph of $y = f(-x)$ from $y = f(x)$, we multiply each of the x-coordinates of each of the points on the graph of $y = f(x)$ by (-1).



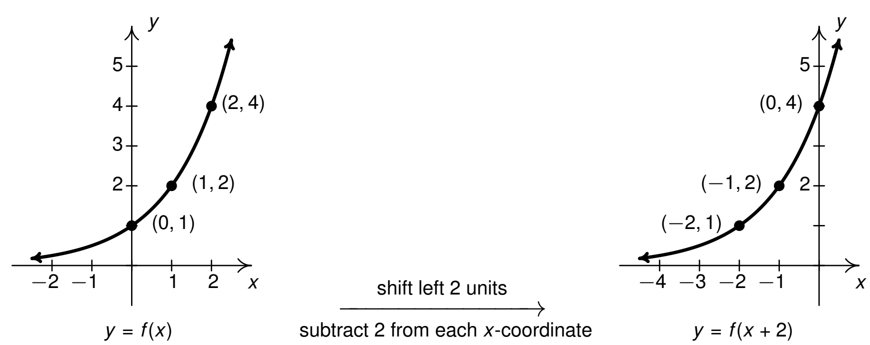
Next, we use Theorem 5.6 to obtain the graph of $y = f(-x) + 1$ from the graph of $y = f(-x)$ by adding 1 to each of the y-coordinates of each of the points on the graph of $y = f(-x)$. This shifts the graph of $y = f(-x)$ up one unit. Note, the horizontal asymptote $y = 0$ is also shifted up 1 unit to $y = 1$.



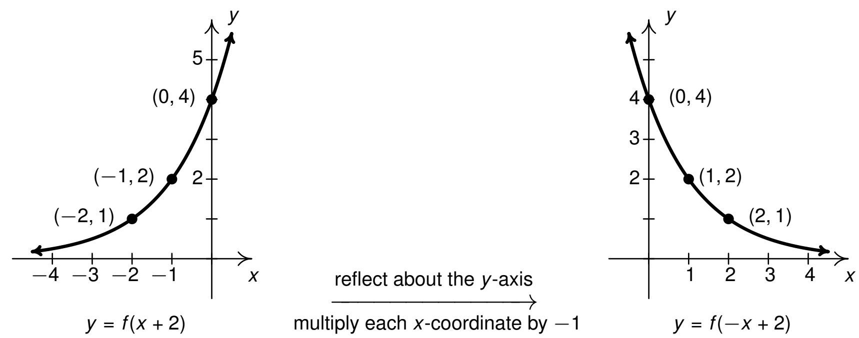
To check our answer, we begin with the point $(0, 2)$. Substituting $x = 0$ into $y = f(-x) + 1$, we get $y = f(-0)+1 = f(0)+1$. Since the point $(0, 1)$ is on the graph of f, we know $f(0) = 1$. Hence, $y = f(0) + 1 = 1 + 1 = 2$, so $(0, 2)$ is, indeed, on the graph of $y = f(-x) + 1$. We leave it to the reader to check the remaining points.

(b) In order to graph $F(x) = 1 - f(2 - x)$, we first rewrite as $F(x) = -f(-x + 2) + 1$ and note there are *four* modifications to the formula $f(x)$ indicated here.

Working from the inside out, we see we have a reflection about the y-axis indicated as well as a horizontal shift. From our work above, we know we first handle the shift: that is, we apply Theorem 5.7 to graph $y = f(x + 2) = f(x - (-2))$ by adding -2 to (subtracting 2 from) the x-coordinates of the points on the graph of $y = f(x)$.



Next, we use Theorem 5.8 to graph $y = f(-x + 2)$ starting with the graph of $y = f(x + 2)$ by multiplying each of the x-coordinates of the points of the graph of $y = f(x + 2)$ by -1. This reflects the graph of $f(x + 2)$ about the y-axis.

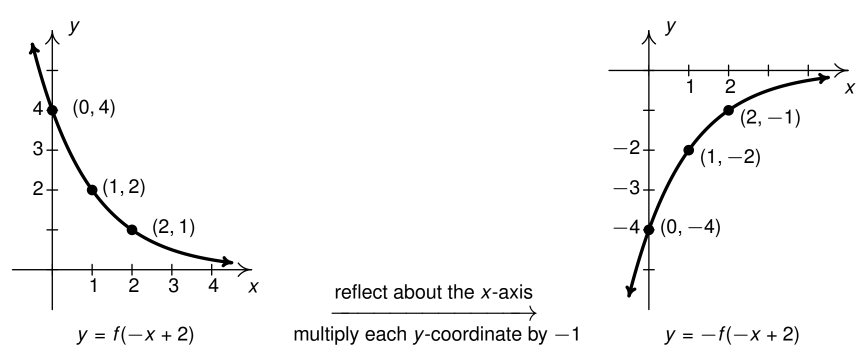


We have the graph of $y = f(-x+2)$ and need to build towards the graph of $y = -f(-x+2)+1$. The transformations that remain are a reflection about the x-axis and a vertical shift. The question is which to do first.

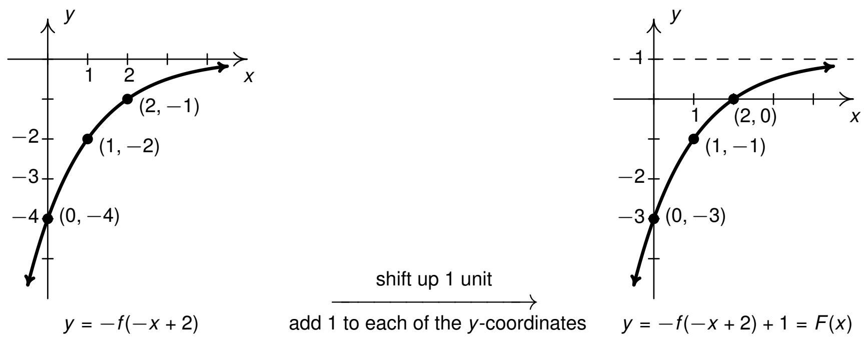
Once again, we can think algebraically about the problem. We know the point $(0, 1)$ is on the graph of f which means $f(0) = 1$. This point corresponds to the point $(2, 1)$ on the graph of $f(-x+2)$. Indeed, when we substitute $x = 2$ into $y = f(-x+2)$, we get $y = f(-2+2) = f(0) = 1$.

If we substitute $x = 2$ into the formula $y = -f(-x + 2) + 1$, we get $y = -f(-2 + 2) + 1 = -f(0) + 1 = -1(1) + 1 = 0$. That is, we first multiply the y-coordinate of $(2, 1)$ by -1 then add 1. This suggests we take care of the reflection about the x-axis first, then the vertical shift.

We proceed below to obtain the graph of $y = -f(-x + 2)$ from $y = f(-x + 2)$ by multiplying each of the y-coordinates on the graph of $y = f(-x + 2)$ by -1. Note the horizontal asymptote remains unchanged: $y = (-1)(0) = 0$.



Finally, we take care of the vertical shift. Per Theorem 5.6, we graph $y = -f(-x + 2) + 1$ by adding 1 to the y-coordinates of each of the points on the graph of $y = -f(-x+2)$. This moves the graph up one unit, including the horizontal asymptote: $y = 0 + 1 = 1$.



To check, we begin with the point $(2, 0)$. Substituting $x = 2$ into $y = 1 - f(2 - x)$, we obtain $y = 1 - f(2 - 2) = 1 - f(0)$. Since $(0, 1)$ is on the graph of f, we know $f(0) = 1$. This means $y = 1 - f(2 - 2) = 1 - f(0) = 1 - 1 = 0$. This proves $(2, 0)$ is on the graph of $y = 1 - f(2 - x)$, and we recommend the reader check the remaining points.

5. With the transformations at our disposal, our task amounts to finding values of h and k and choosing between signs $\pm$ so that $g(x) = \pm f(\pm x - h) + k$.

   Based on the horizontal asymptote, $y = 4$, we choose $k = 4$. Note, however, in the graph of $y = f(x) + 4$, the entire graph is *above* the line $y = 4$. Since the graph of g approaches the asymptote from below, we know $y = -f(\pm x - h) + 4$.

   Hence, two of transformations applied to the graph of f are a reflection across the x-axis followed by a shift up 4 units. This means the point $(0, 1)$ on the graph of f must correspond to the point $(-1, 3)$ on the graph of g, since these are the points closest to the asymptote on each graph.

   Likewise, the points $(1, 2)$ and $(2, 4)$ on the graph of f must correspond to $(0, 2)$ and $(1, 0)$, respectively, on the graph of g. Looking at the x-coordinates only, we have $x = 0$ moves to $x = -1$, $x = 1$ moves to $x = 0$, and $x = 2$ moves to $x = 1$. Hence, the net effect on the x-values is a shift left 1 unit. Hence, we guess the formula for $g(x)$ to be $g(x) = -f(x + 1) + 4$.

   We can readily check by going through the transformations: first, shift left 1 unit; next, reflect across the x-axis; finally, shift up 4. We leave it to the reader to verify that tracking each of the points on the graph of f along with the horizontal asymptote through this sequence of transformations results in the graph of g.

   One way to recover the graph of f from the graph of g is to reverse the process by which we obtained g from f. The challenge here comes from the fact that two different operations were done which affected the y-values: reflection and shifting - and the order in which these are done matters.

   To motivate our methodology, let's consider a more down-to-earth example like putting on socks and then putting on shoes. Unless we're very talented, to reverse this process, we take off the shoes first, then the socks - that is, we undo each step in the reverse order.[2] In the same way, when we

[2]We'll have more to say about this sort of thing in Section 5.6.

think about reversing the steps transforming the graph of f to the graph of g, we need to undo each transformation in the opposite order.

To review, we obtained the graph of g from the graph of f by first shifting the graph to the left 1 unit, then reflecting the graph about the x-axis, then, finally, shifting the graph up 4 units. Hence, we first undo the vertical shift. Instead of shifting the graph *up* four units, we shift the graph *down* four units. This takes the graph of $y = g(x)$ to $y = g(x) - 4$.

Next, we have to undo the refection across the x-axis. Thinking at the level of points, to recover the point (a, b) from its reflection across the x-axis, $(a, -b)$, we simply reflect across the x-axis again: $(a, -(-b)) = (a, b)$. Per Theorem 5.8, this takes the graph the graph of $y = g(x) - 4$ to the graph of $y = -[g(x) - 4] = -g(x) + 4$.[3]

Last, to undo moving the graph to the *left* 1 unit, we move the graph of $y = -g(x) + 4$ to the *right* 1 unit. Per Theorem 5.7, we accomplish this by graphing $y = -g(x - 1) + 4$. We leave it to the reader to start with the graph of $y = g(x)$ and graph $y = -g(x - 1) + 4$ and show it matches the graph of $y = f(x)$. $\square$

Some remarks about Example 5.4.2 are in order. In number 1c above, to find a point on the graph of $y = f(-x+8)$, we took the given x-coordinate on our starting graph, 2, and subtracted 8 first then multiplied by -1. If this seems somehow 'backwards' it should.

When *evaluating* the expression $-x + 8$, the order of operations mandates we multiply by -1 first then add 8. Here, however, we weren't *evaluating* an expression - we were *solving* an equation: $-x + 8 = 2$, which meant we did the exact opposite steps in the opposite order.[4] This exemplifies a larger theme with transformations: when adjusting inputs, the resulting points on the graph are obtained by applying the opposite operations indicated by the formula in the opposite order of operations.

On the other hand, when it came to multiple transformations involving the y-coordinates, we followed the order of operations. As in 4b above, when it came to applying a reflection about the x-axis and a vertical shift, we applied the reflection first, then the shift. This is because instead of *solving* an *equation* to find the new y-coordinates, we were *simplifying* an *expression*. Again, this is an example of a much larger theme: when adjusting outputs, the resulting points on the graph are obtained by applying the stated operations in the usual order.

Last but not least, in number 5, to find f in terms of g, we reversed the steps used to transform f into g. Another tact is to approach the problem in the same way we approached transforming f into g: namely, starting with the graph of g, determine values h and k and signs $\pm$ so that $f(x) = \pm g(\pm x - h) + k$. We leave this to the reader.

[3]To see this better, let us temporarily write $F(x) = g(x) - 4$. Theorem 5.8 tells us to reflect the graph of F about the x-axis, graph $y = -F(x) = -[g(x) - 4] = -g(x) + 4$.

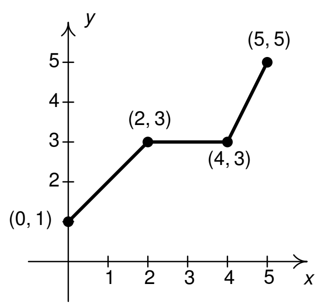
[4]Note that dividing by -1 is the same as multiplying by -1, so to keep with the 'opposite steps in opposite order' theme, we could more precisely say we subtracted 8 and *divided* by -1.

5.4.3 Scalings

We now turn our attention to our last class of transformations: **scalings**. A thorough discussion of scalings can get complicated because they are not as straight-forward as the previous transformations. A quick review of what we've covered so far, namely vertical shifts, horizontal shifts and reflections, will show you why those transformations are known as **rigid transformations**.

Simply put, rigid transformations preserve the distances between points on the graph - only their position and orientation in the plane change.[5] If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be affecting the distance between points. These sorts of transformations are hence called **non-rigid**. As always, we motivate the general theory with an example.

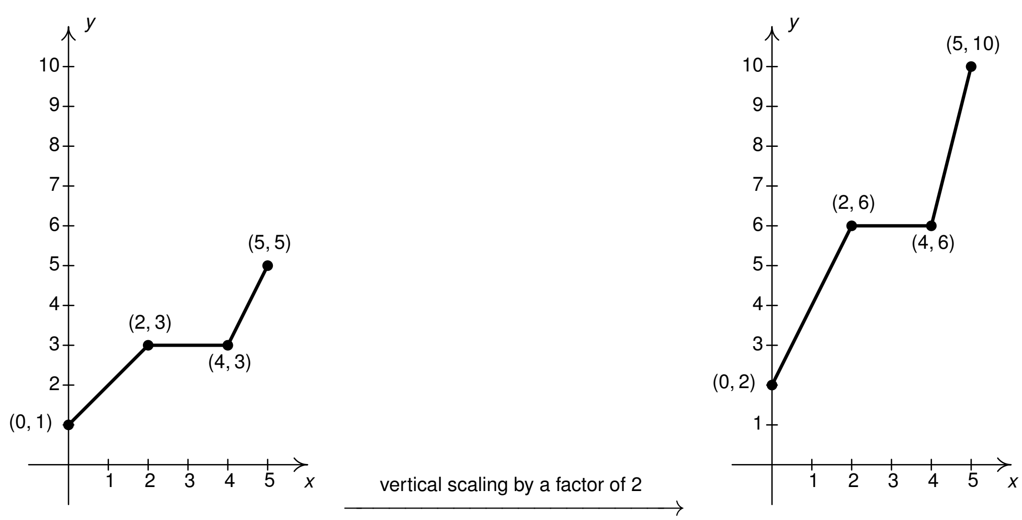
Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given at the beginning of the section. From its graph, we can build a table of values for g as before.



$y = f(x)$

x	$(x, f(x))$	$f(x)$	$g(x) = 2f(x)$	$(x, g(x))$
0	$(0,1)$	1	2	$(0,2)$
2	$(2,3)$	3	6	$(2,6)$
4	$(4,3)$	3	6	$(4,6)$
5	$(5,5)$	5	10	$(5,10)$

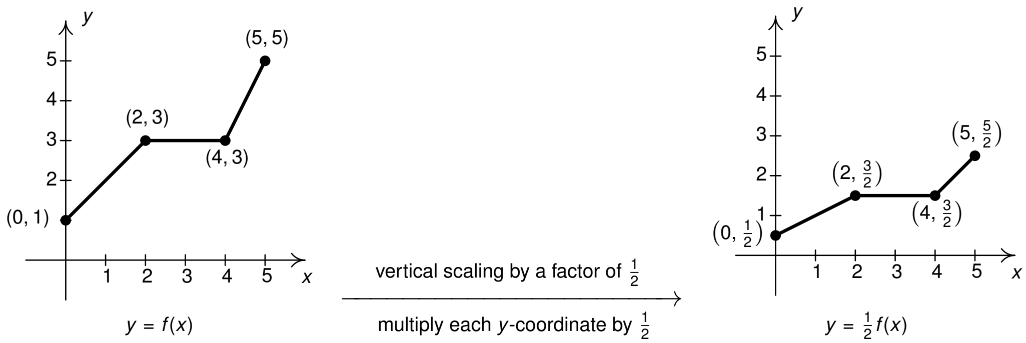
Graphing, we get:



[5] Another word that can be used here instead of 'rigid transformation' is 'isometry' - meaning 'same distance.'

In general, if (a, b) is on the graph of f, then $f(a) = b$ so that $g(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of g. In other words, to obtain the graph of g, we multiply all of the y-coordinates of the points on the graph of f by 2. Multiplying all of the y-coordinates of all of the points on the graph of f by 2 causes what is known as a 'vertical scaling[6] by a factor of 2.'

If we wish to graph $y = \frac{1}{2}f(x)$, we multiply the all of the y-coordinates of the points on the graph of f by $\frac{1}{2}$. This creates a 'vertical scaling[7] by a factor of $\frac{1}{2}$' as seen below.



These results are generalized in the following theorem.

Theorem 5.9. Vertical Scalings. Suppose f is a function and $a > 0$ is a real number.
To graph $F(x) = af(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by a.

- If $a > 1$, we say the graph of f has undergone a vertical stretch[a] by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrink[b] by a factor of $\frac{1}{a}$.

[a]expansion, dilation
[b]compression, contraction

The proof of Theorem 5.9 mimics the proofs of Theorems 5.6 and 5.8. If c is in the domain of f, then $(c, f(c))$ is on the graph of f and the corresponding point on the graph of $F(x) = af(x)$ is $(c, F(c)) = (c, af(c))$. Comparing the points $(c, f(c))$ and $(c, af(c))$ proves the theorem.

A few remarks about Theorem 5.9 are in order. First, a note about the verbiage. To the authors, the words 'stretch', 'expansion', and 'dilation' all indicate something getting bigger. Hence, 'stretched by a factor of 2' makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrink', 'compression' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it 'shrinks by a factor of 2' - not 'shrinks by a factor of $\frac{1}{2}$'. This is why we have written the descriptions 'stretch by a factor of a' and 'shrink by a factor of $\frac{1}{a}$' in the statement of the theorem.

Second, in terms of inputs and outputs, Theorem 5.9 says multiplying the *outputs* from a function by positive number a causes the graph to be vertically scaled by a factor of a. It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

[6]Also called a 'vertical stretch,' 'vertical expansion' or 'vertical dilation' by a factor of 2.
[7]Also called 'vertical shrink,' 'vertical compression' or 'vertical contraction' by a factor of 2.

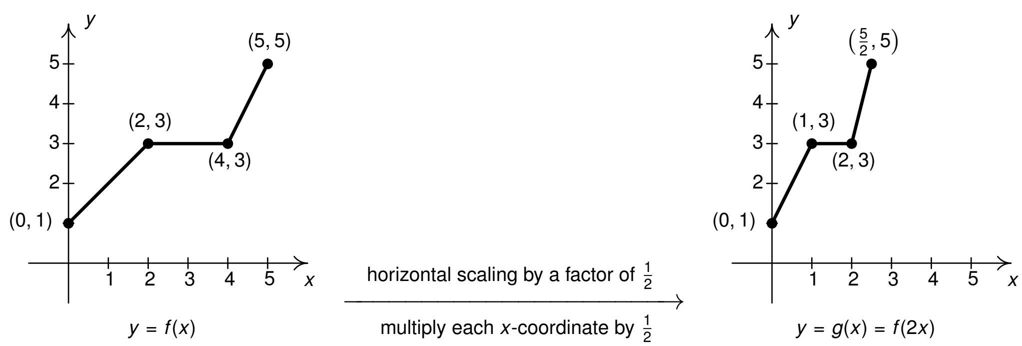
Referring to the graph of f given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to f by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem 5.7, as seen in the table on the left below.

We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on g which corresponds to the point $(2,3)$ on the graph of f, we set $2x = 2$ so that $x = 1$. Substituting $x = 1$ into $g(x)$, we obtain $g(1) = f(2 \cdot 1) = f(2) = 3$, so that $(1,3)$ is on the graph of g. Continuing in this fashion, we obtain the table on the lower right.

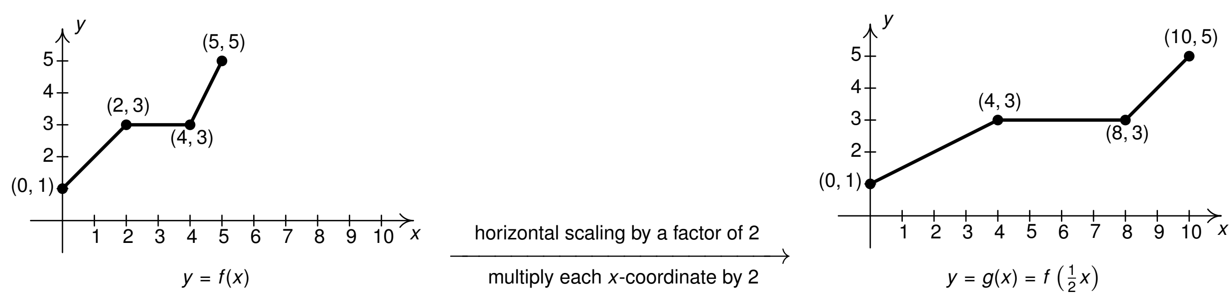
x	$(x, f(x))$	$f(x)$	$g(x) = f(2x)$	$(x, g(x))$
0	$(0,1)$	1	$f(2 \cdot 0) = f(0) = 1$	$(0,1)$
2	$(2,3)$	3	$f(2 \cdot 2) = f(4) = 3$	$(2,3)$
4	$(4,3)$	3	$f(2 \cdot 4) = f(8) = ?$	
5	$(5,5)$	5	$f(2 \cdot 5) = f(10) = ?$	

x	$2x$	$g(x) = f(2x)$	$(x, g(x))$
0	0	$g(0) = f(2 \cdot 0) = f(0) = 1$	$(0,0)$
1	2	$g(1) = f(2 \cdot 1) = f(2) = 3$	$(1,3)$
2	4	$g(2) = f(2 \cdot 2) = f(4) = 3$	$(2,3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f\left(2 \cdot \frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$

In general, if (a,b) is on the graph of f, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2 \cdot \frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of g. In other words, to graph g we divide the x-coordinates of the points on the graph of f by 2. This results in a horizontal scaling[8] by a factor of $\frac{1}{2}$.



If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling[9] by a factor of 2, as demonstrated below.



We have the following theorem.

[8]Also called 'horizontal shrink,' 'horizontal compression' or 'horizontal contraction' by a factor of 2.
[9]Also called 'horizontal stretch,' 'horizontal expansion' or 'horizontal dilation' by a factor of 2.

Theorem 5.10. Horizontal Scalings. Suppose f is a function and $b > 0$ is a real number. To graph $F(x) = f(bx)$, divide each of the x-coordinates of the points on the graph of $y = f(x)$ by b.

- If $0 < b < 1$, we say the graph of f has undergone a horizontal stretch[a] by a factor of $\frac{1}{b}$.
- If $b > 1$, we say the graph of f has undergone a horizontal shrink[b] by a factor of b.

[a] expansion, dilation
[b] compression, contraction

The proof of Theorem 5.10 follows closely the spirit of the proof of Theorems 5.7 and 5.8. If c is an element of the domain of f, them the number $\frac{c}{b}$ corresponds to a domain element of $F(x) = f(bx)$ since $F\left(\frac{c}{b}\right) = f\left(b \cdot \frac{c}{b}\right) = f(c)$. Hence, there is a correspondence between the point $(c, f(c))$ on the graph of f and the point $\left(\frac{c}{b}, F\left(\frac{c}{b}\right)\right) = \left(\frac{c}{b}, f(c)\right)$ on the graph of F. We can obtain $\left(\frac{c}{b}, f(c)\right)$ by dividing the x-coordinate of $(c, f(c))$ by b and the result follows.

Theorem 5.10 tells us that if we multiply the input to a function by b, the resulting graph is scaled horizontally by a factor of $\frac{1}{b}$. The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

Example 5.4.3. Use Theorems 5.6, 5.7, 5.8, 5.9 and 5.10 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1, 4)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (a) $y = 3f(x-2)$ (b) $y = f\left(-\frac{1}{2}x\right)$ (c) $f(2x-3)+1$

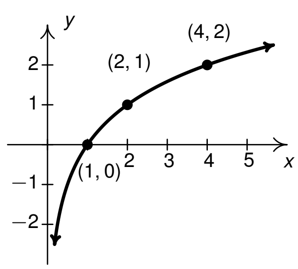
2. Find a formula for a function $G(t)$ whose graph is the same as $y = g(t) = \frac{2t+1}{t-1}$ but is vertically stretched by a factor of 4.

3. Predict how the graph of $H(s) = 8s^3 - 12s^2$ relates to the graph of $h(s) = s^3 - 3s^2$.

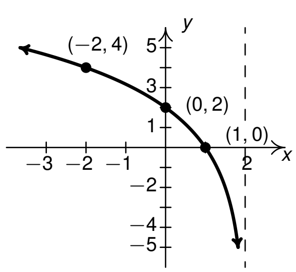
4. Below on the left is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = \dfrac{1 - f(x)}{2}$ (b) $F(x) = f\left(\dfrac{1-x}{2}\right)$

5. Below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.



$y = f(x)$



$y = g(x)$

NOTE: The y-axis, $x = 0$, is a vertical asymptote to the graph of $y = f(x)$ and the line $x = 2$ is a vertical asymptote to the graph of $y = g(x)$.

Solution.

1. (a) As we examine the formula $y = 3f(x-2)$, we note two modifications from $y = f(x)$. Building from the inside out, we start with obtaining a point on the graph of $y = f(x-2)$.

   Per Theorem 5.7, this shifts all of the points on the graph of $y = f(x)$ 2 units to the right. Hence, the point $(-1, 4)$ on the graph of $y = f(x)$ moves to the point $(-1+2, 4) = (1, 4)$ on the graph of $y = f(x-2)$.

   To get a point on the graph of $y = 3f(x-2) = af(x-3)$, we apply Theorem 5.9 with $a = 3$ to the point $(1, 4)$ on the graph of $y = f(x-2)$ to get the point $(1, 3(4)) = (1, 12)$ on the graph of $y = 3f(x-2)$.

   To check, we note that since $(-1, 4)$ is on the graph of $y = f(x)$, we know $f(-1) = 4$. Hence, when we substitute $x = 1$ into the $y = 3f(x-2)$, we get $y = 3f(1-2) = 3f(-1) = 3(4) = 12$.

   (b) The formula $y = f\left(-\frac{1}{2}x\right)$ also indicates two transformations: a horizontal scaling, indicated by $\frac{1}{2}$ factor, as well as a reflection across the y-axis. The question before us is which to do first.

   If we return to algebra for inspiration, we know $f(-1) = 4$, so we match up the arguments of $f\left(-\frac{1}{2}x\right)$ and $f(-1)$ and get the equation $-\frac{1}{2}x = -1$. We solve this equation by multiplying both sides by -2: $x = (-2)(-1) = 2$. That is, we take the original x-value on the graph of $y = f(x)$ and multiply it by -2.

   If we think of $-2 = (-1)(2)$ then multiplying by the '2' in '$(-1)(2)$' produces a horizontal stretch by a factor of 2 while multiplying by the '-1' reflects the point across the y-axis.

   Applying the horizontal stretch first, we use Theorem 5.10 and start with the point $(-1, 4)$ on the graph of $y = f(x)$ and multiply the x-coordinate by 2 to obtain a point on the graph of $y = f\left(\frac{1}{2}x\right)$: $(-1(2), 4) = (-2, 4)$.

   Next, we take care of the reflection about the y-axis using Theorem 5.8 . Starting with $(-2, 4)$ on the graph of $y = f\left(\frac{1}{2}x\right)$, we multiply the x-coordinate by -1 to obtain a point on the graph of $y = f\left(\frac{1}{2}(-x)\right) = f\left(-\frac{1}{2}x\right)$: $((-1)(-2), 4) = (2, 4)$.

   To check, note when $x = 2$ is substituted into $y = f\left(-\frac{1}{2}x\right)$, we get $y = f\left(-\frac{1}{2}(2)\right) = f(-1) = 4$.

   Of course, we could have equally written the multiple $-2 = (2)(-1)$ and reversed these steps: doing the reflection first, then the horizontal scaling.

   Proceeding this way, we start with the point $(-1, 4)$ on the graph of $y = f(x)$ and reflect across the y-axis to obtain the point $((-1)(-1), 4) = (1, 4)$ on the graph of $y = f(-x)$.

   Next, we stretch the graph of $y = f(-x)$ by a factor of 2 by multiplying the x-coordinates of the points on the graph by 2 and obtain $(2(1), 4) = (2, 4)$ on the graph of $y = f\left(-\frac{1}{2}x\right)$.

   In general when it comes to reflections and scalings, whether horizontal or, as we'll see soon, vertical, either order will produce the same results.

   (c) The formula $f(2x-3)+1$ indicates *three* transformations: a horizontal shift, a horizontal scaling, and a vertical shift. As usual, we appeal to algebra to give us guidance on which horizontal transformation to apply first.

Since we know $f(-1) = 4$, we set $2x - 3 = -1$ and solve. Our first step is to add 3 to both sides: $2x = (-1) + 3 = 2$. Since we are adding 3 to the given x-value -1, this corresponds to a shift to the right 3 units, so the point $(-1, 4)$ is moved to the point $(2, 4)$.

Next, to solve $2x = 2$, we divide this new x-coordinate 2 by 2 and get $x = \frac{2}{2} = 1$ which corresponds to a horizontal compression by a factor of 2. This moves the point $(2, 4)$ to $(1, 4)$.

Hence, the algebra suggests we use Theorem 5.7 first and follow it up with Theorem 5.10. Starting with $(-1, 4)$ on the graph of $y = f(x)$, we shift to the right 3 units to obtain the point $(-1 + 3, 4) = (2, 4)$ on the graph of $y = f(x - 3)$.

Next, we start with the point $(2, 4)$ on the graph of $y = f(x-3)$ and horizontally shrink the x-axis by a factor of 2 to get the point $\left(\frac{2}{2}, 4\right) = (1, 4)$ on the graph of $y = f(2x - 3)$.

Last, but not least, we take care of the vertical shift using Theorem 5.6. Starting with the point $(1, 4)$ on the graph of $y = f(2x-3)$, we add 1 to the y-coordinate to get the point $(1, 4+1) = (1, 5)$ on the graph of $y = f(2x - 3) + 1$.

To check, we substitute $x = 1$ into the formula $y = f(2x - 3) + 1$ and get $y = f(2(1) - 3) + 1 = f(-1) + 1 = 4 + 1 = 5$, as required.

2. To vertically stretch the graph of $y = g(t)$ by 4, we use Theorem 5.9 with $a = 4$ to get

$$G(t) = 4g(t) = 4\frac{2t+1}{t-1} = \frac{4(2t+1)}{t-1} = \frac{8t+4}{t-1}.$$

We check our answer below on the left.

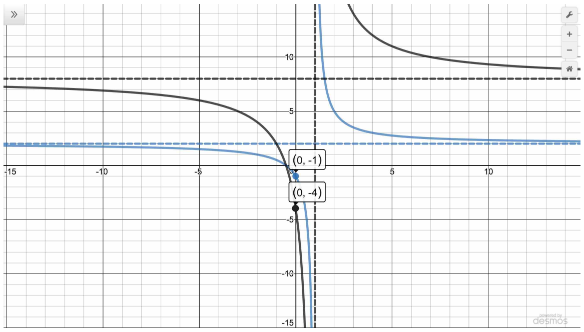
3. When comparing the formulas for $H(s) = 8s^3 - 12s^2$ and $h(s) = s^3 - 3s^2$, it doesn't appear as if any shifting or reflecting is going on (why not?)

We also note that since the coefficient of s^3 in the expression of $H(s)$ is 8 times that of the coefficient of s^3 in $h(s)$, but the coefficient of s^2 in $H(s)$ is only 4 times the coefficient of s^2 in $h(s)$, the change is not the result of a vertical scaling (again, why not?)

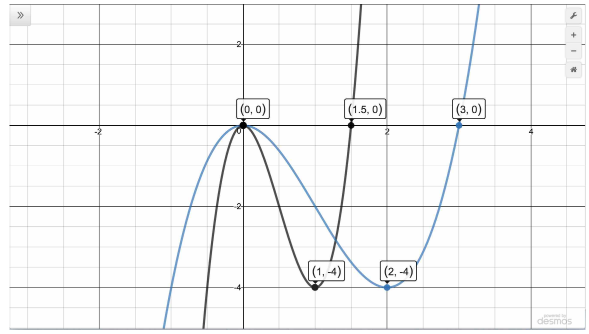
Hence, if anything, we are looking for a horizontal scaling. In other words, we are looking for a real number $b > 0$ so $h(bs) = H(s)$, that is, $(bs)^3 - 3(bs)^2 = b^3s^3 - 3b^2s^2 = 8s^3 - 12s^2$.

Matching up coefficients of s^3 gives $b^3 = 8$ so $b = 2$ which checks with the coefficients of s^2 : $3b^2 = 3(2)^2 = 12$.

Hence, we predict the graph of $y = H(s)$ to be the same as $y = h(s)$ except horizontally compressed by a factor of 2. Our check is below on the right.



$y = g(t) = \frac{2t+1}{t-1}$ (lighter color) and $y = 4g(t) = \dfrac{8t+4}{t-1}$



$y = h(s) = s^3 - 3s^2$ (lighter color) and $y = H(s) = 8s^3 - 12s^2$

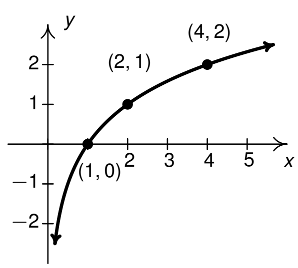
4. (a) We first rewrite the expression for $F(x) = \frac{1-f(x)}{2} = -\frac{1}{2}f(x) + \frac{1}{2}$ in order to use the theorems available to us. Note we have two modifications to the formula of $f(x)$ which correspond to three transformations.

Multiplying $f(x)$ by $-\frac{1}{2}$ indicates a vertical compression by a factor of 2 along with a reflection about the x-axis. Adding $\frac{1}{2}$ indicates a vertical shift up $\frac{1}{2}$ units.

As always the question is which to do first. Once again, we look to algebra for the answer. Picking the point $(1, 0)$ on the graph of $f(x)$, we know $f(1) = 0$. To see which point this corresponds to on the graph of $y = F(x)$, we find $F(1) = -\frac{1}{2}f(1) + \frac{1}{2} = -\frac{1}{2}(0) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$.

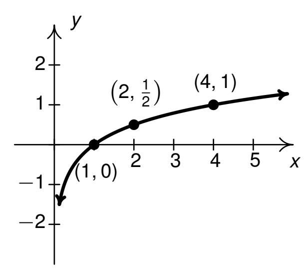
Hence, we first multiplied the y-value 0 by $-\frac{1}{2}$. As above, we can think of $-\frac{1}{2} = (-1)\frac{1}{2}$ so that multiplying by $-\frac{1}{2}$ amounts to a vertical compression by a factor of 2 first, then the refection about the x-axis second. Lastly, adding the $\frac{1}{2}$ is the vertical shift up $\frac{1}{2}$ unit.

Beginning with the vertical scaling by a factor of $\frac{1}{2}$, we use Theorem 5.9 to graph $y = \frac{1}{2}f(x)$ starting from $y = f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = f(x)$ by $\frac{1}{2}$.



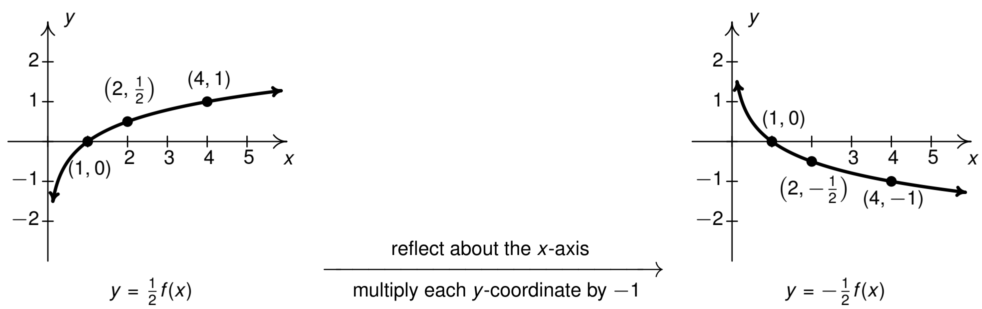
$y = f(x)$

vertical scaling by a factor of $\frac{1}{2}$
$\xrightarrow{\hspace{4cm}}$
multiply each y-coordinate by $\frac{1}{2}$

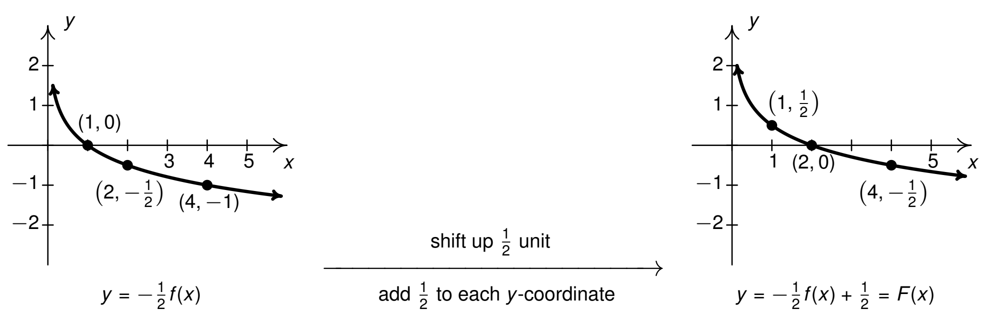


$y = \frac{1}{2}f(x)$

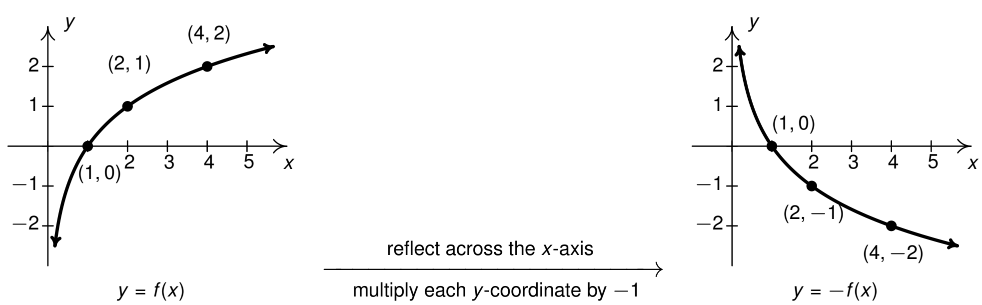
Next, we reflect the graph of $y = \frac{1}{2}f(x)$ across the x-axis to produce the graph of $y = -\frac{1}{2}f(x)$ by multiplying each of the y-coordinates of the points on the graph of $y = \frac{1}{2}f(x)$ by -1:



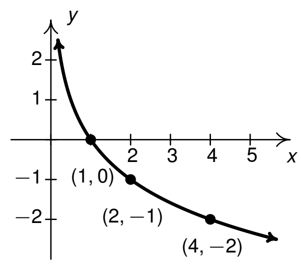
Finally, we shift the graph of $y = -\frac{1}{2}f(x)$ vertically up $\frac{1}{2}$ unit by adding $\frac{1}{2}$ to each of the y-coordinates of each of the points to obtain the graph of $y = -\frac{1}{2}f(x) + \frac{1}{2} = F(x)$.



Note that as with horizontal scalings and reflections about the y-axis, the order of vertical scalings and reflections across the x-axis is interchangeable. Had we decided to think of the factor $-\frac{1}{2} = \frac{1}{2} \cdot (-1)$, we could have just as well started with the graph of $y = f(x)$ and produced the graph of $y = -f(x)$ first:

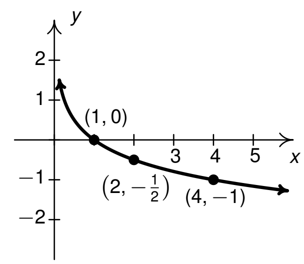


Next, we vertically scale the graph of $y = -f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = -f(x)$ by $\frac{1}{2}$ to obtain the graph of $y = -\frac{1}{2}f(x)$:



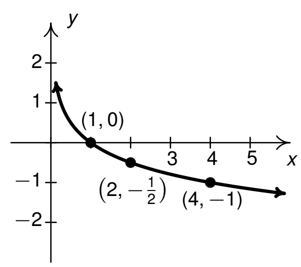
$y = -f(x)$

vertical scaling by a factor of $\frac{1}{2}$ $\longrightarrow$ multiply each y-coordinate by $\frac{1}{2}$



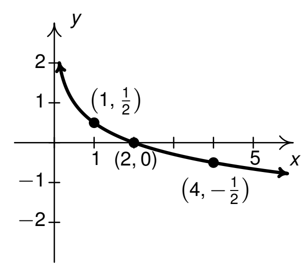
$y = -\frac{1}{2}f(x)$

Notice we've reached the same graph of $y = -\frac{1}{2}f(x)$ that we had before, and, hence we arrive at the same final answer as before:



$y = -\frac{1}{2}f(x)$

shift up $\frac{1}{2}$ unit $\longrightarrow$ add $\frac{1}{2}$ to each y-coordinate



$y = -\frac{1}{2}f(x) + \frac{1}{2} = F(x)$

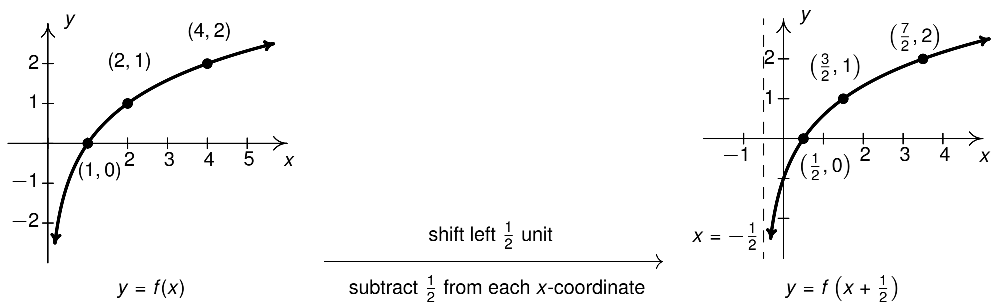
We check our answer as we have so many times before. We start with the point $\left(1, \frac{1}{2}\right)$ and substitute $x = 1$ into $y = \frac{1-f(x)}{2}$ to get $y = \frac{1-f(1)}{2}$. From the graph of f, we know $f(1) = 0$, so we get $y = \frac{1-f(1)}{2} = \frac{1-0}{2} = \frac{1}{2}$. This proves $\left(1, \frac{1}{2}\right)$ is on the graph of $y = \frac{1-f(x)}{2}$. We invite the reader to check the remaining points.

Note that in the preceding example, since none of the transformations included adjusting the x-coordinates of points, the vertical asymptote, $x = 0$ remained in place.

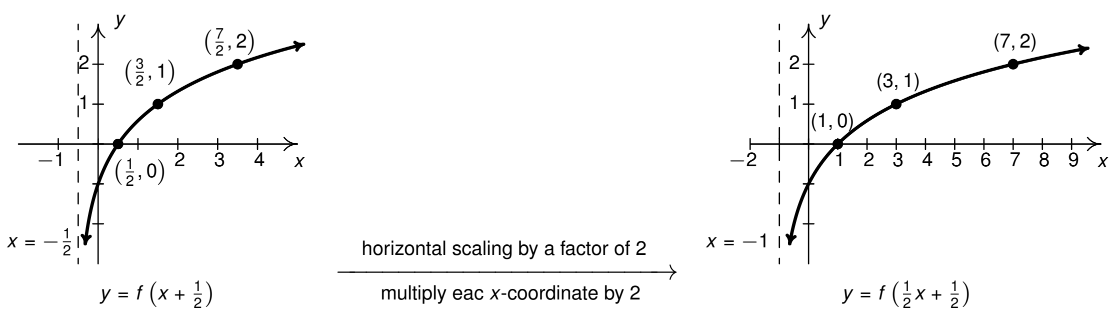
(b) As with the previous example, we first rewrite $F(x) = f\left(\frac{1-x}{2}\right) = F\left(-\frac{1}{2}x + \frac{1}{2}\right)$. Here again, we have two modifications to the formula $f(x)$, the $-\frac{1}{2}$ multiple indicating a horizontal scaling and a reflection across the y-axis and a horizontal shift.

Based on our experience from previous examples, we do the horizontal shift first, with the order of the scaling and reflection more or less irrelevant.

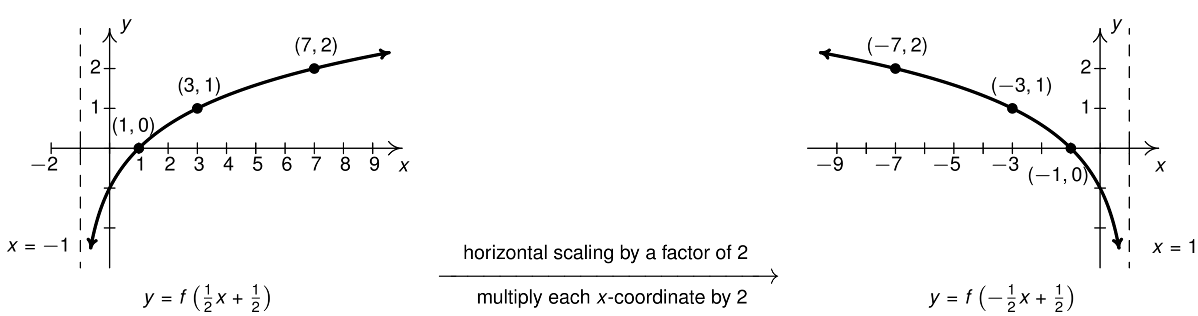
To produce the graph of $y = f\left(x + \frac{1}{2}\right)$ we subtract $\frac{1}{2}$ from each of the x-coordinates of each of the points on the graph of $y = f(x)$. This moves the graph to the left $\frac{1}{2}$ unit, including the vertical asymptote $x = 0$ which moves to $x = -\frac{1}{2}$.



Next, we graph $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ starting with $y = f\left(x + \frac{1}{2}\right)$ by horizontally expanding the graph by a factor of 2. That is, we multiply each x-coordinates on the graph of $y = f\left(x + \frac{1}{2}\right)$ by 2, including the vertical asymptote, $x = -\frac{1}{2}$ which moves to $x = 2\left(-\frac{1}{2}\right) = -1$.



Finally, we reflect the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ about the y-axis to graph $y = f\left(-\frac{1}{2}x + \frac{1}{2}\right)$. We accomplish this by multiplying each of the x-coordinates of each of the points on the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ by -1. This includes the vertical asymptote which is moved to $x = (-1)(-1) = 1$.



To check our answer, we begin with the point $(-1, 0)$ and substitute $x = -1$ into $y = f\left(\frac{1-x}{2}\right)$. We get $y = f\left(\frac{1-(-1)}{2}\right) = f\left(\frac{2}{2}\right) = f(1)$. From the graph of f, we know $f(1) = 0$, hence we have $y = f(1) = 0$, proving $(-1, 0)$ is on the graph of $y = f\left(\frac{1-x}{2}\right)$. The reader is encouraged to check the remaining points.

As mentioned previously, instead of doing the horizontal scaling first, then the reflection, we could have done the reflection first, then the scaling. We leave this to the reader to check.

5. To write $g(x)$ in terms of $f(x)$, we assume we can find real numbers a, b, h, and k and choose signs $\pm$ so that $g(x) = \pm af(\pm bx - h) + k$.

   The most notable change we see is the vertical asymptote $x = 0$ has moved to $x = 2$. Moreover, instead of the graph increasing off to the right, it is decreasing coming in from the left. This suggests a horizontal shift of 2 units as well as a reflection across the y-axis.

   Since we always shift first then reflect, we have a shift *left* of 2 units followed by a reflection about the y-axis. In other words, $g(x) = \pm af(-x + 2) + k$.

   Comparing y-values, the y-values on the graph of g appear to be exactly twice the corresponding values on the graph of f, indicating a vertical stretch by a factor of 2. Hence, we get $g(x) = 2f(-x+2)$. We leave it to the reader to check the graph of $y = 2f(-x + 2)$ matches the graph of $y = g(x)$.

   To write $f(x)$ in terms of $g(x)$, we reverse the steps done in obtaining the graph of $g(x)$ from $f(x)$ in the reverse order.

   Since to get from the graph of f to the graph of g, we: first, shifted left 2 units; second reflected across the y-axis; third, vertically stretched by a factor of 2, our first step in taking g back to f is to implement a vertical compression by a factor of 2. Hence, starting with the graph of $y = g(x)$, our first step results in the formula $y = \frac{1}{2}g(x)$.

   Next, we need to undo the reflection abut the y-axis. If the point (a, b) is reflected about the y-axis, we obtain the point $(-a, b)$. To return to the point (a, b), we reflect $(-a, b)$ across the y-axis again: $(-(-a), b) = (a, b)$. Hence, we take the graph of $y = \frac{1}{2}g(x)$ and reflect it across the y-axis to obtain $y = \frac{1}{2}g(-x)$.

   Our last step is to undo a horizontal shift to the left 2 units. The reverse of this process is shifting the graph to the *right* two units, so we get $y = \frac{1}{2}g(-(x - 2)) = \frac{1}{2}g(-x + 2)$.[10]

   We leave it to the reader to start with the graph of $y = g(x)$ and check the graph of $y = \frac{1}{2}g(-x + 2)$ matches the graph of $y = f(x)$. $\square$

5.4.4 Transformations in Sequence

Now that we have studied three basic classes of transformations: shifts, reflections, and scalings, we present a result below which provides one algorithm to follow to transform the graph of $y = f(x)$ into the graph of $y = af(bx - h) + k$ without the need of using Theorems 5.6, 5.7, 5.8, 5.9 and 5.10 individually.

Theorem 5.11 is the ultimate generalization of Theorems 1.2, 1.3, 2.1, 3.1, 4.1 and 4.4. We note the underlying assumption here is that regardless of the order or number of shifts, reflections and scalings applied to the graph of a function f, we can always represent the final result in the form $g(x) = af(bx-h)+k$.

[10]To see this better, let $F(x) = \frac{1}{2}g(-x)$. Per Theorem 5.7, the graph of $F(x - 2) = \frac{1}{2}g(-(x - 2)) = \frac{1}{2}g(-x + 2)$ is the same as the graph of F but shifted 2 units to the right.

Since each of these transformations can ultimately be traced back to composing f with linear functions,[11] this fact is verified by showing compositions of linear functions results in a linear function.[12]

Theorem 5.11. Transformations in Sequence. Suppose f is a function. If $a, b \neq 0$, then to graph $g(x) = af(bx - h) + k$ start with the graph of $y = f(x)$ and follow the steps below.

1. Add h to each of the x-coordinates of the points on the graph of f.

   NOTE: This results in a horizontal shift to the left if $h < 0$ or right if $h > 0$.

2. Divide the x-coordinates of the points on the graph obtained in Step 1 by b.

   NOTE: This results in a horizontal scaling, but includes a reflection about the y-axis if $b < 0$.

3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by a.

   NOTE: This results in a vertical scaling, but includes a reflection about the x-axis if $a < 0$.

4. Add k to each of the y-coordinates of the points on the graph obtained in Step 3.

   NOTE: This results in a vertical shift up if $k > 0$ or down if $k < 0$.

Theorem 5.11 can be established by generalizing the techniques developed in this section. Suppose $(c, f(c))$ is on the graph of f. To match up the inputs of $f(bx - h)$ and $f(c)$, we solve $bx - h = c$ and solve.

We first add the h (causing the horizontal shift) and then divide by b. If b is a positive number, this induces only a horizontal scaling by a factor of $\frac{1}{b}$. If $b < 0$, then we have a factor of -1 in play, and dividing by it induces a reflection about the y-axis. So we have $x = \frac{c+h}{b}$ as the input to g which corresponds to the input $x = c$ to f.

We now evaluate $g\left(\frac{c+h}{b}\right) = af\left(b \cdot \frac{c+h}{b} - h\right) + K = af(c + h - h) = af(c) + k$. We notice that the output from f is first multiplied by a. As with the constant b, if $a > 0$, this induces only a vertical scaling. If $a < 0$, then the -1 induces a reflection across the x-axis. Finally, we add k to the result, which is our vertical shift.

A less precise, but more intuitive way to paraphrase Theorem 5.11 is to think of the quantity $bx - h$ is the 'inside' of the function f. What's happening inside f affects the inputs or x-coordinates of the points on the graph of f. To find the x-coordinates of the corresponding points on g, we undo what has been done to x in the same way we would solve an equation.

What's happening to the output can be thought of as things happening 'outside' the function, f. Things happening outside affect the outputs or y-coordinates of the points on the graph of f. Here, we follow the usual order of operations to simplify the new y-value: we first multiply by a then add k to find the corresponding y-coordinates on the graph of g.

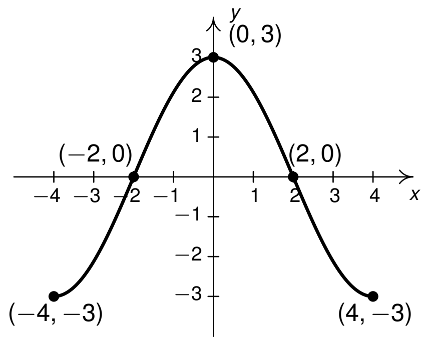
It needs to be stressed that our approach to handling multiple transformations, as summarized in Theorem 5.11 is only one approach. Your instructor may have a different algorithm. As always, the more you understand, the less you'll ultimately need to memorize, so whatever algorithm you choose to follow, it is worth thinking through each step both algebraically and geometrically.

[11] See the remarks at the beginning of the section.

[12] See Exercise 72.

We make good use of Theorem 5.11 in the following example.

Example 5.4.4. Below is the complete graph of $y = f(x)$. Use Theorem 5.11 to graph $g(x) = \dfrac{4 - 3f(1-2x)}{2}$.



Solution. We use Theorem 5.11 to track the five 'key points' $(-4,-3)$, $(-2,0)$, $(0,3)$, $(2,0)$ and $(4,-3)$ indicated on the graph of f to their new locations.

We first rewrite $g(x)$ in the form presented in Theorem 5.11, $g(x) = -\frac{3}{2}f(-2x+1)+2$. We set $-2x+1$ equal to the x-coordinates of the key points and solve.

For example, solving $-2x+1 = -4$, we first subtract 1 to get $-2x = -5$ then divide by -2 to get $x = \frac{5}{2}$. Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by -2 can be thought of as a two step process: dividing by 2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by -1 which causes a reflection across the y-axis. We summarize the results in a table below on the left.

Next, we take each of the x values and substitute them into $g(x) = -\frac{3}{2}f(-2x+1)+2$ to get the corresponding y-values. Substituting $x = \frac{5}{2}$, and using the fact that $f(-4) = -3$, we get

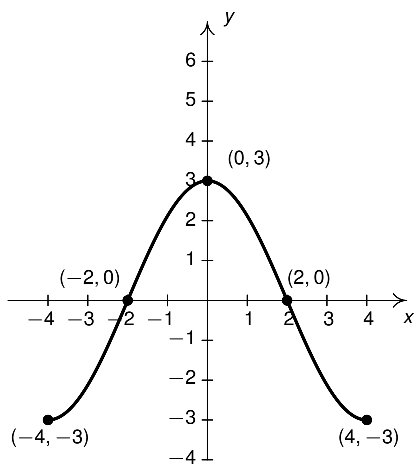
$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right)+1\right)+2 = -\frac{3}{2}f(-4)+2 = -\frac{3}{2}(-3)+2 = \frac{9}{2}+2 = \frac{13}{2}$$

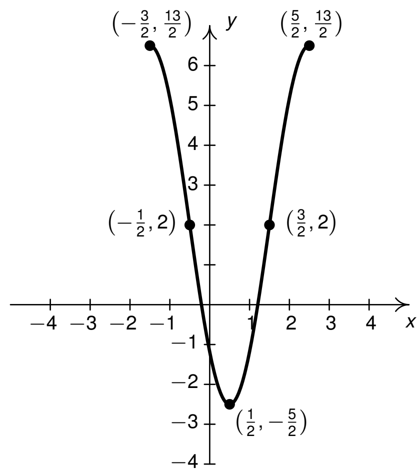
We see that the output from f is first multiplied by $-\frac{3}{2}$. Thinking of this as a two step process, multiplying by $\frac{3}{2}$ then by -1, we have a vertical stretching by a factor of $\frac{3}{2}$ followed by a reflection across the x-axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get the table below on the right.

$(c, f(c))$	c	$-2x+1=c$	x
$(-4,-3)$	-4	$-2x+1=-4$	$x=\frac{5}{2}$
$(-2,0)$	-2	$-2x+1=-2$	$x=\frac{3}{2}$
$(0,3)$	0	$-2x+1=0$	$x=\frac{1}{2}$
$(2,0)$	2	$-2x+1=2$	$x=-\frac{1}{2}$
$(4,-3)$	4	$-2x+1=4$	$x=-\frac{3}{2}$

x	$g(x)$	$(x, g(x))$
$\frac{5}{2}$	$\frac{13}{2}$	$\left(\frac{5}{2},\frac{13}{2}\right)$
$\frac{3}{2}$	2	$\left(\frac{3}{2},2\right)$
$\frac{1}{2}$	$-\frac{5}{2}$	$\left(\frac{1}{2},-\frac{5}{2}\right)$
$-\frac{1}{2}$	2	$\left(-\frac{1}{2},2\right)$
$-\frac{3}{2}$	$\frac{13}{2}$	$\left(-\frac{3}{2},\frac{13}{2}\right)$

To graph g, we plot each of the points in the table above and connect them in the same order and fashion as the points to which they correspond. Plotting f and g side-by-side gives





□

The reader is strongly encouraged to graph the series of functions which shows the gradual transformation of the graph of f into the graph of g in Example 5.4.4. We have outlined the sequence of transformations in the above exposition; all that remains is to plot the five intermediate stages. Our next example turns the tables and asks for the formula of a function given a desired sequence of transformations.

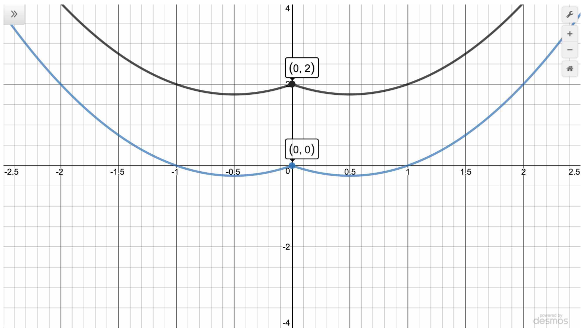
Example 5.4.5. Let $f(x) = x^2 - |x|$. Find and simplify the formula of the function $g(x)$ whose graph is the result of the graph of $y = f(x)$ undergoing the following sequence of transformations. Check your answer to each step using a graphing utility.

1. Vertical shift up 2 units.
2. Reflection across the x-axis.
3. Horizontal shift right 1 unit.
4. Horizontal compression by a factor of 2.
5. Vertical shift up 3 units.
6. Reflection across the y-axis.

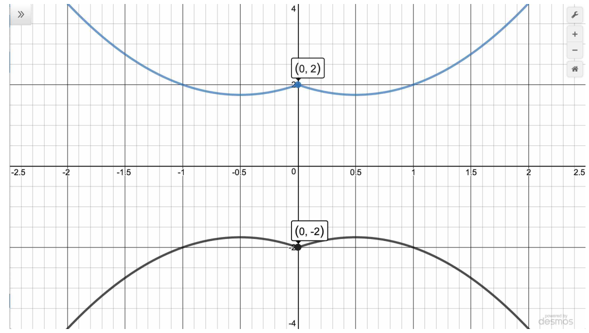
Solution. To help keep us organized we will label each intermediary function. The function g_1 will be the result of applying the first transformation to f. The function g_2 will be the result of applying the first two transformations to f - which is also the result of applying the second transformation to g_1, and so on.[13]

1. Per Theorem 5.6, $g_1(x) = f(x) + 2 = x^2 - |x| + 2$.
2. Per Theorem 5.8, $g_2(x) = -g_1(x) = -[x^2 - |x| + 2] = -x^2 + |x| - 2$.

[13]So, we can think of $g_0 = f$ and $g_6 = g$.



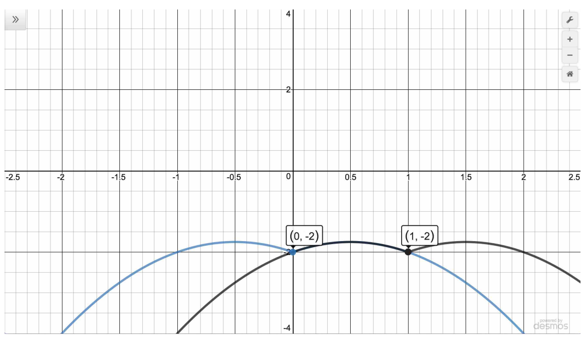
$y = f(x)$ (lighter color) and $y = g_1(x) = f(x) + 2$



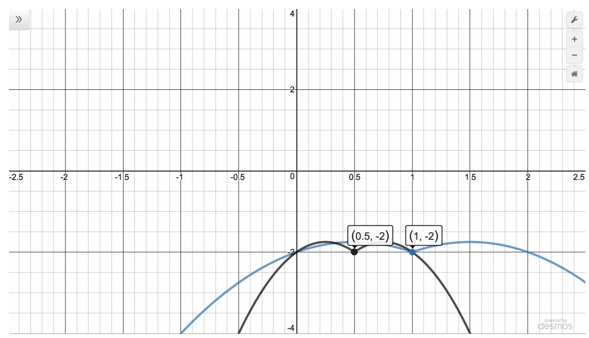
$y = g_1(x)$ (lighter color) and $y = g_2(x) = -g_1(x)$

3. Per Theorem 5.7, $g_3(x) = g_2(x-1) = -(x-1)^2 + |x-1| - 2$.

4. Per Theorem 5.10, $g_4(x) = g_3(2x) = -(2x-1)^2 + |2x-1| - 2$.



$y = g_2(x)$ (lighter color) and $y = g_3(x) = g_2(x-1)$

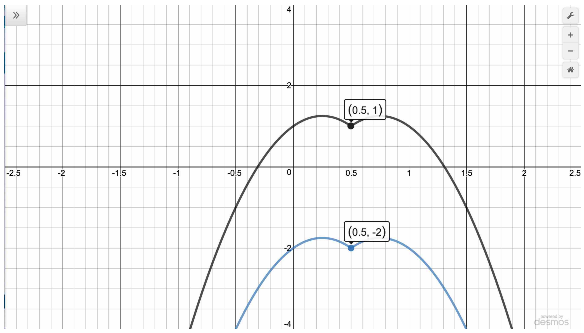


$y = g_3(x)$ (lighter color) and $y = g_4(x) = g_3(2x)$

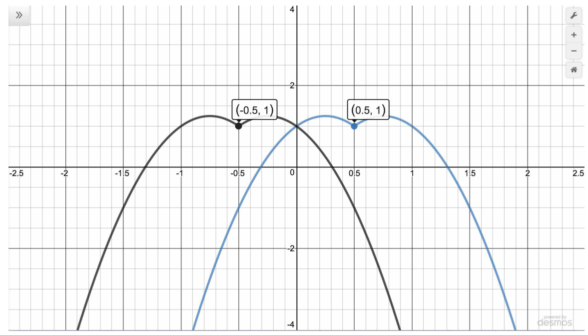
5. Per Theorem 5.6, $g_5(x) = g_4(x) + 3 = -(2x-1)^2 + |2x-1| - 2 + 3 = -(2x-1)^2 + |2x-1| + 1$.

6. Per Theorem 5.8, $g_6(x) = g_5(-x)$:

$$\begin{array}{rcl} g_6(x) & = & g_5(-x) \\ & = & -(2(-x)-1)^2 + |2(-x)-1| + 1 \\ & = & -(-2x-1)^2 + |-2x-1| + 1 \\ & = & -[(-1)(2x+1)]^2 + |[(-1)(2x+1)| + 1 \\ & = & -(-1)^2(2x+1)^2 + |-1||2x+1| + 1 \\ & = & -(2x+1)^2 + |2x+1| + 1 \end{array}$$



$y = g_4(x)$ (lighter color) and $y = g_5(x) = g_4(x) + 3$



$y = g_5(x)$ (lighter color) and $y = g_6(x) = g_5(-x)$

Hence, $g(x) = g_6(x) = -(2x+1)^2 + |2x+1| + 1$. $\square$

It is instructive to show that the expression $g(x)$ in Example 5.4.4 can be written as $g(x) = af(bx-h)+k$.

One way is to compare the graphs of f and g and work backwards. A more methodical way is to repeat the work of Example 5.4.4, but never substitute the formula for $f(x)$ as follows:

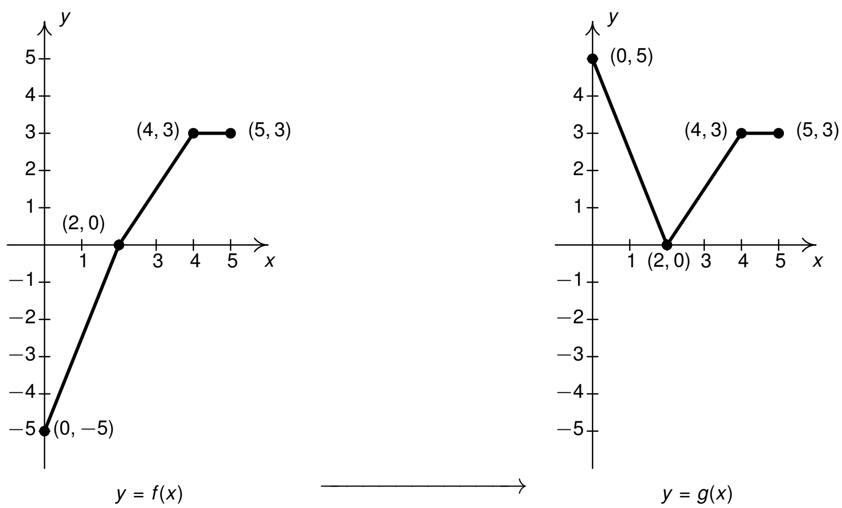
1. Per Theorem 5.6, $g_1(x) = f(x) + 2$.
2. Per Theorem 5.8, $g_2(x) = -g_1(x) = -[f(x)+2] = -f(x) - 2$.
3. Per Theorem 5.7, $g_3(x) = g_2(x-1) = -f(x-1) - 2$.
4. Per Theorem 5.10, $g_4(x) = g_3(2x) = -f(2x-1) - 2$.
5. Per Theorem 5.6, $g_5(x) = g_4(x) + 3 = -f(2x-1) - 2 + 3 = -f(2x-1) + 1$.
6. Per Theorem 5.8, $g_6(x) = g_5(-x) = -f(2(-x)-1) + 1 = -f(-2x-1) + 1$.

Hence $g(x) = -f(-2x-1)+1$. Note we can show f is even,[14] so $f(-2x-1) = f(-(2x+1)) = f(2x+1)$ and obtain $g(x) = -f(2x+1)+1$.

At the beginning of this section, we discussed how all of the transformations we'd be discussing are the result of composing given functions with linear functions. Not all transformations, not even all rigid transformations,[15] fall into these categories.

For example, consider the graphs of $y = f(x)$ and $y = g(x)$ below.

[14] Recall this means $f(-x) = f(x)$.
[15] See Section 14.2.



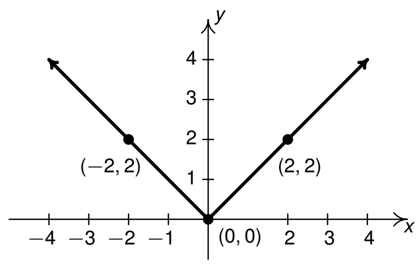
In Exercise 76, we explore a non-linear transformation and revisit the pair of functions f and g then.

5.4.5 Exercises

Suppose $(2, -3)$ is on the graph of $y = f(x)$. In Exercises 1 - 18, use Theorem 5.11 to find a point on the graph of the given transformed function.

1. $y = f(x) + 3$
2. $y = f(x + 3)$
3. $y = f(x) - 1$
4. $y = f(x - 1)$
5. $y = 3f(x)$
6. $y = f(3x)$
7. $y = -f(x)$
8. $y = f(-x)$
9. $y = f(x - 3) + 1$
10. $y = 2f(x + 1)$
11. $y = 10 - f(x)$
12. $y = 3f(2x) - 1$
13. $y = \frac{1}{2}f(4 - x)$
14. $y = 5f(2x + 1) + 3$
15. $y = 2f(1 - x) - 1$
16. $y = f\left(\dfrac{7 - 2x}{4}\right)$
17. $y = \dfrac{f(3x) - 1}{2}$
18. $y = \dfrac{4 - f(3x - 1)}{7}$

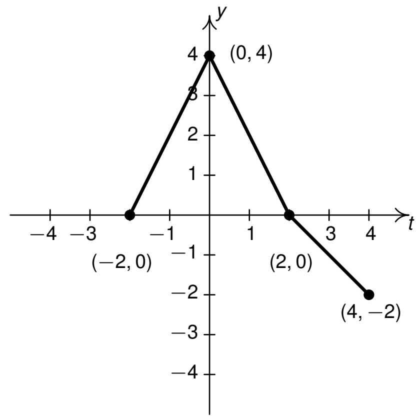
The complete graph of $y = f(x)$ is given below. In Exercises 19 - 27, use it and Theorem 5.11 to graph the given transformed function.



The graph of $y = f(x)$ for Ex. 19 - 27

19. $y = f(x) + 1$
20. $y = f(x) - 2$
21. $y = f(x + 1)$
22. $y = f(x - 2)$
23. $y = 2f(x)$
24. $y = f(2x)$
25. $y = 2 - f(x)$
26. $y = f(2 - x)$
27. $y = 2 - f(2 - x)$
28. Some of the answers to Exercises 19 - 27 above should be the same. Which ones match up? What properties of the graph of $y = f(x)$ contribute to the duplication?
29. The function f used in Exercises 19 - 27 should look familiar. What is $f(x)$? How does this this explain some of the duplication in the answers to Exercises 19 - 27 mentioned in Exercise 28?

The complete graph of $y = g(t)$ is given below. In Exercises 30 - 38, use it and Theorem 5.11 to graph the given transformed function.



The graph of $y = g(t)$ for Ex. 30 - 38

30. $y = g(t) - 1$

31. $y = g(t + 1)$

32. $y = \frac{1}{2}g(t)$

33. $y = g(2t)$

34. $y = -g(t)$

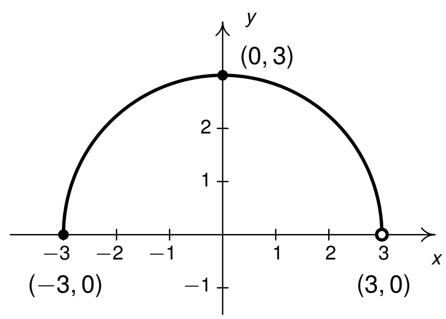
35. $y = g(-t)$

36. $y = g(t + 1) - 1$

37. $y = 1 - g(t)$

38. $y = \frac{1}{2}g(t + 1) - 1$

The complete graph of $y = f(x)$ is given below. In Exercises 39 - 50, use it and Theorem 5.11 to graph the given transformed function.



The graph of $y = f(x)$ for Ex. 39 - 50

39. $g(x) = f(x) + 3$

40. $h(x) = f(x) - \frac{1}{2}$

41. $j(x) = f\left(x - \frac{2}{3}\right)$

42. $a(x) = f(x + 4)$

43. $b(x) = f(x + 1) - 1$

44. $c(x) = \frac{3}{5}f(x)$

45. $d(x) = -2f(x)$

46. $k(x) = f\left(\frac{2}{3}x\right)$

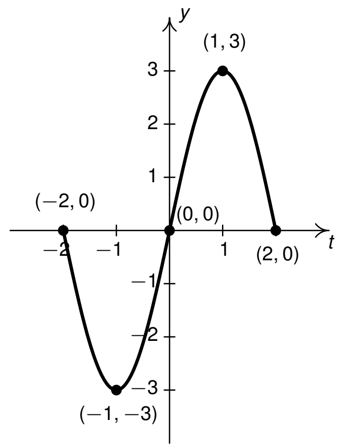
47. $m(x) = -\frac{1}{4}f(3x)$

48. $n(x) = 4f(x - 3) - 6$

49. $p(x) = 4 + f(1 - 2x)$

50. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3$

The complete graph of $y = S(t)$ is given below.



The graph of $y = S(t)$

The purpose of Exercises 51 - 54 is to build up to the graph of $y = \frac{1}{2}S(-t+1)+1$ one step at a time.

51. $y = S_1(t) = S(t+1)$

52. $y = S_2(t) = S_1(-t) = S(-t+1)$

53. $y = S_3(t) = \frac{1}{2}S_2(t) = \frac{1}{2}S(-t+1)$

54. $y = S_4(t) = S_3(t) + 1 = \frac{1}{2}S(-t+1)+1$

Let $f(x) = \sqrt{x}$. Find a formula for a function g whose graph is obtained from f from the given sequence of transformations.

55. (1) shift right 2 units; (2) shift down 3 units

56. (1) shift down 3 units; (2) shift right 2 units

57. (1) reflect across the x-axis; (2) shift up 1 unit

58. (1) shift up 1 unit; (2) reflect across the x-axis

59. (1) shift left 1 unit; (2) reflect across the y-axis; (3) shift up 2 units

60. (1) reflect across the y-axis; (2) shift left 1 unit; (3) shift up 2 units

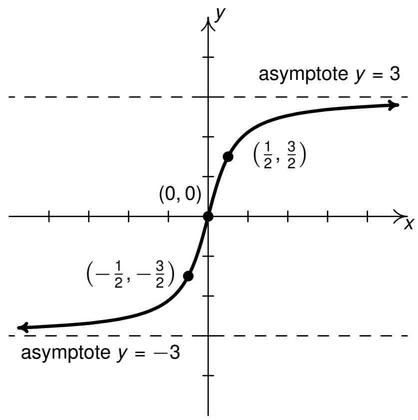
61. (1) shift left 3 units; (2) vertical stretch by a factor of 2; (3) shift down 4 units

62. (1) shift left 3 units; (2) shift down 4 units; (3) vertical stretch by a factor of 2

63. (1) shift right 3 units; (2) horizontal shrink by a factor of 2; (3) shift up 1 unit

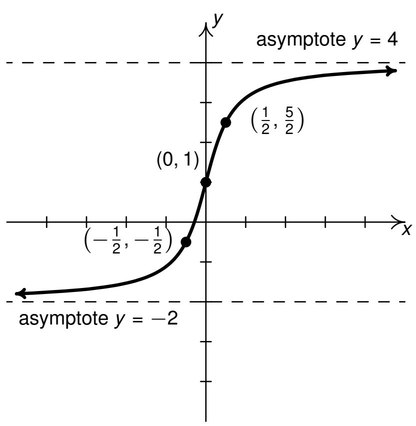
64. (1) horizontal shrink by a factor of 2; (2) shift right 3 units; (3) shift up 1 unit

For Exercises 65 - 70, use the given of $y = f(x)$ to write each function in terms of $f(x)$.

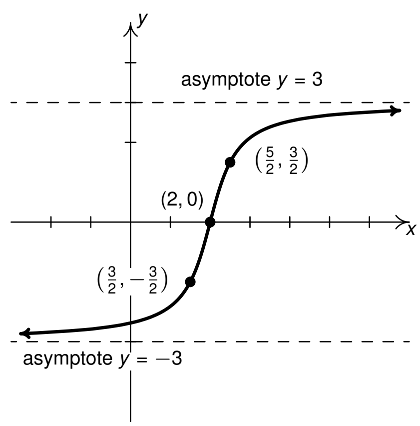


The graph of $y = f(x)$ for Ex. 65 - 70.

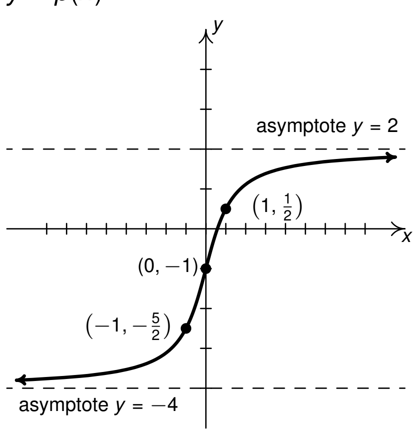
65. $y = g(x)$



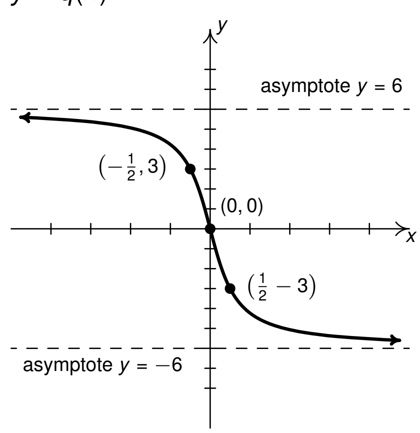
66. $y = h(x)$



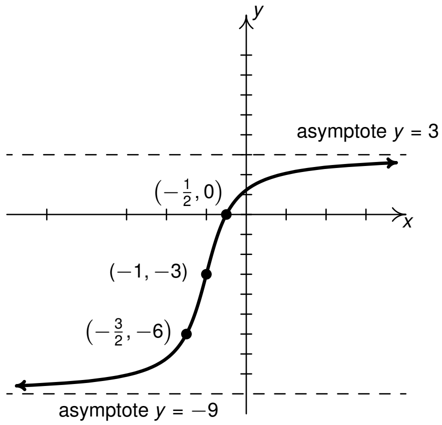
67. $y = p(x)$



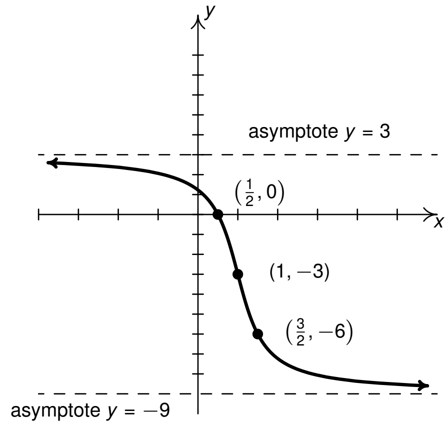
68. $y = q(x)$



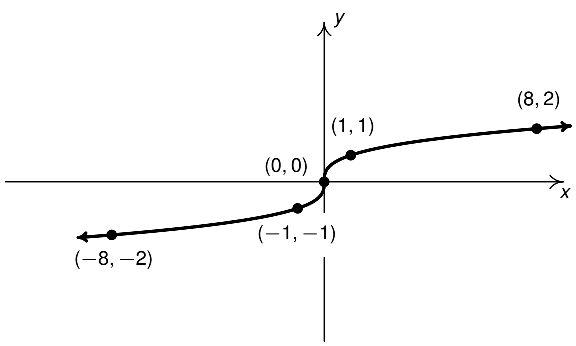
69. $y = r(x)$



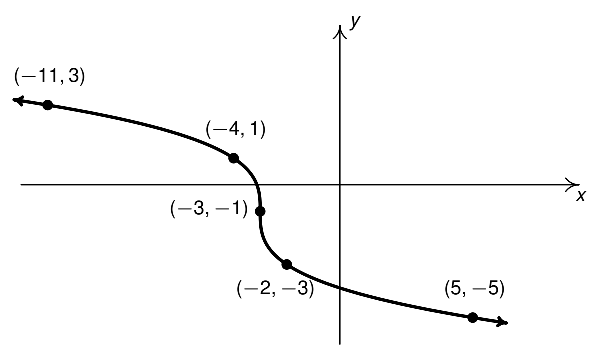
70. $y = s(x)$



71. The graph of $y = f(x) = \sqrt[3]{x}$ is given below on the left and the graph of $y = g(x)$ is given on the right. Find a formula for g based on transformations of the graph of f. Check your answer by confirming that the points shown on the graph of g satisfy the equation $y = g(x)$.



$y = \sqrt[3]{x}$



$y = g(x)$

72. Show that the composition of two linear functions is a linear function. Hence any (finite) sequence of transformations discussed in this section can be combined into the form given in Theorem 5.11.

(HINT: Let $f(x) = ax + b$ and $g(x) = cx + d$. Find $(f \circ g)(x)$.)

73. For many common functions, the properties of Algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, $\sqrt{9x} = 3\sqrt{x}$, so a horizontal compression of $y = \sqrt{x}$ by a factor of 9 results in the same graph as a vertical stretch of $y = \sqrt{x}$ by a factor of 3.

With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings $y = (2x)^3$, $y = |5x|$, $y = \sqrt[3]{27x}$ and $y = \left(\frac{1}{2}x\right)^2$.

What about $y = (-2x)^3$, $y = |-5x|$, $y = \sqrt[3]{-27x}$ and $y = \left(-\frac{1}{2}x\right)^2$?

74. Discuss the following questions with your classmates.

    - If f is even, what happens when you reflect the graph of $y = f(x)$ across the y-axis?
    - If f is odd, what happens when you reflect the graph of $y = f(x)$ across the y-axis?
    - If f is even, what happens when you reflect the graph of $y = f(x)$ across the x-axis?
    - If f is odd, what happens when you reflect the graph of $y = f(x)$ across the x-axis?
    - How would you describe symmetry about the origin in terms of reflections?

75. We mentioned earlier in the section that, in general, the order in which transformations are applied matters, yet in our first example with two transformations the order did not matter. (You could perform the shift to the left followed by the shift down or you could shift down and then left to achieve the same result.) With the help of your classmates, determine the situations in which order does matter and those in which it does not.

76. This Exercise is a follow-up to Exercise 11 in Section 1.3.

    (a) Fill in the table below.

$f(x)$	$\lvert f(x)\rvert$	$f\left(\lvert x\rvert\right)$
$x+2$		
x^2-4x		
x^3-3x^2		
$(x+1)^{-1}$		
$\sqrt{x+2}-3$		

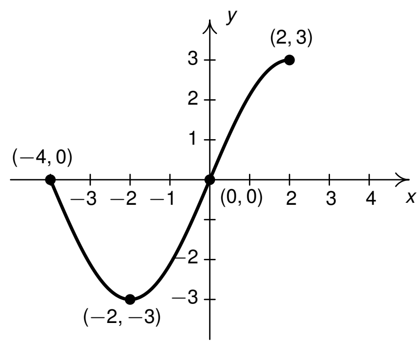
    (b) For each function f above, graph $y = f(x)$ and $y = \lvert f(x)\rvert$ using a graphing utility.

        i. Write a sentence (or two!) explaining how to obtain the graph of $y = \lvert f(x)\rvert$ from $y = f(x)$.
        ii. How does your explanation relate to Definition 1.9?

    (c) For each function f above, graph $y = f(x)$ and $y = f(\lvert x\rvert)$ using a graphing utility.

        i. Write a sentence (or two!) explaining how to obtain the graph of $y = f(\lvert x\rvert)$ from $y = f(x)$.
        ii. How does your explanation relate to Definition 1.9?

(d) Use the graph of $y = f(x)$ below to graph $y = |f(x)|$ and $y = f(|x|)$.

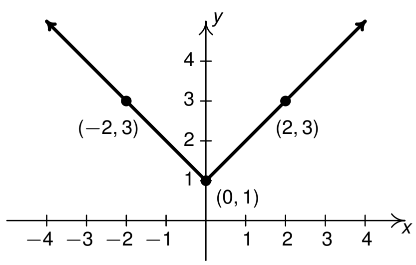


$y = f(x)$

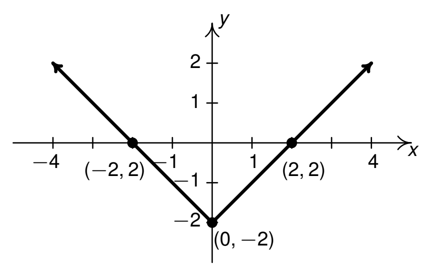
(e) Referring to the functions f and g graphed on page 439, write g in terms of f.

5.4.6 Answers

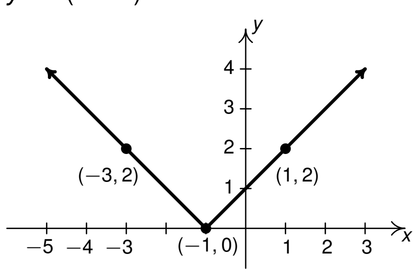
1. $(2, 0)$
2. $(-1, -3)$
3. $(2, -4)$
4. $(3, -3)$
5. $(2, -9)$
6. $\left(\frac{2}{3}, -3\right)$
7. $(2, 3)$
8. $(-2, -3)$
9. $(5, -2)$
10. $(1, -6)$
11. $(2, 13)$
12. $y = (1, -10)$
13. $\left(2, -\frac{3}{2}\right)$
14. $\left(\frac{1}{2}, -12\right)$
15. $(-1, -7)$
16. $\left(-\frac{1}{2}, -3\right)$
17. $\left(\frac{2}{3}, -2\right)$
18. $(1, 1)$
19. $y = f(x) + 1$



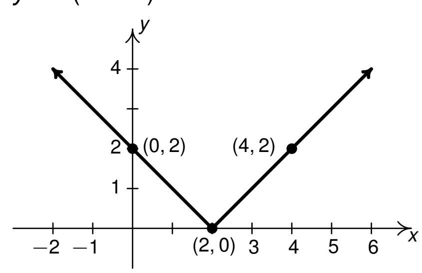
20. $y = f(x) - 2$



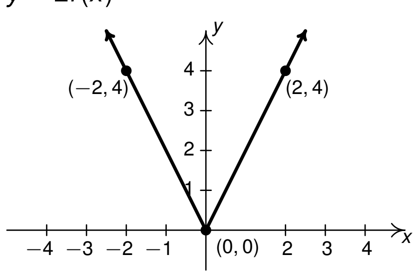
21. $y = f(x + 1)$



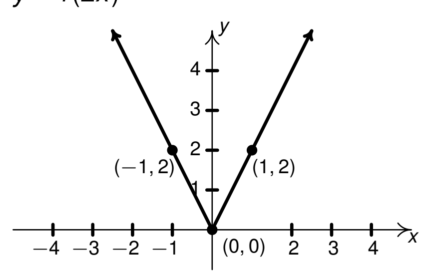
22. $y = f(x - 2)$



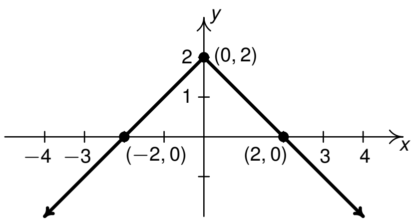
23. $y = 2f(x)$



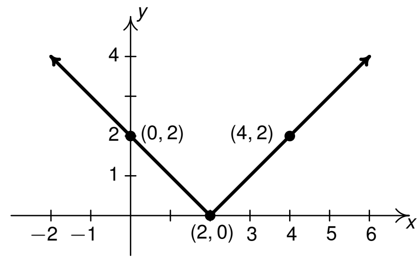
24. $y = f(2x)$



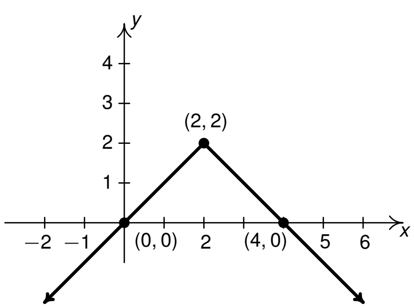
25. $y = 2 - f(x)$



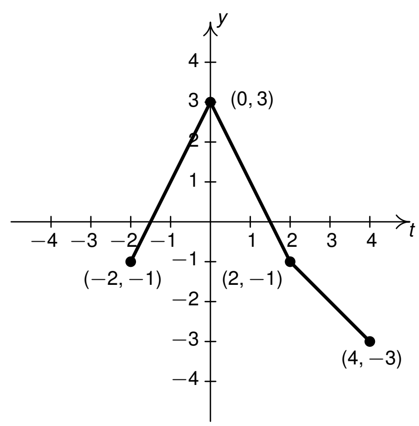
26. $y = f(2 - x)$



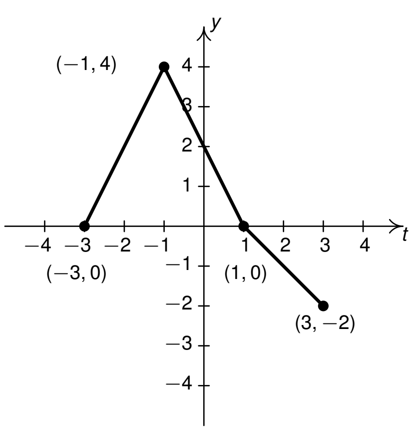
27. $y = 2 - f(2 - x)$



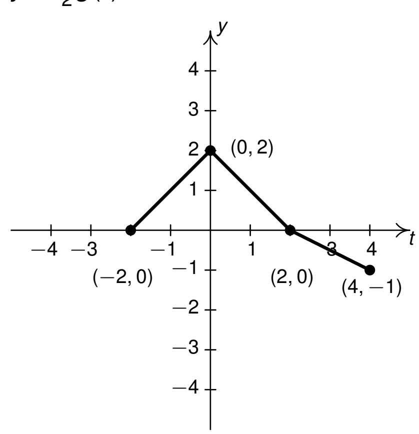
30. $y = g(t) - 1$



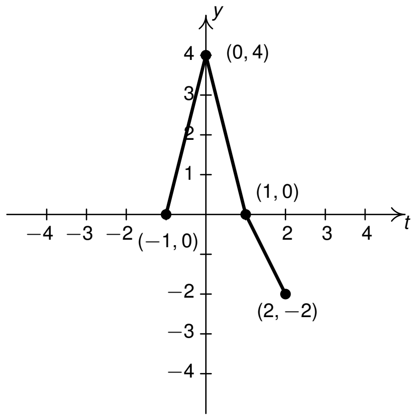
31. $y = g(t + 1)$



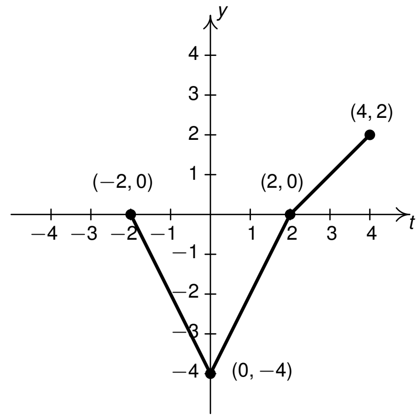
32. $y = \frac{1}{2}g(t)$



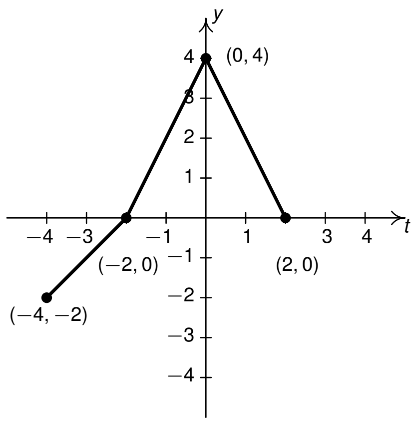
33. $y = g(2t)$



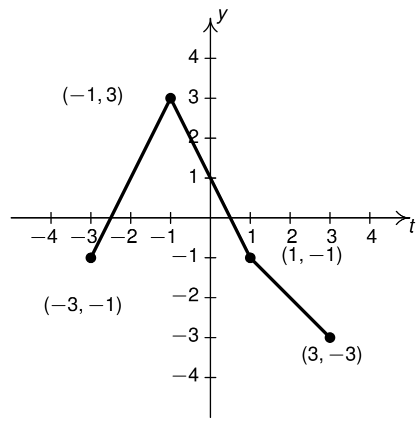
34. $y = -g(t)$



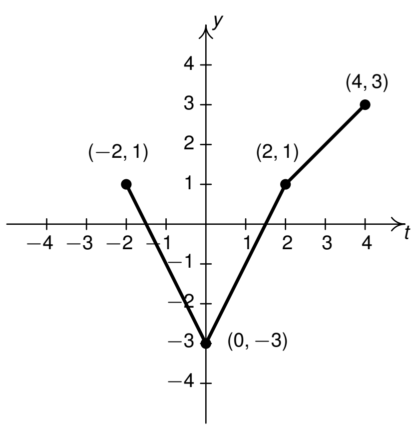
35. $y = g(-t)$



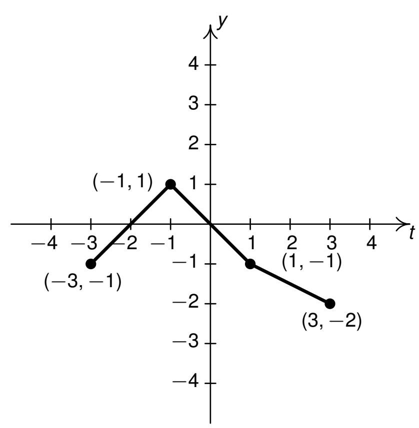
36. $y = g(t + 1) - 1$



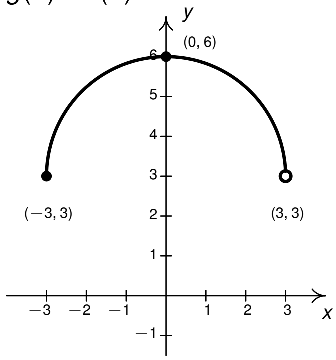
37. $y = 1 - g(t)$



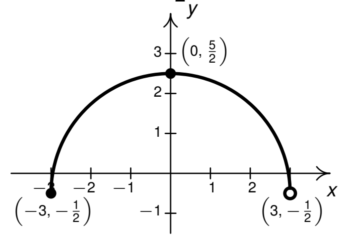
38. $y = \frac{1}{2}g(t + 1) - 1$



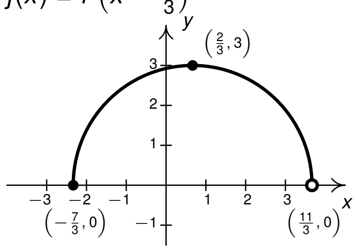
39. $g(x) = f(x) + 3$



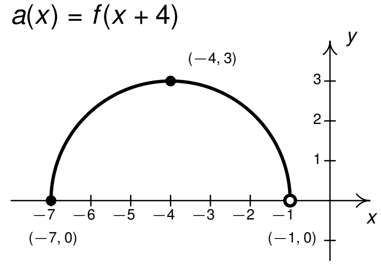
40. $h(x) = f(x) - \frac{1}{2}$



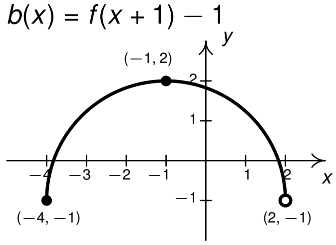
41. $j(x) = f\left(x - \frac{2}{3}\right)$



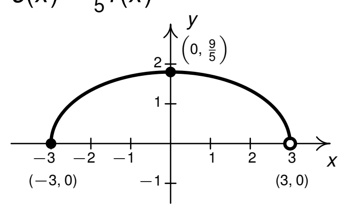
42. $a(x) = f(x + 4)$



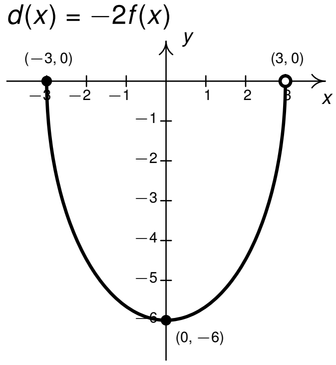
43. $b(x) = f(x + 1) - 1$



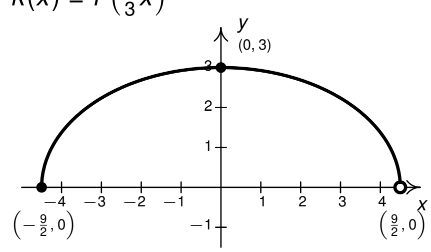
44. $c(x) = \frac{3}{5}f(x)$



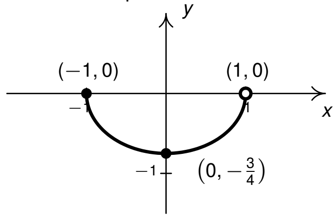
45. $d(x) = -2f(x)$



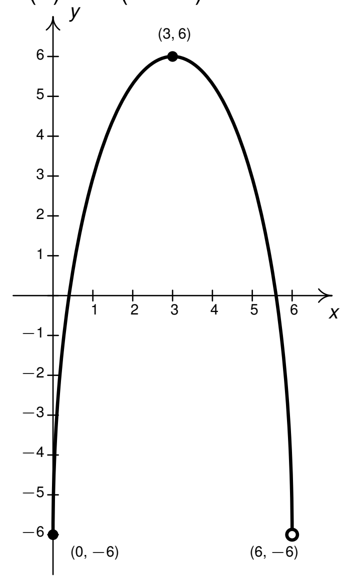
46. $k(x) = f\left(\frac{2}{3}x\right)$



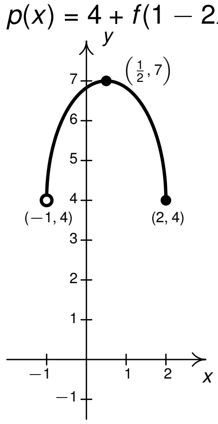
47. $m(x) = -\frac{1}{4}f(3x)$



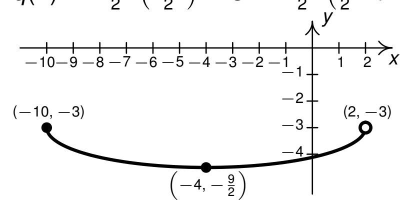
48. $n(x) = 4f(x-3) - 6$



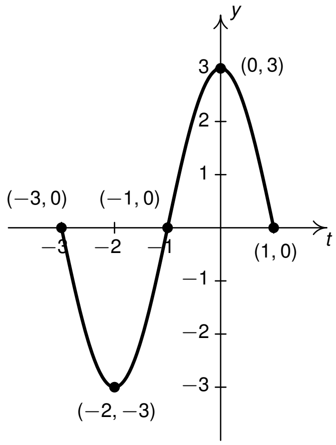
49. $p(x) = 4 + f(1-2x) = f(-2x+1) + 4$



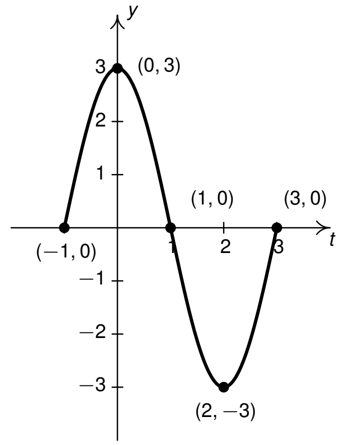
50. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3 = -\frac{1}{2}f\left(\frac{1}{2}x+2\right) - 3$



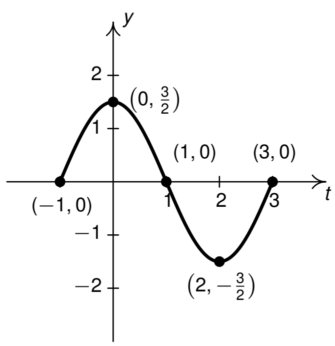
51. $y = S_1(t) = S(t+1)$



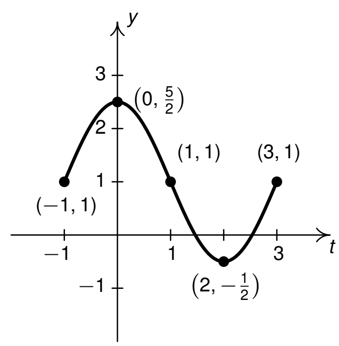
52. $y = S_2(t) = S_1(-t) = S(-t+1)$



53. $y = S_3(t) = \frac{1}{2}S_2(t) = \frac{1}{2}S(-t+1)$



54. $y = S_4(t) = S_3(t) + 1 = \frac{1}{2}S(-t+1) + 1$



55. $g(x) = \sqrt{x-2} - 3$

56. $g(x) = \sqrt{x-2} - 3$

57. $g(x) = -\sqrt{x} + 1$

58. $g(x) = -(\sqrt{x} + 1) = -\sqrt{x} - 1$

59. $g(x) = \sqrt{-x+1} + 2$

60. $g(x) = \sqrt{-(x+1)} + 2 = \sqrt{-x-1} + 2$

61. $g(x) = 2\sqrt{x+3} - 4$

62. $g(x) = 2\left(\sqrt{x+3} - 4\right) = 2\sqrt{x+3} - 8$

63. $g(x) = \sqrt{2x-3} + 1$

64. $g(x) = \sqrt{2(x-3)} + 1 = \sqrt{2x-6} + 1$

65. $g(x) = f(x) + 1$

66. $h(x) = f(x-2)$

67. $p(x) = f\left(\frac{x}{2}\right) - 1$

68. $q(x) = -2f(x) = 2f(-x)$

69. $r(x) = 2f(x+1) - 3$

70. $s(x) = 2f(-x+1) - 3 = -2f(x-1) + 3$

71. $g(x) = -2\sqrt[3]{x+3} - 1$ or $g(x) = 2\sqrt[3]{-x-3} - 1$

76. (a)

$f(x)$	$\lvert f(x)\rvert$	$f\left(\lvert x\rvert\right)$
$x+2$	$\lvert x+2\rvert$	$\lvert x\rvert+2$
x^2-4x	$\lvert x^2-4x\rvert$	$\lvert x\rvert^2-3\lvert x\rvert = x^2-4\lvert x\rvert$
x^3-3x^2	$\lvert x^3-3x^2\rvert$	$\lvert x\rvert^3-3\lvert x\rvert^2 = \lvert x\rvert^3-3x^2$
$(x+1)^{-1}$	$\lvert (x+1)^{-1}\rvert$	$(\lvert x\rvert+1)^{-1}$
$\sqrt{x+2}-3$	$\lvert\sqrt{x+2}-3\rvert$	$\sqrt{\lvert x\rvert+2}-3$

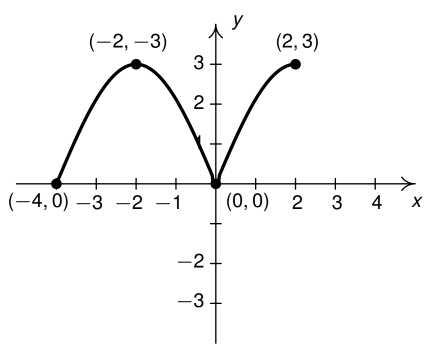
(b) i. To graph $y = |f(x)|$ from the graph of $y = f(x)$, reflect about the x-axis any portion of the graph of $y = f(x)$ which is below the x-axis.

ii. If the graph is below the x-axis, then $f(x) < 0$. Since $|f(x)| = -f(x)$ if $f(x) < 0$, we are graphing $y = -f(x)$ for these values of x which is a reflection across the x-axis.

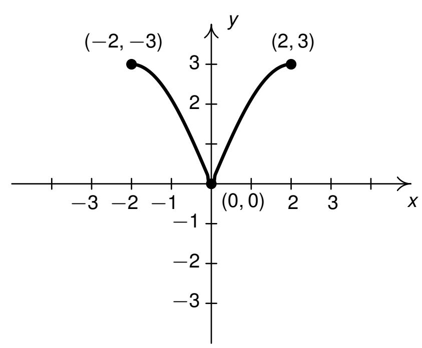
(c) i. To graph $y = f(|x|)$ from the graph of $y = f(x)$, replace the graph of $y = f(x)$ for $x \leq 0$ with the reflection about the y-axis of the graph of $y = f(x)$ for $x \geq 0$.

ii. If $x < 0$, then $|x| = -x$, so $f(|x|) = f(-x)$. Since if $x < 0$, $-x > 0$, this means we reflect the graph of $y = f(x)$ about the y-axis for $x > 0$ only.

(d)



$y = |f(x)|$



$y = f(|x|)$ - note the domain

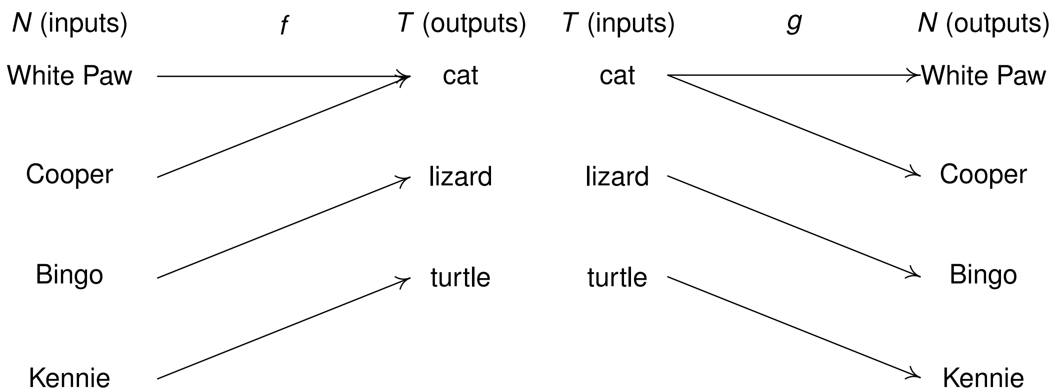
(e) $g(x) = |f(x)|$.

5.5 Relations and Implicit Functions

Up until now in this text, we have been exclusively special kinds of mappings called *functions*. In this section, we broaden our horizons to study more general mappings called *relations*. The reader is encouraged to revisit Definition 1.1 in Section 1.1 before proceeding with the definition of *relation* below.

> **Definition 5.3.** Given two sets A and B, a **relation** from A to B is a process by which elements of A are matched with (or 'mapped to') elements of B.

Unlike Definition 1.1, Definition 5.3 puts no conditions on the process which maps elements of A to elements of B. This means that while all functions are relations, not all relations need be functions. For example, consider the mappings f and g below from Section 1.1.



Both f and g are relations. More specifically, f is a *function* from N to T while g is merely *relation* from T to N. As with functions, we may describe general relations in a variety of different ways: verbally, as mapping diagrams, or a set of ordered pairs. For example, just as we may describe the function f above as

$$f = \{(\text{White Paw}, \text{cat}), (\text{Cooper}, \text{cat}), (\text{Bingo}, \text{lizard}), (\text{Kennie}, \text{turtle})\},$$

we may represent g as

$$g = \{(\text{cat}, \text{White Paw}), (\text{cat}, \text{Cooper}), (\text{lizard}, \text{Bingo}), (\text{turtle}, \text{Kennie})\}.$$

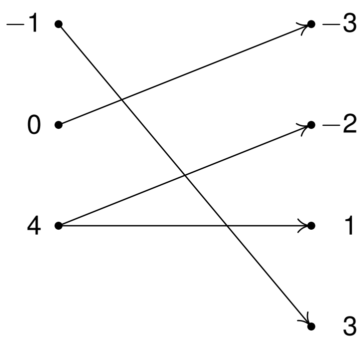
Note here the grammar 'g is a relation *from T to N*' is evidenced by the elements of T being listed first in the ordered pairs (i.e., the abscissae) and the elements of N being listed second (i.e., the ordinates.)

Unlike functions, we do not use function notation when describing the input/output relationship for general relations. For example, we may write '$f(\text{White Paw}) = \text{cat}$' since f maps the input 'White Paw' to only one output, 'cat.' However, $g(\text{cat})$ is ambiguous since it could mean 'White Paw' or 'Cooper.'[1]

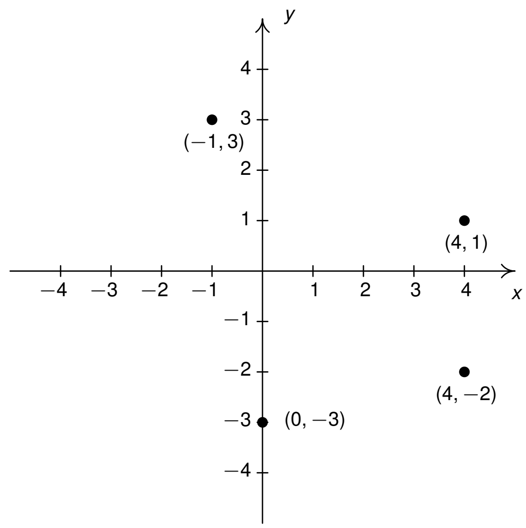
As with functions, our focus in this course will rest with relations of real numbers. Consider the relation R described as follows: $R = \{(-1, 3), (0, -3), (4, -2), (4, 1)\}$. Below on the left is a mapping diagram of R.

[1] In more advanced texts, we would write 'cat g White Paw' and 'cat g Cooper' to indicate g maps 'cat' to both 'White Paw' and 'Cooper.' Our study of relations, however, isn't deep enough to necessitate introducing and using this notation. Similarly, we won't introduce the notions of 'domain,' 'codomain,' and 'range' for relations, either.

However, since R relates real numbers, we can also create the graph of R in the same way we graphed functions - by interpreting the ordered pairs which comprise R as points in the plane. Since we have no context, we use the default labels 'x' for the horizontal axis and 'y' for the vertical axis.



A Mapping Diagram of R.



The graph of R.

Our next example focuses on using relations to describe sets of points in the plane and vice-versa.

Example 5.5.1.

1. Graph the following relations.

   (a) $S = \{(k, 2^k) \mid k = 0, \pm 1, \pm 2\}$

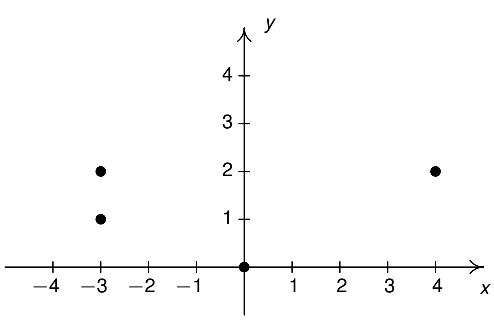
   (b) $P = \{(j, j^2) \mid j \text{ is an integer}\}$

   (c) $V = \{(3, y) \mid y \text{ is a real number}\}$

   (d) $R = \{(x, y) \mid x \text{ is a real number}, 1 < y \leq 3\}$

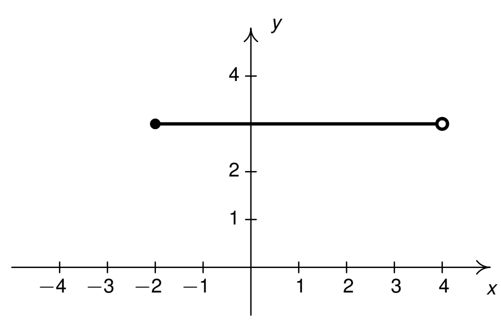
2. Find a roster or set-builder description for each of the relations below.

   (a)



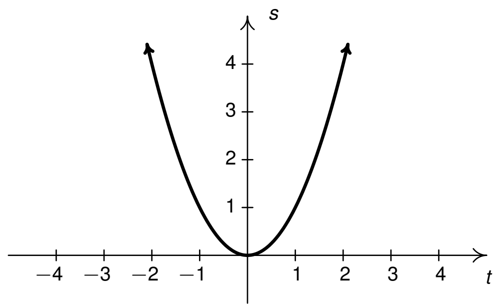
The graph of A

   (b)



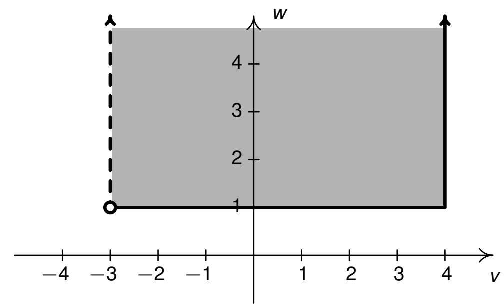
The graph of H

(c)



The graph of Q

(d)



The graph of T

Solution.

1. (a) The relation S is described using *set-builder notation*.[2] To generate the ordered pairs which belong to S, we substitute the given values of k, $k = 0, \pm 1, \pm 2$, into the formula $\left(k, 2^k\right)$.

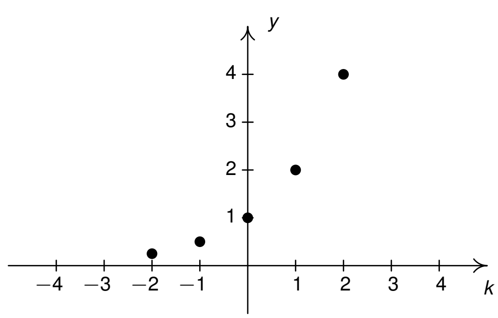
   Starting with $k = 0$, we get $\left(0, 2^0\right) = (0, 1)$. For $k = 1$, we get $\left(1, 2^1\right) = (1, 2)$, and for $k = -1$, we get $\left(-1, 2^{-1}\right) = \left(-1, \frac{1}{2}\right)$. Continuing, we get $\left(2, 2^2\right) = (2, 4)$ for $k = 2$ and, finally $\left(-2, 2^{-2}\right) = \left(-2, \frac{1}{4}\right)$ for $k = -2$. Hence, a roster description of S is $S = \left\{\left(-2, \frac{1}{4}\right), \left(-1, \frac{1}{2}\right), (0, 1), (1, 2), (2, 4)\right\}$.

   When we graph S, we label the horizontal axis as the k-axis, since 'k' was the variable chosen used to generate the ordered pairs and keep the default label 'y' for the vertical axis. The graph of S is below on the left.

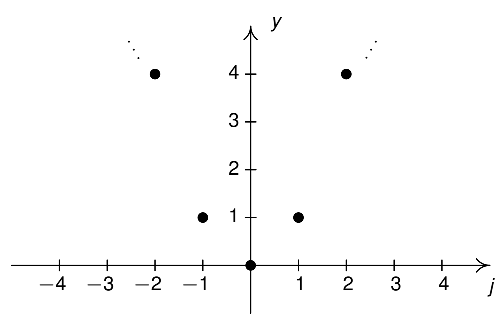
   (b) To graph the relation $P = \left\{\left(j, j^2\right) \mid j \text{ is an integer}\right\}$, we proceed as above when we graphed the relation S. Here, j is restricted to being an integer, which means $j = 0, \pm 1, \pm 2$, etc.

   Plugging in these sample values for j, we obtain the ordered pairs $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(2, 4)$, $(-2, 4)$, etc. Since the variable j takes on only integer values, we could write P using the roster notation: $P = \{(0, 0), (\pm 1, 1), (\pm 2, 4), \ldots\}$.

   We plot a few of these points and use some periods of ellipsis to indicate the complete graph contains additional points not in the current field of view. The graph of P is below on the right.



The graph of S



The graph of P

[2]See Section A.1 to review this, if needed.

(c) Next, we come to the relation V, described, once again, using set-builder notation. In this case, V consists of all ordered pairs of the form $(3, y)$ where y is free to be whatever real number we like, without any restriction.[3] For example, $(3, 0)$, $(3, -1)$, and $(3, 117)$ all belong to V as do $\left(3, \frac{1}{2}\right)$, $(3, -1.0342)$, $(3, \sqrt{2})$, etc.

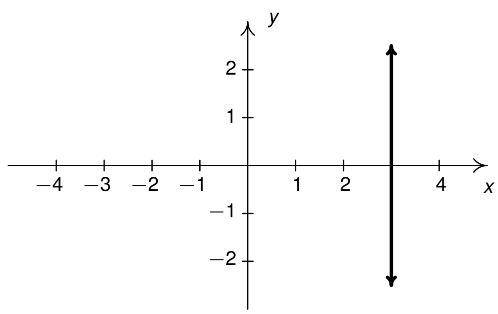
After plotting some sample points, becomes apparent that the ordered pairs which belong to V correspond to points which lie on the vertical line $x = 3$, and vice-versa. That is, every point on the line $x = 3$ has coordinates which correspond to an ordered pair belonging to V. The graph of V is below on the left.

(d) In the relation $R = \{(x, y) \mid 1 < y \leq 3\}$, we see y is restricted by the inequality $1 < y \leq 3$, but x is free to be whatever it likes.

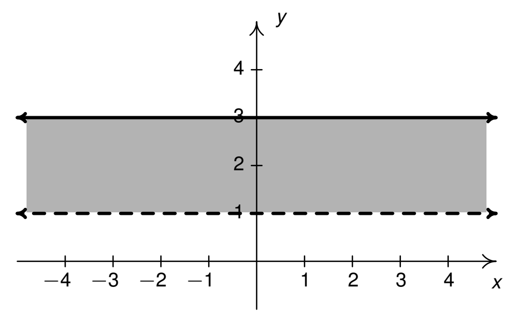
Since x is unrestricted, this means whatever the graph of R is, it will extend indefinitely off to the right and left. The restriction $y > 1$ means all points on the graph of R have a y-coordinate larger than one, so they are *above* the horizontal line $y = 1$. The restriction $y \leq 3$, on the other hand, means all the points on the graph of R have a y-coordinate less than or equal to 3, meaning they are either *on* or *below* the horizontal line $y = 3$.

In other words, the graph of R is the region in the plane between $y = 1$ and $y = 3$, including $y = 3$ but not $y = 1$. We signify this by *shading* the region between these two horizontal lines.

How do we communicate $y = 1$ is not part of the graph? One way is to visualize putting 'holes' all along the line $y = 1$ to indicate this is not part of the graph. In practice, however, this looks cluttered and could be confusing. Instead, we 'dash' the line $y = 1$ as seen below on the right.



The graph of V



The graph of R

2. (a) Since A consists of finitely many points, we can describe A using the roster method:

$$A = \{(-3, 2), (-3, 1), (0, 0), (4, 2)\}.$$

(b) The graph of H appears to be a portion of the horizontal line $y = 3$ from $x = -2$ (including $x = -2$) up to, but not including $x = 4$. Since it is impossible[4] to *list* each and every one of these points, we'll opt to describe H using set-builder as opposed to the roster method. Taking a cue from the description of the relations V and R above, we write $H = \{(x, 3) \mid -2 \leq x < 4\}$.

[3] We'll revisit the concept of a 'free variable' in Section 9.1.

[4] Really impossible. The interested reader is encouraged to research **countable** versus **uncountable** sets.

(c) The graph of Q appears to be the graph of the function $s = f(t) = t^2$. Again, as the graph consists of infinitely many points, we will use set-builder notation to describe Q out of necessity.

There are a couple of different ways to do this. Taking a cue from the relation P above, we could write $Q = \{(t, t^2) \,|\, t \text{ is a real number}\}$. Alternatively, we could introduce the dependent variable, s into the description by writing $Q = \{(t, s) \,|\, s = t^2\}$ where here the assumption is x takes in all real number values.

(d) As with the relation R above, the relation T describes a region in the plane, The v-values appear to range between -3 (not including -3) and up to, and including, $v = 4$. The only restriction on the w-values is that $w \geq 1$, so we have $T = \{(v, w) \,|\, -3 < v \leq 4, \, w \geq 1\}$. □

As with functions, we can describe relations algebraically using equations. For example, the equation $v^2 + w^3 = 1$ relates two variables v and w each of which represent real numbers. More formally, we can express this sentiment by defining the relation $R = \{(v, w) \,|\, v^2 + w^3 = 1\}$. An ordered pair $(v, w) \in R$ means v and w are *related* by the equation $v^2 + w^3 = 1$; that is, the pair (v, w) *satisfy* the equation.

For example, to show $(3, -2) \in R$, we check that when we substitute $v = 3$ and $w = -2$, the equation $v^2 + w^3 = 1$ is true. Sure enough, $(3)^2 + (-2)^3 = 9 - 8 = 1$. Hence, R maps 3 to -2. Note, however, that $(-2, 3) \notin R$ since $(-2)^2 + (3)^3 = -8 + 27 \neq 1$ which means R does not map -2 to 3.

When asked to 'graph the equation' $v^2 + w^3 = 1$, we really have two options. We could graph the relation R above. In this case, we would be graphing $v^2 + w^3 = 1$ on the vw-plane.[5] Alternatively, we could define $S = \{(w, v) \,|\, v^2 + w^3 = 1\}$ and graph S. This is equivalent to graphing $v^2 + w^3 = 1$ on the wv-plane. We do both in our next example.

Example 5.5.2. Graph the equation $v^2 + w^3 = 1$ in the vw- and wv-planes. Include the axis-intercepts.
Solution.

- *graphing in the vw-plane:* We begin by finding the axis intercepts of the graph. To obtain a point on the v-axis, we require $w = 0$. To see if we have any v-intercepts on the graph of the equation $v^2 + w^3 = 1$, we substitute $w = 0$ into the equation and solve for v: $v^2 + (0)^3 = 1$. We get $v^2 = 1$ or $v = \pm 1$ so our two v-intercepts, as described in the vw-plane, are $(1, 0)$ and $(-1, 0)$.

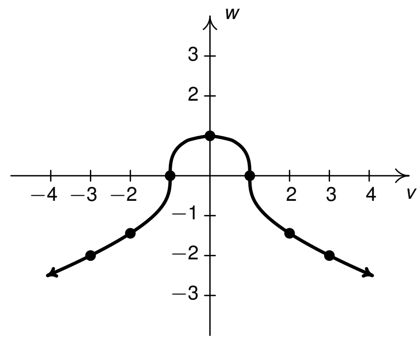
  Likewise, to find w-intercepts of the graph, we substitute $v = 0$ into the equation $v^2 + w^3 = 1$ and get $w^3 = 1$ or $w = 1$. Hence, he have only one w-intercept, $(0, 1)$.

  One way to efficiently produce additional points is to solve the equation $v^2 + w^3 = 1$ for one of the variables, say w, in terms of the other, v. In this way, we are treating w as the dependent variable and v as the independent variable. From $v^2 + w^3 = 1$, we get $w^3 = 1 - v^2$ or $w = \sqrt[3]{1 - v^2}$.

  We now substitute a value in for v, determine the corresponding value w, and plot the resulting point (v, w). We summarize our results below on the left. By plotting additional points (or getting help from a graphing utility), we produce the graph below on the right.

[5] Recall this means the horizontal axis is labeled 'v' and the vertical axis is labeled 'w.'

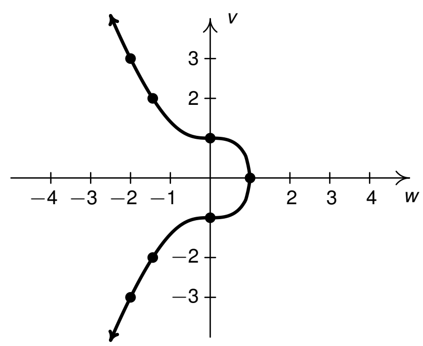
v	w	(v, w)
-3	-2	$(-3, -2)$
-2	$-\sqrt[3]{3}$	$(-2, -\sqrt[3]{3})$
-1	0	$(-1, 0)$
0	1	$(0, 1)$
1	0	$(1, 0)$
2	$-\sqrt[3]{3}$	$(2, -\sqrt[3]{3})$
3	-2	$(3, -2)$



$v^2 + w^3 = 1$

- *graphing in the wv-plane:* To graph $v^2 + w^3 = 1$ in the wv-plane, all we need to do is reverse the coordinates of the ordered pairs we obtained for our graph in the vw-plane. In particular, the v-intercepts are written $(0, 1)$ and $(0, -1)$ and the w-intercept is written $(1, 0)$. Using the table below on the left we produce the graph below on the right.

v	w	(w, v)
-3	-2	$(-2, -3)$
-2	$-\sqrt[3]{3}$	$(-\sqrt[3]{3}, -2)$
-1	0	$(0, -1)$
0	1	$(1, 0)$
1	0	$(0, 1)$
2	$-\sqrt[3]{3}$	$(-\sqrt[3]{3}, 2)$
3	-2	$(-2, 3)$



$v^2 + w^3 = 1$

□

Note that regardless of which geometric depiction we choose for $v^2 + w^3 = 1$, the graph appears to be symmetric about the w-axis. To prove this is the case, consider a generic point (v, w) on the graph of $v^2 + w^3 = 1$ in the vw-plane.

To show the point symmetric about the w-axis, $(-v, w)$ is also on the graph of $v^2 + w^3 = 1$, we need to show that the coordinates of the point $(-v, w)$ satisfy the equation $v^2 + w^3 = 1$. That is, we need to show $(-v)^2 + w^3 = 1$. Since $(-v)^2 + w^3 = v^2 + w^3$, and we know by assumption $v^2 + w^3 = 1$, we get $(-v)^2 + w^3 = v^2 + w^3 = 1$, proving $(-v, w)$ is also on the graph of the equation.

The key reason our proof above is successful is that algebraically, the equation $v^2 + w^3 = 1$ is unchanged if v is replaced with $-v$. Geometrically, this means the graph is the same if it undergoes a reflection across the w-axis. We generalize this reasoning in the following result. Note that, as usual, we default to the more common x and y-axis labels.

Theorem 5.12. Testing the Graph of an Equation for Symmetry:
To test the graph of an equation in the xy-plane for symmetry:

- about the x-axis: substitute $(x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the x-axis.
- about the y-axis: substitute $(-x, y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the y-axis.
- about the origin: substitute $(-x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the origin.

Parts of Theorem 5.12 should look familiar from our work with even and odd functions. Indeed if a function f is even, $f(-x) = f(x)$. Hence, the equation $y = f(-x)$ reduces to the equation $y = f(x)$, so the graph of f is symmetric about the y-axis.

Likewise if f is odd, then $f(-x) = -f(x)$. In this case, the equation $-y = f(-x)$ reduces to $-y = -f(x)$, or $y = f(x)$, proving the graph is symmetric about the origin.

When it comes to symmetry about the x-axis, most of the time this indicates a violation of the Vertical Line Test, which is why we haven't discussed that particular kind of symmetry until now.

We put Theorem 5.12 to good use in the following example.

Example 5.5.3. Graph each of the equations below in the xy-plane. Find the axis intercepts, if any, and prove any symmetry suggested by the graphs.

1. $x^2 - y^2 = 4$
2. $(x-1)^2 + 4y^2 = 16$

Solution.

1. We begin graphing $x^2 - y^2 = 4$ by checking for axis intercepts. To check for x-intercepts, we set $y = 0$ and solve $x^2 - (0)^2 = 4$. We get $x = \pm 2$ and obtain two x-intercepts $(-2, 0)$ and $(2, 0)$.

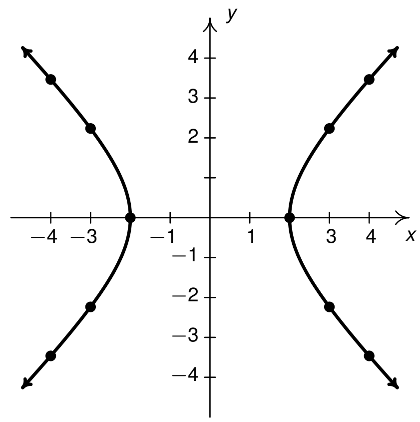
   When looking for y-intercepts, we set $x = 0$ and get $(0)^2 - y^2 = 4$ or $y^2 = -4$. Since this equation has no real number solutions, we have no y-intercepts.

   In order to produce more points on the graph, we solve $x^2 - y^2 = 4$ for y and obtain $y = \pm\sqrt{x^2 - 4}$. Since we know $x^2 - 4 \geq 0$ in order to produce real number results for y, we restrict our attention to $x \leq -2$ and $x \geq 2$. Doing so produces the table below on the left. Using these, we construct the graph below the right.

   The graph certainly appears to be symmetric about both axes and the origin. To prove this, we note that the equation $x^2 - (-y)^2 = 4$ quickly reduces to $x^2 - y^2 = 4$, proving the graph is symmetric about the x-axis.

   Likewise, the equations $(-x)^2 - y^2 = 4$ and $(-x)^2 - (-y)^2 = 4$ also reduce to $x^2 - y^2 = 4$, proving the graph is, indeed, symmetric about the y-axis and origin, respectively.

x	y	(x, y)
-4	$\pm 2\sqrt{3}$	$(-4, \pm 2\sqrt{3})$
-3	$\pm\sqrt{5}$	$(-3, \pm\sqrt{5})$
-2	0	$(-2, 0)$
2	0	$(2, 0)$
3	$\pm\sqrt{5}$	$(3, \pm\sqrt{5})$
4	$\pm 2\sqrt{3}$	$(4, \pm 2\sqrt{3})$



$x^2 - y^2 = 4$

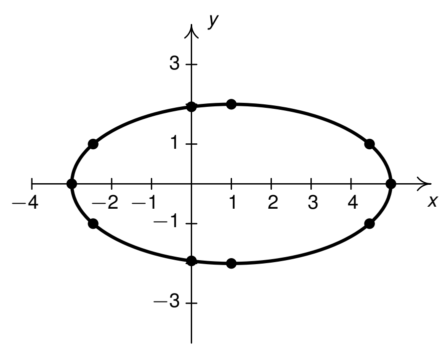
2. To determine if there are any x-intercepts on the graph of $(x-1)^2 + 4y^2 = 16$, we set $y = 0$ and solve $(x-1)^2 + 4(0)^2 = 16$. This reduces to $(x-1)^2 = 16$ which gives $x = -3$ and $x = 5$. Hence, we have two x-intercepts, $(-3, 0)$ and $(5, 0)$.

   Looking for y-intercepts, we set $x = 0$ and solve $(0-1)^2 + 4y^2 = 16$ or $1 + 4y^2 = 16$. This gives $y^2 = \frac{15}{4}$ so $y = \pm\frac{\sqrt{15}}{2}$. Hence, we have two y-intercepts: $\left(0, \pm\frac{\sqrt{15}}{2}\right)$.

   In this case, it is slightly easier[6] to solve for x in terms of y. From $(x-1)^2 + 4y^2 = 16$ we get $(x-1)^2 = 16 - 4y^2$ which gives $x = 1 \pm \sqrt{16 - 4y^2}$.

   Since we know $16 - 4y^2 \geq 0$ to produce real number results for x, we require $-2 \leq y \leq 2$. Selecting values in that range produces the table below on the left. Plotting these points, along with the y-intercepts produces the graph on the right.

y	x	(x, y)
-2	1	$(1, -2)$
-1	$1 \pm 2\sqrt{3}$	$(1 \pm 2\sqrt{3}, -1)$
0	$1 \pm 4 = -3, 5$	$(-3, 0), (5, 0)$
1	$1 \pm 2\sqrt{3}$	$(1 \pm 2\sqrt{3}, 1)$
2	1	$(1, 2)$



$(x-1)^2 + 4y^2 = 16$

   The graph certainly appears to be symmetric about the x-axis. To check, we substitute $(-y)$ in for y and get $(x-1)^2 + 4(-y)^2 = 16$ which reduces to $(x-1)^2 + 4y^2 = 16$.

   Owing to the placement of the x-intercepts, $(-3, 0)$ and $(5, 0)$, the graph is most certainly not symmetric about the y-axis nor about the origin. □

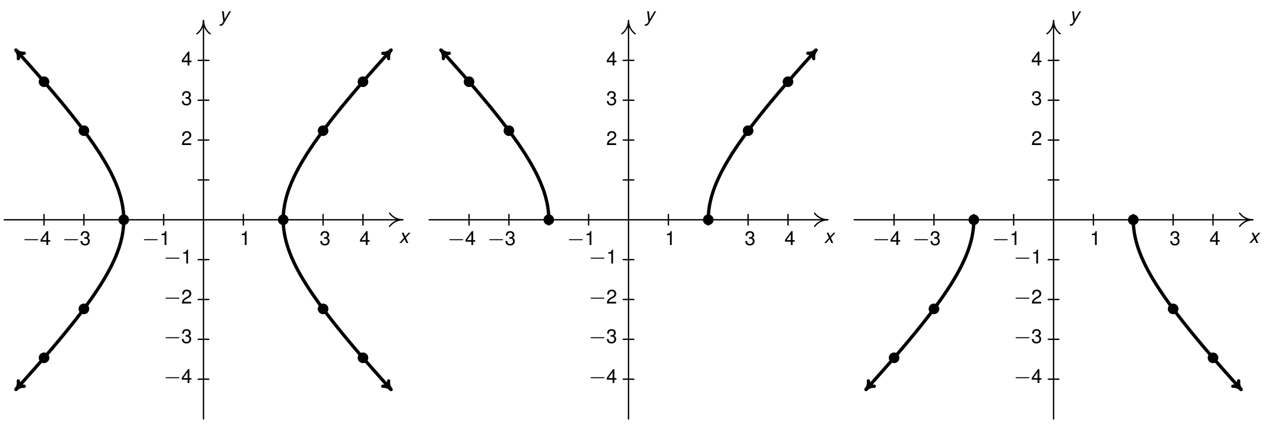
[6]Read this as we're avoiding fractions.

Looking at the graphs of the equations $x^2 - y^2 = 4$ and $(x-1)^2 + 4y^2 = 16$ in Example 5.5.3, it is evident neither of these equations represents y as a function of x nor x as a function of y. (Do you see why?)

With the concept of 'function' being touted in the opening remarks of Section 1.1 as being one of the 'universal tools' with which scientists and engineers solve a wide variety of problems, you may well wonder if we can't somehow apply what we know about functions to these sorts of relations. It turns out that while, taken all at once, these equations do not describe functions, taken in parts, they do.

For example, consider the equation $x^2 - y^2 = 4$. Solving for y, we obtained $y = \pm\sqrt{x^2-4}$. Defining $f_1(x) = \sqrt{x^2-4}$ and $f_2(x) = -\sqrt{x^2-4}$, we get a functional description for the upper and lower halves, or *branches* of the curve, respectively.[7]

If, for instance, we wanted to analyze this curve near $(3, -\sqrt{5})$, we could use the *function* f_2 and all the associated function tools[8] to do just that.



$x^2 - y^2 = 4$ $f_1(x) = \sqrt{x^2-4}$ $f_2(x) = -\sqrt{x^2-4}$

In this way we say the equation $x^2 - y^2 = 4$ *implicitly* describes y as a function of x meaning that given any point (x_0, y_0) on $x^2 - y^2 = 4$, we can find a function f defined (on an interval) containing x_0 so that $f(x_0) = y_0$ and whose graph lies on the curve $x^2 - y^2 = 4$.

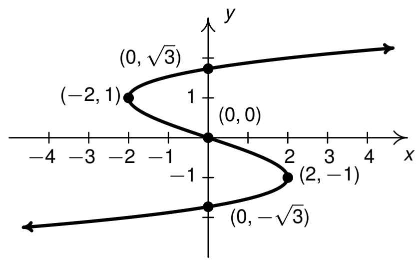
Note that in this case, we are fortunate to have two *explicit* formulas for functions that cover the entire curve, namely $f_1(x) = \sqrt{x^2-4}$ and $f_2(x) = -\sqrt{x^2-4}$. We explore this concept further in the next example.

Example 5.5.4. Consider the graph of the relation R below.

1. Explain why this curve does not represent y as a function of x.
2. Resolve the graph of R into two or more graphs of implicitly defined functions.
3. Explain why this curve represents x as a function of y and find a formula for $x = g(y)$.

[7]There are many more ways to break this relation into functional parts. We could, for instance, go piecewise and take portions of the graph which lie in Quadrants I and III as one function and leave the parts in Quadrants II and IV as the other; we could look at this as being comprised of *four* functions, and so on.

[8]including, when the time comes, Calculus



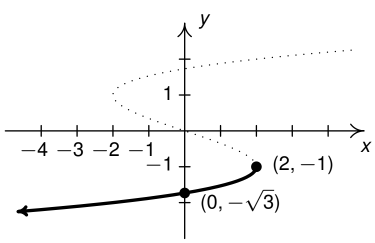
The graph of R

Solution.

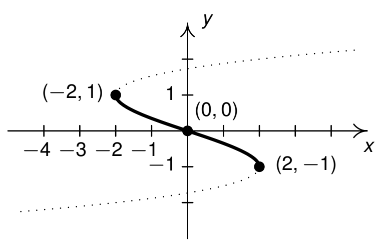
1. Using the Vertical Line Test, Theorem 1.1, we find several instances where vertical lines intersect the graph of R more than once. The y-axis, $x = 0$ is one such line. We have $x = 0$ matched with *three* different y-values: $-\sqrt{3}$, 0, and $\sqrt{3}$.

2. Since the maximum number of times a vertical line intersects the graph of R is three, it stands to reason we need to resolve the graph of R into at least three pieces.

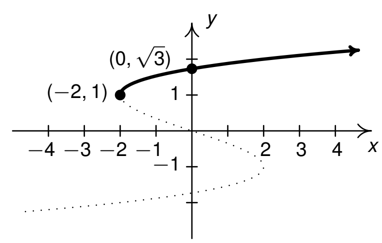
   One strategy is to begin at the far left and begin tracing the graph until it begins to 'double back' and repeat y-coordinates. Doing so we get three functions (represented by the bold solid lines) below.



$y = f_1(x)$



$y = f_2(x)$



$y = f_3(x)$

3. To verify that R represents x as a function of y, we check to see if any y-value has more than one x associated with it. One way to do this is to employ the the Horizontal Line Test (Exercise 57 in Section 1.1.) Since every horizontal line intersects the graph at most once, x is a function of y.

   Using Theorem 2.16 from Chapter 2, we get $x = (1)y(y - \sqrt{3})(y + \sqrt{3}) = y^3 - 3y$, a fact we can readily check using a graphing utility. □

Not all equations implicitly define y as a function of x. For a quick example, take $x = 117$ or any other vertical line. Even if an equation implicitly describes y as a function of x near one point, there's no guarantee we can find an explicit algebraic representation for that function.[9]

While the theory of implicit functions is well beyond the scope of this text, we will nevertheless see this concept come into play in Section 5.6. For our purposes, it suffices to know that just because a relation is not a function doesn't mean we cannot find a way to apply what we know about functions to analyze the relation locally through a functional lens.

[9]An example of this is $y^5 - y - x = 1$ near $(-1, 0)$.

5.5.1 Exercises

In Exercises 1 - 20, graph the given relation in the *xy*-plane.

1. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$

2. $\{(-2,0),\ (-1,1),\ (-1,-1),\ (0,2),\ (0,-2),\ (1,3),\ (1,-3)\}$

3. $\{(m, 2m) \mid m = 0, \pm 1, \pm 2\}$

4. $\left\{\left(\frac{6}{k}, k\right) \mid k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}$

5. $\left\{\left(n, 4-n^2\right) \mid n = 0, \pm 1, \pm 2\right\}$

6. $\left\{\left(\sqrt{j}, j\right) \mid j = 0, 1, 4, 9\right\}$

7. $\{(x, -2) \mid x > -4\}$

8. $\{(x, 3) \mid x \leq 4\}$

9. $\{(-1, y) \mid y > 1\}$

10. $\{(2, y) \mid y \leq 5\}$

11. $\{(-2, y) \mid -3 < y \leq 4\}$

12. $\{(3, y) \mid -4 \leq y < 3\}$

13. $\{(x, 2) \mid -2 \leq x < 3\}$

14. $\{(x, -3) \mid -4 < x \leq 4\}$

15. $\{(x, y) \mid x > -2\}$

16. $\{(x, y) \mid x \leq 3\}$

17. $\{(x, y) \mid y < 4\}$

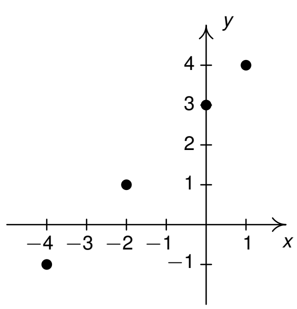
18. $\{(x, y) \mid x \leq 3,\ y < 2\}$

19. $\{(x, y) \mid x > 0,\ y < 4\}$

20. $\{(x, y) \mid -\sqrt{2} \leq x \leq \frac{2}{3},\ \pi < y \leq \frac{9}{2}\}$

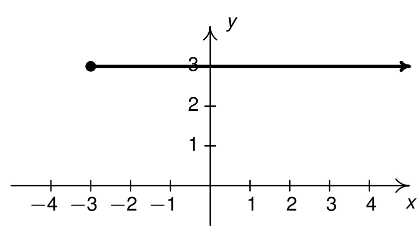
In Exercises 21 - 30, describe the given relation using either the roster or set-builder method.

21.



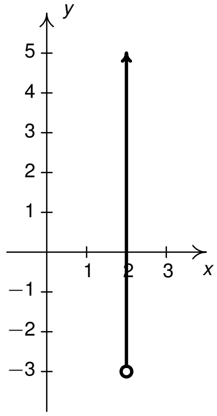
Relation *A*

22.



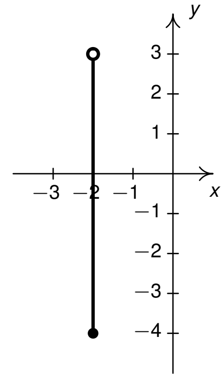
Relation *B*

23.



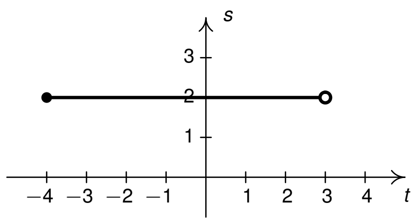
Relation C

24.



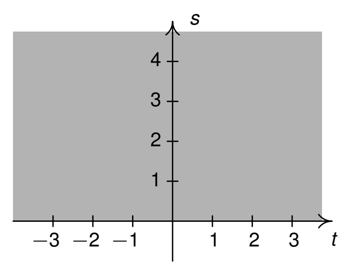
Relation D

25.



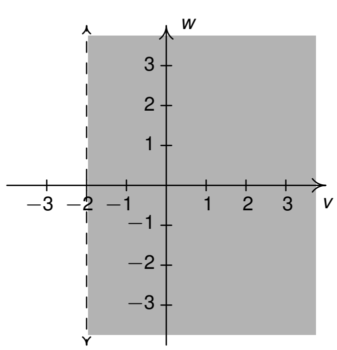
Relation E

26.



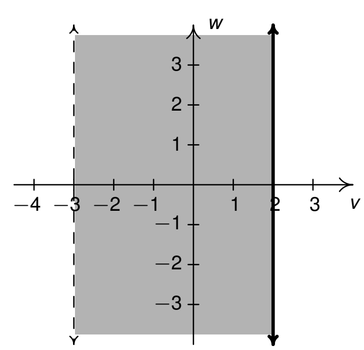
Relation F

27.



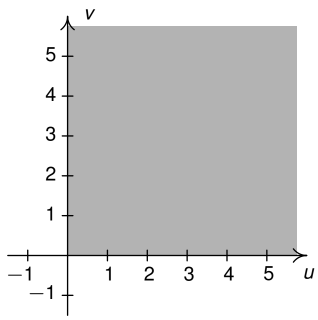
Relation G

28.



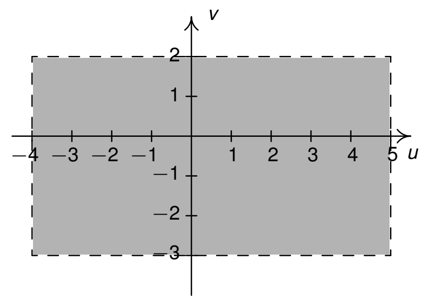
Relation H

29.



Relation I

30.



Relation J

Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. Discuss with your classmates how you might graph the relations given in Exercises 31 - 34. Note that in the notation below we are using the ellipsis, '. . . ,' to denote that the list does not end, but rather, continues to follow the established pattern indefinitely.

For the relations in Exercises 31 and 32, give two examples of points which belong to the relation and two points which do not belong to the relation.

31. $\{(x, y) \mid x$ is an odd integer, and y is an even integer.$\}$

32. $\{(x, 1) \mid x$ is an irrational number $\}$

33. $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), ...\}$

34. $\{... , (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), ...\}$

For each equation given in Exercises 35 - 38:

- Graph the equation in the xy-plane by creating a table of points.
- Find the axis intercepts, if they exist.
- Test the equation for symmetry. If the equation fails a symmetry test, find a point on the graph of the equation whose symmetric point is not on the graph of the equation.
- Determine if the equation describes y as a function of x. If not, describe the graph of the equation using two or more explicit functions of x. Check your answers using a graphing utility.

35. $(x + 2)^2 + y^2 = 16$

36. $x^2 - y^2 = 1$

37. $4y^2 - 9x^2 = 36$

38. $x^3y = -4$

For each equation given in Exercises 39 - 42:

- Graph the equation in the vw-plane by creating a table of points.
- Find the axis intercepts, if they exist.
- Test the equation for symmetry. If the equation fails a symmetry test, find a point on the graph of the equation whose symmetric point is not on the graph of the equation.
- Determine if the equation describes w as a function of v. If not, describe the graph of the equation using two or more explicit functions of v. Check your answers using a graphing utility.

39. $v + w^2 = 4$

40. $v^3 + w^3 = 8$

41. $v^2 w^3 = 8$

42.[10] $v^4 - 2v^2 w + w^2 = 16$

The procedures which we have outlined in the Examples of this section and used in Exercises 35 - 38 all rely on the fact that the equations were "well-behaved". Not everything in Mathematics is quite so tame, as the following equations will show you. Discuss with your classmates how you might approach graphing the equations given in Exercises 43 - 46. What difficulties arise when trying to apply the various tests and procedures given in this section? For more information, including pictures of the curves, each curve name is a link to its page at www.wikipedia.org. For a much longer list of fascinating curves, click here.

43. $x^3 + y^3 - 3xy = 0$ Folium of Descartes

44. $x^4 = x^2 + y^2$ Kampyle of Eudoxus

45. $y^2 = x^3 + 3x^2$ Tschirnhausen cubic

46. $(x^2 + y^2)^2 = x^3 + y^3$ Crooked egg

47. With the help of your classmates, find examples of equations whose graphs possess

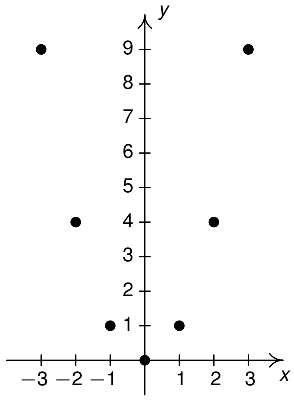
    - symmetry about the x-axis only
    - symmetry about the y-axis only
    - symmetry about the origin only
    - symmetry about the x-axis, y-axis, and origin

    Can you find an example of an equation whose graph possesses exactly *two* of the symmetries listed above? Why or why not?

[10] HINT: $v^4 - 2v^2 w + w^2 = \left(v^2 - w\right)^2 \ldots$

5.5.2 Answers

1.



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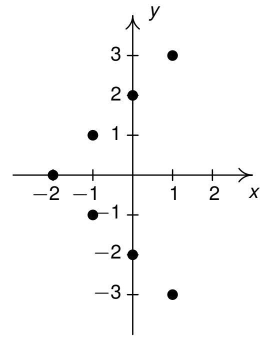

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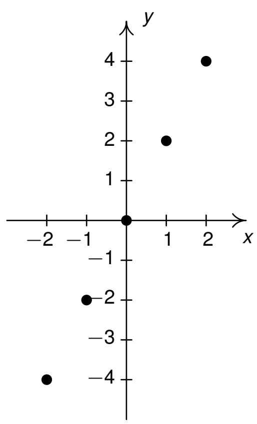

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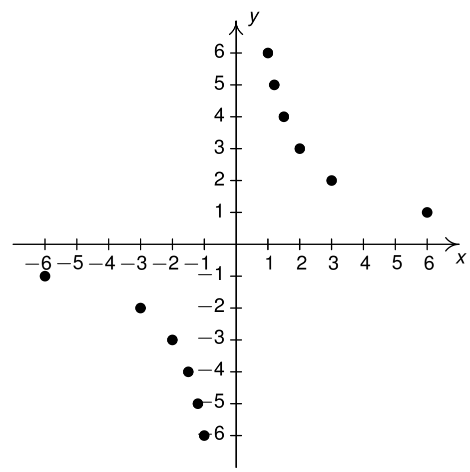

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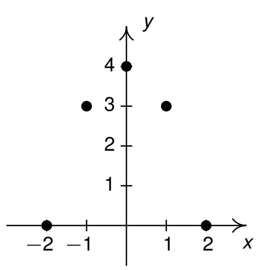

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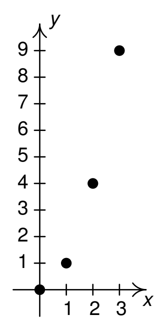

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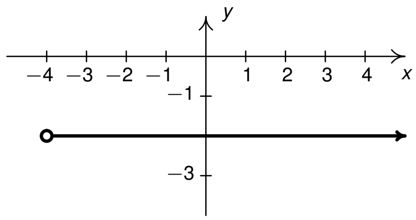
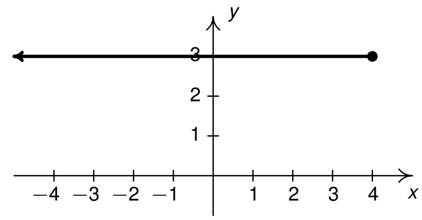
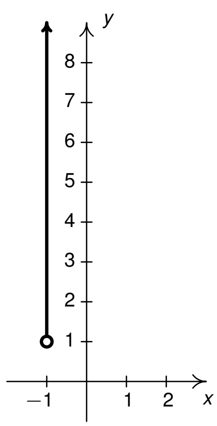

y
−4 −3 −2 −1 1 2 3 4 x
−1
−3

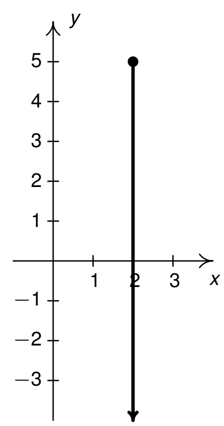
8.


y
3
2
1
−4 −3 −2 −1 1 2 3 4 x

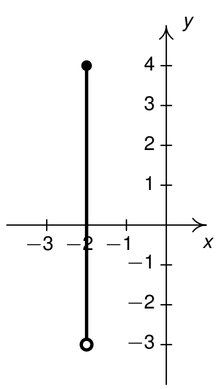
9.


y
8
7
6
5
4
3
2
1
−1 1 2 x

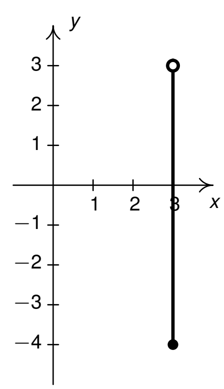
10.


y
5
4
3
2
1
1 2 3 x
−1
−2
−3

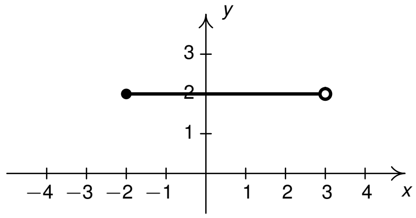
11.


y
4
3
2
1
−3 −2 −1 x
−1
−2
−3

12.


y
3
2
1
1 2 3 x
−1
−2
−3
−4

13.



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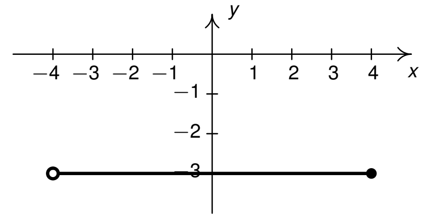

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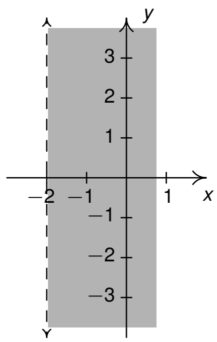

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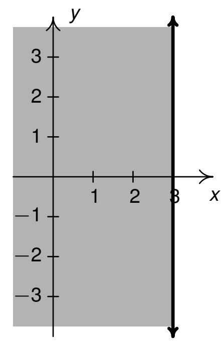

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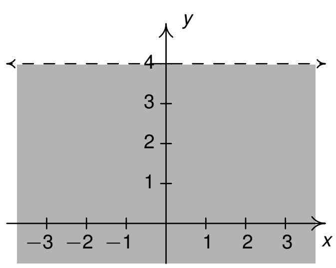

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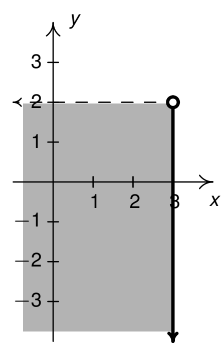

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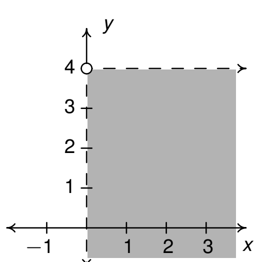
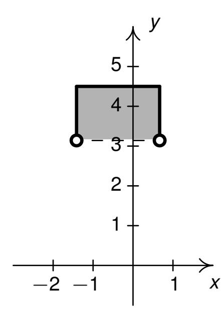


20.



21. $A = \{(-4,-1),(-2,1),(0,3),(1,4)\}$

22. $B = \{(x,3) \mid x \geq -3\}$

23. $C = \{\left(2,y\right) \mid y > -3\}$

24. $D = \{\left(-2,y\right) \mid -4 \leq y < 3\}$

25. $E = \{(t,2) \mid -4 < t \leq 3\}$

26. $F = \{\left(t,s\right) \mid s \geq 0\}$

27. $G = \{(v,w) \mid v > -2\}$

28. $H = \{(v,w) \mid -3 < v \leq 2\}$

29. $I = \{\left(u,v\right) \mid u \geq 0, v \geq 0\}$

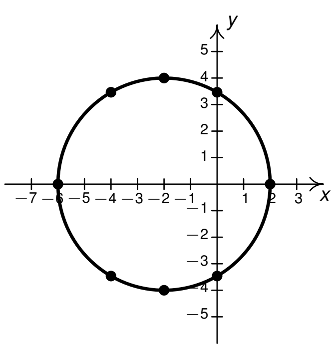
30. $J = \{(u,v) \mid -4 < u < 5,\ -3 < v < 2\}$

35. $(x+2)^2 + y^2 = 16$
Re-write as $y = \pm\sqrt{16-(x+2)^2}$.

x-intercepts: $(-6,0)$, $(2,0)$

y-intercepts: $\left(0, \pm 2\sqrt{3}\right)$

x	y	(x,y)
-6	0	$(-6,0)$
-4	$\pm 2\sqrt{3}$	$\left(-4, \pm 2\sqrt{3}\right)$
-2	± 4	$(-2, \pm 4)$
0	$\pm 2\sqrt{3}$	$\left(0, \pm 2\sqrt{3}\right)$
2	0	$(2,0)$



The graph is symmetric about the x-axis

The graph is not symmetric about the y-axis: $(-6,0)$ is on the graph but $(6,0)$ is not.

The graph is not symmetric about the origin: $(-6,0)$ is on the graph but $(6,0)$ is not.

The equation does not describe y as a function of x.

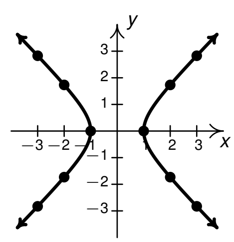
The graph of the equation is the graphs of $f_1(x) = \sqrt{16-(x+2)^2}$ together with $f_2(x) = -\sqrt{16-(x+2)^2}$.

36. $x^2 - y^2 = 1$
Re-write as: $y = \pm\sqrt{x^2-1}$.

x-intercepts: $(-1,0),(1,0)$

The graph has no y-intercepts

x	y	(x,y)
-3	$\pm\sqrt{8}$	$(-3, \pm\sqrt{8})$
-2	$\pm\sqrt{3}$	$(-2, \pm\sqrt{3})$
-1	0	$(-1,0)$
1	0	$(1,0)$
2	$\pm\sqrt{3}$	$(2, \pm\sqrt{3})$
3	$\pm\sqrt{8}$	$(3, \pm\sqrt{8})$



The graph is symmetric about the x-axis.

The graph is symmetric about the y-axis.

The graph is symmetric about the origin.

The equation does not describe y as a function of x.

The graph of the equation is the graphs of $f_1(x) = \sqrt{x^2-1}$ together with $f_2(x) = -\sqrt{x^2-1}$.

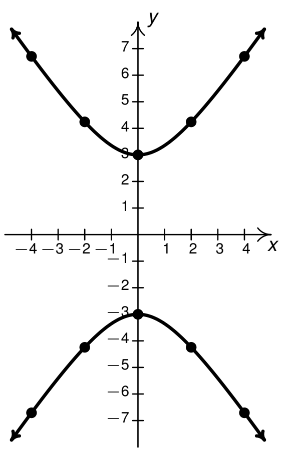
37. $4y^2 - 9x^2 = 36$

Re-write as: $y = \pm\dfrac{\sqrt{9x^2+36}}{2}$.

The graph has no x-intercepts

y-intercepts: $(0, \pm 3)$

x	y	(x, y)
-4	$\pm 3\sqrt{5}$	$\left(-4, \pm 3\sqrt{5}\right)$
-2	$\pm 3\sqrt{2}$	$\left(-2, \pm 3\sqrt{2}\right)$
0	± 3	$(0, \pm 3)$
2	$\pm 3\sqrt{2}$	$\left(2, \pm 3\sqrt{2}\right)$
4	$\pm 3\sqrt{5}$	$\left(4, \pm 3\sqrt{5}\right)$



The graph is symmetric about the x-axis.

The graph is symmetric about the y-axis.

The graph is symmetric about the origin.

The equation does not describe y as a function of x.

The graph of the equation is the graphs of $f_1(x) = \dfrac{\sqrt{9x^2+36}}{2}$ together with $f_2(x) = -\dfrac{\sqrt{9x^2+36}}{2}$.

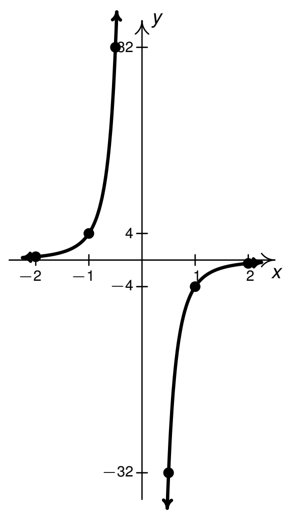
38. $x^3 y = -4$

Re-write as: $y = -\dfrac{4}{x^3} = -4x^{-3}$.

The graph has no x-intercepts

The graph has no y-intercepts

x	y	(x, y)
-2	$\frac{1}{2}$	$(-2, \frac{1}{2})$
-1	4	$(-1, 4)$
$-\frac{1}{2}$	32	$(-\frac{1}{2}, 32)$
$\frac{1}{2}$	-32	$(\frac{1}{2}, -32)$
1	-4	$(1, -4)$
2	$-\frac{1}{2}$	$(2, -\frac{1}{2})$



The graph is not symmetric about the x-axis: $(1, -4)$ is on the graph but $(1, 4)$ is not.

The graph is not symmetric about the y-axis: $(1, -4)$ is on the graph but $(-1, -4)$ is not.

The graph is symmetric about the origin.

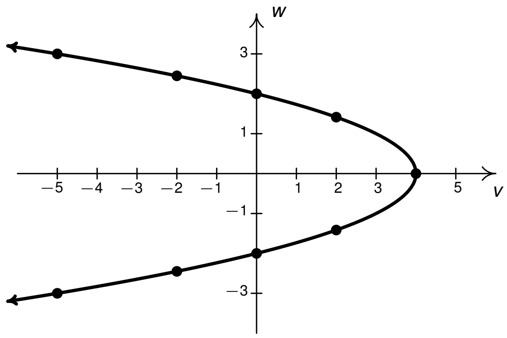
The equation does describe y as a function of x, namely $y = f(x) = -4x^{-3}$.

39. $v + w^2 = 4$
Re-write as $w = \pm\sqrt{4 - v}$.

v-intercept: $(4, 0)$

w-intercepts: $(0, \pm 2)$

v	w	(x, y)
-5	± 3	$(-5, \pm 3)$
-2	$\pm\sqrt{6}$	$\left(-2, \pm\sqrt{6}\right)$
0	± 2	$(0, \pm 2)$
2	$\pm\sqrt{2}$	$\left(1, \pm\sqrt{3}\right)$
4	0	$(4, 0)$



The graph is symmetric about the v-axis

The graph is not symmetric about the w-axis: $(4, 0)$ is on the graph but $(-4, 0)$ is not.

The graph is not symmetric about the origin: $(4, 0)$ is on the graph but $(-4, 0)$ is not.

The equation does not describe w as a function of v.

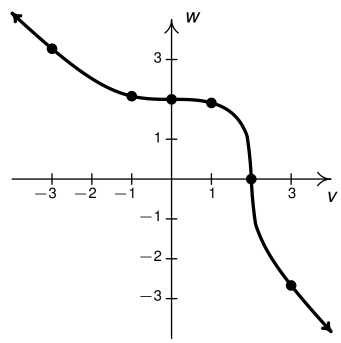
The graph of the equation is the graphs of $f_1(v) = \sqrt{4 - v}$ together with $f_2(v) = -\sqrt{4 - v}$.

40. $v^3 + w^3 = 8$
Re-write as: $w = \sqrt[3]{8 - v^3}$.

v-intercept: $(2, 0)$

w-intercept: $(0, 2)$

v	w	(v, w)
-3	$\sqrt[3]{35}$	$(-3, \sqrt[3]{35})$
-1	$\sqrt[3]{9}$	$(-1, \sqrt[3]{9})$
0	2	$(0, 2)$
1	$\sqrt[3]{7}$	$(1, \sqrt[3]{7})$
2	0	$(2, 0)$
3	$-\sqrt[3]{19}$	$(3, -\sqrt[3]{19})$



The graph is not symmetric about the v-axis: $(0, 2)$ is on the graph but $(0, -2)$ is not.

The graph is not symmetric about the w-axis: $(2, 0)$ is on the graph but $(-2, 0)$ is not.

The graph is not symmetric about the origin: $(0, 2)$ is on the graph but $(0, -2)$ is not.

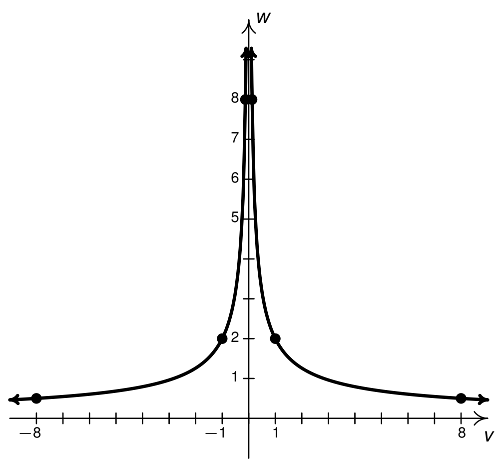
The equation does describe w as a function of v, namely $w = f(v) = \sqrt[3]{8 - v^3}$.

41. $v^2w^3 = 8$
Re-write as $w = \dfrac{2}{\sqrt[3]{v^2}} = 2v^{-\frac{2}{3}}$.

The graph has no v-intercepts.

The graph has no w-intercepts.

v	w	(x, y)
-8	$\frac{1}{2}$	$\left(-8, \frac{1}{2}\right)$
-1	2	$(-1, 2)$
$-\frac{1}{8}$	8	$\left(-\frac{1}{8}, 8\right)$
$\frac{1}{8}$	8	$\left(\frac{1}{8}, 8\right)$
1	2	$(1, 2)$
8	$\frac{1}{2}$	$\left(8, \frac{1}{2}\right)$



The graph is not symmetric about the v-axis: $(-1, 2)$ is on the graph but $(-1, -2)$ is not.

The graph is symmetric about the w-axis.

The graph is not symmetric about the origin: $(-1, 2)$ is on the graph but $(-1, -2)$ is not.

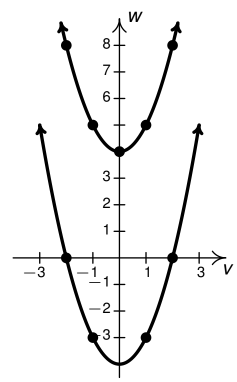
The equation does describe w as a function of v, namely $w = f(v) = 2v^{-\frac{2}{3}}$.

42. $v^4 - 2v^2w + w^2 = 16$
Re-write as: $\left(v^2 - w\right)^2 = 16$
Extracting square roots gives:
$w = v^2 + 4$ and $w = v^2 - 4$

v-intercepts: $(-2, 0), (2, 0)$.

w-intercepts: $(0, -4), (0, 4)$

v	w	(v, w)
-2	8	$(-2, 8)$
-2	0	$(-2, 0)$
-1	5	$(-1, 5)$
-1	-3	$(-1, -3)$
0	± 4	$(0, \pm 4)$
1	5	$(1, 5)$
1	-3	$(1, -3)$
2	8	$(2, 8)$
2	0	$(2, 0)$



The graph is not symmetric about the v-axis: $(1, 5)$ is on the graph but $(1, -5)$ is not.

The graph is symmetric about the w-axis.

The graph is not symmetric about the origin: $(1, 5)$ is on the graph but $(-1, -5)$ is not.

The equation does not describe w as a function of v.

The graph of the equation is the graphs of $f_1(v) = v^2 + 4$ together with $f_2(v) = v^2 - 4$.

5.6 Inverse Functions

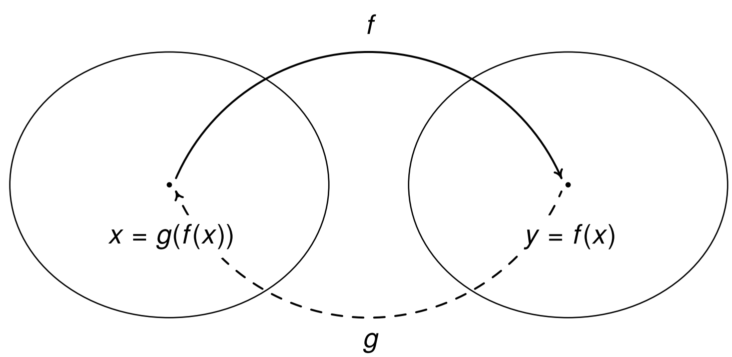
In Section 1.1, we defined functions as processes. In this section, we seek to reverse, or 'undo' those processes. As in real life, we will find that some processes (like putting on socks and shoes) are reversible while some (like baking a cake) are not.

Consider the function $f(x) = 3x+4$. Starting with a real number input x, we apply two steps in the following sequence: first we multiply the input by 3 and, second, we add 4 to the result.

To reverse this process, we seek a function g which will undo each of these steps and take the output from f, $3x+4$, and return the input x. If we think of the two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes and then we take off the socks. In much the same way, the function g should undo each step of f but in the opposite order. That is, the function g should first *subtract* 4 from the input x then *divide* the result by 3. This leads us to the formula $g(x) = \frac{x-4}{3}$.

Let's check to see if the function g does the job. If $x = 5$, then $f(5) = 3(5) + 4 = 15 + 4 = 19$. Taking the output 19 from f, we substitute it into g to get $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$, which is our original input to f. To check that g does the job for all x in the domain of f, we take the generic output from f, $f(x) = 3x + 4$, and substitute that into g. That is, we simplify $g(f(x)) = g(3x + 4) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$, which is our original input to f. If we carefully examine the arithmetic as we simplify $g(f(x))$, we actually see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3.

Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x) = \frac{x-4}{3}$, and substitute that into f, we get $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right) + 4 = (x - 4) + 4 = x$. Using the language of function composition developed in Section 5.3, the statements $g(f(x)) = x$ and $f(g(x)) = x$ can be written as $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$, respectively.[1] Abstractly, we can visualize the relationship between f and g in the diagram below.



The main idea to get from the diagram is that g takes the outputs from f and returns them to their respective inputs, and conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the central definition of the section.

[1] At the level of functions, $g \circ f = f \circ g = I$, where I is the identity function as defined as $I(x) = x$ for all real numbers, x.

Definition 5.4. Suppose f and g are two functions such that

1. $(g \circ f)(x) = x$ for all x in the domain of f

   and

2. $(f \circ g)(x) = x$ for all x in the domain of g

then f and g are **inverses** of each other and the functions f and g are said to be **invertible**.

If we abstract one step further, we can express the sentiment in Definition 5.4 by saying that f and g are inverses if and only if $g \circ f = I_1$ and $f \circ g = I_2$ where I_1 is the identity function restricted[2] to the domain of f and I_2 is the identity function restricted to the domain of g.

In other words, $I_1(x) = x$ for all x in the domain of f and $I_2(x) = x$ for all x in the domain of g. Using this description of inverses along with the properties of function composition listed in Theorem 5.4, we can show that function inverses are unique.[3]

Suppose g and h are both inverses of a function f. By Theorem 5.13, the domain of g is equal to the domain of h, since both are the range of f. This means the identity function I_2 applies both to the domain of h and the domain of g. Thus $h = h \circ I_2 = h \circ (f \circ g) = (h \circ f) \circ g = I_1 \circ g = g$, as required.

We summarize the important properties of invertible functions in the following theorem.[4] Apart from introducing notation, each of the results below are immediate consequences of the idea that inverse functions map the outputs from a function f back to their corresponding inputs.

Theorem 5.13. Properties of Inverse Functions: Suppose f is an invertible function.

- There is exactly one inverse function for f, denoted f^{-1} (read 'f-inverse')
- The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1}
- $f(a) = c$ if and only if $a = f^{-1}(c)$

  NOTE: In particular, for all y in the range of f, the solution to $f(x) = y$ is $x = f^{-1}(y)$.
- (a, c) is on the graph of f if and only if (c, a) is on the graph of f^{-1}

  NOTE: This means graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across $y = x$.[a]
- f^{-1} is an invertible function and $(f^{-1})^{-1} = f$.

[a]See Example A.3.5 in Section A.3 and Example A.5.5 in Section A.5.

[2]The identity function I, first introduced in Exercise 35 in Section 1.2 and mentioned in Theorem 5.4, has a domain of all real numbers. Since the domains of f and g may not be all real numbers, we need the restrictions listed here.

[3]In other words, invertible functions have exactly one inverse.

[4]In the interests of full disclosure, the authors would like to admit that much of the discussion in the previous paragraphs could have easily been avoided had we appealed to the description of a function as a set of ordered pairs. We make no apology for our discussion from a function composition standpoint, however, since it exposes the reader to more abstract ways of thinking of functions and inverses. We will revisit this concept again in Chapter 9.

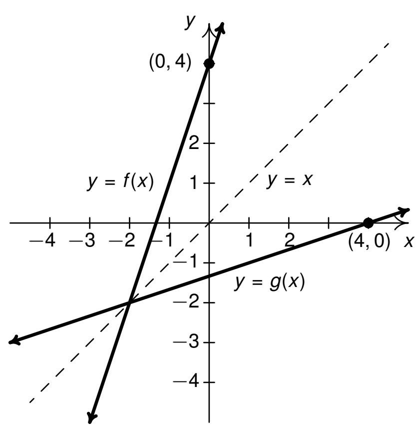
The notation f^{-1} is an unfortunate choice since you've been programmed since Elementary Algebra to think of this as $\frac{1}{f}$. This is most definitely *not* the case since, for instance, $f(x) = 3x + 4$ has as its inverse $f^{-1}(x) = \frac{x-4}{3}$, which is certainly different than $\frac{1}{f(x)} = \frac{1}{3x+4}$.

Why does this confusing notation persist? As we mentioned in Section 5.3, the identity function I is to function composition what the real number 1 is to real number multiplication. The choice of notation f^{-1} alludes to the property that $f^{-1} \circ f = I_1$ and $f \circ f^{-1} = I_2$, in much the same way as $3^{-1} \cdot 3 = 1$ and $3 \cdot 3^{-1} = 1$.

Before we embark on an example, we demonstrate the pertinent parts of Theorem 5.13 to the inverse pair $f(x) = 3x + 4$ and $g(x) = f^{-1}(x) = \frac{x-4}{3}$. Suppose we wanted to solve $3x + 4 = 7$. Going through the usual machinations, we obtain $x = 1$.

If we view this equation as $f(x) = 7$, however, then we are looking for the input x corresponding to the output $f(x) = 7$. This is exactly the question f^{-1} was built to answer. In other words, the solution to $f(x) = 7$ is $x = f^{-1}(7) = 1$. In other words, the formula $f^{-1}(x)$ encodes all of the algebra required to 'undo' what the formula $f(x)$ does to x. More generally, any time you have ever solved an equation, you have really been working through an inverse problem.

We also note the graphs of $f(x) = 3x + 4$ and $g(x) = f^{-1}(x) = \frac{x-4}{3}$ are easily seen to be reflections across the line $y = x$ as seen below. In particular, note that the y-intercept $(0, 4)$ on the graph of $y = f(x)$ corresponds to the x-intercept on the graph of $y = f^{-1}(x)$. Indeed, the point $(0, 4)$ on the graph of $y = f(x)$ can be interpreted as $(0, 4) = (0, f(0)) = (f^{-1}(4), 4)$ just as the point $(4, 0)$ on the graph of $y = f^{-1}(x)$ can be interpreted as $(4, 0) = (4, f^{-1}(4)) = (f(0), 0)$.



Graphs of inverse functions $y = f(x) = 3x + 4$ and $y = f^{-1}(x) = \dfrac{x-4}{3}$.

Example 5.6.1. For each pair of functions f and g below:

1. Verify each pair of functions f and g are inverses: (a) algebraically and (b) graphically.

2. Use the fact f and g are inverses to solve $f(x) = 5$ and $g(x) = -3$

- $f(x) = \sqrt[3]{x-1} + 2$ and $g(x) = (x-2)^3 + 1$
- $f(t) = \dfrac{2t}{t+1}$ and $g(t) = \dfrac{t}{2-t}$

Solution.

Solution for $f(x) = \sqrt[3]{x-1} + 2$ *and* $g(x) = (x-2)^3 + 1$*.*

1. (a) To verify $f(x) = \sqrt[3]{x-1} + 2$ and $g(x) = (x-2)^3 + 1$ are inverses, we appeal to Definition 5.4 and show $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$ for all real numbers, x.

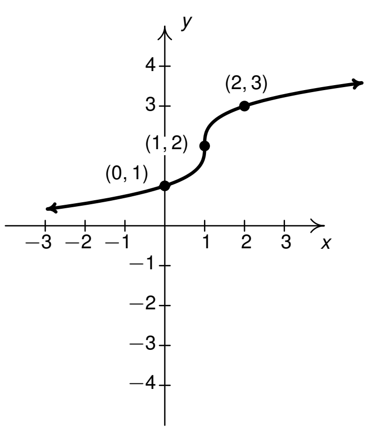
$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(\sqrt[3]{x-1} + 2) \\ &= [(\sqrt[3]{x-1} + 2) - 2]^3 + 1 \\ &= (\sqrt[3]{x-1})^3 + 1 \\ &= x - 1 + 1 \\ &= x \checkmark \end{aligned} \qquad \begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f((x-2)^3 + 1) \\ &= \sqrt[3]{[(x-2)^3 + 1] - 1} + 2 \\ &= \sqrt[3]{(x-2)^3} + 2 \\ &= x - 4 + 4 \\ &= x \checkmark \end{aligned}$$

Since the root here, 3, is odd, Theorem 4.2 gives $(\sqrt[3]{x-1})^3 = x - 1$ and $\sqrt[3]{(x-2)^3} = x - 2$.

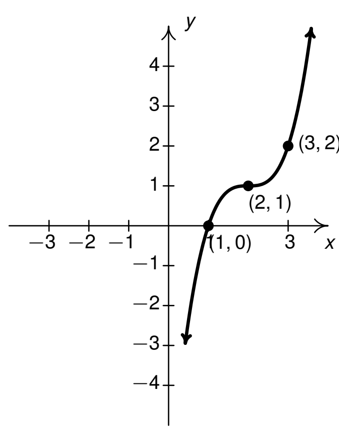
(b) To show f and g are inverses graphically, we graph $y = f(x)$ and $y = g(x)$ on the same set of axes and check to see if they are reflections about the line $y = x$.

The graph of $y = f(x) = \sqrt[3]{x-1} + 2$ appears below on the left courtesy of Theorem 4.1 in Section 4.1. The graph of $y = g(x) = (x-2)^3 + 1$ appears below in the middle thanks to Theorem 2.1 in Section 2.1.

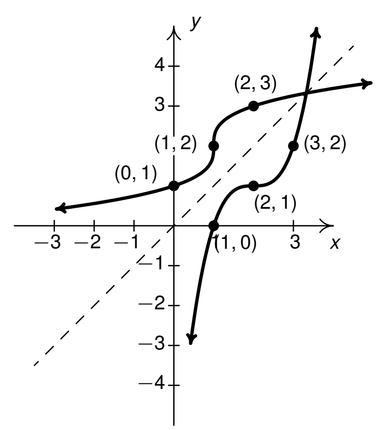
We can immediately see three pairs of corresponding points: $(0,1)$ and $(1,0)$, $(1,2)$ and $(2,1)$, $(2,3)$ and $(3,2)$. When graphed on the same pair of axes, the two graphs certainly appear to be symmetric about the line $y = x$, as required.



$y = f(x)$



$y = g(x)$



$y = f(x)$, $y = g(x)$, $y = x$

2. Since f and g are inverses, the solution to $f(x) = 5$ is $x = f^{-1}(5) = g(5) = (5-2)^3 + 1 = 28$. To check, we find $f(28) = \sqrt[3]{28-1} + 2 = \sqrt[3]{27} + 2 = 3 + 2 = 5$, as required.

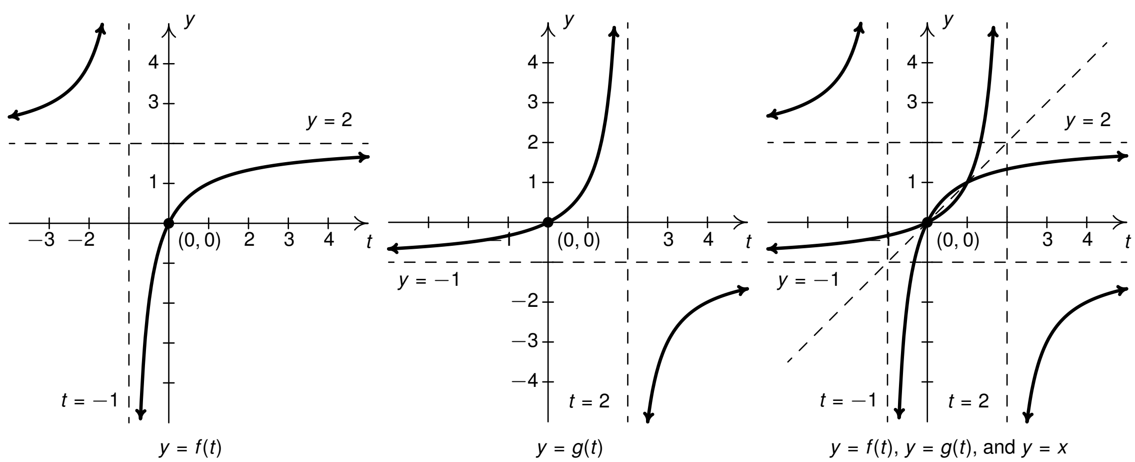
   Likewise, the solution to $g(x) = -3$ is $x = g^{-1}(-3) = f(-3) = \sqrt[3]{(-3)-1} + 2 = 2 - \sqrt[3]{4}$. Once again, to check, we find $g(2 - \sqrt[3]{4}) = (2 - \sqrt[3]{4} - 2)^3 + 1 = (-\sqrt[3]{4})^3 + 1 = -4 + 1 = -3$.

Solution for $f(t) = \dfrac{2t}{t+1}$ *and* $g(t) = \dfrac{t}{2-t}$.

1. (a) Note the domain of f excludes $t = -1$ and the domain of g excludes $t = 2$. Hence, when simplifying $(g \circ f)(t)$ and $(f \circ g)(t)$, we tacitly assume $t \neq -1$ and $t \neq 2$, respectively.

$$\begin{aligned}
(g \circ f)(t) &= g(f(t)) \\
&= g\left(\frac{2t}{t+1}\right) \\
&= \frac{\frac{2t}{t+1}}{2 - \frac{2t}{t+1}} \\
&= \frac{\frac{2t}{t+1}}{2 - \frac{2t}{t+1}} \cdot \frac{(t+1)}{(t+1)} \\
&= \frac{2t}{2(t+1) - 2t} \\
&= \frac{2t}{2t + 2 - 2t} \\
&= \frac{2t}{2} \\
&= t \checkmark
\end{aligned}
\qquad
\begin{aligned}
(f \circ g)(t) &= f(g(t)) \\
&= f\left(\frac{t}{2-t}\right) \\
&= \frac{2\left(\frac{t}{2-t}\right)}{\left(\frac{t}{2-t}\right) + 1} \\
&= \frac{2\left(\frac{t}{2-t}\right)}{\left(\frac{t}{2-t}\right) + 1} \cdot \frac{(2-t)}{(2-t)} \\
&= \frac{2t}{t + (1)(2-t)} \\
&= \frac{2t}{t + 2 - t)} \\
&= \frac{2t}{2} \\
&= t \checkmark
\end{aligned}$$

(b) We graph $y = f(t)$ and $y = g(t)$ using the techniques discussed in Sections 3.1 and 3.2.



We find the graph of f has a vertical asymptote $t = -1$ and a horizontal asymptote $y = 2$. Corresponding to the *vertical* asymptote $t = -1$ on the graph of f, we find the graph of g has a *horizontal* asymptote $y = -1$.

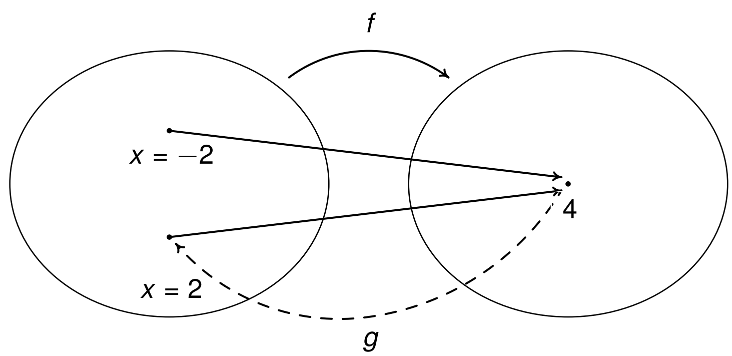
Likewise, the *horizontal* asymptote $y = 2$ on the graph of f corresponds to the *vertical* asymptote $t = 2$ on the graph of g. Both graphs share the intercept $(0, 0)$. When graphed together on the same set of axes, the graphs of f and g do appear to be symmetric about the line $y = t$.

2. Don't let the fact that f and g in this case were defined using the independent variable, 't' instead of 'x' deter you in your efforts to solve $f(x) = 5$. Remember that, ultimately, the function f here is the *process* represented by the formula $f(t)$, and is the same process (with the same inverse!) regardless of the letter used as the independent variable. Hence, the solution to $f(x) = 5$ is $x = f^{-1}(1) = g(5)$. We get $g(5) = \frac{5}{2-5} = -\frac{5}{3}$.

   To check, we find $f\left(-\frac{5}{3}\right) = \left(-\frac{10}{3}\right) / \left(-\frac{2}{3}\right) = 5$. Similarly, we solve $g(x) = -3$ by finding $x = g^{-1}(-3) = f(-3) = \frac{-6}{-2} = 3$. Sure enough, we find $g(3) = \frac{3}{2-3} = -3$. □

We now investigate under what circumstances a function is invertible. As a way to motivate the discussion, we consider $f(x) = x^2$. A likely candidate for the inverse is the function $g(x) = \sqrt{x}$. However, $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$, which is not equal to x unless $x \geq 0$.

For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$. That is, g failed to return the input -2 from its output 4. Instead, g matches the output 4 to a *different* input, namely 2, which satisfies $f(2) = 4$. Schematically:



We see from the diagram that since both $f(-2)$ and $f(2)$ are 4, it is impossible to construct a *function* which takes 4 back to *both* $x = 2$ and $x = -2$ since, by definition, a function can match 4 with only *one* number.

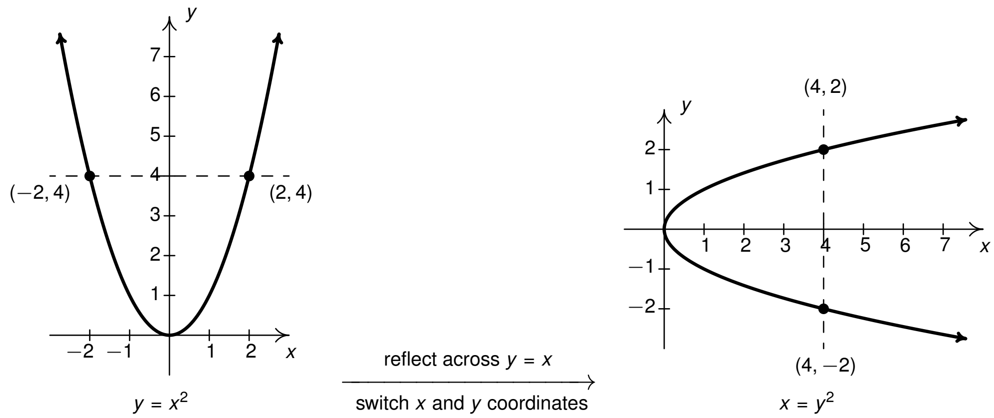
In general, in order for a function to be invertible, each output can come from only *one* input. Since, by definition, a function matches up each input to only *one* output, invertible functions have the property that they match one input to one output and vice-versa. We formalize this concept below.

Definition 5.5. A function f is said to be **one-to-one** if whenever $f(a) = f(b)$, then $a = b$.

Note that an equivalent way to state Definition 5.5 is that a function is one-to-one if *different* inputs go to *different* outputs. That is, if $a \neq b$, then $f(a) \neq f(b)$.

Before we solidify the connection between invertible functions and one-to-one functions, we take a moment to see what goes wrong graphically when trying to find the inverse of $f(x) = x^2$.

Per Theorem 5.13, the graph of $y = f^{-1}(x)$, if it exists, is obtained from the graph of $y = x^2$ by reflecting $y = x^2$ about the line $y = x$. Procedurally, this is accomplished by interchanging the x and y coordinates of each point on the graph of $y = x^2$. Algebraically, we are swapping the variables 'x' and 'y' which results in the equation $x = y^2$ whose graph is below on the right.



We see immediately the graph of $x = y^2$ fails the Vertical Line Test, Theorem 1.1. In particular, the vertical line $x = 4$ intersects the graph at two points, $(4,-2)$ and $(4,2)$ meaning the relation described by $x = y^2$ matches the x-value 4 with two different y-values, -2 and 2.

Note that the *vertical* line $x = 4$ and the points $(4,\pm 2)$ on the graph of $x = y^2$ correspond to the *horizontal* line $y = 4$ and the points $(\pm 2, 4)$ on the graph of $y = x^2$ which brings us right back to the concept of one-to-one. The fact that both $(-2,4)$ and $(2,4)$ are on the graph of f means $f(-2) = f(2) = 4$. Hence, f takes different inputs, -2 and 2, to the same output, 4, so f is not one-to-one.

Recall the Horizontal Line Test from Exercise 57 in Section 1.1. Applying that result to the graph of f we say the graph of f 'fails' the Horizontal Line Test since the horizontal line $y = 4$ intersects the graph of $y = x^2$ more than once. This means that the equation $y = x^2$ does not represent x is not a function of y.

Said differently, the Horizontal Line Test detects when there is at least one y-value (4) which is matched to more than one x-value (± 2). In other words, the Horizontal Line Test can be used to detect whether or not a function is one-to-one.

So, to review, $f(x) = x^2$ is not invertible, not one-to-one, and its graph fails the Horizontal Line Test. It turns out that these three attributes: being invertible, one-to-one, and having a graph that passes the Horizontal Line Test are mathematically equivalent. That is to say if one if these things is true about a function, then they all are; it also means that, as in this case, if one of these things *isn't* true about a function, then *none* of them are. We summarize this result in the following theorem.

Theorem 5.14. Equivalent Conditions for Invertibility:
For a function f, either all of the following statements are true or none of them are:

- f is invertible.
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.[a]

[a] i.e., no horizontal line intersects the graph more than once.

To prove Theorem 5.14, we first suppose f is invertible. Then there is a function g so that $g(f(x)) = x$ for all x in the domain of f. If $f(a) = f(b)$, then $g(f(a)) = g(f(b))$. Since $g(f(x)) = x$, the equation $g(f(a)) = g(f(b))$ reduces to $a = b$. We've shown that if $f(a) = f(b)$, then $a = b$, proving f is one-to-one.

Next, assume f is one-to-one. Suppose a horizontal line $y = c$ intersects the graph of $y = f(x)$ at the points (a, c) and (b, c). This means $f(a) = c$ and $f(b) = c$ so $f(a) = f(b)$. Since f is one-to-one, this means $a = b$ so the points (a, c) and (b, c) are actually one in the same. This establishes that each horizontal line can intersect the graph of f at most once, so the graph of f passes the Horizontal Line Test.

Last, but not least, suppose the graph of f passes the Horizontal Line Test. Let c be a real number in the range of f. Then the horizontal line $y = c$ intersects the graph of $y = f(x)$ just *once*, say at the point $(a, c) = (a, f(a))$. Define the mapping g so that $g(c) = g(f(a)) = a$. The mapping g is a *function* since each horizontal line $y = c$ where c is in the range of f intersects the graph of f only *once*. By construction, we have the domain of g is the range of f and that for all x in the domain of f,$g(f(x)) = x$. We leave it to the reader to show that for all x in the domain of g, $f(g(x)) = x$, too.

Hence, we've shown: first, if f invertible, then f is one-to-one; second, if f is one-to-one, then the graph of f passes the Horizontal Line Test; and third, if f passes the Horizontal Line Test, then f is invertible. Hence if f is satisfies any one of these three conditions, we can show f must satisfy the other two.[5]

We put this result to work in the next example.

Example 5.6.2. Determine if the following functions are one-to-one: (a) analytically using Definition 5.5 and (b) graphically using the Horizontal Line Test. For the functions that are one-to-one, graph the inverse.

1. $f(x) = x^2 - 2x + 4$
2. $g(t) = \dfrac{2t}{1-t}$
3. $F = \{(-1, 1), (0, 2), (1, -3), (2, 1)\}$
4. $G = \{(t^3 + 1, 2t) \mid t \text{ is a real number.}\}$

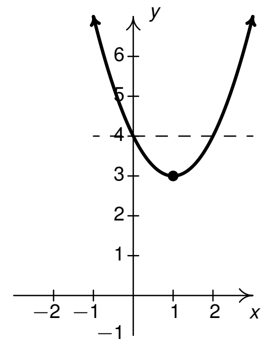
Solution.

1. (a) To determine whether or not f is one-to-one analytically, we assume $f(a) = f(b)$ and work to see if we can deduce $a = b$. As we work our way through the problem below on the left, we encounter a quadratic equation. We rewrite the equation so it equals 0 and factor by grouping. We get $a = b$ as one possibility, but we also get the possibility that $a = 2 - b$. This suggests that f may not be one-to-one. Taking $b = 0$, we get $a = 0$ or $a = 2$. Since $f(0) = 4$ and $f(2) = 4$, we have two different inputs with the same output, proving f is neither one-to-one nor invertible.

   (b) We note that f is a quadratic function and we graph $y = f(x)$ using the techniques presented in Section 1.4 below on the right. We see the graph fails the Horizontal Line Test quite often - in particular, crossing the line $y = 4$ at the points $(0, 4)$ and $(2, 4)$.

[5] For example, if we know f is one-to-one, we showed the graph of f passes the HLT which, in turn, guarantees f is invertible.

$$\begin{aligned}
f(a) &= f(b) \\
a^2 - 2a + 4 &= b^2 - 2b + 4 \\
a^2 - 2a &= b^2 - 2b \\
a^2 - b^2 - 2a + 2b &= 0 \\
(a+b)(a-b) - 2(a-b) &= 0 \\
(a-b)((a+b)-2) &= 0 \\
a - b = 0 \quad &\text{or} \quad a + b - 2 = 0 \\
a = b \quad &\text{or} \quad a = 2 - b
\end{aligned}$$

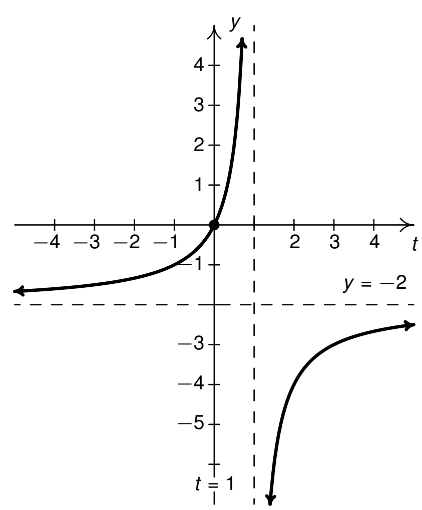


$y = f(x)$

2. (a) We begin with the assumption that $g(a) = g(b)$ for a, b in the domain of g (That is, we assume $a \neq 1$ and $b \neq 1$.) Through our work below on the left, we deduce $a = b$, proving g is one-to-one.

   (b) We graph $y = g(t)$ below on the right using the procedure outlined in Section 3.2. We find the sole intercept is $(0, 0)$ with asymptotes $t = 1$ and $y = -2$. Based on our graph, the graph of g appears to pass the Horizontal Line Test, verifying g is one-to-one.

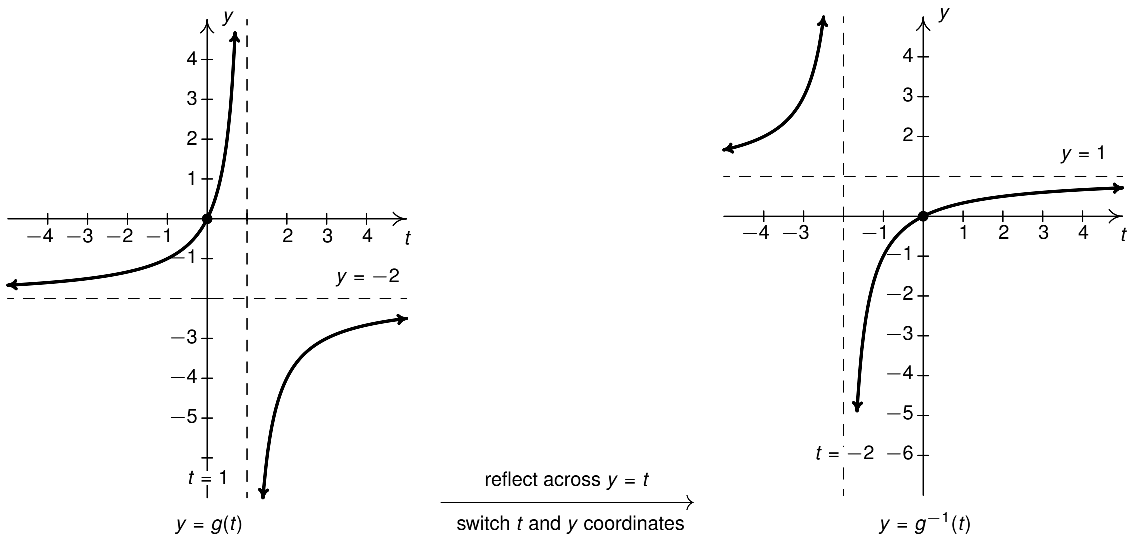
$$\begin{aligned}
g(a) &= g(b) \\
\frac{2a}{1-a} &= \frac{2b}{1-b} \\
2a(1-b) &= 2b(1-a) \\
2a - 2ab &= 2b - 2ba \\
2a &= 2b \\
a &= b \checkmark
\end{aligned}$$



$y = g(t)$

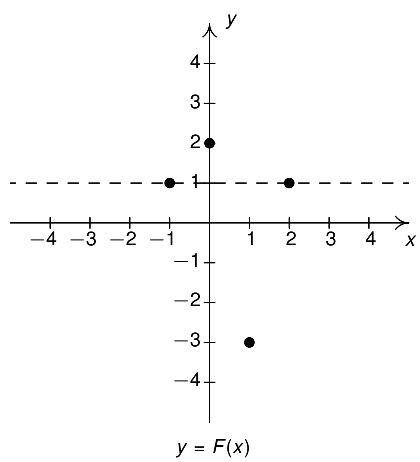
Since g is one-to-one, g is invertible. Even though we do not have a *formula* for $g^{-1}(t)$, we can nevertheless sketch the graph of $y = g^{-1}(t)$ by reflecting the graph of $y = g(t)$ across $y = t$.

Corresponding to the *vertical* asymptote $t = 1$ on the graph of g, the graph of $y = g^{-1}(t)$ will have a *horizontal* asymptote $y = 1$. Similarly, the *horizontal* asymptote $y = -2$ on the graph of g corresponds to a *vertical* asymptote $t = -2$ on the graph of g^{-1}. The point $(0, 0)$ remains unchanged when we switch the t and y coordinates, so it is on both the graph of g and g^{-1}.



3. (a) The function F is given to us as a set of ordered pairs. Recall each ordered pair is of the form $(a, F(a))$. Since $(-1, 1)$ and $(2, 1)$ are both elements of F, this means $F(-1) = 1$ and $F(2) = 1$. Hence, we have two distinct inputs, -1 an 2 with the same output, 1, so F is not one-to-one and, hence, not invertible.

   (b) To graph F, we plot the points in F below on the left. We see the horizontal line $y = 1$ crosses the graph more than once. Hence, the graph of F fails the Horizontal Line Test.

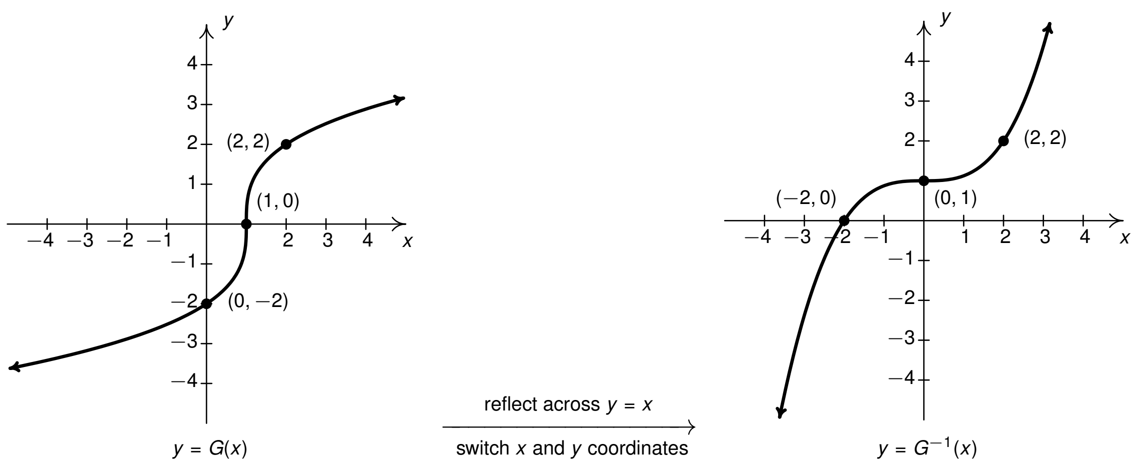


4. Like the function F above, the function G is described as a set of ordered pairs. Before we set about determining whether or not G is one-to-one, we take a moment to show G is, in fact, a function. That is, we must show that each real number input to G is matched to only one output.

   We are given $G = \{(t^3 + 1, 2t) \mid t \text{ is a real number.}\}$. and we know that when represented in this way, each ordered pair is of the form (input, output). Hence, the inputs to G are of the form $t^3 + 1$ and the outputs from G are of the form $2t$. To establish G is a function, we must show that each input produces only one output. If it should happen that $a^3 + 1 = b^3 + 1$, then we must show $2a = 2b$. The equation $a^3 + 1 = b^3 + 1$ gives $a^3 = b^3$, or $a = b$. From this it follows that $2a = 2b$ so G is a function.

(a) To show G is one-to-one, we must show that if two outputs from G are the same, the corresponding inputs must also be the same. That is, we must show that if $2a = 2b$, then $a^3 + 1 = b^3 + 1$. We see almost immediately that if $2a = 2b$ then $a = b$ so $a^3 + 1 = b^3 + 1$ as required. This shows G is one-to-one and, hence, invertible.

(b) We graph G below on the left by plotting points in the default xy-plane by choosing different values for t. For instance, $t = 0$ corresponds to the point $(0^3+1, 2(0)) = (1, 0)$, $t = 1$ corresponds to the point $(1^3 + 1, 2(1)) = (2, 2)$, $t = -1$ corresponds to the point $((-1)^3 + 1, 2(-1)) = (0, -2)$, etc.[6] Our graph appears to pass the Horizontal Line Test, confirming G is one-to-one. We obtain the graph of G^{-1} below on the right by reflecting the graph of G about the line $y = x$.



□

In Example 5.6.2, we showed the functions G and g are invertible and graphed their inverses. While graphs are perfectly fine representations of functions, we have seen where they aren't the most accurate. Ideally, we would like to represent G^{-1} and g^{-1} in the same manner in which G and g are presented to us. The key to doing this is to recall that inverse functions take outputs back to their associated inputs.

Consider $G = \{(t^3 + 1, 2t) \mid t \text{ is a real number.}\}$. As mentioned in Example 5.6.2, the ordered pairs which comprise G are in the form (input, output). Hence to find a compatible description for G^{-1}, we simply interchange the expressions in each of the coordinates to obtain $G^{-1} = \{(2t, t^3 + 1) \mid t \text{ is a real number.}\}$.

Since the function g was defined in terms of a formula we would like to find a formula representation for g^{-1}. We apply the same logic as above. Here, the input, represented by the independent variable t, and the output, represented by the dependent variable y, are related by the equation $y = g(t)$. Hence, to

[6] Foreshadowing Section 14.5, we could let $x = t^3 + 1$ so that $t = \sqrt[3]{x - 1}$. Hence, $y = 2t = 2\sqrt[3]{x-1}$.

exchange inputs and outputs, we interchange the 't' and 'y' variables. Doing so, we obtain the equation $t = g(y)$ which is an *implicit* description for g^{-1}. Solving for y gives an explicit formula for g^{-1}, namely $y = g^{-1}(t)$. We demonstrate this technique below.

$$\begin{array}{rcll}
y & = & g(t) & \\
y & = & \dfrac{2t}{1-t} & \\
t & = & \dfrac{2y}{1-y} & \text{interchange variables: } t \text{ and } y \\
t(1-y) & = & 2y & \\
t - ty & = & 2y & \\
t & = & ty + 2y & \\
t & = & y(t+2) & \text{factor} \\
y & = & \dfrac{t}{t+2} &
\end{array}$$

We claim $g^{-1}(t) = \frac{t}{t+2}$, and leave the algebraic verification of this to the reader.

We generalize this approach below. As always, we resort to the default 'x' and 'y' labels for the independent and dependent variables, respectively.

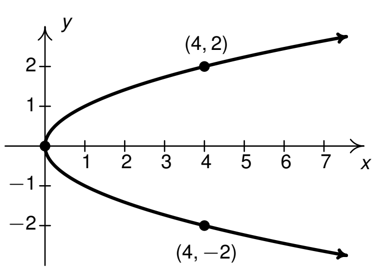
Steps for finding a formula for the Inverse of a one-to-one function

1. Write $y = f(x)$
2. Interchange x and y
3. Solve $x = f(y)$ for y to obtain $y = f^{-1}(x)$

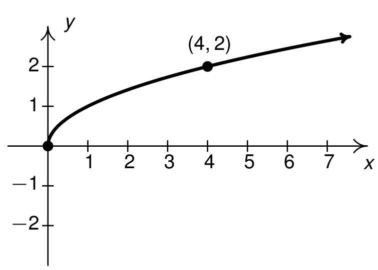
We now return to $f(x) = x^2$. We know that f is not one-to-one, and thus, is not invertible, but our goal here is to see what way to see what goes wrong algebraically.

If we attempt to follow the algorithm above to find a formula for $f^{-1}(x)$, we start with the equation $y = x^2$ and interchange the variables 'x' and 'y' to produce the equation $x = y^2$. Solving for y gives $y = \pm\sqrt{x}$. It's this '$\pm$' which is causing the problem for us since this produces *two* y-values for any $x > 0$.

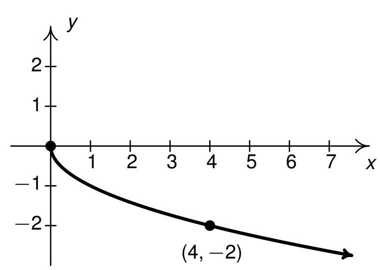
Using the language of Section 5.5, the equation $x = y^2$ implicitly defines *two* functions, $g_1(x) = \sqrt{x}$ and $g_2(x) = -\sqrt{x}$, each of which represents the top and bottom halves, respectively, of the graph of $x = y^2$.



$x = y^2$

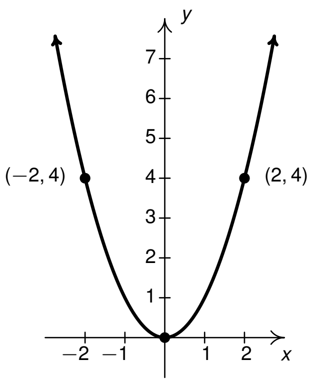


$y = g_1(x) = \sqrt{x}$

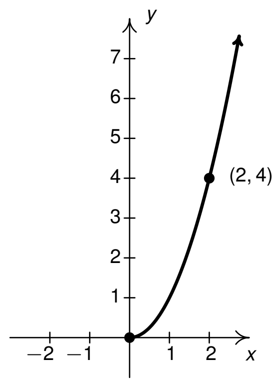


$y = g_2(x) = -\sqrt{x}$

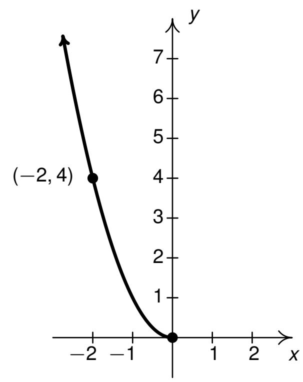
Hence, in some sense, we have two *partial* inverses for $f(x) = x^2$: $g_1(x) = \sqrt{x}$ returns the *positive* inputs from f and $g_2(x) = -\sqrt{x}$ returns the *negative* inputs to f. In order to view each of these functions as strict inverses, however, we need to split f into two parts: $f_1(x) = x^2$ for $x \geq 0$ and $f_2(x) = x^2$ for $x \leq 0$.



$y = f(x) = x^2$



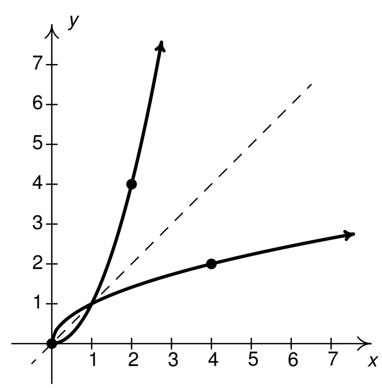
$y = f_1(x) = x^2,\ x \geq 0$



$y = f_2(x) = x^2,\ x \leq 0$

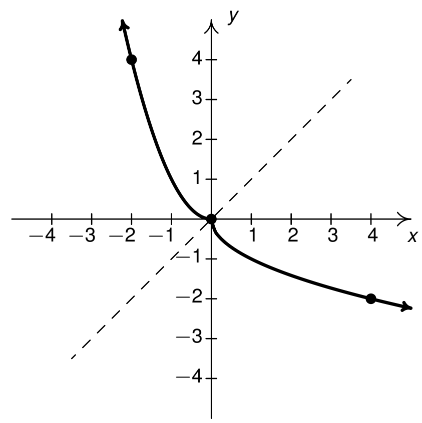
We claim that f_1 and g_1 are an inverse function pair as are f_2 and g_2. Indeed, we find:

$$\begin{array}{rcl} (g_1 \circ f_1)(x) & = & g_1(f_1(x)) \\ & = & g_1(x^2) \\ & = & \sqrt{x^2} \\ & = & |x| = x, \text{ as } x \geq 0. \end{array} \qquad \begin{array}{rcl} (f_1 \circ g_1)(x) & = & f_1(g_1(x)) \\ & = & f_1(\sqrt{x}) \\ & = & (\sqrt{x})^2 \\ & = & x \end{array}$$



$y = f_1(x) = x^2,\ x \geq 0$ and $y = g_1(x) = \sqrt{x}$

$$\begin{array}{rcl} (g_2 \circ f_2)(x) & = & g_2(f_2(x)) \\ & = & g_2(x^2) \\ & = & -\sqrt{x^2} \\ & = & -|x| \\ & = & -(-x) = x, \text{ as } x \leq 0. \end{array} \qquad \begin{array}{rcl} (f_2 \circ g_2)(x) & = & f_2(g_2(x)) \\ & = & f_2(-\sqrt{x}) \\ & = & (-\sqrt{x})^2 \\ & = & (\sqrt{x})^2 \\ & = & x \end{array}$$



$y = f_2(x) = x^2,\ x \leq 0$ and $y = g_2(x) = -\sqrt{x}$

Hence, by restricting the domain of f we are able to produce invertible functions. Said differently, in much the same way the equation $x = y^2$ implicitly describes a pair of *functions*, the equation $y = x^2$ implicitly describes a pair of *invertible* functions.

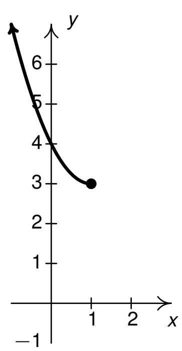
Our next example continues the theme of restricting the domain of a function to find inverse functions.

Example 5.6.3. Graph the following functions to show they are one-to-one and find their inverses. Check your answers analytically using function composition and graphically.

1. $j(x) = x^2 - 2x + 4$, $x \leq 1$.
2. $k(t) = \sqrt{t+2} - 1$

Solution.

1. The function j is a restriction of the function f from Example 5.6.2. Since the domain of j is restricted to $x \leq 1$, we are selecting only the 'left half' of the parabola. Hence, the graph of j, seen below on the left, passes the Horizontal Line Test and thus j is invertible. Below on the right, we find an explicit formula for $j^{-1}(x)$ using our standard algorithm.[7]



$y = j(x)$

$$\begin{array}{rcll} y & = & j(x) & \\ y & = & x^2 - 2x + 4, \quad x \leq 1 & \\ x & = & y^2 - 2y + 4, \quad y \leq 1 & \text{switch } x \text{ and } y \\ 0 & = & y^2 - 2y + 4 - x & \\ y & = & \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} & \text{quadratic formula, } c = 4 - x \\ y & = & \dfrac{2 \pm \sqrt{4x - 12}}{2} & \\ y & = & \dfrac{2 \pm \sqrt{4(x-3)}}{2} & \\ y & = & \dfrac{2 \pm 2\sqrt{x-3}}{2} & \\ y & = & \dfrac{2\left(1 \pm \sqrt{x-3}\right)}{2} & \\ y & = & 1 \pm \sqrt{x-3} & \\ y & = & 1 - \sqrt{x-3} & \text{since } y \leq 1. \end{array}$$

Hence, $j^{-1}(x) = 1 - \sqrt{x-3}$.

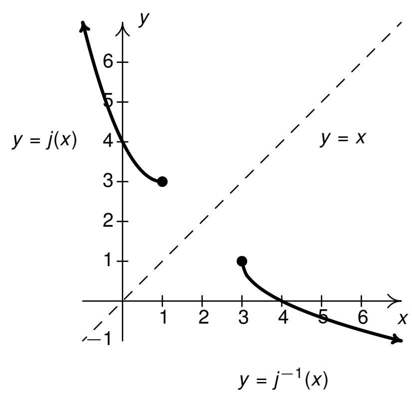
To check our answer algebraically, we simplify $(j^{-1} \circ j)(x)$ and $(j \circ j^{-1})(x)$ Note the importance of the domain restriction $x \leq 1$ when simplifying $(j^{-1} \circ j)(x)$.

[7]Here, we use the Quadratic Formula to solve for y. For 'completeness,' we note you can (and should!) also consider solving for y by 'completing' the square.

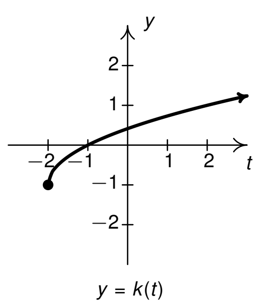
$$\begin{aligned}
\left(j^{-1} \circ j\right)(x) &= j^{-1}(j(x)) \\
&= j^{-1}\left(x^2 - 2x + 4\right), \quad x \leq 1 \\
&= 1 - \sqrt{\left(x^2 - 2x + 4\right) - 3} \\
&= 1 - \sqrt{x^2 - 2x + 1} \\
&= 1 - \sqrt{(x-1)^2} \\
&= 1 - |x - 1| \\
&= 1 - (-(x-1)) \text{ since } x \leq 1 \\
&= x \ \checkmark
\end{aligned}$$

$$\begin{aligned}
\left(j \circ j^{-1}\right)(x) &= j\left(j^{-1}(x)\right) \\
&= j\left(1 - \sqrt{x-3}\right) \\
&= \left(1 - \sqrt{x-3}\right)^2 - 2\left(1 - \sqrt{x-3}\right) + 4 \\
&= 1 - 2\sqrt{x-3} + \left(\sqrt{x-3}\right)^2 - 2 \\
&\quad + 2\sqrt{x-3} + 4 \\
&= 1 + x - 3 - 2 + 4 \\
&= x \ \checkmark
\end{aligned}$$

We graph both j and j^{-1} on the axes below. They appear to be symmetric about the line $y = x$.



2. Graphing $y = k(t) = \sqrt{t+2} - 1$, we see k is one-to-one, so we proceed to find an formula for k^{-1}.



$$\begin{aligned}
y &= k(t) \\
y &= \sqrt{t+2} - 1 \\
t &= \sqrt{y+2} - 1 \quad \text{switch } t \text{ and } y \\
t + 1 &= \sqrt{y+2} \\
(t+1)^2 &= \left(\sqrt{y+2}\right)^2 \\
t^2 + 2t + 1 &= y + 2 \\
y &= t^2 + 2t - 1
\end{aligned}$$

We have $k^{-1}(t) = t^2 + 2t - 1$. Based on our experience, we know something isn't quite right. We determined k^{-1} is a quadratic function, and we have seen several times in this section that these are not one-to-one unless their domains are suitably restricted.

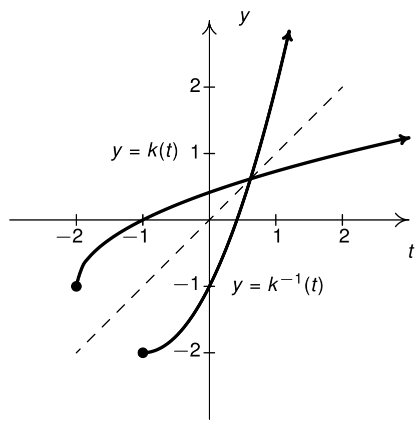
Theorem 5.13 tells us that the domain of k^{-1} is the range of k. From the graph of k, we see that the range is $[-1, \infty)$, which means we restrict the domain of k^{-1} to $t \geq -1$.

We now check that this works in our compositions. Note the importance of the domain restriction, $t \geq -1$ when simplifying $(k \circ k^{-1})(t)$.

$$\begin{aligned}
\left(k^{-1} \circ k\right)(t) &= k^{-1}(k(t)) \\
&= k^{-1}\left(\sqrt{t+2}-1\right) \\
&= \left(\sqrt{t+2}-1\right)^2 + 2\left(\sqrt{t+2}-1\right) - 1 \\
&= \left(\sqrt{t+2}\right)^2 - 2\sqrt{t+2} + 1 \\
&\quad + 2\sqrt{t+2} - 2 - 1 \\
&= t + 2 - 2 \\
&= t \checkmark
\end{aligned}$$

$$\begin{aligned}
\left(k \circ k^{-1}\right)(t) &= k\left(t^2 + 2t - 1\right), \quad t \geq -1 \\
&= \sqrt{\left(t^2 + 2t - 1\right) + 2} - 1 \\
&= \sqrt{t^2 + 2t + 1} - 1 \\
&= \sqrt{(t+1)^2} - 1 \\
&= |t+1| - 1 \\
&= t + 1 - 1, \text{ since } t \geq -1 \\
&= t \checkmark
\end{aligned}$$

Graphically, everything checks out, provided that we remember the domain restriction on k^{-1} means we take the right half of the parabola.



□

Our last example of the section gives an application of inverse functions. Recall in Example 1.2.4 in Section 1.2, we modeled the demand for PortaBoy game systems as the price per system, $p(x)$ as a function of the number of systems sold, x. In the following example, we find $p^{-1}(x)$ and interpret what it means.

Example 5.6.4. Recall the price-demand function for PortaBoy game systems is modeled by the formula $p(x) = -1.5x + 250$ for $0 \leq x \leq 166$ where x represents the number of systems sold (the demand) and $p(x)$ is the price per system, in dollars.

1. Explain why p is one-to-one and find a formula for $p^{-1}(x)$. State the restricted domain.
2. Find and interpret $p^{-1}(220)$.
3. Recall from Section 1.4 that the profit P, in dollars, as a result of selling x systems is given by $P(x) = -1.5x^2 + 170x - 150$. Find and interpret $\left(P \circ p^{-1}\right)(x)$.
4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example 1.4.3.

Solution.

1. Recall the graph of $p(x) = -1.5x + 250$, $0 \leq x \leq 166$, is a line segment from $(0, 250)$ to $(166, 1)$, and as such passes the Horizontal Line Test. Hence, p is one-to-one. We find the expression for $p^{-1}(x)$ as usual and get $p^{-1}(x) = \frac{500-2x}{3}$. The domain of p^{-1} should match the range of p, which is $[1, 250]$, and as such, we restrict the domain of p^{-1} to $1 \leq x \leq 250$.

2. We find $p^{-1}(220) = \frac{500-2(220)}{3} = 20$. Since the function p took as inputs the number of systems sold and returned the price per system as the output, p^{-1} takes the price per system as its input and returns the number of systems sold as its output. Hence, $p^{-1}(220) = 20$ means 20 systems will be sold in if the price is set at \$220 per system.

3. We compute $\left(P \circ p^{-1}\right)(x) = P\left(p^{-1}(x)\right) = P\left(\frac{500-2x}{3}\right) = -1.5\left(\frac{500-2x}{3}\right)^2 + 170\left(\frac{500-2x}{3}\right) - 150$. After a hefty amount of Elementary Algebra,[8] we obtain $\left(P \circ p^{-1}\right)(x) = -\frac{2}{3}x^2 + 220x - \frac{40450}{3}$.

   To understand what this means, recall that the original profit function P gave us the profit as a function of the number of systems sold. The function p^{-1} gives us the number of systems sold as a function of the price. Hence, when we compute $(P \circ p^{-1})(x) = P(p^{-1}(x))$, we input a price per system, x into the function p^{-1}.

   The number $p^{-1}(x)$ is the number of systems sold at that price. This number is then fed into P to return the profit obtained by selling $p^{-1}(x)$ systems. Hence, $\left(P \circ p^{-1}\right)(x)$ gives us the profit (in dollars) as a function of the price per system, x.

4. We know from Section 1.4 that the graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards. The maximum profit is realized at the vertex. Since we are concerned only with the price per system, we need only find the x-coordinate of the vertex. Identifying $a = -\frac{2}{3}$ and $b = 220$, we get, by the Vertex Formula, Equation 1.2, $x = -\frac{b}{2a} = 165$.

   Hence, weekly profit is maximized if we set the price at \$165 per system. Comparing this with our answer from Example 1.4.3, there is a slight discrepancy to the tune of \$0.50. We leave it to the reader to balance the books appropriately. □

[8] It is good review to actually do this!

5.6.1 Exercises

In Exercises 1 - 8, verify the given pairs of functions are inverses algebraically and graphically.

1. $f(x) = 2x + 7$ and $g(x) = \dfrac{x-7}{2}$
2. $f(x) = \dfrac{5-3x}{4}$ and $g(x) = -\dfrac{4}{3}x + \dfrac{5}{3}$.
3. $f(t) = \dfrac{5}{t-1}$ and $g(t) = \dfrac{t+5}{t}$
4. $f(t) = \dfrac{t}{t-1}$ and $g(t) = f(t) = \dfrac{t}{t-1}$
5. $f(x) = \sqrt{4-x}$ and $g(x) = -x^2 + 4$, $x \geq 0$
6. $f(x) = 1 - \sqrt{x+1}$ and $g(x) = x^2 - 2x$, $x \leq 1$.
7. $f(t) = (t-1)^3 + 5$ and $g(t) = \sqrt[3]{t-5} + 1$
8. $f(t) = -\sqrt[4]{t-2}$ and $g(t) = t^4 + 2$, $t \leq 0$.

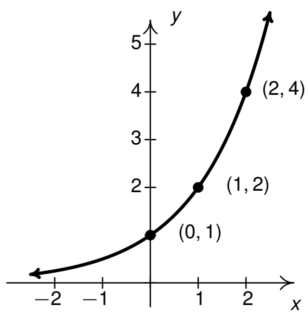
In Exercises 9 - 28, show that the given function is one-to-one and find its inverse. Check your answers algebraically and graphically. Verify the range of the function is the domain of its inverse and vice-versa.

9. $f(x) = 6x - 2$
10. $f(x) = 42 - x$
11. $g(t) = \dfrac{t-2}{3} + 4$
12. $g(t) = 1 - \dfrac{4+3t}{5}$
13. $f(x) = \sqrt{3x-1} + 5$
14. $f(x) = 2 - \sqrt{x-5}$
15. $g(t) = 3\sqrt{t-1} - 4$
16. $g(t) = 1 - 2\sqrt{2t+5}$
17. $f(x) = \sqrt[5]{3x-1}$
18. $f(x) = 3 - \sqrt[3]{x-2}$
19. $g(t) = t^2 - 10t$, $t \geq 5$
20. $g(t) = 3(t+4)^2 - 5$, $t \leq -4$
21. $f(x) = x^2 - 6x + 5$, $x \leq 3$
22. $f(x) = 4x^2 + 4x + 1$, $x < -1$
23. $g(t) = \dfrac{3}{4-t}$
24. $g(t) = \dfrac{t}{1-3t}$
25. $f(x) = \dfrac{2x-1}{3x+4}$
26. $f(x) = \dfrac{4x+2}{3x-6}$
27. $g(t) = \dfrac{-3t-2}{t+3}$
28. $g(t) = \dfrac{t-2}{2t-1}$
29. Explain why each set of ordered pairs below represents a one-to-one function and find the inverse.
    (a) $F = \{(0,0), (1,1), (2,-1), (3,2), (4,-2), (5,3), (6,-3)\}$
    (b) $G = \{(0,0), (1,1), (2,-1), (3,2), (4,-2), (5,3), (6,-3), \ldots\}$
    NOTE: The difference between F and G is the '....'
    (c) $P = \{(2t^5, 3t-1) \mid t \text{ is a real number.}\}$
    (d) $Q = \{(n, n^2) \mid n \text{ is a } \textit{natural} \text{ number.}\}$[9]

[9] Recall this means $n = 0, 1, 2, \ldots$.

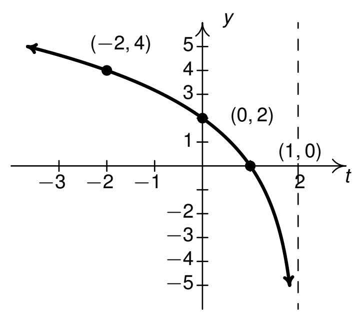
In Exercises 30 - 33, explain why each graph represents[10] a one-to-one function and graph its inverse.

30. $y = f(x)$



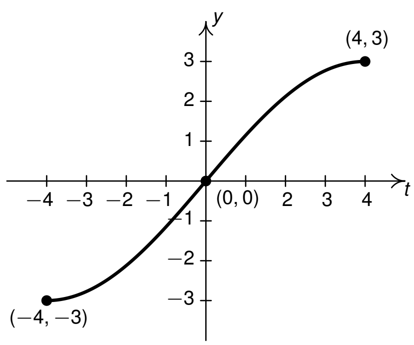
Asymptote: $y = 0$.

31. $y = g(t)$



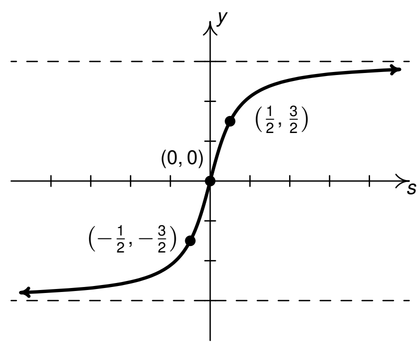
Asymptote: $t = 2$.

32. $y = S(t)$



Domain: $[-4, 4]$.

33. $y = R(s)$



Asymptotes: $y = \pm 3$.

34. The price of a dOpi media player, in dollars per dOpi, is given as a function of the weekly sales x according to the formula $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.

    (a) Find $p^{-1}(x)$ and state its domain.

    (b) Find and interpret $p^{-1}(105)$.

    (c) The profit (in dollars) made from producing and selling x dOpis per week is given by the formula $P(x) = -15x^2 + 350x - 2000$, for $0 \leq x \leq 30$. Find $\left(P \circ p^{-1}\right)(x)$ and determine what price per dOpi would yield the maximum profit. What is the maximum profit? How many dOpis need to be produced and sold to achieve the maximum profit?

35. Show that the Fahrenheit to Celsius conversion function found in Exercise 26 in Section 1.2.2 is invertible and that its inverse is the Celsius to Fahrenheit conversion function.

36. Analytically show that the function $f(x) = x^3 + 3x + 1$ is one-to-one. Use Theorem 5.13 to help you compute $f^{-1}(1)$, $f^{-1}(5)$, and $f^{-1}(-3)$. What happens when you attempt to find a formula for $f^{-1}(x)$?

[10] or, more precisely, *appears* to represent . . .

37. Let $f(x) = \dfrac{2x}{x^2 - 1}$.

    (a) Graph $y = f(x)$ using the techniques in Section 3.2. Check your answer using a graphing utility.
    (b) Verify that f is one-to-one on the interval $(-1, 1)$.
    (c) Use the procedure outlined on Page 486 to find the formula for $f^{-1}(x)$ for $-1 < x < 1$.
    (d) Since $f(0) = 0$, it should be the case that $f^{-1}(0) = 0$. What goes wrong when you attempt to substitute $x = 0$ into $f^{-1}(x)$? Discuss with your classmates how this problem arose and possible remedies.

38. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.

39. If f is odd and invertible, prove that f^{-1} is also odd.

40. Let f and g be invertible functions. With the help of your classmates show that $(f \circ g)$ is one-to-one, hence invertible, and that $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

With help from your classmates, find the inverses of the functions in Exercises 41 - 44.

41. $f(x) = ax + b, \ a \neq 0$

42. $f(x) = a\sqrt{x - h} + k, \ a \neq 0, x \geq h$

43. $f(x) = ax^2 + bx + c$ where $a \neq 0, \ x \geq -\dfrac{b}{2a}$.

44. $f(x) = \dfrac{ax + b}{cx + d}$, (See Exercise 45 below.)

45. What conditions must you place on the values of a, b, c and d in Exercise 44 in order to guarantee that the function is invertible?

46. The function given in number 4 is an example of a function which is its own inverse.

    (a) Algebraically verify every function of the form: $f(x) = \dfrac{ax + b}{cx - a}$ is its own inverse.
    What assumptions do you need to make about the values of a, b, and c?
    (b) Under what conditions is $f(x) = mx + b$, $m \neq 0$ its own inverse? Prove your answer.

5.6.2 Answers

9. $f^{-1}(x) = \dfrac{x+2}{6}$

10. $f^{-1}(x) = 42 - x$

11. $g^{-1}(t) = 3t - 10$

12. $g^{-1}(t) = -\frac{5}{3}t + \frac{1}{3}$

13. $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3}$, $x \geq 5$

14. $f^{-1}(x) = (x-2)^2 + 5$, $x \leq 2$

15. $g^{-1}(t) = \frac{1}{9}(t+4)^2 + 1$, $t \geq -4$

16. $g^{-1}(t) = \frac{1}{8}(t-1)^2 - \frac{5}{2}$, $t \leq 1$

17. $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$

18. $f^{-1}(x) = -(x-3)^3 + 2$

19. $g^{-1}(t) = 5 + \sqrt{t+25}$

20. $g^{-1}(t) = -\sqrt{\frac{t+5}{3}} - 4$

21. $f^{-1}(x) = 3 - \sqrt{x+4}$

22. $f^{-1}(x) = -\frac{\sqrt{x}+1}{2}$, $x > 1$

23. $g^{-1}(t) = \dfrac{4t-3}{t}$

24. $g^{-1}(t) = \dfrac{t}{3t+1}$

25. $f^{-1}(x) = \dfrac{4x+1}{2-3x}$

26. $f^{-1}(x) = \dfrac{6x+2}{3x-4}$

27. $g^{-1}(t) = \dfrac{-3t-2}{t+3}$

28. $g^{-1}(t) = \dfrac{t-2}{2t-1}$

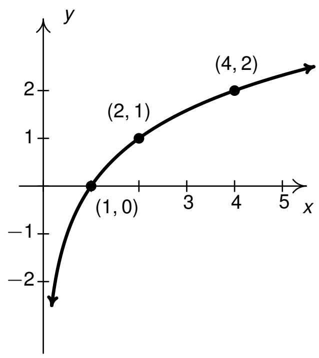
29. (a) None of the first coordinates of the ordered pairs in F are repeated, so F is a function and none of the second coordinates of the ordered pairs of F are repeated, so F is one-to-one. $F^{-1} = \{(0,0), (1,1), (-1,2), (2,3), (-2,4), (3,5), (-3,6)\}$

(b) Because of the '...' it is helpful to determine a formula for the matching. For the even numbers n, $n = 0, 2, 4, ...$, the ordered pair $\left(n, -\frac{n}{2}\right)$ is in G. For the odd numbers $n = 1, 3, 5, ...$, the ordered pair $\left(n, \frac{n+1}{2}\right)$ is in G. Hence, given any input to G, n, whether it be even or odd, there is only one output from G, either $-\frac{n}{2}$ or $\frac{n+1}{2}$, both of which are functions of n. To show G is one to one, we note that if the output from G is 0 or less, then it must be of the form $-\frac{n}{2}$ for an even number n. Moreover, if $-\frac{n}{2} = -\frac{m}{2}$, then $n = m$. In the case we are looking at outputs from G which are greater than 0, then it must be of the form $\frac{n+1}{2}$ for an odd number n. In this, too, if $\frac{n+1}{2} = \frac{m+1}{2}$, then $n = m$. Hence, in any case, if the outputs from G are the same, then the inputs to G had to be the same so G is one-to-one and $G^{-1} = \{(0,0), (1,1), (-1,2), (2,3), (-2,4), (3,5), (-3,6), ...\}$

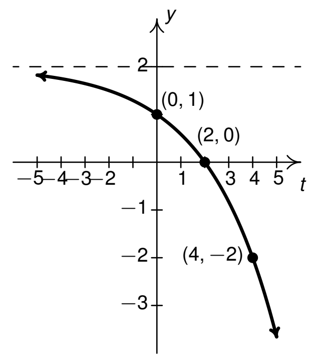
(c) To show P is a function we note that if we have the same inputs to P, say $2t^5 = 2u^5$, then $t = u$. Hence the corresponding outputs, $2t - 1$ and $3u - 1$, are equal, too. To show P is one-to-one, we note that if we have the same outputs from P, $3t - 1 = 3u - 1$, then $t = u$. Hence, the corresponding inputs $2t^5$ and $2u^5$ are equal, too. Hence P is one-to-one and $P^{-1} = \{(3t-1, 2t^5) \mid t \text{ is a real number.}\}$

(d) To show Q is a function, we note that if we have the same inputs to Q, say $n = m$, then the outputs from Q, namely n^2 and m^2 are equal. To show Q is one-to-one, we note that if we get the same output from Q, namely $n^2 = m^2$, then $n = \pm m$. However since n and m are *natural* numbers, both n and m are positive so $n = m$. Hence Q is one-to-one and $Q^{-1} = \{(n^2, n) \mid n \text{ is a } \textit{natural} \text{ number.}\}$.

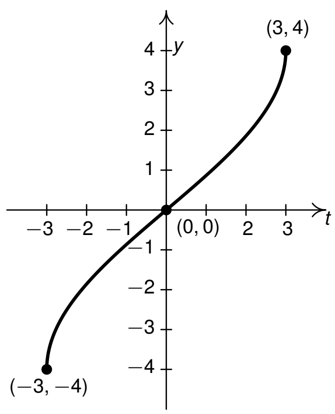
30. $y = f^{-1}(x)$. Asymptote: $x = 0$.



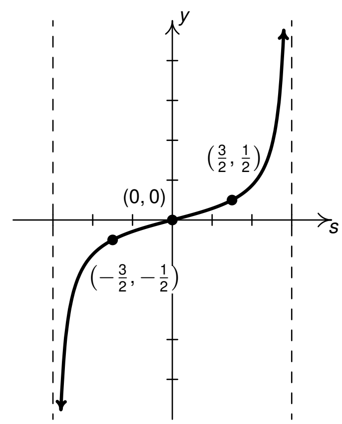
31. $y = g^{-1}(t)$. Asymptote: $y = 2$.



32. $y = S^{-1}(t)$. Domain $[-3, 3]$.



33. $y = R^{-1}(s)$. Asymptotes: $s = \pm 3$.



34. (a) $p^{-1}(x) = \frac{450-x}{15}$. The domain of p^{-1} is the range of p which is $[0, 450]$

(b) $p^{-1}(105) = 23$. This means that if the price is set to \$105 then 23 dOpis will be sold.

(c) $\left(P \circ p^{-1}\right)(x) = -\frac{1}{15}x^2 + \frac{110}{3}x - 5000$, $0 \leq x \leq 450$.

The graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards with vertex $\left(275, \frac{125}{3}\right) \approx (275, 41.67)$. This means that the maximum profit is a whopping \$41.67 when the price per dOpi is set to \$275. At this price, we can produce and sell $p^{-1}(275) = 11.\overline{6}$ dOpis. Since we cannot sell part of a system, we need to adjust the price to sell either 11 dOpis or 12 dOpis. We find $p(11) = 285$ and $p(12) = 270$, which means we set the price per dOpi at either \$285 or \$270, respectively. The profits at these prices are $\left(P \circ p^{-1}\right)(285) = 35$ and $\left(P \circ p^{-1}\right)(270) = 40$, so it looks as if the maximum profit is \$40 and it is made by producing and selling 12 dOpis a week at a price of \$270 per dOpi.

36. Given that $f(0) = 1$, we have $f^{-1}(1) = 0$. Similarly $f^{-1}(5) = 1$ and $f^{-1}(-3) = -1$

46. (b) If $b = 0$, then $m = \pm 1$. If $b \neq 0$, then $m = -1$ and b can be any real number.

Chapter 6

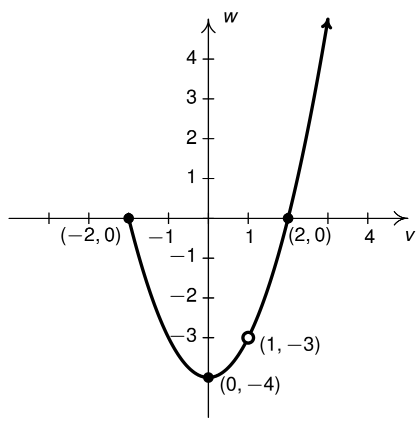
First Steps into Calculus

6.1 A (more) Formal Introduction to Limits

In this chapter, we take some more steps towards[1] Calculus. We first revisit the concept of **limit** . We've primarily used limits as a way to analyze and codify function behavior in places where we simplify could not evaluate the function.[2] We first focus on how the concept is expressed graphically.

6.1.1 Limits from Graphs

Even though we didn't introduce the limit concept or notation until Chapter 2, we first encounter the underlying concept much earlier. Recall in Example 1.1.4 we were given the graph of a function $w = F(v)$:



The hole in the graph tells us that even though $F(1)$ is undefined, we'd **expect** $F(1)$ to be -3 based on what's happening with the graph **near** the point $(1, -3)$. Using limit notation, we'd write $\displaystyle\lim_{v \to 1} F(v) = -3$. We take a moment below to better define what we mean when use the limit notation.

[1] into?

[2] Whether it be describing end behavior as $x \to -\infty$ or $x \to \infty$ or places where we'd be dividing by '0.'

Definition 6.1. Informal Definition of Limit: Given a function f defined on an open interval containing $x = a$, except possibly at $x = a$, the notation $\lim\limits_{x \to a} f(x) = L$, means as input values, x, approach[a] the number a, the output values, $f(x)$, approach the number L. The notation '$\lim\limits_{x \to a} f(x) = L$' is read read 'the **limit** as x approaches a of $f(x)$ equals L.'

[a]ignoring what is happening at $x = a$

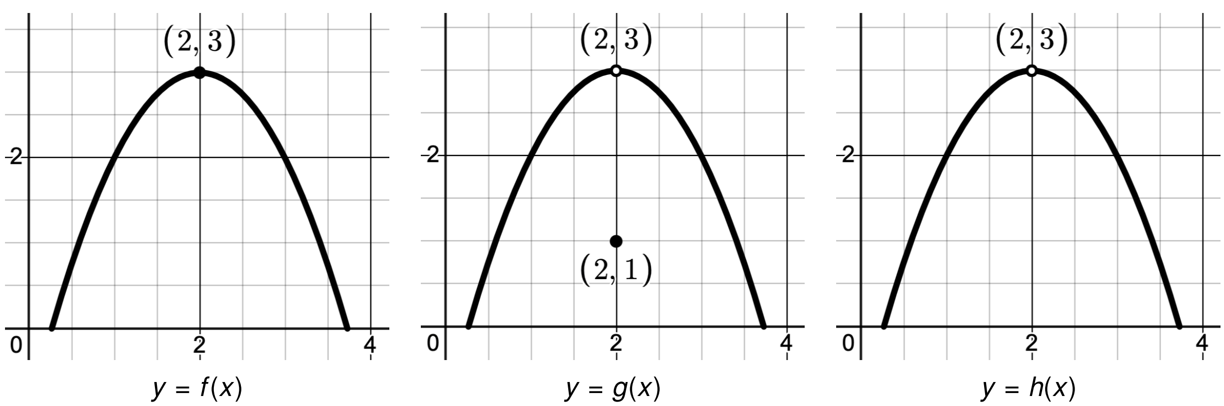
Some remarks about Definition 6.1 are in order. Note that the business about f being defined on 'an open interval containing $x = a$' is there to guarantee that we have the appropriate 'room' for inputs x to approach a from either direction.[3] (For now, we'll just assume we all understand what the word 'approach' means in this context and let a Calculus class explain how this is more precisely codified mathematically.)

The phrase 'except possibly at $x = a$' which immediately follows means the limit doesn't concern itself with what is actually happening at $x = a$. The function f may or may not be defined at $x = a$. Indeed, if $\lim\limits_{x \to a} f(x) = L$, $f(a)$ could be L, $f(a)$ could be a number different than L or $f(a)$ could not be defined.

This drives home the principle difference between the precalculus notion of '$f(a)$' and the Calculus notion of '$\lim\limits_{x \to a} f(x)$': '$\lim\limits_{x \to a} f(x)$' is what we **expect** $f(a)$ to be - which may or may not agree with what $f(a)$, if $f(a)$ is even defined.

For example, using the graph from Example 1.1.4, we write $\lim\limits_{v \to 0} F(v) = -4$ since as $v \to 0$, we see $w = F(v) \to -4$. In this particular case, $F(0) = -4$ so we get from F at $v = 0$ what we expect to get.[4]

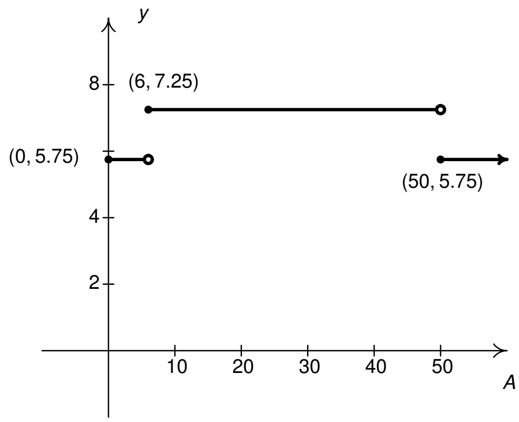
For another example, consider the graphs of the functions f, g, and h below near $x = 2$. Through a precalculus lens, each of these functions is different at $x = 2$: $f(2) = 3$, $g(2) = 1$, and $h(2)$ is undefined. Through a Calculus lens, however, all three of these functions are behaving identically as x approaches 2: $\lim\limits_{x \to 2} f(x) = 3$, $\lim\limits_{x \to 2} g(x) = 3$, and $\lim\limits_{x \to 2} h(x) = 3$.



Next let's head to Section 1.2 and revisit Example 1.2.1 in a piecewise-defined function is used to model matinee admission prices at a local theater:

[3]We'll get to 'one-sided' limits here shortly.

[4]This is another way to describe the notion of **continuity**. (See Definition 6.4.)



$$y = p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50 \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$$

What can be said about $\lim\limits_{A \to 6} p(A)$? Remember, $\lim\limits_{A \to 6} p(A)$ is what we would **expect** $p(6)$ to be by analyzing p as $A \to 6$, ignoring what is happening at $A = 6$. If $A < 6$, $p(A)$ is always 5.75, so, based on this information, we'd **expect** $p(6)$ to be 5.75. If $A > 6$, then $p(A)$ is always 7.25, so we'd expect $p(6)$ to be 7.25. Since Definition 6.1 requires the $p(A)$ values to approach a **single** value L as $A \to 6$, we'd say in this case that $\lim\limits_{A \to 6} p(A)$ does not exist.

Even though $\lim\limits_{A \to 6} p(A)$ does not exist, we've used so-called 'one-sided' limit notation in Chapters 3 and 4 which we can apply here. Specifically, we write $\lim\limits_{A \to 6^-} p(A) = 5.75$ and $\lim\limits_{A \to 6^+} p(A) = 7.25$ to more precisely record the behavior of p as we approach $A = 6$ from either direction.[5]

Definition 6.2. One-sided Limits:

- If f is defined on an open interval for $x < a$ except possibly at $x = a$, the notation $\lim\limits_{x \to a^-} f(x) = L$, read 'the **limit** as x approaches a **from the left** of $f(x)$ equals L' means as input values, x, $x < a$, approach the number a (ignoring what is happening at $x = a$), the output values, $f(x)$, approach the number L.
- If f is defined on an open interval for $x > a$ except possibly at $x = a$, the notation $\lim\limits_{x \to a^+} f(x) = L$, read 'the **limit** as x approaches a **from the right** of $f(x)$ equals L' means as input values, x, $x > a$, approach the number a (ignoring what is happening at $x = a$), the output values, $f(x)$, approach the number L.

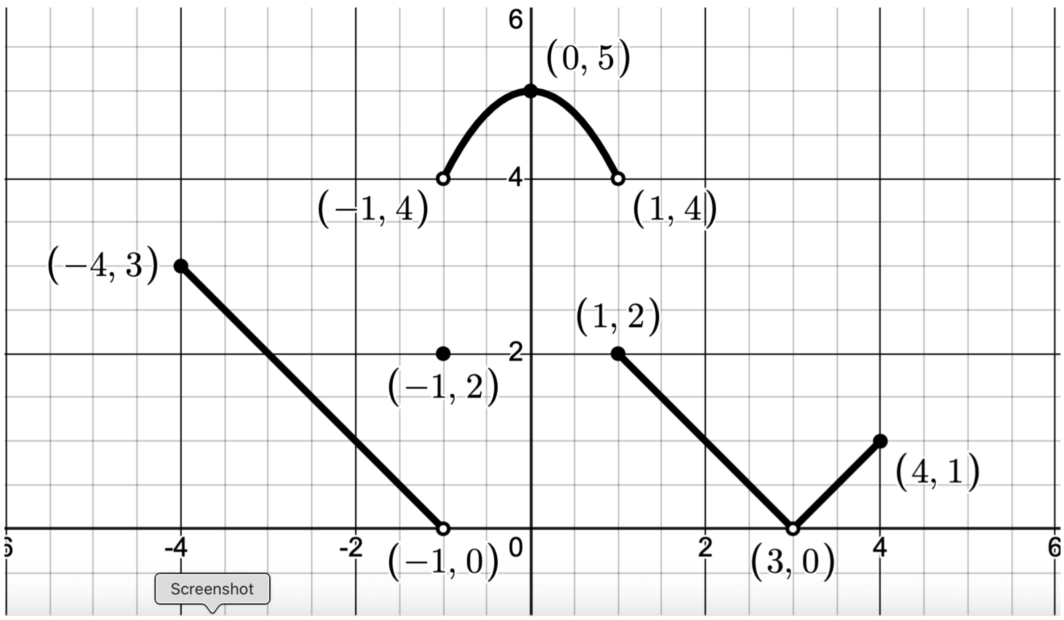
In order for the (two-sided) limit to exist, both one-sided limits need to exist, be equal, and vice-versa. This is recorded in the following theorem.

[5]Note that $\lim\limits_{A \to 6^+} p(A) = 7.25 = p(6)$ in this case. So at least we 'get' what we 'expect to get' when approaching 6 from the right.

Theorem 6.1. Given a function f defined on an open interval containing $x = a$, except possibly at $x = a$, $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

It's time for an example.

Example 6.1.1. Use the **complete** graph[6] of $y = f(x)$ below to answer the following questions.



1. State the domain and range of f using interval notation.

2. Find the following values. Explain your reasoning.

• $f(-1)$	• $\lim_{x\to -1^-} f(x)$	• $\lim_{x\to -1^+} f(x)$	• $\lim_{x\to -1} f(x)$
• $f(1)$	• $\lim_{x\to 1^-} f(x)$	• $\lim_{x\to 1^+} f(x)$	• $\lim_{x\to 1} f(x)$
• $f(0)$	• $\lim_{x\to 0^-} f(x)$	• $\lim_{x\to 0^+} f(x)$	• $\lim_{x\to 0} f(x)$
• $f(3)$	• $\lim_{x\to 3^-} f(x)$	• $\lim_{x\to 3^+} f(x)$	• $\lim_{x\to 3} f(x)$

[6]Recall this means this is the entire graph of f. There's nothing hidden offscreen.

3. Explain why $\lim_{x\to -4} f(x)$ does not exist and find $\lim_{x\to -4^+} f(x)$

4. Explain why $\lim_{x\to 4} f(x)$ does not exist and find $\lim_{x\to 4^-} f(x)$

Solution.

1. Projecting the graph of f to the $x-$axis, we find that everything from -4 to 4, inclusive, is covered except for $x = 3$. Hence the domain is $[-4, 3) \cup (3, 4]$. When projecting the graph of f to the y-axis, we see everything between 0 and 3 is covered (excluding 0 and including 3) then everything between 4 and 5 is covered (excluding 4 and including 5). Hence the range is $(0, 3] \cup (4, 5]$.

2.
   - Regarding $x = -1$: since the point $(-1, 2)$ is on the graph of f, we know $f(-1) = 2$. To determine $\lim_{x\to -1^-} f(x)$, we see that the graph to the left of $x = -1$ is headed towards the point $(-1, 0)$ as x approaches -1. Hence, $\lim_{x\to -1^-} f(x) = 0$. To determine $\lim_{x\to -1^+} f(x)$, we see that the graph to the right of $x = -1$ is headed towards the point $(-1, 4)$ as x approaches -1. Hence, $\lim_{x\to -1^+} f(x) = 4$. Since $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$ are different, $\lim_{x\to -1} f(x)$ does not exist.
   - Regarding $x = 1$: since the point $(1, 2)$ is on the graph of f, we know $f(1) = 2$. To determine $\lim_{x\to 1^-} f(x)$, we see that the graph to the left of $x = 1$ is headed towards the point $(1, 4)$ as x approaches 1. Hence, $\lim_{x\to 1^-} f(x) = 4$. To determine $\lim_{x\to 1^+} f(x)$, we see that the graph to the right of $x = 1$ is headed towards the point $(1, 2)$ as x approaches 1. Hence, $\lim_{x\to 1^+} f(x) = 2$. Since $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$ are different, $\lim_{x\to 1} f(x)$ does not exist.
   - Regarding $x = 0$: since the point $(0, 5)$ is on the graph of f, we know $f(0) = 5$. To determine $\lim_{x\to 0^-} f(x)$, we see that the graph to the left of $x = 0$ is headed towards the point $(0, 5)$ as x approaches 0. Hence, $\lim_{x\to 0^-} f(x) = 5$. To determine $\lim_{x\to 0^+} f(x)$, we see that the graph to the right of $x = 0$ is headed towards the point $(0, 5)$ as x approaches 0. Hence, $\lim_{x\to 0^+} f(x) = 5$. Since $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ are both 5, $\lim_{x\to 0} f(x) = 5$.
   - Regarding $x = 3$: since there is no point on the graph of f with an x-coordinate of 3, $f(3)$ is undefined. To determine $\lim_{x\to 3^-} f(x)$, we see that the graph to the left of $x = 3$ is headed towards the point $(3, 0)$ as x approaches 3. Hence, $\lim_{x\to 3^-} f(x) = 0$. To determine $\lim_{x\to 3^+} f(x)$, we see that the graph to the right of $x = 3$ is headed towards the point $(3, 0)$ as x approaches 3. Hence, $\lim_{x\to 3^+} f(x) = 0$. Since $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$ are both 0, $\lim_{x\to 3} f(x) = 0$.

3. In order to find $\lim_{x\to -4} f(x)$, we need to analyze the graph of f from both the left an right of $x = -4$. Since there is no graph to the left of $x = -4$, $\lim_{x\to -4^-} f(x)$, and hence $\lim_{x\to -4} f(x)$ does not exist. However, $\lim_{x\to -4^+} f(x) = 3$, since, when coming from the right, the graph approaches the point $(-4, 3)$ as x approaches -4.

4. In order to find $\lim\limits_{x \to 4} f(x)$, we need to analyze the graph of f from both the left an right of $x = 4$. Since there is no graph to the right of $x = 4$, $\lim\limits_{x \to 4^+} f(x)$, and hence $\lim\limits_{x \to 4} f(x)$ does not exist. However, $\lim\limits_{x \to 4^-} f(x) = 1$, since, when coming from the left, the graph approaches the point $(4, 1)$ as x approaches 4. □

Another use of limits we've seen is to codify unbounded behavior. Since ∞ and $-\infty$ aren't real numbers, we used limit notation to help us describe end behavior (as $x \to -\infty$ or $x \to \infty$) and unbounded function behavior ($f(x) \to -\infty$ or $f(x) \to \infty$.) Let's take a moment to think about what it means to write $\lim\limits_{x \to \infty} f(x) = \infty$. How does one 'approach' infinity anyhow?

Let's consider $\lim\limits_{x \to \infty} x^2 = \infty$. What me mean here is that as x grows larger and larger (without bound), $f(x) = x^2$ follows suit. To prove something like this, we'd need to show that for any 'arbitrarily large' real number, N, we can find some threshold M so that if the inputs, $x > M$, the outputs, $f(x) > N$. For example, if we set $N = 10000$, then to guarantee $f(x) = x^2 > 10000$, we can solve and get $x > \sqrt{10000} = 100$. So provided $x > 100$, $f(x) > 10000$. In this case, $N = 10000$ and $M = \sqrt{10000} = 100$. In general, if $x > \sqrt{N}$, $x^2 > N$, which justifies us writing $\lim\limits_{x \to \infty} x^2 = \infty$.

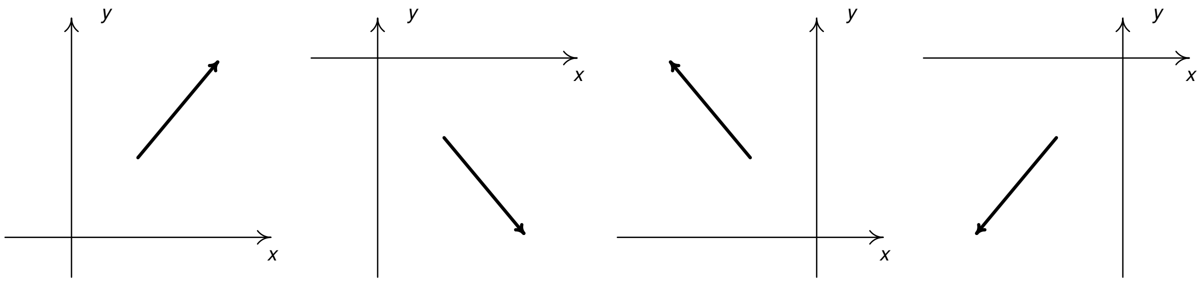
We can adjust the inequality signs in the sort of argument[7] above to direct x or $f(x)$ to either ∞ or $-\infty$. Doing so gives us the (formal) definitions of below.

Definition 6.3.

1. Given a function f defined on an open interval (a, ∞):
   - the notation '$\lim\limits_{x \to \infty} f(x) = \infty$' means that for any real number N there is a real number M so that if $x > M$, $f(x) > N$.
   - the notation '$\lim\limits_{x \to \infty} f(x) = -\infty$' means that for any real number N there is a real number M so that if $x > M$, $f(x) < N$.
2. Given a function f defined on an open interval $(-\infty, a)$:
   - the notation '$\lim\limits_{x \to -\infty} f(x) = \infty$' means that for any real number N there is a real number M so that if $x < M$, $f(x) > N$.
   - the notation '$\lim\limits_{x \to -\infty} f(x) = -\infty$' means that for any real number N there is a real number M so that if $x < M$, $f(x) < N$.

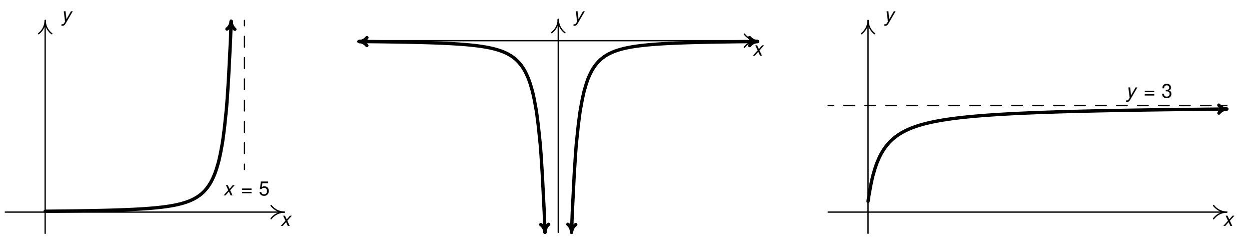
We'll explore Definition 6.3 more in the Exercises. In the meantime, the reader is encouraged to take some time and think about the inequalities in Definition 6.3 and how they force the corresponding graphical behavior showcased below:

[7] If this sort of argument seems familiar, it should! Replacing '$>$' with '$=$' is how we proved the ranges of the monomial functions and Laurent monomials in Sections 2.1 and 3.1, respectively.



$\lim_{x\to\infty} f(x) = \infty$ $\quad$ $\lim_{x\to\infty} f(x) = -\infty$ $\quad$ $\lim_{x\to-\infty} f(x) = \infty$ $\quad$ $\lim_{x\to-\infty} f(x) = -\infty$

Combining the ideas of what it means for x or $f(x)$ to approach (finite) real numbers along with our (more precise notion) of what it means for x or $f(x)$ to approach $-\infty$ or ∞, we can mix and match to produce expressions and graphs containing vertical and horizontal asymptotes such as the ones depicted below:



$\lim_{x\to 5^-} f(x) = \infty$ $\quad$ $\lim_{x\to 0} f(x) = -\infty$ $\quad$ $\lim_{x\to\infty} f(x) = 3$

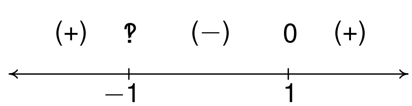
We would be remiss in our duties as (pre)Calculus instructors if we failed to point out that even though we've used notation '$= \infty$' in expressions like $\lim_{x\to 5^-} f(x) = \infty$ above, since ∞ is not a real number, technically, $\lim_{x\to 5^-} f(x)$ does not exist. The '$= \infty$' here just codifies better **the manner in which** the limit fails to exist.

Our last example of this section turns the tables and has you construct the graph of function given information provided by limits.

Example 6.1.2. Sketch the graph of a function f which satisfies all of the following criteria:

- $\lim_{x\to-\infty} f(x) = 0$
- $\lim_{x\to-1^-} f(x) = \infty$
- $\lim_{x\to-1^+} f(x) = -\infty$
- $\lim_{x\to 1^-} f(x) = -\frac{1}{2}$
- $\lim_{x\to 1^+} f(x) = 0$
- $\lim_{x\to\infty} f(x) = \infty$

The sign diagram for f is:



A sign diagram for $f(x)$

Solution. Each piece of information given describes a portion of the graph of $y = f(x)$. The strategy is to sketch each individual portion and connect them together.

First off, $\displaystyle\lim_{x\to-\infty} f(x) = 0$ tells us that $y = 0$ is a horizontal asymptote to the graph. This means as we head off to the left, the graph approaches the x-axis. Since the sign diagram tells is $f(x) > 0$ for $x < -1$, we know the graph must approach the x-axis from above.

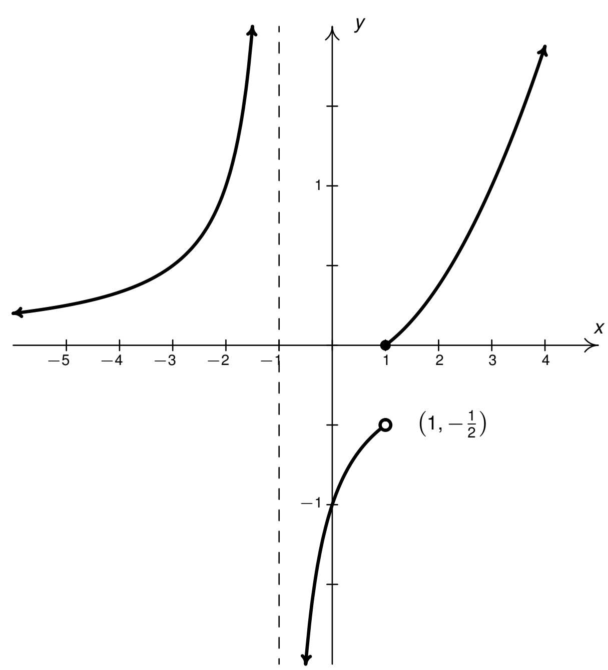
Next, we have $\displaystyle\lim_{x\to-1^-} f(x) = \infty$ and $\displaystyle\lim_{x\to-1^+} f(x) = -\infty$ which tells us $x = -1$ is a vertical asymptote to the graph. These behaviors agree with the sign diagram both in sign ('+' ∞ for $x < -1$ and '$-$' ∞ for $x > -1$) and the fact that f is undefined at $x = -1$.

Moving on we are given information about f near $x = 1$. The limit $\displaystyle\lim_{x\to1^-} f(x) = -\frac{1}{2}$ means as we approach $x = 1$ from the left, the y-values approach $-\frac{1}{2}$. Likewise, $\displaystyle\lim_{x\to1^+} f(x) = 0$ means as we approach $x = 1$ from the right, the y-values approach 0 (the x-axis).

The sign diagram tells us that, indeed, $f(1) = 0$. Hence, as $x \to 1^-$, the graph of f approaches a hole at $\left(0, -\frac{1}{2}\right)$. As $x \to 0^+$, the graph of f approaches an x-intercept, $(1, 0)$, which is included in the graph.

Finally, $\displaystyle\lim_{x\to\infty} f(x) = \infty$ means that as we move farther to the right, the graph moves farther up which we indicate, as usual, with an arrow up to the right.

Connecting these pieces together (careful to not violate the Vertical Line Test, Theorem 1.1) we get:



□

6.1.2 Limit Properties and an Introduction to Continuity

Let $f(x) = 6$. Consider $\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5} 6$. Since the function values are unchanging, there is no other value other than '6' to expect from f so it stands to reason that $\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5} 6 = 6$. Indeed, for any real number a, $\lim\limits_{x\to a} 6 = 6$. In general, if $f(x) = c$ is a constant function, $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} c = c$. The formal proof of this fact requires a formal definition of limit,but for now, we'll just take it as true.

Next, let's consider $f(x) = x$. Consider $\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5} x$. What do we expect the value of 'x' to be as $x \to 5$? Well, '5'. Indeed, it can be proved that $\lim\limits_{x\to a} x = a$ for all real numbers, a.

What about $\lim\limits_{x\to 5}(x+6)$? Since $\lim\limits_{x\to 5} x = 5$ and $\lim\limits_{x\to 5} 6 = 6$, it stands to reason that

$$\lim_{x\to 5}(x+6) = \lim_{x\to 5} x + \lim_{x\to 5} 6 = 5 + 6 = 11,$$

which is indeed the case. It turns out that in most cases, limits do respect arithmetic:

Theorem 6.2. (Some of the) Properties of Limits:

1. **Constant Rule:** If c is a constant, then $\lim\limits_{x\to a} c = c$ for every real number a.
2. **Identity Rule:** $\lim\limits_{x\to a} x = a$ for every real number a.
3. **Limits Respect Function Arithmetic:** If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = K$, then:
   (a) **Sum and Difference Rule:** $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = L \pm K$
   (b) **Product Rule:** $\lim\limits_{x\to a}[f(x)\,g(x)] = \left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} g(x)\right] = L\,K$
      i. **Scalar Multiple Rule:** If c is a real number, $\lim\limits_{x\to a}[c\,f(x)] = c\left[\lim\limits_{x\to a} f(x)\right] = c\,L$
      ii. **Power Rule:** If n is any natural number,[a] $\lim\limits_{x\to a}[f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n = L^n$.
   (c) **Quotient Rule:** $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{L}{K}$, provided $K \neq 0$.
   (d) **Rules for Radicals:**
      - If n is **odd**, $\lim\limits_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)} = \sqrt[n]{L}$.
      - If n is **even** and $L > 0$, $\lim\limits_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)} = \sqrt[n]{L}$.
      - If n is **even** and $L = 0$, $\lim\limits_{x\to a}\sqrt[n]{f(x)} = 0$ provided $f(x) \geq 0$ for all x near a.
   (e) **Real Number Powers:** If $L > 0$ and p is a real number, $\lim\limits_{x\to a}[f(x)]^p = \left[\lim\limits_{x\to a} f(x)\right]^p = L^p$.

[a] Recall this means $n = 1, 2, 3, \ldots$

For those interested, the Scalar Multiple Rule and Power Rule are grouped with the Product Rule since they both follow directly from the Product Rule. For instance, using the Product Rule,

$$\lim_{x \to a} [c\, f(x)] = \lim_{x \to a} c \, \lim_{x \to a} f(x) = c \lim_{x \to a} f(x).$$

For powers, note that $[f(x)]^2 = f(x)\, f(x)$ so that

$$\lim_{x \to a} [f(x)]^2 = \lim_{x \to a} [f(x)\, f(x)] = \lim_{x \to a} f(x) \, \lim_{x \to a} f(x) = L\,L = L^2.$$

Once this is established, we can use the fact that $[f(x)]^3 = f(x)\ [f(x)]^2$ and the product rule again to get

$$\lim_{x \to a} [f(x)]^3 = \lim_{x \to a} \left[f(x)\ [f(x)]^2\right] = \lim_{x \to a} f(x) \, \lim_{x \to a} [f(x)]^2 = L\,L^2 = L^3.$$

Continuing in this manner gives us the Power Rule.[8]

A note regarding the Rules for Radicals: since $\sqrt{N}$ is not real if $N < 0$, we have to be careful about limits involving even-indexed radicals (or exponents which indicate even-indexed radicals.) For example, consider $\lim\limits_{x \to 5} \sqrt{5-x}$. Since this is a 'two-sided' limit, we must consider both $x \to 5^-$ and $x \to 5^+$.

As $x \to 5^-$, the radicand, $(5-x) > 0$ so $\sqrt{5-x}$ is defined as a real number. More specifically, as $x \to 5^-$, the quantity $(5-x) \to 0^+$ so $\lim\limits_{x \to 5^-} \sqrt{5-x} = 0$. On the other hand, if $x \to 5^+$, the quantity $(5-x) < 0$, and $\sqrt{5-x}$ is no longer a real number. Therefore, $\lim\limits_{x \to 5^+} \sqrt{5-x}$, and, hence, $\lim\limits_{x \to 5} \sqrt{5-x}$ does not exist.

Note the Real Number Powers rule can be thought as a generalization of the Power Rule, Quotient Rule, and Rules for Radicals for the case $L > 0$. Recall that positive rational number exponents can be defined in terms of natural number powers and radicals as: $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m$. Negative exponents can be defined in terms of quotients: $x^{-\frac{m}{n}} = \frac{1}{x^{\frac{m}{n}}}$. For the Real Number Exponents rule, we are generalizing the exponents to any real number but keeping the stipulation that $L > 0$ to make sure the resulting answer is defined.[9]

We put the limit properties to good use in the following example.

[8]See Exercise 10 in Section 10.3 for more details.

[9]See the discussion of real number exponents in Section 4.2.

Example 6.1.3. Let $f(x) = \dfrac{x\sqrt{x+1}}{x^2+2x-4}$. Use Theorem 6.2 to find $\displaystyle\lim_{x\to 3} f(x)$.

Solution.

$$\begin{aligned}
\lim_{x\to 3} \frac{x\sqrt{x+1}}{x^2+2x-4} &= \frac{\lim_{x\to 3} x\sqrt{x+1}}{\lim_{x\to 3}\left(x^2+2x-4\right)} && \text{Quotient Rule} \\
&= \frac{\left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3}\sqrt{x+1}\right)}{\lim_{x\to 3}\left(x^2\right) + \lim_{x\to 3}(2x) - \lim_{x\to 3} 4} && \frac{\text{Product Rule}}{\text{Sum and Difference Rule}} \\
&= \frac{3\sqrt{\lim_{x\to 3}(x+1)}}{\left(\lim_{x\to 3} x\right)^2 + 2\lim_{x\to 3} x - 4} && \frac{\text{Identity and Radical Rules}}{\text{Power, Constant Multiple, and Constant Rules}} \\
&= \frac{3\sqrt{\lim_{x\to 3} x + \lim_{x\to 3} 1}}{(3)^2+2(3)-4} && \frac{\text{Sum Rule}}{\text{Identity Rule}} \\
&= \frac{3\sqrt{3+1}}{9+6-4} = \frac{6}{11} && \text{Identity and Constant Rules, Simplify} \quad \Box
\end{aligned}$$

It is worth noting that we could have arrived at the same (correct) answer to Example 6.1.3 by evaluating $f(3)$: $f(3) = \frac{3\sqrt{3+1}}{(3)^2+6-4} = \frac{6}{11}$. That's really the power of Theorem 6.2. Under 'nice' circumstances,[10] Theorem 6.2 allows us to compute limits using direct substitution. Functions with this property have a familiar name.

> **Definition 6.4.** A function f is said to be **continuous** at an input $x = a$ if $\displaystyle\lim_{x\to a} f(x) = f(a)$.
>
> Said differently, a function f is said to be continuous at a real number a if what we **get**, $f(a)$, exactly what we **expect to get**, $\displaystyle\lim_{x\to a} f(x)$.

This is not the first time we've mentioned this property of functions. Indeed, we've discussed continuity albeit in graphical terms throughout Chapters 1 through 4. In those chapters, we described continuous functions as those whose graphs are connected meaning they have 'no holes or breaks' in them. It is a great exercise to compare the description given in Definition 6.4 to the graphical description to see how those two ideas mesh.

In a standard Calculus course, you'll explore properties of continuous functions more extensively. For our purposes here, polynomial, and, more generally, rational functions are continuous **on their domains**, as well as the functions we encountered in Chapter 4. Indeed, so long as we avoid the usual domain pitfalls, combining continuous functions via the standard four operation function arithmetic or using function composition results in a continuous function. This means in order to **evaluate limits** of these functions, we may use Definition 6.4 and simply **evaluate the function** at the corresponding value.

[10] primarily those in which we're not dealing with piecewise-defined functions or dividing by 0 . . .

Example 6.1.4. Let $f(x) = \begin{cases} 2x-1 & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$.

1. Is f is continuous at $x = 2$? Explain.

2. Find a constant 'm' so that $g(x) = \begin{cases} mx-1 & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$ is continuous at $x = 2$.

Solution.

1. To determine if f is continuous at $x = 2$, we need to check to see if $\displaystyle\lim_{x\to 2} f(x) = f(2)$. We note that f is defined at $x = 2$ and that $f(2) = (2)^2 = 4$ so we set about determining $\displaystyle\lim_{x\to 2} f(x)$.

   Since f is a piecewise-defined function which has different formulas on either side of 2, we need to check $\displaystyle\lim_{x\to 2} f(x)$ from both directions. To find $\displaystyle\lim_{x\to 2^-} f(x)$, we note that as $x \to 2^-$, $x < 2$ so $f(x) = 2x - 1$. Hence, $\displaystyle\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x-1) = 2(2) - 1 = 3$, the last step coming from the fact that for $x < 2$, $f(x) = 2x - 1$ is a linear function (a polynomial) and is continuous.

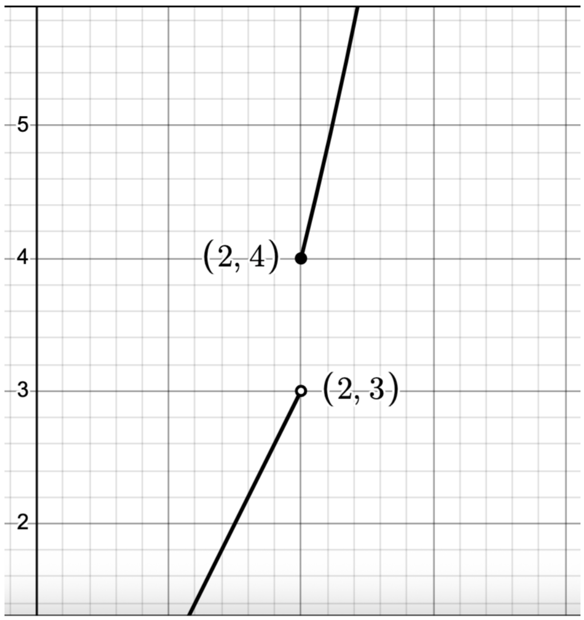
   Now on to $\displaystyle\lim_{x\to 2^+} f(x)$. Here, $x \to 2^+$, so $x > 2$ and $f(x) = x^2$. Hence, $\displaystyle\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = (2)^2 = 4$, the last step courtesy of the fact that for $x > 2$, $f(x) = x^2$ is a quadratic function (a polynomial) and is continuous.

   Since $\displaystyle\lim_{x\to 2^-} f(x) = 3$ and $\displaystyle\lim_{x\to 2^+} f(x) = 4$, we have that $\displaystyle\lim_{x\to 2} f(x)$ does not exist per Theorem 6.1. Hence, f is not continuous. If we graph f near $x = 2$ using desmos, we can see the vertical gap or 'jump' occurring at $x = 2$.

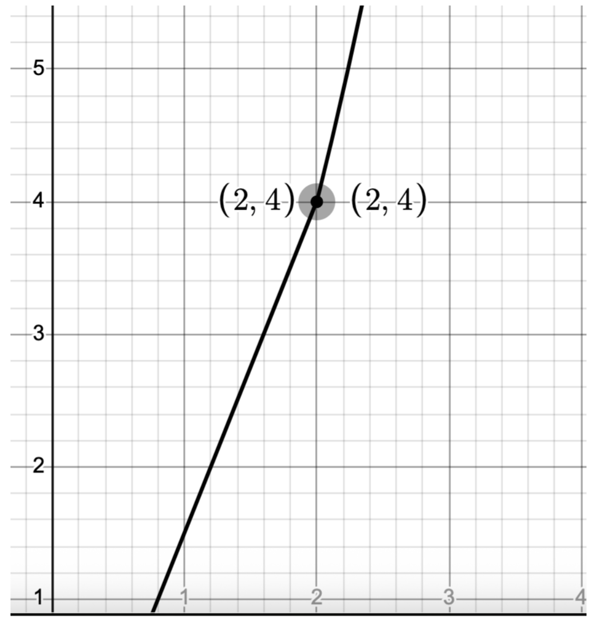
2. In this problem,[11] we're given a parameter 'm' to help us adjust the left hand side of the graph to meet the right hand side at $x = 2$. If $x < 2$, $g(x) = mx - 1$ so $\displaystyle\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} (mx-1) = 2m - 1$.

   To ensure the limit exists, we need $\displaystyle\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x)$. Since $g(x) = f(x) = x^2$ for $x \geq 2$, we know $\displaystyle\lim_{x\to 2^+} g(x) = 4$. Solving $2m - 1 = 4$, we get $m = \frac{5}{2} = 2.5$. Sure enough, $\displaystyle\lim_{x\to 2^-} (2.5x - 1) = 5 - 1 = 4$. Since $g(2) = 4$, we have $\displaystyle\lim_{x\to 2} g(x) = g(2)$, so g is continuous at $x = 2$.

[11] For a graphically interactive take on this problem, check out this desmos worksheet.



$y = f(x)$ near $x = 2$



$y = g(x)$ near $x = 2$

□

It is worth noting that despite each 'piece' of the piecewise-defined function f in Example 6.1.4 being continuous, the pieces don't match up at $x = 2$ causing what is called a **discontinuity**. A discontinuity is a place where a function is **not** continuous. The particular variety of discontinuity appearing here is usually called a 'jump' discontinuity - a type of discontinuity belonging to a larger class of 'non-removable' or 'essential' discontinuities. We'll point out other types of discontinuities as we encounter them.[12]

We close this section with an example that ties (most of) the fundamental concepts of limits and their calculations together.

Example 6.1.5. Let $f(x) = \frac{x^2-2x-3}{x^2-1}$. Find $\displaystyle\lim_{x\to-1} f(x)$ analytically[13] and interpret graphically.

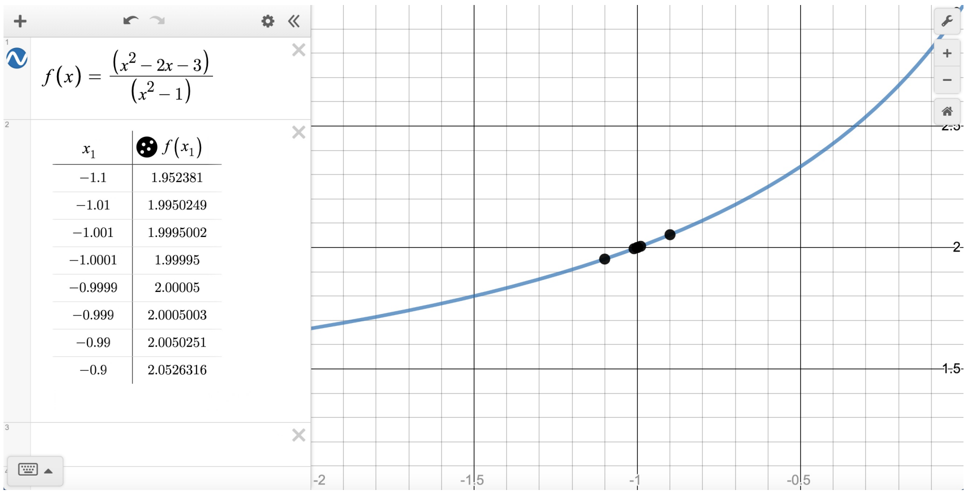
Solution. Since f is a rational function, and rational functions are continuous on their domains, it's worth checking to see what happens when we attempt to find $f(-1)$. We get $f(-1) = \frac{(-1)^2-2(-1)-3}{(-1)^2-1} = \frac{0}{0}$, which is undefined because of the '0' in the denominator. However, the '0' in the numerator signals to us[14] there are common factors of $(x-(-1)) = (x+1)$ which will cancel and potentially help us determine the indeterminate form. To that end, we simplify: $f(x) = \frac{x^2-2x-3}{x^2-1} = \frac{(x-3)(x+1)}{(x-1)(x+1)} = \frac{(x-3)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = \frac{x-3}{x-1}, \quad x \neq -1$.

That is, for all real numbers **except** $x = -1$, $\frac{x^2-2x-3}{x^2-1} = \frac{x-3}{x-1}$. Since $\displaystyle\lim_{x\to-1} f(x)$ is concerned only with what's happening **near** $x = -1$, but not with what's happening **at** $x = -1$, it seems reasonable to suggest that $\displaystyle\lim_{x\to-1} f(x) = \lim_{x\to-1} \frac{x^2-2x-3}{x^2-1} = \lim_{x\to-1} \frac{x-3}{x-1}$.

[12]Probably in the footnotes because, after all, this is (supposedly) a <u>pre</u>calculus book, not a Calculus book ...
[13]i.e, using properties of limits
[14]courtesy of the Factor Theorem (Theorem 2.8) ...

Note that $x = -1$ is in the domain of the function $g(x) = \frac{x-3}{x-1}$, hence g **is** continuous at $x = -1$. This means $\lim\limits_{x\to-1} g(x) = g(-1)$, that is, $\lim\limits_{x\to-1} \frac{x-3}{x-1} = \frac{-1-3}{-1-1} = 2$.

Putting all of this work together, we get $\lim\limits_{x\to-1} f(x) = \lim\limits_{x\to-1} \frac{x-3}{x-1} = \frac{-1-3}{-1-1} = 2$. Graphically, this means there is a hole in the graph of f at the location $(-1, 2)$.[15] The table and graph below confirm our answer.[16]



$y = f(x)$ near $x = -1$

□

The above reasoning is sound and is true in general. We'll be getting a lot of use out of the following:

Theorem 6.3. If f and g are two functions which agree on an open interval containing $x = a$, except possibly at $x = a$, then if $\lim\limits_{x\to a} g(x)$ exists, $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$.

[15]Since $f(-1)$ is undefined but $\lim\limits_{x\to-1} f(x)$ exists means the discontinuity here is classified as 'removable.' We can 'remove' the discontinuity by 'patching' the 'hole' by defining $f(-1) = 2$. We do such repairs in Calculus.

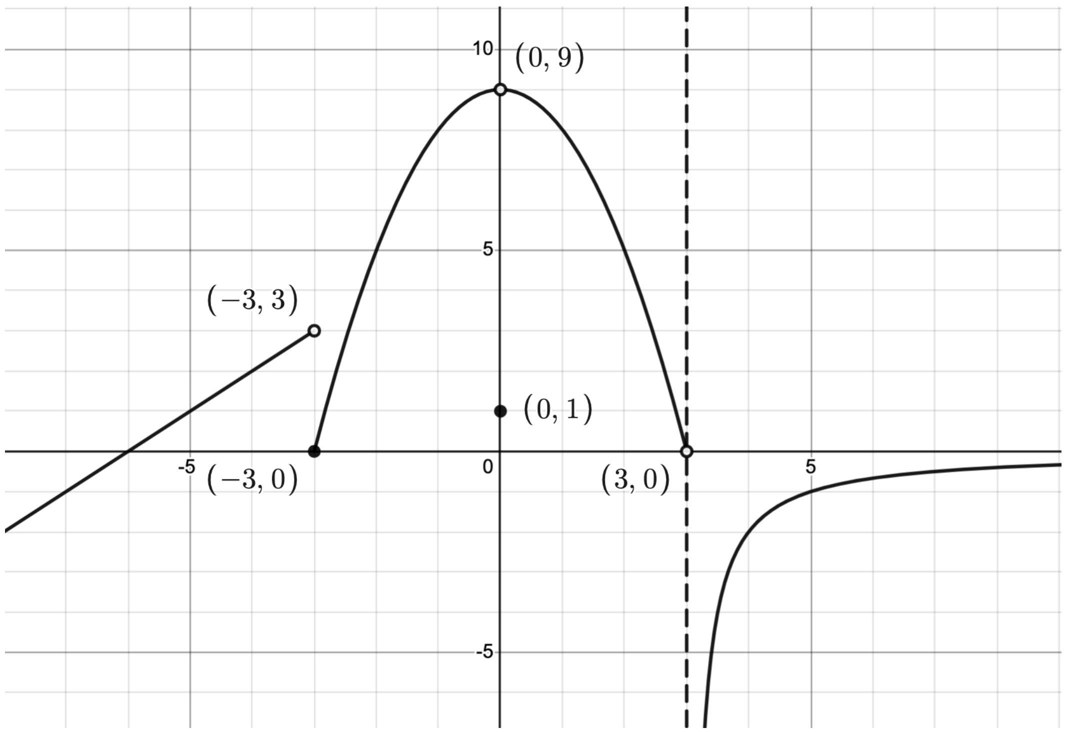
[16]Note that depending on which graphing utility is used, the 'hole' at $(-1, 2)$ may or may not be immediately visible on the graph.

6.1.3 Exercises

1. Consider the complete graph of the function f below. Use the graph to find the indicated values.

   If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.

   NOTE: The graph has a vertical asymptote $x = 3$.



   - $\lim\limits_{x \to -3^-} f(x)$
   - $\lim\limits_{x \to -3^+} f(x)$
   - $\lim\limits_{x \to -3} f(x)$
   - $f(-3)$
   - $\lim\limits_{x \to 0} f(x)$
   - $f(0)$
   - $\lim\limits_{x \to 3^-} f(x)$
   - $\lim\limits_{x \to 3^+} f(x)$

2. (a) Explain why if $\lim\limits_{x \to a} f(x)$ exist, then so do $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$.

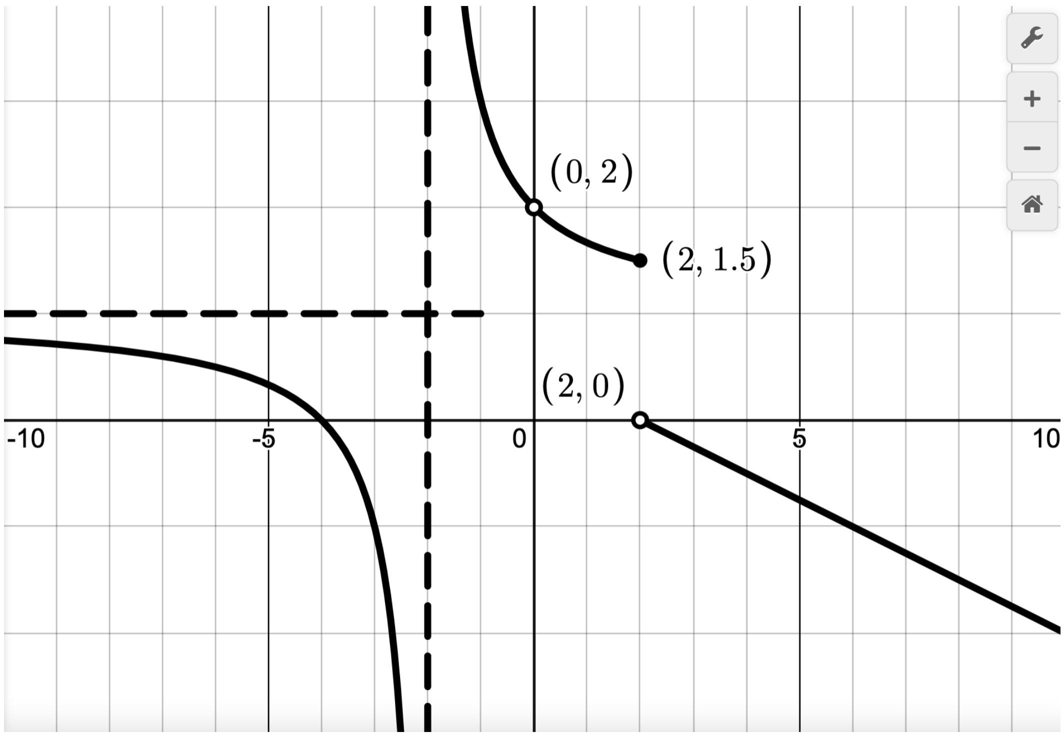
   (b) Find an instance[17] where $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist but $\lim\limits_{x \to a} f(x)$ does not.

[17] There are a few examples to be found in Example 6.1.1 . . .

3. Consider the complete graph of the function g below. Use the graph to find the indicated values.

   If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.

   NOTE: The graph has a vertical asymptote $x = -2$ and a horizontal asymptote $y = 1$.



- $\lim\limits_{x \to -\infty} g(x)$
- $\lim\limits_{x \to -2^-} g(x)$
- $\lim\limits_{x \to -2^+} g(x)$
- $\lim\limits_{x \to \infty} g(x)$
- $\lim\limits_{x \to 0} g(x)$
- $\lim\limits_{x \to 2^-} g(x)$
- $g(2)$
- $\lim\limits_{x \to 2^+} g(x)$

For Exercises 4 - 9, find the limit analytically using Exercise 6.1.5 as a guide. If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.

4. $\lim\limits_{x \to 2} \dfrac{2x^2 + x - 3}{x^2 - 1}$

5. $\lim\limits_{x \to 1} \dfrac{2x^2 + x - 3}{x^2 - 1}$

6. $\lim\limits_{x \to -1} \dfrac{2x^2 + x - 3}{x^2 - 1}$

7. $\displaystyle\lim_{x \to -1^+} \frac{2x^2 + x - 3}{x^2 - 1}$

8. $\displaystyle\lim_{x \to 3} \frac{x^2 - 3x}{x^2 - x - 6}$

9. $\displaystyle\lim_{x \to 3} \frac{x^2 - 6x}{x^2 - 6x + 9}$

For Exercises 10 - 11, use the piecewise definition of absolute value, Definition 1.9 to help you find the limit analytically. If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.

10. $\displaystyle\lim_{x \to 3^-} \frac{|3x - x^2|}{x - 3}$

11. $\displaystyle\lim_{x \to 2^+} \frac{|6 - 3x|}{x^2 - 4x + 4}$

For Exercises 12 - 14, simplify the complex fraction in order to help you find the limit analytically.[18] If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.

12. $\displaystyle\lim_{x \to 1} \frac{\frac{x}{x-2} + 1}{x - 1}$

13. $\displaystyle\lim_{x \to 2} \frac{\frac{2x}{x+2} - 1}{x - 2}$

14. $\displaystyle\lim_{h \to 0} \frac{\frac{1}{2(x+h) - 1} - \frac{1}{2x - 1}}{h}$

For Exercises 15 - 17, rationalize the numerator of the fraction in order to help you find the limit analytically.[19] If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.

15. $\displaystyle\lim_{x \to -1} \frac{\sqrt{x+5} - 2}{x + 1}$

16. $\displaystyle\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}$

17. $\displaystyle\lim_{h \to 0} \frac{\sqrt{2x + 2h - 1} - \sqrt{2x - 1}}{h}$

In Exercises 18 - 23, find the limit analytically. Use the symbols '∞' and '$-\infty$' as appropriate.

18. $\displaystyle\lim_{x \to \infty} \frac{3x - 4}{2x + 1}$

19. $\displaystyle\lim_{x \to -\infty} \frac{1 - 2x}{x - 5}$

20. $\displaystyle\lim_{x \to -\infty} \frac{2x - 1}{x^2 + 4}$

21. $\displaystyle\lim_{x \to \infty} \frac{\sqrt{4x^2 + x - 1}}{1 - x}$

22. $\displaystyle\lim_{x \to -\infty} \frac{\sqrt{4x^2 + x - 1}}{1 - x}$

23. $\displaystyle\lim_{x \to \infty} \frac{x^2 + 2x + 3}{4 - x}$.

24. Let $f(x) = \dfrac{x}{\lfloor x \rfloor}$ where '$\lfloor x \rfloor$' is the greatest integer (or floor) function.[20]

    Fill in the blanks below to help you analyze $\displaystyle\lim_{x \to 0} f(x)$.

    (a) If $-1 < x < 0$, then $\lfloor x \rfloor =$ ____________. So we can rewrite $f(x) = \dfrac{x}{\lfloor x \rfloor} =$ ____________.

    (b) Using part (a), we can find $\displaystyle\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}$ ____________ $=$ ____________.

    (c) If $0 < x < 1$, then $\lfloor x \rfloor =$ ____________, hence $f(x) = \dfrac{x}{\lfloor x \rfloor}$ does not exist as $x \to 0^+$.

    (d) Putting parts (b) and (c) together, we have that $\displaystyle\lim_{x \to 0} f(x)$ ________________

    (e) Graph $f(x) = \dfrac{x}{\lfloor x \rfloor}$ on desmos near $x = 0$ to confirm your answers.

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[18] A review of Section A.12 may be in order.

[19] A review of Section A.13.1 may be in order.

[20] See Example 1.2.2 in Section 1.2 for review, if needed.

25. Sketch the graph of a function which satisfies all of the following criteria:

- $\displaystyle\lim_{x\to-\infty} f(x) = \infty$
- $\displaystyle\lim_{x\to4^-} f(x) = 6$
- $\displaystyle\lim_{x\to4^+} f(x) = -\infty$
- $\displaystyle\lim_{x\to\infty} f(x) = 0$

26. Sketch the graph of a function f which satisfies all of the following criteria:

- $\displaystyle\lim_{x\to-\infty} f(x) = 2$
- $\displaystyle\lim_{x\to0^-} f(x) = \infty$
- $\displaystyle\lim_{x\to0^+} f(x) = -\infty$
- $\displaystyle\lim_{x\to2^-} f(x) = 3$
- $\displaystyle\lim_{x\to2^+} f(x) = 0$
- $\displaystyle\lim_{x\to\infty} f(x) = -\infty$

27. A function is said to be **continuous from the left** at $x = a$ if $\displaystyle\lim_{x\to a^-} f(x) = f(a)$. Likewise, a function is said to be **continuous from the right** at $x = a$ if $\displaystyle\lim_{x\to a^+} f(x) = f(a)$.

   (a) Explain why $r(x) = \sqrt{5-x}$ is not continuous at $x = 5$. Is r continuous from the left at $x = 5$? From the right? Explain.

   (b) Explain why the floor function[21] $F(x) = \lfloor x \rfloor$ is not continuous at $x = 117$. Is F continuous from the left at $x = 117$? From the right? Explain.

   (c) If a function f is continuous at $x = a$, explain why f is continuous from both the left and the right at $x = a$. Is the converse true? That is, if f is continuous from both the left and the right at $x = a$, is f continuous at $x = a$?

   (d) Compare and contrast your answers in this exercise to those in Exercise 2.

28. Consider the table of values below:

x	$f(x)$
-0.001	1.9
-0.0001	1.99
-0.00001	1.999
-0.000001	1.9999
0.000001	-10000
0.00001	-1000
0.0001	-100
0.001	-10

It turns out that $\displaystyle\lim_{x\to0} f(x) = 117$. How is this possible assuming the data in the table is correct?

[21] See Example 1.2.2 in Section 1.2 for review, if needed.

29. In this exercise, we use Definition 6.3 to prove $\lim\limits_{x \to \infty} x^3 = \infty$ and $\lim\limits_{x \to -\infty} x^3 = -\infty$:

   (a) Solve the following inequalities:

   - $x^3 > 1000$
   - $x^3 > 100000$
   - $x^3 > 10^{99}$
   - $x^3 > N$

   (b) Show that for $N > 0$, if $x > \sqrt[3]{N}$, then $x^3 > N$. Write a sentence (or two!) which uses your work and Definition 6.3 to prove $\lim\limits_{x \to \infty} x^3 = \infty$.

   (c) Repeat a similar argument to prove $\lim\limits_{x \to -\infty} x^3 = -\infty$

6.1.4 Answers

1. • $\lim\limits_{x\to -3^-} f(x) = 3$ • $\lim\limits_{x\to -3^+} f(x) = 0$ • $\lim\limits_{x\to -3} f(x)$ d.n.e. • $f(-3) = 0$

   • $\lim\limits_{x\to 0} f(x) = 9$ • $f(0)$ • $\lim\limits_{x\to 3^-} f(x) = 0$ • $\lim\limits_{x\to 3^+} f(x) = -\infty$

2. (a) If $\lim\limits_{x\to a} f(x)$ exists, say $\lim\limits_{x\to a} f(x) = L$ then the $f(x)$ values approach L as $x \to a$ from both directions. Hence both one-sided limits exist. In particular, $\lim\limits_{x\to a^-} f(x) = L$ and $\lim\limits_{x\to a^+} f(x) = L$.

   (b) In Example 6.1.1, $\lim\limits_{x\to -1^-} f(x) = 0$ and $\lim\limits_{x\to -1^+} f(x) = 4$ both exist but $\lim\limits_{x\to -1} f(x)$ does not because the two one-sided limits are not equal.

3. • $\lim\limits_{x\to -\infty} g(x) = 1$ • $\lim\limits_{x\to -2^-} g(x) = -\infty$ • $\lim\limits_{x\to -2^+} g(x) = \infty$ • $\lim\limits_{x\to \infty} g(x) = -\infty$

   • $\lim\limits_{x\to 0} g(x) = 2$ • $\lim\limits_{x\to 2^-} g(x) = 1.5$ • $g(2) = 1.5$ • $\lim\limits_{x\to 2^+} g(x) = 0$

4. $\lim\limits_{x\to 2} \dfrac{2x^2 + x - 3}{x^2 - 1} = \dfrac{7}{3}$

5. $\lim\limits_{x\to 1} \dfrac{2x^2 + x - 3}{x^2 - 1} = \dfrac{5}{2}$

6. $\lim\limits_{x\to -1} \dfrac{2x^2 + x - 3}{x^2 - 1}$ does not exist

7. $\lim\limits_{x\to -1^+} \dfrac{2x^2 + x - 3}{x^2 - 1} = \infty$

8. $\lim\limits_{x\to 3} \dfrac{x^2 - 3x}{x^2 - x - 6} = \dfrac{3}{5}$

9. $\lim\limits_{x\to 3} \dfrac{x^2 - 6x}{x^2 - 6x + 9} = -\infty$

10. $\displaystyle\lim_{x \to 3^-} \frac{|3x - x^2|}{x - 3} = -3$

11. $\displaystyle\lim_{x \to 2} \frac{|6 - 3x|}{x^2 - 4x + 4} = \infty$

12. $\displaystyle\lim_{x \to 1} \frac{\frac{x}{x-2} + 1}{x - 1} = -2$

13. $\displaystyle\lim_{x \to 2} \frac{\frac{2x}{x+2} - 1}{x - 2} = \frac{1}{4}$

14. $\displaystyle\lim_{h \to 0} \frac{\frac{1}{2(x+h) - 1} - \frac{1}{2x - 1}}{h} = -\frac{2}{(2x-1)^2}$

15. $\displaystyle\lim_{x \to -1} \frac{\sqrt{x + 5} - 2}{x + 1} = \frac{1}{4}$

16. $\displaystyle\lim_{h \to 0} \frac{\sqrt{4 + h} - 2}{h} = \frac{1}{4}$

17. $\displaystyle\lim_{h \to 0} \frac{\sqrt{2x + 2h - 1} - \sqrt{2x - 1}}{h} = \frac{1}{\sqrt{2x - 1}}$

18. $\displaystyle\lim_{x \to \infty} \frac{3x - 4}{2x + 1} = \frac{3}{2}$

19. $\displaystyle\lim_{x \to -\infty} \frac{1 - 2x}{x - 5} = -2$

20. $\displaystyle\lim_{x \to -\infty} \frac{2x - 1}{x^2 + 4} = 0$

21. $\displaystyle\lim_{x \to \infty} \frac{\sqrt{4x^2 + x - 1}}{1 - x} = -2$

22. $\displaystyle\lim_{x \to -\infty} \frac{\sqrt{4x^2 + x - 1}}{1 - x} = 2$

23. $\lim_{x\to\infty} \dfrac{x^2+2x+3}{4-x} = -\infty$

24. (a) If $-1 < x < 0$, then $\lfloor x \rfloor = -1$. So we can rewrite $f(x) = \dfrac{x}{\lfloor x \rfloor} = -x$.

    (b) Using part (a), we can find $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x = 0$.

    (c) If $0 < x < 1$, then $\lfloor x \rfloor = 0$, hence $f(x) = \dfrac{x}{\lfloor x \rfloor}$ does not exist as $x \to 0^+$.

    (d) Putting parts (b) and (c) together, we have that $\lim_{x\to 0} f(x)$ does not exist.

25. Answers vary.

26. Answers vary.

27. (a) The function $r(x) = \sqrt{5-x}$ is not continuous at $x = 5$ since r is undefined if $x > 5$, so $\lim_{x\to 5^+} r(x)$ does not exist. However, r is continuous from the left at $x = 5$ since $\lim_{x\to 5^-} r(x) = \lim_{x\to 5^-} \sqrt{5-x} = 0 = \sqrt{5-5} = r(5)$.

    (b) The function $F(x) = \lfloor x \rfloor$ is not continuous at $x = 117$ since $\lim_{x\to 117^-} \lfloor x \rfloor = 116$ but $\lim_{x\to 117^+} \lfloor x \rfloor = 117$. However, F is continuous from the right at $x = 117$ since $\lim_{x\to 117^+} \lfloor x \rfloor = 117 = \lfloor 117 \rfloor = F(117)$.

    (c) If f is continuous at $x = a$, then $\lim_{x\to a} f(x) = f(a)$. This means $\lim_{x\to a^-} f(x) = f(a)$ and $\lim_{x\to a^+} f(x) = f(a)$, so f is continuous from both directions at $x = a$. The converse is also true since ff $\lim_{x\to a^-} f(x) = f(a)$ and $\lim_{x\to a^+} f(x) = f(a)$, then $\lim_{x\to a} f(x) = f(a)$.

    (d) The difference between the scenario here and that in Exercise 2 is that here, we know what each of the one-sided limits are: $f(a)$. They can't be different numbers like they could be in Example 6.1.1

28. We are told nothing of the function on the interval $(-0.000001, 0.000001)$ so the function has plenty of opportunities to approach 117.

29. (a)
    - $x^3 > 1000$ for $x > 10$.
    - $x^3 > 1000000$ for $x > 100$.
    - $x^3 > 10^{99}$ for $x > 10^{33}$.
    - $x^3 > N$ for $x > \sqrt[3]{N}$.

    (b) If $x > \sqrt[3]{N}$, then $x^3 > \left(\sqrt[3]{N}\right)^3 = N$. Per Definition 6.3, given $N > 0$, choose $M = \sqrt[3]{N}$. If $x > M$, then $x^3 > M^3 = N$. Hence, $\lim_{x\to\infty} x^3 = \infty$.

    (c) Given $N < 0$, choose $M = \sqrt[3]{N}$. If $x < M$, then $x^3 < M^3 = \left(\sqrt[3]{N}\right)^3 = N$. Hence, $\lim_{x\to-\infty} x^3 = -\infty$.

6.2 Introduction to Derivatives

6.2.1 Average and Instantaneous Velocity, Revisited

We begin this section by revisiting (again!) the notion of **average velocity** - a concept we first encountered in Example 1.2.8 in Section 1.2 and later revisited in Example 3.1.3 in Section 3.1.

In this scenario, the position function[1] $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. The average rate of change of s over an interval is the **average velocity** of the rocket over that interval. The average velocity provides two pieces of information: the average speed of the rocket along with the rocket's direction. We formalized the average velocity in Definition 3.5 in Section 3.1:

> **Definition.** Let $s(t)$ be the position of an object at time t and t_0 a fixed time in the domain of s. The **average velocity** between time t and time t_0 for $t \neq t_0$ is given by
>
> $$\overline{v}(t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t) - s(t_0)}{t - t_0}.$$

If we define the change in time, $\Delta t = t - t_0$, we get $t = t_0 + \Delta t$ which gives:

$$\overline{v}(\Delta t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}, \quad \Delta t \neq 0.$$

The above formula measures the average velocity between time t_0 and time $t_0 + \Delta t$ as a function of Δt.

We now revisit Example 3.1.3 in Section 3.1 using this new formulation.[2]

Example 6.2.1. Let $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ give the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff.

1. Find, and simplify: $\overline{v}(\Delta t) = \dfrac{s(15 + \Delta t) - s(15)}{\Delta t}$, for $\Delta t \neq 0$.
2. Find and interpret $\overline{v}(-1)$.
3. Find and interpret $\displaystyle\lim_{\Delta t \to 0} \overline{v}(\Delta t)$.
4. Graph $y = \overline{v}(\Delta t)$ and interpret your answer to part 3 graphically.

Solution.

1. To find $\overline{v}(\Delta t)$, we first find $s(15 + \Delta t)$:

$$\begin{aligned} s(15 + \Delta t) &= -5(15 + \Delta t)^2 + 100(15 + \Delta t) \\ &= -5(225 + 30\Delta t + (\Delta t)^2) + 1500 + 100\Delta t \\ &= -5(\Delta t)^2 - 50\Delta t + 375 \end{aligned}$$

[1] So named because $s(t)$ provides information about **where** the rocket is at time t.
[2] Along with some of the new tools we learned in Section 6.1.

Since $s(15) = -5(15)^2 + 100(15) = 375$, we get:

$$\begin{aligned}
\overline{v}(\Delta t) &= \frac{s(15+\Delta t) - s(15)}{\Delta t} \\
&= \frac{(-5(\Delta t)^2 - 50\Delta t + 375) - 375}{\Delta t} \\
&= \frac{\Delta t(-5\Delta t - 50)}{\Delta t} \\
&= \frac{\cancel{\Delta t}(-5\Delta t - 50)}{\cancel{\Delta t}} \\
&= -5\Delta t - 50 \qquad\qquad \Delta t \neq 0
\end{aligned}$$

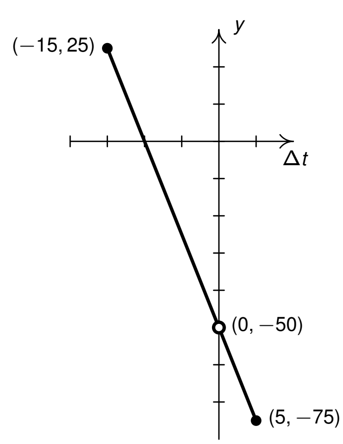
In addition to $\Delta t \neq 0$, the domain of s is restricted to $0 \leq t \leq 20$. Hence, we require $0 \leq 15+\Delta t \leq 20$ or $-15 \leq \Delta t \leq 5$. Our final answer is $\overline{v}(\Delta t) = -5\Delta t - 50$, for $\Delta t \in [-15, 0) \cup (0, 5]$.

2. We find $\overline{v}(-1) = -5(-1) - 50 = -45$. This means the average velocity over between time $t = 15 + (-1) = 14$ seconds and $t = 15$ seconds is -45 feet per second. This indicates the rocket is, on average, heading *downwards* at a rate of 45 feet per second.

3. Since $\overline{v}(\Delta t) = -5\Delta t - 50$, for all values of Δt near $\Delta t = 0$ (excluding $\Delta t = 0$), Theorem 6.3 applies. We get $\displaystyle\lim_{\Delta t \to 0} \overline{v}(\Delta t) = \lim_{\Delta t \to 0} -5\Delta t - 50 = -5(0) - 50 = -50$, where we have used the fact that the function $f(\Delta t) = -5\Delta t - 50$ is continuous to evaluate the limit.

   Recall from Example 3.1.3 that the limit value here, -50 is the so-called *instantaneous velocity* of the rocket *at* $t = 15$ seconds. That is, 15 seconds after lift-off, the rocket is heading back towards the surface of the moon at a rate of 50 feet per second.

4. Since the domain of $\overline{v}$ is $[-15, 0) \cup (0, 5]$, the graph of $y = \overline{v}(\Delta t) = -5\Delta t - 50$ is a line **segment** from $(-15, 25)$ to $(5, -75)$ with a hole at $(0, -50)$.



$y = \overline{v}(\Delta t)$

□

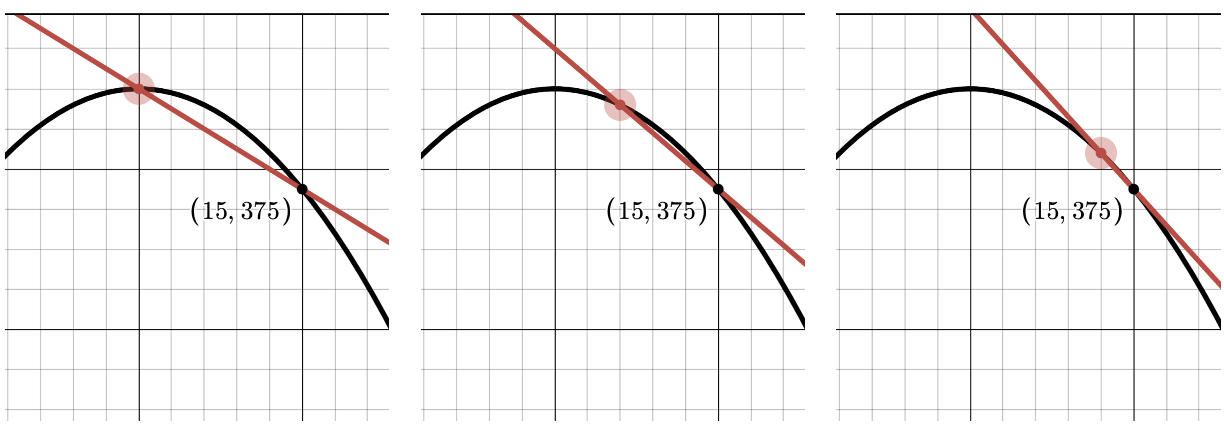
The reader is invited to compare Example 3.1.3 in Section 3.1 with Exercise 6.2.1 above. We obtain the exact same information because we are asking the *exact same* questions - they are just framed differently. We now take the time to formally define **instantaneous velocity**:

Definition 6.5. Let $s(t)$ be the position of an object at time t and t_0 a fixed time in the domain of s. The **instantaneous velocity** at t_0 is given by:

$$v(t_0) = \lim_{\Delta t \to 0} \frac{\Delta[s(t)]}{\Delta t} = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}, \quad \text{provided this limit exists.}$$

Based on our work in Examples 3.1.3 and 6.2.1, we have $v(15) = -50$. In both of those examples, we've seen what $v(15)$ means on the graph of $\overline{v}$, but there is a more important interpretation when we analyze the graph of s. Recall that the average velocity, and, more generally, average rates of change can be visualized as slopes of **secant lines**.[3]

Below is a sequence of secant lines along with the graph of $y = s(t)$. In each case, the secant line is graphed between $(15, s(15)) = (15, 375)$ and another point on the graph.[4] As the points on the parabola approach $(15, 375)$ the secant lines approach what is known as the **tangent line**.



To find the equation of the tangent line in this case, which we'll call $L(t)$, we refer to the point-slope form of a line, Equation 1.1:

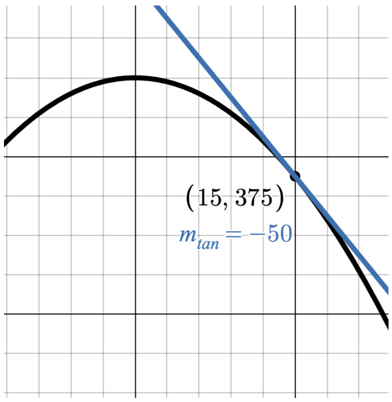
$$L(t) = s(t_0) + m(t - t_0) = s(15) + v(15)(t - 15) = 375 - 50(t - 15) = -50t + 1125.$$

The tangent line can best be thought of as 'the best linear approximation' to the graph of $y = s(t)$ at $(15, 375)$. That is, if we zoom in near $(15, 375)$, the graph of $y = s(t)$ and this tangent line become indistinguishable. This geometric property of $y = s(t)$ is called **local linearity**[5] and is foundational to the analysis of functions. Below we graph $y = s(t) = -5t^2 + 100t$ along with $y = L(t) = -50t + 1125$ and observe the local linearity near $(15, 375)$.

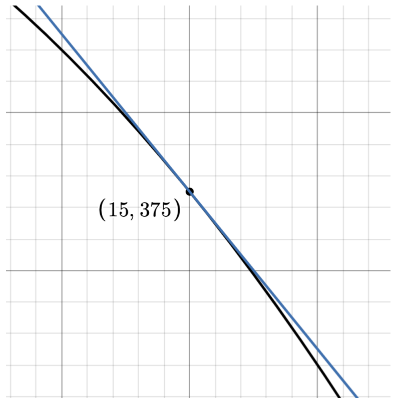
[3] See Section 1.2.4 for a refresher, if needed.

[4] For an interactive demonstration of this process, check out this desmos worksheet.

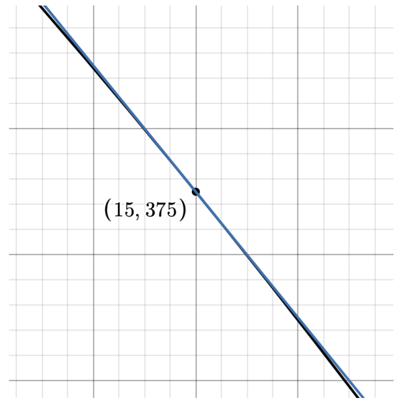
[5] a.k.a. **differentiability**



The tangent line at $(15, 375)$.



Zooming in near $(15, 375)$.



Zooming in closer to $(15, 375)$.

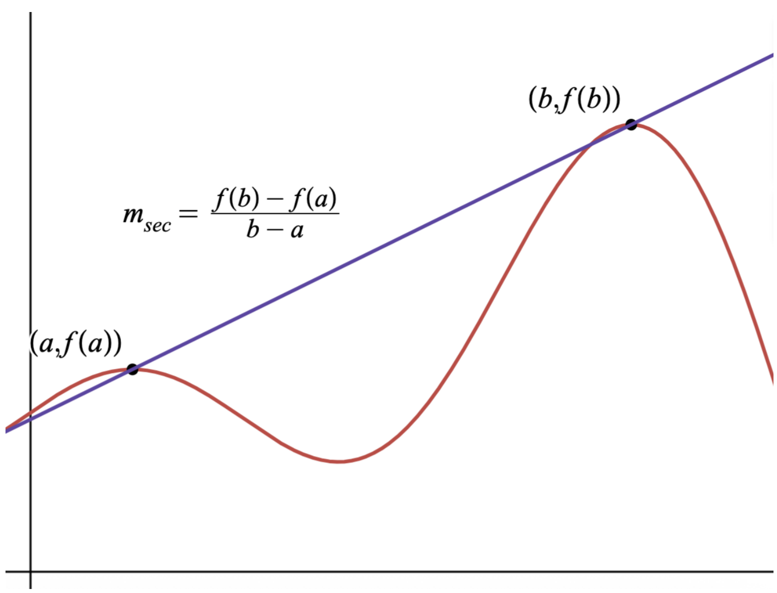
Our next step is to generalize these notions to all functions.

6.2.2 Difference Quotients and Derivatives

Recall in Section 1.2.4 the concept of the average rate of change of a function over the interval $[a, b]$ is the slope between the two points $(a, f(a))$ and $(b, f(b))$ and is given by

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, the average rate of change is the slope of the so-called **secant line** which 'cuts' through the graph of $y = f(x)$ at the points $(a, f(a))$ and $(b, f(b))$:



Consider a function f defined over an interval containing x and $x+h$ where $h \neq 0$. The average rate of change of f over the interval $[x, x+h]$ is thus given by the formula:[6]

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+h) - f(x)}{h}, \quad h \neq 0.$$

The above is an example of what is traditionally called the **difference quotient** or **Newton quotient** of f, since it is the *quotient* of two *differences*, namely $\Delta[f(x)]$ and Δx. Another formula for the difference quotient (as seen in Section 5.2.1) keeps with the notation Δx instead of h:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0.$$

It is important to understand that in this formulation of the difference quotient, the variables 'x' and 'Δx' are distinct - that is they do not combine as like terms.

Note that, regardless of which form the difference quotient takes, when h, Δx, or Δt is 0, the difference quotient returns the indeterminate form '$\frac{0}{0}$.' As we've seen with rational functions in Section 3.1, when this happens, we can use a limit to help us **determine** the **indeterminate** form.

In Section 6.2.1, taking the limit of average velocity as $\Delta t \to 0$ produced instantaneous velocity. More generally, taking the limit of the average rate of change as the denominator approaches 0 produces the **instantaneous rate of change** of the function at that point. The instantaneous rate of change of a function, called the **derivative** of the function, is defined below.

Definition 6.6. Given a function f defined on an open interval containing $x = a$, the **derivative** of f at a, denoted $f'(a)$, is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \quad \text{provided this limit exists.}$$

The number $f'(a)$ represents the **instantaneous rate of change** of f with respect to x at the input $x = a$. If $f'(a)$ exists, we say f is **differentiable** at $x = a$.

Using the language of derivatives, Examples 3.1.3 and 6.2.1 have us computing $v(15) = s'(15)$. Moreover, since the derivative is a rate of change, it's important to note that the associated units of $f'(a)$ are $\frac{\text{units of } f(x)}{\text{units of } x}$. This tracks with the units of $v(15) = s'(15)$ being $\frac{\text{feet}}{\text{second}}$, a velocity.

As in Section 6.2.1, $f'(a)$ represents the slope of the tangent line at the point $(a, f(a))$. We use this to formally define the **tangent line** below.

Definition 6.7. If f is differentiable at $x = a$, then $f'(a) = m_{\text{tan}}$, the slope of the **tangent linetangent line** to $y = f(x)$ at $(a, f(a))$. The equation of the tangent line is therefore: $y = f'(a)(x - a) + f(a)$.

We put these definitions to good use in the following example.

[6] assuming $h > 0$; otherwise, the interval is $[x+h, x]$. We get the same formula for the difference quotient either way.

Example 6.2.2. Let $f(x) = -x^2 + 3x - 1$.

1. Find the equation of the tangent line to $y = f(x)$ at $x = -2$. Check your answer graphically.

2. If f represents the temperature (in degrees Celsius) x hours after Noon on a particular day, interpret $f'(-2)$ in terms of time and temperature.

Solution.

1. We first find $m_{\tan} = f'(-2)$ using Definition 6.6 with $a = -2$: $f'(-2) = \displaystyle\lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$.

   First we find $f(-2+h)$ and are careful to apply the exponent in the expression $-(-2+h)^2$:

$$\begin{aligned} f(-2+h) &= -(-2+h)^2 + 3(-2+h) - 1 \\ &= -(4 - 4h + h^2) - 6 + 3h - 1 \\ &= -4 + 4h - h^2 - 6 + 3h - 1 \\ &= -h^2 + 7h - 11 \end{aligned}$$

   Next, we find $f(-2) = -(-2)^2 + 3(-2) - 1 = -11$, so the difference quotient is:

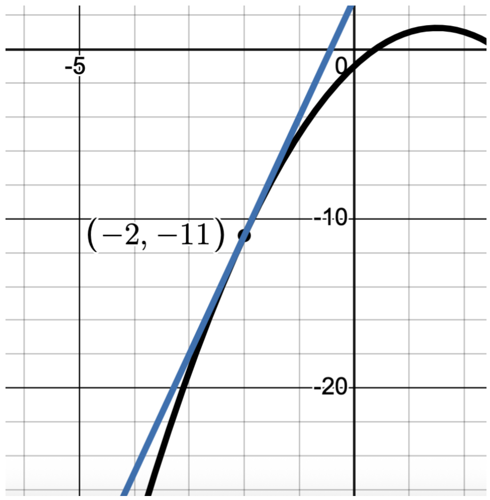
$$\begin{aligned} \frac{f(-2+h) - f(-2)}{h} &= \frac{(-h^2 + 7h - 11) - (-11)}{h} \\ &= \frac{-h^2 + 7h}{h} && \text{simplify} \\ &= \frac{h(-h+7)}{h} && \text{factor} \\ &= \frac{\cancel{h}(-h+7)}{\cancel{h}} && \text{cancel} \\ &= -h + 7 \end{aligned}$$

   Finally, we get $f'(-2) = \displaystyle\lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0} (-h+7) = -(0) + 7 = 7$.

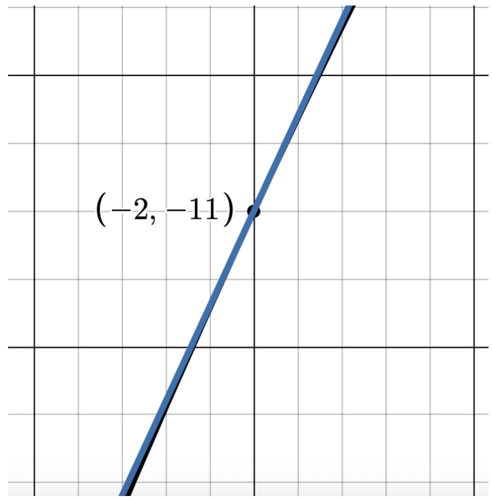
   Hence, the slope of the tangent line is $m_{\tan} = f'(-2) = 7$. The equation of the tangent line is:

$$\begin{aligned} y &= f'(-2)(x - (-2)) + f(-2) \\ &= 7(x+2) - 11 \\ &= 7x + 14 - 11 \\ y &= 7x + 3 \end{aligned}$$

   Graphing $y = 7x + 3$ and $y = f(x)$ near $(-2, -11)$ reveals the local linearity we would expect:



The tangent line at $(-2, -11)$.



Zooming in near $(-2, -11)$.

2. Since x represents the number of hours **after** Noon, $x = -2$ corresponds to 2 hours **before** Noon, or 10 AM. Since the units of $f(x)$ are degrees Celsius and the units of x are hours, the units of $f'(-2)$ are degrees Celsius per hour. Since $f'(-2)$ is positive, we know the slope is positive, so the temperature is increasing. Putting all this together, $f'(-2) = 7$ means that at 10 AM, the temperature is rising at a rate of 7 degrees Celsius per hour. □

What if we wanted to find the equation of the tangent line to the graph of the function in Example 6.2.2 at $x = 0$? $x = 1$? $x = 5$? We'd ostensibly need to run through difference quotients and limit calculations for each and every input value: $x = 0$, $x = 1$, and $x = 5$. Or we could do a single limit with a generic 'x', simplify the difference quotient and take the limit once, and substitute in particular values of x:

Definition 6.8. Given a function f defined on an open interval, the **derivative** of f, denoted $f'(x)$, is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{provided the limit exists.}$$

It is worth noting that if we set $h = \Delta x$, and consider the graph $y = f(x)$, we get:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta[f(x)]}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

which is why sometimes the derivative is denoted[7] $\frac{dy}{dx}$.

[7]This is the so-called Leibniz notation ...

Example 6.2.3. Let $f(x) = -x^2 + 3x - 1$.

1. Find an expression for $f'(x)$.
2. Find $f'(-2)$ using your answer to part 1 and compare that to what you obtained in Example 6.2.2.
3. Find the equation of the tangent line to the graph of $y = f(x)$ at $x = 0$. Check your answer graphically.
4. Solve $f'(x) = 0$ and interpret your answer graphically.

Solution.

1. To find $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ we first find $f(x+h)$:

$$\begin{array}{rcl} f(x+h) & = & -(x+h)^2 + 3(x+h) - 1 \\ & = & -(x^2 + 2xh + h^2) + 3x + 3h - 1 \\ & = & -x^2 - 2xh - h^2 + 3x + 3h - 1 \end{array}$$

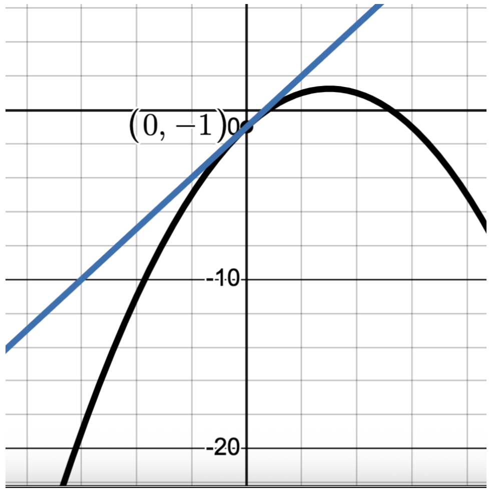
The difference quotient simplifies as follows:

$$\begin{array}{rcll} \dfrac{f(x+h) - f(x)}{h} & = & \dfrac{(-x^2 - 2xh - h^2 + 3x + 3h - 1) - (-x^2 + 3x - 1)}{h} & \\ & = & \dfrac{-x^2 - 2xh - h^2 + 3x + 3h - 1 + x^2 - 3x + 1}{h} & \\ & = & \dfrac{-2xh - h^2 + 3h}{h} & \text{simplify} \\ & = & \dfrac{h(-2x - h + 3)}{h} & \text{factor} \\ & = & \dfrac{\cancel{h}(-2x - h + 3)}{\cancel{h}} & \text{cancel} \\ & = & -2x - h + 3 & \end{array}$$

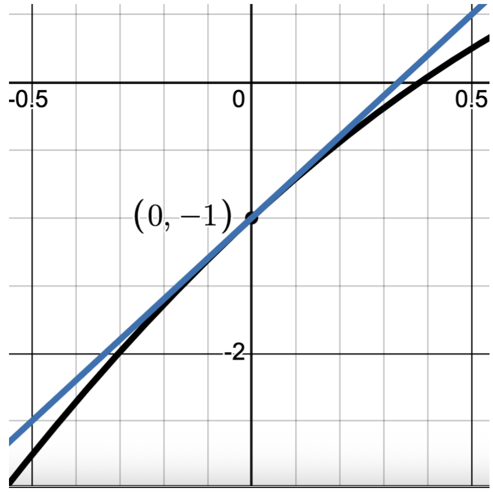
Our last step is to take the limit: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (-2x - h + 3)$. Notice here that we have two variables, x and h, in the limit. Of these two variables, we are taking the limit on the h: $h \to 0$. As far as h is concerned, x may as well be just another constant like the '3'. Hence, $f'(x) = \lim\limits_{h \to 0} (-2x - h + 3) = -2x - 0 + 3 = -2x + 3$.

2. Evaluating our formula for $f'(x)$ at $x = -2$ gives $f'(-2) = -2(-2) + 3 = 7$ which matches with what we obtained in Example 6.2.2.

3. The equation of the tangent line to the graph of $y = f(x)$ at $x = 0$ is $y = f'(0)(x - 0) + f(0)$. We have $f'(0) = 2(0) + 3 = 3$ and $f(0) = -(0)^2 + 3(0) - 1 = -1$. We get $y = 3(x - 0) + (-1)$ so $y = 3x - 1$. Our graph bears this out.

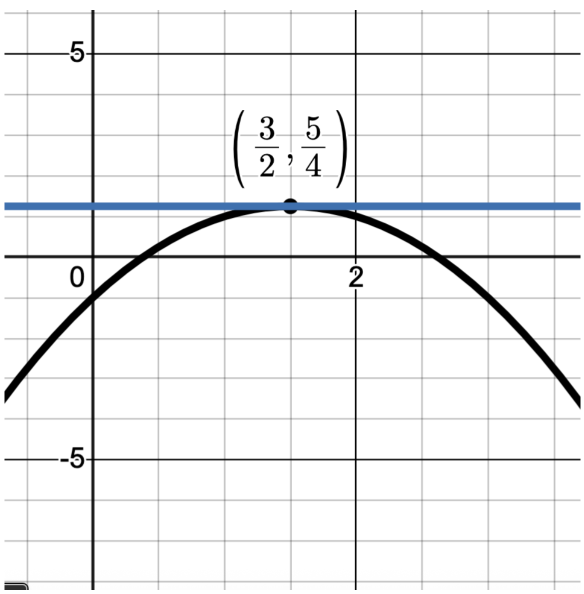


The tangent line at $(0, -1)$.

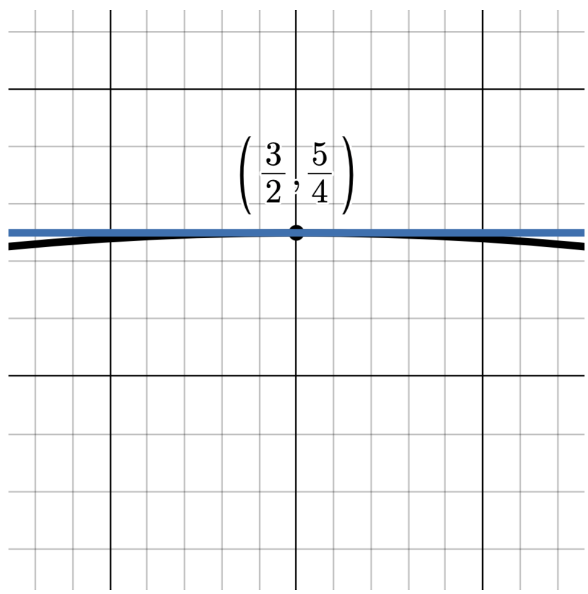


Zooming in near $(0, -1)$.

4. Solving $f'(x) = 0$ gives $-2x + 3 = 0$ so $x = \frac{3}{2}$. This means the slope of the tangent line at the point $\left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right)$ is 0, so the tangent line there is horizontal. We find $f\left(\frac{3}{2}\right) = -\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) - 1 = \ldots = \frac{5}{4}$. Hence, the tangent line at $\left(\frac{3}{2}, \frac{5}{4}\right)$ is $y = \frac{5}{4}$. Graphically, this checks out.



The tangent line at $\left(\frac{3}{2}, \frac{5}{4}\right)$.



Zooming in near $\left(\frac{3}{2}, \frac{5}{4}\right)$.

□

The astute reader will note that the graph of $f(x) = -x^2 + 3x - 1$ in Example 6.2.3 is a parabola and finding where $f'(x) = 0$ lead us right back to the vertex. Using a derivative to find the vertex may seem a bit excessive given that we've algebraically derived a handy 'vertex formula' in Section 1.4. However, as the functions we aim to analyze become more and more sophisticated, the tools we use to analyze them must also become more sophisticated. The derivative is one such tool that has a near universal application.[8]

Example 6.2.4.

1. For $f(x) = x^2 - x - 2$, find and simplify:
   (a) $f'(3)$
   (b) The equation of the tangent line to the graph $y = f(x)$ at $(3, f(3))$. Check your answer graphically.
   (c) $f'(x)$
2. For $g(x) = \dfrac{3}{2x+1}$, find and simplify:[9]
   (a) $g'(0)$
   (b) The equation of the tangent line to the graph $y = g(x)$ at $(0, g(0))$. Check your answer graphically.
   (c) $g'(x)$
3. $r(t) = \sqrt{t}$, find and simplify:[10]
   (a) $r'(9)$
   (b) The equation of the tangent line to the graph $y = r(t)$ at $(9, r(9))$. Check your answer graphically.
   (c) $r'(t)$

Solution.

1. (a) To find $f'(3) = \lim\limits_{h \to 0} \frac{f(3+h)-f(3)}{h}$ we first simplify $f(3+h)$:

$$\begin{array}{rcl} f(3+h) & = & (3+h)^2 - (3+h) - 2 \\ & = & 9 + 6h + h^2 - 3 - h - 2 \\ & = & 4 + 5h + h^2 \end{array}$$

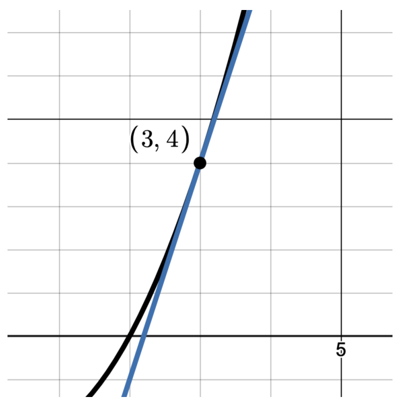
Since $f(3) = (3)^2 - 3 - 2 = 4$, we get for the difference quotient:

[8] As we'll see in Section **??**.
[9] A review of Section A.12 may be in order for this problem.
[10] A review of Section A.13.1 may be in order for this problem.

$$\begin{aligned}
\frac{f(3+h)-f(3)}{h} &= \frac{(4+5h+h^2)-4}{h} \\
&= \frac{5h+h^2}{h} \\
&= \frac{h(5+h)}{h} && \text{factor} \\
&= \frac{\cancel{h}(5+h)}{\cancel{h}} && \text{cancel} \\
&= 5+h
\end{aligned}$$

Hence, $f'(3) = \lim_{h \to 0} (5+h) = 5+0 = 5$.

(b) The equation of the tangent line at $x = 3$ is: $y = f'(3)(x-3) + f(3) = 5(x-3) + 4$, or $y = 5x - 11$. We check graphically below.



$y = f(x)$ and $y = 5x - 11$ near $(3, 4)$

(c) To find $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, we first find $f(x+h)$:

$$\begin{aligned}
f(x+h) &= (x+h)^2 - (x+h) - 2 \\
&= x^2 + 2xh + h^2 - x - h - 2.
\end{aligned}$$

So the difference quotient is

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\left(x^2+2xh+h^2-x-h-2\right)-\left(x^2-x-2\right)}{h} \\
&= \frac{x^2+2xh+h^2-x-h-2-x^2+x+2}{h} \\
&= \frac{2xh+h^2-h}{h}
\end{aligned}$$

$$\begin{aligned}
&= \frac{h\,(2x+h-1)}{h} && \text{factor} \\
&= \frac{\cancel{h}\,(2x+h-1)}{\cancel{h}} && \text{cancel} \\
&= 2x+h-1.
\end{aligned}$$

Hence, $f'(x) = \lim\limits_{h\to 0}(2x+h-1) = 2x+0-1$ so $f'(x) = 2x-1$. Note that using this formula, we get $f'(3) = 2(3)-1 = 5$ which checks our answer above.

2. (a) Next we find $g'(0) = \lim\limits_{h\to 0}\frac{g(0+h)-g(0)}{h} = \frac{g(h)-g(0)}{h}$.

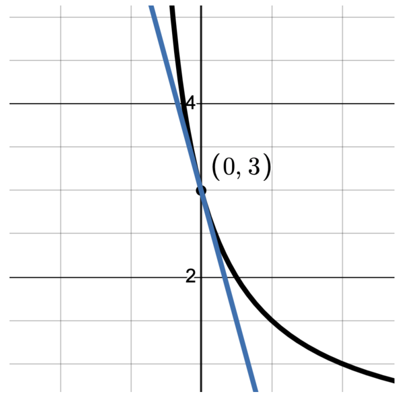
Since $g(h) = \frac{3}{2h+1}$ and $g(0) = \frac{3}{2(0)+1} = 3$, our difference quotient contains a complex fraction. Thinking ahead, we need to (eventually) be able to cancel the factor 'h' from the denominator $\frac{g(h)-g(0)}{h}$, so we begin by simplifying the complex fraction and see where that takes us:

$$\begin{aligned}
\frac{g(0+h)-g(0)}{h} &= \frac{\frac{3}{2h+1}-3}{h} \\
&= \frac{\frac{3}{2h+1}-3}{h}\cdot\frac{(2h+1)}{(2h+1)} \\
&= \frac{3-3(2h+1)}{h(2h+1)} \\
&= \frac{3-6h-3}{h(2h+1)} \\
&= \frac{-6h}{h(2h+1)} \\
&= \frac{-6\cancel{h}}{\cancel{h}(2h+1)} && \text{cancel} \\
&= \frac{-6}{2h+1}.
\end{aligned}$$

We are now ready to take the limit:

$$g'(0) = \lim_{h\to 0}\frac{-6}{2h+1} = \frac{-6}{2(0)+1} = -6.$$

(b) The equation of the tangent line when $x = 0$ is: $y = g'(0)(x-0)+g(0) = (-6)(x-0)+3$ or $y = -6x+3$, which checks graphically below.



$y = g(x)$ and $y = -6x + 3$ near $(0, 3)$

(c) To find $g'(x)$, we first find $g(x + h)$:

$$\begin{aligned} g(x+h) &= \frac{3}{2(x+h)+1} \\ &= \frac{3}{2x+2h+1} \end{aligned}$$

Simplifying the difference quotient involves simplifying the resulting complex fraction, as above, keeping an eye out for an opportunity to cancel the factor 'h' from the denominator:

$$\begin{aligned} \frac{g(x+h)-g(x)}{h} &= \frac{\frac{3}{2x+2h+1} - \frac{3}{2x+1}}{h} \\ &= \frac{\frac{3}{2x+2h+1} - \frac{3}{2x+1}}{h} \cdot \frac{(2x+2h+1)(2x+1)}{(2x+2h+1)(2x+1)} \\ &= \frac{3(2x+1) - 3(2x+2h+1)}{h(2x+2h+1)(2x+1)} \\ &= \frac{6x+3-6x-6h-3}{h(2x+2h+1)(2x+1)} \\ &= \frac{-6h}{h(2x+2h+1)(2x+1)} \\ &= \frac{-6\cancel{h}}{\cancel{h}(2x+2h+1)(2x+1)} \qquad \text{cancel} \\ &= \frac{-6}{(2x+2h+1)(2x+1)}. \end{aligned}$$

Hence,

$$g'(x) = \lim_{h \to 0} \frac{-6}{(2x+2h+1)(2x+1)} = \frac{-6}{(2x+2(0)+1)(2x+1)} = -\frac{6}{(2x+1)^2}.$$

We check $g'(0) = -\frac{6}{(2(0)+1)^2} = \ldots = -6$, as required.

3. (a) To find $r'(9) = \lim\limits_{h \to 0} \frac{r(9+h)-r(9)}{h}$, we start with $r(9+h) = \sqrt{9+h}$ and $r(9) = \sqrt{9} = 3$. Hence our difference quotient is:

$$\frac{r(9+h) - r(9)}{h} = \frac{\sqrt{9+h} - 3}{h}.$$

In order for us to determine the limit as $h \to 0$, we need to somehow cancel the factor of h from the **denominator**. To do so, we set about **rationalizing the numerator** by multiplying both numerator and denominator by the conjugate[11] of the numerator, $\sqrt{9+h} - 3$:

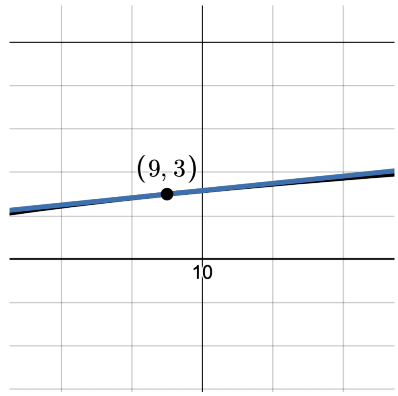
$$\begin{array}{rcll}
\dfrac{r(9+h) - r(9)}{h} & = & \dfrac{\sqrt{9+h} - 3}{h} & \\
& = & \dfrac{\left(\sqrt{9+h} - 3\right)}{h} \cdot \dfrac{\left(\sqrt{9+h} + 3\right)}{\left(\sqrt{9+h} + 3\right)} & \text{Multiply by the conjugate.} \\
& = & \dfrac{\left(\sqrt{9+h}\right)^2 - (3)^2}{h\left(\sqrt{9+h} + 3\right)} & \text{Difference of Squares.} \\
& = & \dfrac{(9+h) - 9}{h\left(\sqrt{9+h} + 3\right)} & \\
& = & \dfrac{h}{h\left(\sqrt{9+h} + 3\right)} & \\
& = & \dfrac{\cancelto{1}{h}}{\cancel{h}\left(\sqrt{9+h} + 3\right)} & \text{cancel} \\
& = & \dfrac{1}{\sqrt{9+h} + 3} &
\end{array}$$

Hence,

$$r'(9) = \lim_{h \to 0} \frac{1}{\sqrt{9+h} + 3} = \frac{1}{\sqrt{9+0} + 3} = \frac{1}{6}.$$

(b) The equation of the tangent line to $y = r(t)$ at $(9, 3)$ is therefore: $y = r'(9)(t-9)+r(9) = \frac{1}{6}(t-9)+3$ or $y = \frac{1}{6}\,t + \frac{3}{2}$. The graph below confirms this.

[11] Again, see Section A.13.1 for a review of these sorts of machinations.



$y = r(t)$ and $y = \frac{1}{6}\,t + \frac{3}{2}$ near $(9, 3)$

(c) As one might expect, we use this same strategy of rationalizing the numerator to simplify the difference quotient to find $r'(t)$:

$$\begin{aligned}
\frac{r(t+h) - r(t)}{h} &= \frac{\sqrt{t+h} - \sqrt{t}}{h} \\
&= \frac{\left(\sqrt{t+h} - \sqrt{t}\right)}{h} \cdot \frac{\left(\sqrt{t+h} + \sqrt{t}\right)}{\left(\sqrt{t+h} + \sqrt{t}\right)} && \text{Multiply by the conjugate.} \\
&= \frac{\left(\sqrt{t+h}\right)^2 - \left(\sqrt{t}\right)^2}{h\left(\sqrt{t+h} + \sqrt{t}\right)} && \text{Difference of Squares.} \\
&= \frac{(t+h) - t}{h\left(\sqrt{t+h} + \sqrt{t}\right)} \\
&= \frac{h}{h\left(\sqrt{t+h} + \sqrt{t}\right)} \\
&= \frac{\cancelto{1}{h}}{\cancel{h}\left(\sqrt{t+h} + \sqrt{t}\right)} \\
&= \frac{1}{\sqrt{t+h} + \sqrt{t}}
\end{aligned}$$

We get

$$r'(t) = \lim_{h \to 0} \frac{1}{\sqrt{t+h} + \sqrt{t}} = \frac{1}{\sqrt{t+0} + \sqrt{t}} = \frac{1}{2\sqrt{t}}.$$

We check $r'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}$ as above. □

6.2.3 Exercises

In Exercises 1 - 6, find the limit of the following difference quotients.

- $\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$
- $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

1. $f(x) = 2x - 5$
2. $f(x) = -3x + 5$
3. $f(x) = 6$
4. $f(x) = 3x^2 - x$
5. $f(x) = -x^2 + 2x - 1$
6. $f(x) = 4x^2$

In Exercises 7 - 8, find:

- $f'(2) = \lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$
- The equation of the tangent line at $(2, f(2))$. Check your answer graphically.
- $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
- The equation of the tangent line at $(0, f(0))$. Check your answer graphically.

7. $f(x) = x - x^2$
8. $f(x) = x^3 + 1$
9. Find $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ for $f(x) = mx + b$ where $m \neq 0$
10. (a) Find $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ for $f(x) = ax^2 + bx + c$ where $a \neq 0$.
    (b) Solve $f'(x) = 0$ for x. Does this look familiar? Explain.

In Exercises 11 - 14, find the limit of the following difference quotients:

- $\lim\limits_{\Delta x \to 0} \dfrac{f(-1+\Delta x) - f(-1)}{\Delta x}$
- $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

11. $f(x) = \dfrac{2}{x}$
12. $f(x) = \dfrac{3}{1-x}$
13. $f(x) = \dfrac{1}{x^2}$
14. $f(x) = \dfrac{2}{x+5}$

In Exercises 15 - 18, find the limit of the following:

- $f'(-1) = \lim\limits_{\Delta x \to 0} \dfrac{f(-1+\Delta x) - f(-1)}{\Delta x}$
- The equation of the tangent line at $(-1, f(-1))$.
- $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$
- The equation of the tangent line at $(0, f(0))$.

15. $f(x) = \dfrac{1}{4x - 3}$

16. $f(x) = \dfrac{3x}{x+2}$

17. $f(x) = \dfrac{x}{x - 9}$

18. $f(x) = \dfrac{x^2}{2x + 1}$

In Exercises 19 - 20, find the limit of the following difference quotients:

- $\displaystyle\lim_{\Delta t \to 0} \frac{g(\Delta t) - g(0)}{\Delta t}$
- $\displaystyle\lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}$

19. $g(t) = \sqrt{9 - t}$

20. $g(t) = \sqrt{2t + 1}$

In Exercises 21 - 22, find the following:

- $g'(0) = \displaystyle\lim_{\Delta t \to 0} \frac{g(\Delta t) - g(0)}{\Delta t}$
- The equation of the tangent line at $(0, g(0))$.
- $g'(t) = \displaystyle\lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}$
- The equation of the tangent line at $(1, g(1))$.

21. $g(t) = \sqrt{-4t + 5}$

22. $g(t) = \sqrt{4 - t}$

23. For $g(t) = t\sqrt{t}$:

    (a) Explain why $g'(0) = \displaystyle\lim_{\Delta t \to 0} \frac{g(\Delta t) - g(0)}{\Delta t}$ does not exist.

    (b) Find the derivative **from the right** at $t = 0$: $g'_+(0) = \displaystyle\lim_{\Delta t \to 0^+} \frac{g(\Delta t) - g(0)}{\Delta t}$

    (c) Find $y = g'_+(0)(x - 0) + g(0)$ and interpret.

    (d) Find $g'(t) = \displaystyle\lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}$. Assume $t > 0$.

24. (a) Find $g'(t) = \displaystyle\lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}$ for $g(t) = \sqrt{at + b}$, $a \neq 0$.

    (b) What restrictions do you place on t so your formula is valid?

25. Let $f(x) = |x|$.

    (a) Explain why f is continuous at $x = 0$.

    (b) Show $f'(0)$ does not exist by showing $\displaystyle\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = -1$ but $\displaystyle\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = 1$.

(c) Graph $y = f(x)$ near $(0,0)$. Interpret your answer to number 25b graphically.

(d) Find and simplify $f'(x) = \lim\limits_{h \to 0} \dfrac{|x+h| - |x|}{h}$ assuming $x \neq 0$.

HINT: Consider the two cases $x > 0$ and $x < 0$ ...

26. Let $g(t) = \sqrt[3]{t}$.

(a) Explain why g is continuous at $t = 0$.

(b) Show $g'(0)$ does not exist by showing $\lim\limits_{\Delta t \to 0} \dfrac{g(\Delta t) - g(0)}{\Delta t} = \infty$.

(c) Graph $y = g(t)$ near $(0,0)$. Interpret your answer to number 26b graphically.

(d) Find and simplify $g'(t) = \lim\limits_{\Delta t \to 0} \dfrac{g(t + \Delta t) - g(t)}{\Delta t}$ assuming $t \neq 0$.

HINT: $(a-b)\left(a^2 + ab + b^2\right) = a^3 - b^3$

27. Let $h(x) = x^{\frac{2}{3}}$.

(a) Explain why h is continuous at $x = 0$.

(b) Show $h'(0)$ does not exist by showing $\lim\limits_{\Delta x \to 0^-} \dfrac{h(\Delta x) - h(0)}{\Delta x} = -\infty$ and $\lim\limits_{\Delta x \to 0^+} \dfrac{h(\Delta x) - h(0)}{\Delta x} = \infty$

(c) Graph $y = h(x)$ near $(0,0)$. Interpret your answer to number 27b graphically.

(d) Find and simplify $h'(x) = \lim\limits_{\Delta x \to 0} \dfrac{h(x + \Delta x) - h(x)}{\Delta x}$ assuming $x \neq 0$.

HINT: $(a-b)\left(a^2 + ab + b^2\right) = a^3 - b^3$ and $a^2 - b^2 = (a+b)(a-b)$.

28. Recall from Exercise 48 in Section 1.4 that Carl's friend Jason participates in the Highland Games. In one event, the hammer throw, the height $h(t)$ in feet of the hammer above the ground t seconds after Jason lets it go is modeled by the function $h(t) = -16t^2 + 22.08t + 6$.

(a) Find and simplify a formula for the velocity of the hammer, $v(t) = h'(t) = \lim\limits_{\Delta t \to 0} \dfrac{h(t + \Delta t) - h(t)}{\Delta t}$.

(b) Find and interpret $v(0)$.

(c) Solve $v(t) = 0$ and interpret.

(d) Find the velocity of the hammer when it hits the ground, rounded to three decimal places.

29. In Exercise 40 in Section 1.4, the average fuel economy $F(t)$ in miles per gallon (mpg) for passenger cars in the US t years after 1980 is modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$.

(a) Find and simplify a formula for $F'(t) = \lim\limits_{h \to 0} \dfrac{F(t+h) - F(t)}{h}$.

(b) Find and interpret $F'(0)$, $F'(5)$ and $F'(10)$.

(c) Interpret the trend you observe in your answers to part 29b.

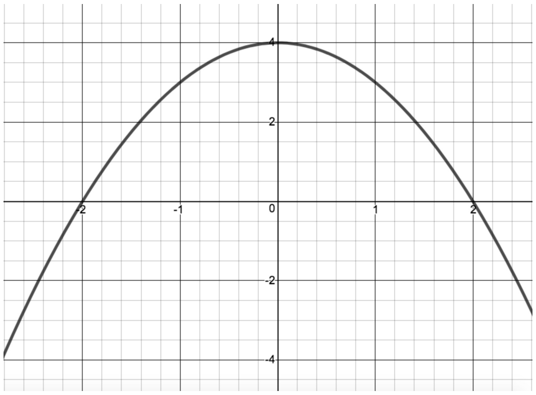
30. Let us return to Example 5.2.4 where $C(x) = .03x^3 - 4.5x^2 + 225x + 250$ denotes the cost, in dollars, of producing x PortaBoy game systems.

(a) Find and interpret $C'(75) = \lim\limits_{h \to 0} \dfrac{C(75+h) - C(75)}{h}$.

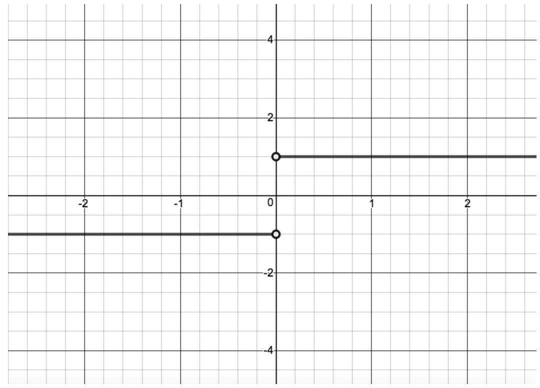
(b) Recall in Exercise 53 in Section 5.2, we found the marginal cost, $MC(75) = 58.53$, which means it will cost an additional \$58.53 to produce the 76th item. Compare $C'(75)$ and $MC(75)$.

In Exercises 31 - 33, match the graph of the function with a plausible graph of its derivative.

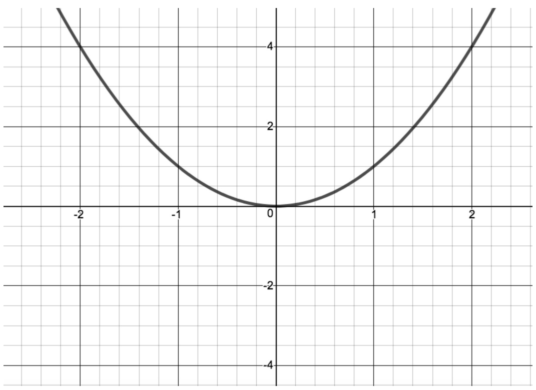
31. $y = f(x)$:



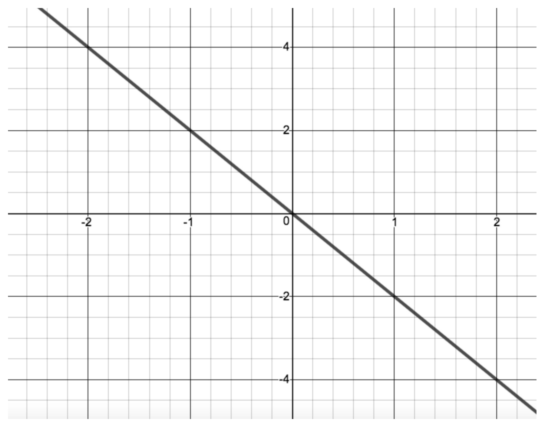
Graph A:



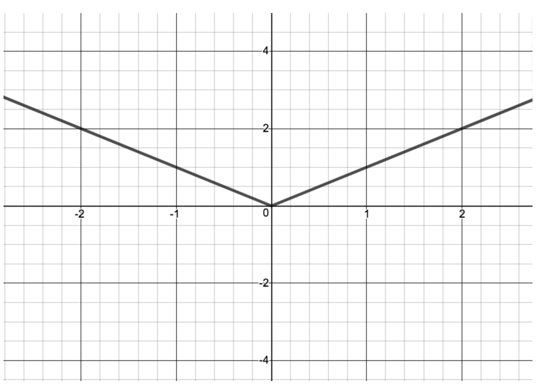
32. $y = g(x)$:



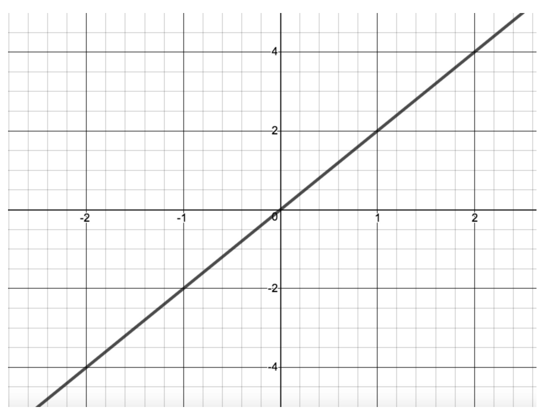
Graph B:



33. $y = h(x)$:

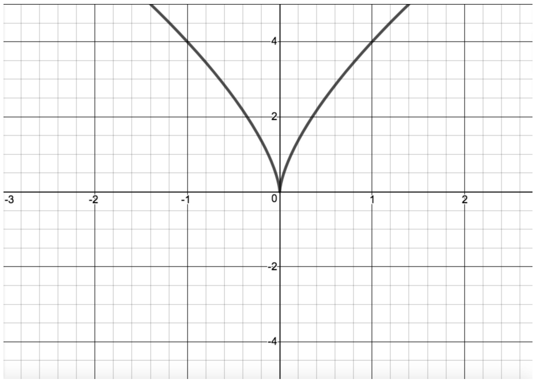


Graph C:

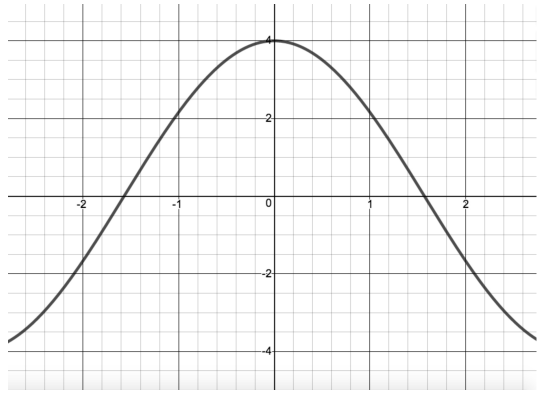


In Exercises 34 - 36, match the graph of the function with a plausible graph of its derivative.

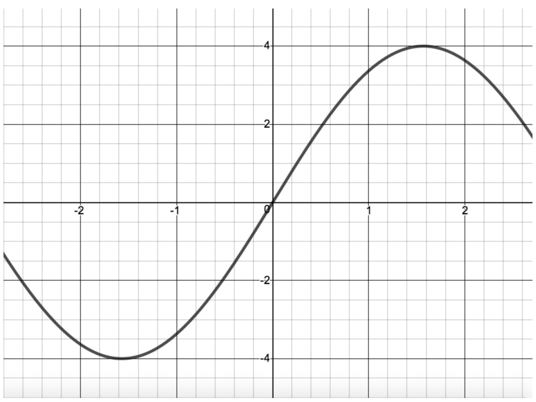
34. $y = f(x)$:



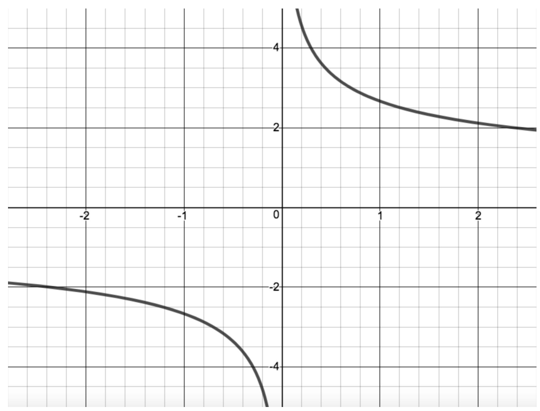
Graph A:



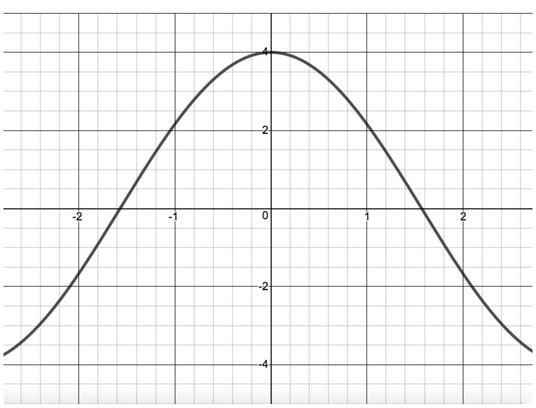
35. $y = g(x)$:



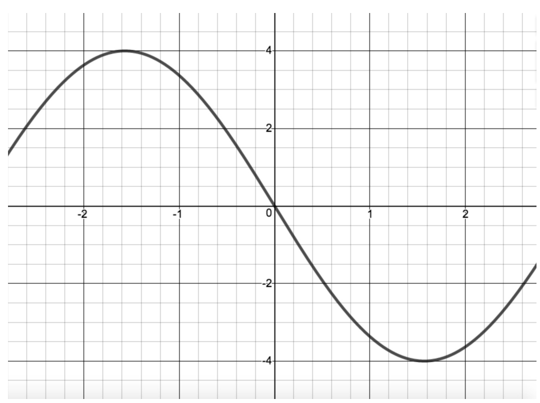
Graph B:



36. $y = h(x)$:



Graph C:



6.2.4 Answers

1. $\lim\limits_{h\to 0} 2 = 2$, $\lim\limits_{h\to 0} 2 = 2$

2. $\lim\limits_{h\to 0} -3 = -3$, $\lim\limits_{h\to 0} -3 = -3$,

3. $\lim\limits_{h\to 0} 0 = 0$, $\lim\limits_{h\to 0} 0 = 0$

4. $\lim\limits_{h\to 0}(3h+11) = 11$, $\lim\limits_{h\to 0}(6x+3h-1) = 6x-1$

5. $\lim\limits_{h\to 0}(-h-2) = -2$, $\lim\limits_{h\to 0}(-2x-h+2) = -2x+2$

6. $\lim\limits_{h\to 0}(4h+16) = 16$, $\lim\limits_{h\to 0}(8x+4h) = 8x$

7. - $f'(2) = \lim\limits_{h\to 0}(-h-3) = -3$
   - $y = f'(2)(x-2)+f(2) = (-3)(x-2)+(-2)$ so $y = -3x+4$.
   - $f'(x) = \lim\limits_{h\to 0}(-2x-h+1) = -2x+1$
   - $y = f'(0)(x-0)+f(0) = (1)(x-0)+0$ so $y = x$.

8. - $f'(2) = \lim\limits_{h\to 0}\left(h^2+6h+12\right) = 12$
   - $y = f'(2)(x-2)+f(2) = 12(x-2)+9$ so $y = 12x-15$.
   - $f'(x) = \lim\limits_{h\to 0}\left(3x^2+3xh+h^2\right) = 3x^2$
   - $y = f'(0)(x-0)+f(0) = 0(x-0)+1$ so $y = 1$.

9. $f'(x) = \lim\limits_{h\to 0} m = m$.

10. - $f'(x) = \lim\limits_{h\to 0}(2ax+ah+b) = 2ax+b$.
    - Solving $f'(x) = 2ax+b = 0$ for x gives $x = -\frac{b}{2a}$ which is the formula for the x-coordinate of the vertex of the parabola $y = f(x)$. If we zoom in near the the vertex of a parabola, the graph becomes locally flat so it makes sense the slope of the tangent line, $f'(x) = 0$ there.

11. $\lim\limits_{\Delta x\to 0} \dfrac{2}{\Delta x - 1} = -2$, $\lim\limits_{\Delta x\to 0} \dfrac{-2}{x(x+\Delta x)} = -\dfrac{2}{x^2}$

12. $\lim\limits_{\Delta x\to 0} \dfrac{-3}{2(\Delta x - 2)} = \dfrac{3}{4}$, $\lim\limits_{\Delta x\to 0} \dfrac{3}{(x+\Delta x-1)(x-1)} = \dfrac{3}{(x-1)^2}$

13. $\lim\limits_{\Delta x\to 0} \dfrac{2-\Delta x}{(\Delta x - 1)^2} = 2$, $\lim\limits_{\Delta x\to 0} \dfrac{-(2x+\Delta x)}{x^2(x+\Delta x)^2} = -\dfrac{2}{x^3}$

14. $\lim\limits_{\Delta x\to 0} \dfrac{-1}{2(\Delta x+4)} = -\dfrac{1}{8}$, $\lim\limits_{\Delta x\to 0} \dfrac{-2}{(x+5)(x+\Delta x+5)} = -\dfrac{2}{(x+5)^2}$

15.
- $f'(-1) = \lim\limits_{\Delta x\to 0} \dfrac{4}{7(4\Delta x-7)} = -\dfrac{4}{49}$
- $y = f'(-1)(x-(-1)) + f(-1) = -\frac{4}{49}(x+1) + \left(-\frac{1}{7}\right)$ so $y = -\frac{4}{49}x - \frac{11}{49}$.
- $f'(x) = \lim\limits_{\Delta x\to 0} \dfrac{-4}{(4x-3)(4x+4\Delta x-3)} = -\dfrac{4}{(4x-3)^2}$
- $y = f'(0)(x-0) + f(0) = -\frac{4}{9}(x-0) + \left(-\frac{1}{3}\right)$ so $y = -\frac{4}{9}x - \frac{1}{3}$.

16.
- $f'(-1) = \lim\limits_{\Delta x\to 0} \dfrac{6}{\Delta x+1} = 6$
- $y = f'(-1)(x-(-1)) + f(-1) = 6(x+1) + (-3)$ so $y = 6x+3$.
- $f'(x) = \lim\limits_{\Delta x\to 0} \dfrac{6}{(x+2)(x+\Delta x+2)} = \dfrac{6}{(x+2)^2}$
- $y = f'(0)(x-0) + f(0) = \frac{3}{2}(x-0) + 0$ so $y = \frac{3}{2}x$.

17.
- $f'(-1) = \lim\limits_{\Delta x\to 0} \dfrac{9}{10(\Delta x-10)} = -\dfrac{9}{100}$
- $y = f'(-1)(x-(-1)) + f(-1) = -\frac{9}{100}(x+1) + \frac{1}{10}$ so $y = -\frac{9}{100}x + \frac{1}{100}$.
- $f'(x) = \lim\limits_{\Delta x\to 0} \dfrac{-9}{(x-9)(x+\Delta x-9)} = -\dfrac{9}{(x-9)^2}$
- $y = f'(0)(x-0) + f(0) = -\frac{1}{9}(x-0) + 0$ so $y = -\frac{1}{9}x$.

18.
- $f'(-1) = \lim\limits_{\Delta x\to 0} \dfrac{\Delta x}{2\Delta x-1} = 0$
- $y = f'(-1)(x-(-1)) + f(-1) = (0)(x+1) + (-1)$ so $y = -1$.
- $f'(x) = \lim\limits_{\Delta x\to 0} \dfrac{2x^2+2x\Delta x+2x+\Delta x}{(2x+1)(2x+2\Delta x+1)} = \dfrac{2x^2+2x}{(2x+1)^2}$
- $y = f'(0)(x-0) + f(0) = (0)(x-0) + 0$ so $y = 0$.

19. $\lim\limits_{\Delta t\to 0} \dfrac{-1}{\sqrt{9-\Delta t}+3} = -\dfrac{1}{6}$, $\lim\limits_{\Delta t\to 0} \dfrac{-1}{\sqrt{9-t-\Delta t}+\sqrt{9-t}} = -\dfrac{1}{2\sqrt{9-t}}$

20. $\lim\limits_{\Delta t\to 0} \dfrac{2}{\sqrt{2\Delta t+1}+1} = 1$, $\lim\limits_{\Delta t\to 0} \dfrac{2}{\sqrt{2t+2\Delta t+1}+\sqrt{2t+1}} = \dfrac{2}{2\sqrt{2t+1}}$

21.
- $g'(0) = \lim\limits_{\Delta t\to 0} \dfrac{-4}{\sqrt{5-4\Delta t}+\sqrt{5}} = -\dfrac{2}{\sqrt{5}}$

- $y = g'(0)(x-0) + g(0) = -\frac{2}{\sqrt{5}}(x-0) + \sqrt{5}$ so $y = -\frac{2}{\sqrt{5}}x + \sqrt{5}$.
- $g'(t) = \lim\limits_{\Delta t \to 0} \dfrac{-4}{\sqrt{-4t - 4\Delta t + 5} + \sqrt{-4t+5}} = -\dfrac{2}{\sqrt{-4t+5}}$
- $y = g'(1)(x-1) + g(1) = (-2)(x-1) + 1$ so $y = -2x + 3$.

22.
- $g'(0) = \lim\limits_{\Delta t \to 0} \dfrac{-1}{\sqrt{4 - \Delta t} + 2} = -\dfrac{1}{4}$
- $y = g'(0)(x-0) + g(0) = -\frac{1}{4}(x-0) + 2$ so $y = -\frac{1}{4}x + 2$.
- $g'(t) = \lim\limits_{\Delta t \to 0} \dfrac{-1}{\sqrt{4 - t - \Delta t} + \sqrt{4-t}} = -\dfrac{1}{2\sqrt{4-t}}$
- $y = g'(1)(x-1) + g(1) = -\frac{1}{2\sqrt{3}}(x-1) + \sqrt{3}$ so $y = -\frac{1}{2\sqrt{3}}x + \frac{7\sqrt{3}}{6}$.

23. (a) If $\Delta t < 0$, then $\sqrt{\Delta t}$ is not a real number. Hence, $g'(0) = \lim\limits_{\Delta t \to 0} \dfrac{g(\Delta t) - g(0)}{\Delta t}$ does not exist.

(b) $g'_+(0) = \lim\limits_{\Delta t \to 0^+} (\Delta t)^{\frac{1}{2}} = 0$

(c) $y = g'_+(0)(x-0) + g(0) = (0)(x-0) + 0$ so $y = 0$.
This line is a tangent line to the graph of $y = t\sqrt{t}$ at $(0,0)$ for $t \geq 0$.

(d) $g'(t) = \lim\limits_{\Delta t \to 0} \dfrac{3t^2 + 3t\Delta t + (\Delta t)^2}{(t + \Delta t)^{3/2} + t^{3/2}} = \dfrac{3t^2}{2t^{3/2}} = \dfrac{3}{2}t^{1/2}$ provided $t > 0$.

24. (a) $g'(t) = \lim\limits_{\Delta t \to 0} \dfrac{a}{\sqrt{at + a\Delta t + b} + \sqrt{at+b}} = \dfrac{a}{2\sqrt{at+b}}$.

(b) We are told $a \neq 0$. In order for the square roots to be happy, we assume $at + b > 0$. Hence, $t > -\frac{b}{a}$ if $a > 0$ and $t < -\frac{b}{a}$ if $a < 0$.

25. (a) f is continuous at $x = 0$ since $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} |x| = 0 = |0| = f(0)$.

(b) $\lim\limits_{h \to 0^-} \dfrac{f(h) - f(0)}{h} = \lim\limits_{h \to 0^-} \dfrac{-h}{h} = -1$, $\lim\limits_{h \to 0^+} \dfrac{f(h) - f(0)}{h} = \lim\limits_{h \to 0^+} \dfrac{h}{h} = 1$

(c) Near $x = 0$, the graph of $y = f(x)$ looks like a '$\vee$' shape[12] consisting of a line of slope -1 to the left of $x = 0$ and a line with a slope of +1 to the right of $x = 0$.

(d) $f'(x) = \lim\limits_{h \to 0} \dfrac{|x+h| - |x|}{h} = -1$ if $x < 0$ and $f'(x) = \lim\limits_{h \to 0} \dfrac{|x+h| - |x|}{h} = 1$ of $x > 0$.

26. (a) g is continuous at $t = 0$ since $\lim\limits_{t \to 0} g(t) = \lim\limits_{t \to 0} \sqrt[3]{t} = 0 = \sqrt[3]{0} = g(0)$.

[12] We say the graph of f has a **corner** at $(0,0)$ since we have two different, but finite slopes meeting at a point.

(b) $\lim_{\Delta t \to 0} \frac{g(\Delta t) - g(0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^{\frac{2}{3}}} = \infty$

(c) The graph of $y = g(t)$ near $(0,0)$ is a vertical line.[13] Since g is increasing through $(0,0)$, the 'slope' of this vertical line could be seen as $+\infty$.

(d) $g'(t) = \lim_{\Delta t \to 0} \frac{1}{(t+\Delta t)^{\frac{2}{3}} + (t+\Delta t)^{\frac{1}{3}} t^{\frac{1}{3}} + t^{\frac{2}{3}}} = \frac{1}{3t^{\frac{2}{3}}}$, $t \neq 0$.

27. (a) $\lim_{x \to 0} h(x) = \lim_{x \to 0} x^{\frac{2}{3}} = 0 = 0^{\frac{2}{3}} = h(0)$.

(b) $\lim_{\Delta x \to 0^-} \frac{h(\Delta x) - h(0)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{1}{(\Delta x)^{\frac{1}{3}}} = -\infty$ and $\lim_{\Delta x \to 0^+} \frac{h(\Delta x) - h(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{1}{(\Delta x)^{\frac{1}{3}}} = \infty$

(c) The graph $y = h(x)$ near $(0,0)$ shows a steeply decreasing graph as we approach $(0,0)$ from the left followed by a steeply increasing graph as we approach $(0,0)$ from the right.[14]

(d) $h'(x) = \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x + \Delta x}{(x+\Delta x)^{\frac{4}{3}} + (x+\Delta x)^{\frac{2}{3}} x^{\frac{2}{3}} + x^{\frac{4}{3}}} = \frac{2}{3x^{\frac{1}{3}}}$, $x \neq 0$.

28. (a) $v(t) = \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{-16(\Delta t)^2 - 32t\,\Delta t + 22.08\Delta t}{\Delta t} = -32t + 22.08$.

(b) $v(0) = -32(0) + 22.08 = 22.08$. This means initially (when Jason lets go of the hammer), the hammer is traveling upwards at 22.08 feet per second.

(c) $v(t) = 0$ when $t = \frac{22.08}{32} = 0.69$. This means the (vertical) velocity zeros out 0.69 seconds after Jason lets go of the hammer. In this scenario, this corresponds to when the hammer reaches its peak height

(d) We first find when the hammer hits the ground by solving $h(t) = 0$. The positive answer here is $t \approx 1.612$ seconds. The velocity of the hammer is: $v(1.612) = -32(1.612) + 22.08 = -29.504$. The hammer hits the ground going (approximately) 29.504 feet per second.[15]

29. (a) $F'(t) = \lim_{h \to 0} \frac{F(t+h) - F(t)}{h} = \lim_{h \to 0} \frac{-0.0076h^2 - 0.0152th + 0.45h}{h} = -0.0152t + 0.45$.

(b) $F'(0) = 0.45$, so fuel economy was increasing at a rate of 0.45 mpg per year in 1980.[16]

$F'(5) = 0.374$, so fuel economy was increasing at a rate of 0.374 mpg per year in 1985.

$F'(10) = 0.298$, so fuel economy was increasing at a rate of 0.298 mpg per year in 1990.

(c) Based on the model, we have that during the years 1980 - 1990, fuel economy was increasing, but less so as the decade wore on. Technical and cost limitations could be at work here.

[13] We say the graph of g has a **vertical tangent line** at $(0,0)$. This is the more formal way to describe all of the 'unusual steepness' we saw back in Chapter 4.

[14] We say the graph of f has a **cusp** at $(0,0)$ since we have two different, infinite slopes meeting at a point.

[15] The negative '$-$' here on $v(1.612)$ indicates the hammer is heading **downwards** when it strikes the ground.

[16] Since the domain of F is $0 \leq t \leq 28$, $F'(0)$ is actually $F'_+(0)$.

30. (a) $C'(75) = \lim_{h \to 0} \dfrac{C(75+h) - C(75)}{h} = \lim_{h \to 0} \dfrac{0.03h^3 + 2.25h^2 + 56.25h}{h} = 56.25$. This means when producing 75 systems, the cost is increasing at a rate of \$56.25 per system.

(b) We see that $C'(75)$ is numerically close to $MC(75)$ but the former is a rate of change (measured in dollars per system) where the latter is a change (measured in dollars). Note that if we set $h = 1$ in the difference quotient:

$$\frac{C(75+h) - C(75)}{h} = \frac{C(75+1) - C(75)}{1} = C(76) - C(75)$$

we see these two quantities can be used to approximate each other.

31. $f'(x)$ is Graph B

32. $g'(x)$ is Graph C

33. $h'(x)$ is Graph A

34. $f'(x)$ is Graph B

35. $g'(x)$ is Graph A

36. $h'(x)$ is Graph C

6.3 The Shape of Graphs

We know if f is differentiable at $x = a$ then the graph of f is **locally linear** at $x = a$ and $f'(a)$ is the **slope** of the tangent line at the point $(a, f(a))$. In this section, we explore how local behavior near a point can be extrapolated to global behavior over an interval. First, we review Definition 1.7 from Section 1.2:

Definition. Let f be a function defined on an interval I. Then f is said to be:

- **increasing** on I if, whenever $a < b$, then $f(a) < f(b)$. (i.e., as inputs increase, outputs **increase**.)
  NOTE: The graph of an increasing function **rises** as one moves from left to right.
- **decreasing** on I if, whenever $a < b$, then $f(a) > f(b)$. (i.e., as inputs increase, outputs **decrease**.)
  NOTE: The graph of a decreasing function **falls** as one moves from left to right.
- **constant** on I if $f(a) = f(b)$ for all a, b in I. (i.e., outputs don't change with inputs.)
  NOTE: The graph of a function that is constant over an interval is a horizontal line.

Suppose a function satisfies $f'(x) > 0$ for all x in an open interval[1] I. Then we know that not only is the graph of f locally linear on I, but the slopes of all of the tangent lines are positive. This means that all of the tangent lines are increasing so it stands to reason that the function f is likewise increasing on I. In other words, if a function is **locally** increasing on I, then it is **globally** increasing on I as well.

We can apply the same reasoning above to situations where $f'(x) < 0$ for all x in I, which implies f is decreasing on I or $f'(x) = 0$ on I, which implies f is constant on I. In Calculus, you'll learn this fact is a consequence of the Mean Value Theorem.[2] In this text, we'll just accept the following theorem is true and hope we've done enough hand-waving to deem it reasonable.

Theorem 6.4. Suppose f is differentiable on an open interval I:

- If $f'(x) > 0$ for all x in I, then f is increasing on I.
- If $f'(x) < 0$ for all x in I, then f is decreasing on I.
- If $f'(x) = 0$ for all x in I, then f is constant on I.

[1] We've defined derivatives as two-sided limits, so an open interval here guarantees enough 'room' on either side of any given number to take such a limit.

[2] which Carl thinks is the actual 'Fundamental Theorem of Calculus' since it relates local and global behavior . . .

Theorem 6.4 may be visualized as follows:

- $f'(x) > 0$ for all x in I:

- $f'(x) < 0$ for all x in I:

- $f'(x) = 0$ for all x in I:

We can use Theorem 6.4 to help us determine the (open) intervals over which a function f is increasing, decreasing, and constant by making a sign diagram for the derivative f'.

In order to avoid us having to go through the (somewhat lengthy) process of finding $f'(x)$ using Definition 6.8, we'll just use some properties of derivatives from Calculus behind the scenes and present you with both a function and its derivative. It's time for an example.

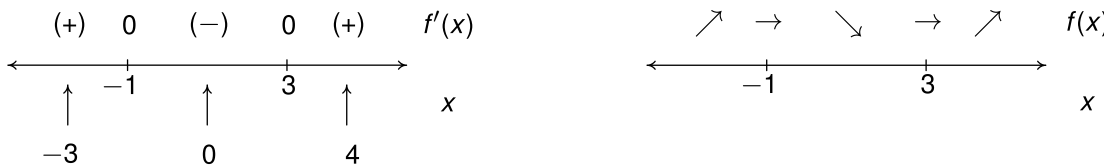
Example 6.3.1. Let $f(x) = x^3 - 3x^2 - 9x + 5$. Use the fact that $f'(x) = 3x^2 - 6x - 9$ to find the open intervals over which f is increasing, decreasing, and constant. Check your answer graphically.

Solution. To make use of Theorem 6.4, we make a sign diagram for $f'(x)$. Since f' is a polynomial, f' is continuous so the per the Intermediate Value Theorem, Theorem 2.14, f' will only change sign on either side of zeros. Hence, our first step is to solve $f'(x) = 0$.

We are given $f'(x) = 3x^2 - 6x - 9$. Solving $f'(x) = 3x^2 - 6x - 9 = 0$ gives $3(x^2 - 2x - 3) = 0$ or $3(x-3)(x+1) = 0$. We get two solutions: $x = -1$ and $x = 3$ which divides the x-axis into three regions: $x < -1$, $-1 < x < 3$ and $x > 3$.

Next we select a test value in each of these three regions to determine the sign of $f'(x)$. For the interval $x < -1$, we select $x = -3$: $f'(-3) = 3(-3)^2 - 6(-3) - 9 = (+)$. For $-1 < x < 3$, we select $x = 0$: $f'(0) = 3(0)^2 - 6(0) - 9 = (-)$. Finally, for $x > 3$, we select $x = 4$: $f'(4) = 3(4)^2 - 6(4) - 9 = (+)$.

Below on the left is a sign diagram for $f'(x)$ and on the right is what this means for the graph of $y = f(x)$:

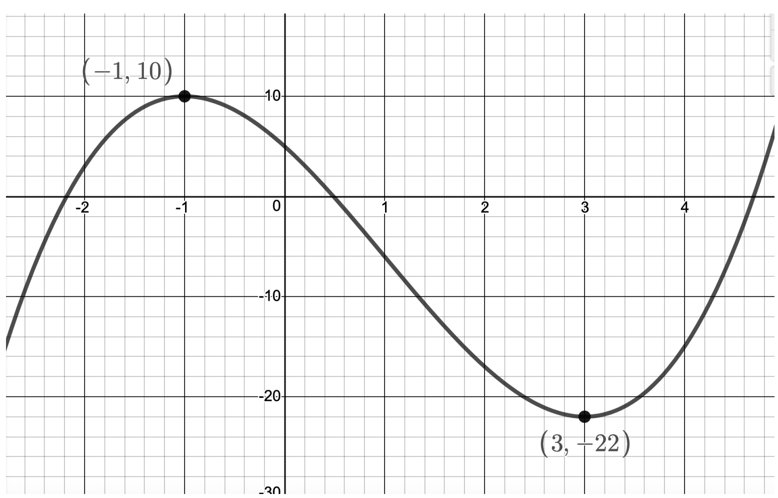


We find f is increasing on $(-\infty, -1)$ and again on $(3, \infty)$ while f is decreasing on $(-1, 3)$. At the points $x = -1$ and $x = 3$, we have $f'(x) = 0$ so the graph of f is locally flat there.

Since f changes from increasing just to the left of $x = -1$ to decreasing just to the right of $x = -1$, it stands to reason that f has a local maximum at $x = -1$. This is indeed the case and we find that the local maximum value is $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = 10$.

Similarly, since f changes from decreasing just to the left of $x = 3$ to increasing just to the right of $x = 3$, f has a local minimum at $x = 3$. The local minimum value is $f(3) = (3)^3 - 3(3)^2 - 9(3) + 5 = -22$.

A quick check using desmos confirms our results.



□

We generalize our observations about local extrema in the following result.

Theorem 6.5. The (First)[a] Derivative Test for Local Extema: Let f be continuous on an open interval I containing a critical number c.[b] If f is differentiable on I, except possibly at c, then

- If $f'(x)$ changes from (+) for $x < c$ to $(-)$ for $x > c$, f has a local maximum at $x = c$.
- If $f'(x)$ changes from $(-)$ for $x < c$ to (+) for $x > c$, f has a local minimum at $x = c$.
- If $f'(x)$ doesn't change sign going from $x < c$ to $x > c$, f does not have a local extremum at $x = c$.

[a]Why 'First'? Stay tuned . . .
[b]Recall this means $f'(c) = 0$ or $f'(c)$ does not exist.

Example 6.3.2. Let $f(x) = x^{4/3} - 4x^{1/3}$. Use the fact that $f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$ to help you find:

1. the open intervals over which f is increasing, decreasing, and constant.
2. the local extrema.

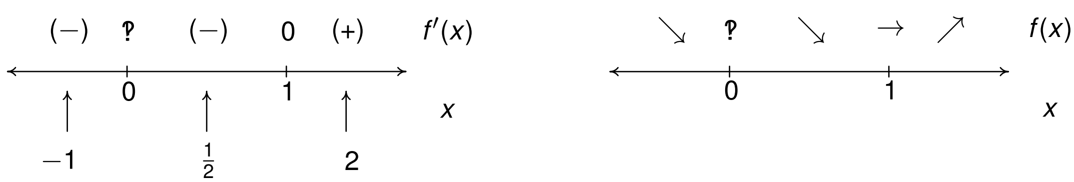
Solution.

1. In order to make a sign diagram for $f'(x)$, we rewrite $f'(x)$ as a single fraction:

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4x^{1/3}}{3} - \frac{4}{3x^{2/3}} = \frac{4x^{1/3}}{3} \cdot \frac{x^{2/3}}{x^{2/3}} - \frac{4}{3x^{2/3}} = \frac{4x}{3x^{2/3}} - \frac{4}{3x^{2/3}} = \frac{4x-4}{3x^{2/3}}.$$

Unlike the derivative in Example 6.3.1, $f'(x) = \frac{4x-4}{3x^{2/3}}$ is undefined when $3x^{2/3} = 0$, that is, when $x = 0$, so we need to record this on our sign diagram with the customary '?.'

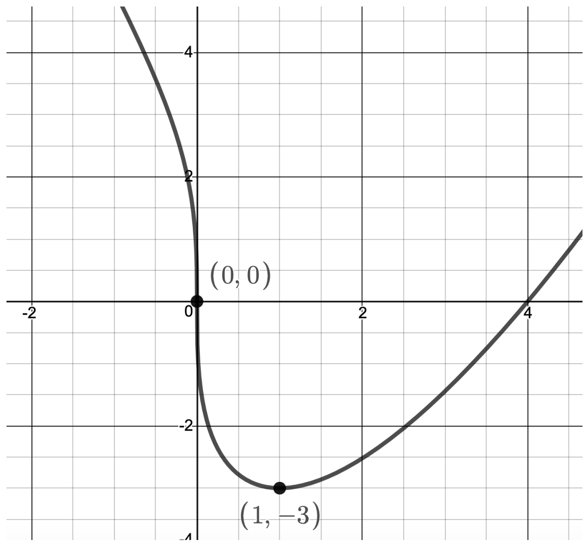
Next, we solve $f'(x) = \frac{4x-4}{3x^{2/3}} = 0$ to get $4x - 4 = 0$ or $x = 1$. The usual machinations produces the sign diagram for $f'(x)$ below on the left. We interpret what this means for f below on the right.



We get f is decreasing for $x < 0$ as well as from $0 < x < 1$. Since 0 is in the domain of f, we splice the two intervals together so f is decreasing from $(-\infty, 1)$. We see f is increasing from $(1, \infty)$.

2. We note that f satisfies the conditions of Theorem 6.5 since f is continuous everywhere and f' exists for all $x \neq 0$. Since f changes from decreasing just to the left of $x = 1$ to increasing just to the right of $x = 1$, the graph of f has a local minimum at $x = 1$. The local minimum value is $f(1) = (1)^{4/3} - 4(1)^{1/3} = -3$.

What is happening at $x = 0$? Since $f'(x)$ doesn't change sign on either side of 0, the graph of f doesn't have a local extremum there. The sign diagram indicates f is decreasing through that point. A quick check using desmos reveals 'unsual steepness' at $x = 0$, a phenomenon which is called a **vertical tangent**. This means the function locally resembles a vertical line.[3]



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6.3.1 Concavity and the Second Derivative

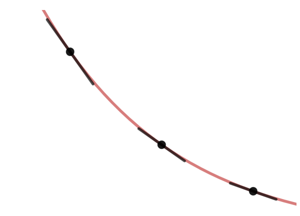
In section Section 4.2, we introduced the notion of **concavity**. In that section, we described curves as being **concave up** over an interval if it resembles a portion of a '◡' shape and **concave down** over an interval if resembles part of a '⌒' shape. Now that we've had some exposure to Calculus, we can more precisely define these notions.

Definition 6.9. Let f be a differentiable function on an open interval I. Then f is said to be:

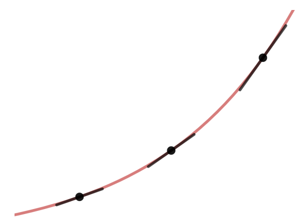
- **concave up** on I if the tangent lines lie **below** the graph on I.
- **concave down** on I if the tangent lines lie **above** the graph on I.

[3] See Example 4.1.2 for another such example and discussion.

If we take the time to study a generic concave up curve, the '$\smile$' shape can be divided into a decreasing and increasing arc:



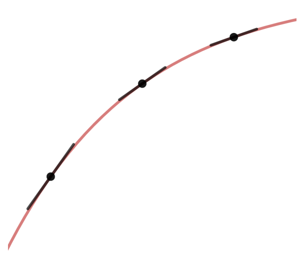
slopes are increasing towards 0



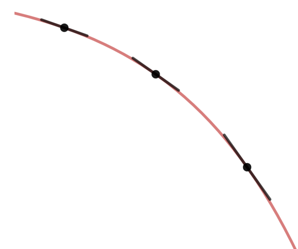
slopes are increasing away from 0

In both of these cases, the **slopes** of the tangent line are **increasing**.

Likewise, we can dissect a generic '$\frown$' shape curve into an increasing and decreasing arc:



slopes are decreasing towards 0



slopes are decreasing away from 0

Here, the **slopes** of the tangent line are **decreasing**.

We know from Theorem 6.4 that the derivative of a function can tell us where that function is increasing and decreasing. Since the function which gives us the slopes of tangent lines is the derivative, $f'(x)$, we could use the derivative of $f'(x)$ to determine where the slopes of the tangent lines were increasing and decreasing. This leads us to define the **second derivative**, $f''(x)$ as the derivative of $f'(x)$.

We present the following theorem without proof, but hopefully sufficiently motivated.

Theorem 6.6. Suppose f is twice differentiable on an open interval I:

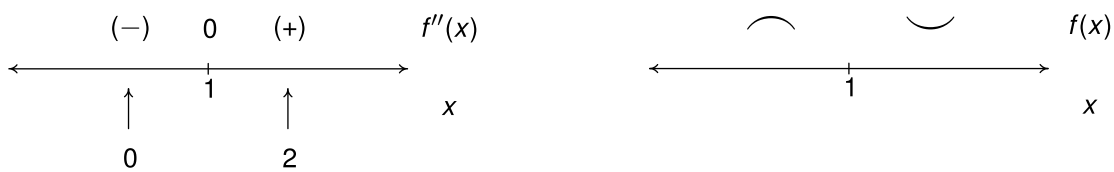
- If $f''(x) > 0$ for all x in I, then **slopes** are **increasing** and f is **concave up** on I.
- If $f''(x) < 0$ for all x in I, then **slopes** are **decreasing** and f is **concave down** on I.

Example 6.3.3. Let $f(x) = x^3 - 3x^2 - 9x + 5$. Use the fact that $f''(x) = 6x - 6$ to find the intervals over which the graph of f is concave up and concave down.

Solution. To analyze the concavity of the graph of f, we need to make a sign diagram for $f''(x)$.

Solving $f''(x) = 6x - 6 = 0$ gives $x = 1$. We find $f''(0) = 6(0) - 6 = (-)$ and $f''(2) = 6(2) - 6 = (+)$.

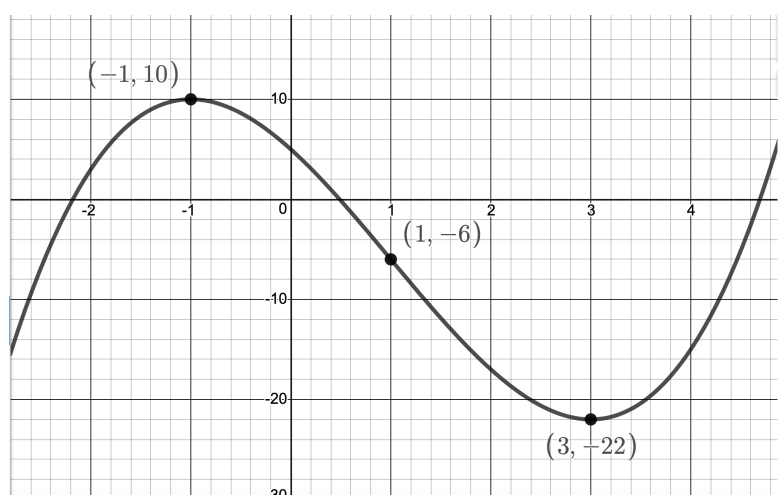
We have our sign diagram below on the left and our interpretation below on the right.



We find f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

At $x = 1$, the concavity changes. We find $f(1) = (1)^3 - 3(1)^2 - 9(1) + 5 = -6$ and we call the point $(1, -6)$ an **inflection point**. In this case since the concavity changes from concave down to concave up, the point $(1, -6)$ is the point on the graph of $y = f(x)$ were the slopes stop decreasing and start to increase.

A quick check using desmos confirms our results.



$\square$

Note that we can use concavity to help us distinguish local extrema.

For the function above, both $f'(-1) = 0$ and $f'(3) = 0$. Note that $f''(-1) < 0$ which means f is concave down there. This forces f to have a local maximum at $(-1, 6)$. Likewise, $f''(3) > 0$ which means f is concave up there. This forces f to have a local minimum at $(3, -22)$. We generalize this observation below.

Theorem 6.7. The Second[a] Derivative Test for Local Extrema: Suppose f is differentiable on an open interval I containing c and $f'(c) = 0$:

- If $f''(c) > 0$ then f has a local minimum at $x = c$.
- If $f''(c) < 0$ then f has a local maximum at $x = c$.
- If $f''(c) = 0$ then the test is inconclusive. f may or may not have a local extremum at $x = c$. (In this case, we would appeal to the first derivative test.)

[a] Now you know why we titled Theorem 6.5 the 'First' Derivative Test for Local Extrema.

Example 6.3.4. Let $f(x) = x^{4/3} - 4x^{1/3}$. Use the fact that $f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3}$ to help you find:

1. the open intervals over which the graph of f is concave up and concave down.
2. the inflection points in the graph.

Solution.

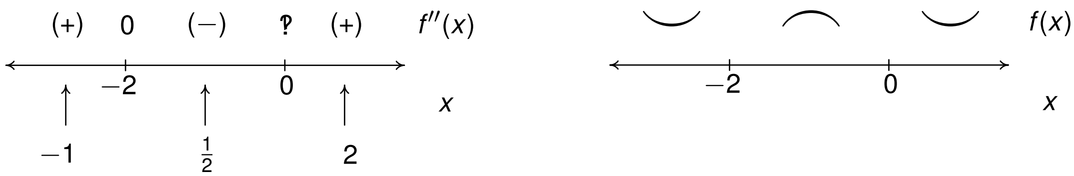
1. As in Example 6.3.2, our first step is to rewrite $f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3}$ as a single fraction:

$$f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4}{9x^{2/3}} + \frac{8}{9x^{5/3}} = \frac{4}{9x^{2/3}} \cdot \frac{x^{3/3}}{x^{3/3}} + \frac{8}{9x^{5/3}} = \frac{4x}{9x^{5/3}} + \frac{8}{9x^{5/3}} = \frac{4x+8}{9x^{5/3}}$$

We see $f''(x) = \frac{4x+8}{9x^{5/3}}$ is undefined when $9x^{5/3} = 0$, that is, when $x = 0$.

Solving $f''(x) = \frac{4x+8}{9x^{5/3}} = 0$ gives $4x + 8 = 0$ so $x = -2$.

Going through the usual routine, we obtain our sign diagram for $f''(x)$ is below.

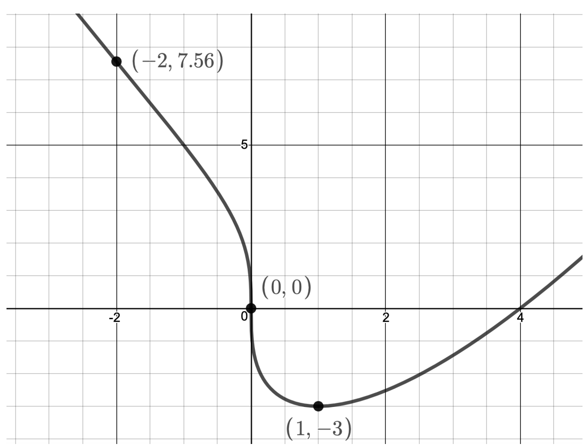


We see f is concave up on $(-\infty, -2)$ and again from $(0, \infty)$. f is concave down on $(-2, 0)$.

2. Since f changes concavity at both $x = -2$ and $x = 0$, we have inflection point at both of these values.

   We find: $f(-2) = (-2)^{4/3} - 4(-2)^{1/3} = 2(2)^{1/3} + 4(2)^{1/3} = 6(2)^{1/3}$. So $\left(-2, 6(2)^{1/3}\right)$ is one inflection point. When $x = 0$, $f(0) = (0)^{4/3} - 4(0)^{1/3} = 0$, so $(0, 0)$ is the other inflection point.

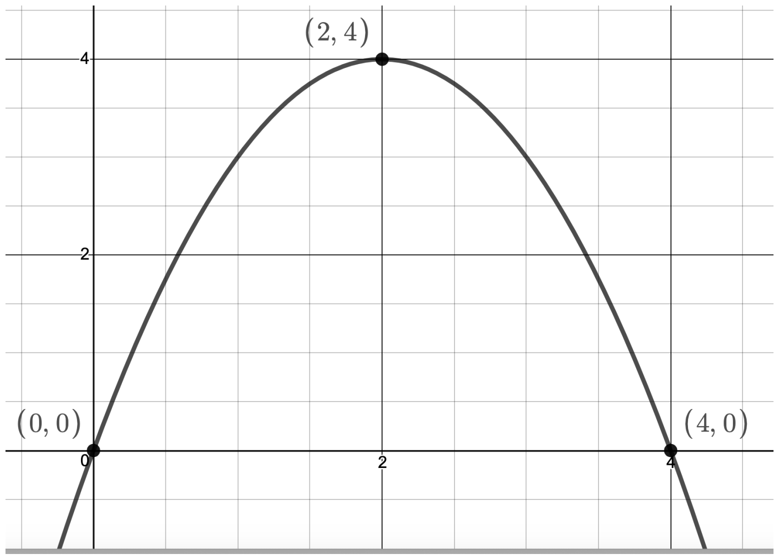
   Checking with desmos, it's not apparent that the graph of $y = f(x)$ is concave up for $x < -2$. We invite the reader to graph f and zoom out to see that characteristic of the graph.



□

Our last example offers a twist on these sorts of curve-sketching problems.

Example 6.3.5. Below is the graph of the **derivative** of a function. Assume as $x \to \pm\infty$, $f'(x) \to -\infty$.



The graph of $y = f'(x)$

1. Use the graph of $y = f'(x)$ to determine the open intervals where f is increasing and decreasing.

   Find the x-coordinates of the local extrema.

2. Use the graph of $y = f'(x)$ make a sign diagram for $y = f''(x)$.

3. List the open intervals over which the graph of f is concave up and concave down.

   Find the x-coordinates of the inflection points.

4. Sketch a possible graph of $y = f(x)$.

Solution.

1. Recall from algebra, the solutions to $f'(x) < 0$ are the x-values where the graph of $y = f'(x)$ is below the x-axis. This happens on the intervals $(-\infty, 0)$ and $(4, \infty)$, so this means f is decreasing here.

   Likewise, the solutions to $f'(x) > 0$ are the x-values where $y = f'(x)$ is above the x-axis. This happens on the interval $(0, 4)$, so f is increasing here.

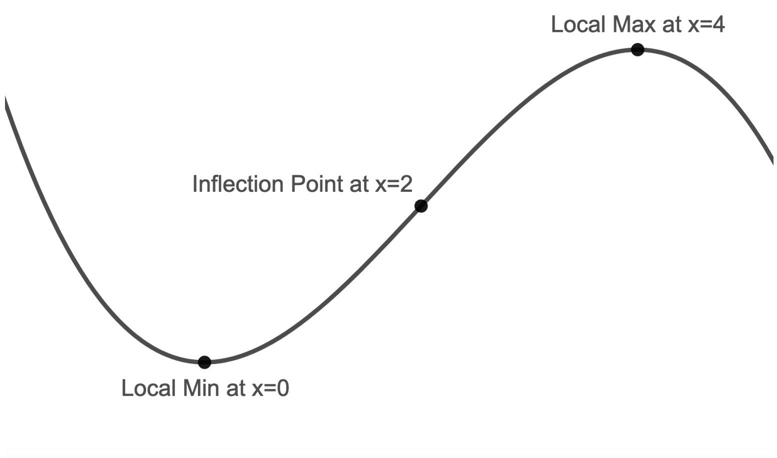
   Since f goes from decreasing to the left of $x = 0$ to increasing to the right of $x = 0$, f has a local minimum at $x = 0$. Since f goes from increasing to the left of $x = 4$ to decreasing to the right of $x = 4$, f has a local maximum at $x = 4$.

2. Since $f''(x)$ is the derivative of $f'(x)$, we know $f''(x) > 0$ on $(-\infty, 2)$ since $f'(x)$ is increasing there. We see $f''(2) = 0$ since $f'(x)$ is locally flat at $(2, 4)$. Lastly, we see $f''(x) < 0$ on $(2, \infty)$ since $f'(x)$ is decreasing there. We put all this together in a sign diagram below.

3. We have f is concave up on $(-\infty, 2)$ and concave down on $(2, \infty)$.

   Since f changes concavity at $x = 2$, there is an inflection point there.

4. A plausible graph of $y = f(x)$ is below. We cannot determine any y-coordinates (why not?)



A possible graph of $y = f(x)$

□

6.3.2 Exercises

In Exercises 1 - 3, use the given function f and its (first) derivative f' to help you find:

- the open intervals over which f is increasing, decreasing, and constant.
- the local extrema.

Check your answers using a graphing utility.

1. $f(x) = 2x^3 - 3x^2 - 12x + 1$, $f'(x) = 6x^2 - 6x - 12$
2. $f(x) = \dfrac{10x}{x^2+1}$, $f'(x) = \dfrac{10-10x^2}{\left(x^2+1\right)^2}$
3. $f(x) = x\sqrt[3]{x-2}$, $f'(x) = \dfrac{4x-6}{3(x-2)^{\frac{2}{3}}}$

In Exercises 4 - 6, use the given function f and its second derivative f'' to help you find:

- the open intervals over which the graph of f is concave up and concave down.
- the inflection points in the graph.

Check your answers using a graphing utility.

4. $f(x) = 2x^3 - 3x^2 - 12x + 1$, $f''(x) = 12x - 6$
5. $f(x) = \dfrac{10x}{x^2+1}$, $f''(x) = \dfrac{20x^3-60x}{\left(x^2+1\right)^3}$
6. $f(x) = x\sqrt[3]{x-2}$, $f''(x) = \dfrac{4(x-3)}{9(x-2)^{\frac{5}{3}}}$
7. If $a \neq 0$, we showed in Exercise 10 in Section 6.2 that if $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$. Solving $f'(x) = 0$ produced $x = -\frac{b}{2a}$, the x-coordinate of the vertex of the parabola $y = f(x)$. This Exercise shows this is part of a pattern.

   (a) If $a \neq 0$, show the x-coordinate of the x-intercept of the graph of $y = ax + b$ is $x = -\frac{b}{a} = -\frac{b}{1\,a}$.

   (b) If $a \neq 0$, for $f(x) = ax^3 + bx^2 + cx + d$ it turns out that $f''(x) = 6ax + 2b$. Show $x = -\frac{b}{3a}$ is the x-coordinate of the inflection point of the graph of $y = f(x)$.

8. In Exercise 53 in Section 5.2, we observed that average cost appeared to be minimized when average cost was approximately equal to marginal cost. In this Exercise, we use Calculus and the tools from this section to show this.

   Recall if $C(x)$ is the cost to produce x items, the **average cost** is defined as $\overline{C}(x) = \frac{C(x)}{x}$, $x > 0$, is the cost per item.

   (a) It turns out that $\overline{C}'(x) = \dfrac{x\,C'(x) - C(x)}{x^2}$. Show $\overline{C}'(x) = 0$ when $C'(x) = \overline{C}(x)$.

   (b) It turns out that $\overline{C}''(x) = \dfrac{x^2\,C''(x) - 2x\,C'(x) + 2C(x)}{x^3}$.

   Show we can rewrite this as: $\overline{C}''(x) = \dfrac{x\,C''(x) - 2\,C'(x) + 2\overline{C}(x)}{x^2}$.

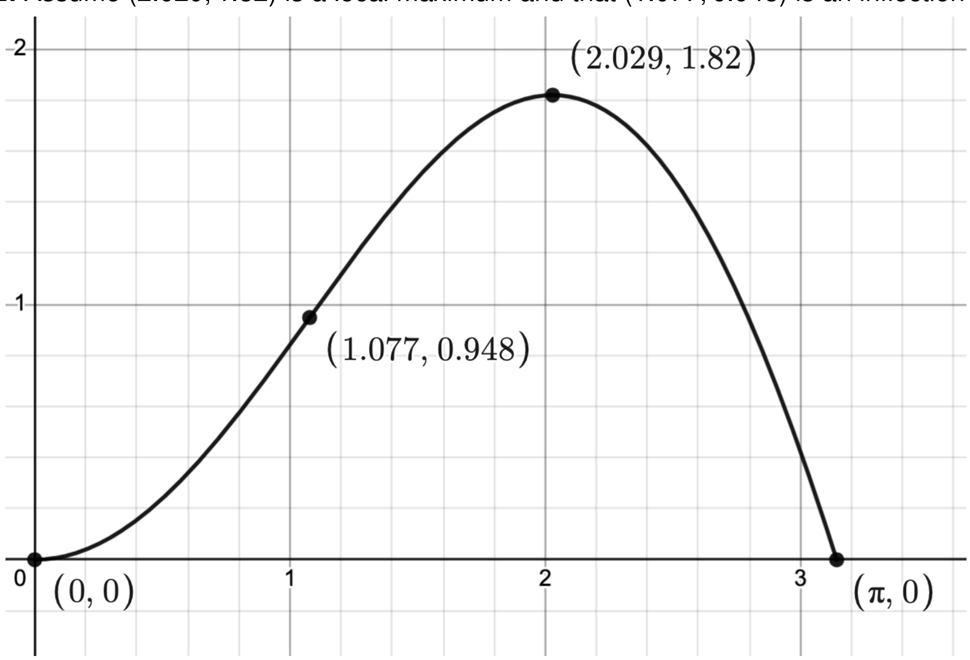
   (c) Show that when $C'(x) = \overline{C}(x)$, then $\overline{C}''(x) = \frac{C''(x)}{x}$.

   (d) It is usually assumed in most economic settings that for cost functions,[4] $C''(x) > 0$. Use this and your results from parts 8a and 8c to prove that a minimum is produced when $C'(x) = \overline{C}(x)$.

   NOTE: In Exercise 30 in Section 6.2, we saw how $C'(x)$ can be used to approximate the marginal cost, $MC(x)$, so we have established that in order to minimize average cost, we should look where the average cost matches the marginal cost.

9. The complete graph of $y = f(x)$ is shown below. Use the graph to answer the following questions.

   NOTE: Assume $(2.029, 1.82)$ is a local maximum and that $(1.077, 0.948)$ is an inflection point.



[4]Can you think of reasons why?

(a) Determine the x-values where:

$f(x) = 0$:

$f'(x) = 0$:

(b) List the open intervals over which:

$f(x) > 0$:

$f(x) < 0$:

$f'(x) > 0$:

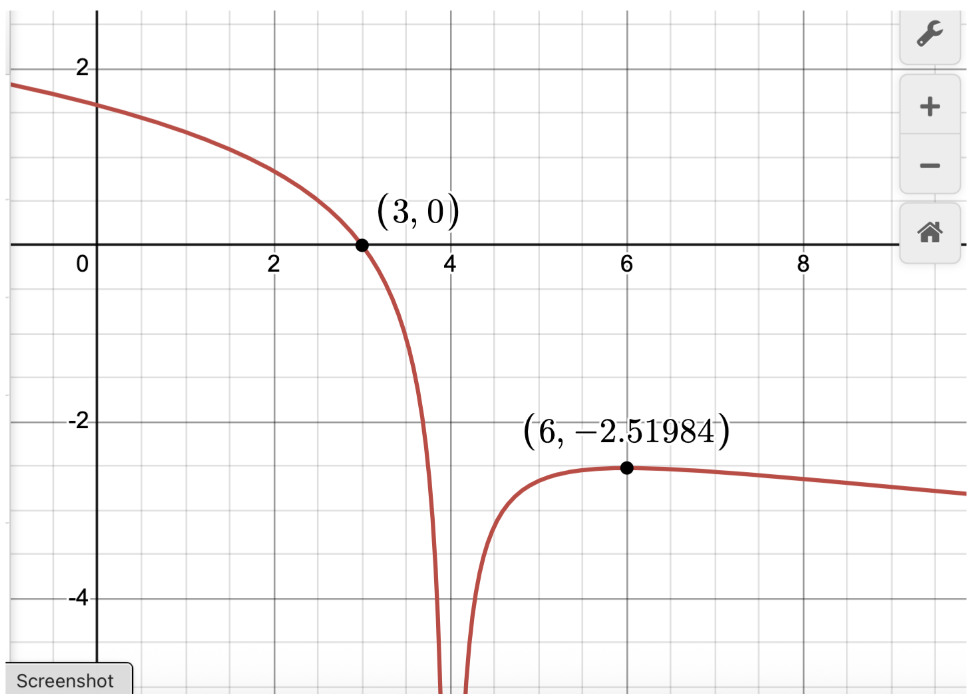
$f'(x) < 0$:

$f''(x) > 0$:

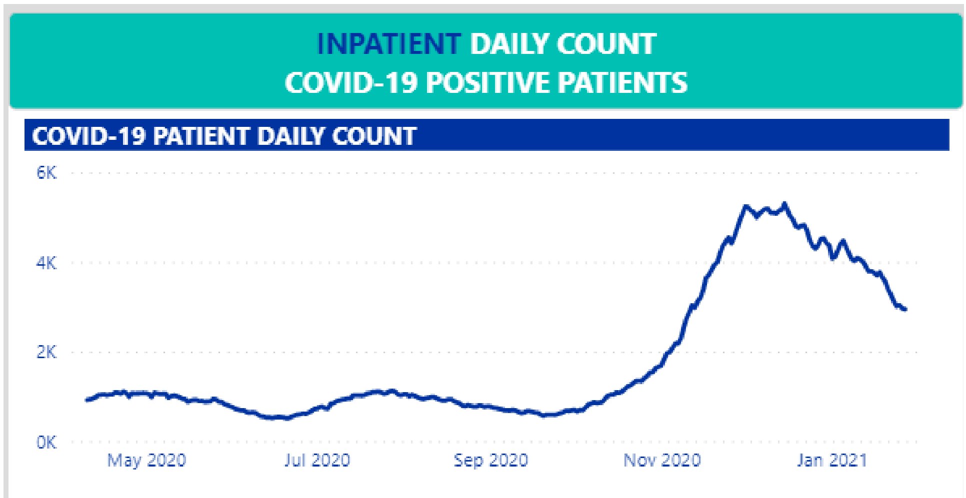
$f''(x) < 0$:

10. Below is the graph of $y = f'(x)$ for a **continuous** function f.

Using Example 6.3.5 as a guide, sketch a probable graph of $y = f(x)$.



11. The graph below was taken from https://ohiohospitals.org/covid19data on January 28th, 2021:



With help from your classmate, highlight and label one segment on the graph which (roughly) represents the following scenarios.

For brevity, we'll use 'patients' to mean 'inpatient COVID positive patients' and 'the rate of change' to mean 'the rate of change of inpatient COVID positive patients with respect to time.'

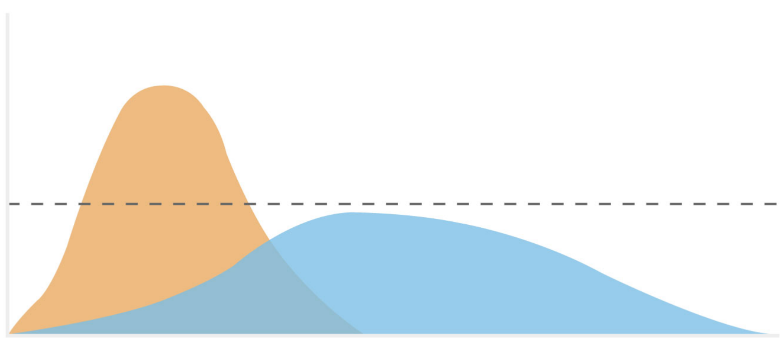
(a) The number of patients is **decreasing** <u>and</u> the rate of change is **decreasing**. (Label this 'a.')

(b) The number of patients is **decreasing** <u>and</u> the rate of change is **increasing**. (Label this 'b.')

(c) The number of patients is **increasing** <u>and</u> the rate of change is **increasing**. (Label this 'c.')

(d) The number of patients is **increasing** <u>and</u> the rate of change is **decreasing**. (Label this 'd.')

(e) Discuss with your classmates what the phrase 'flatten the curve' could mean in terms of first and second derivatives. (See below for an illustration.)



6.3.3 Answers

1. increasing: $(-\infty, -1)$, $(2, \infty)$; decreasing: $(-1, 2)$; local max: $(-1, 8)$; local min: $(2, -19)$.

2. increasing: $(-1, 1)$; decreasing: $(-\infty, -1)$, $(1, \infty)$; local max: $(1, 5)$; local min: $(-1, -5)$.

3. increasing: $\left(\frac{3}{2}, \infty\right)$; decreasing: $\left(-\infty, \frac{3}{2}\right)$; local (absolute) min: $\left(\frac{3}{2}, -\frac{3}{2\sqrt[3]{2}}\right)$

4. concave up: $\left(\frac{1}{2}, \infty\right)$; concave down: $\left(-\infty, \frac{1}{2}\right)$; inflection point: $\left(\frac{1}{2}, \frac{11}{2}\right)$

5. concave up: $\left(-\sqrt{3}, 0\right)$, $\left(\sqrt{3}, \infty\right)$; concave down: $\left(-\infty, -\sqrt{3}\right)$, $\left(0, \sqrt{3}\right)$; inflection points: $\left(-\sqrt{3}, -\frac{5\sqrt{3}}{2}\right)$, $(0, 0)$, $\left(\sqrt{3}, \frac{5\sqrt{3}}{2}\right)$

6. concave up: $(-\infty, 2)$, $(3, \infty)$; concave down: $(2, 3)$; inflection points: $(2, 0)$, $(3, 3)$.

7. (a) To find the x-intercept, we set $ax + b = 0$ and get $x = -\frac{b}{a}$ provided $a \neq 0$.

   (b) Solving $f''(x) = 6ax + 2b = 0$, we get $x = -\frac{2b}{6a} = -\frac{b}{3a}$, provided $a \neq 0$.

   The graph of $f''(x) = 6ax + 2b$ is a line so we know on one side of $x = -\frac{b}{3a}$, $f''(x) > 0$ and on the other side, $f''(x) < 0$.

   Hence, the graph of the original function $y = f(x)$ changes concavity at $x = -\frac{b}{3a}$.

8. (a) To solve $\overline{C}'(x) = \dfrac{x\,C'(x) - C(x)}{x^2} = 0$, we set the numerator, $x\,C'(x) - C(x) = 0$. We get $x\,C'(x) = C(x)$ so $C'(x) = \frac{C(x)}{x} = \overline{C}(x)$.

   (b) Divide both numerator and denominator of $\overline{C}''(x) = \dfrac{x^2\,C''(x) - 2x\,C'(x) + 2C(x)}{x^3}$ by x:

   $\overline{C}''(x) = \dfrac{x\,C''(x) - 2\,C'(x) + 2\frac{C(x)}{x}}{x^2}$ and substitute $\frac{C(x)}{x} = \overline{C}(x)$.

   (c) If $C'(x) = \overline{C}(x)$, then:

   $$\overline{C}''(x) = \frac{x\,C''(x) - 2\,C'(x) + 2\overline{C}(x)}{x^2} = \frac{x\,C''(x) - 2\overline{C}(x) + 2\overline{C}(x)}{x^2} = \frac{x\,C''(x)}{x^2} = \frac{C''(x)}{x}.$$

   (d) When $C'(x) = \overline{C}(x)$, we have that $\overline{C}'(x) = 0$ and $\overline{C}''(x) > 0$. By the Second Derivative Test for Local Extrema, Theorem 6.7, the average cost $\overline{C}(x)$ has a minimum when $C'(x) = \overline{C}(x)$

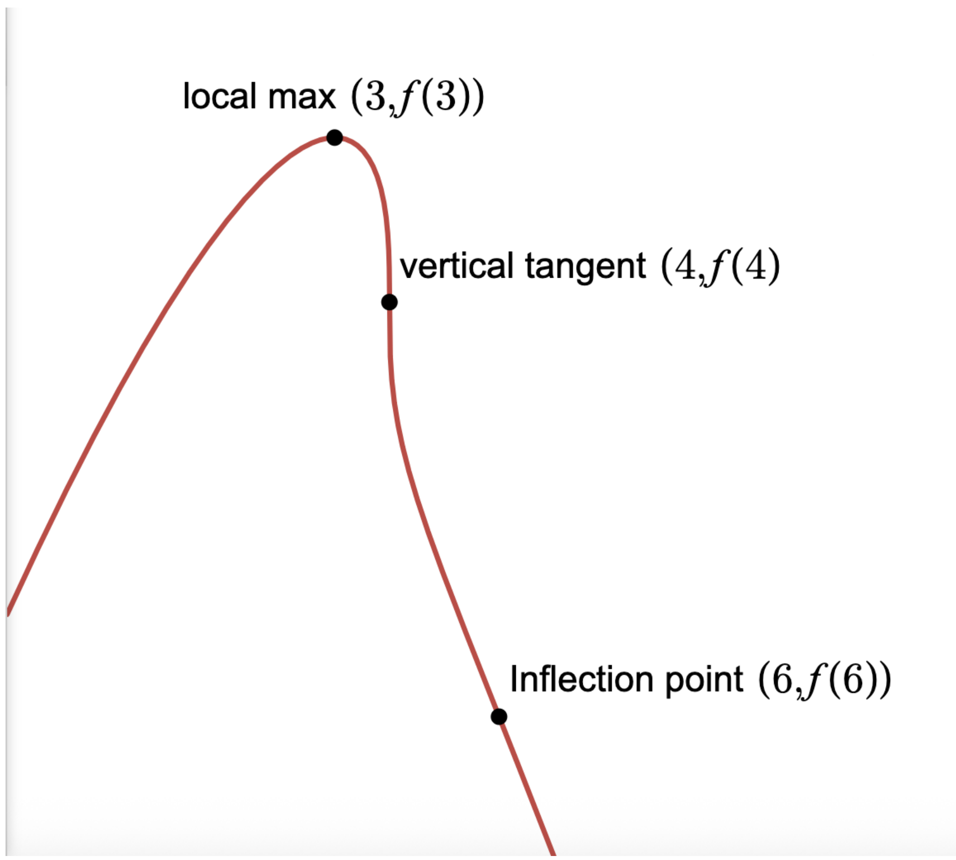
9. (a) $f(x) = 0$: $x = 0, \pi$ $\qquad$ $f'(x) = 0$: $x = 2.029$ ($f'_+(0) = 0.$, too.)

(b) $f(x) > 0$: $(0, \pi)$ $\qquad$ $f(x) < 0$: none

$f'(x) > 0$: $(0, 2.029)$ $\qquad$ $f'(x) < 0$: $(2.029, \pi)$

$f''(x) > 0$: $(0, 1.077)$ $\qquad$ $f''(x) < 0$: $(1.077, \pi)$.

10. Answers vary. below is a sketch of the key features that should be included:



Chapter 7

Exponential and Logarithmic Functions

7.1 Exponential Functions

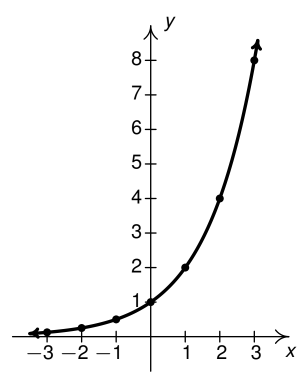
Of all of the functions we study in this text, exponential functions are possibly the ones which impact everyday life the most. This section introduces us to these functions while the rest of the chapter will more thoroughly explore their properties.

Up to this point, we have dealt with functions which involve terms like x^3, $x^{\frac{3}{2}}$, or x^{π} - in other words, terms of the form x^p where the base of the term, x, varies but the exponent of each term, p, remains constant.

In this chapter, we study functions of the form $f(x) = b^x$ where the base b is a constant and the exponent x is the variable. We start our exploration of these functions with the time-honored classic, $f(x) = 2^x$.

We make a table of function values, plot enough points until we are more or less confident with the shape of the curve, and connect the dots in a pleasing fashion.

x	$f(x)$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$
2	$2^2 = 4$	$(2, 4)$
3	$2^3 = 8$	$(3, 8)$



$y = f(x) = 2^x$

A few remarks about the graph of $f(x) = 2^x$ are in order. As $x \to -\infty$ and takes on values like $x = -100$ or $x = -1000$, the function $f(x) = 2^x$ takes on values like $f(-100) = 2^{-100} = \frac{1}{2^{100}}$ or $f(-1000) = 2^{-1000} = \frac{1}{2^{1000}}$. In other words, as $x \to -\infty$, $2^x \approx \frac{1}{\text{very big }(+)} \approx \text{very small }(+)$ That is, as $x \to -\infty$, $2^x \to 0^+$, so $\displaystyle\lim_{x \to -\infty} 2^x = 0$. This produces the x-axis, $y = 0$ as a horizontal asymptote to the graph as $x \to -\infty$.

On the flip side, as $x \to \infty$, we find $f(100) = 2^{100}$, $f(1000) = 2^{1000}$, and so on, thus $\lim\limits_{x \to \infty} 2^x = \infty$.

We note that by 'connecting the dots in a pleasing fashion,' we are implicitly using the fact that $f(x) = 2^x$ is not only defined for all real numbers,[1] but is also *continuous*. Moreover, we are assuming $f(x) = 2^x$ is increasing: that is, if $a < b$, then $2^a < 2^b$. While these facts are true, the proofs of these properties are best left to Calculus. For us, we assume these properties in order to state the domain of f is $(-\infty, \infty)$, the range of f is $(0, \infty)$ and, since f is increasing, f is one-to-one, hence invertible.

Suppose we wish to study the family of functions $f(x) = b^x$. Which bases b make sense to study? We find that we run into difficulty if $b < 0$. For example, if $b = -2$, then the function $f(x) = (-2)^x$ has trouble, for instance, at $x = \frac{1}{2}$ since $(-2)^{1/2} = \sqrt{-2}$ is not a real number. In general, if x is any rational number with an even denominator,[2] then $(-2)^x$ is not defined, so we must restrict our attention to bases $b \geq 0$.

What about $b = 0$? The function $f(x) = 0^x$ is undefined for $x \leq 0$ because we cannot divide by 0 and 0^0 is an indeterminate form. For $x > 0$, $0^x = 0$ so the function $f(x) = 0^x$ is the same as the function $f(x) = 0$, $x > 0$. Since we know everything about this function, we ignore this case.

The only other base we exclude is $b = 1$, since the function $f(x) = 1^x = 1$ for all real numbers x, since, once again, a function we have already studied. We are now ready for our definition of exponential functions.

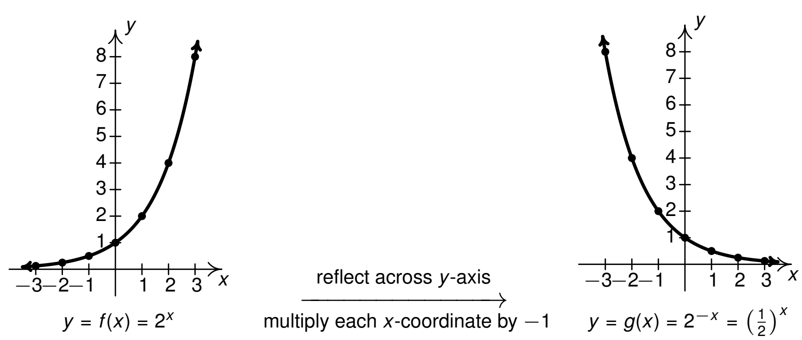
Definition 7.1. An **exponential function** is the function of the form

$$f(x) = b^x$$

where b is a real number, $b > 0$, $b \neq 1$. The domain of an exponential function $(-\infty, \infty)$.
NOTE: More specifically, $f(x) = b^x$ is called the '*base b exponential function*.'

We leave it to the reader to verify[3] that if $b > 1$, then the exponential function $f(x) = b^x$ will share the same basic shape and characteristics as $f(x) = 2^x$.

What if $0 < b < 1$? Consider $g(x) = \left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and note that $g(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = f(-x)$, where $f(x) = 2^x$. Per Section 5.4, the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the y-axis.



We see that the domain and range of g match that of f, namely $(-\infty, \infty)$ and $(0, \infty)$, respectively. Like f, g is also one-to-one. Whereas f is always increasing, g is always decreasing. As a result, as

[1]See the discussion of real number exponents in Section 4.2.
[2]or, as we defined real number exponents in Section 4.2, if x is an irrational number ...
[3]Meaning, graph some more examples on your own.

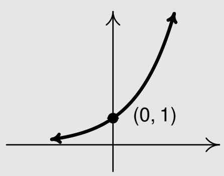
$\lim_{x\to-\infty} g(x) = \infty$, and on the flip side, $\lim_{x\to\infty} g(x) = 0$. (More specifically, $x \to \infty$, $g(x) \to 0^+$.) It shouldn't be too surprising that for all choices of the base $0 < b < 1$, the graph of $y = b^x$ behaves similarly to the graph of g.

We summarize the basic properties of exponential functions in the following theorem.

Theorem 7.1. Properties of Exponential Functions: Suppose $f(x) = b^x$.

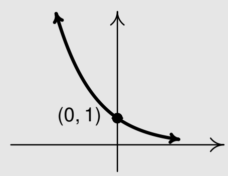
- The domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$.
- $f(x) > 0$ for all x.
- $(0, 1)$ is on the graph of f and $y = 0$ is a horizontal asymptote to the graph of f.
- f is one-to-one, continuous and smooth[a]

- If $b > 1$:
  - f is always increasing
  - $\lim_{x\to-\infty} f(x) = 0$
  - $\lim_{x\to\infty} f(x) = \infty$
  - The graph of f resembles:



$y = b^x$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - $\lim_{x\to-\infty} f(x) = \infty$
  - $\lim_{x\to\infty} f(x) = 0$
  - The graph of f resembles:



$y = b^x$, $0 < b < 1$

[a] Recall that this means the graph of f has no sharp turns or corners.

Exponential functions also inherit the basic properties of exponents from Theorem 4.3. We formalize these below and use them as needed in the coming examples.

Theorem 7.2. (Algebraic Properties of Exponential Functions) Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $f(u + w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u - w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $(b^u)^w = b^{uw}$

In addition to base 2 which is important to computer scientists,[4] two other bases are used more often than not in scientific and economic circles. The first is base 10. Base 10 is called the '**common base**' and is important in the study of intensity (sound intensity, earthquake intensity, acidity, etc.)

The second base is an irrational number, e. Like $\sqrt{2}$ or π, the decimal expansion of e neither terminates nor repeats, so we represent this number by the letter 'e.' A decimal approximation of e is $e \approx 2.718$, so the function $f(x) = e^x$ is an increasing exponential function.

The number e is called the '**natural base**' for lots of reasons, one of which is that it 'naturally' arises in the study of growth functions in Calculus. We will more formally discuss the origins of e in Section 7.6.

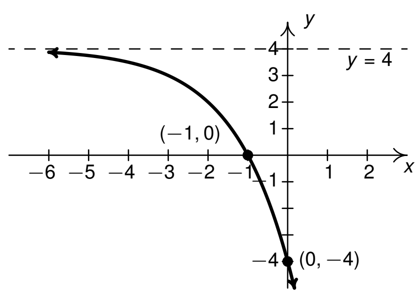
It is time for an example.

Example 7.1.1.

1. Graph the following functions by starting with a basic exponential function and using transformations, Theorem 5.11. Track at least three points and the horizontal asymptote through the transformations.

   (a) $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$ (b) $G(t) = 2 - e^{-t}$

2. Find a formula for the graph of the function below. Assume the base of the exponential is 2.



$y = F(x)$

Solution.

1. (a) Since the base of the exponent in $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$ is $\frac{1}{3}$, we start with the graph of $f(x) = \left(\frac{1}{3}\right)^x$.

   To use Theorem 5.11, we first need to choose some 'control points' on the graph of $f(x) = \left(\frac{1}{3}\right)^x$. Since we are instructed to track three points (and the horizontal asymptote, $y = 0$) through the transformations, we choose the points corresponding to $x = -1$, $x = 0$, and $x = 1$: $(-1, 3)$, $(0, 1)$, and $\left(1, \frac{1}{3}\right)$, respectively.

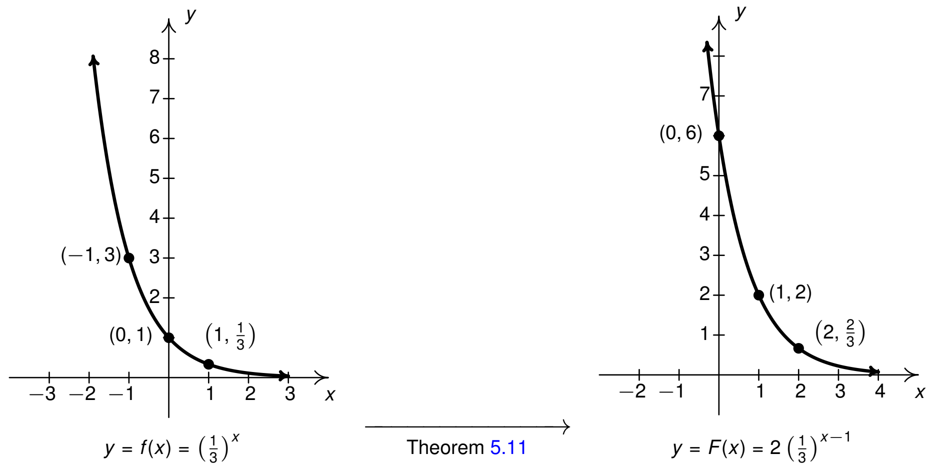
   Next, we need determine how to modify $f(x) = \left(\frac{1}{3}\right)^x$ to obtain $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$. The key is to recognize the argument, or 'inside' of the function is the exponent and the 'outside' is anything outside the base of $\frac{1}{3}$. Using these principles as a guide, we find $F(x) = 2f(x-1)$.

[4] The digital world is comprised of bytes which take on one of *two* values: 0 or 'off' and 1 or 'on.'

Per Theorem 5.11, we first add 1 to the x-coordinates of the points on the graph of $y = f(x)$, shifting the graph to the right 1 unit. Next, multiply the y-coordinates of each point on this new graph by 2, vertically stretching the graph by a factor of 2.

Looking point by point, we have $(-1, 3) \to (0, 3) \to (0, 6)$, $(0, 1) \to (1, 1) \to (1, 2)$, and $\left(1, \frac{1}{3}\right) \to \left(2, \frac{1}{3}\right) \to \left(2, \frac{2}{3}\right)$. The horizontal asymptote, $y = 0$ remains unchanged under the horizontal shift and the vertical stretch since $2 \cdot 0 = 0$.

Below we graph $y = f(x) = \left(\frac{1}{3}\right)^x$ on the left $y = F(x) = 2\left(\frac{1}{3}\right)^{x-1}$ on the right.



As always we can check our answer by verifying each of the points $(0, 6)$, $(1, 2)$, $\left(2, \frac{2}{3}\right)$ is on the graph of $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$ by checking $F(0) = 6$, $F(1) = 2$, and $F(2) = \frac{2}{3}$.

We can check the end behavior as well, that is, $\displaystyle\lim_{x \to -\infty} F(x) = \infty$ and $\displaystyle\lim_{x \to \infty} F(x) = 0$. We leave these calculations to the reader.

(b) Since the base of the exponential in $G(t) = 2 - e^{-t}$ is e, we start with the graph of $g(t) = e^t$.

Note that since e is an irrational number, we will use the approximation $e \approx 2.718$ when *plotting* points. However, when it comes to tracking and labeling said points, we do so with *exact* coordinates, that is, in terms of e.

We choose points corresponding to $t = -1$, $t = 0$, and $t = 1$: $(-1, e^{-1}) \approx (-1, 0.368)$, $(0, 1)$, and $(1, e) \approx (1, 2.718)$, respectively.

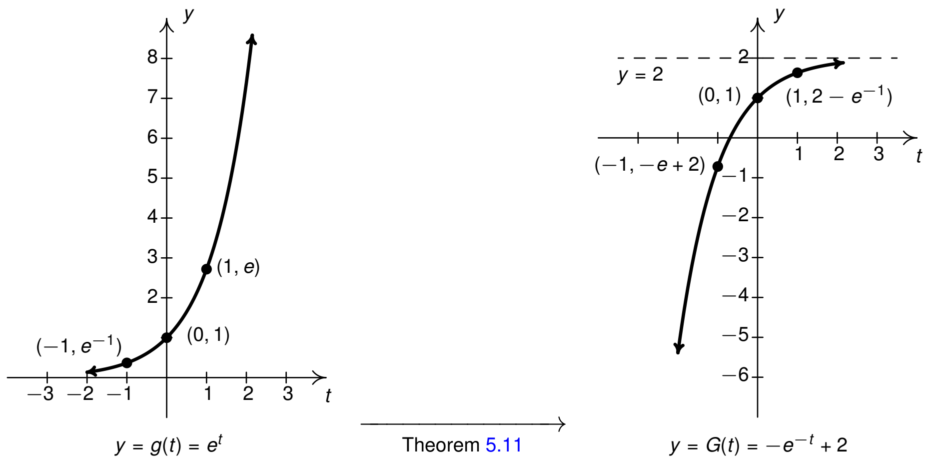
Next, we need to determine how the formula for $G(t) = 2 - e^{-t}$ can be obtained from the formula $g(t) = e^t$. Rewriting $G(t) = -e^{-t} + 2$, we find $G(t) = -g(-t) + 2$.

Following Theorem 5.11, we first multiply the t-coordinates of the graph of $y = g(t)$ by -1, effecting a reflection across the y-axis. Next, we multiply each of the y-coordinates by -1 which reflects the graph about the t-axis. Finally, we add 2 to each of the y-coordinates of the graph from the second step which shifts the graph up 2 units.

Tracking points, we have $(-1, e^{-1}) \to (1, e^{-1}) \to (1, -e^{-1}) \to (1, -e^{-1} + 2) \approx (1, 1.632)$, $(0, 1) \to (0, 1) \to (0, -1) \to (0, 1)$, and $(1, e) \to (-1, e) \to (-1, -e) \to (-1, -e + 2) \approx$

$(-1, -0.718)$. The horizontal asymptote is unchanged by the reflections, but is shifted up 2 units $y = 0 \rightarrow y = 2$.

We graph $g(t) = e^t$ below on the left and the transformed function $G(t) = -e^{-t} + 2$ below on the left. As usual, we can check our answer by verifying the indicated points do, in fact, lie on the graph of $y = G(t)$ along with checking end behavior. We leave these details to the reader.



2. Since we are told to assume the base of the exponential function is 2, we assume the function $F(x)$ is the result of the transforming the graph of $f(x) = 2^x$ using Theorem 5.11. This means we are tasked with finding values for a, b, h, and k so that $F(x) = af(bx - h) + k = a \cdot 2^{bx-h} + k$.

   Since the horizontal asymptote to the graph of $y = f(x) = 2^x$ is $y = 0$ and the horizontal asymptote to the graph $y = F(x)$ is $y = 4$, we know the vertical shift is 4 units up, so $k = 4$.

   Next, looking at how the graph of F approaches the vertical asymptote, it stands to reason the graph of $f(x) = 2^x$ undergoes a reflection across x-axis, meaning $a < 0$. For simplicity, we assume $a = -1$ and set see if we can find values for b and h that go along with this choice.

   Since $(-1, 0)$ and $(0, -4)$ on the graph of $F(x) = -2^{bx-h} + 4$, we know $F(-1) = 0$ and $F(0) = -4$. From $F(-1) = 0$, we have $-2^{-b-h} + 4 = 0$ or $2^{-b-h} = 4 = 2^2$. Hence, $-b - h = 2$ is one solution.[5]

   Next, using $F(0) = -4$, we get $-2^{-h} + 4 = -4$ or $2^{-h} = 8 = 2^3$. From this, we have $-h = 3$ so $h = -3$. Putting this together with $-b - h = 2$, we get $-b + 3 = 2$ so $b = 1$.

   Hence, one solution to the problem is $F(x) = -2^{x+3} + 4$. To check our answer, we leave it to the reader verify $F(-1) = 0$, $F(0) = -4$, $\displaystyle\lim_{x \to -\infty} F(x) = 4$, $\displaystyle\lim_{x \to \infty} F(x) = -\infty$.

   Since we made a simplifying assumption ($a = -1$), we may well wonder if our solution is the *only* solution. Indeed, we started with what amounts to three pieces of information and set out to determine the value of four constants. We leave this for a thoughtful discussion in Exercise 14.

Our next example showcases an important application of exponential functions: economic depreciation.

[5]This is the *only* solution. Since $f(x) = 2^x$, the equation $2^{-b-h} = 2^2$ is equivalent to the functional equation $f(-b - h) = f(2)$. Since f is one-to-one, we know this is true *only* when $-b - h = 2$.

Example 7.1.2. The value of a car can be modeled by $V(t) = 25(0.8)^t$, where $t \geq 0$ is number of years the car is owned and $V(t)$ is the value in thousands of dollars.

1. Find and interpret $V(0)$, $V(1)$, and $V(2)$.
2. Find and interpret the average rate of change of V over the intervals $[0, 1]$ and $[0, 2]$ and $[1, 2]$.
3. Find and interpret $\dfrac{V(1)}{V(0)}$, $\dfrac{V(2)}{V(1)}$ and $\dfrac{V(2)}{V(0)}$.
4. For $t \geq 0$, find and interpret $\dfrac{V(t+1)}{V(t)}$ and $\dfrac{V(t+k)}{V(t)}$.
5. Find and interpret $\dfrac{V(1) - V(0)}{V(0)}$, $\dfrac{V(2) - V(1)}{V(1)}$, and $\dfrac{V(2) - V(0)}{V(0)}$.
6. For $t \geq 0$, find and interpret $\dfrac{V(t+1) - V(t)}{V(t)}$ and $\dfrac{V(t+k) - V(t)}{V(t)}$.
7. Graph $y = V(t)$ starting with the graph of $y = V(t)$ and using transformations.
8. Interpret the horizontal asymptote of the graph of $y = V(t)$.
9. Using a graphing utility, determine how long it takes for the car to depreciate to (a) one half its original value and (b) one quarter of its original value. Round your answers to the nearest hundredth.

Solution.

1. We find $V(0) = 25(0.8)^0 = 25 \cdot 1 = 25$, $V(1) = 25(0.8)^1 = 25 \cdot 0.8 = 20$ and $V(2) = 25(0.8)^2 = 25 \cdot 0.64 = 16$. Since t represents the number of years the car has been owned, $t = 0$ corresponds to the purchase price of the car. Since $V(t)$ returns the value of the car in *thousands* of dollars, $V(0) = 25$ means the car is worth \$25,000 when first purchased. Likewise, $V(1) = 20$ and $V(2) = 16$ means the car is worth \$20,000 after one year of ownership and \$16,000 after two years, respectively.

2. Recall to find the average rate of change of V over an interval $[a, b]$, we compute: $\frac{V(b) - V(a)}{b - a}$. For the interval $[0, 1]$, we find $\frac{V(1) - V(0)}{1 - 0} = \frac{20 - 25}{1} = -5$, which means over the course of the first year of ownership, the value of the car depreciated, on average, at a rate of \$5000 per year.

   For the interval $[0, 1]$, we compute $\frac{V(2) - V(0)}{2 - 0} = \frac{16 - 25}{2} = -4.5$, which means over the course of the first two years of ownership, the car lost, on average, \$4500 per year in value.

   Finally, we find for the interval $[1, 2]$, $\frac{V(2) - V(1)}{2 - 1} = \frac{16 - 20}{1} = -4$, meaning the car lost, on average, \$4000 in value per year between the first and second years.

   Notice that the car lost more value over the first year (\$5000) than it did the second year (\$4000), and these losses average out to the average yearly loss over the first two years (\$4500 per year.)[6]

[6] It turns out for any function f, the average rate of change over the interval $[x, x + 2]$ is the average of the average rates of change of f over $[x, x + 1]$ and $[x + 1, x + 2]$. See Exercise 23.

3. We compute: $\frac{V(1)}{V(0)} = \frac{20}{25} = 0.8$, $\frac{V(2)}{V(1)} = \frac{16}{20} = 0.8$, and $\frac{V(2)}{V(0)} = \frac{16}{25} = 0.64$.

   The ratio $\frac{V(1)}{V(0)} = 0.8$ can be rewritten as $V(1) = 0.8V(0)$ which means that the value of the car after 1 year, $V(1)$ is 0.8 times, or 80% the initial value of the car, $V(0)$.

   Similarly, the ratio $\frac{V(2)}{V(1)} = 0.8$ rewritten as $V(2) = 0.8V(1)$ means the value of the car after 2 years, $V(2)$ is 0.8 times, or 80% the value of the car after one year, $V(1)$.

   Finally, the ratio $\frac{V(2)}{V(0)} = 0.64$, or $V(2) = 0.64V(0)$ means the value of the car after 2 years, $V(2)$ is 0.64 times, or 64% of the initial value of the car, $V(0)$.

   Note that this last result tracks with the previous answers. Since $V(1) = 0.8V(0)$ and $V(2) = 0.8V(1)$, we get $V(2) = 0.8V(1) = 0.8(0.8V(0)) = 0.64V(0)$. Also note it is no coincidence that the base of the exponential, 0.8 has shown up in these calculations, as we'll see in the next problem.

4. Using properties of exponents, we find

   $$\frac{V(t+1)}{V(t)} = \frac{25(0.8)^{t+1}}{25(0.8)^{t}} = (0.8)^{t+1-t} = 0.8$$

   Rewriting, we have $V(t+1) = 0.8V(t)$. This means after one year, the value of the car $V(t+1)$ is only 80% of the value it was a year ago, $V(t)$.

   Similarly, we find

   $$\frac{V(t+k)}{V(t)} = \frac{25(0.8)^{t+k}}{25(0.8)^{t}} = (0.8)^{t+k-t} = (0.8)^{k}$$

   which, rewritten, says $V(t+k) = V(t)(0.8)^k$. This means in k years' time, the value of the car $V(t+k)$ is only $(0.8)^k$ times what it was worth k years ago, $V(t)$.

   These results shouldn't be too surprising. Verbally, the function $V(t) = 25(0.8)^t$ says to multiply 25 by 0.8 multiplied by itself t times. Therefore, for each additional year, we are multiplying the value of the car by an additional factor of 0.8.

5. We compute $\frac{V(1)-V(0)}{V(0)} = \frac{20-25}{25} = -0.2$, $\frac{V(2)-V(1)}{V(1)} = \frac{16-20}{20} = -0.2$, and $\frac{V(2)-V(0)}{V(0)} = \frac{16-25}{25} = -0.36$.

   The ratio $\frac{V(1)-V(0)}{V(0)}$ computes the ratio of *difference* in the value of the car after the first year of ownership, $V(1) - V(0)$, to the initial value, $V(0)$. We find this to be -0.2 or a 20% decrease in value. This makes sense since we know from our answer to number 3, the value of the car after 1 year, $V(1)$ is 80% of the initial value, $V(0)$. Indeed:

   $$\frac{V(1) - V(0)}{V(0)} = \frac{V(1)}{V(0)} - \frac{V(0)}{V(0)} = \frac{V(1)}{V(0)} - 1,$$

   and since $\frac{V(1)}{V(0)} = 0.8$, we get $\frac{V(1)-V(0)}{V(0)} = 1 - 0.8 = -0.2$.

   Likewise, the ratio $\frac{V(2)-V(1)}{V(1)} = -0.2$ means the value of the car has lost 20% of its value over the course of the second year of ownership.

Finally, the ratio $\frac{V(2)-V(0)}{V(0)} = -0.36$ means that over the first two years of ownership, the car value has depreciated 36% of its initial purchase price. Again, this tracks with the result of number 3 which tells us that after two years, the car is only worth 64% of its initial purchase price.

6. Using properties of fractions and exponents, we get:

$$\frac{V(t+1)-V(t)}{V(t)} = \frac{25(0.8)^{t+1}-25(0.8)^t}{25(0.8)^t} = \frac{25(0.8)^{t+1}}{25(0.8)^t} - \frac{25(0.8)^t}{25(0.8)^t} = 0.8-1 = -0.2,$$

so after one year, the value of the car $V(t+1)$ has lost 20% of the value it was a year ago, $V(t)$.

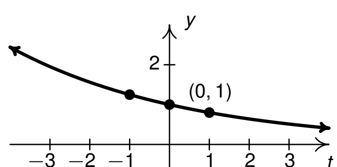
Similarly, we find:

$$\frac{V(t+k)-V(t)}{V(t)} = \frac{25(0.8)^{t+k}-25(0.8)^t}{25(0.8)^t} = \frac{25(0.8)^{t+1}}{25(0.8)^t} - \frac{25(0.8)^t}{25(0.8)^t} = (0.8)^k - 1,$$

so after k years' time, the value of the car $V(t)$ has decreased by $((0.8)^k - 1)\cdot 100\%$ of the value k years ago, $V(t)$.

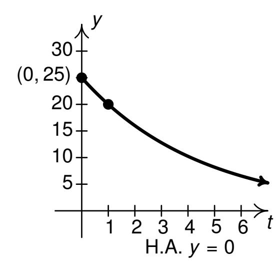
7. To graph $y = 25(0.8)^t$, we start with the basic exponential function $f(t) = (0.8)^t$. Since the base $b = 0.8$ satisfies $0 < b < 1$, the graph of $y = f(t)$ is decreasing. We plot the y-intercept $(0,1)$ and two other points, $(-1, 1.25)$ and $(1, 0.8)$, and label the horizontal asymptote $y = 0$.

To obtain the graph of $y = 25(0.8)^t = 25f(t)$, we multiply all of the y values in the graph by 25 (including the y value of the horizontal asymptote) in accordance with Theorem 5.9 to obtain the points $(-1, 31.25)$, $(0, 25)$ and $(1, 20)$. The horizontal asymptote remains the same, since $25 \cdot 0 = 0$. Finally, we restrict the domain to $[0,\infty)$ to fit with the applied domain given to us.



$y = f(t) = (0.8)^t$

$\xrightarrow[\text{multiply each } y\text{-coordinate by 25}]{\text{vertical scale by a factor of 25}}$



$y = V(t) = 25f(t),\ t \geq 0$

8. We see from the graph of V that its horizontal asymptote is $y = 0$. This means as the car gets older, its value diminishes to 0.

9. We know the value of the car, brand new, is \$25,000, so when we are asked to find when the car depreciates to one half and one quarter of this value, we are trying to find when the value of the car dips to \$12,500 and \$6,125, respectively. Since $V(t)$ is measured in *thousands* of dollars, we this translates to solving the equations $V(t) = 12.5$ and $V(t) = 6.125$.

Since we have yet to develop any analytic means to solve equations like $25(0.8)^t = 12.5$ (since t is in the exponent here), we are forced to approximate solutions to this equation numerically[7] or use a graphing utility. Choosing the latter, we graph $y = V(t)$ along with the lines $y = 12.5$ and $y = 6.125$ and look for intersection points.

We find $y = V(t)$ and $y = 12.5$ intersect at (approximately) (3.10612.5) which means the car depreciates to half its initial value in (approximately) 3.11 years. Similarly, we find the car depreciates to one-quarter its initial value after (approximately) 6.23 years.[8]

□

Some remarks about Example 7.1.2 are in order. First the function in the previous example is called a 'decay curve'. Increasing exponential functions are used to model 'growth curves' and we shall see several different examples of those in Section 7.6.

Second, as seen in numbers 3 and 4, $V(t+1) = 0.8V(t)$. That is to say, the function V has a *constant unit multiplier*, in this case, 0.8 because to obtain the function value $V(t+1)$, we *multiply* the function value $V(t)$ by b. It is not coincidence that the multiplier here is the base of the exponential, 0.8.

Indeed, exponential functions of the form $f(x) = a \cdot b^x$ have a constant unit multiplier, b. To see this, note

$$\frac{f(x+1)}{f(x)} = \frac{a \cdot b^{x+1}}{a \cdot b^x} = b^1 = b.$$

[7]Since exponential functions are continuous we could use the Bisection Method to solve $f(t) = 25(0.8)^t - 12.5 = 0$. See the discussion on page 192 in Section 2.3 for more details.

[8]It turns out that it takes exactly twice as long for the car to depreciate to one-quarter of its initial value as it takes to depreciate to half its initial value. Can you see why?

Hence $f(x+1) = f(x) \cdot b$. This will prove useful to us in Section 7.6 when making decisions about whether or not a data set represents exponential growth or decay.

More generally, one can show (see Exercise 24) for any real number x_0 that $f(x_0 + \Delta x) = f(x_0)b^{\Delta x}$. That is, to obtain $f(x_0 + \Delta x)$ from $f(x_0)$, we *multiply* by Δx factors of the constant unit multiplier, b. This is at the heart of what it means to be an exponential function.

If this discussion seems familiar, it should. For linear functions, $f(x) = mx + b$, we can obtain the slope m by computing $f(x+1) - f(x)$. To see this, note $f(x+1) - f(x) = (m(x+1) + b) - (mx + b) = m$ so that $f(x+1) = f(x) + m$. In this way, we see that the slope m is the constant unit *addend* in that in order to obtain $f(x+1)$, we *add* m to the function value $f(x)$.

This notion is solidified in the point-slope form of a linear function, Equation 1.1. For for any real numbers x and x_0, we have $f(x) = f(x_0) + m(x - x_0)$. If we let $x = x_0 + \Delta x$, we get $f(x_0 + \Delta x) = f(x_0) + m\Delta x$. In other words, to obtain $f(x_0 + \Delta x)$ from $f(x_0)$, we *add* m times Δx.

Taking inspiration from linear functions, we define the 'point-base' form of an exponential function below.

Definition 7.2. The **point-base form** of the exponential function $f(x) = b^x$ is

$$f(x) = f(x_0)b^{x-x_0}$$

Just as the point-slope form of a linear function is helpful in building linear models, the point-base form of an exponential function will prove useful in building exponential models.

Next, while we saw in Example 7.1.2 number 2, exponential functions, unlike linear functions, do not have a constant rate of change. However, in numbers 5 and 6, we see that in some cases, they do have a constant *relative* rate of change. We define this notion below.

Definition 7.3. Let f be a function defined on the interval $[a, b]$ where $f(a) \neq 0$.
The **relative rate of change** of f over $[a, b]$ is defined as:

$$\frac{\Delta[f(x)]}{f(a)} = \frac{f(b) - f(a)}{f(a)}.$$

For exponential functions of the form $f(x) = a \cdot b^x$, we compute the relative rate of change over the interval $[x, x+1]$ and find it is constant:

$$\frac{f(x+1) - f(x)}{f(x)} = \frac{f(x+1)}{f(x)} - \frac{f(x)}{f(x)} = b - 1,$$

where we are using the fact that $\frac{f(x+1)}{f(x)} = b$.

One way to interpret this result is when comparing $f(x)$ to $f(x+1)$, the exponential function grows (if $b > 1$) or decays (if $b < 1$) by $(b-1) \cdot 100\%$. In our example, $V(t) = 25(0.8)^t$ so $b = 0.8$ and, as we saw, the relative rate of change from $V(t)$ to $V(t+1)$ was $0.8 - 1 = -0.2$, meaning the value of the car over the course of one year depreciates by 20%.

We close this section with another important application of exponential functions, Newton's Law of Cooling.

Example 7.1.3. According to Newton's Law of Cooling[9] the temperature of coffee $T(t)$ (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$.

1. Find and interpret $T(0)$.
2. Sketch the graph of $y = T(t)$ using transformations.
3. Find and interpret $\displaystyle\lim_{t\to\infty} T(t)$.

Solution.

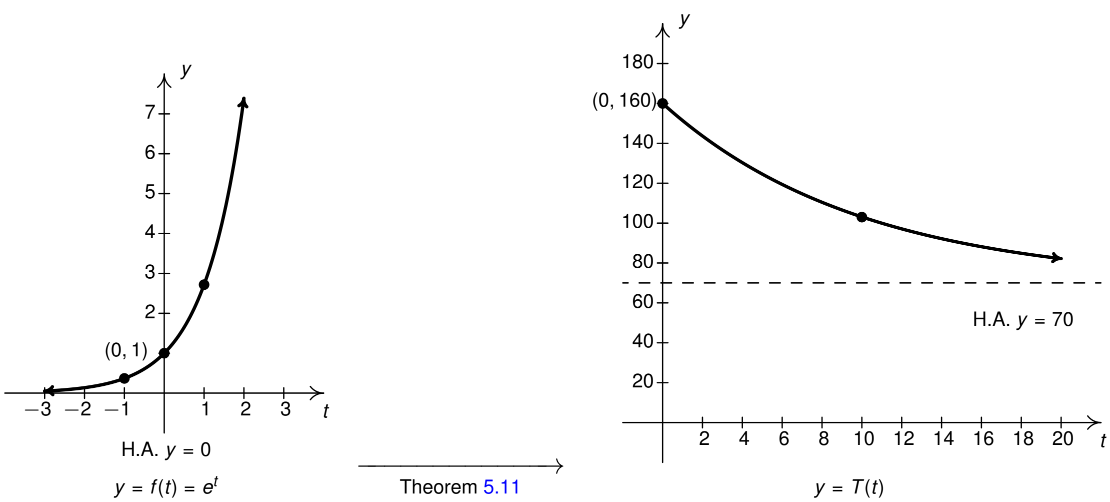
1. Since $T(0) = 70 + 90e^{-0.1(0)} = 160$, the temperature of the coffee when it is served is 160°F.

2. To graph $y = T(t)$ using transformations, we start with the basic function, $f(t) = e^t$. As in Example 7.1.1, we track the points $(-1, e^{-1}) \approx (-1, 0.368)$, $(0, 1)$, and $(1, e) \approx (1, 2.718)$, along wth the horizontal asymptote $y = 0$ through each of transformations.

   To use Theorem 5.11, we rewrite $T(t) = 70 + 90e^{-0.1t} = 90e^{-0.1t} + 70 = 90f(-0.1t) + 70$. Following Theorem 5.11, we first divide the t-coordinates of each point on the graph of $y = f(t)$ by -0.1 which results in a horizontal expansion by a factor of 10 as well as a reflection about the y-axis.

   Next, we multiply the y-values of the points on this new graph by 90 which effects a vertical stretch by a factor of 90. Last but not least, we add 70 to all of the y-coordinates of the points on this second graph, which shifts the graph upwards 70 units.

   Tracking points, we have $(-1, e^{-1}) \to (10, e^{-1}) \to (10, 90e^{-1}) \to (10, 90e^{-1} + 70) \approx (10, 103.112)$, $(0, 1) \to (0, 1) \to (0, 90) \to (0, 160)$, and $(1, e) \to (-10, e) \to (-10, 90e) \to (-10, 90e + 70) \approx (-10, 314.62)$. The horizontal asymptote $y = 0$ is unaffected by the horizontal expansion, reflection about the y-axis, and the vertical stretch. The vertical shift moves the horizontal asymptote up 70 units, $y = 0 \to y = 70$. After restricting the domain to $t \geq 0$, we get the graph below on the right.



[9] We will discuss this in greater detail in Section 7.6.

3. We can determine $\lim\limits_{t\to\infty} T(t)$ two ways. First, we can employ the 'number sense' developed in Chapter 3.

   That is, as $t \to \infty$, We get $T(t) = 70 + 90e^{-0.1t} \approx 70 + 90e^{\text{very big }(-)}$. Since $e > 1$, $e^{\text{very big }(-)} \approx$ very small (+) The larger t becomes, the smaller $e^{-0.1t}$ becomes, so the term $90e^{-0.1t} \approx$ very small (+). Hence, $T(t) = 70 + 90e^{-0.1t} \approx 70 +$ very small (+) ≈ 70.

   Alternatively, we can look to the graph of $y = T(t)$. We know the horizontal asymptote is $y = 70$ which means as $t \to \infty$, $T(t) \approx 70$.

   In either case, we find that as time goes by, the temperature of the coffee is cooling to 70° Fahrenheit, ostensibly room temperature. $\square$

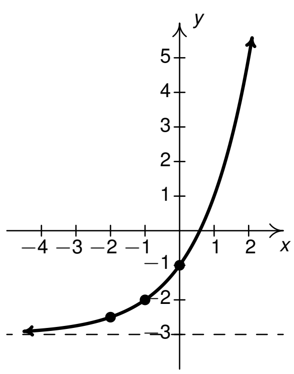
7.1.1 Exercises

In Exercises 1 - 8, sketch the graph of g by starting with the graph of f and using transformations. Track at least three points of your choice and the horizontal asymptote through the transformations. State the domain and range of g.

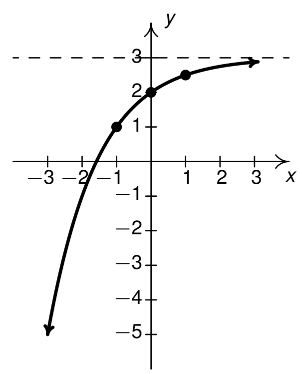
1. $f(x) = 2^x$, $g(x) = 2^x - 1$
2. $f(x) = \left(\frac{1}{3}\right)^x$, $g(x) = \left(\frac{1}{3}\right)^{x-1}$
3. $f(x) = 3^x$, $g(x) = 3^{-x} + 2$
4. $f(x) = 10^x$, $g(x) = 10^{\frac{x+1}{2}} - 20$
5. $f(t) = (0.5)^t$, $g(t) = 100(0.5)^{0.1t}$
6. $f(t) = (1.25)^t$, $g(t) = 1 - (1.25)^{t-2}$
7. $f(t) = e^t$, $g(t) = 8 - e^{-t}$
8. $f(t) = e^t$, $g(t) = 10e^{-0.1t}$

In Exercises, 9 - 12, the graph of an exponential function is given. Find a formula for the function in the form $F(x) = a \cdot 2^{bx-h} + k$.

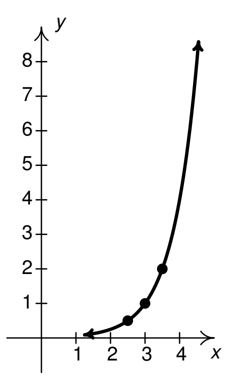
9. Points: $\left(-2, -\frac{5}{2}\right)$, $(-1, -2)$, $(0, -1)$,
Asymptote: $y = -3$.



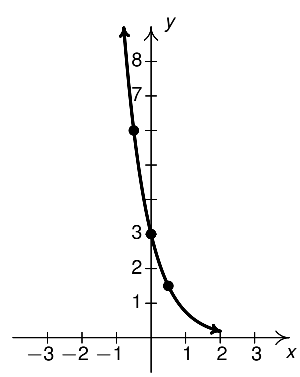
10. Points: $(-1, 1)$, $(0, 2)$, $\left(1, \frac{5}{2}\right)$,
Asymptote: $y = 3$.



11. Points: $\left(\frac{5}{2}, \frac{1}{2}\right)$, $(3, 1)$, $\left(\frac{7}{2}, 2\right)$,
Asymptote: $y = 0$.



12. Points: $\left(-\frac{1}{2}, 6\right)$, $(0, 3)$, $\left(\frac{1}{2}, \frac{3}{2}\right)$,
Asymptote: $y = 0$.



13. Find a formula for each graph in Exercises 9 - 12 of the form $G(x) = a \cdot 4^{bx-h} + k$. Did you change your solution methodology? What is the relationship between your answers for $F(x)$ and $G(x)$ for each graph?

14. In Example 7.1.1 number 2, we obtained the solution $F(x) = -2^{x+3} + 4$ as one formula for the given graph by making a simplifying assumption that $a = -1$. This exercises explores if there are any other solutions for different choices of a.

    (a) Show $G(x) = -4 \cdot 2^{x+1} + 4$ also fits the data for the given graph, and use properties of exponents to show $G(x) = F(x)$. (Use the fact that $4 = 2^2 \ldots$)

    (b) With help from your classmates, find solutions to Example 7.1.1 number 2 using $a = -8$, $a = -16$ and $a = -\frac{1}{2}$. Show all your solutions can be rewritten as: $F(x) = -2^{x+3} + 4$.

    (c) Using properties of exponents and the fact that the range of 2^x is $(0, \infty)$, show that any function of the form $f(x) = -a \cdot 2^{bx-h} + k$ for $a > 0$ can be rewritten as $f(x) = -2^c\, 2^{bx-h} + k = -2^{bx-h+c} + k$. Relabeling, this means every function of the form $f(x) = -a \cdot 2^{bx-h} + k$ with four parameters (a, b, h, and k) can be rewritten as $f(x) = -2^{bx-H} + k$, a formula with just three parameters: b, H, and k. Conclude that *every* solution to Example 7.1.1 number 2 reduces to $F(x) = -2^{x+3} + 4$.

In Exercises 15 - 20, write the given function as a nontrivial decomposition of functions as directed.

15. For $f(x) = e^{-x} + 1$, find functions g and h so that $f = g + h$.

16. For $f(x) = e^{2x} - x$, find functions g and h so that $f = g - h$.

17. For $f(t) = t^2 e^{-t}$, find functions g and h so that $f = gh$.

18. For $r(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, find functions f and g so $r = \dfrac{f}{g}$.

19. For $k(x) = e^{-x^2}$, find functions f and g so that $k = g \circ f$.

20. For $s(x) = \sqrt{e^{2x} - 1}$, find functions f and g so $s = g \circ f$.

21. The amount of money in a savings account, $A(t)$, in dollars, t years after an initial investment is made is given by: $A(t) = 500(1.05)^t$, for $t \geq 0$.

    (a) Find and interpret $A(0)$, $A(1)$, and $A(2)$.

    (b) Find and interpret the relative rate of change of A over the intervals $[0, 1]$, $[1, 2]$, $[0, 2]$.

    (c) Find, simplify, and interpret the relative rate of change of A over the $[t, t + 1]$. Assume $t \geq 0$.

    (d) Use a graphing utility to estimate how long until the savings account is worth \$1500. Round your answer to the nearest year.

22. Based on census data,[10] the population of Lake County, Ohio, in 2010 was 230,041 and in 2015, the population was 229,437.

    (a) Show the percentage change in the population from 2010 to 2015 is approximately -0.263%.

    (b) If this percentage change remains constant, predict the population of Lake County in 2020.

    (c) Assuming this percentage change per five years remains constant, find an expression for the population $P(t)$ of Lake County where t is the number of five year intervals after 2010. (So $t = 0$ corresponds to 2010, $t = 1$ corresponds to 2015, $t = 2$ corresponds to 2020, etc.)

        HINT: Definitions 7.2 and 7.3 and ensuing discussion on that page is useful here.

    (d) Use your answer to 22c to predict the population of Lake County in the year 2017.

    (e) Let $A(t)$ represent the population of Lake County t years after 2010 where the we approximate the percentage change in population per year as $-\frac{0.263\%}{5} = -0.0526\%$. Find a formula for $A(t)$ and compare your predictions with $A(t)$ to those given by $P(t)$. In particular, what population does each model give for the year 2050? Discuss any discrepancies with your classmates.

23. Show that the average rate of change of a function over the interval $[x, x+2]$ is average of the average rates of change of the function over the intervals $[x, x+1]$ and $[x+1, x+2]$. Can the same be said for the average rate of change of the function over $[x, x+3]$ and the average of the average rates of change over $[x, x+1]$, $[x+1, x+2]$, and $[x+2, x+3]$? Generalize.

24. If $f(x) = b^x$ where $b > 0$, $b \neq 1$, show $f(x_0 + \Delta x) = f(x_0)b^{\Delta x}$.

25. Which is larger: e^π or π^e? How do you know? Can you find a proof that doesn't use technology?

26. (a) Use properties of exponential functions to show that if $f(x) = e^x$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} e^x \left(\frac{e^h - 1}{h}\right)$$

    (b) Numerically and graphically investigate the limit: $\lim\limits_{h\to 0} \frac{e^h-1}{h}$.

    (c) Use parts 26a and 26b to show to your surprise and delight that the derivative of e^x is $\ldots e^x$.

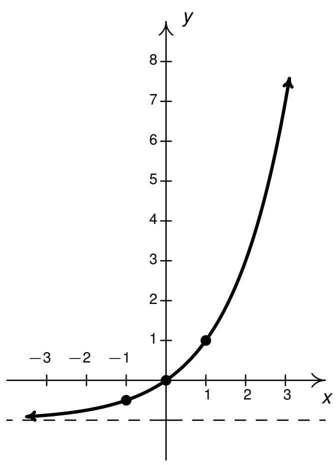
    (d) Write the equations of the tangent lines to $y = e^x$ at the following points:

        i. $(0, 1)$
        ii. $(1, e)$
        iii. $\left(-1, e^{-1}\right)$

[10] See here.

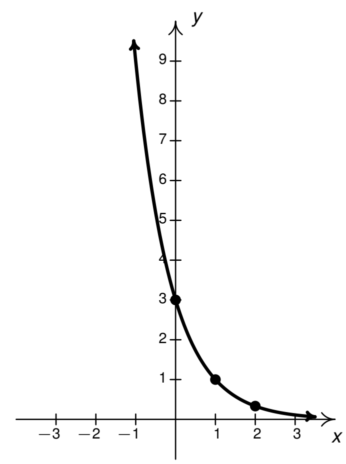
7.1.2 Answers

1. Domain of g: $(-\infty, \infty)$
   Range of g: $(-1, \infty)$
   Points: $\left(-1, -\frac{1}{2}\right)$, $(0, 0)$, $(1, 1)$
   Asymptote: $y = -1$



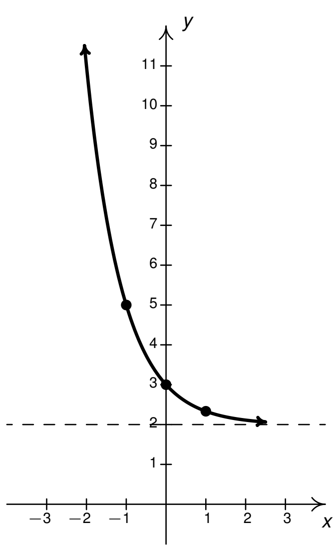
$y = g(x) = 2^x - 1$

2. Domain of g: $(-\infty, \infty)$
   Range of g: $(0, \infty)$
   Points: $(0, 3)$, $(1, 1)$, $\left(2, \frac{1}{3}\right)$
   Asymptote: $y = 0$



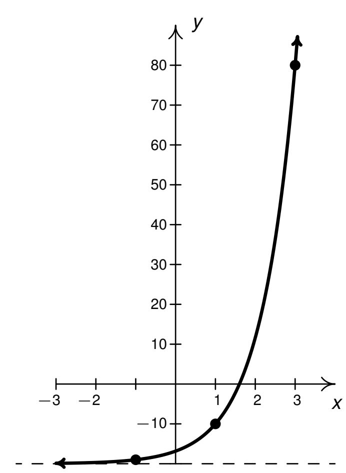
$y = g(x) = \left(\frac{1}{3}\right)^{x-1}$

3. Domain of g: $(-\infty, \infty)$
   Range of g: $(2, \infty)$
   Points: $\left(1, \frac{7}{3}\right)$, $(0, 3)$, $(-1, 5)$
   Asymptote: $y = 2$



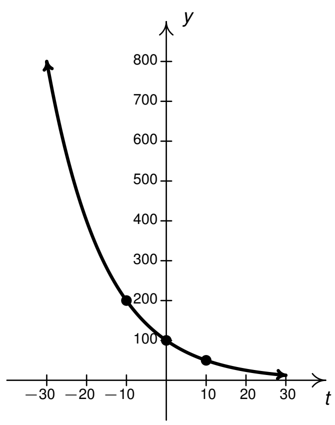
$y = g(x) = 3^{-x} + 2$

4. Domain of g: $(-\infty, \infty)$
   Range of g: $(-20, \infty)$
   Points: $(-1, -19)$, $(1, -10)$, $(3, 80)$
   Asymptote: $y = -20$



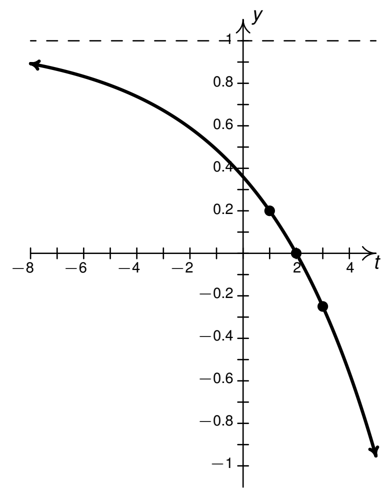
$y = g(x) = 10^{\frac{x+1}{2}} - 20$

5. Domain of g: $(-\infty, \infty)$
   Range of g: $(0, \infty)$
   Points: $(-10, 200)$, $(0, 100)$, $(10, 50)$
   Asymptote: $y = 0$



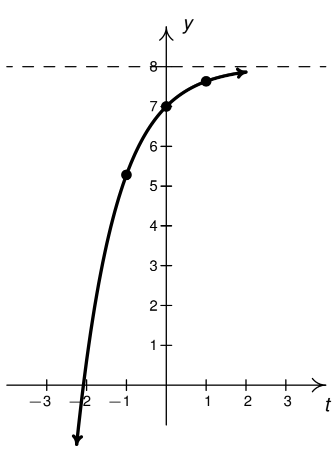
$y = g(t) = 100(0.5)^{0.1t}$

6. Domain of g: $(-\infty, \infty)$
   Range of g: $(-\infty, 1)$
   Points: $(1, 0.2)$, $(2, 0)$, $(3, -0.25)$
   Asymptote: $y = 1$



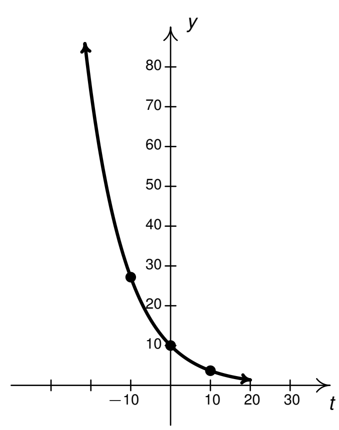
$y = g(t) = 1 - (1.25)^{t-2}$

7. Domain of g: $(-\infty, \infty)$
   Range of g: $(-\infty, 8)$
   Points: $\left(1, 8 - e^{-1}\right) \approx (1, 7.63)$,
   $(0, 7)$, $(-1, 8 - e) \approx (1, 5.28)$
   Asymptote: $y = 8$



$y = g(t) = 8 - e^{-t}$

8. Domain of g: $(-\infty, \infty)$
   Range of g: $(0, \infty)$
   Points: $\left(10, 10e^{-1}\right) \approx (10, 3.68)$
   $(0, 10)$, $(-10, 10e) \approx (-10, 27.18)$
   Asymptote: $y = 0$



$y = g(t) = 10e^{-0.1t}$

9. $F(x) = 2^{x+1} - 3$ 10. $F(x) = -2^{-x} + 3$ 11. $F(x) = 2^{2x-6}$ 12. $F(x) = 3 \cdot 2^{-2x}$

13. Since $2 = 4^{\frac{1}{2}}$, one way to obtain the formulas for $G(x)$ is to use properties of exponents. For example, $F(x) = 2^{x+1} - 3 = \left(4^{\frac{1}{2}}\right)^{x+1} - 3 = 4^{\frac{1}{2}(x+1)} - 3 = 4^{\frac{1}{2}x+\frac{1}{2}} - 3$. In order, the formulas for $G(x)$ are:
   - $G(x) = 4^{\frac{1}{2}x+\frac{1}{2}} - 3$
   - $G(x) = -4^{-\frac{1}{2}x} + 3$
   - $G(x) = 4^{x-3}$
   - $G(x) = 3 \cdot 4^{-x}$

15. [11]$g(x) = e^{-x}$ and $h(x) = 1$.

16. $g(x) = e^{2x}$ and $h(x) = x$.

17. $g(t) = t^2$ and $h(t) = e^{-t}$.

18. $f(x) = e^x - e^{-x}$ and $g(x) = e^x + e^{-x}$.

19. $f(x) = -x^2$ and $g(x) = e^x$.

20. $f(x) = e^{2x} - 1$ and $g(x) = \sqrt{x}$.

21. (a) $A(0) = 500$, so the principal is \$500. $A(1) = 525$, so after 1 year, there is \$525 in the savings account. $A(2) = 551.25$, so after 2 years, there is \$551.25 in the savings account.
    (b) The relative rate of change of A over the intervals $[0, 1]$ and $[1, 2]$ is 0.05 which means the savings account is growing by 5% each year for those two years. Over the interval $[0, 2]$, the relative rate of change is 0.1025 meaning the account has grown by 10.25% over the course of the first two years. Note this is greater than the sum of the two rates 5% + 5% = 10%. This is due to the 'compounding effect' and will be discussed in greater detail in Section 7.6.
    (c) The relative rate of change of A over the $[t, t + 1]$ is 0.05. This means over the course of one year, the savings account grows by 5%.
    (d) Graphing $y = A(t)$ and $y = 1500$, we find they intersect when $t \approx 22.5$ so it takes approximately $22 - 23$ years for the savings account to grow to \$1500 in value.

22. (a) $\frac{229437-230041}{230041} \approx 0.263\%$.
    (b) Since 2020 is five years after 2015, we expect the population to decrease by 0.263% of 229437, or approximately 603 people. Hence, we approximate the population in 2020 as 228834.
    (c) $P(t) = 230041(1 - 0.00263)^t = 230041(0.99737)^t$, $t \geq 0$.
    (d) Since 2017 is 7 years after 2010, we set $t = \frac{7}{5} = 1.4$ and find $P(1.4) \approx 229194$. So the population is approximately 229, 194 in 2017.
    (e) $A(t) = 230041(1 - 0.0005626)^t = 230041(0.999474)^t$, $t \geq 0$. Since 2050 is 40 years after 2010, using the model $P(t)$, we divide $\frac{40}{5} = 8$ and find $P(8) \approx 225,245$. On the other hand, $A(40) \approx 225,250$. This is more than roundoff error. There is a compounding effect which makes the functions $A(t)$ and $P(t)$ different. [12]

26. (d) i. $y = x + 1$ ii. $y = e\,x$ iii. $y = e^{-1}\,x + 2e^{-1}$

[11] Answers for Exercises 15 - 20 vary. We list one solution for each problem.
[12] See number 21 above or, for more, see Section 7.6.

7.2 Logarithmic Functions

In Section 7.1, we saw exponential functions $f(x) = b^x$ are are one-to-one which means they are invertible. In this section, we explore their inverses, the *logarithmic functions* which are called 'logs' for short.

Definition 7.4. For the exponential function $f(x) = b^x$, $f^{-1}(x) = \log_b(x)$ is called the **base *b* logarithm function**. We read '$\log_b(x)$' as 'log base b of x.'

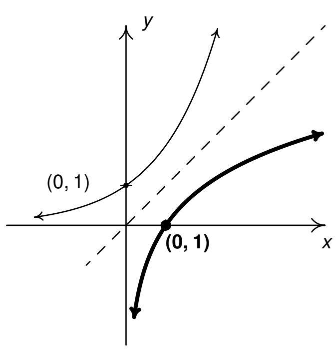
We have special notations for the common base, $b = 10$, and the natural base, $b = e$.

Definition 7.5.

- The **common logarithm** of a real number x is $\log_{10}(x)$ and is usually written $\log(x)$.
- The **natural logarithm** of a real number x is $\log_e(x)$ and is usually written $\ln(x)$.

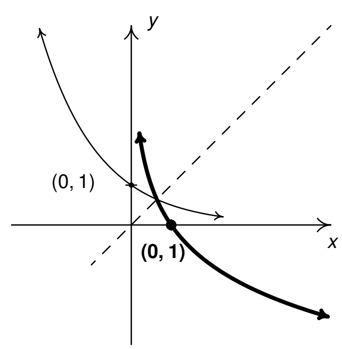
Since logs are defined as the inverses of exponential functions, we can use Theorems 5.13 and 7.1 to tell us about logarithmic functions. For example, we know that the domain of a log function is the range of an exponential function, namely $(0, \infty)$, and that the range of a log function is the domain of an exponential function, namely $(-\infty, \infty)$.

Moreover, since we know the basic shapes of $y = f(x) = b^x$ for the different cases of b, we can obtain the graph of $y = f^{-1}(x) = \log_b(x)$ by reflecting the graph of f across the line $y = x$. The y-intercept $(0, 1)$ on the graph of f corresponds to an x-intercept of $(1, 0)$ on the graph of f^{-1}. The horizontal asymptotes $y = 0$ on the graphs of the exponential functions become vertical asymptotes $x = 0$ on the log graphs.



$y = b^x$, $b > 1$

$\boldsymbol{y = \log_b(x)}$, $b > 1$



$y = b^x$, $0 < b < 1$

$\boldsymbol{y = \log_b(x)}$, $0 < b < 1$

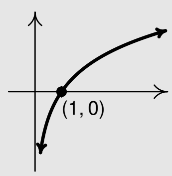
Procedurally, logarithmic functions 'undo' the exponential functions. Consider the function $f(x) = 2^x$. When we evaluate $f(3) = 2^3 = 8$, the input 3 becomes the exponent on the base 2 to produce the real number 8. The function $f^{-1}(x) = \log_2(x)$ then takes the number 8 as its input and returns the exponent 3 as its output. In symbols, $\log_2(8) = 3$.

More generally, $\log_2(x)$ is the exponent you put on 2 to get x. Thus, $\log_2(16) = 4$, because $2^4 = 16$. The following theorem summarizes the basic properties of logarithmic functions, all of which come from the fact that they are inverses of exponential functions.

Theorem 7.3. Properties of Logarithmic Functions: Suppose $f(x) = \log_b(x)$.

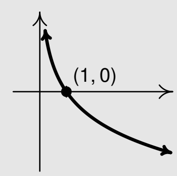
- The domain of f is $(0, \infty)$ and the range of f is $(-\infty, \infty)$.
- $(1, 0)$ is on the graph of f and $x = 0$ is a vertical asymptote of the graph of f.
- f is one-to-one, continuous and smooth
- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b\left(b^x\right) = x$ for all real numbers x and $b^{\log_b(x)} = x$ for all $x > 0$

- If $b > 1$:
  - f is always increasing
  - $\lim\limits_{x \to 0^+} f(x) = -\infty$
  - $\lim\limits_{x \to \infty} f(x) = \infty$
  - The graph of f resembles:



$y = \log_b(x)$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - $\lim\limits_{x \to 0^+} f(x) = \infty$
  - $\lim\limits_{x \to \infty} f(x) = -\infty$
  - The graph of f resembles:



$y = \log_b(x)$, $0 < b < 1$

As we have mentioned, Theorem 7.3 is a consequence of Theorems 5.13 and 7.1. However, it is worth the reader's time to understand Theorem 7.3 from an exponent perspective.

As an example, we know that the domain of $g(x) = \log_2(x)$ is $(0, \infty)$. Why? Because the range of $f(x) = 2^x$ is $(0, \infty)$. In a way, this says everything, but at the same time, it doesn't.

To really *understand* why the domain of $g(x) = \log_2(x)$ is $(0, \infty)$, consider trying to compute $\log_2(-1)$. We are searching for the exponent we put on 2 to give us -1. In other words, we are looking for x that satisfies $2^x = -1$. There is no such real number, since all powers of 2 are positive.

While what we have said is exactly the same thing as saying 'the domain of $g(x) = \log_2(x)$ is $(0, \infty)$ because the range of $f(x) = 2^x$ is $(0, \infty)$', we feel it is in a student's best interest to understand the statements in Theorem 7.3 at this level instead of just merely memorizing the facts.

Our first example gives us practice computing logarithms as well as constructing basic graphs.

Example 7.2.1.

1. Simplify the following.

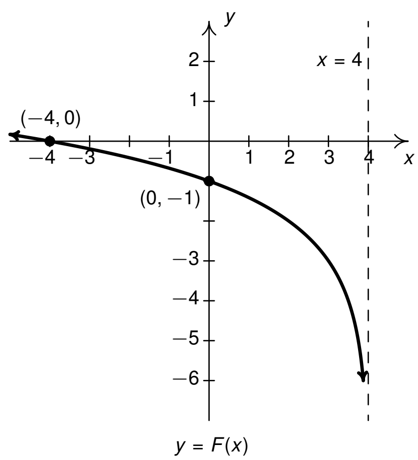
   (a) $\log_3(81)$ (b) $\log_2\left(\dfrac{1}{8}\right)$ (c) $\log_{\sqrt{5}}(25)$ (d) $\ln\left(\sqrt[3]{e^2}\right)$

   (a) $\log(0.001)$ (b) $2^{\log_2(8)}$ (c) $117^{-\log_{117}(6)}$

2. Graph the following functions by starting with a basic logarithmic function and using transformations, Theorem 5.11. Track at least three points and the vertical asymptote through the transformations.

   (a) $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ (b) $G(t) = -\ln(2 - t)$

3. Find a formula for the graph of the function below. Assume the base of the logarithm is 2.



Solution.

1. (a) The number $\log_3(81)$ is the exponent we put on 3 to get 81. As such, we want to write 81 as a power of 3. We find $81 = 3^4$, so that $\log_3(81) = 4$.

   (b) To find $\log_2\left(\frac{1}{8}\right)$, we need rewrite $\frac{1}{8}$ as a power of 2. We find $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.

   (c) To determine $\log_{\sqrt{5}}(25)$, we need to express 25 as a power of $\sqrt{5}$. We know $25 = 5^2$, and $5 = \left(\sqrt{5}\right)^2$, so we have $25 = \left(\left(\sqrt{5}\right)^2\right)^2 = \left(\sqrt{5}\right)^4$. We get $\log_{\sqrt{5}}(25) = 4$.

   (d) First, recall that the notation $\ln\left(\sqrt[3]{e^2}\right)$ means $\log_e\left(\sqrt[3]{e^2}\right)$, so we are looking for the exponent to put on e to obtain $\sqrt[3]{e^2}$. Rewriting $\sqrt[3]{e^2} = e^{2/3}$, we find $\ln\left(\sqrt[3]{e^2}\right) = \ln\left(e^{2/3}\right) = \frac{2}{3}$.

   (e) Rewriting $\log(0.001)$ as $\log_{10}(0.001)$, we see that we need to write 0.001 as a power of 10. We have $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$. Hence, $\log(0.001) = \log\left(10^{-3}\right) = -3$.

(f) We can use Theorem 7.3 directly to simplify $2^{\log_2(8)} = 8$.

We can also understand this problem by first finding $\log_2(8)$. By definition, $\log_2(8)$ is the exponent we put on 2 to get 8. Since $8 = 2^3$, we have $\log_2(8) = 3$.

We now substitute to find $2^{\log_2(8)} = 2^3 = 8$.

(g) From Theorem 7.3, we know $117^{\log_{117}(6)} = 6$,[1] but we cannot directly apply this formula to the expression $117^{-\log_{117}(6)}$ without first using a property of exponents. (Can you see why?)

Rather, we find: $117^{-\log_{117}(6)} = \frac{1}{117^{\log_{117}(6)}} = \frac{1}{6}$.

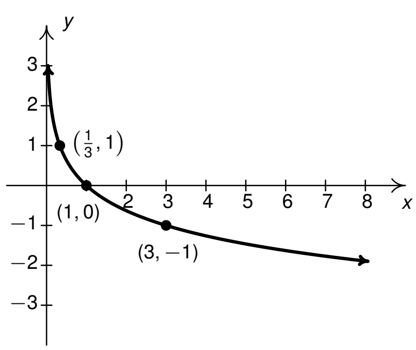
2. (a) To graph $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ we start with the graph of $f(x) = \log_{\frac{1}{3}}(x)$. and use Theorem 5.11.

First we choose some 'control points' on the graph of $f(x) = \log_{\frac{1}{3}}(x)$. Since we are instructed to track three points (and the vertical asymptote, $x = 0$) through the transformations, we choose the points corresponding to powers of $\frac{1}{3}$: $\left(\frac{1}{3}, 1\right)$, $(1, 0)$, and $(3, -1)$, respectively.

Next, we note $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1 = f\left(\frac{x}{2}\right) + 1$. Per Theorem 5.11, we first multiply the x-coordinates of the points on the graph of $y = f(x)$ by 2, horizontally expanding the graph by a factor of 2. Next, we add 1 to the y-coordinates of each point on this new graph, vertically shifting the graph up 1.

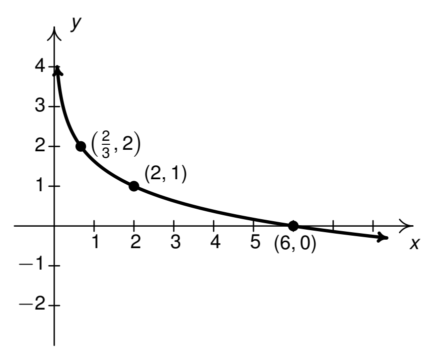
Looking at each point, we get $\left(\frac{1}{3}, 1\right) \to \left(\frac{2}{3}, 1\right) \to \left(\frac{2}{3}, 2\right)$, $(1, 0) \to (2, 0) \to (2, 1)$, and $(3, -1) \to (6, -1) \to (6, 0)$. The horizontal asymptote, $x = 0$ remains unchanged under the horizontal stretch and the vertical shift.

Below we graph $y = f(x) = \log_{\frac{1}{3}}(x)$ on the left and $y = F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ on the right.



$y = f(x) = \log_{\frac{1}{3}}(x)$

$\xrightarrow{\text{Theorem 5.11}}$



$y = F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$

As always we can check our answer by verifying each of the points $\left(\frac{2}{3}, 2\right)$, $(2, 1)$, , and $(6, 0)$, is on the graph of $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ by checking $F\left(\frac{2}{3}\right) = 2$, $F(2) = 1$, and $F(6) = 0$. We can check the end behavior as well, that is, $\lim\limits_{x\to 0^+} F(x) = \infty$ and as $\lim\limits_{x\to\infty} F(x) \to -\infty$. We leave these calculations to the reader.

[1] It is worth a moment of your time to think your way through why $117^{\log_{117}(6)} = 6$. By definition, $\log_{117}(6)$ is the exponent we put on 117 to get 6. What are we doing with this exponent? We are putting it on 117, so we get 6.

(b) Since the base of $G(t) = -\ln(2-t)$ is e, we start with the graph of $g(t) = \ln(t)$. As usual, since e is an irrational number, we use the approximation $e \approx 2.718$ when plotting points, but label points using exact coordinates in terms of e.

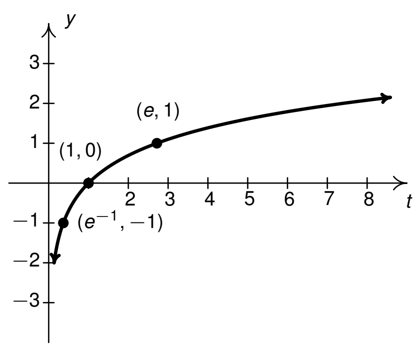
We choose points corresponding to powers of e on the graph of $g(t) = \ln(t)$: $(e^{-1}, -1) \approx (0.368, -1)$, $(1, 0)$, and $(e, 1) \approx (2.718, 1)$, respectively.

Since $G(t) = -\ln(2-t) = -\ln(-t+2) = -g(-t+2)$, Theorem 5.11 instructs us to first subtract 2 from each of the t-coordinates of the points on the graph of $g(t) = \ln(t)$, shifting the graph to the left two units.

Next, we multiply (divide) the t-coordinates of points on this new graph by -1 which reflects the graph across the y-axis. Lastly, we multiply each of the y-coordinates of this second graph by -1, reflecting it across the t-axis.

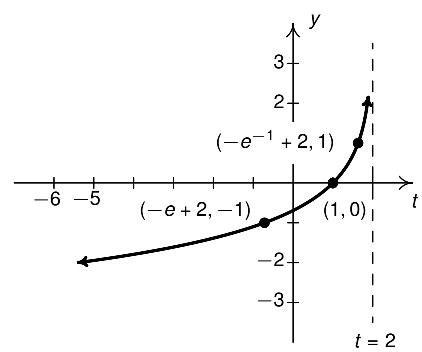
Tracking points, we have $(e^{-1}, -1) \to (e^{-1} - 2, -1) \to (-e^{-1} + 2, -1) \to (-e^{-1} + 2, 1) \approx (1.632, 1)$, $(1, 0) \to (-1, 0) \to (1, 0) \to (1, 0)$, and $(e, 1) \to (e - 2, 1) \to (-e + 2, 1) \to (-e + 2, -1) \approx (-0.718, -1)$. The vertical asymptote is affected by the horizontal shift and the reflection about the y-axis only: $t = 0 \to t = -2 \to t = 2$.

We graph $g(t) = \ln(t)$ below on the left and the transformed function $G(t) = -\ln(-t+2)$ below on the right. As usual, we can check our answer by verifying the indicated points do, in fact, lie on the graph of $y = G(t)$ along with checking the behavior as $t \to -\infty$ and $t \to 2^{-}$.



$y = g(t) = \ln(t)$

$\xrightarrow{\hspace{2cm}}$ Theorem 5.11



$y = G(t) = -\ln(-t+2)$

3. Since we are told to assume the base of the exponential function is 2, we assume the function $F(x)$ is the result of the transforming the graph of $f(x) = \log_2(x)$ using Theorem 5.11. This means we are tasked with finding values for a, b, h, and k so that $F(x) = af(bx - h) + k = a\log_2(bx - h) + k$.

   Since the vertical asymptote to the graph of $y = f(x) = \log_2(x)$ is $x = 0$ and the vertical asymptote to the graph $y = F(x)$ is $x = 4$, we know we have a vertical shift of 4 units. Moreover, since the curve approaches the vertical asymptote from the *left*, we also know we have a reflection about the y-axis, so $b < 0$. Since the recipe in Theorem 5.11 instructs us to perform the vertical shift *before* the reflection across the y-axis, we take $h = -4$ and assume for simplicity $b = -1$ so $F(x) = a\log_2(-x + 4) + k$.

To determine a and k, we make use of the two points on the graph. Since $(-4, 0)$ is on the graph of F, $F(-4) = a\log_2(-(-4) + 4) + k = 0$. This reduces to $a\log_2(8) + k = 0$ or $3a + k = 0$. Next, we use the point $(0, -1)$ to get $F(0) = a\log_2(-(0) + 4) + k = -1$. This reduces to $a\log_2(4) + k = -1$ or $2a + k = -1$. From $3a + k = 0$, we get $k = -3a$ which when substituted into $2a + k = -1$ gives $2a + (-3a) = -1$ or $a = 1$. Hence, $k = -3a = -3(1) = -3$.

Putting all of this work together we find $F(x) = \log_2(-x+4)-3$. As always, we can check our answer by verifying $F(-4) = 0$, $F(0) = -1$, $\displaystyle\lim_{x\to-\infty} F(x) = \infty$, and $\displaystyle\lim_{x\to 4^-} F(x) = -\infty$. We leave these details to the reader.[2]

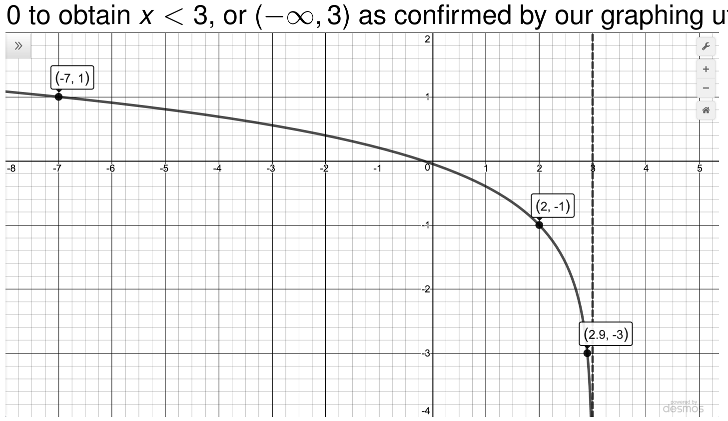
Up until this point, restrictions on the domains of functions came from avoiding division by zero and keeping negative numbers from beneath even indexed radicals. With the introduction of logs, we now have another restriction. Since the domain of $f(x) = \log_b(x)$ is $(0, \infty)$, the argument of the log[3] must be strictly positive.

Example 7.2.2. Find the domain each function analytically and check your answer using a graphing utility.

1. $f(x) = 2\log(3 - x) - 1$
2. $g(x) = \ln\left(\dfrac{x}{x-1}\right)$

Solution.

1. We set $3 - x > 0$ to obtain $x < 3$, or $(-\infty, 3)$ as confirmed by our graphing utility below.



Note that in this case, we can graph f using transformations, which we do so here for extra practice.

Taking a cue from Theorem 5.11, we rewrite $f(x) = 2\log_{10}(-x + 3) - 1$ and view this function as a transformed version of $h(x) = \log_{10}(x)$.

To graph $y = \log(x) = \log_{10}(x)$, We select three points to track corresponding to powers of 10: $(0.1, -1)$, $(1, 0)$ and $(10, 1)$, along with the vertical asymptote $x = 0$.

[2]As with Exercise 7.1.1 in Section 7.1, we may well wonder if our solution to this problem is the *only* solution since we made a simplifying assumption that $b = -1$. We leave this for a thoughtful discussion in Exercise 40 in Section 7.3.

[3]that is, what's 'inside' the log

Since $f(x) = 2h(-x + 3) - 1$, Theorem 5.11 tells us that to obtain the destinations of these points, we first subtract 3 from the x-coordinates (shifting the graph left 3 units), then divide (multiply) by the x-coordinates by -1 (causing a reflection across the y-axis).

Next, we multiply the y-coordinates by 2 which results in a vertical stretch by a factor of 2, then we finish by subtracting 1 from the y-coordinates which shifts the graph down 1 unit.

Tracking points, we find: $(0.1, -1) \to (-2.9, -1) \to (2.9, -1) \to (2.9, -2) \to (2.9, -3)$, $(1, 0) \to (-2, 0) \to (2, 0) \to (2, 0) \to (2, -1)$, and $(10, 1) \to (7, 1) \to (-7, 1) \to (-7, 2) \to (-7, 1)$. The vertical shift and reflection about the y-axis affects the vertical asymptote: $x = 0 \to x = -3 \to x = 3$.

Plotting these three points along with the vertical asymptote produces the graph of f as seen above.

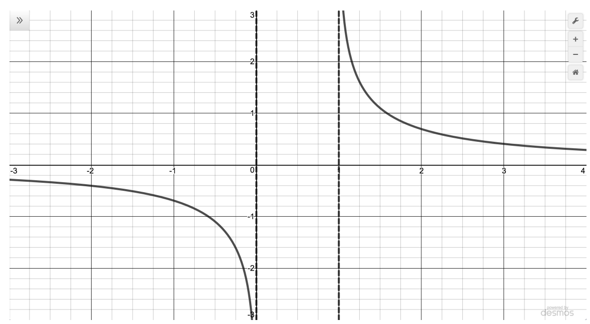
2. To find the domain of g, we need to solve the inequality $\frac{x}{x-1} > 0$ using a sign diagram.[4]

   If we define $r(x) = \frac{x}{x-1}$, we find r is undefined at $x = 1$ and $r(x) = 0$ when $x = 0$. Choosing some test values, we generate the sign diagram below on the left.

   We find $\frac{x}{x-1} > 0$ on $(-\infty, 0) \cup (1, \infty)$ which is the domain of g. The graph below confirms this.

   (+) 0 (−) ? (+)

   0 1



   We can tell from the graph of g that it is not the result of Section 5.4 transformations being applied to the graph $y = \ln(x)$, (do you see why?) so barring a more detailed analysis using Calculus, producing a graph using a graphing utility is the best we can do.

   One thing worthy of note, however, is the end behavior of g. The graph suggests that $\lim\limits_{x\to-\infty} g(x) = 0$ and $\lim\limits_{x\to\infty} g(x) = 0$. We can verify this analytically. Using results from Chapter 3 and continuity, we know that $\lim\limits_{x\to-\infty} \dfrac{x}{x+1} = 1$ and $\lim\limits_{x\to\infty} \dfrac{x}{x+1} = 1$. Hence, it stands to reason that

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \ln\left(\frac{x}{x-1}\right) = ln(1) = 0$$

   and, likewise, $\lim\limits_{x\to-\infty} g(x) = ln(1) = 0$. □

While logarithms have some interesting applications of their own which you'll explore in the exercises, their primary use to us will be to undo exponential functions. (This is, after all, how they were defined.) Our last example reviews not only the major topics of this section, but reviews the salient points from Section 5.6.

[4] See Section 3.2 for a review of this process, if needed.

Example 7.2.3. Let $f(x) = 2^{x-1} - 3$.

1. Graph f using transformations and state the domain and range of f.
2. Explain why f is invertible and find a formula for $f^{-1}(x)$.
3. Graph f^{-1} using transformations and state the domain and range of f^{-1}.
4. Verify $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
5. Graph f and f^{-1} on the same set of axes and check for symmetry about the line $y = x$.
6. Use f or f^{-1} to solve the following equations. Check your answers algebraically.

   (a) $2^{x-1} - 3 = 4$ (b) $\log_2(t+3) + 1 = 0$

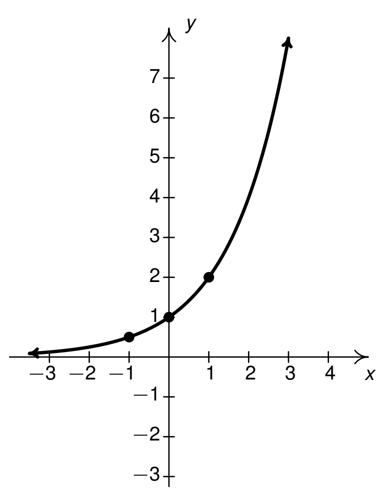
Solution.

1. To graph $f(x) = 2^{x-1} - 3$ using Theorem 5.11, we first identify $g(x) = 2^x$ and note $f(x) = g(x-1) - 3$. Choosing the 'control points' of $\left(-1, \frac{1}{2}\right)$, $(0,1)$ and $(1,2)$ on the graph of g along with the horizontal asymptote $y = 0$, we implement the algorithm set forth in Theorem 5.11.

   First, we first add 1 to the x-coordinates of the points on the graph of g which shifts the the graph of g to the right one unit. Next, we subtract 3 from each of the y-coordinates on this new graph, shifting the graph down 3 units to get the graph of f.

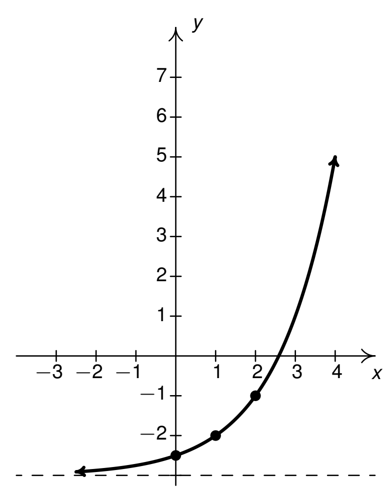
   Looking point-by-point, we have $\left(-1, \frac{1}{2}\right) \to \left(0, \frac{1}{2}\right) \to \left(0, -\frac{5}{2}\right)$, $(0,1) \to (1,1) \to (1,-2)$, and, finally, $(1,2) \to (2,2) \to (2,-1)$. The horizontal asymptote is affected only by the vertical shift, $y = 0 \to y = -3$.

   From the graph of f, we get the domain is $(-\infty, \infty)$ and the range is $(-3, \infty)$.



$y = h(x) = 2^x$

$\longrightarrow$



$y = f(x) = 2^{x-1} - 3$

2. The graph of f passes the Horizontal Line Test so f is one-to-one, hence invertible.

   To find a formula for $f^{-1}(x)$, we normally set $y = f(x)$, interchange the x and y, then proceed to solve for y. Doing so in this situation leads us to the equation $x = 2^{y-1} - 3$. We have yet to discuss how to solve this kind of equation, so we will attempt to find the formula for f^{-1} procedurally.

   Thinking of f as a process, the formula $f(x) = 2^{x-1} - 3$ takes an input x and applies the steps: first subtract 1. Second put the result of the first step as the exponent on 2. Last, subtract 3 from the result of the second step.

   Clearly, to undo subtracting 1, we will add 1, and similarly we undo subtracting 3 by adding 3. How do we undo the second step? The answer is we use the logarithm.

   By definition, $\log_2(x)$ undoes exponentiation by 2. Hence, f^{-1} should: first, add 3. Second, take the logarithm base 2 of the result of the first step. Lastly, add 1 to the result of the second ste. In symbols, $f^{-1}(x) = \log_2(x+3) + 1$.

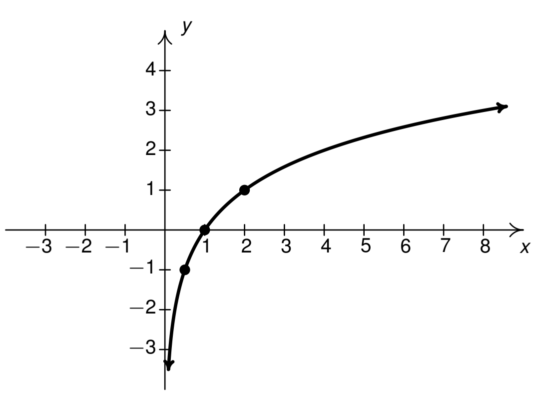
3. To graph $f^{-1}(x) = \log_2(x+3) + 1$ using Theorem 5.11, we start with $j(x) = \log_2(x)$ and track the points $\left(\frac{1}{2}, -1\right)$, $(1, 0)$ and $(2, 1)$ on the graph of j along with the vertical asymptote $x = 0$ through the transformations.

   Since $f^{-1}(x) = j(x+3) + 1$, we first subtract 3 from each of the x-coordinates of each of the points on the graph of $y = j(x)$ shifting the graph of j to the left three units. We then add 1 to each of the y-coordinates of the points on this new graph, shifting the graph up one unit.

   Tracking points, we get $\left(\frac{1}{2}, -1\right) \rightarrow \left(-\frac{5}{2}, -1\right) \rightarrow \left(-\frac{5}{2}, 0\right)$, $(1, 0) \rightarrow (-2, 1) \rightarrow (-2, 2)$, and $(2, 1) \rightarrow (-1, 1) \rightarrow (-1, 2)$.

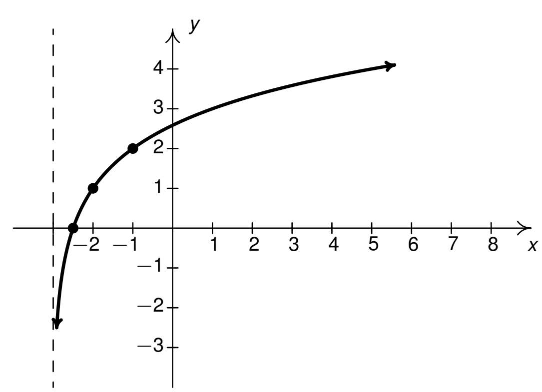
   The vertical asymptote is only affected by the horizontal shift, so we have $x = 0 \rightarrow x = -3$.

   From the graph below, we get the domain of f^{-1} is $(-3, \infty)$, which matches the range of f, and the range of f^{-1} is $(-\infty, \infty)$, which matches the domain of f, in accordance with Theorem 5.13.



$y = j(x) = \log_2(x)$

$\longrightarrow$



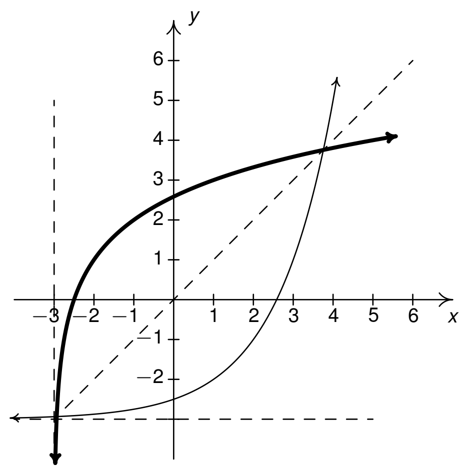
$y = f^{-1}(x) = \log_2(x+3) + 1$

4. We now verify that $f(x) = 2^{x-1} - 3$ and $f^{-1}(x) = \log_2(x+3)+1$ satisfy the composition requirement for inverses. When simplifying $(f^{-1} \circ f)(x)$ we assume x can be any real number while when simplifying $(f \circ f^{-1})(x)$, we restrict our attention to $x > -3$. (Do you see why?)

   Note the use of the inverse properties of exponential and logarithmic functions from Theorem 7.3 when it comes to simplifying expressions of the form $\log_2(2^u)$ and $2^{\log_2(u)}$.

$$\begin{array}{rcl} \left(f^{-1} \circ f\right)(x) & = & f^{-1}(f(x)) \\ & = & f^{-1}\left(2^{x-1} - 3\right) \\ & = & \log_2\left(\left[2^{x-1} - 3\right] + 3\right) + 1 \\ & = & \log_2\left(2^{x-1}\right) + 1 \\ & = & (x-1) + 1 \\ & = & x \checkmark \end{array} \qquad \begin{array}{rcl} \left(f \circ f^{-1}\right)(x) & = & f\left(f^{-1}(x)\right) \\ & = & f\left(\log_2(x+3) + 1\right) \\ & = & 2^{\left(\log_2(x+3)+1\right)-1} - 3 \\ & = & 2^{\log_2(x+3)} - 3 \\ & = & (x+3) - 3 \\ & = & x \checkmark \end{array}$$

5. Last, but certainly not least, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes and observe the symmetry about the line $y = x$.



$y = f(x) = 2^{x-1} - 3$

$\boldsymbol{y = f^{-1}(x) = \log_2(x+3) + 1}$

1. Viewing $2^{x-1} - 3 = 4$ as $f(x) = 4$, we apply f^{-1} to 'undo' f to get $f^{-1}(f(x)) = f^{-1}(4)$, which reduces to $x = f^{-1}(4)$. Since we have shown (algebraically and graphically!) that $f^{-1}(x) = \log_2(x+3) + 1$, we get $x = f^{-1}(4) = \log_2(4+3) + 1 = \log_2(7) + 1$.

   Alternatively, we know from Theorem 5.13 that $f(x) = 4$ is equivalent to $x = f^{-1}(4)$ directly.

   Note that since, by definition, $2^{\log_2(7)} = 7$, $2^{(\log_2(7)+1)-1} - 3 = 2^{\log_2(7)} - 3 = 7 - 3 = 4$, as required.

2. Since we may think of the equation $\log_2(t+3) + 1 = 0$ as $f^{-1}(t) = 0$, we can solve this equation by applying f to both sides to get $f(f^{-1}(t)) = f(0)$ or $t = 2^{0-1} - 3 = \frac{1}{2} - 3 = -\frac{5}{2}$.

   Since $\log_2(2^{-1}) = -1$, we get $\log_2\left(-\frac{5}{2} + 3\right) + 1 = \log_2\left(\frac{1}{2}\right) + 1 = \log_2(2^{-1}) - 1 + 1 = 0$, as required. $\square$

7.2.1 Exercises

In Exercises 1 - 15, use the property: $b^a = c$ if and only if $\log_b(c) = a$ from Theorem 7.3 to rewrite the given equation in the other form. That is, rewrite the exponential equations as logarithmic equations and rewrite the logarithmic equations as exponential equations.

1. $2^3 = 8$
2. $5^{-3} = \frac{1}{125}$
3. $4^{5/2} = 32$
4. $\left(\frac{1}{3}\right)^{-2} = 9$
5. $\left(\frac{4}{25}\right)^{-1/2} = \frac{5}{2}$
6. $10^{-3} = 0.001$
7. $e^0 = 1$
8. $\log_5(25) = 2$
9. $\log_{25}(5) = \frac{1}{2}$
10. $\log_3\left(\frac{1}{81}\right) = -4$
11. $\log_{\frac{4}{3}}\left(\frac{3}{4}\right) = -1$
12. $\log(100) = 2$
13. $\log(0.1) = -1$
14. $\ln(e) = 1$
15. $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$

In Exercises 16 - 42, evaluate the expression without using a calculator.

16. $\log_3(27)$
17. $\log_6(216)$
18. $\log_2(32)$
19. $\log_6\left(\frac{1}{36}\right)$
20. $\log_8(4)$
21. $\log_{36}(216)$
22. $\log_{\frac{1}{5}}(625)$
23. $\log_{\frac{1}{6}}(216)$
24. $\log_{36}(36)$
25. $\log\left(\frac{1}{1000000}\right)$
26. $\log(0.01)$
27. $\ln\left(e^3\right)$
28. $\log_4(8)$
29. $\log_6(1)$
30. $\log_{13}\left(\sqrt{13}\right)$
31. $\log_{36}\left(\sqrt[4]{36}\right)$
32. $7^{\log_7(3)}$
33. $36^{\log_{36}(216)}$
34. $\log_{36}\left(36^{216}\right)$
35. $\ln\left(e^5\right)$
36. $\log\left(\sqrt[9]{10^{11}}\right)$
37. $\log\left(\sqrt[3]{10^5}\right)$
38. $\ln\left(\frac{1}{\sqrt{e}}\right)$
39. $\log_5\left(3^{\log_3(5)}\right)$
40. $\log\left(e^{\ln(100)}\right)$
41. $\log_2\left(3^{-\log_3(2)}\right)$
42. $\ln\left(42^{6\log(1)}\right)$

In Exercises 43 - 57, find the domain of the function.

43. $f(x) = \ln(x^2 + 1)$
44. $f(x) = \log_7(4x + 8)$
45. $g(t) = \ln(4t - 20)$
46. $g(t) = \log\left(t^2 + 9t + 18\right)$

47. $f(x) = \log\left(\dfrac{x+2}{x^2-1}\right)$

48. $f(x) = \log\left(\dfrac{x^2+9x+18}{4x-20}\right)$

49. $g(t) = \ln(7-t) + \ln(t-4)$

50. $g(t) = \ln(4t-20) + \ln\left(t^2+9t+18\right)$

51. $f(x) = \log\left(x^2+x+1\right)$

52. $f(x) = \sqrt[4]{\log_4(x)}$

53. $g(t) = \log_9(|t+3|-4)$

54. $g(t) = \ln(\sqrt{t-4}-3)$

55. $f(x) = \dfrac{1}{3-\log_5(x)}$

56. $f(x) = \dfrac{\sqrt{-1-x}}{\log_{\frac{1}{2}}(x)}$

57. $f(x) = \ln(-2x^3 - x^2 + 13x - 6)$

In Exercises 58 - 65, sketch the graph of g by starting with the graph of f and using transformations. Track at least three points of your choice and the vertical asymptote through the transformations. State the domain and range of g.

58. $f(x) = \log_2(x)$, $g(x) = \log_2(x+1)$

59. $f(x) = \log_{\frac{1}{3}}(x)$, $g(x) = \log_{\frac{1}{3}}(x) + 1$

60. $f(x) = \log_3(x)$, $g(x) = -\log_3(x-2)$

61. $f(x) = \log(x)$, $g(x) = 2\log(x+20) - 1$

62. $f(t) = \log_{0.5}(t)$, $g(t) = 10\log_{0.5}\left(\frac{t}{100}\right)$

63. $f(t) = \log_{1.25}(t)$, $g(t) = \log_{1.25}(-t+1) + 2$

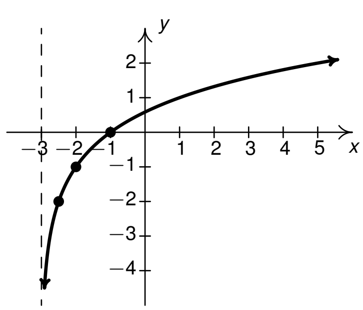
64. $f(t) = \ln(t)$, $g(t) = -\ln(8-t)$

65. $f(t) = \ln(t)$, $g(t) = -10\ln\left(\frac{t}{10}\right)$

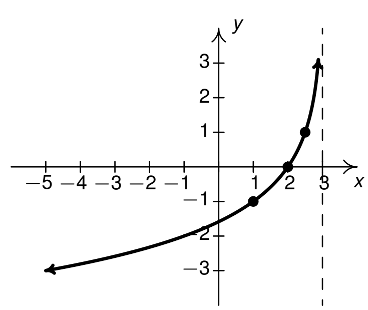
66. Verify that each function in Exercises 58 - 65 is the inverse of the corresponding function in Exercises 1 - 8 in Section 7.1. (Match up #1 and #58, and so on.)

In Exercises, 67 - 70, the graph of a logarithmic function is given. Find a formula for the function in the form $F(x) = a \cdot \log_2(bx - h) + k$.

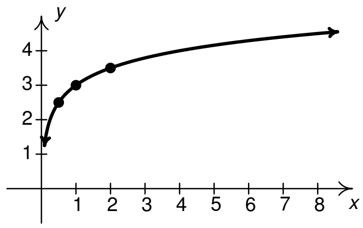
67. Points: $\left(-\frac{5}{2}, -2\right)$, $(-2, -1)$, $(-1, 0)$,
Asymptote: $x = -3$.



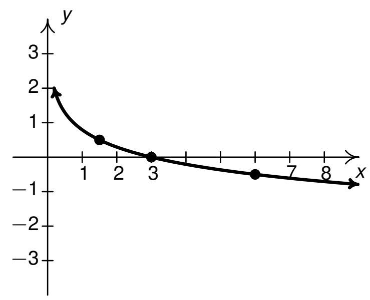
68. Points: $(1, -1)$, $(2, 0)$, $\left(\frac{5}{2}, 1\right)$
Asymptote: $x = 3$.



69. Points: $\left(\frac{1}{2}, \frac{5}{2}\right)$, $(1,3)$, $\left(2, \frac{7}{2}\right)$,
Asymptote: $x = 0$.



70. Points: $\left(6, -\frac{1}{2}\right)$, $(3,0)$, $\left(\frac{3}{2}, \frac{1}{2}\right)$,
Asymptote: $x = 0$.



71. Find a formula for each graph in Exercises 67 - 70 of the form $G(x) = a \cdot \log_4(bx - h) + k$.

In Exercises 72 - 75, find the inverse of the function from the 'procedural perspective' discussed in Example 7.2.3 and graph the function and its inverse on the same set of axes.

72. $f(x) = 3^{x+2} - 4$

73. $f(x) = \log_4(x - 1)$

74. $g(t) = -2^{-t} + 1$

75. $g(t) = 5\log(t) - 2$

In Exercises 76 - 81, write the given function as a nontrivial decomposition of functions as directed.

76. For $f(x) = \log_2(x + 3) + 4$, find functions g and h so that $f = g + h$.

77. For $f(x) = \log(2x) - e^{-x}$, find functions g and h so that $f = g - h$.

78. For $f(t) = 3t\log(t)$, find functions g and h so that $f = gh$.

79. For $r(x) = \dfrac{\ln(x)}{x}$, find functions f and g so $r = \dfrac{f}{g}$.

80. For $k(t) = \ln(t^2 + 1)$, find functions f and g so that $k = g \circ f$.

81. For $p(z) = (\ln(z))^2$, find functions f and g so $p = g \circ f$.

(Logarithmic Scales) In Exercises 82 - 84, we introduce three widely used measurement scales which involve common logarithms: the Richter scale, the decibel scale and the pH scale. The computations involved in all three scales are nearly identical so pay attention to the subtle differences.

82. Earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a solid course in Geology[5] or the U.S. Geological Survey's Earthquake Hazards Program found here and present only a simplified version of the Richter scale. The Richter scale measures the magnitude of an earthquake by comparing the amplitude of the seismic waves of the given earthquake to those of a "magnitude 0 event",

[5] Rock-solid, perhaps?

which was chosen to be a seismograph reading of 0.001 millimeters recorded on a seismometer 100 kilometers from the earthquake's epicenter. Specifically, the magnitude of an earthquake is given by

$$M(x) = \log\left(\frac{x}{0.001}\right)$$

where x is the seismograph reading in millimeters of the earthquake recorded 100 kilometers from the epicenter.

(a) Show that $M(0.001) = 0$.

(b) Compute $M(80,000)$.

(c) Show that an earthquake which registered 6.7 on the Richter scale had a seismograph reading ten times larger than one which measured 5.7.

(d) Find two news stories about recent earthquakes which give their magnitudes on the Richter scale. How many times larger was the seismograph reading of the earthquake with larger magnitude?

83. While the decibel scale can be used in many disciplines,[6] we shall restrict our attention to its use in acoustics, specifically its use in measuring the intensity level of sound. The Sound Intensity Level L (measured in decibels) of a sound intensity I (measured in watts per square meter) is given by

$$L(I) = 10\log\left(\frac{I}{10^{-12}}\right).$$

Like the Richter scale, this scale compares I to baseline: $10^{-12}\,\frac{W}{m^2}$ is the threshold of human hearing.

(a) Compute $L(10^{-6})$.

(b) Damage to your hearing can start with short term exposure to sound levels around 115 decibels. What intensity I is needed to produce this level?

(c) Compute $L(1)$. How does this compare with the threshold of pain which is around 140 decibels?

84. The pH of a solution is a measure of its acidity or alkalinity. Specifically, $\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles per liter. A solution with a pH less than 7 is an acid, one with a pH greater than 7 is a base (alkaline) and a pH of 7 is regarded as neutral.

(a) The hydrogen ion concentration of pure water is $[\text{H}^+] = 10^{-7}$. Find its pH.

(b) Find the pH of a solution with $[\text{H}^+] = 6.3 \times 10^{-13}$.

(c) The pH of gastric acid (the acid in your stomach) is about 0.7. What is the corresponding hydrogen ion concentration?

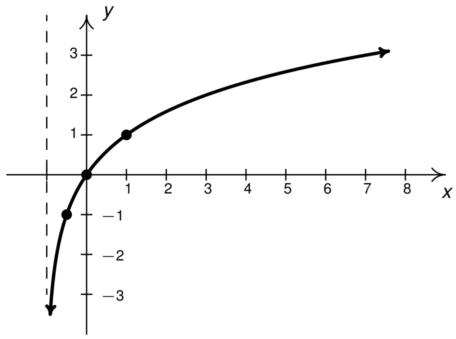
85. Use the definition of logarithm to explain why $\log_b 1 = 0$ and $\log_b b = 1$ for every $b > 0$, $b \neq 1$.

[6] See this webpage for more information.

7.2.2 Answers

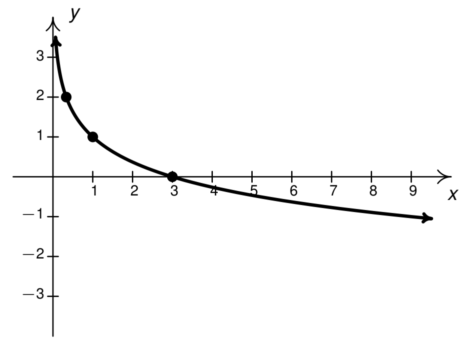
1. $\log_2(8) = 3$
2. $\log_5\left(\frac{1}{125}\right) = -3$
3. $\log_4(32) = \frac{5}{2}$
4. $\log_{\frac{1}{3}}(9) = -2$
5. $\log_{\frac{4}{25}}\left(\frac{5}{2}\right) = -\frac{1}{2}$
6. $\log(0.001) = -3$
7. $\ln(1) = 0$
8. $5^2 = 25$
9. $(25)^{\frac{1}{2}} = 5$
10. $3^{-4} = \frac{1}{81}$
11. $\left(\frac{4}{3}\right)^{-1} = \frac{3}{4}$
12. $10^2 = 100$
13. $10^{-1} = 0.1$
14. $e^1 = e$
15. $e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$
16. $\log_3(27) = 3$
17. $\log_6(216) = 3$
18. $\log_2(32) = 5$
19. $\log_6\left(\frac{1}{36}\right) = -2$
20. $\log_8(4) = \frac{2}{3}$
21. $\log_{36}(216) = \frac{3}{2}$
22. $\log_{\frac{1}{5}}(625) = -4$
23. $\log_{\frac{1}{6}}(216) = -3$
24. $\log_{36}(36) = 1$
25. $\log \frac{1}{1000000} = -6$
26. $\log(0.01) = -2$
27. $\ln\left(e^3\right) = 3$
28. $\log_4(8) = \frac{3}{2}$
29. $\log_6(1) = 0$
30. $\log_{13}\left(\sqrt{13}\right) = \frac{1}{2}$
31. $\log_{36}\left(\sqrt[4]{36}\right) = \frac{1}{4}$
32. $7^{\log_7(3)} = 3$
33. $36^{\log_{36}(216)} = 216$
34. $\log_{36}\left(36^{216}\right) = 216$
35. $\ln(e^5) = 5$
36. $\log\left(\sqrt[9]{10^{11}}\right) = \frac{11}{9}$
37. $\log\left(\sqrt[3]{10^5}\right) = \frac{5}{3}$
38. $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$
39. $\log_5\left(3^{\log_3 5}\right) = 1$
40. $\log\left(e^{\ln(100)}\right) = 2$
41. $\log_2\left(3^{-\log_3(2)}\right) = -1$
42. $\ln\left(42^{6\log(1)}\right) = 0$
43. $(-\infty, \infty)$
44. $(-2, \infty)$
45. $(5, \infty)$
46. $(-\infty, -6) \cup (-3, \infty)$
47. $(-2, -1) \cup (1, \infty)$
48. $(-6, -3) \cup (5, \infty)$
49. $(4, 7)$
50. $(5, \infty)$
51. $(-\infty, \infty)$
52. $[1, \infty)$
53. $(-\infty, -7) \cup (1, \infty)$
54. $(13, \infty)$
55. $(0, 125) \cup (125, \infty)$
56. No domain
57. $(-\infty, -3) \cup \left(\frac{1}{2}, 2\right)$

58. Domain of g: $(-1, \infty)$
Range of g: $(-\infty, \infty)$
Points: $\left(-\frac{1}{2}, -1\right)$, $(0, 0)$, $(1, 1)$
Asymptote: $x = -1$



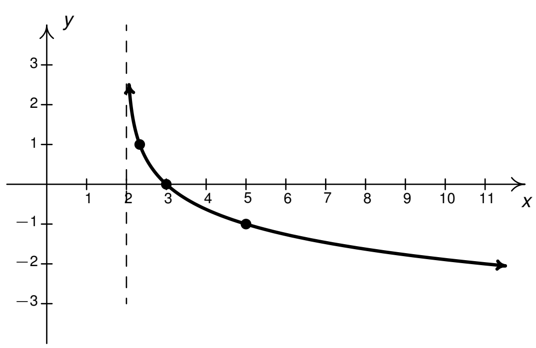
$y = g(x) = \log_2(x + 1)$

59. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$
Points: $\left(\frac{1}{3}, 2\right)$, $(1, 1)$, $(3, 0)$
Asymptote: $x = 0$



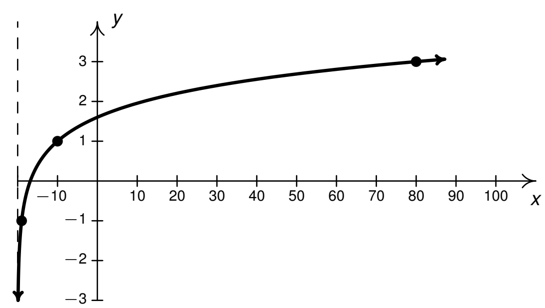
$y = g(x) = \log_{\frac{1}{3}}(x) + 1$

60. Domain of g: $(2, \infty)$
Range of g: $(-\infty, \infty)$
Points: $\left(\frac{7}{3}, 1\right)$, $(3, 0)$, $(5, -1)$
Asymptote: $x = 2$



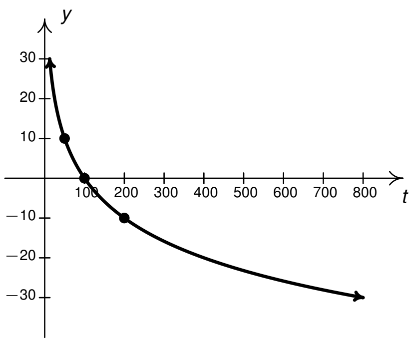
$y = g(x) = -\log_3(x - 2)$

61. Domain of g: $(-20, \infty)$
Range of g: $(-\infty, \infty)$
Points: $(-19, -1)$, $(-10, 1)$, $(80, 3)$
Asymptote: $x = -20$



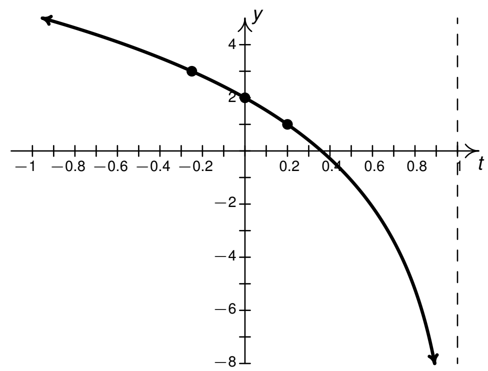
$y = g(x) = 2\log(x + 20) - 1$

62. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$
Points: $(50, 10)$, $(100, 0)$, $(200, -10)$
Asymptote: $t = 0$



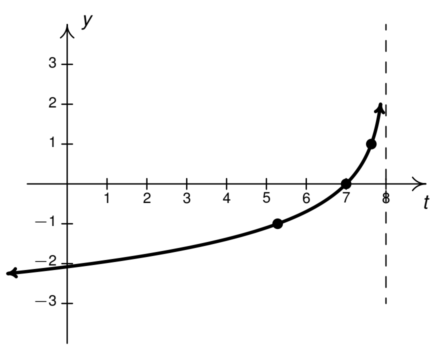
$y = g(t) = 10\log_{0.5}\left(\frac{t}{100}\right)$

63. Domain of g: $(-\infty, 1)$
Range of g: $(-\infty, \infty)$
Points: $(-0.25, 3)$, $(0, 2)$, $(0.2, 1)$
Asymptote: $t = 1$



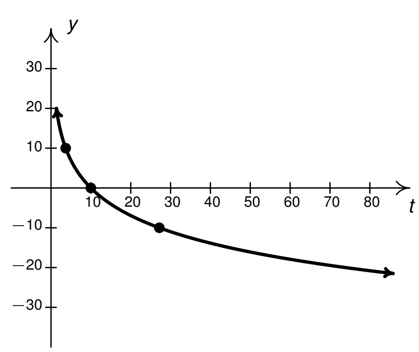
$y = g(t) = \log_{1.25}(-t + 1) + 2$

64. Domain of g: $(-\infty, 8)$
Range of g:$(-\infty, \infty)$
Points: $(8 - e, -1) \approx (5.28, -1)$,
$(7, 0)$, $(8 - e^{-1}, 1) \approx (7.63, 1)$
Asymptote: $t = 8$



$y = g(t) = -\ln(8 - t)$

65. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$
Points: $(10e^{-1}, 10) \approx (3.68.10)$
$(10, 0)$, $(10e, -10) \approx (27.18, -10)$
Asymptote: $t = 0$



$y = g(t) = -10\ln\left(\frac{t}{10}\right)$

67. $F(x) = \log_2(x+3) - 1$

68. $F(x) = -\log_2(-x+3)$

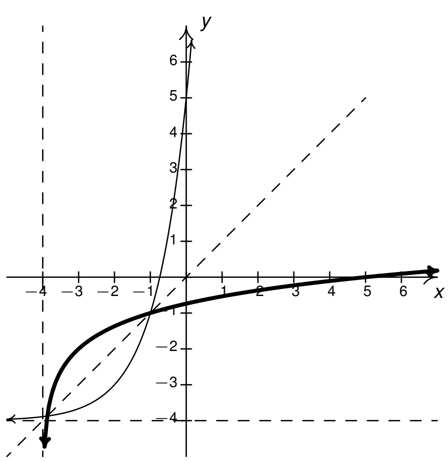
69. $F(x) = \frac{1}{2}\log_2(x) + 3$

70. $F(x) = -\frac{1}{2}\log_2\left(\frac{x}{3}\right)$

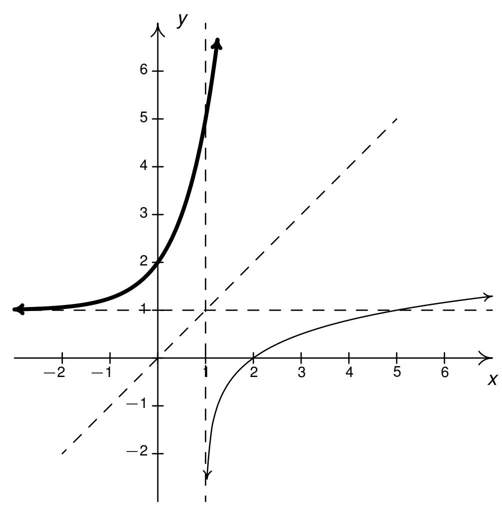
71. In order, the formulas for $G(x)$ are:

- $G(x) = 2\log_4(x+3) - 1$
- $G(x) = -2\log_4(-x+3)$
- $G(x) = \log_4(x) + 3$
- $G(x) = -\log_4\left(\frac{x}{3}\right)$

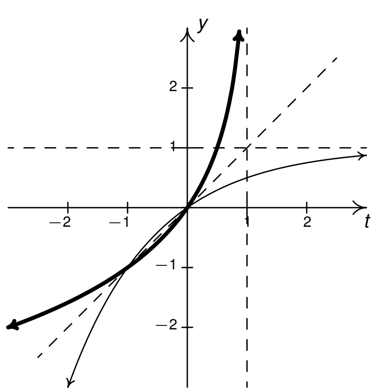
72. $y = f(x) = 3^{x+2} - 4$
$y = f^{-1}(x) = \log_3(x+4) - 2$



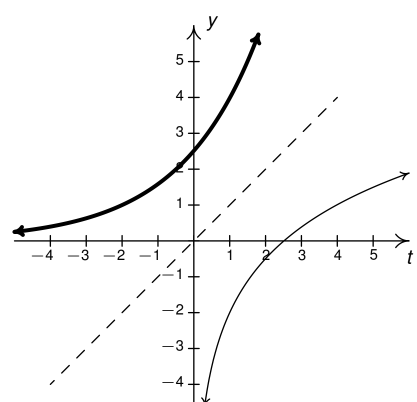
73. $y = f(x) = \log_4(x-1)$
$y = f^{-1}(x) = 4^x + 1$



74. $y = g(t) = -2^{-t} + 1$
$y = g^{-1}(t) = -\log_2(-t+1)$



75. $y = g(t) = 5\log(t) - 2$
$y = g^{-1}(t) = 10^{\frac{t+2}{5}}$



76. One solution is $g(x) = \log_2(x+3)$ and $h(x) = 4$.

77. One solution is $g(x) = \log(2x)$ and $h(x) = e^{-x}$.

78. One solution is $g(t) = 3t$ and $h(t) = \log(t)$.

79. One solution is $f(x) = \ln(x)$ and $g(x) = x$.

80. One solution is $f(t) = t^2 + 1$ and $g(t) = \ln(t)$.

81. One solution is $f(z) = \ln(z)$ and $g(z) = z^2$.

82. (a) $M(0.001) = \log\left(\frac{0.001}{0.001}\right) = \log(1) = 0$.
    (b) $M(80,000) = \log\left(\frac{80,000}{0.001}\right) = \log(80,000,000) \approx 7.9$.

83. (a) $L(10^{-6}) = 60$ decibels.
    (b) $I = 10^{-.5} \approx 0.316$ watts per square meter.
    (c) Since $L(1) = 120$ decibels and $L(100) = 140$ decibels, a sound with intensity level 140 decibels has an intensity 100 times greater than a sound with intensity level 120 decibels.

84. (a) The pH of pure water is 7.
    (b) If $[H^+] = 6.3 \times 10^{-13}$ then the solution has a pH of 12.2.
    (c) $[H^+] = 10^{-0.7} \approx .1995$ moles per liter.

7.3 Properties of Logarithms

In Section 7.2, we introduced the logarithmic functions as inverses of exponential functions and discussed a few of their functional properties from that perspective. In this section, we explore the algebraic properties of logarithms. Historically, these have played a huge role in the scientific development of our society since, among other things, they were used to develop analog computing devices called slide rules which enabled scientists and engineers to perform accurate calculations leading to such things as space travel and the moon landing.

As we shall see shortly, logs inherit analogs of all of the properties of exponents you learned in Elementary and Intermediate Algebra. We first extract two properties from Theorem 7.3 to remind us of the definition of a logarithm as the inverse of an exponential function.

Theorem 7.4. (Inverse Properties of Exponential and Logarithmic Functions)
Let $b > 0$, $b \neq 1$.

- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b\left(b^x\right) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$

Next, we spell out what it means for exponential and logarithmic functions to be one-to-one.

Theorem 7.5. (One-to-one Properties of Exponential and Logarithmic Functions)
Let $f(x) = b^x$ and $g(x) = \log_b(x)$ where $b > 0$, $b \neq 1$. Then f and g are one-to-one and

- $b^u = b^w$ if and only if $u = w$ for all real numbers u and w.
- $\log_b(u) = \log_b(w)$ if and only if $u = w$ for all real numbers $u > 0$, $w > 0$.

Next, we re-state Theorem 7.2 for reference below.

Theorem 7.6. (Algebraic Properties of Exponential Functions) Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $f(u+w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u-w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $\left(b^u\right)^w = b^{uw}$

To each of these properties of listed in Theorem 7.2, there corresponds an analogous property of logarithmic functions. We list these below in our next theorem.

Theorem 7.7. (Algebraic Properties of Logarithmic Functions) Let $g(x) = \log_b(x)$ be a logarithmic function ($b > 0$, $b \neq 1$) and let $u > 0$ and $w > 0$ be real numbers.

- **Product Rule:** $g(uw) = g(u) + g(w)$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$
- **Quotient Rule:** $g\left(\dfrac{u}{w}\right) = g(u) - g(w)$. In other words, $\log_b\left(\dfrac{u}{w}\right) = \log_b(u) - \log_b(w)$
- **Power Rule:** $g\left(u^w\right) = wg(u)$. In other words, $\log_b\left(u^w\right) = w\log_b(u)$

There are a couple of different ways to understand why Theorem 7.7 is true. For instance, consider the product rule: $\log_b(uw) = \log_b(u) + \log_b(w)$.

Let $a = \log_b(uw)$, $c = \log_b(u)$, and $d = \log_b(w)$. Then, by definition, $b^a = uw$, $b^c = u$ and $b^d = w$. Hence, $b^a = uw = b^c b^d = b^{c+d}$, so that $b^a = b^{c+d}$.

By the one-to-one property of b^x, $b^a = b^{c+d}$ gives $a = c + d$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$. The remaining properties are proved similarly.

From a purely functional approach, we can see the properties in Theorem 7.7 as an example of how inverse functions interchange the roles of inputs in outputs.

For instance, the Product Rule for exponential functions given in Theorem 7.2, $f(u + w) = f(u)f(w)$, says that adding inputs results in multiplying outputs.

Hence, whatever f^{-1} is, it must take the products of outputs from f and return them to the sum of their respective inputs. Since the outputs from f are the inputs to f^{-1} and vice-versa, we have that that f^{-1} must take products of its inputs to the sum of their respective outputs. This is precisely one way to interpret the Product Rule for Logarithmic functions: $g(uw) = g(u) + g(w)$.

The reader is encouraged to view the remaining properties listed in Theorem 7.7 similarly.

The following examples help build familiarity with these properties. In our first example, we are asked to 'expand' the logarithms. This means that we read the properties in Theorem 7.7 from left to right and rewrite products inside the log as sums outside the log, quotients inside the log as differences outside the log, and powers inside the log as factors outside the log.[1]

Example 7.3.1. Expand the following using the properties of logarithms and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\log_2\left(\dfrac{8}{x}\right)$
2. $\log_{0.1}\left(10x^2\right)$
3. $\ln\left(\dfrac{3}{et}\right)^2$
4. $\log\sqrt[3]{\dfrac{100x^2}{yz^5}}$
5. $\log_{117}\left(x^2 - 4\right)$

[1] Interestingly enough, it is the exact *opposite* process (which we will practice later) that is most useful in Algebra, the utility of expanding logarithms becomes apparent in Calculus.

Solution.

1. To expand $\log_2\left(\frac{8}{x}\right)$, we use the Quotient Rule identifying $u = 8$ and $w = x$ and simplify.

$$\begin{array}{rcll} \log_2\left(\dfrac{8}{x}\right) & = & \log_2(8) - \log_2(x) & \text{Quotient Rule} \\ & = & 3 - \log_2(x) & \text{Since } 2^3 = 8 \\ & = & -\log_2(x) + 3 & \end{array}$$

2. In the expression $\log_{0.1}\left(10x^2\right)$, we have a power (the x^2) and a product, and the question becomes which property, Power Rule or Product Rule to use first.

   In order to use the Power Rule, the *entire* quantity inside the log must be raised to the same exponent. Since the exponent 2 applies only to the x, we first apply the Product Rule with $u = 10$ and $w = x^2$. Once the x^2 is by itself inside the log, we apply the Power Rule with $u = x$ and $w = 2$.

$$\begin{array}{rcll} \log_{0.1}\left(10x^2\right) & = & \log_{0.1}(10) + \log_{0.1}\left(x^2\right) & \text{Product Rule} \\ & = & \log_{0.1}(10) + 2\log_{0.1}(x) & \text{Power Rule} \\ & = & -1 + 2\log_{0.1}(x) & \text{Since } (0.1)^{-1} = 10 \\ & = & 2\log_{0.1}(x) - 1 & \end{array}$$

3. We have a power, quotient and product occurring in $\ln\left(\frac{3}{et}\right)^2$. Since the exponent 2 applies to the entire quantity inside the logarithm, we begin with the Power Rule with $u = \frac{3}{et}$ and $w = 2$.

   Next, we see the Quotient Rule is applicable, with $u = 3$ and $w = et$, so we replace $\ln\left(\frac{3}{et}\right)$ with the quantity $\ln(3) - \ln(et)$.

   Since $\ln\left(\frac{3}{et}\right)$ is being multiplied by 2, the entire quantity $\ln(3) - \ln(et)$ is multiplied by 2.

   Finally, we apply the Product Rule with $u = e$ and $w = x$, and replace $\ln(et)$ with the quantity $\ln(e) + \ln(t)$, and simplify, keeping in mind that the natural log is log base e.

$$\begin{array}{rcll} \ln\left(\dfrac{3}{et}\right)^2 & = & 2\ln\left(\dfrac{3}{et}\right) & \text{Power Rule} \\ & = & 2\left[\ln(3) - \ln(et)\right] & \text{Quotient Rule} \\ & = & 2\ln(3) - 2\ln(et) & \\ & = & 2\ln(3) - 2\left[\ln(e) + \ln(t)\right] & \text{Product Rule} \\ & = & 2\ln(3) - 2\ln(e) - 2\ln(t) & \\ & = & 2\ln(3) - 2 - 2\ln(t) & \text{Since } e^1 = e \\ & = & -2\ln(t) + 2\ln(3) - 2 & \end{array}$$

4. In Theorem 7.7, there is no mention of how to deal with radicals. However, thinking back to Definition A.9, we can rewrite the cube root as a $\frac{1}{3}$ exponent. We begin by using the Power Rule[2], and we keep in mind that the common log is log base 10.

$$\begin{array}{rcll}
\log \sqrt[3]{\dfrac{100x^2}{yz^5}} & = & \log\left(\dfrac{100x^2}{yz^5}\right)^{1/3} & \\
& = & \frac{1}{3}\log\left(\dfrac{100x^2}{yz^5}\right) & \text{Power Rule} \\
& = & \frac{1}{3}\left[\log\left(100x^2\right) - \log\left(yz^5\right)\right] & \text{Quotient Rule} \\
& = & \frac{1}{3}\log\left(100x^2\right) - \frac{1}{3}\log\left(yz^5\right) & \\
& = & \frac{1}{3}\left[\log(100) + \log\left(x^2\right)\right] - \frac{1}{3}\left[\log(y) + \log\left(z^5\right)\right] & \text{Product Rule} \\
& = & \frac{1}{3}\log(100) + \frac{1}{3}\log\left(x^2\right) - \frac{1}{3}\log(y) - \frac{1}{3}\log\left(z^5\right) & \\
& = & \frac{1}{3}\log(100) + \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) & \text{Power Rule} \\
& = & \frac{2}{3} + \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) & \text{Since } 10^2 = 100 \\
& = & \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) + \frac{2}{3} &
\end{array}$$

5. At first it seems as if we have no means of simplifying $\log_{117}\left(x^2-4\right)$, since none of the properties of logs addresses the issue of expanding a difference *inside* the logarithm. However, we may factor $x^2 - 4 = (x+2)(x-2)$ thereby introducing a product which gives us license to use the Product Rule. Assuming both $x+2 > 0$ and $x-2 > 0$, that is, $x > 2$ we expand as follows.

$$\begin{array}{rcll}
\log_{117}\left(x^2-4\right) & = & \log_{117}\left[(x+2)(x-2)\right] & \text{Factor} \\
& = & \log_{117}(x+2) + \log_{117}(x-2) & \text{Product Rule}
\end{array}$$

□

A couple of remarks about Example 7.3.1 are in order. First, if we take a step back and look at each problem in the foregoing example, a general rule of thumb to determine which log property to apply first when faced with a multi-step problem is to apply the logarithm properties in the 'reverse order of operations.' For example, if we were to substitute a number for x into the expression $\log_{0.1}\left(10x^2\right)$, we would first square the x, then multiply by 10. The last step is the multiplication, which tells us the first log property to apply is the Product Rule. The last property of logarithm to apply would the be the power rule applied to $\log_{0.1}(x^2)$. Second, the equivalence $\log_{117}\left(x^2-4\right) = \log_{117}(x+2) + \log_{117}(x-2)$ is valid only if $x > 2$. Indeed, the functions $f(x) = \log_{117}\left(x^2-4\right)$ and $g(x) = \log_{117}(x+2) + \log_{117}(x-2)$ have different domains, and, hence, are different functions.[3] In general, when using log properties to expand a logarithm, we may very well be restricting the domain as we do so.

[2] At this point in the text, the reader is encouraged to carefully read through each step and think of which quantity is playing the role of u and which is playing the role of w as we apply each property.

[3] We leave it to the reader to verify the domain of f is $(-\infty,-2)\cup(2,\infty)$ whereas the domain of g is $(2,\infty)$.

One last comment before we move to reassembling logs from their various bits and pieces. The authors are well aware of the propensity for some students to become overexcited and invent their own properties of logs like $\log_{117}\left(x^2-4\right) = \log_{117}\left(x^2\right) - \log_{117}(4)$, which simply isn't true, in general. The unwritten[4] property of logarithms is that if it isn't written in a textbook, it probably isn't true.

Example 7.3.2. Use the properties of logarithms to write the following as a single logarithm.

1. $\log_3(x-1) - \log_3(x+1)$
2. $\log(x) + 2\log(y) - \log(z)$
3. $4\log_2(x) + 3$
4. $-\ln(t) - \frac{1}{2}$

Solution. Whereas in Example 7.3.1 we read the properties in Theorem 7.7 from left to right to expand logarithms, in this example we read them from right to left.

1. The difference of logarithms requires the Quotient Rule: $\log_3(x-1) - \log_3(x+1) = \log_3\left(\frac{x-1}{x+1}\right)$.
2. In the expression, $\log(x) + 2\log(y) - \log(z)$, we have both a sum and difference of logarithms.

   Before we use the product rule to combine $\log(x) + 2\log(y)$, we note that we need to apply the Power Rule to rewrite the coefficient 2 as the power on y. We then apply the Product and Quotient Rules as we move from left to right.

$$\begin{array}{rcll} \log(x) + 2\log(y) - \log(z) & = & \log(x) + \log\left(y^2\right) - \log(z) & \text{Power Rule} \\ & = & \log\left(xy^2\right) - \log(z) & \text{Product Rule} \\ & = & \log\left(\dfrac{xy^2}{z}\right) & \text{Quotient Rule} \end{array}$$

3. We begin rewriting $4\log_2(x) + 3$ by applying the Power Rule: $4\log_2(x) = \log_2\left(x^4\right)$.

   In order to continue, we need to rewrite 3 as a logarithm base 2. From Theorem 7.4, we know $3 = \log_2\left(2^3\right)$. Rewriting 3 this way paves the way to use the Product Rule.

$$\begin{array}{rcll} 4\log_2(x) + 3 & = & \log_2\left(x^4\right) + 3 & \text{Power Rule} \\ & = & \log_2\left(x^4\right) + \log_2\left(2^3\right) & \text{Since } 3 = \log_2\left(2^3\right) \\ & = & \log_2\left(x^4\right) + \log_2(8) & \\ & = & \log_2\left(8x^4\right) & \text{Product Rule} \end{array}$$

4. To get started with $-\ln(t) - \frac{1}{2}$, we rewrite $-\ln(t)$ as $(-1)\ln(t)$. We can then use the Power Rule to obtain $(-1)\ln(t) = \ln\left(t^{-1}\right)$.

   As in the previous problem, in order to continue, we need to rewrite $\frac{1}{2}$ as a natural logarithm. Theorem 7.4 gives us $\frac{1}{2} = \ln\left(e^{1/2}\right) = \ln\left(\sqrt{e}\right)$. Hence,

[4] The authors relish the irony involved in writing what follows.

$$\begin{aligned}
-\ln(t)-\tfrac{1}{2} &= (-1)\ln(t)-\tfrac{1}{2} && \\
&= \ln\left(t^{-1}\right)-\tfrac{1}{2} && \text{Power Rule} \\
&= \ln\left(t^{-1}\right)-\ln\left(e^{1/2}\right) && \text{Since } \tfrac{1}{2}=\ln\left(e^{1/2}\right) \\
&= \ln\left(t^{-1}\right)-\ln\left(\sqrt{e}\right) && \\
&= \ln\left(\frac{t^{-1}}{\sqrt{e}}\right) && \text{Quotient Rule} \\
&= \ln\left(\frac{1}{t\sqrt{e}}\right) && \Box
\end{aligned}$$

As we would expect, the rule of thumb for re-assembling logarithms is the opposite of what it was for dismantling them. That is, to rewrite an expression as a single logarithm, we apply log properties following the usual order of operations: first, rewrite coefficients of logs as powers using the Power Rule, then rewrite addition and subtraction using the Product and Quotient Rules, respectively, as written from left to right.

Additionally, we find that using log properties in this fashion can increase the domain of the expression. For example, we leave it to the reader to verify the domain of $f(x) = \log_3(x-1) - \log_3(x+1)$ is $(1,\infty)$ but the domain of $g(x) = \log_3\left(\frac{x-1}{x+1}\right)$ is $(-\infty,-1)\cup(1,\infty)$. We'll need to keep this in mind in Section 7.5 since such manipulations can result in extraneous solutions.

The two logarithm buttons commonly found on calculators are the 'LOG' and 'LN' buttons which correspond to the common and natural logs, respectively. Suppose we wanted an approximation to $\log_2(7)$. The answer should be a little less than 3, (Can you explain why?) but how do we coerce the calculator into telling us a more accurate answer? We need the following theorem.

Theorem 7.8. (Change of Base Formulas) Let $a, b > 0$, $a, b \neq 1$.

- $a^x = b^{x\log_b(a)}$ for all real numbers x.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x > 0$.

To prove these formulas, consider $b^{x\log_b(a)}$. Using the Power Rule, we can rewrite $x\log_b(a)$ as $\log_b\left(a^x\right)$. Following this with the Inverse Properties in Theorem 7.4, we get

$$b^{x\log_b(a)} = b^{\log_b\left(a^x\right)} = a^x.$$

To verify the logarithmic form of the property, we use the Power Rule and an Inverse Property to get:

$$\log_a(x)\cdot\log_b(a) = \log_b\left(a^{\log_a(x)}\right) = \log_b(x).$$

We get the result by dividing both sides of the equation $\log_a(x)\cdot\log_b(a) = \log_b(x)$ by $\log_b(a)$.

Of course, the authors can't help but point out the inverse relationship between these two change of base formulas. To change the base of an exponential expression, we *multiply* the *input* by the factor $\log_b(a)$. To change the base of a logarithmic expression, we *divide* the *output* by the factor $\log_b(a)$.

While, in the grand scheme of things, both change of base formulas are really saying the same thing, the logarithmic form is the one usually encountered in Algebra while the exponential form isn't usually introduced until Calculus.

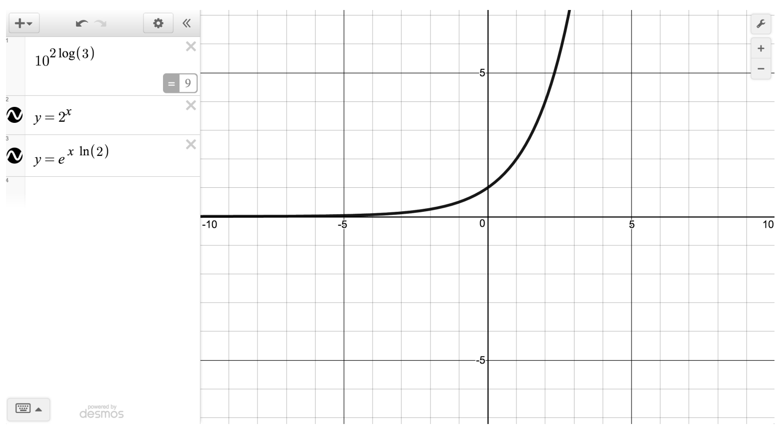
Example 7.3.3. Use an appropriate change of base formula to convert the following expressions to ones with the indicated base. Verify your answers using a graphing utility, as appropriate.

1. 3^2 to base 10
2. 2^x to base e
3. $\log_4(5)$ to base e
4. $\ln(x)$ to base 10

Solution.

1. We apply the Change of Base formula with $a = 3$ and $b = 10$ to obtain $3^2 = 10^{2\log(3)}$. Typing the latter into a graphing utility produces an answer of 9 as seen below.

2. Here, $a = 2$ and $b = e$ so we have $2^x = e^{x\ln(2)}$. Using a graphing utility, we find the graphs of $f(x) = 2^x$ and $g(x) = e^{x\ln(2)}$ appear to overlap perfectly.

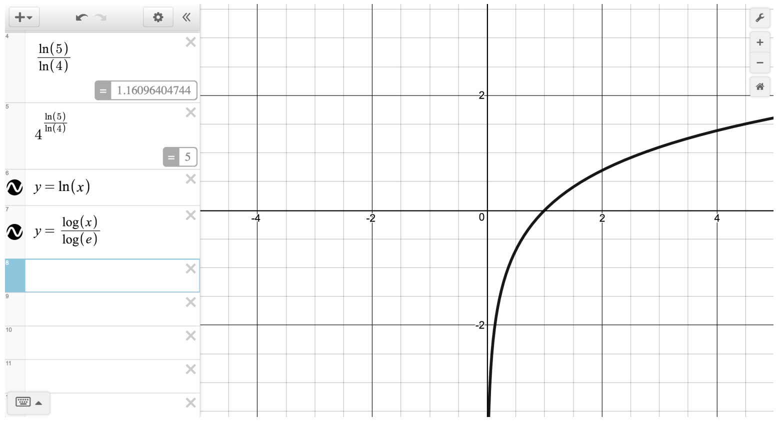


3. Applying the change of base with $a = 4$ and $b = e$ leads us to write $\log_4(5) = \frac{\ln(5)}{\ln(4)}$. Evaluating this gives the numerical approximation $\frac{\ln(5)}{\ln(4)} \approx 1.16$.

   To check our answer we know that, by definition, $\log_4(5)$ is the exponent we put on 4 to get 5, so a number a little larger than 1 seems reasonable.

   Taking this one step further, we use a graphing utility and find $4^{\frac{\ln(5)}{\ln(4)}} = 5$, which means if the machine is lying to us about the first answer it gave us, at least it is being consistent.

4. We write $\ln(x) = \log_e(x) = \frac{\log(x)}{\log(e)}$. We graph both $f(x) = \ln(x)$ and $g(x) = \frac{\log(x)}{\log(e)}$ and find both graphs appear to be identical.



□

What Theorem 7.8 really tells us is that all exponential and logarithmic functions are just scalings of one another. Not only does this explain why their graphs have similar shapes, but it also tells us that we could do all of mathematics with a single base, be it 10, 0.42, π, or 117.

As mentioned in Section 7.1, the 'natural' base, base e, features prominently in mathematical applications as we'll see in Section 7.6. Hence, we conclude this section by specifying Theorem 7.8 to this case.

Theorem 7.9. Conversion to the Natural Base: Suppose $b > 0$, $b \neq 1$. Then

- $b^x = e^{x \ln(b)}$ for all real numbers x.
- $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$ for all real numbers $x > 0$.

7.3.1 Exercises

In Exercises 1 - 15, expand the given logarithm and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\ln(x^3y^2)$
2. $\log_2\left(\dfrac{128}{x^2+4}\right)$
3. $\log_5\left(\dfrac{z}{25}\right)^3$
4. $\log(1.23\times 10^{37})$
5. $\ln\left(\dfrac{\sqrt{z}}{xy}\right)$
6. $\log_5\left(x^2-25\right)$
7. $\log_{\sqrt{2}}\left(4x^3\right)$
8. $\log_{\frac{1}{3}}(9x(y^3-8))$
9. $\log\left(1000x^3y^5\right)$
10. $\log_3\left(\dfrac{x^2}{81y^4}\right)$
11. $\ln\left(\sqrt[4]{\dfrac{xy}{ez}}\right)$
12. $\log_6\left(\dfrac{216}{x^3y}\right)^4$
13. $\log\left(\dfrac{100x\sqrt{y}}{\sqrt[3]{10}}\right)$
14. $\log_{\frac{1}{2}}\left(\dfrac{4\sqrt[3]{x^2}}{y\sqrt{z}}\right)$
15. $\ln\left(\dfrac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

In Exercises 16 - 29, use the properties of logarithms to write the expression as a single logarithm.

16. $4\ln(x)+2\ln(y)$
17. $\log_2(x)+\log_2(y)-\log_2(z)$
18. $\log_3(x)-2\log_3(y)$
19. $\frac{1}{2}\log_3(x)-2\log_3(y)-\log_3(z)$
20. $2\ln(x)-3\ln(y)-4\ln(z)$
21. $\log(x)-\frac{1}{3}\log(z)+\frac{1}{2}\log(y)$
22. $-\frac{1}{3}\ln(x)-\frac{1}{3}\ln(y)+\frac{1}{3}\ln(z)$
23. $\log_5(x)-3$
24. $3-\log(x)$
25. $\log_7(x)+\log_7(x-3)-2$
26. $\ln(x)+\frac{1}{2}$
27. $\log_2(x)+\log_4(x)$
28. $\log_2(x)+\log_4(x-1)$
29. $\log_2(x)+\log_{\frac{1}{2}}(x-1)$

In Exercises 30 - 33, use the appropriate change of base formula to convert the given expression to an expression with the indicated base.

30. 7^{x-1} to base e
31. $\log_3(x+2)$ to base 10
32. $\left(\dfrac{2}{3}\right)^x$ to base e
33. $\log(x^2+1)$ to base e

In Exercises 34 - 39, use the appropriate change of base formula to approximate the logarithm.

34. $\log_3(12)$
35. $\log_5(80)$
36. $\log_6(72)$
37. $\log_4\left(\dfrac{1}{10}\right)$
38. $\log_{\frac{3}{5}}(1000)$
39. $\log_{\frac{2}{3}}(50)$

40. In Example 7.2.1 number 3 in Section 7.2, we obtained the solution $F(x) = \log_2(-x + 4) - 3$ as one formula for the given graph by making a simplifying assumption that $b = -1$. This exercises explores if there are any other solutions for different choices of b.

    (a) Show $G(x) = \log_2(-2x + 8) - 4$ also fits the data for the given graph.

    (b) Use properties of logarithms to show $G(x) = \log_2(-2x + 8) - 4 = \log_2(-x + 4) - 3 = F(x)$.

    (c) With help from your classmates, find solutions to Example 7.2.1 number 3 in Section 7.2 by assuming $b = -4$ and $b = -8$. In each case, use properties of logarithms to show the solutions reduce to $F(x) = \log_2(-x + 4) - 3$.

    (d) Using properties of logarithms and the fact that the range of $\log_2(x)$ is all real numbers, show that any function of the form $f(x) = a\log_2(bx - h) + k$ where $a \neq 0$ can be rewritten as:

    $$f(x) = a\left(\log_2(bx - h) + \frac{k}{a}\right) = a(\log_2(bx - h) + \log_2(p)) = a\log_2(p(bx - h)) = a\log_2(pbx - ph),$$

    where $\frac{k}{a} = \log_2(p)$ for some positive real number p. Relabeling, we get every function of the form $f(x) = a\log_2(bx - h) + k$ with four parameters (a, b, h, and k) can be rewritten as $f(x) = a\log_2(Bx - H)$, a formula with just three parameters: a, B, and H.

    Show *every* solution to Example 7.2.1 number 3 in Section 7.2 can be written in the form $f(x) = \log_2\left(-\frac{1}{8}x + \frac{1}{2}\right)$ and that, in particular, $F(x) = \log_2(-x + 4) - 3 = \log_2\left(-\frac{1}{8}x + \frac{1}{2}\right) = f(x)$. Hence, there is really just one solution to Example 7.2.1 number 3 in Section 7.2.

41. The Henderson-Hasselbalch Equation: Suppose HA represents a weak acid. Then we have a reversible chemical reaction

    $$HA \rightleftharpoons H^+ + A^-.$$

    The acid disassociation constant, K_a, is given by

    $$K_a = \frac{[H^+][A^-]}{[HA]} = [H^+]\frac{[A^-]}{[HA]},$$

    where the square brackets denote the concentrations just as they did in Exercise 84 in Section 7.2. The symbol pK_a is defined similarly to pH in that p$K_a = -\log(K_a)$. Using the definition of pH from Exercise 84 and the properties of logarithms, derive the Henderson-Hasselbalch Equation:

    $$\text{pH} = \text{p}K_a + \log\frac{[A^-]}{[HA]}$$

42. Compare and contrast the graphs of $y = \ln(x^2)$ and $y = 2\ln(x)$.

43. Prove the Quotient Rule and Power Rule for Logarithms.

44. Give numerical examples to show that, in general,

(a) $\log_b(x + y) \neq \log_b(x) + \log_b(y)$

(b) $\log_b(x - y) \neq \log_b(x) - \log_b(y)$

(c) $\log_b\left(\frac{x}{y}\right) \neq \frac{\log_b(x)}{\log_b(y)}$

45. Research the history of logarithms including the origin of the word 'logarithm' itself. Why is the abbreviation of natural log 'ln' and not 'nl'?

46. There is a scene in the movie 'Apollo 13' in which several people at Mission Control use slide rules to verify a computation. Was that scene accurate? Look for other pop culture references to logarithms and slide rules.

47. (a) Use properties of logarithm functions to show that if $f(x) = \ln(x)$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h}$$

(b) Numerically and graphically investigate the limit: $\lim\limits_{h \to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{h}$ for various positive numbers x to convince yourself that $\lim\limits_{h \to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{h} = \frac{1}{x}$

(c) Use parts 47a and 47b to show to show that the derivative of $\ln(x)$ is $\ldots \frac{1}{x}$.

(d) Write the equations of the tangent lines to $y = \ln(x)$ at the following points. Check your answers graphically. Compare your answers the tangent lines at the corresponding points in Exercise 26 in Section 7.1.

i. $(1, 0)$

ii. $(e, 1)$

iii. $\left(e^{-1}, -1\right)$

7.3.2 Answers

1. $3\ln(x) + 2\ln(y)$
2. $7 - \log_2(x^2 + 4)$
3. $3\log_5(z) - 6$
4. $\log(1.23) + 37$
5. $\frac{1}{2}\ln(z) - \ln(x) - \ln(y)$
6. $\log_5(x - 5) + \log_5(x + 5)$
7. $3\log_{\sqrt{2}}(x) + 4$
8. $-2 + \log_{\frac{1}{3}}(x) + \log_{\frac{1}{3}}(y - 2) + \log_{\frac{1}{3}}(y^2 + 2y + 4)$
9. $3 + 3\log(x) + 5\log(y)$
10. $2\log_3(x) - 4 - 4\log_3(y)$
11. $\frac{1}{4}\ln(x) + \frac{1}{4}\ln(y) - \frac{1}{4} - \frac{1}{4}\ln(z)$
12. $12 - 12\log_6(x) - 4\log_6(y)$
13. $\frac{5}{3} + \log(x) + \frac{1}{2}\log(y)$
14. $-2 + \frac{2}{3}\log_{\frac{1}{2}}(x) - \log_{\frac{1}{2}}(y) - \frac{1}{2}\log_{\frac{1}{2}}(z)$
15. $\frac{1}{3}\ln(x) - \ln(10) - \frac{1}{2}\ln(y) - \frac{1}{2}\ln(z)$
16. $\ln(x^4 y^2)$
17. $\log_2\left(\frac{xy}{z}\right)$
18. $\log_3\left(\frac{x}{y^2}\right)$
19. $\log_3\left(\frac{\sqrt{x}}{y^2 z}\right)$
20. $\ln\left(\frac{x^2}{y^3 z^4}\right)$
21. $\log\left(\frac{x\sqrt{y}}{\sqrt[3]{z}}\right)$
22. $\ln\left(\sqrt[3]{\frac{z}{xy}}\right)$
23. $\log_5\left(\frac{x}{125}\right)$
24. $\log\left(\frac{1000}{x}\right)$
25. $\log_7\left(\frac{x(x-3)}{49}\right)$
26. $\ln\left(x\sqrt{e}\right)$
27. $\log_2\left(x^{3/2}\right)$
28. $\log_2\left(x\sqrt{x-1}\right)$
29. $\log_2\left(\frac{x}{x-1}\right)$
30. $7^{x-1} = e^{(x-1)\ln(7)}$
31. $\log_3(x + 2) = \frac{\log(x+2)}{\log(3)}$
32. $\left(\frac{2}{3}\right)^x = e^{x\ln\left(\frac{2}{3}\right)}$
33. $\log(x^2 + 1) = \frac{\ln(x^2+1)}{\ln(10)}$
34. $\log_3(12) \approx 2.26186$
35. $\log_5(80) \approx 2.72271$
36. $\log_6(72) \approx 2.38685$
37. $\log_4\left(\frac{1}{10}\right) \approx -1.66096$
38. $\log_{\frac{3}{5}}(1000) \approx -13.52273$
39. $\log_{\frac{2}{3}}(50) \approx -9.64824$

47. (d) i. $y = x - 1$ ii. $y = \frac{1}{e}x$ iii. $y = ex - 2$

7.4 Equations and Inequalities involving Exponential Functions

In this section we will develop techniques for solving equations involving exponential functions. Consider the equation $2^x = 128$. After a moment's calculation, we find $128 = 2^7$, so we have $2^x = 2^7$. The one-to-one property of exponential functions, detailed in Theorem 7.5, tells us that $2^x = 2^7$ if and only if $x = 7$. This means that not only is $x = 7$ a solution to $2^x = 2^7$, it is the *only* solution.

Now suppose we change the problem ever so slightly to $2^x = 129$. We could use one of the inverse properties of exponentials and logarithms listed in Theorem 7.4 to write $129 = 2^{\log_2(129)}$. We'd then have $2^x = 2^{\log_2(129)}$, which means our solution is $x = \log_2(129)$.

After all, the definition of $\log_2(129)$ is 'the exponent we put on 2 to get 129.' Indeed we could have obtained this solution directly by rewriting the equation $2^x = 129$ in its logarithmic form $\log_2(129) = x$. Either way, in order to get a reasonable decimal approximation to this number, we'd use the change of base formula, Theorem 7.8, to give us something more calculator friendly. Typically this means we convert our answer to base 10 or base e, and we choose the latter: $\log_2(129) = \frac{\ln(129)}{\ln(2)} \approx 7.011$.

Still another way to obtain this answer is to 'take the natural log' of both sides of the equation. Since $f(x) = \ln(x)$ is a *function*, as long as two quantities are equal, their natural logs are equal.[1]

We then use the Power Rule to write the exponent x as a factor then divide both sides by the constant $\ln(2)$ to obtain our answer.[2]

$$\begin{array}{rcll} 2^x & = & 129 & \\ \ln\left(2^x\right) & = & \ln(129) & \text{Take the natural log of both sides.} \\ x\ln(2) & = & \ln(129) & \text{Power Rule} \\ x & = & \dfrac{\ln(129)}{\ln(2)} & \end{array}$$

We summarize our two strategies for solving equations featuring exponential functions below.

Steps for Solving an Equation involving Exponential Functions

1. Isolate the exponential function.
2. (a) If convenient, express both sides with a common base and equate the exponents.
   (b) Otherwise, take the natural log of both sides of the equation and use the Power Rule.

Example 7.4.1. Solve the following equations. Check your answer using a graphing utility.

1. $2^{3x} = 16^{1-x}$
2. $2000 = 1000 \cdot 3^{-0.1t}$
3. $9 \cdot 3^x = 7^{2x}$
4. $75 = \frac{100}{1+3e^{-2t}}$
5. $25^x = 5^x + 6$
6. $\frac{e^x - e^{-x}}{2} = 5$

[1] This is also the 'if' part of the statement $\log_b(u) = \log_b(w)$ if and only if $u = w$ in Theorem 7.5.

[2] Please resist the temptation to divide both sides by 'ln' instead of $\ln(2)$. Just like it wouldn't make sense to divide both sides by the square root symbol '$\sqrt{\ }$' when solving $x\sqrt{2} = 5$, it makes no sense to divide by 'ln'.

Solution.

1. Since 16 is a power of 2, we can rewrite $2^{3x} = 16^{1-x}$ as $2^{3x} = \left(2^4\right)^{1-x}$. Using properties of exponents, we get $2^{3x} = 2^{4(1-x)}$.

   Using the one-to-one property of exponential functions, we get $3x = 4(1-x)$ which gives $x = \frac{4}{7}$.

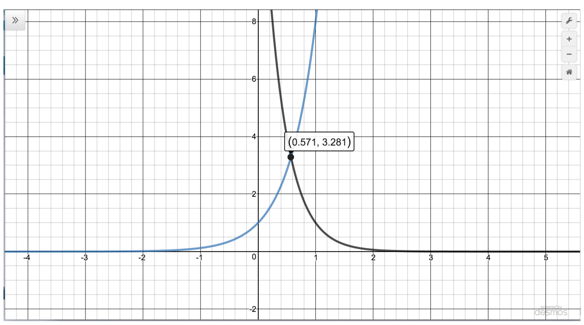
   Graphing $f(x) = 2^{3x}$ and $g(x) = 16^{1-x}$ and see that they intersect at $x \approx 0.571 \approx \frac{4}{7}$.

2. We begin solving $2000 = 1000 \cdot 3^{-0.1t}$ by dividing both sides by 1000 to isolate the exponential which yields $3^{-0.1t} = 2$.

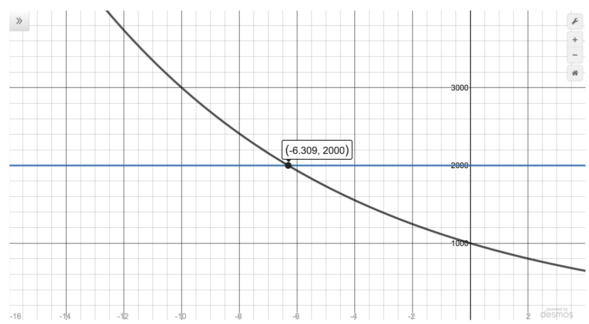
   Since it is inconvenient to write 2 as a power of 3, we use the natural log to get $\ln\left(3^{-0.1t}\right) = \ln(2)$.

   Using the Power Rule, we get $-0.1t\ln(3) = \ln(2)$, so we divide both sides by $-0.1\ln(3)$ and obtain $t = -\frac{\ln(2)}{0.1\ln(3)} = -\frac{10\ln(2)}{\ln(3)}$.

   We see the graphs of $f(x) = 2000$ and $g(x) = 1000 \cdot 3^{-0.1x}$ intersect at $x \approx -6.309 \approx -\frac{10\ln(2)}{\ln(3)}$.



Checking $2^{3x} = 16^{1-x}$



Checking $2000 = 1000 \cdot 3^{-0.1t}$

3. We first note that we can rewrite the equation $9 \cdot 3^x = 7^{2x}$ as $3^2 \cdot 3^x = 7^{2x}$ to obtain $3^{x+2} = 7^{2x}$.

   Since it is not convenient to express both sides as a power of 3 (or 7 for that matter) we use the natural log: $\ln\left(3^{x+2}\right) = \ln\left(7^{2x}\right)$.

   The power rule gives $(x+2)\ln(3) = 2x\ln(7)$. Even though this equation appears very complicated, keep in mind that $\ln(3)$ and $\ln(7)$ are just constants.

   The equation $(x+2)\ln(3) = 2x\ln(7)$ is actually a linear equation (do you see why?) and as such we gather all of the terms with x on one side, and the constants on the other. We then divide both sides by the coefficient of x, which we obtain by factoring.

$$\begin{array}{rcll}
(x+2)\ln(3) & = & 2x\ln(7) & \\
x\ln(3) + 2\ln(3) & = & 2x\ln(7) & \\
2\ln(3) & = & 2x\ln(7) - x\ln(3) & \\
2\ln(3) & = & x(2\ln(7) - \ln(3)) & \text{Factor.} \\
x & = & \frac{2\ln(3)}{2\ln(7)-\ln(3)} &
\end{array}$$

We see the graphs of $f(x) = 9 \cdot 3^x$ and $g(x) = 7^{2x}$ intersect at $x \approx 0.787 \approx \frac{2\ln(3)}{2\ln(7)-\ln(3)}$.

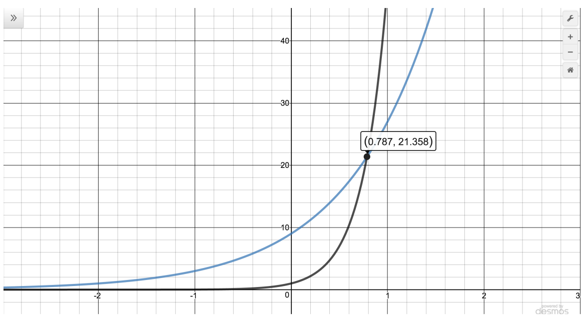
4. Our objective in solving $75 = \frac{100}{1+3e^{-2t}}$ is to first isolate the exponential.

   To that end, we clear denominators and get $75\left(1+3e^{-2t}\right) = 100$, or $75 + 225e^{-2t} = 100$. We get $225e^{-2t} = 25$, so finally, $e^{-2t} = \frac{1}{9}$.

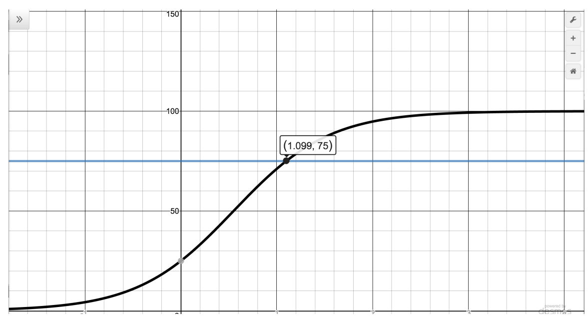
   Taking the natural log of both sides gives $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$. Since natural log is log base e, $\ln\left(e^{-2t}\right) = -2t$. Likewise, we use the Power Rule to rewrite $\ln\left(\frac{1}{9}\right) = -\ln(9)$.

   Putting these two steps together, we simplify $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$ to $-2t = -\ln(9)$. We arrive at our solution, $t = \frac{\ln(9)}{2}$ which simplifies to $t = \ln(3)$. (Can you explain why?)

   To check, we see the graphs of $f(x) = 75$ and $g(x) = \frac{100}{1+3e^{-2x}}$, intersect at $x \approx 1.099 \approx \ln(3)$.



Checking $9 \cdot 3^x = 7^{2x}$



Checking $75 = \frac{100}{1+3e^{-2t}}$

5. We start solving $25^x = 5^x + 6$ by rewriting $25 = 5^2$ so that we have $\left(5^2\right)^x = 5^x + 6$, or $5^{2x} = 5^x + 6$.

   Even though we have a common base, having two terms on the right hand side of the equation foils our plan of equating exponents or taking logs.

   If we stare at this long enough, we notice that we have three terms with the exponent on one term exactly twice that of another. To our surprise and delight, we have a 'quadratic in disguise'.

   Letting $u = 5^x$, we have $u^2 = (5^x)^2 = 5^{2x}$ so the equation $5^{2x} = 5^x + 6$ becomes $u^2 = u + 6$. Solving this as $u^2 - u - 6 = 0$ gives $u = -2$ or $u = 3$. Since $u = 5^x$, we have $5^x = -2$ or $5^x = 3$.

   Since $5^x = -2$ has no real solution,[3] we focus on $5^x = 3$. Since it isn't convenient to express 3 as a power of 5, we take natural logs and get $\ln\left(5^x\right) = \ln(3)$ so that $x\ln(5) = \ln(3)$ or $x = \frac{\ln(3)}{\ln(5)}$.

   We see the graphs of $f(x) = 25^x$ and $g(x) = 5^x + 6$ intersect at $x \approx 0.683 \approx \frac{\ln(3)}{\ln(5)}$.

6. Clearing the denominator in $\frac{e^x - e^{-x}}{2} = 5$ gives $e^x - e^{-x} = 10$, at which point we pause to consider how to proceed. Rewriting $e^{-x} = \frac{1}{e^x}$, we see we have another denominator to clear: $e^x - \frac{1}{e^x} = 10$.

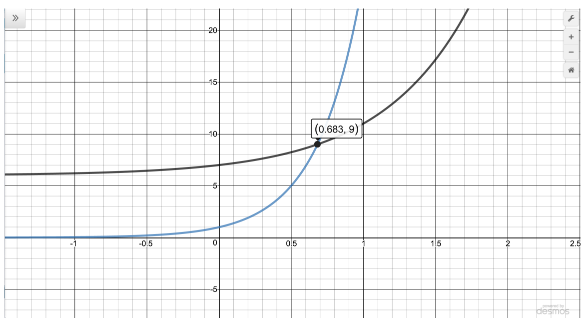
[3] Why not?

Doing so gives $e^{2x} - 1 = 10e^x$, which, once again fits the criteria of being a 'quadratic in disguise.'

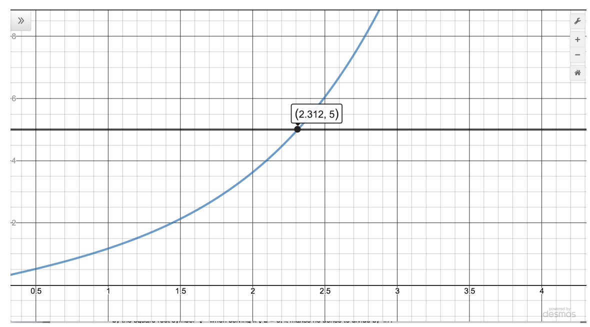
If we let $u = e^x$, then $u^2 = e^{2x}$ so the equation $e^{2x} - 1 = 10e^x$ can be viewed as $u^2 - 1 = 10u$. Solving $u^2 - 10u - 1 = 0$ using the quadratic formula gives $u = 5 \pm \sqrt{26}$.

From this, we have $e^x = 5 \pm \sqrt{26}$. Since $5 - \sqrt{26} < 0$, we get no real solution to $e^x = 5 - \sqrt{26}$ (why not?) but for $e^x = 5 + \sqrt{26}$, we take natural logs to obtain $x = \ln\left(5 + \sqrt{26}\right)$.

We see the graphs of $f(x) = \frac{e^x - e^{-x}}{2}$ and $g(x) = 5$ intersect at $x \approx 2.312 \approx \ln\left(5 + \sqrt{26}\right)$.



Checking $25^x = 5^x + 6$



Checking $\frac{e^x - e^{-x}}{2} = 5$

$\square$

Note that verifying our solutions to the equations in Example 7.4.1 *analytically* holds great educational value, since it reviews many of the properties of logarithms and exponents in tandem.

For example, to verify our solution to $2000 = 1000 \cdot 3^{-0.1t}$, we substitute $t = -\frac{10\ln(2)}{\ln(3)}$ and check:

$$\begin{array}{rcll}
2000 & \stackrel{?}{=} & 1000 \cdot 3^{-0.1\left(-\frac{10\ln(2)}{\ln(3)}\right)} & \\
2000 & \stackrel{?}{=} & 1000 \cdot 3^{\frac{\ln(2)}{\ln(3)}} & \\
2000 & \stackrel{?}{=} & 1000 \cdot 3^{\log_3(2)} & \text{Change of Base} \\
2000 & \stackrel{?}{=} & 1000 \cdot 2 & \text{Inverse Property} \\
2000 & \stackrel{\checkmark}{=} & 2000 &
\end{array}$$

We strongly encourage the reader to check the remaining equations analytically as well.

Since exponential functions are continuous on their domains, the Intermediate Value Theorem 2.14 applies. This allows us to solve inequalities using sign diagrams as demonstrated below.

Example 7.4.2. Solve the following inequalities. Check your answer graphically.

1. $2^{x^2-3x} - 16 \geq 0$
2. $\dfrac{e^x}{e^x - 4} \leq 3$
3. $te^{2t} < 4t$

Solution.

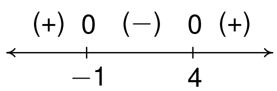
1. Since we already have 0 on one side of the inequality, we set $r(x) = 2^{x^2-3x} - 16$.

   The domain of r is all real numbers, so to construct our sign diagram, we need to find the zeros of r.

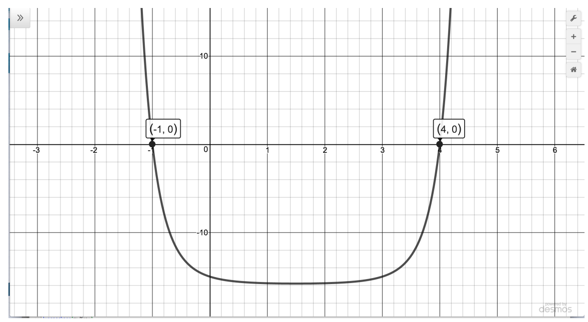
   Setting $r(x) = 0$ gives $2^{x^2-3x} - 16 = 0$ or $2^{x^2-3x} = 16$. Since $16 = 2^4$ we have $2^{x^2-3x} = 2^4$. By the one-to-one property of exponential functions, $x^2 - 3x = 4$ which gives $x = 4$ and $x = -1$.

   From the sign diagram, we see $r(x) \geq 0$ on $(-\infty, -1] \cup [4, \infty)$, which is our solution.

   Graphing $r(x) = 2^{x^2-3x} - 16$, we find it is on or above the line $y = 0$ (the x-axis) precisely on the intervals $(-\infty, -1]$ and $[4, \infty)$ which checks our answer.



A Sign Diagram for $r(x) = 2^{x^2-3x} - 16$



Checking $2^{x^2-3x} - 16 \geq 0$

2. The first step we need to take to solve $\frac{e^x}{e^x-4} \leq 3$ is to get 0 on one side of the inequality. To that end, we subtract 3 from both sides and get a common denominator

$$\begin{aligned} \frac{e^x}{e^x - 4} &\leq 3 \\ \frac{e^x}{e^x - 4} - 3 &\leq 0 \\ \frac{e^x}{e^x - 4} - \frac{3\left(e^x - 4\right)}{e^x - 4} &\leq 0 \quad \text{Common denomintors.} \\ \frac{12 - 2e^x}{e^x - 4} &\leq 0 \end{aligned}$$

   We set $r(x) = \frac{12-2e^x}{e^x-4}$ and we note that r is undefined when its denominator $e^x - 4 = 0$, or when $e^x = 4$. Solving this gives $x = \ln(4)$, so the domain of r is $(-\infty, \ln(4)) \cup (\ln(4), \infty)$.

   To find the zeros of r, we solve $r(x) = 0$ and obtain $12 - 2e^x = 0$. We find $e^x = 6$, or $x = \ln(6)$.

   When we build our sign diagram, finding test values may be a little tricky since we need to check values around $\ln(4)$ and $\ln(6)$.

   Recall that the function $\ln(x)$ is increasing[4] which means $\ln(3) < \ln(4) < \ln(5) < \ln(6) < \ln(7)$.

[4]This is because the base of $\ln(x)$ is $e > 1$. If the base b were in the interval $0 < b < 1$, then $\log_b(x)$ would decreasing.

To determine the sign of $r(\ln(3))$, we remember that $e^{\ln(3)} = 3$ and get

$$r(\ln(3)) = \frac{12 - 2e^{\ln(3)}}{e^{\ln(3)} - 4} = \frac{12 - 2(3)}{3 - 4} = -6.$$

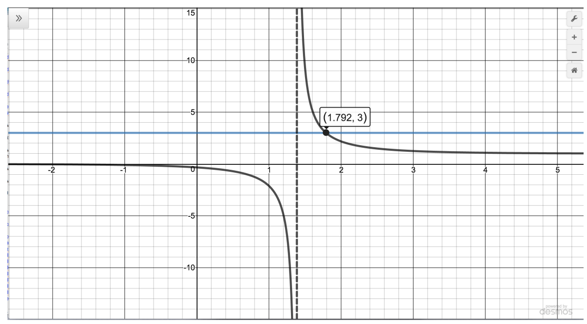
We determine the signs of $r(\ln(5))$ and $r(\ln(7))$ similarly.[5] From the sign diagram, we find our answer to be $(-\infty, \ln(4)) \cup [\ln(6), \infty)$.

Using a graphing utility, we find the graph of $f(x) = \frac{e^x}{e^x - 4}$ is below the graph of $g(x) = 3$ on $(-\infty, \ln(4)) \cup (\ln(6), \infty)$, and they intersect at $x \approx 1.792 \approx \ln(6)$.

$(-)$? $(+)$ 0 $(-)$

$\ln(4)$ $\ln(6)$

A Sign Diagram for $r(x) = \dfrac{12 - 2e^x}{e^x - 4}$



Checking $\dfrac{e^x}{e^x - 4} \leq 3$

3. As before, we start solving $te^{2t} < 4t$ by getting 0 on one side of the inequality, $te^{2t} - 4t < 0$.

   We set $r(t) = te^{2t} - 4t$ and since there are no denominators, even-indexed radicals, or logs, the domain of r is all real numbers.

   Setting $r(t) = 0$ produces $te^{2t} - 4t = 0$. We factor to get $t\left(e^{2t} - 4\right) = 0$ which gives $t = 0$ or $e^{2t} - 4 = 0$.

   To solve the latter, we isolate the exponential and take logs to get $2t = \ln(4)$, or $t = \frac{\ln(4)}{2}$ which simplifies to $t = \ln(2)$. (Can you see why?)

   As in the previous example, we need to be careful about choosing test values. Since $\ln(1) = 0$, we choose $\ln\left(\frac{1}{2}\right)$, $\ln\left(\frac{3}{2}\right)$ and $\ln(3)$. Evaluating,[6] we get

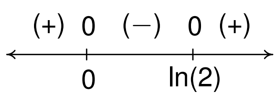
$$\begin{array}{rcll}
r\left(\ln\left(\frac{1}{2}\right)\right) & = & \ln\left(\frac{1}{2}\right) e^{2\ln\left(\frac{1}{2}\right)} - 4\ln\left(\frac{1}{2}\right) & \\
& = & \ln\left(\frac{1}{2}\right) e^{\ln\left(\frac{1}{2}\right)^2} - 4\ln\left(\frac{1}{2}\right) & \text{Power Rule} \\
& = & \ln\left(\frac{1}{2}\right) e^{\ln\left(\frac{1}{4}\right)} - 4\ln\left(\frac{1}{2}\right) & \\
& = & \frac{1}{4}\ln\left(\frac{1}{2}\right) - 4\ln\left(\frac{1}{2}\right) = -\frac{15}{4}\ln\left(\frac{1}{2}\right) &
\end{array}$$

[5] We could, of course, use the calculator, but what fun would that be?

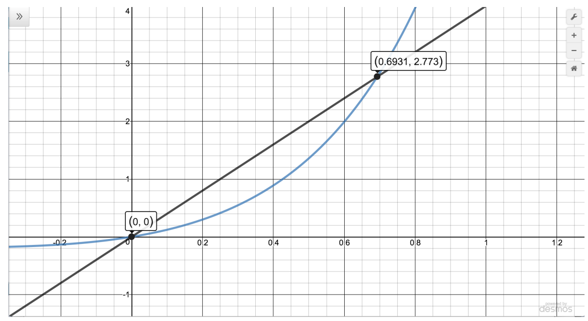
[6] A calculator can be used at this point. As usual, we proceed without apologies, with the analytical method.

Since $\frac{1}{2} < 1$, $\ln\left(\frac{1}{2}\right) < 0$ and we get $r(\ln\left(\frac{1}{2}\right))$ is $(+)$. Proceeding similarly, we find $r\left(\ln\left(\frac{3}{2}\right)\right) < 0$ and $r(\ln(3)) > 0$. Our solution corresponds to $r(t) < 0$ which occurs on $(0, \ln(2))$.

The graphing utility confirms that the graph of $f(t) = te^{2t}$ is below the graph of $g(t) = 4t$ on $(0, \ln(2))$.[7]



A Sign Diagram for $r(t) = te^{2t} - 4t$



Checking $te^{2t} < 4t$

□

We note here that while sign diagrams will *always* work for solving inequalities involving exponential functions, as we've seen previously, there are circumstances in which we can short-cut this method.

For example, consider number 1 from Example 7.4.2 above: $2^{x^2-3x} - 16 \geq 0$. Since the base $2 > 1$, $\log_2(x)$ is an *increasing* function meaning it preserves inequalities.

We can use this to our advantage in this case and eliminate the exponential from the inequality altogether:

$$\begin{aligned} 2^{x^2-3x} - 16 &\geq 0 \\ 2^{x^2-3x} &\geq 16 \\ \log_2\left(2^{x^2-3x}\right) &\geq \log_2(16) \quad f(x) = \log_2(x) \text{ is increasing so if } b \geq a, \log_2(b) \geq \log_2(a). \\ x^2 - 3x &\geq 4 \end{aligned}$$

Hence, we've reduced our given inequality to $x^2 - 3x \geq 4$. As seen in Section 1.4, we can solve this inequality by completing the square, graphing, or a sign diagram, whichever strikes the reader's fancy.

Our next example is a follow-up to Example 7.1.3 in Section 7.1.

Example 7.4.3. Recall from Example 7.1.3 the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$. When will the coffee be warmer than 100°F?

Solution. We need to find when $T(t) > 100$, that is, we need to solve $70 + 90e^{-0.1t} > 100$.

To use a sign diagram, we need to get 0 on one side of the inequality. Subtracting 100 from both sides of $70 + 90e^{-0.1t} > 100$ produces $90e^{-0.1t} - 30 > 0$.

Identifying $r(t) = 90e^{-0.1t} - 30$, we note from the context of the problem the domain of r is $[0, \infty)$, so to build the sign diagram, we proceed to find the zeros of r.

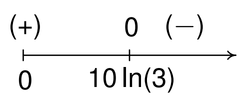
[7]Note: $\ln(2) \approx 0.693$.

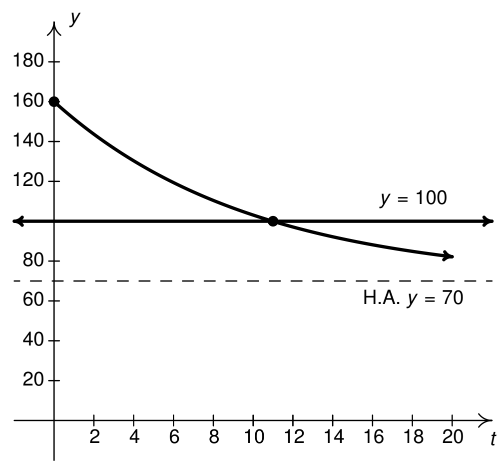
Solving $90e^{-0.1t} - 30 = 0$ results in $e^{-0.1t} = \frac{1}{3}$ so that $t = -10\ln\left(\frac{1}{3}\right)$ which reduces to $t = 10\ln(3)$.

If we wish to avoid using the calculator to choose test values, we note that $f(x) = \ln(x)$ is increasing. As a result, since $1 < 3$, $0 = \ln(1) < \ln(3)$ which proves $10\ln(3) > 0$. Hence, we may choose $t = 0$ as a test value in $[0, 10\ln(3))$. Since $3 < 4$, $\ln(3) < \ln(4)$, so $10\ln(3) < 10\ln(4)$. Hence, we may choose $10\ln(4)$ as test value for the interval $(10\ln(3), \infty)$.

We find $r(0) > 0$ and $r(10\ln(4)) < 0$ which gives the sign diagram below. We see $r(t) > 0$ on $[0, 10\ln(3))$.

We graph $y = T(t)$ from Example 7.1.3 below on the right along with with the horizontal line $y = 100$. We see the graph of T is above the horizontal line to the left of the intersection point, which we leave to the reader to show is $(10\ln(3), 100)$.





$y = T(t)$

Hence, the coffee is warmer than 100°F up to $10\ln(3) \approx 11$ minutes after it is served, or, said differently, it takes approximately 11 minutes for the coffee to cool to under 100°F. □

We note that, once again, we can short-cut the sign diagram in Example 7.4.3 to solve $70+90e^{-0.1t} > 100$. Since $\ln(x)$ is increasing, it preserves inequality. This means we can solve this inequality as follows.

$$\begin{array}{rcll}
70 + 90e^{-0.1t} & > & 100 & \\
90e^{-0.1t} & > & 30 & \\
e^{-0.1t} & > & \frac{1}{3} & \\
\ln\left(e^{-0.1t}\right) & > & \ln\left(\frac{1}{3}\right) & f(x) = \ln(x) \text{ is increasing so if } b \geq a, \ln(b) \geq \ln(a). \\
-0.1t & > & -\ln(3) & \ln\left(\frac{1}{3}\right) = \ln\left(3^{-1}\right) = -\ln(3). \\
t & < & \frac{-\ln(3)}{-0.1} = 10\ln(3) &
\end{array}$$

Since we are given $t \geq 0$, we arrive at the same answer $0 \leq t < 10\ln(3)$ or $[0, 10\ln(3))$.

Note the importance, once again, of having a base larger than 1 so that the corresponding logarithmic function is *increasing*. We can still adapt this strategy to exponential functions whose base is less than 1, but we need to remember the corresponding logarithmic function is *decreasing* so it *reverses* inequalities.

Example 7.4.4. The function $f(x) = \dfrac{5e^x}{e^x + 1}$ is one-to-one.

1. Find a formula for $f^{-1}(x)$.
2. Solve $\dfrac{5e^x}{e^x + 1} = 4$.

Solution.

1. We start by writing $y = f(x)$, and interchange the roles of x and y. To solve for y, we first clear denominators and then isolate the exponential function.

$$\begin{array}{rcll} y & = & \dfrac{5e^x}{e^x + 1} & \\ x & = & \dfrac{5e^y}{e^y + 1} & \text{Switch } x \text{ and } y \\ x\left(e^y + 1\right) & = & 5e^y & \\ xe^y + x & = & 5e^y & \\ x & = & 5e^y - xe^y & \\ x & = & e^y(5 - x) & \\ e^y & = & \dfrac{x}{5 - x} & \\ \ln\left(e^y\right) & = & \ln\left(\dfrac{x}{5 - x}\right) & \\ y & = & \ln\left(\dfrac{x}{5 - x}\right) & \end{array}$$

We claim $f^{-1}(x) = \ln\left(\frac{x}{5-x}\right)$. To verify this analytically, we would need to verify the compositions $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}. We leave this, as well as a graphical check, to the reader in Exercise 56.

2. We recognize the equation $\frac{5e^x}{e^x+1} = 4$ as $f(x) = 4$. Hence, our solution is $x = f^{-1}(4) = \ln\left(\frac{4}{5-4}\right) = \ln(4)$.

   We can check this fairly quickly algebraically. Using $e^{\ln(4)} = 4$, we find $\frac{5e^{\ln(4)}}{e^{\ln(4)}+1} = \frac{5(4)}{4+1} = \frac{20}{5} = 4$. $\square$

Our last example uses the tools of this section along with those developed in Section 6.3.

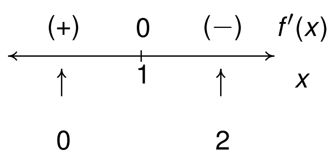
Example 7.4.5. Let $f(x) = 3xe^{-x}$.

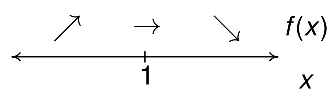
1. Given that $f'(x) = 3e^{-x} - 3xe^{-x}$, find the open intervals over which f is increasing and decreasing.
2. Find the local extrema.[8]
3. Given that $f''(x) = 3xe^{-x} - 6e^{-x}$, find the open intervals over which the graph of f is concave up and concave down.
4. Locate the inflection points.
5. Check your answers using a graphing utility.

Solution.

1. To determine where f is increasing and decreasing, we need to make a sign diagram for $f'(x)$. Since the domain of f' is all real numbers, we just need to find the zeros of f'.

   Solving $f'(x) = 3e^{-x} - 3xe^{-x} = 0$ gives $3e^{-x}(1 - x) = 0$ so $3e^{-x} = 0$, which has no solution, or $1 - x = 0$ so $x = 1$. We make a sign diagram for $f'(x)$ below.

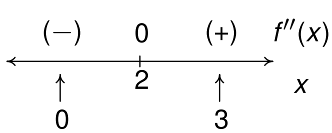
   

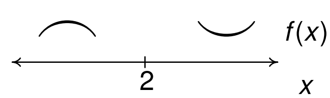
   We get f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

2. Using the sign diagram for $f'(x)$, we see we have a local maximum at $(1, f(1)) = \left(1, 3e^{-1}\right)$.

3. Once again, the domain of f'' is all real numbers, so our first step in constructing a sign diagram for $f''(x)$ is to find the zeros.

   Solving $f''(x) = 3xe^{-x} - 6e^{-x} = 0$ gives $3e^{-x}(x - 2) = 0$ so either $3e^{-x} = 0$, which has no solution, or $x - 2 = 0$, so $x = 2$. We make the sign diagram for $f''(x)$ below.

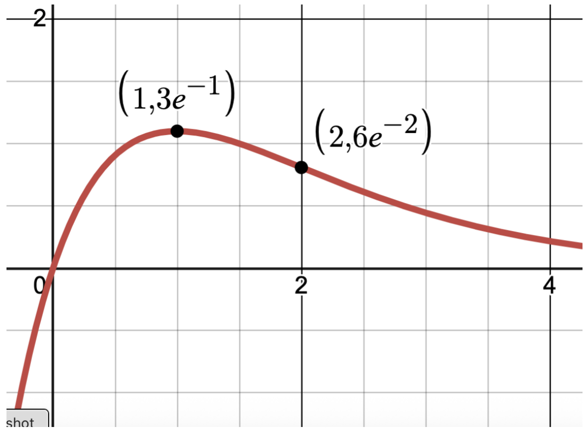
   

   We see the graph of f is concave up on $(-\infty, 2)$ and concave down on $(2, \infty)$.

---
[8]Recall this means the local maximums and minimums.

4. Since f changes concavity at $x = 2$, we have an inflection point at $(2, f(2)) = \left(2, 6e^{-2}\right)$.

5. We check using desmos:



□

Note that the graph of $f(x) = 3xe^{-x}$ produced by the graphing utility in Example 7.4.5 suggests the graph of f has a horizontal asymptote as $x \to \infty$.

Indeed, it is the case that $\lim\limits_{x \to \infty} f(x) = 0$, however if we try to reason this analytically, we get another instance of an indeterminate form.[9] As $x \to \infty$, $3x \to \infty$ but $e^{-x} \to 0$. Hence, as $x \to \infty$, we get the indeterminate form '$\infty \cdot 0$.' Depending on how quickly the first factor approaches '∞' and how quickly the second factor approaches '0', we could end up with '∞', '0,' or some number in between.[10]

We'll explore more of this phenomenon in Exercise 57.[11] For now, we take it as true that exponential functions dominate polynomial functions so in the above indeterminate form, the factor e^{-x} determines[12] the end behavior of f, so $\lim\limits_{x \to \infty} f(x) = 0$.

[9] Previously, we've seen the indeterminate form '$\frac{0}{0}$.'

[10] It is important here to understand that the factor e^{-x} which results in the '0' in the indeterminate form '$\infty \cdot 0$' is **approaching** 0 and is not actually **equal** to 0. An incorrect reasoning that the form $\infty \cdot 0 \to 0$ in this case is that 'anything times 0 is 0.' Again, we'll have more to say about this in the Exercises.

[11] With formal proofs found in Calculus ...

[12] Less formally, the factor $e^{-x} \to 0$ 'faster' than the factor $3x \to \infty$ which forces the product $3xe^{-x} \to 0$.

7.4.1 Exercises

In Exercises 1 - 33, solve the equation analytically.

1. $2^{4x} = 8$
2. $3^{(x-1)} = 27$
3. $5^{2x-1} = 125$
4. $4^{2t} = \frac{1}{2}$
5. $8^{t} = \frac{1}{128}$
6. $2^{(t^3-t)} = 1$
7. $3^{7x} = 81^{4-2x}$
8. $9 \cdot 3^{7x} = \left(\frac{1}{9}\right)^{2x}$
9. $3^{2x} = 5$
10. $5^{-t} = 2$
11. $5^{t} = -2$
12. $3^{(t-1)} = 29$
13. $(1.005)^{12x} = 3$
14. $e^{-5730k} = \frac{1}{2}$
15. $2000e^{0.1t} = 4000$
16. $500\left(1 - e^{2t}\right) = 250$
17. $70 + 90e^{-0.1t} = 75$
18. $30 - 6e^{-0.1t} = 20$
19. $\dfrac{100e^{x}}{e^{x} + 2} = 50$
20. $\dfrac{5000}{1 + 2e^{-3t}} = 2500$
21. $\dfrac{150}{1 + 29e^{-0.8t}} = 75$
22. $25\left(\frac{4}{5}\right)^{x} = 10$
23. $e^{2x} = 2e^{x}$
24. $7e^{2t} = 28e^{-6t}$
25. $3^{(x-1)} = 2^{x}$
26. $3^{(x-1)} = \left(\frac{1}{2}\right)^{(x+5)}$
27. $7^{3+7x} = 3^{4-2x}$
28. $e^{2t} - 3e^{t} - 10 = 0$
29. $e^{2t} = e^{t} + 6$
30. $4^{t} + 2^{t} = 12$
31. $e^{x} - 3e^{-x} = 2$
32. $e^{x} + 15e^{-x} = 8$
33. $3^{x} + 25 \cdot 3^{-x} = 10$

In Exercises 34 - 41, solve the inequality analytically.

34. $e^{x} > 53$
35. $1000\,(1.005)^{12t} \geq 3000$
36. $2^{(x^3-x)} < 1$
37. $25\left(\dfrac{4}{5}\right)^{x} \geq 10$
38. $\dfrac{150}{1 + 29e^{-0.8t}} \leq 130$
39. $70 + 90e^{-0.1t} \leq 75$
40. $e^{-x} - xe^{-x} \geq 0$
41. $(1 - e^{t})t^{-1} \leq 0$

In Exercises 42 - 47, use a graphing utility to help you solve the equation or inequality.

42. $2^{x} = x^{2}$
43. $e^{t} = \ln(t) + 5$
44. $e^{\sqrt{x}} = x + 1$
45. $e^{-2t} - te^{-t} \leq 0$
46. $3^{(x-1)} < 2^{x}$
47. $e^{t} < t^{3} - t$

In Exercises 48 - 53, find the domain of the function.

48. $T(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

49. $C(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

50. $s(t) = \sqrt{e^{2t} - 3}$

51. $c(t) = \sqrt[3]{e^{2t} - 3}$

52. $L(x) = \log\left(3 - e^x\right)$

53. $\ell(x) = \ln\left(\dfrac{e^{2x}}{e^x - 2}\right)$

54. Since $f(x) = \ln(x)$ is a strictly increasing function, if $0 < a < b$ then $\ln(a) < \ln(b)$. Use this fact to solve the inequality $e^{(3x-1)} > 6$ without a sign diagram. Use this technique to solve the inequalities in Exercises 34 - 41. (NOTE: Isolate the exponential function first!)

55. Compute the inverse of $f(x) = \dfrac{e^x - e^{-x}}{2}$. State the domain and range of both f and f^{-1}.

56. In Example 7.4.4, we found that the inverse of $f(x) = \dfrac{5e^x}{e^x + 1}$ was $f^{-1}(x) = \ln\left(\dfrac{x}{5 - x}\right)$ but we left a few loose ends for you to tie up.

    (a) Algebraically check our answer by verifying: $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.

    (b) Find the range of f by finding the domain of f^{-1}.

    (c) With help of a graphing utility, graph $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same set of axes. How does this help to verify our answer?

    (d) Let $g(x) = \dfrac{5x}{x + 1}$ and $h(x) = e^x$. Show that $f = g \circ h$ and that $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$.
    NOTE: We know this is true in general by Exercise 40 in Section 5.6, but it's nice to see a specific example of the property.

57. (a) With the help of your classmates, numerically and graphically investigate $\lim\limits_{x \to \infty} \frac{x^p}{e^x}$ for various real number powers, p.

    (b) What does part 57a suggest about the relative growth rates of powers of x as opposed to e^x?

    (c) For each power p you investigated in part 57a, solve the inequality: $\frac{x^p}{e^x} < \frac{1}{x}$.

    (d) Use your results from part 57c to show that for each real number p you investigated in part 57a, there is a real number M so that if $x > M$, $0 < \frac{x^p}{e^x} < \frac{1}{x}$.

    Since $\lim\limits_{x \to \infty} \frac{1}{x} = 0$, what do you conclude about $\lim\limits_{x \to \infty} \frac{x^p}{e^x}$?

    (This Exercise foreshadows the celebrated **Squeeze Theorem**, Theorem 10.2 which we'll formally introduce in Section 10.1.)

In Exercises 58 - 59 a function f along with its derivatives f' and f'' are given.

- Find the x- and y-intercepts of the graph of each function, if any.
- Use limits to determine the end behavior.
- Use f' to determine the open intervals over which f is increasing or decreasing.
- Determine the local extrema, if any.
- Use f'' to determine the open intervals over which the graph of f is concave up or concave down.
- Determine the inflection points of the graph, if any.

58. $f(x) = \dfrac{5}{1+e^{-x}}$, $f'(x) = \dfrac{5e^{-x}}{\left(1+e^{-x}\right)^2}$, $f''(x) = \dfrac{5e^{-x}\left(e^{-x}-1\right)}{\left(1+e^{-x}\right)^3}$

59. $f(x) = e^{-x} - e^{-2x}$, $f'(x) = 2e^{-2x} - e^{-x}$, $f''(x) = e^{-x} - 4e^{-2x}$

7.4.2 Answers

1. $x = \frac{3}{4}$
2. $x = 4$
3. $x = 2$
4. $t = -\frac{1}{4}$
5. $t = -\frac{7}{3}$
6. $t = -1, 0, 1$
7. $x = \frac{16}{15}$
8. $x = -\frac{2}{11}$
9. $x = \dfrac{\ln(5)}{2\ln(3)}$
10. $t = -\dfrac{\ln(2)}{\ln(5)}$
11. No solution.
12. $t = \dfrac{\ln(29)+\ln(3)}{\ln(3)}$
13. $x = \dfrac{\ln(3)}{12\ln(1.005)}$
14. $k = \dfrac{\ln\left(\frac{1}{2}\right)}{-5730} = \dfrac{\ln(2)}{5730}$
15. $t = \dfrac{\ln(2)}{0.1} = 10\ln(2)$
16. $t = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{1}{2}\ln(2)$
17. $t = \dfrac{\ln\left(\frac{1}{18}\right)}{-0.1} = 10\ln(18)$
18. $t = -10\ln\left(\frac{5}{3}\right) = 10\ln\left(\frac{3}{5}\right)$
19. $x = \ln(2)$
20. $t = \frac{1}{3}\ln(2)$
21. $t = \dfrac{\ln\left(\frac{1}{29}\right)}{-0.8} = \dfrac{5}{4}\ln(29)$
22. $x = \dfrac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)} = \dfrac{\ln(2)-\ln(5)}{\ln(4)-\ln(5)}$
23. $x = \ln(2)$
24. $t = -\frac{1}{8}\ln\left(\frac{1}{4}\right) = \frac{1}{4}\ln(2)$
25. $x = \dfrac{\ln(3)}{\ln(3)-\ln(2)}$
26. $x = \dfrac{\ln(3)+5\ln\left(\frac{1}{2}\right)}{\ln(3)-\ln\left(\frac{1}{2}\right)} = \dfrac{\ln(3)-5\ln(2)}{\ln(3)+\ln(2)}$
27. $x = \dfrac{4\ln(3)-3\ln(7)}{7\ln(7)+2\ln(3)}$
28. $t = \ln(5)$
29. $t = \ln(3)$
30. $t = \dfrac{\ln(3)}{\ln(2)}$
31. $x = \ln(3)$
32. $x = \ln(3), \ln(5)$
33. $x = \dfrac{\ln(5)}{\ln(3)}$
34. $(\ln(53), \infty)$
35. $\left[\dfrac{\ln(3)}{12\ln(1.005)}, \infty\right)$
36. $(-\infty, -1) \cup (0, 1)$
37. $\left(-\infty, \dfrac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)}\right] = \left(-\infty, \dfrac{\ln(2)-\ln(5)}{\ln(4)-\ln(5)}\right]$
38. $\left(-\infty, \dfrac{\ln\left(\frac{2}{377}\right)}{-0.8}\right] = \left(-\infty, \frac{5}{4}\ln\left(\dfrac{377}{2}\right)\right]$
39. $\left[\dfrac{\ln\left(\frac{1}{18}\right)}{-0.1}, \infty\right) = [10\ln(18), \infty)$

40. $(-\infty, 1]$

41. $(-\infty, 0) \cup (0, \infty)$

42. $x \approx -0.76666$, $x = 2$, $x = 4$

43. $x \approx 0.01866$, $x \approx 1.7115$

44. $x = 0$

45. $\approx [0.567, \infty)$

46. $\approx (-\infty, 2.7095)$

47. $\approx (2.3217, 4.3717)$

48. $(-\infty, \infty)$

49. $(-\infty, 0) \cup (0, \infty)$

50. $\left(\frac{1}{2}\ln(3), \infty\right)$

51. $(-\infty, \infty)$

52. $(-\infty, \ln(3))$

53. $(\ln(2), \infty)$

54. $x > \frac{1}{3}(\ln(6) + 1)$, so $\left(\frac{1}{3}(\ln(6) + 1), \infty\right)$

55. $f^{-1} = \ln\left(x + \sqrt{x^2 + 1}\right)$. Both f and f^{-1} have domain $(-\infty, \infty)$ and range $(-\infty, \infty)$.

58.
- There are no x-intercepts; the y-intercept is $(0, 5)$.
- $\lim\limits_{x \to -\infty} f(x) = 0$ and $\lim\limits_{x \to \infty} f(x) = 5$; we have two horizontal asymptotes: $y = 0$ and $y = 5$.
- f is always increasing: $(-\infty, \infty)$.
- There are no local extrema.
- The graph of f is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.
- The inflection point is $(0, 5)$.

59.
- The x- and y-intercept is $(0, 0)$.
- $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to \infty} f(x) = 0$; we have a horizontal asymptote $y = 0$.
- f is increasing on $(-\infty, \ln(2))$ and decreasing on $(\ln(2), \infty)$.
- There is a local (absolute) maximum at $\left(\ln(2), \frac{1}{4}\right)$.
- The graph of f is concave up on $(\ln(4), \infty)$ and concave down on $(-\infty, \ln(4))$.
- The inflection point is $\left(\ln(4), \frac{3}{16}\right)$.

7.5 Equations and Inequalities involving Logarithmic Functions

In Section 7.4 we solved equations and inequalities involving exponential functions using one of two basic strategies. We now turn our attention to equations and inequalities involving logarithmic functions, and not surprisingly, there are two basic strategies to choose from.

For example, per Theorem 7.5, the *only* solution to $\log_2(x) = \log_2(5)$ is $x = 5$. Now consider $\log_2(x) = 3$. To use Theorem 7.5, we need to rewrite 3 as a logarithm base 2. Theorem 7.4 gives us $3 = \log_2\left(2^3\right) = \log_2(8)$. Hence, $\log_2(x) = 3$ is equivalent to $\log_2(x) = \log_2(8)$ so that $x = 8$.

A second approach to solving $\log_2(x) = 3$ us to apply the corresponding exponential function, $f(x) = 2^x$ to both sides: $2^{\log_2(x)} = 2^3$ so $x = 2^3 = 8$.

A third approach to solving $\log_2(x) = 3$ is to use Theorem 7.4 to rewrite $\log_2(x) = 3$ as $2^3 = x$, so $x = 8$.

In the grand scheme of things, all three approaches we have presented to solve $\log_2(x) = 3$ are mathematically equivalent, so we opt to choose the last approach in our summary below.

Steps for Solving an Equation involving Logarithmic Functions

1. Isolate the logarithmic function.
2. (a) If convenient, express both sides as logs with the same base and equate arguments.
   (b) Otherwise, rewrite the log equation as an exponential equation.

Example 7.5.1. Solve the following equations. Check your solutions graphically using a calculator.

1. $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$
2. $2 - \ln(t-3) = 1$
3. $\log_6(x+4) + \log_6(3-x) = 1$
4. $\log_7(1-2t) = 1 - \log_7(3-t)$
5. $\log_2(x+3) = \log_2(6-x) + 3$
6. $1 + 2\log_4(t+1) = 2\log_2(t)$

Solution.

1. Since we have the same base on both sides of the equation $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$, we equate the arguments (what's inside) of the logs to get $1 - 3x = x^2 - 3$. Solving $x^2 + 3x - 4 = 0$ gives $x = -4$ and $x = 1$.

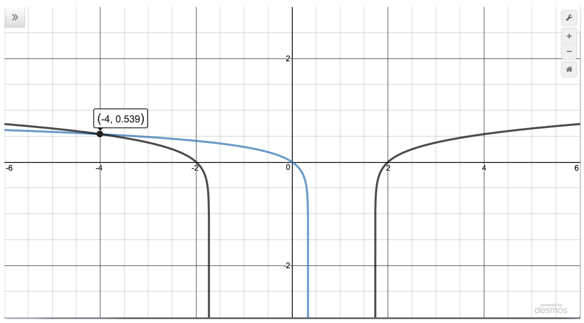
   To check these answers using a graphing utility, we make use of the change of base formula and graph $f(x) = \frac{\ln(1-3x)}{\ln(117)}$ and $g(x) = \frac{\ln\left(x^2-3\right)}{\ln(117)}$. We see these graphs intersect only at $x = -4$. however.

   To see what happened to the solution $x = 1$, we substitute it into our original equation to obtain $\log_{117}(-2) = \log_{117}(-2)$. While these expressions look identical, neither is a real number,[1] which means $x = 1$ is not in the domain of the original equation, and is not a solution.

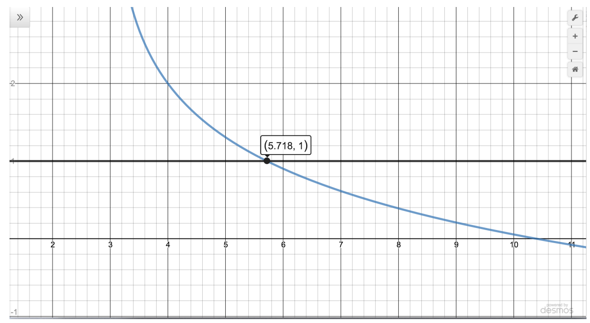
[1] They do, however, represent the same **family** of complex numbers. We refer the reader to a course in Complex Variables.

2. To solve $2 - \ln(t-3) = 1$, we first isolate the logarithm and get $\ln(t-3) = 1$. Rewriting $\ln(t-3) = 1$ as an exponential equation, we get is $e^1 = t - 3$, so $t = e + 3$.

   A graphing utility shows the graphs of $f(t) = 2 - \ln(t-3)$ and $g(t) = 1$ intersect at $t \approx 5.718 \approx e + 3$.



Checking $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$



Checking $2 - \ln(t-3) = 1$

3. We start solving $\log_6(x+4) + \log_6(3-x) = 1$ by using the Product Rule for logarithms to rewrite the equation as $\log_6\left[(x+4)(3-x)\right] = 1$.

   Rewriting as an exponential equation gives $6^1 = (x+4)(3-x)$ which reduces to $x^2 + x - 6 = 0$. We get two solutions: $x = -3$ and $x = 2$.

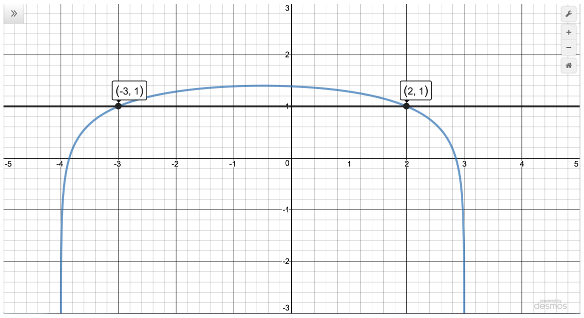
   Using the change of base formula, we graph $y = f(x) = \frac{\ln(x+4)}{\ln(6)} + \frac{\ln(3-x)}{\ln(6)}$ and $y = g(x) = 1$ and we see the graphs intersect twice, at $x = -3$ and $x = 2$, as required.

4. Taking a cue from the previous problem, we begin solving $\log_7(1-2t) = 1 - \log_7(3-t)$ by first collecting the logarithms on the same side, $\log_7(1-2t) + \log_7(3-t) = 1$, and then using the Product Rule to get $\log_7[(1-2t)(3-t)] = 1$.

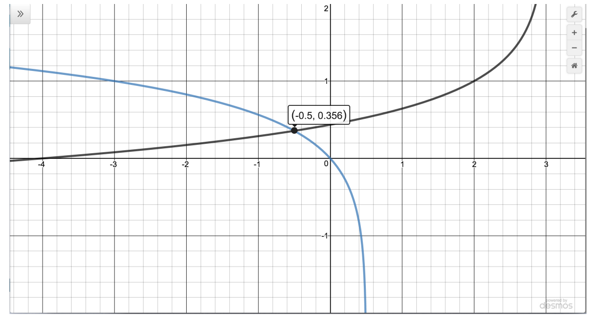
   Rewriting as an exponential equation gives $7^1 = (1-2t)(3-t)$ which gives the quadratic equation $2t^2 - 7t - 4 = 0$. Solving, we find $t = -\frac{1}{2}$ and $t = 4$.

   Once again, we use the change of base formula and find the graphs of $y = f(t) = \frac{\ln(1-2t)}{\ln(7)}$ and $y = g(t) = 1 - \frac{\ln(3-t)}{\ln(7)}$ intersect only at $t = -\frac{1}{2}$.

   Checking $t = 4$ in the original equation produces $\log_7(-7) = 1 - \log_7(-1)$, showing $t = 4$ is not in the domain of f nor g.



Checking $\log_6(x+4) + \log_6(3-x) = 1$



Checking $\log_7(1-2t) = 1 - \log_7(3-t)$

5. Our first step in solving $\log_2(x+3) = \log_2(6-x) + 3$ is to gather the logarithms to one side of of the equation: $\log_2(x+3) - \log_2(6-x) = 3$.

   The Quotient Rule gives $\log_2\left(\frac{x+3}{6-x}\right) = 3$ which, as an exponential equation is $2^3 = \frac{x+3}{6-x}$.

   Clearing denominators, we get $8(6-x) = x+3$, which reduces to $x = 5$.

   Using the change of base once again, we graph $f(x) = \frac{\ln(x+3)}{\ln(2)}$ and $g(x) = \frac{\ln(6-x)}{\ln(2)} + 3$ and find they intersect at $x = 5$.

6. Our first step in solving $1 + 2\log_4(t+1) = 2\log_2(t)$ is to gather the logs on one side of the equation. We obtain $1 = 2\log_2(t) - 2\log_4(t+1)$ but find we need a common base to combine the logs.

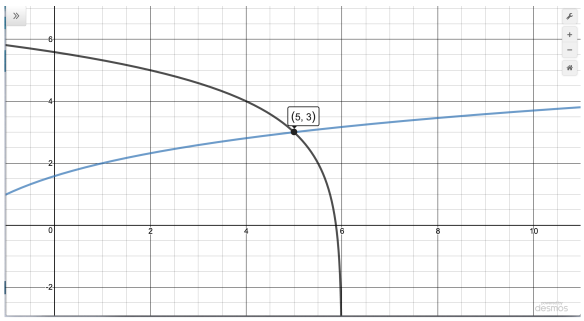
   Since 4 is a power of 2, we use change of base to convert $\log_4(t+1) = \frac{\log_2(t+1)}{\log_2(4)} = \frac{1}{2}\log_2(t+1)$. Hence, our original equation becomes

$$\begin{array}{rcll} 1 & = & 2\log_2(t) - 2\left(\frac{1}{2}\log_2(t+1)\right) & \\ 1 & = & 2\log_2(t) - \log_2(t+1) & \\ 1 & = & \log_2\left(t^2\right) - \log_2(t+1) & \text{Power Rule} \\ 1 & = & \log_2\left(\dfrac{t^2}{t+1}\right) & \text{Quotient Rule} \end{array}$$

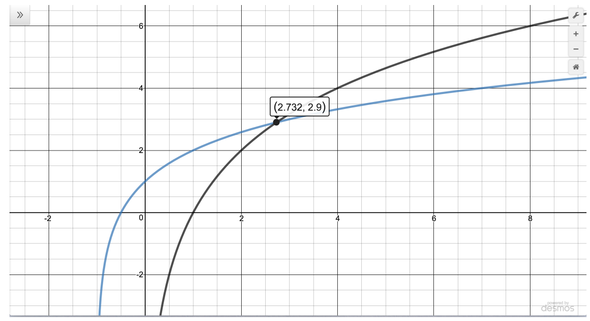
   Rewriting $1 = \log_2\left(\frac{t^2}{t+1}\right)$ in exponential form gives $\frac{t^2}{t+1} = 2$ or $t^2 - 2t - 2 = 0$. Using the quadratic formula, we obtain $t = 1 \pm \sqrt{3}$.

   One last time, we use the change of base formula and graph $f(t) = 1 + \frac{2\ln(t+1)}{\ln(4)}$ and $g(t) = \frac{2\ln(t)}{\ln(2)}$. We see the graphs intersect only at $t \approx 2.732 \approx 1 + \sqrt{3}$.

   Note the solution $t = 1-\sqrt{3} < 0$ Hence if substituted into the original equation, the term $2\log_2\left(1-\sqrt{3}\right)$ is undefined, which explains why the graphs below intersect only once.



Checking $\log_2(x+3) = \log_2(6-x)+3$



Checking $1 + 2\log_4(t+1) = 2\log_2(t)$

□

If nothing else, Example 7.5.1 demonstrates the importance of checking for extraneous solutions[2] when solving equations involving logarithms. Even though we checked our answers graphically, extraneous solutions are easy to spot: any supposed solution which causes the argument of a logarithm to be negative must be discarded.

While identifying extraneous solutions is important, it is equally important to understand which machinations create the opportunity for extraneous solutions to appear. In the case of Example 7.5.1, extraneous solutions, by and large, result from using the Power, Product, or Quotient Rules. We encourage the reader to take the time to track each extraneous solution found in Example 7.5.1 backwards through the solution process to see at precisely which step it fails to be a solution.

As with the equations in Example 7.4.1, much can be learned from checking all of the answers in Example 7.5.1 analytically. We leave this to the reader and turn our attention to inequalities involving logarithmic functions. Since logarithmic functions are continuous on their domains, we can use sign diagrams.

Example 7.5.2. Solve the following inequalities. Check your answer graphically using a calculator.

1. $\dfrac{1}{\ln(x)+1} \leq 1$
2. $\left(\log_2(x)\right)^2 < 2\log_2(x) + 3$
3. $t\log(t+1) \geq t$

Solution.

1. We start solving $\frac{1}{\ln(x)+1} \leq 1$ by getting 0 on one side of the inequality: $\frac{1}{\ln(x)+1} - 1 \leq 0$.

   Getting a common denominator yields $\frac{1}{\ln(x)+1} - \frac{\ln(x)+1}{\ln(x)+1} \leq 0$ which reduces to $\frac{-\ln(x)}{\ln(x)+1} \leq 0$, or $\frac{\ln(x)}{\ln(x)+1} \geq 0$.

   We define $r(x) = \frac{\ln(x)}{\ln(x)+1}$ and set about finding the domain and the zeros of r. Due to the appearance of the term $\ln(x)$, we require $x > 0$. In order to keep the denominator away from zero, we solve $\ln(x) + 1 = 0$ so $\ln(x) = -1$, so $x = e^{-1} = \frac{1}{e}$. Hence, the domain of r is $\left(0, \frac{1}{e}\right) \cup \left(\frac{1}{e}, \infty\right)$.

   To find the zeros of r, we set $r(x) = \frac{\ln(x)}{\ln(x)+1} = 0$ so that $\ln(x) = 0$, and we find $x = e^0 = 1$.

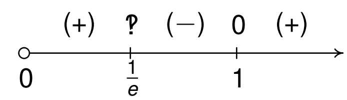
[2] Recall that an extraneous solution is an answer obtained analytically which does not satisfy the original equation.

In order to determine test values for r without resorting to the calculator, we need to find numbers between 0, $\frac{1}{e}$, and 1 which have a base of e. Since $e \approx 2.718 > 1$, $0 < \frac{1}{e^2} < \frac{1}{e} < \frac{1}{\sqrt{e}} < 1 < e$.

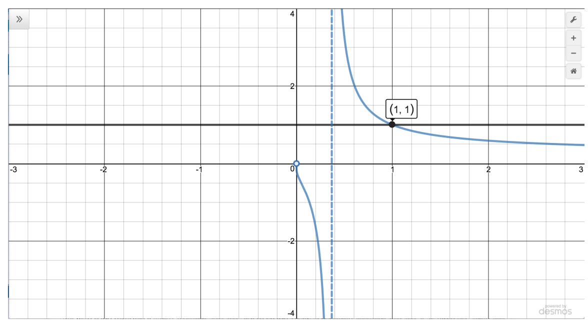
To determine the sign of $r\left(\frac{1}{e^2}\right)$, note $\ln\left(\frac{1}{e^2}\right) = \ln\left(e^{-2}\right) = -2$. Hence, $r\left(\frac{1}{e^2}\right) = \frac{-2}{-2+1} = 2 > 0$. The rest of the test values are determined similarly.

From our sign diagram, we find $r(x) \geq 0$ on $\left(0, \frac{1}{e}\right) \cup [1, \infty)$, which is our solution.

Graphing $f(x) = \frac{1}{\ln(x)+1}$ and $g(x) = 1$, we see the graph of f is below the graph of g on these intervals, and that the graphs intersect at $x = 1$.



A sign diagram for $r(x) = \dfrac{\ln(x)}{\ln(x)+1}$



Checking $\dfrac{1}{\ln(x)+1} \leq 1$

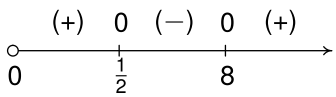
2. Moving all of the nonzero terms of $\left(\log_2(x)\right)^2 < 2\log_2(x) + 3$ to one side of the inequality in order to make use of a sign diagram, we have $\left(\log_2(x)\right)^2 - 2\log_2(x) - 3 < 0$.

   Defining $r(x) = \left(\log_2(x)\right)^2 - 2\log_2(x) - 3$, we get the domain of r is $(0, \infty)$, due to the presence of the logarithm. To find the zeros of r, we set $r(x) = \left(\log_2(x)\right)^2 - 2\log_2(x) - 3 = 0$ which we identify as a 'quadratic in disguise.'

   Setting $u = \log_2(x)$, our equation becomes $u^2 - 2u - 3 = 0$. Factoring gives us $u = -1$ and $u = 3$. Since $u = \log_2(x)$, we get $\log_2(x) = -1$, or $x = 2^{-1} = \frac{1}{2}$, and $\log_2(x) = 3$, which gives $x = 2^3 = 8$.

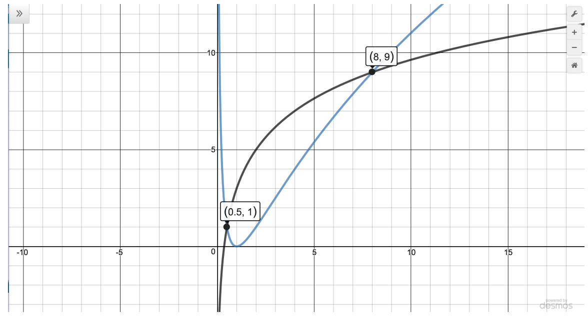
   We use test values which are powers of 2: $0 < \frac{1}{4} < \frac{1}{2} < 1 < 8 < 16$ to create the sign diagram below. From our sign diagram, we see $r(x) < 0$, which corresponds to our solution, on $\left(\frac{1}{2}, 8\right)$.

   Geometrically, the graph of $f(x) = \left(\frac{\ln(x)}{\ln(2)}\right)^2$ is below the graph of $y = g(x) = \frac{2\ln(x)}{\ln(2)} + 3$ on $\left(\frac{1}{2}, 8\right)$.



A sign diagram for

$r(x) = \left(\log_2(x)\right)^2 - 2\log_2(x) - 3$



Checking $\left(\log_2(x)\right)^2 < 2\log_2(x) + 3$

3. We begin to solve $t\log(t+1) \geq t$ by subtracting t from both sides to get $t\log(t+1) - t \geq 0$.

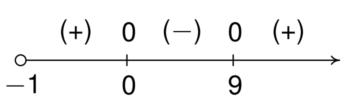
   We define $r(t) = t\log(t+1) - t$ and due to the presence of the logarithm, we require $t > -1$.

   To find the zeros of r, we set $r(t) = t\log(t+1) - t = 0$. Factoring, we get $t\left(\log(t+1) - 1\right) = 0$, which gives $t = 0$ or $\log(t+1) - 1 = 0$.

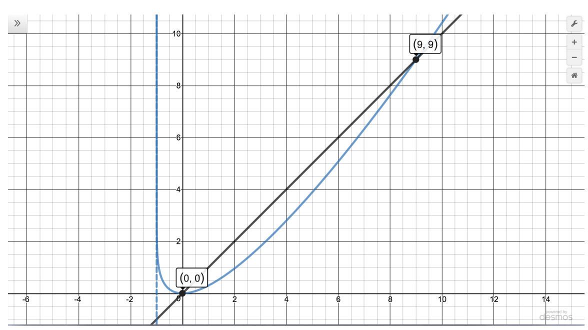
   From $\log(t+1) - 1 = 0$ we get $\log(t+1) = 1$, which we rewrite as $t + 1 = 10^1$. Hence, $t = 9$.

   We select test values t so that $t + 1$ is a power of 10. Using $-1 < -0.9 < 0 < \sqrt{10} - 1 < 9 < 99$, our sign diagram gives the solution as $(-1, 0] \cup [9, \infty)$.

   We find the graphs of $y = f(t) = t\log(t+1)$ and $y = g(t) = t$ intersect at $t = 0$ and $t = 9$ with the graph of f above the graph of g on the given solution intervals.



A sign diagram for
$r(t) = t\log(t+1) - t$



Checking $t\log(t+1) \geq t$

□

Our next example revisits the concept of pH first seen in Exercise 84 in Section 7.2.

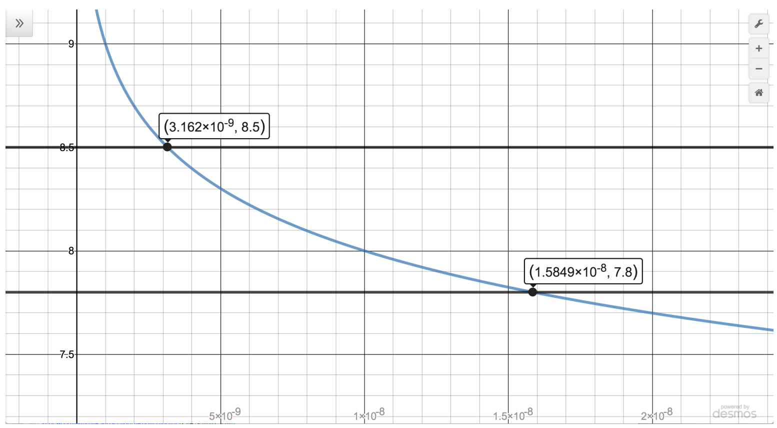
Example 7.5.3. In order to successfully breed Ippizuti fish the pH of a freshwater tank must be at least 7.8 but can be no more than 8.5. Determine the corresponding range of hydrogen ion concentration, and check your answer using a calculator.

Solution. Recall from Exercise 84 in Section 7.2 that $\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles per liter.

We require $7.8 \leq -\log[\text{H}^+] \leq 8.5$ or $-8.5 \leq \log[\text{H}^+] \leq -7.8$. One way to proceed is to break this compound inequality into two inequalities, solve each using a sign diagram, and take the intersection of the solution sets.[3]

On the other hand, we take advantage of the fact that $F(x) = 10^x$ is an increasing function, meaning that if $a \leq b \leq c$, then $10^a \leq 10^b \leq 10^c$. This property allows us to solve our inequality in one step: from $-8.5 \leq \log[\text{H}^+] \leq -7.8$, we get $10^{-8.5} \leq 10^{\log[\text{H}^+]} \leq 10^{-7.8}$, so our solution is $\leq [\text{H}^+] \leq 10^{-7.8}$. (Your Chemistry professor may want the answer written as $3.16 \times 10^{-9} \leq [\text{H}^+] \leq 1.58 \times 10^{-8}$.) Using interval notation, our answer is $\left[10^{-8.5}, 10^{-7.8}\right]$.

After *very* carefully adjusting the viewing window on the graphing utility, we see the graph of $f(x) = -\log(x)$ lies between the lines $y = 7.8$ and $y = 8.5$ on the interval $[3.162 \times 10^{-9}, 1.5849 \times 10^{-8}]$.



□

We close this section by finding an inverse of a one-to-one function which involves logarithms.

Example 7.5.4. The function $f(x) = \dfrac{\log(x)}{1 - \log(x)}$ is one-to-one.

1. Find a formula for $f^{-1}(x)$ and check your answer graphically using a graphing utility.

2. Solve $\dfrac{\log(x)}{1 - \log(x)} = 1$

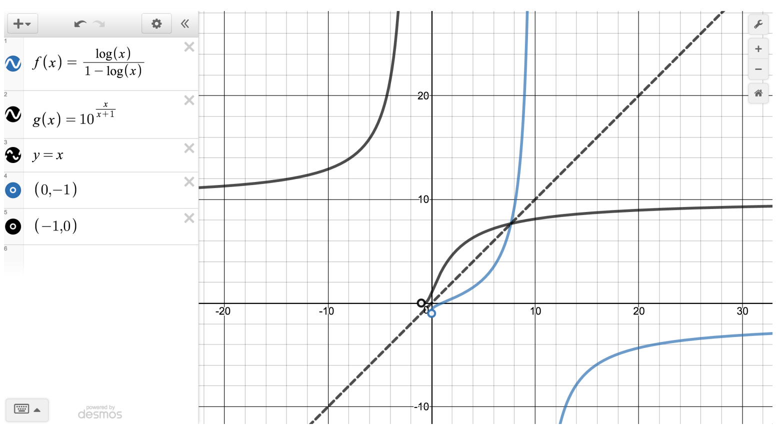
[3]Refer to page 1318 for a discussion of what this means.

Solution.

1. We first write $y = f(x)$ then interchange the x and y and solve for y.

$$\begin{array}{rcll}
y & = & f(x) & \\
y & = & \dfrac{\log(x)}{1-\log(x)} & \\
x & = & \dfrac{\log(y)}{1-\log(y)} & \text{Interchange } x \text{ and } y. \\
x\,(1-\log(y)) & = & \log(y) & \\
x - x\log(y) & = & \log(y) & \\
x & = & x\log(y)+\log(y) & \\
x & = & (x+1)\log(y) & \\
\dfrac{x}{x+1} & = & \log(y) & \\
y & = & 10^{\frac{x}{x+1}} & \text{Rewrite as an exponential equation.}
\end{array}$$

We have $f^{-1}(x) = 10^{\frac{x}{x+1}}$. Graphing f and f^{-1} on the same viewing window produces the required symmetry about $y = x$.



2. Recognizing $\frac{\log(x)}{1-\log(x)} = 1$ as $f(x) = 1$, we have $x = f^{-1}(1) = 10^{\frac{1}{1+1}} = 10^{\frac{1}{2}} = \sqrt{10}$.

   To check our answer algebraically, first recall $\log(\sqrt{10}) = \log_{10}(\sqrt{10})$. Next, we know $\sqrt{10} = 10^{\frac{1}{2}}$. Hence, $\log_{10}\left(10^{\frac{1}{2}}\right) = \frac{1}{2} = 0.5$. It follows that $\frac{\log(\sqrt{10})}{1-\log(\sqrt{10})} = \frac{0.5}{1-0.5} = \frac{0.5}{0.5} = 1$, as required. $\square$

Our last example uses the tools from this section along with Section 6.3.

Example 7.5.5. Let $f(x) = x^2 \ln(x)$.

1. Find the domain of f.
2. Find the x-intercepts, if any.
3. Explain why $\displaystyle\lim_{x\to 0^+} x^2 \ln(x)$ results in an indeterminate form.

   Use a table of values to approximate $\displaystyle\lim_{x\to 0^+} x^2 \ln(x)$. What does your answer mean graophically?
4. Find $\displaystyle\lim_{x\to\infty} x^2 \ln(x)$
5. Given $f'(x) = 2x\ln(x) + x$, find the open intervals over which f is increasing and decreasing.
6. Find the local extrema.
7. Given $f''(x) = 2\ln(x) + 3$, find the open intervals over which the graph of f is concave up or concave down.
8. Find the inflection points.
9. Check your answers using a graphing utility.

Solution.

1. Owing to the presence of the '$\ln(x)$' factor, we have the domain of f is $(0, \infty)$.
2. To find the x-intercepts, we set $f(x) = x^2 \ln(x) = 0$. This gives $x^2 = 0$ or $\ln(x) = 0$, so $x = 0$ or $x = 1$. Since $x = 0$ is not in the domain of f, our only solution is $x = 1$. We have just one x-intercept, $(1, 0)$.
3. To analyze $\displaystyle\lim_{x\to 0^+} x^2 \ln(x)$, we note that as $x \to 0^+$, $x^2 \to 0$ but $\ln(x) \to -\infty$. Hence we obtain the indeterminate form[4] '$0 \cdot (-\infty)$.' Making a table of values of $f(x)$ as $x \to 0^+$ suggests $\displaystyle\lim_{x\to 0^+} x^2 \ln(x) = 0$. Since $x = 0$ is not in the domain of f, there is a hole in the graph at $(0, 0)$.

$f(x) = x^2 \ln(x)$

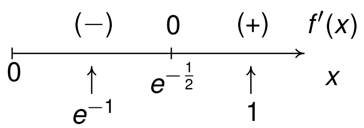
x_1	$f(x_1)$
0.1	-0.023025851
0.01	-4.60517×10^{-4}
0.001	-6.907755×10^{-6}
0.0001	-9.21034×10^{-8}
0.00001	-1.151293×10^{-9}

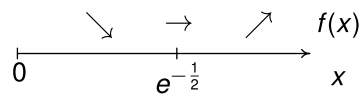
[4] See the remarks following Example 7.4.5.

4. To determine the intervals over which f is increasing and decreasing, we make a sign diagram for $f'(x) = 2x\ln(x) + x$. Solving $f'(x) = 2x\ln(x) + x = 0$ we factor: $x(2\ln(x)+1) = 0$. Since the domain of f is $(0,\infty)$, we focus on the factor $2\ln(x)+1 = 0$. We get $\ln(x) = -\frac{1}{2}$ so $x = e^{-\frac{1}{2}}$.

   When choosing test values, we could opt to find a decimal approximation for $e^{-\frac{1}{2}}$ or we could use more 'log-friendly' values. In this case we note that $e^{-1} < e^{-\frac{1}{2}} < e^{0} = 1$ so we choose e^{-1} and 1 as our test values.

   We find $f'\left(e^{-1}\right) = 2\ln\left(e^{-1}\right) + 1 = 2(-1) + 1 = -1 < 0$ and $f'(1) = 2\ln(1) + 1 = 1 > 0$. We fill out our sign diagram and interpret accordingly.



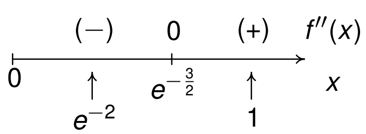


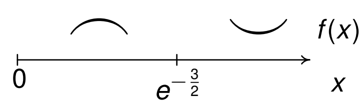
   We find f is decreasing on $\left(0, e^{-\frac{1}{2}}\right)$ and increasing on $\left(e^{-\frac{1}{2}}, \infty\right)$.

5. We see f has a local (and in this case, global) minimum when $x = e^{-\frac{1}{2}}$. The minimum value is $f\left(e^{-\frac{1}{2}}\right) = \left(e^{-\frac{1}{2}}\right)^2 \ln\left(e^{-\frac{1}{2}}\right) = e^{-1}\left(-\frac{1}{2}\right) = -\frac{e^{-1}}{2}$. The local minimum is $\left(e^{-\frac{1}{2}}, -\frac{e^{-1}}{2}\right)$.

6. To find the intervals over which the graph of f is concave up and concave down, we make a sign diagram for $f''(x) = 2\ln(x) + 3$. Solving $f''(x) = 2\ln(x) + 3 = 0$ we get $\ln(x) = -\frac{3}{2}$ so $x = e^{-\frac{3}{2}}$. As with our first derivative analysis, we elect to choose some 'log-friendly' test values and note $e^{-2} < e^{-\frac{3}{2}} < e^{0} = 1$.

   We find $f''\left(e^{-2}\right) = 2\ln\left(e^{-2}\right) + 3 = 2(-2) + 3 = -1 < 0$ and $f''(1) = 2\ln(1) + 3 = 3 > 0$.



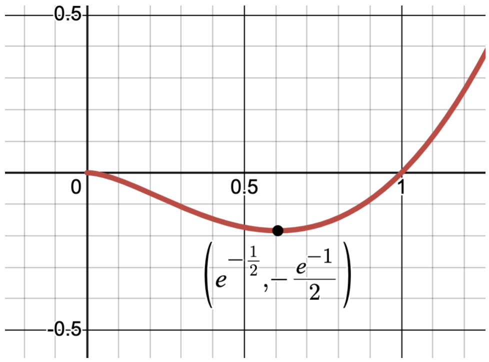


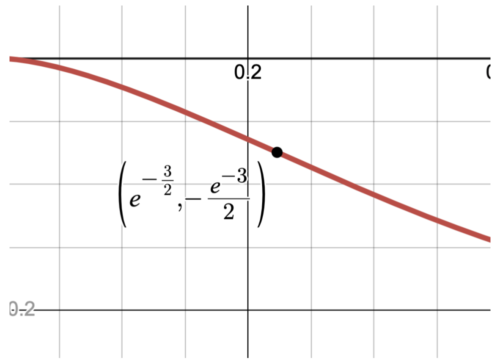
   We see the graph of f is concave down on $\left(0, e^{-\frac{3}{2}}\right)$ and concave up on $\left(e^{-\frac{3}{2}}, \infty\right)$.

7. Since the concavity changes at $x = e^{-\frac{3}{2}}$, we have an inflection point at $\left(e^{-\frac{3}{2}}, f\left(e^{-\frac{3}{2}}\right)\right)$. We find $f\left(e^{-\frac{3}{2}}\right) = \left(f\left(e^{-\frac{3}{2}}\right)\right)^2 \ln\left(f\left(e^{-\frac{3}{2}}\right)\right) = e^{-3}\left(-\frac{3}{2}\right) = -\frac{3e^{-3}}{2}$. Our inflection point is $\left(e^{-\frac{3}{2}}, -\frac{3e^{-3}}{2}\right)$.

8. Using desmos, we confirm our calculations.[5]

[5] It may require some window adjustment to capture the inflection point.





□

When determining $\displaystyle\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 \ln(x)$ in Example 7.5.5, we encountered the indeterminate form '$0 \cdot (-\infty)$.' This is a similar scenario to what we encountered in the remarks following Example 7.4.5 in Section 7.4. I this case, the factor $x^2 \to 0$ and the factor $\ln(x) \to -\infty$ and so we have a tug-of-war to see which factor's behavior will win out over the other. Here, $x^2 \to 0$ dominates as the table suggests $\displaystyle\lim_{x \to 0^+} x^2 \ln(x) = 0$. This is indeed the case and, in general, polynomials (and, in general, all positive powers of x) dominate logarithms as we'll explore in the Exercise 48b.

7.5.1 Exercises

In Exercises 1 - 24, solve the equation analytically.

1. $\log(3x-1) = \log(4-x)$
2. $\log_2\left(x^3\right) = \log_2(x)$
3. $\ln\left(8-t^2\right) = \ln(2-t)$
4. $\log_5\left(18-t^2\right) = \log_5(6-t)$
5. $\log_3(7-2x) = 2$
6. $\log_{\frac{1}{2}}(2x-1) = -3$
7. $\ln\left(t^2-99\right) = 0$
8. $\log(t^2-3t) = 1$
9. $\log_{125}\left(\dfrac{3x-2}{2x+3}\right) = \dfrac{1}{3}$
10. $\log\left(\dfrac{x}{10^{-3}}\right) = 4.7$
11. $-\log(x) = 5.4$
12. $10\log\left(\dfrac{x}{10^{-12}}\right) = 150$
13. $6-3\log_5(2t) = 0$
14. $3\ln(t)-2 = 1-\ln(t)$
15. $\log_3(t-4)+\log_3(t+4) = 2$
16. $\log_5(2t+1)+\log_5(t+2) = 1$
17. $\log_{169}(3x+7)-\log_{169}(5x-9) = \dfrac{1}{2}$
18. $\ln(x+1)-\ln(x) = 3$
19. $2\log_7(t) = \log_7(2)+\log_7(t+12)$
20. $\log(t)-\log(2) = \log(t+8)-\log(t+2)$
21. $\log_3(x) = \log_{\frac{1}{3}}(x)+8$
22. $\ln(\ln(x)) = 3$
23. $(\log(t))^2 = 2\log(t)+15$
24. $\ln(t^2) = (\ln(t))^2$

In Exercises 25 - 30, solve the inequality analytically.

25. $\dfrac{1-\ln(t)}{t^2} < 0$
26. $t\ln(t)-t > 0$
27. $10\log\left(\dfrac{x}{10^{-12}}\right) \geq 90$
28. $5.6 \leq \log\left(\dfrac{x}{10^{-3}}\right) \leq 7.1$
29. $2.3 < -\log(x) < 5.4$
30. $\ln(t^2) \leq (\ln(t))^2$

In Exercises 31 - 34, use a graphing utility to help you solve the equation or inequality.

31. $\ln(t) = e^{-t}$

32. $\ln(x) = \sqrt[4]{x}$

33. $\ln(t^2 + 1) \geq 5$

34. $\ln(-2x^3 - x^2 + 13x - 6) < 0$

In Exercises 35 - 40, find the domain of the function.

35. $r(x) = \dfrac{x}{1 - \ln(x)}$

36. $R(x) = \dfrac{x\ln(x)}{1 - \ln(x)}$

37. $s(t) = \sqrt{2 - \log(t)}$

38. $c(t) = (2\ln(t) - 1)^{\frac{2}{3}}$

39. $\ell(t) = \ln(\ln(t))$

40. $L(x) = \log\left(\dfrac{x\ln(x)}{1 - \ln(x)}\right)$

41. Since $f(x) = e^x$ is a strictly increasing function, if $a < b$ then $e^a < e^b$. Use this fact to solve the inequality $\ln(2x + 1) < 3$ without a sign diagram. Use this technique to solve the inequalities in Exercises 27 - 29. (Compare this to Exercise 54 in Section 7.4.)

42. Solve $\ln(3 - y) - \ln(y) = 2x + \ln(5)$ for y.

43. In Example 7.5.4 we found the inverse of $f(x) = \dfrac{\log(x)}{1 - \log(x)}$ to be $f^{-1}(x) = 10^{\frac{x}{x+1}}$.

    (a) Algebraically check our answer by verifying $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.

    (b) Find the range of f by finding the domain of f^{-1}.

    (c) Let $g(x) = \dfrac{x}{1 - x}$ and $h(x) = \log(x)$. Show that $f = g \circ h$ and $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$.

    NOTE: We know this is true in general by Exercise 40 in Section 5.6, but it's nice to see a specific example of the property.

44. Let $f(x) = \dfrac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$. Compute $f^{-1}(x)$ and find its domain and range.

45. Explain the equation in Exercise 10 and the inequality in Exercise 28 above in terms of the Richter scale for earthquake magnitude. (See Exercise 82 in Section 7.1.)

46. Explain the equation in Exercise 12 and the inequality in Exercise 27 above in terms of sound intensity level as measured in decibels. (See Exercise 83 in Section 7.1.)

47. Explain the equation in Exercise 11 and the inequality in Exercise 29 above in terms of the pH of a solution. (See Exercise 84 in Section 7.1.)

48. (a) With the help of your classmates, numerically and graphically investigate $\lim_{x \to \infty} \frac{\ln(x)}{x^p}$ for various real number powers, $p > 0$.

    (b) With the help of your classmates, numerically and graphically investigate $\lim_{x \to 0^+} x^p \ln(x)$ for various real number powers, $p > 0$.

    (c) What do 48a and 48b suggest about the relative growth rates of powers of x and $\ln(x)$?

In Exercises 49 - 50 a function f along with its derivatives f' and f'' are given.

- Find the domain of f.
- Find the x- and y-intercepts of the graph of each function, if any.
- Use limits to determine the end behavior and behavior at the endpoints of the domain.
- Use f' to determine the open intervals over which f is increasing or decreasing.
- Determine the local extrema, if any.
- Use f'' to determine the open intervals over which the graph of f is concave up or concave down.
- Determine the inflection points of the graph, if any.

49. $f(x) = \ln(x) - \ln(5-x)$, $f'(x) = \dfrac{1}{x} + \dfrac{1}{5-x}$, $f''(x) = \dfrac{1}{(5-x)^2} - \dfrac{1}{x^2}$

50. $f(x) = \dfrac{\ln(x)}{x}$, $f'(x) = \dfrac{1-\ln(x)}{x^2}$, $f''(x) = \dfrac{2\ln(x)-3}{x^3}$.

7.5.2 Answers

1. $x = \frac{5}{4}$
2. $x = 1$
3. $t = -2$
4. $t = -3,\ 4$
5. $x = -1$
6. $x = \frac{9}{2}$
7. $t = \pm 10$
8. $t = -2,\ 5$
9. $x = -\frac{17}{7}$
10. $x = 10^{1.7}$
11. $x = 10^{-5.4}$
12. $x = 10^{3}$
13. $t = \frac{25}{2}$
14. $t = e^{3/4}$
15. $t = 5$
16. $t = \frac{1}{2}$
17. $x = 2$
18. $x = \frac{1}{e^3 - 1}$
19. $t = 6$
20. $t = 4$
21. $x = 81$
22. $x = e^{e^3}$
23. $t = 10^{-3},\ 10^{5}$
24. $t = 1,\ x = e^2$
25. (e, ∞)
26. (e, ∞)
27. $\left[10^{-3}, \infty\right)$
28. $\left[10^{2.6}, 10^{4.1}\right]$
29. $\left(10^{-5.4}, 10^{-2.3}\right)$
30. $(0, 1] \cup [e^2, \infty)$
31. $t \approx 1.3098$
32. $x \approx 4.177,\ x \approx 5503.665$
33. $\approx (-\infty, -12.1414) \cup (12.1414, \infty)$
34. $\approx (-3.0281, -3) \cup (0.5, 0.5991) \cup (1.9299, 2)$
35. $(-\infty, e) \cup (e, \infty)$
36. $(0, e) \cup (e, \infty)$
37. $(0, 100]$
38. $(0, \infty)$
39. $(1, \infty)$
40. $(1, e)$
41. $-\dfrac{1}{2} < x < \dfrac{e^3 - 1}{2}$, so $\left(-\dfrac{1}{2}, \dfrac{e^3 - 1}{2}\right)$
42. $y = \dfrac{3}{5e^{2x} + 1}$
44. $f^{-1}(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$.

    To see why we rewrite this in this form, see Exercise 51 in Section 14.5.

    The domain of f^{-1} is $(-\infty, \infty)$ and its range is the same as the domain of f, namely $(-1, 1)$.

49.
- Domain: $(0, 5)$.
- x-intercept: $\left(\frac{5}{2}, 0\right)$; there is no y-intercept.
- $\lim_{x\to 0^+} f(x) = -\infty$, $\lim_{x\to 5^-} f(x) = \infty$; we have two vertical asymptotes: $x = 0$ and $x = 5$.
- f is always increasing: $(0, 5)$.
- There are no local extrema.
- The graph of f is concave up on $\left(0, \frac{5}{2}\right)$ and concave down on $\left(\frac{5}{2}, 5\right)$.
- The inflection point is $\left(\frac{5}{2}, 0\right)$.

50.
- Domain: $(0, \infty)$.
- x-intercept: $(1, 0)$; there is no y-intercept.
- $\lim_{x\to 0^+} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 0$; we have a vertical and horizontal asymptote: $x = 0$ and $y = 0$.
- f is increasing on $(0, e)$ and decreasing on (e, ∞).
- There is a local (absolute) max at $\left(e, \frac{1}{e}\right)$.
- The graph of f is concave up on $\left(e^{\frac{3}{2}}, \infty\right)$ and concave down on $\left(0, e^{\frac{3}{2}}\right)$.
- The inflection point is $\left(e^{\frac{3}{2}}, \frac{3}{2e^{\frac{3}{2}}}\right)$.

7.6 Applications of Exponential and Logarithmic Functions

As we mentioned in Sections 7.1 and 7.2, exponential and logarithmic functions are used to model a wide variety of behaviors in the real world. In the examples that follow, note that while the applications are drawn from many different disciplines, the mathematics remains essentially the same. Due to the applied nature of the problems we will examine in this section, we will often express our final answers as decimal approximations (after finding exact answers first, of course!)

7.6.1 Applications of Exponential Functions

Perhaps the most well-known application of exponential functions comes from the financial world. Suppose you have \$100 to invest at your local bank and they are offering a whopping 5 % annual percentage interest rate. This means that after one year, the bank will pay *you* 5% of that \$100, or $\$100(0.05) = \$5$ in interest, so you now have \$105. This is in accordance with the formula for *simple interest* which you have undoubtedly run across at some point before.

Equation 7.1. Simple Interest:
The amount of interest I accrued at an annual rate r on an investment[a] P after t years is

$$I = Prt$$

The amount in the account after t years, $A(t)$ is given by

$$A(t) = P + I = P + Prt = P(1 + rt)$$

[a]Called the **principal**

Suppose, however, that six months into the year, you hear of a better deal at a rival bank.[1] Naturally, you withdraw your money and try to invest it at the higher rate there. Since six months is one half of a year, that initial \$100 yields $\$100(0.05)\left(\frac{1}{2}\right) = \$2.50$ in interest.

You take your \$102.50 off to the competitor and find out that those restrictions which *may* apply actually do apply, so you return to your bank and re-deposit the \$102.50 for the remaining six months of the year.

To your surprise and delight, at the end of the year your statement reads \$105.06, not \$105 as you had expected.[2] Where did those extra six cents come from?

For the first six months of the year, interest was earned on the original principal of \$100, but for the second six months, interest was earned on \$102.50, that is, you earned interest on your interest. This is the basic concept behind **compound interest**.

In the previous discussion, we would say that the interest was compounded twice per year, or semiannually.[3] If more money can be earned by earning interest on interest already earned, one wonders what happens if the interest is compounded more often, say every three months - 4 times a year, or 'quarterly.'

[1]Some restrictions may apply.
[2]Actually, the final balance should be \$105.0625.
[3]Using this convention, simple interest after one year is the same as compounding the interest only once.

In this case, the money is in the account for three months, or $\frac{1}{4}$ of a year, at a time. After the first quarter, we have $A = P(1 + rt) = \$100\left(1 + 0.05 \cdot \frac{1}{4}\right) = \$101.25$. We now invest the \$101.25 for the next three months and find that at the end of the second quarter, we have $A = \$101.25\left(1 + 0.05 \cdot \frac{1}{4}\right) \approx \$102.51$. Continuing in this manner, the balance at the end of the third quarter is \$103.79, and, at last, we obtain \$105.08. The extra two cents hardly seems worth it, but we see that we do in fact get more money the more often we compound.

In order to develop a formula for this phenomenon, we need to do some abstract calculations. Suppose we wish to invest our principal P at an annual rate r and compound the interest n times per year. This means the money sits in the account $\frac{1}{n}^{\text{th}}$ of a year between compoundings. Let A_k denote the amount in the account after the k^{th} compounding.

Then $A_1 = P\left(1 + r\left(\frac{1}{n}\right)\right)$ which simplifies to $A_1 = P\left(1 + \frac{r}{n}\right)$. After the second compounding, we use A_1 as our new principal and get $A_2 = A_1\left(1 + \frac{r}{n}\right) = \left[P\left(1 + \frac{r}{n}\right)\right]\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2$. Continuing in this fashion, we get $A_3 = P\left(1 + \frac{r}{n}\right)^3$, $A_4 = P\left(1 + \frac{r}{n}\right)^4$, and so on, so that $A_k = P\left(1 + \frac{r}{n}\right)^k$.

Since we compound the interest n times per year, after t years, we have nt compoundings. We have just derived the general formula for compound interest below.

Equation 7.2. Compounded Interest:
If an initial principal P is invested at an annual rate r and the interest is compounded n times per year, the amount in the account after t years, $A(t)$ is given by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

If we take $P = 100$, $r = 0.05$, and $n = 4$, Equation 7.2 becomes $A(t) = 100\left(1 + \frac{0.05}{4}\right)^{4t}$ which reduces to $A(t) = 100(1.0125)^{4t}$. To check this new formula against our previous calculations, we find $A\left(\frac{1}{4}\right) = 100(1.0125)^{4\left(\frac{1}{4}\right)} = 101.25$, $A\left(\frac{1}{2}\right) \approx \$102.51$, $A\left(\frac{3}{4}\right) \approx \$103.79$, and $A(1) \approx \$105.08$.

Example 7.6.1. Suppose \$2000 is invested in an account which offers 7.125% compounded monthly.

1. Express the amount $A(t)$ in the account as a function of the term of the investment t in years.
2. How much is in the account after 5 years?
3. How long will it take for the initial investment to double?
4. Find and interpret the average rate of change[4] of the amount in the account:
   - from the end of the fourth year to the end of the fifth year
   - from the end of the thirty-fourth year to the end of the thirty-fifth year.

[4]See Definition 1.8 in Section 1.2.

5. Find and interpret the relative rate of change[5] of the amount in the account:

   - from the end of the fourth year to the end of the fifth year
   - from the end of the thirty-fourth year to the end of the thirty-fifth year.

Solution.

1. Substituting $P = 2000$, $r = 0.07125$, and $n = 12$ (since interest is compounded *monthly*) into Equation 7.2 yields $A(t) = 2000\left(1 + \frac{0.07125}{12}\right)^{12t} = 2000(1.0059375)^{12t}$.

2. To find the amount in the account after 5 years, we compute $A(5) = 2000(1.0059375)^{12(5)} \approx 2852.92$. After 5 years, we have approximately \$2852.92.

3. Our initial investment is \$2000, so to find the time it takes this to double, we need to find t when $A(t) = 4000$. That is, we need to solve $2000(1.0059375)^{12t} = 4000$, or $(1.0059375)^{12t} = 2$.

   Taking natural logs as in Section 7.4, we get $t = \frac{\ln(2)}{12\ln(1.0059375)} \approx 9.75$. Hence, it takes approximately 9 years 9 months for the investment to double.

4. Recall to find the average rate of change of A over an interval $[a, b]$, we compute $\frac{A(b)-A(a)}{b-a}$.

   - The average rate of change of A from the end of the fourth year to the end of the fifth year is $\frac{A(5)-A(4)}{5-4} \approx 195.63$.
     This means that the value of the investment is increasing at a rate of approximately \$195.63 per year between the end of the fourth and fifth years.
   - Likewise, the average rate of change of A from the end of the thirty-fourth year to the end of the thirty-fifth year is $\frac{A(35)-A(34)}{35-34} \approx 1648.21$, so the value of the investment is increasing at a rate of approximately \$1648.21 per year during this time.

   So, not only is it true that the longer you wait, the more money you have, but also the longer you wait, the faster the money increases.[6]

5. Recall to find the relative rate of change of A over an interval $[a, b]$, we compute $\frac{A(b)-A(a)}{A(a)}$.

   - The relative rate of change of A from the end of the fourth year to the end of the fifth year is $\frac{A(5)-A(4)}{A(4)} \approx 0.07362$.
     This means that the amount in the account is increasing at a rate of approximately 7.362% per year between the end of the fourth and fifth years.
   - Similarly, we find the relative rate of change of A from the end of the thirty-fourth year to the end of the thirty-fifth year to be $\frac{A(35)-A(34)}{A(34)} \approx 0.07362$ as well.
     This means that the percentage growth from the thirty-fourth to thirty-fifth year is 7.362%, the same as the percentage growth from the fourth to the fifth year.

[5] See Definition 7.3 in Section 7.1.

[6] In fact, the rate of increase of the amount in the account is exponential as well. This is the quality that really defines exponential functions and we refer the reader to a course in Calculus.

We know from the remarks following Definition 7.3 that for exponential functions, the relative rate of change over an interval of length 1 is constant and, moreover, is equal to $b - 1$ where b is the base of the exponential function, $f(x) = a \cdot b^x$.

In our scenario, $A(t) = 2000(1.0059375)^{12t} = 2000\left[(1.0059375)^{12}\right]^t = 2000 \cdot (1.07362\ldots)^t$. Hence, the base is $b = 1.07362\ldots$ and the relative rate of change is $b - 1 = 0.07362\ldots$.

Note that the interest rate quoted to us at the beginning of this problem is 7.125% per year. The rate 7.362% is called the '*effective*' interest rate which factors in the effect of the compounding on the growth of the investment. □

We have observed that the more times you compound the interest per year, the more money you will earn in a year. Let's push this notion to the limit.[7]

Consider an investment of \$1 invested at 100% interest for 1 year compounded n times a year. Equation 7.2 tells us that the amount of money in the account after 1 year is $A = \left(1 + \frac{1}{n}\right)^n$. Below is a table of values relating n and A.

n	A
1	2
2	2.25
4	≈ 2.4414
12	≈ 2.6130
360	≈ 2.7145
1000	≈ 2.7169
10000	≈ 2.7181
100000	≈ 2.7182

As promised, the more compoundings per year, the more money there is in the account, but we also observe that the increase in money is greatly diminishing.

We are witnessing a mathematical 'tug of war'. While we are compounding more times per year, and hence getting interest on our interest more often, the amount of time between compoundings is getting smaller and smaller, so there is less time to build up additional interest.

With Calculus, we can show[8] that $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$, where e is the natural base first presented in Section 7.1. Taking the number of compoundings per year to infinity results in what is called **continuously** compounded interest.

Theorem 7.10. Investing \$1 at 100% interest compounded continuously for one year returns \$$e$.

Using the limit definition of e along with some limit properties, we can derive a general formula for continuously compounded interest.

[7]See Section 6.1 to understand this pun.

[8]Or define, depending on your point of view.

Consider the limit $\displaystyle\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}$. In order to use the limit definition of 'e,' we need to make a substitution to make '$\left(1+\frac{r}{n}\right)$' look like '$\left(1+\frac{1}{\text{something}}\right)$.'

If we let $u = \frac{n}{r}$, we get $n = r\,u$, so $1+\frac{r}{n} = 1+\frac{r}{ru} = 1+\frac{1}{u}$ and $nt = (ru)t = urt$. In the context of compound interest, $r > 0$, so $n \to \infty$ implies $u \to \infty$ and vice-versa so the limit becomes:

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = \lim_{u\to\infty} P\left(1+\frac{1}{u}\right)^{urt}.$$

Using Properties of Limits, Theorem 6.2, we get:

$$\begin{aligned}
\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} &= \lim_{u\to\infty} P\left(1+\frac{1}{u}\right)^{urt} & \\
&= P\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{urt} & \text{Scalar Multiple Rule} \\
&= P\lim_{u\to\infty}\left[\left(1+\frac{1}{u}\right)^{u}\right]^{rt} & \text{Properties of Exponents} \\
&= P\left[\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{u}\right]^{rt} & \text{Real Number Powers} \\
&= Pe^{rt} & \text{Limit Definition of '}e\text{'.}
\end{aligned}$$

A couple of remarks are in order. First, in the limit $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$, the variable here, n, takes on natural number values, and as such, is a **discrete** variable, not a **continuous** one.[9] We'll revisit limits of discrete variables in Section 10.1.1.

Second, when we use the limit definition of 'e' in the above argument, we used $\displaystyle\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{u} = e$. Like n, the variable u is a discrete variable (not necessarily natural numbers, but discrete nonetheless). Since both n and u are tending to infinity, this change in dummy variable doesn't affect the limit.[10]

Last, we note that the Real Numbers Power rule applies since $e \approx 2.718 > 0$. We codify this result in the following theorem.

> **Equation 7.3. Continuously Compounded Interest:**
> If an initial principal P is invested at an annual rate r and the interest is compounded continuously, the amount in the account after t years, $A(t)$ is given by
>
> $$A(t) = Pe^{rt}$$

It is worth noting that if we take the scenario of Example 7.6.1 and compare monthly compounding to continuous compounding over 35 years, we find that monthly compounding yields $A(35) = 2000(1.0059375)^{12(35)}$

[9] See Section 1.1.3 for a discussion of the difference, if needed!

[10] Both n and u are taking discrete steps to infinity - one is doing so at just a slower pace.

which is about \$24,035.28, whereas continuously compounding gives $A(35) = 2000e^{0.07125(35)}$ which is about \$24,213.18 - a difference of less than 1%.

Equations 7.2 and **??** both use exponential functions to describe the growth of an investment. It turns out, the same principles which govern compound interest are also used to model short term growth of populations. As with many concepts in this text, these notions are best formalized using the language of Calculus. Nevertheless, we do our best here.

In Biology, **The Law of Uninhibited Growth** states as its premise that the *instantaneous* rate at which a population increases at any time is directly proportional to the population at that time. In other words, the more organisms there are at a given moment, the faster they reproduce. Formulating the law as stated results in a differential equation, which requires Calculus to solve. Solving said differential equation gives us the formula below.

Equation 7.4. Uninhibited Growth:
If a population increases according to The Law of Uninhibited Growth, the number of organisms at time t, $N(t)$ is given by the formula

$$N(t) = N_0 e^{kt},$$

where $N(0) = N_0$ (read 'N nought') is the initial number of organisms and $k > 0$ is the constant of proportionality which satisfies the equation:

$$\frac{dN}{dt} = k\,N(t),$$

where $\dfrac{dN}{dt}$ is the Leibniz notation[a] notation for the derivative of N with respect to t.

[a] See the remark following Definition 6.8 on Section 6.2.

It is worth taking some time to compare Equations **??** and 7.4. In Equation **??**, we use P to denote the initial investment; in Equation 7.4, we use N_0 to denote the initial population. In Equation **??**, r denotes the annual interest rate, and so it shouldn't be too surprising that the k in Equation 7.4 corresponds to a growth rate as well. While Equations **??** and 7.4 look entirely different, they both represent the same mathematical concept.

Example 7.6.2. In order to perform arthrosclerosis research, epithelial cells are harvested from discarded umbilical tissue and grown in the laboratory. A technician observes that a culture of twelve thousand cells grows to five million cells in one week. Assuming that the cells follow The Law of Uninhibited Growth, find a formula for the number of cells, in thousands, after t days, $N(t)$.

Solution. We begin with $N(t) = N_0 e^{kt}$. Since $N(t)$ is to give the number of cells *in thousands*, we have $N_0 = 12$, so $N(t) = 12e^{kt}$.
Next, we need to determine the growth rate k. We know that after one week, the number of cells has grown to five million. Since t measures days and the units of $N(t)$ are in thousands, this translates mathematically to $N(7) = 5000$ or $12e^{7k} = 5000$. Solving, we get $k = \frac{1}{7}\ln\left(\frac{1250}{3}\right)$, so $N(t) = 12e^{\frac{t}{7}\ln\left(\frac{1250}{3}\right)}$.
Of course, in practice, we would approximate k to some desired accuracy, say $k \approx 0.8618$, which we can interpret as an 86.18% daily growth rate for the cells. □

Whereas Equations **??** and 7.4 model the growth of quantities, we can use equations like them to describe the decline of quantities.

One example we've seen already is Example 7.1.2 in Section 7.1. There, the value of a car decreased from its purchase price of \$25,000 to nothing at all.

Another real world phenomenon which follows suit is radioactive decay. There are elements which are unstable and emit energy spontaneously. In doing so, the amount of the element itself diminishes.

The assumption behind this model is that the rate of decay of an element at a particular time is directly proportional to the amount of the element present at that time. In other words, the more of the element there is, the faster the element decays.

This is precisely the same kind of hypothesis which drives The Law of Uninhibited Growth, and as such, the equation governing radioactive decay is hauntingly similar to Equation 7.4 with the exception that the rate constant k is negative.

Equation 7.5. Radioactive Decay:
The amount of a radioactive element at time t, $A(t)$ is given by the formula

$$A(t) = A_0 e^{kt},$$

where $A(0) = A_0$ is the initial amount of the element and $k < 0$ is the constant of proportionality which satisfies the equation

$$\frac{dA}{dt} = k\,A(t)$$

Example 7.6.3. Iodine-131 is a commonly used radioactive isotope used to help detect how well the thyroid is functioning. Suppose the decay of Iodine-131 follows the model given in Equation 7.5, and that the half-life[11] of Iodine-131 is approximately 8 days. If 5 grams of Iodine-131 is present initially, find a function which gives the amount of Iodine-131, A, in grams, t days later.

Solution. Since we start with 5 grams initially, Equation 7.5 gives $A(t) = 5e^{kt}$.

Since the half-life is 8 days, it takes 8 days for half of the Iodine-131 to decay, leaving half of it behind. Mathematically, this translates to $A(8) = 2.5$, or $5e^{8k} = 2.5$. We get $k = \frac{1}{8}\ln\left(\frac{1}{2}\right) = -\frac{\ln(2)}{8} \approx -0.08664$, which we can interpret as a loss of material at a rate of 8.664% daily.

Hence, our final answer is $A(t) = 5e^{-\frac{t\ln(2)}{8}} \approx 5e^{-0.08664t}$. □

We now turn our attention to some more mathematically sophisticated models. One such model is Newton's Law of Cooling, which we first encountered in Example 7.1.3 of Section 7.1.

In that example we had a cup of coffee cooling from 160°F to room temperature 70°F according to the formula $T(t) = 70+90e^{-0.1t}$, where t was measured in minutes. In that situation, we knew the physical limit of the temperature of the coffee was room temperature,[12] and the differential equation which gives rise to our formula for $T(t)$ takes this into account.

[11] The time it takes for half of the substance to decay.

[12] The Second Law of Thermodynamics states that heat can spontaneously flow from a hotter object to a colder one, but not the other way around. Thus, the coffee could not continue to release heat into the air so as to cool below room temperature.

Whereas the radioactive decay model had a rate of decay at time t directly proportional to the amount of the element which remained at time t, Newton's Law of Cooling states that the rate of cooling of the coffee at a given time t is directly proportional to how much of a temperature *gap* exists between the coffee at time t and room temperature, not the temperature of the coffee itself. In other words, the coffee cools faster when it is first served, and as its temperature nears room temperature, the coffee cools ever more slowly.

Of course, if we take an item from the refrigerator and let it sit out in the kitchen, the object's temperature will rise to room temperature, and since the physics behind warming and cooling is the same, we combine both cases in the equation below.

Equation 7.6. Newton's Law of Cooling (Warming):
The temperature of an object at time t, $T(t)$ is given by the formula

$$T(t) = T_a + (T_0 - T_a)\, e^{-kt},$$

where $T(0) = T_0$ is the initial temperature of the object, T_a is the ambient temperature[a] and $k > 0$ is the constant of proportionality which satisfies the equation

$$\frac{dT}{dt} = k\ (T(t) - T_a)$$

[a]That is, the temperature of the surroundings.

If we re-examine the situation in Example 7.1.3 with $T_0 = 160$, $T_a = 70$, and $k = 0.1$, we get, according to Equation 7.6, $T(t) = 70 + (160 - 70)e^{-0.1t}$ which reduces to the original formula given in that example. The rate constant $k = 0.1$ in this case indicates the coffee is cooling at a rate equal to 10% of the difference between the temperature of the coffee and its surroundings.

Note in Equation 7.6 that the constant k is positive for both the cooling and warming scenarios. What determines if the function $T(t)$ is increasing or decreasing is if T_0 (the initial temperature of the object) is greater than T_a (the ambient temperature) or vice-versa, as we see in our next example.

Example 7.6.4. A roast initially at 40°F cooked in a 350°F oven. After 2 hours, the temperature of the roast is 125°F.

1. Assuming the temperature of the roast follows Newton's Law of Warming, find a formula for the temperature of the roast $T(t)$ as a function of its time in the oven, t, in hours.

2. The roast is done when the internal temperature reaches 165°F. When will the roast be done?

Solution.

1. The initial temperature of the roast is 40°F, so $T_0 = 40$. The environment in which we are placing the roast is the 350°F oven, so $T_a = 350$. Newton's Law of Warming gives $T(t) = 350 + (40 - 350)e^{-kt}$, or after some simplification, $T(t) = 350 - 310e^{-kt}$.

   To determine k, we use the fact that after 2 hours, the roast is 125°F, which means $T(2) = 125$. This gives rise to the equation $350 - 310e^{-2k} = 125$ which yields $k = -\frac{1}{2}\ln\left(\frac{45}{62}\right) \approx 0.1602$. The temperature function is

$$T(t) = 350 - 310e^{\frac{t}{2}\ln\left(\frac{45}{62}\right)} \approx 350 - 310e^{-0.1602t}.$$

2. To find when the roast is done, we set $T(t) = 165$. This gives $350-310e^{-0.1602t} = 165$ whose solution is $t = -\frac{1}{0.1602}\ln\left(\frac{37}{62}\right) \approx 3.22$. Hence, the roast is done after roughly 3 hours and 15 minutes. □

If we had taken the time to graph $y = T(t)$ in Example 7.6.4, we would have found the horizontal asymptote to be $y = 350$, which corresponds to the temperature of the oven. We can also arrive at this conclusion analytically by applying 'number sense'.

As $t \to \infty$, $-0.1602t \approx$ very big $(-)$ so that $e^{-0.1602t} \approx$ very small $(+)$. The larger the value of t, the smaller $e^{-0.1602t}$ becomes so that $T(t) \approx 350 -$ very small $(+)$, which suggests the graph of $y = T(t)$ is approaching its horizontal asymptote $y = 350$ from below. Physically, this means the roast will eventually warm up to 350°F.[13]

The function T in this situation is sometimes called a **limited** growth model, since the function T remains bounded as $t \to \infty$. If we apply the principles behind Newton's Law of Cooling to a biological example, it says the growth rate of a population is directly proportional to how much room the population has to grow. In other words, the more room for expansion, the faster the growth rate.

Our final model, the **logistic** growth model combines The Law of Uninhibited Growth with limited growth and states that the rate of growth of a population varies jointly with the population itself as well as the room the population has to grow.

[13] at which point it would be more toast than roast.

Equation 7.7. Logistic Growth:
If a population behaves according to the assumptions of logistic growth, the number of organisms at time t, $N(t)$ is given by

$$N(t) = \frac{L}{1 + Ce^{-kLt}},$$

where $N(0) = N_0$ is the initial population, L is the limiting population,[a] and C is a measure of how much room there is to grow given by

$$C = \frac{L}{N_0} - 1.$$

and $k > 0$ is the constant of proportionality which satisfies the equation

$$\frac{dN}{dt} = k\,N(t)\,(L - N(t))$$

[a]That is, $\lim_{t \to \infty} N(t) = L$

The logistic function is used not only to model the growth of organisms, but is also often used to model the spread of disease and rumors.[14]

Example 7.6.5. The number of people $N(t)$, in hundreds, at a local community college who have heard the rumor 'Carl's afraid of Sasquatch' can be modeled using the logistic equation

$$N(t) = \frac{84}{1 + 2799e^{-t}},$$

where $t \geq 0$ is the number of days after April 1, 2016.

1. Find and interpret $N(0)$.
2. Find and interpret the end behavior of $N(t)$.
3. How long until 4200 people have heard the rumor?
4. Check your answers to 2 and 3 using a graphing utility.

Solution.

1. We find $N(0) = \frac{84}{1+2799e^{0}} = \frac{84}{2800} = 0.03$. Since $N(t)$ measures the number of people who have heard the rumor in hundreds, $N(0)$ corresponds to 3 people. Since $t = 0$ corresponds to April 1, 2016, we may conclude that on that day, 3 people have heard the rumor.[15]
2. We could simply note that $N(t)$ is written in the form of Equation 7.7, and identify $L = 84$. However, to see better *why* the answer is 84, we proceed analytically.

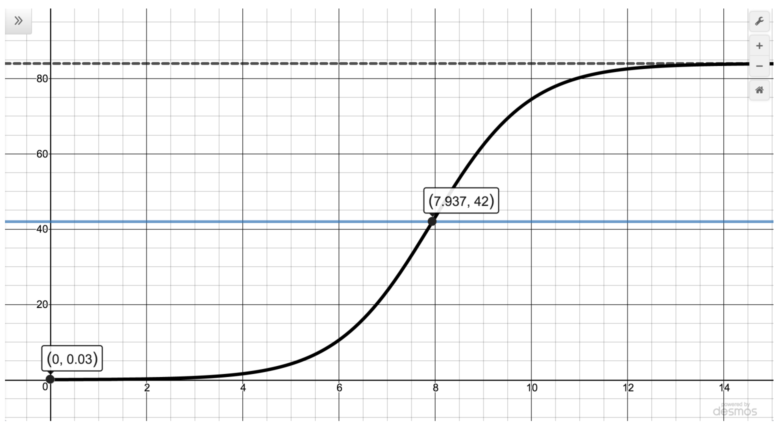
[14]Which can be just as damaging as diseases.
[15]Or, more likely, three people started the rumor. I'd wager Jeffey, Rosie, and JT started it.

Since the domain of N is restricted to $t \geq 0$, the only end behavior of significance is $t \to \infty$. As we've seen before,[16] as $t \to \infty$, we have $1997e^{-t} \to 0^+$ and so $N(t) \approx \frac{84}{1+\text{very small }(+)} \approx 84$.

Hence, $\lim\limits_{t\to\infty} N(t) = 84$. This means that as time goes by, the number of people who will have heard the rumor approaches 8400.

3. To find how long it takes until 4200 people have heard the rumor, we set $N(t) = 42$. Solving $\frac{84}{1+2799e^{-t}} = 42$ gives $t = \ln(2799) \approx 7.937$, so it takes around 8 days until 4200 people have heard the rumor.

4. Graphing $y = N(t)$ below, we see $y = 84$ is the horizontal asymptote of the graph, confirming our answer to number 2, and the graph intersects the line $y = 42$ at $t \approx 7.937 \approx \ln(2799)$, which confirms our answer to number 3.



□

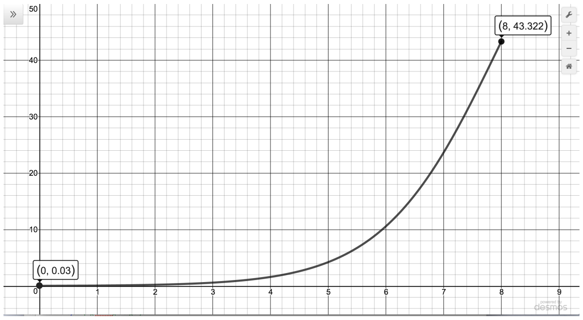
If we take the time to analyze the graph of $y = N(t)$ in Example 7.6.5, we can see *graphically* how logistic growth combines features of uninhibited and limited growth.

We can see graphically that there is an inflection point in the graph. In this case, the inflection point is called the **point of diminishing returns**. Even though the function is still increasing through the inflection point (more people are hearing the rumor), the *rate* at which it does so begins to decrease.

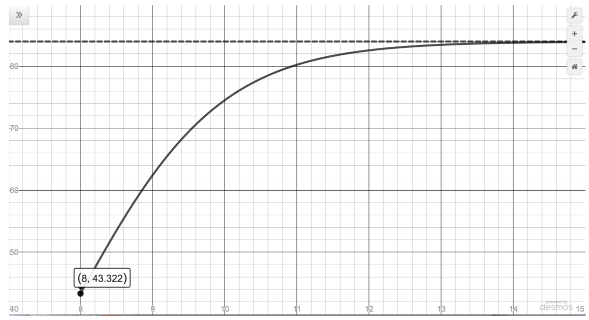
With Calculus, one can show the point of diminishing returns always occurs at half the limiting population.[17] (In our case, when $N(t) = 42$.) So with that in mind, we present two portions of the graph of $y = N(x)$, one on the interval $[0, 8]$, the other on $[8, 15]$. The former looks strikingly like uninhibited growth while the latter like limited growth.

[16] See, for example, Example 7.1.3.

[17] We'll explore this in the Exercises.



$y = N(t)$ for $0 \leq t \leq 8$



$y = N(t)$ for $8 \leq t \leq 15$

7.6.2 Applications of Logarithms

Just as many physical phenomena can be modeled by exponential functions, the same is true of logarithmic functions. In Exercises 82, 83 and 84 of Section 7.2, we showed that logarithms are useful in measuring the intensities of earthquakes (the Richter scale), sound (decibels) and acids and bases (pH). We now present yet a different use of the a basic logarithm function, password strength.

Example 7.6.6. The information entropy H, in bits, of a randomly generated password consisting of L characters is given by $H = L\log_2(N)$, where N is the number of possible symbols for each character in the password. In general, the higher the entropy, the stronger the password.

1. If a 7 character case-sensitive[18] password is comprised of letters and numbers only, find the associated information entropy.

2. How many possible symbol options per character is required to produce a 7 character password with an information entropy of 50 bits?

Solution.

1. There are 26 letters in the alphabet, 52 if upper and lower case letters are counted as different. There are 10 digits (0 through 9) for a total of $N = 62$ symbols. Since the password is to be 7 characters long, $L = 7$. Thus, $H = 7\log_2(62) = \frac{7\ln(62)}{\ln(2)} \approx 41.68$.

2. We have $L = 7$ and $H = 50$ and we need to find N. Solving the equation $50 = 7\log_2(N)$ gives $N = 2^{50/7} \approx 141.323$, so we would need 142 different symbols to choose from.[19] $\Box$

Chemical systems known as buffer solutions have the ability to adjust to small changes in acidity to maintain a range of pH values. Buffer solutions have a wide variety of applications from maintaining a healthy fish tank to regulating the pH levels in blood. Our next example shows how the pH in a buffer solution is a little more complicated than the pH we first encountered in Exercise 84 in Section 7.2.

[18] That is, upper and lower case letters are treated as different characters.

[19] Since there are only 94 distinct ASCII keyboard characters, to achieve this strength, the number of characters in the password should be increased.

Example 7.6.7. Blood is a buffer solution. When carbon dioxide is absorbed into the bloodstream it produces carbonic acid and lowers the pH. The body compensates by producing bicarbonate, a weak base to partially neutralize the acid. The equation[20] which models blood pH in this situation is $\text{pH} = 6.1 + \log\left(\frac{800}{x}\right)$, where x is the partial pressure of carbon dioxide in arterial blood, measured in torr. Find the partial pressure of carbon dioxide in arterial blood if the pH is 7.4.

Solution. We set pH = 7.4 and get $7.4 = 6.1 + \log\left(\frac{800}{x}\right)$, or $\log\left(\frac{800}{x}\right) = 1.3$. We get $x = \frac{800}{10^{1.3}} \approx 40.09$. Hence, the partial pressure of carbon dioxide in the blood is about 40 torr. □

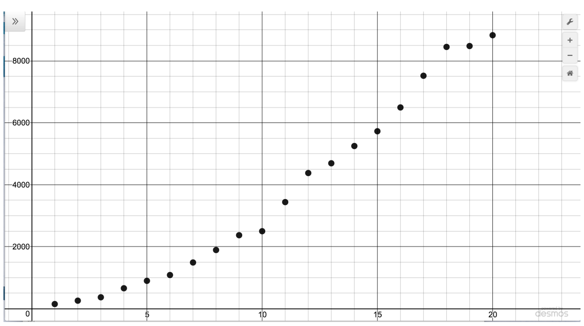
Another place logarithms are used is in data analysis. Suppose, for instance, we wish to model the spread of influenza A (H1N1), the so-called 'Swine Flu'. Below is data taken from the World Health Organization (WHO) where t represents the number of days since April 28, 2009, and N represents the number of confirmed cases of H1N1 virus worldwide.

t	1	2	3	4	5	6	7	8	9	10	11	12	13
N	148	257	367	658	898	1085	1490	1893	2371	2500	3440	4379	4694

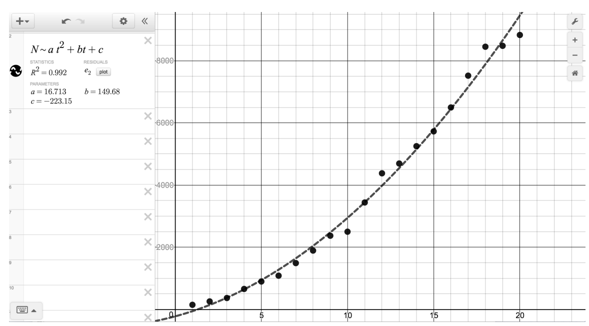
t	14	15	16	17	18	19	20
N	5251	5728	6497	7520	8451	8480	8829

Making a scatter plot of the data treating t as the independent variable and N as the dependent variable gives the plot below on the left. Which models are suggested by the shape of the data?

Thinking back Section 1.4, we try a Quadratic Regression. We find $N(t) \approx 16.713t^2 + 149.68t - 233.15$ with $R^2 = 0.992$, indicating a pretty good fit. However, is there any underlying scientific principle which would account for these data to be quadratic? Are there other models which fit the data better?



Scatterplot of the Data



A quadratic regression model

To answer these questions, scientists often use logarithms in an attempt to 'linearize' non-liner data sets such as the one before us. To see how this could work, suppose we guessed the relationship between N and t is something from Section 4.2, $N(t) = at^p$.

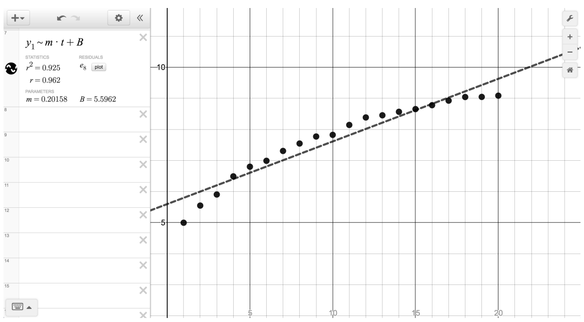
By taking the natural logs of both sides and using the Product and Power Rules, in turn, we find that $\ln(N(t)) = \ln(at^p) = \ln(a) + \ln(t^p) = \ln(a) + p\ln(t) = p\ln(t) + \ln(a)$. If we let $x = \ln(t)$ and $y = \ln(N(t))$, the

[20] Derived from the Henderson-Hasselbalch Equation. See Exercise 41 in Section 7.3. Hasselbalch himself was studying carbon dioxide dissolving in blood - a process called metabolic acidosis.

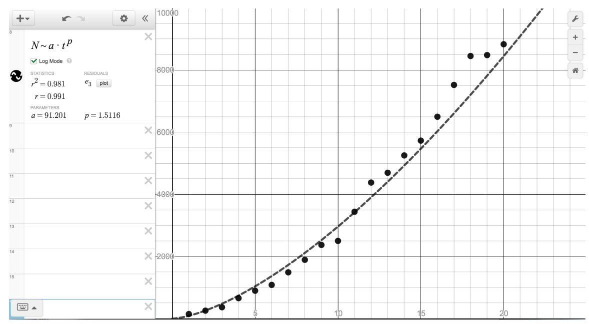
model takes the form $y = px + \ln(a)$ which is a *linear* model with slope p and y-intercept $\ln(a)$. So, instead of plotting $N(t)$ versus t, we plot $y = \ln(N(t))$ versus $x = \ln(t)$ and find a linear regression for this data set.

$\ln(t)$	0	0.693	1.099	1.386	1.609	1.792	1.946	2.079	2.197	2.302	2.398	2.485	2.565
$\ln(N)$	4.997	5.549	5.905	6.489	6.800	6.989	7.306	7.546	7.771	7.824	8.143	8.385	8.454

$\ln(t)$	2.639	2.708	2.773	2.833	2.890	2.944	2.996
$\ln(N)$	8.566	8.653	8.779	8.925	9.042	9.045	9.086



linear regression: $\ln(N(t)) = p\ln(t) + \ln(a)$



power function regression: $N(t) = at^p$

We see $r = 0.991$, which is very close to 1 indicating a very good fit. The slope of the regression line is $m \approx 1.512$ which corresponds to our exponent p. The y-intercept $b \approx 4.513$ corresponds to $\ln(a)$, so that $a \approx 91.201$. Hence, we get the model $N = 91.201t^{1.512}$.

Of interest here is that the graphing utility we used, desmos has its own built-in power regression model. If the 'log mode' square is checked, the graphing utility returns the *same* model we obtained using our linearization (since the routine which determines the coefficients uses logarithms as well.)[21]

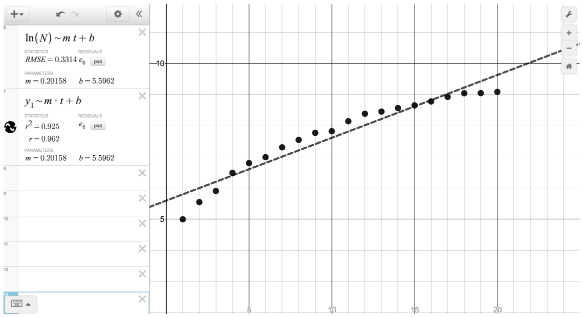
At this point, the quadratic model fits the data better, ostensibly because we have *three* parameters we can adjust in the formula $N(t) = at^2 + bt + c$ to minimize our error as opposed to just *two* parameters in the formula $N(t) = at^p$. Neither model, however, is based on any underlying scientific principle.

If we think about this situation from a scientific perspective, it does seem to make sense that, at least in the early stages of the outbreak, the more people who have the flu, the faster it will spread. This suggests we fit the data to an uninhibited growth model.

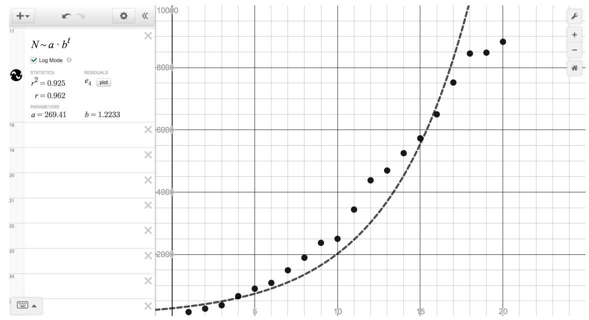
As written, Equation 7.4 gives uninhibited growth as $N(t) = N_0 e^{kt}$. Here, for simplicity's sake, we relabel $N_0 = a$ and $e^k = b$ so that we are looking for parameters a and b so that $N(t) = a \cdot b^t$.

If we assume $N(t) = a \cdot b^t$ then, taking logs as before, we get $\ln(N(t)) = t\ln(b) + \ln(a)$. If we let $y = \ln(N(t))$, then, once again, we get a linear model this time with slope $\ln(b)$ and y-intercept $\ln(a)$. We present the results of the regression below. While there is a strong correlation, $r = 0.962$, the plot doesn't instill the greatest of confidence in this model.

[21] If, however, we uncheck that box, we get a *different* power function model, $N(t) = 62.318t^{1.675}$ which chooses a and p directly to minimize the total squared error. See here for more details.



linear regression: $\ln(N(t)) = t\ln(b) + \ln(a)$

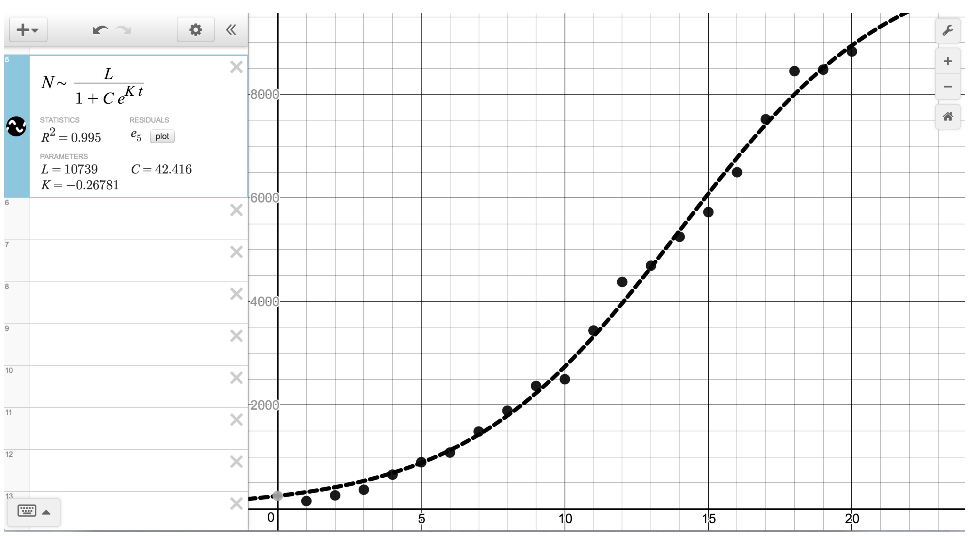


exponential regression: $N(t) = a \cdot b^t$

From the slope of the model, we have $m = \ln(b) \approx 0.202$ so $b \approx 1.223$. From the y-intercept of the model, we get $B = \ln(a) \approx 5.596$ so $a \approx 269.35$, so that our model is $N(t) = 269.35(1.223)^t$. Using the built-in exponential regression (again, with 'log mode' checked) returns the model $N(t) = 269.41(1.223)^t$, the discrepancy between 269.35 and 269.41 stemming ostensibly from round-off error.

The exponential model didn't fit the data as well as the quadratic or power function model, but it stands to reason that, perhaps, the spread of the flu is not unlike that of the spread of a rumor and that a logistic model can be used to model the data. Again, for simplicity, we abbreviate the model given in Equation 7.7 from $N(t) = \frac{L}{1+Ce^{-kLt}}$ to $N(t) = \frac{L}{1+Ce^{Kt}}$.

Running the data, a logistic function appears to be an excellent fit, both judging by the graph as well as the coefficient of determination, $R^2 \approx 0.995$. Moreover, the underlying principles which lead to the formulation of this model seem reasonable enough.



While the quadratic model also fits extremely well, our logistic model takes into account that only a finite number of people will ever get the flu (according to our model, $L = 10{,}739$), whereas the quadratic model predicts no limit to the number of cases. As we have stated several times before in the text, mathematical models, regardless of their sophistication, are just that: models, and they all have their limitations.[22]

[22]Speaking of limitations, as of June 3, 2009, there were 19,273 confirmed cases of influenza A (H1N1). This is well above our

7.6.3 Exercises

For each of the scenarios given in Exercises 1 - 6,

- Find the amount A in the account as a function of the term of the investment t in years.
- To the nearest cent, determine how much is in the account after 5, 10, 30 and 35 years.
- To the nearest year, determine how long will it take for the initial investment to double.
- Find and interpret the average rate of change of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year. Round your answer to two decimal places.

1. \$500 is invested in an account which offers 0.75%, compounded monthly.
2. \$500 is invested in an account which offers 0.75%, compounded continuously.
3. \$1000 is invested in an account which offers 1.25%, compounded monthly.
4. \$1000 is invested in an account which offers 1.25%, compounded continuously.
5. \$5000 is invested in an account which offers 2.125%, compounded monthly.
6. \$5000 is invested in an account which offers 2.125%, compounded continuously.
7. Look back at your answers to Exercises 1 - 6. What can be said about the difference between monthly compounding and continuously compounding the interest in those situations? With the help of your classmates, discuss scenarios where the difference between monthly and continuously compounded interest would be more dramatic. Try varying the interest rate, the term of the investment and the principal. Use computations to support your answer.
8. How much money needs to be invested now to obtain \$2000 in 3 years if the interest rate in a savings account is 0.25%, compounded continuously? Round your answer to the nearest cent.
9. How much money needs to be invested now to obtain \$5000 in 10 years if the interest rate in a CD is 2.25%, compounded monthly? Round your answer to the nearest cent.
10. On May, 31, 2009, the Annual Percentage Rate listed at Jeff's bank for regular savings accounts was 0.25% compounded monthly. Use Equation 7.2 to answer the following.
    (a) If $P = 2000$ what is $A(8)$?
    (b) Solve the equation $A(t) = 4000$ for t.
    (c) What principal P should be invested so that the account balance is \$2000 is three years?

prediction of 10,739. Each time a new report is issued, the data set increases and the model must be recalculated. We leave this recalculation to the reader.

11. Jeff's bank also offers a 36-month Certificate of Deposit (CD) with an APR of 2.25%.

    (a) If $P = 2000$ what is $A(8)$?
    (b) Solve the equation $A(t) = 4000$ for t.
    (c) What principal P should be invested so that the account balance is \$2000 in three years?
    (d) The Annual Percentage Yield is the simple interest rate that returns the same amount of interest after one year as the compound interest does. With the help of your classmates, compute the APY for this investment.

12. A finance company offers a promotion on \$5000 loans. The borrower does not have to make any payments for the first three years, however interest will continue to be charged to the loan at 29.9% compounded continuously. What amount will be due at the end of the three year period, assuming no payments are made? If the promotion is extended an additional three years, and no payments are made, what amount would be due?

13. Use Equation 7.2 to show that the time it takes for an investment to double in value does not depend on the principal P, but rather, depends only on the APR and the number of compoundings per year. Let $n = 12$ and with the help of your classmates compute the doubling time for a variety of rates r. Then look up the Rule of 72 and compare your answers to what that rule says. If you're really interested[23] in Financial Mathematics, you could also compare and contrast the Rule of 72 with the Rule of 70 and the Rule of 69.

In Exercises 14 - 18, we list some radioactive isotopes and their associated half-lives. Assume that each decays according to the formula $A(t) = A_0 e^{kt}$ where A_0 is the initial amount of the material and k is the decay constant. For each isotope:

- Find the decay constant k. Round your answer to four decimal places.
- Find a function which gives the amount of isotope A which remains after time t. (Keep the units of A and t the same as the given data.)
- Determine how long it takes for 90% of the material to decay. Round your answer to two decimal places. (HINT: If 90% of the material decays, how much is left?)

14. Cobalt 60, used in food irradiation, initial amount 50 grams, half-life of 5.27 years.

15. Phosphorus 32, used in agriculture, initial amount 2 milligrams, half-life 14 days.

16. Chromium 51, used to track red blood cells, initial amount 75 milligrams, half-life 27.7 days.

17. Americium 241, used in smoke detectors, initial amount 0.29 micrograms, half-life 432.7 years.

18. Uranium 235, used for nuclear power, initial amount 1 kg, half-life 704 million years.

[23] Awesome pun!

19. With the help of your classmates, show that the time it takes for 90% of each isotope listed in Exercises 14 - 18 to decay does not depend on the initial amount of the substance, but rather, on only the decay constant k. Find a formula, in terms of k only, to determine how long it takes for 90% of a radioactive isotope to decay.

20. In Example 7.1.2 in Section 7.1, the exponential function $V(x) = 25\left(\frac{4}{5}\right)^x$ was used to model the value of a car over time. Use a change of base formula to rewrite the model in the form $V(t) = 25e^{kt}$.

21. The Gross Domestic Product (GDP) of the US (in billions of dollars) t years after the year 2000 can be modeled by:

$$G(t) = 9743.77e^{0.0514t}$$

    (a) Find and interpret $G(0)$.

    (b) According to the model, what should have been the GDP in 2007? In 2010? (According to the US Department of Commerce, the 2007 GDP was $\$14,369.1$ billion and the 2010 GDP was $\$14,657.8$ billion.)

22. The diameter D of a tumor, in millimeters, t days after it is detected is given by:

$$D(t) = 15e^{0.0277t}$$

    (a) What was the diameter of the tumor when it was originally detected?

    (b) How long until the diameter of the tumor doubles?

23. Under optimal conditions, the growth of a certain strain of *E. Coli* is modeled by the Law of Uninhibited Growth $N(t) = N_0e^{kt}$ where N_0 is the initial number of bacteria and t is the elapsed time, measured in minutes. From numerous experiments, it has been determined that the doubling time of this organism is 20 minutes. Suppose 1000 bacteria are present initially.

    (a) Find the growth constant k. Round your answer to four decimal places.

    (b) Find a function which gives the number of bacteria $N(t)$ after t minutes.

    (c) How long until there are 9000 bacteria? Round your answer to the nearest minute.

24. Yeast is often used in biological experiments. A research technician estimates that a sample of yeast suspension contains 2.5 million organisms per cubic centimeter (cc). Two hours later, she estimates the population density to be 6 million organisms per cc. Let t be the time elapsed since the first observation, measured in hours. Assume that the yeast growth follows the Law of Uninhibited Growth $N(t) = N_0e^{kt}$.

    (a) Find the growth constant k. Round your answer to four decimal places.

    (b) Find a function which gives the number of yeast (in millions) per cc $N(t)$ after t hours.

    (c) What is the doubling time for this strain of yeast?

25. The Law of Uninhibited Growth also applies to situations where an animal is re-introduced into a suitable environment. Such a case is the reintroduction of wolves to Yellowstone National Park. According to the National Park Service, the wolf population in Yellowstone National Park was 52 in 1996 and 118 in 1999. Using these data, find a function of the form $N(t) = N_0 e^{kt}$ which models the number of wolves t years after 1996. (Use $t = 0$ to represent the year 1996. Also, round your value of k to four decimal places.) According to the model, how many wolves were in Yellowstone in 2002? (The recorded number is 272.)

26. During the early years of a community, it is not uncommon for the population to grow according to the Law of Uninhibited Growth. According to the Painesville Wikipedia entry, in 1860, the Village of Painesville had a population of 2649. In 1920, the population was 7272. Use these two data points to fit a model of the form $N(t) = N_0 e^{kt}$ were $N(t)$ is the number of Painesville Residents t years after 1860. (Use $t = 0$ to represent the year 1860. Also, round the value of k to four decimal places.) According to this model, what was the population of Painesville in 2010? (The 2010 census gave the population as 19,563) What could be some causes for such a vast discrepancy? For more on this, see Exercise 40.

27. The population of Sasquatch in Bigfoot county is modeled by

$$P(t) = \frac{120}{1 + 3.167e^{-0.05t}}$$

where $P(t)$ is the population of Sasquatch t years after 2010.

(a) Find and interpret $P(0)$.

(b) Find the population of Sasquatch in Bigfoot county in 2013 rounded to the nearest Sasquatch.

(c) To the nearest year, when will the population of Sasquatch in Bigfoot county reach 60?

(d) Find and interpret $\lim_{t \to \infty} P(t)$ analytically. Check your answer using a graphing utility.

28. Let $f(x) = \dfrac{10}{1 + e^{-x+1}}$.

(a) From Calculus, we know the inflection point of the graph of $y = f(x)$ is $(1, 5)$. This means the function is increasing the fastest at $x = 1$, or, equivalently, the slope at $(1, 5)$ is the largest anywhere on the graph. Graph $y = f(x)$ using a graphing utility and convince yourself of the reasonableness of this claim.

(b) Find average rate of change of f over each of the intervals below. What do you guess the slope of the curve is at $(1, 5)$? Zoom in on the graph near $(1, 5)$ to check your guess.

- $[0.75, 1]$
- $[0.9, 1]$
- $[0.99, 1]$
- $[1, 1.01]$
- $[1, 1.1]$
- $[1, 1.25]$

29. The half-life of the radioactive isotope Carbon-14 is about 5730 years.

    (a) Use Equation 7.5 to express the amount of Carbon-14 left from an initial N milligrams as a function of time t in years.

    (b) What percentage of the original amount of Carbon-14 is left after 20,000 years?

    (c) If an old wooden tool is found in a cave and the amount of Carbon-14 present in it is estimated to be only 42% of the original amount, approximately how old is the tool?

    (d) Radiocarbon dating is not as easy as these exercises might lead you to believe. With the help of your classmates, research radiocarbon dating and discuss why our model is somewhat over-simplified.

30. Carbon-14 cannot be used to date inorganic material such as rocks, but there are many other methods of radiometric dating which estimate the age of rocks. One of them, Rubidium-Strontium dating, uses Rubidium-87 which decays to Strontium-87 with a half-life of 50 billion years. Use Equation 7.5 to express the amount of Rubidium-87 left from an initial 2.3 micrograms as a function of time t in *billions* of years. Research this and other radiometric techniques and discuss the margins of error for various methods with your classmates.

31. Find and interpret the relative rate of change of $A(t)$ in Equation 7.2 over the interval $\left[t, t + \frac{1}{n}\right]$.

32. Use Equation 7.5 to show that $k = -\dfrac{\ln(2)}{h}$ where h is the half-life of the radioactive isotope.

33. A pork roast[24] was taken out of a hardwood smoker when its internal temperature had reached 180°F and it was allowed to rest in a 75°F house for 20 minutes after which its internal temperature had dropped to 170°F.

    Assuming that the temperature of the roast follows Newton's Law of Cooling (Equation 7.6),

    (a) Express the temperature T (in °F) as a function of time t (in minutes).

    (b) Find the time at which the roast would have dropped to 140°F had it not been eaten.

34. In reference to Exercise 20 in Section 4.2, if Fritzy the Fox's speed is the same as Chewbacca the Bunny's speed, Fritzy's pursuit curve is given by

$$y(x) = \frac{1}{4}x^2 - \frac{1}{4}\ln(x) - \frac{1}{4}$$

    Graph this path for $x > 0$ using a graphing utility. Investigate $\displaystyle\lim_{x \to 0^+} y(x)$ and interpret.

[24] This roast was enjoyed by Jeff and his family on June 10, 2009. This is real data, folks!

35. The current i measured in amps in a certain electronic circuit with a constant impressed voltage of 120 volts is given by $i(t) = 2 - 2e^{-10t}$ where $t \geq 0$ is the number of seconds after the circuit is switched on. Determine $\displaystyle\lim_{t\to\infty} i(t)$. (This is called the **steady state** current.)

36. If the voltage in the circuit in Exercise 35 above is switched off after 30 seconds, the current is given by the piecewise-defined function

$$i(t) = \begin{cases} 2 - 2e^{-10t} & \text{if } 0 \leq t < 30 \\ \left(2 - 2e^{-300}\right) e^{-10t+300} & \text{if } t \geq 30 \end{cases}$$

With the help of a graphing utility, graph $y = i(t)$ and discuss with your classmates the physical significance of the two parts of the graph $0 \leq t < 30$ and $t \geq 30$.

37. In Exercise 50 in Section 1.4, we stated that the cable of a suspension bridge formed a parabola but that a free hanging cable did not. A free hanging cable forms a catenary and its basic shape is given by $y = \frac{1}{2}\left(e^x + e^{-x}\right)$. Use a graphing utility to graph this function. What are its domain and range? What is its end behavior? Is it invertible? How do you think it is related to the function given in Exercise 55 in Section 7.4 and the one given in the answer to Exercise 44 in Section 7.5?

When flipped upside down, the catenary makes an arch. The Gateway Arch in St. Louis, Missouri has the shape

$$y = 757.7 - \frac{127.7}{2}\left(e^{\frac{x}{127.7}} + e^{-\frac{x}{127.7}}\right)$$

where x and y are measured in feet and $-315 \leq x \leq 315$. Find the highest point on the arch.

38. In Exercise 52a in Section 1.4, we examined the data set given below which showed how two cats and their surviving offspring can produce over 80 million cats in just ten years. Plot x versus $\ln(x)$ as was done on page 658 using a graphing utility.

Find a linear model for this new data and comment on its goodness of fit and find an exponential model for the original data and comment on its goodness of fit.

Year x	1	2	3	4	5	6	7	8	9	10
Number of Cats $N(x)$	12	66	382	2201	12680	73041	420715	2423316	13968290	80399780

39. In Example 4.2.3 in Section 4.2, we fit a power function of the form $L(x) = ax^p$ to a set of data, $(x, L(x))$. In this exercise, we use logs to linearize this data using the same methods presented on page 658, but with a slight difference in interpretation.

   (a) Starting with $L(x) = ax^p$, take natural logs of both sides of the equation and use log properties to rewrite the resulting equation as: $\ln(L(x)) = p\ln(x) + \ln(a)$.

   (b) Use a graphing utility to find a least squares regression line using the data $(\ln(x), \ln(L(x)))$. NOTE: In this situation, we are plotting $\ln(x)$ versus $\ln(L(x))$ instead of x versus $\ln(L(x))$.

(c) Find the slope p of the regression line and the intercept $\ln(a)$. Use these to construct a model of the form $L(x) = ax^p$. Find and interpret $L(90)$.

(d) Graph both the model obtained in Example 4.2.3 and the model obtained in part 39c along with the original data. What do you notice?

40. This exercise is a follow-up to Exercise 26 which more thoroughly explores the population growth of Painesville, Ohio. According to Wikipedia, the population of Painesville, Ohio is given by

Year t	1860	1870	1880	1890	1900	1910	1920	1930	1940	1950
Population	2649	3728	3841	4755	5024	5501	7272	10944	12235	14432

Year t	1960	1970	1980	1990	2000
Population	16116	16536	16351	15699	17503

(a) Use a graphing utility to perform an exponential regression on the data from 1860 through 1920 only, letting $t = 0$ represent the year 1860 as before. How does this model compare with the model you found in Exercise 26? Use the graphing utility's exponential model to predict the population in 2010. (The 2010 census gave the population as 19,563)

(b) The logistic model fit to *all* of the given data points for the population of Painesville t years after 1860 (again, using $t = 0$ as 1860) is

$$P(t) = \frac{18691}{1 + 9.8505e^{-0.03617t}}$$

According to this model, what should the population of Painesville have been in 2010? (The 2010 census gave the population as 19,563.) What is the population limit of Painesville?

41. According to OhioBiz, the census data for Lake County, Ohio is as follows:

Year t	1860	1870	1880	1890	1900	1910	1920	1930	1940	1950
Population	15576	15935	16326	18235	21680	22927	28667	41674	50020	75979

Year t	1960	1970	1980	1990	2000
Population	148700	197200	212801	215499	227511

(a) Use a graphing utility to fit a logistic model to these data with $x = 0$ representing the year 1860.

(b) Graph the data and your model using a graphing utility to judge the reasonableness of the fit.

(c) Use this model to estimate the population of Lake County in 2010. (The 2010 census gave the population to be 230,041.)

(d) According to your model, what is the population limit of Lake County, Ohio?

42. According to facebook, the number of active users of facebook has grown significantly since its initial launch from a Harvard dorm room in February 2004. The chart below has the approximate number $U(x)$ of active users, in millions, x months after February 2004. For example, the first entry $(10, 1)$ means that there were 1 million active users in December 2004 and the last entry $(77, 500)$ means that there were 500 million active users in July 2010.

Month x	10	22	34	38	44	54	59	60	62	65	67	70	72	77
Active Users in Millions $U(x)$	1	5.5	12	20	50	100	150	175	200	250	300	350	400	500

With the help of your classmates, find a model for this data.

43. Each Monday during the registration period before the Fall Semester at LCCC, the Enrollment Planning Council gets a report prepared by the data analysts in Institutional Effectiveness and Planning.[25] While the ongoing enrollment data is analyzed in many different ways, we shall focus only on the overall headcount. Below is a chart of the enrollment data for Fall Semester 2008. It starts 21 weeks before "Opening Day" and ends on "Day 15" of the semester, but we have relabeled the top row to be $x = 1$ through $x = 24$ so that the math is easier. (Thus, $x = 22$ is Opening Day.)

Week x	1	2	3	4	5	6	7	8
Total Headcount	1194	1564	2001	2475	2802	3141	3527	3790

Week x	9	10	11	12	13	14	15	16
Total Headcount	4065	4371	4611	4945	5300	5657	6056	6478

Week x	17	18	19	20	21	22	23	24
Total Headcount	7161	7772	8505	9256	10201	10743	11102	11181

With the help of your classmates, find a model for this data. Unlike most of the phenomena we have studied in this section, there is no single differential equation which governs the enrollment growth. Thus there is no scientific reason to rely on a logistic function even though the data plot may lead us to that model. What are some factors which influence enrollment at a community college and how can you take those into account mathematically?

44. When we wrote this exercise, the Enrollment Planning Report for Fall Semester 2009 had only 10 data points for the first 10 weeks of the registration period. Those numbers are given below.

Week x	1	2	3	4	5	6	7	8	9	10
Total Headcount	1380	2000	2639	3153	3499	3831	4283	4742	5123	5398

With the help of your classmates, find a model for this data and make a prediction for the Opening Day enrollment as well as the Day 15 enrollment. (WARNING: The registration period for 2009 was one week shorter than it was in 2008 so Opening Day would be $x = 21$ and Day 15 is $x = 23$.)

[25]Thanks to Dr. Wendy Marley and her staff for this data and Dr. Marcia Ballinger for the permission to use it in this problem.

7.6.4 Answers

1.
    - $A(t) = 500\left(1 + \frac{0.0075}{12}\right)^{12t}$
    - $A(5) \approx \$519.10$, $A(10) \approx \$538.93$, $A(30) \approx \$626.12$, $A(35) \approx \$650.03$
    - It will take approximately 92 years for the investment to double.
    - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 3.88. This means that the investment is growing at an average rate of \$3.88 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 4.85. This means that the investment is growing at an average rate of \$4.85 per year at this point.

2.
    - $A(t) = 500e^{0.0075t}$
    - $A(5) \approx \$519.11$, $A(10) \approx \$538.94$, $A(30) \approx \$626.16$, $A(35) \approx \$650.09$
    - It will take approximately 92 years for the investment to double.
    - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 3.88. This means that the investment is growing at an average rate of \$3.88 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 4.86. This means that the investment is growing at an average rate of \$4.86 per year at this point.

3.
    - $A(t) = 1000\left(1 + \frac{0.0125}{12}\right)^{12t}$
    - $A(5) \approx \$1064.46$, $A(10) \approx \$1133.07$, $A(30) \approx \$1454.71$, $A(35) \approx \$1548.48$
    - It will take approximately 55 years for the investment to double.
    - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 13.22. This means that the investment is growing at an average rate of \$13.22 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 19.23. This means that the investment is growing at an average rate of \$19.23 per year at this point.

4.
    - $A(t) = 1000e^{0.0125t}$
    - $A(5) \approx \$1064.49$, $A(10) \approx \$1133.15$, $A(30) \approx \$1454.99$, $A(35) \approx \$1548.83$
    - It will take approximately 55 years for the investment to double.
    - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 13.22. This means that the investment is growing at an average rate of \$13.22 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 19.24. This means that the investment is growing at an average rate of \$19.24 per year at this point.

5.
   - $A(t) = 5000\left(1 + \frac{0.02125}{12}\right)^{12t}$
   - $A(5) \approx \$5559.98$, $A(10) \approx \$6182.67$, $A(30) \approx \$9453.40$, $A(35) \approx \$10512.13$
   - It will take approximately 33 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 116.80. This means that the investment is growing at an average rate of \$116.80 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 220.83. This means that the investment is growing at an average rate of \$220.83 per year at this point.

6.
   - $A(t) = 5000e^{0.02125t}$
   - $A(5) \approx \$5560.50$, $A(10) \approx \$6183.83$, $A(30) \approx \$9458.73$, $A(35) \approx \$10519.05$
   - It will take approximately 33 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 116.91. This means that the investment is growing at an average rate of \$116.91 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 221.17. This means that the investment is growing at an average rate of \$221.17 per year at this point.

8. $P = \frac{2000}{e^{0.0025\cdot 3}} \approx \$1985.06$

9. $P = \frac{5000}{\left(1+\frac{0.0225}{12}\right)^{12\cdot 10}} \approx \$3993.42$

10. (a) $A(8) = 2000\left(1 + \frac{0.0025}{12}\right)^{12\cdot 8} \approx \$2040.40$

    (b) $t = \dfrac{\ln(2)}{12\ln\left(1 + \frac{0.0025}{12}\right)} \approx 277.29$ years

    (c) $P = \dfrac{2000}{\left(1 + \frac{0.0025}{12}\right)^{36}} \approx \$1985.06$

11. (a) $A(8) = 2000\left(1 + \frac{0.0225}{12}\right)^{12\cdot 8} \approx \$2394.03$

    (b) $t = \dfrac{\ln(2)}{12\ln\left(1 + \frac{0.0225}{12}\right)} \approx 30.83$ years

    (c) $P = \dfrac{2000}{\left(1 + \frac{0.0225}{12}\right)^{36}} \approx \$1869.57$

    (d) $\left(1 + \frac{0.0225}{12}\right)^{12} \approx 1.0227$ so the APY is 2.27%

12. $A(3) = 5000e^{0.299\cdot 3} \approx \$12,226.18$, $A(6) = 5000e^{0.299\cdot 6} \approx \$30,067.29$

14.
- $k = \frac{\ln(1/2)}{5.27} \approx -0.1315$
- $A(t) = 50e^{-0.1315t}$
- $t = \frac{\ln(0.1)}{-0.1315} \approx 17.51$ years.

15.
- $k = \frac{\ln(1/2)}{14} \approx -0.0495$
- $A(t) = 2e^{-0.0495t}$
- $t = \frac{\ln(0.1)}{-0.0495} \approx 46.52$ days.

16.
- $k = \frac{\ln(1/2)}{27.7} \approx -0.0250$
- $A(t) = 75e^{-0.0250t}$
- $t = \frac{\ln(0.1)}{-0.025} \approx 92.10$ days.

17.
- $k = \frac{\ln(1/2)}{432.7} \approx -0.0016$
- $A(t) = 0.29e^{-0.0016t}$
- $t = \frac{\ln(0.1)}{-0.0016} \approx 1439.11$ years.

18.
- $k = \frac{\ln(1/2)}{704} \approx -0.0009846$
- $A(t) = e^{-0.0009846t}$
- $t = \frac{\ln(0.1)}{-0.0009846} \approx 2338.60$ million years, or 2.339 billion years.

19. $t = \frac{\ln(0.1)}{k} = -\frac{\ln(10)}{k}$

20. $V(t) = 25e^{\ln\left(\frac{4}{5}\right)t} \approx 25e^{-0.22314355t}$

21. (a) $G(0) = 9743.77$ This means that the GDP of the US in 2000 was \$9743.77 billion dollars.
    (b) $G(7) = 13963.24$ and $G(10) = 16291.25$, so the model predicted a GDP of \$13,963.24 billion in 2007 and \$16,291.25 billion in 2010.

22. (a) $D(0) = 15$, so the tumor was 15 millimeters in diameter when it was first detected.
    (b) $t = \frac{\ln(2)}{0.0277} \approx 25$ days.

23. (a) $k = \frac{\ln(2)}{20} \approx 0.0346$
    (b) $N(t) = 1000e^{0.0346t}$
    (c) $t = \frac{\ln(9)}{0.0346} \approx 63$ minutes

24. (a) $k = \frac{1}{2}\frac{\ln(6)}{2.5} \approx 0.4377$
    (b) $N(t) = 2.5e^{0.4377t}$
    (c) $t = \frac{\ln(2)}{0.4377} \approx 1.58$ hours

25. $N_0 = 52$, $k = \frac{1}{3}\ln\left(\frac{118}{52}\right) \approx 0.2731$, $N(t) = 52e^{0.2731t}$. $N(6) \approx 268$.

26. $N_0 = 2649$, $k = \frac{1}{60}\ln\left(\frac{7272}{2649}\right) \approx 0.0168$, $N(t) = 2649e^{0.0168t}$. $N(150) \approx 32923$, so the population of Painesville in 2010 based on this model would have been 32,923.

27. (a) $P(0) = \frac{120}{4.167} \approx 29$. There are 29 Sasquatch in Bigfoot County in 2010.
    (b) $P(3) = \frac{120}{1+3.167e^{-0.05(3)}} \approx 32$ Sasquatch.
    (c) $t = 20\ln(3.167) \approx 23$ years.
    (d) We find $\lim_{t\to\infty} P(t) = 120$. As time goes by, the Sasquatch Population in Bigfoot County will approach 120. Graphically, $y = P(x)$ has a horizontal asymptote $y = 120$.

28. (b) The average rates of change are listed in order below. They suggest slope at $(1, 5)$ is 2.5.

- ≈ 2.487
- ≈ 2.498
- ≈ 2.500
- ≈ 2.500
- ≈ 2.498
- ≈ 2.487

29. (a) $A(t) = Ne^{-\left(\frac{\ln(2)}{5730}\right)t} \approx Ne^{-0.00012097t}$

(b) $A(20000) \approx 0.088978 \cdot N$ so about 8.9% remains

(c) $t \approx \dfrac{\ln(.42)}{-0.00012097} \approx 7171$ years old

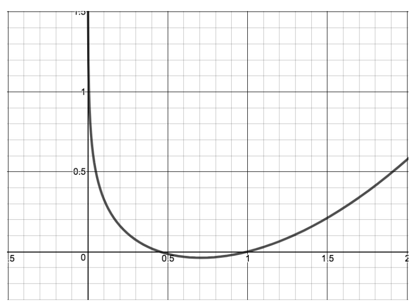
30. $A(t) = 2.3e^{-0.0138629t}$

31. The relative rate of change of $A(t)$ over $\left[t, t + \frac{1}{n}\right]$ is $\frac{r}{n}$ which is the annual percentage rate divided by the number of compoundings per year – that is, the percentage growth rate over one compounding.

33. (a) $T(t) = 75 + 105e^{-0.005005t}$

(b) The roast would have cooled to 140°F in about 95 minutes.

34. From the graph, it appears that $\displaystyle\lim_{x\to 0^+} y(x) = \infty$. This is due to the presence of the $\ln(x)$ term in the function. This means that Fritzy will never catch Chewbacca, which makes sense since Chewbacca has a head start and Fritzy only runs as fast as he does.



$y(x) = \frac{1}{4}x^2 - \frac{1}{4}\ln(x) - \frac{1}{4}$

35. The steady state current is 2 amps.

37. 630 feet.

38. The linear regression on the data below is $y = 1.74899x + 0.70739$ with $r^2 \approx 0.999995$.

This is an excellent fit.

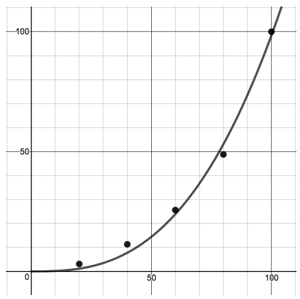
x	1	2	3	4	5	6	7	8	9	10
$\ln(N(x))$	2.4849	4.1897	5.9454	7.6967	9.4478	11.1988	12.9497	14.7006	16.4523	18.2025

$N(x) = 2.02869(5.74879)^x = 2.02869e^{1.74899x}$ with $r^2 \approx 0.999995$. This is also an excellent fit and corresponds to our linearized model because $\ln(2.02869) \approx 0.70739$.

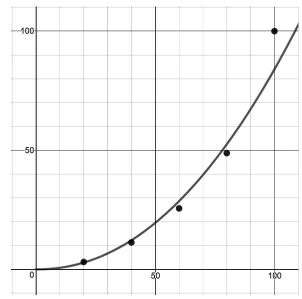
39. (b) The linearized model is: $\ln(L(x)) \approx 2.106\ln(x) - 5.268$ with an $r^2 \approx 0.9914$.

(c) $L(x) = 0.005154x^{2.106}$. $L(90) \approx 67.3$ meaning the bottom 90% of wage earners take home 67.3% of the total national income. Said differently, according to this model, the top 10% of wage earners take home 32.7% of the total national income.

(d) We graph our answer to Example 4.2.3 in Section 4.2, $L(x) = 0.00027901x^{2.7738}$, below on the left. Below on the right is the model we derived in this exercise.



$L(x) = 0.00027901x^{2.7738}$



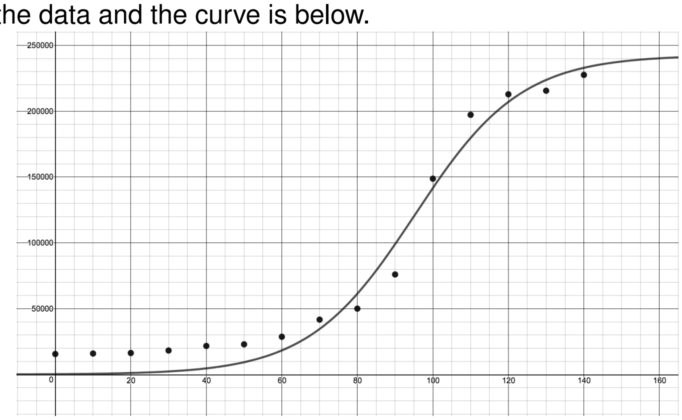
$L(x) = 0.005154x^{2.106}$

40. (a) We get: $y = 2895.06(1.0147)^x$. Graphing this along with our answer from Exercise 26 over the interval $[0, 60]$ shows that they are pretty close. From this model, $y(150) \approx 25840$ which once again overshoots the actual data value.

(b) $P(150) \approx 18717$, so this model predicts 17,914 people in Painesville in 2010, a more conservative number than was recorded in the 2010 census. We have $\displaystyle\lim_{t\to\infty} P(t) = 18691$, so the limiting population of Painesville based on this model is 18,691 people.

41. (a) $y = \dfrac{242526}{1 + 874.63e^{-0.07113x}}$, where x is the number of years since 1860.

(b) The plot of the data and the curve is below.



(c) $y(140) \approx 232884$, so this model predicts 232,884 people in Lake County in 2010.

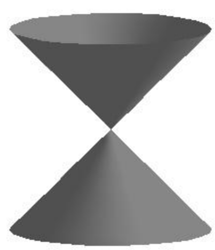
(d) We get $\displaystyle\lim_{x\to\infty} y = 242526$, so the limiting population of Lake County based on this model is 242,526 people.

Chapter 8

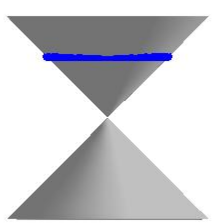
The Conic Sections

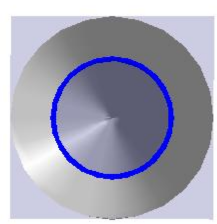
8.1 Introduction to Conics

In this chapter, we study the **Conic Sections** - literally 'sections of a cone'. Imagine a double-napped cone as seen below being 'sliced' by a plane.

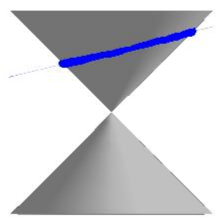


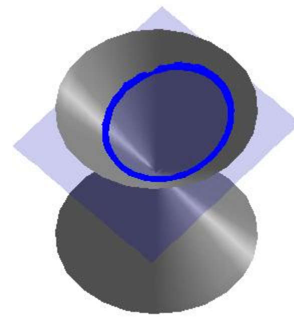
If we slice the cone with a horizontal plane the resulting curve is a **circle**.



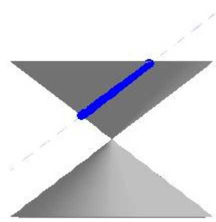


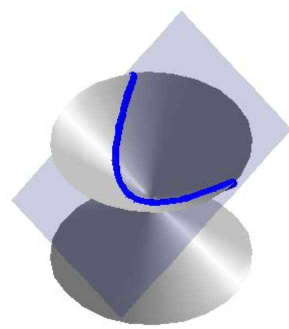
Tilting the plane ever so slightly produces an **ellipse**.



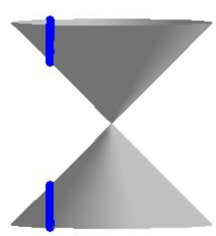


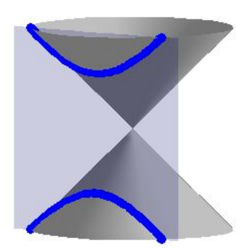
If the plane cuts parallel to the cone, we get a **parabola**.



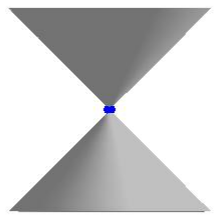
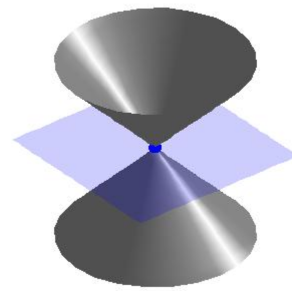


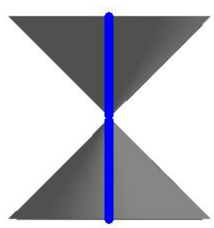
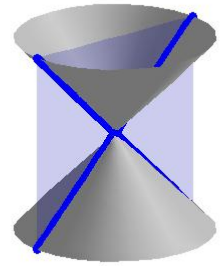
If we slice the cone with a vertical plane, we get a **hyperbola**.

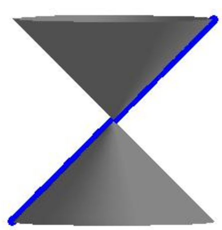
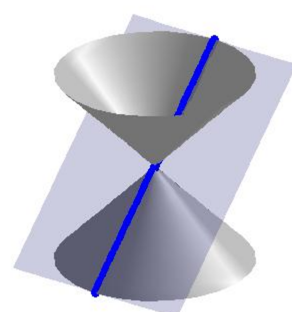




If the slicing plane contains the vertex of the cone, we get the so-called ‘degenerate’ conics: a point, a line, or two intersecting lines.

While this geometric introduction to the conic sections has its uses,[1] in order to study the applications of the conic sections, we require a more analytic approach. It turns out each of the of the conic sections can be described as as a *locus* of points - that is, a set of points which satisfy a certain condition involving distance. The reader is referred to Section A.3 for a review of the distance and related formulas.

As we’ll see, we’ll be able to use the distance formula to algebraically represent the conic sections as graphs of general quadratic equations in two variables. That is, every conic section in the xy-plane can be represented as the graph of an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ for real numbers A, B, C, D, E, and F.

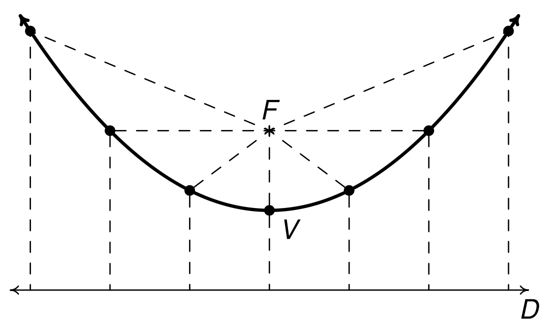
[1] See Pierre Boulle’s *Planet of the Apes* for one example - we’re serious!

8.2 Parabolas

We begin our study of the conic sections with parabolas, since we have already seen parabolas described as graphs of quadratic functions, $f(x) = ax^2+bx+c$ ($a \neq 0$). It turns out that we can also describe parabolas in terms of distances.

Definition 8.1. Let F be a point in the plane and D be a line not containing F. A **parabola** is the set of all points equidistant from F and D. The point F is called the **focus** of the parabola and the line D is called the **directrix** of the parabola.

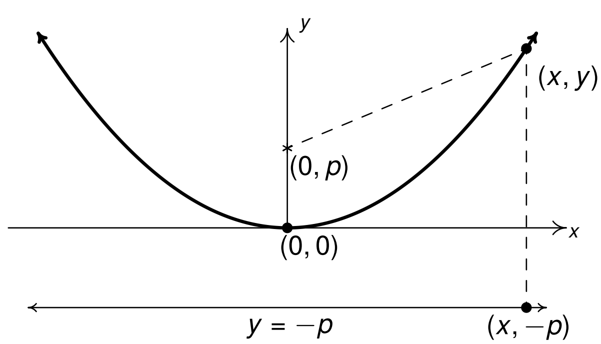
Schematically, we have the following.



Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D. The point suggestively labeled V is, as you may expect, the **vertex**. The vertex is the point on the parabola closest to the focus.

We want to use only the distance definition of parabola to derive the equation of a parabola and, if all is right with the universe, we should get an expression much like those studied in Section 1.4.

For simplicity, assume that the vertex is $(0,0)$ and that the parabola opens upwards. Let p denote the directed[1] distance from the vertex to the focus, which by definition is the same as the distance from the vertex to the directrix. Hence, the focus is $(0,p)$ and the directrix is the line $y=-p$. Our picture becomes



From the definition of parabola, we know the distance from $(0,p)$ to (x,y) is the same as the distance from $(x,-p)$ to (x,y). Using the Distance Formula, Equation A.1, we get

[1]We'll talk more about what 'directed' means later.

$$\begin{aligned}
\sqrt{(x-0)^2+(y-p)^2} &= \sqrt{(x-x)^2+(y-(-p))^2} & \\
\sqrt{x^2+(y-p)^2} &= \sqrt{(y+p)^2} & \\
x^2+(y-p)^2 &= (y+p)^2 & \text{square both sides} \\
x^2+y^2-2py+p^2 &= y^2+2py+p^2 & \text{expand quantities} \\
x^2 &= 4py & \text{gather like terms}
\end{aligned}$$

Solving for y yields $y = \frac{x^2}{4p} = \frac{1}{4p}x^2$, which is a quadratic function of the form found in Equation 1.2 with $a = \frac{1}{4p}$ and vertex $(0, 0)$.

We know from previous experience that if the coefficient of x^2 is negative, the parabola opens downwards. In the equation $y = \frac{1}{4p}x^2$ this happens when $p < 0$. In our formulation, we say that p is a 'directed distance' from the vertex to the focus: if $p > 0$, the focus is above the vertex; if $p < 0$, the focus is below the vertex. The **focal length** of a parabola, that is, the length from the vertex to the focus, is therefore $|p|$.

If we choose to place the vertex at an arbitrary point (h, k), we arrive at the following formula using either transformations from Section 5.4 or re-deriving the formula from Definition 8.1.

Equation 8.1. The Standard Equation of a Vertical[a] Parabola in the xy-plane:
The equation of a (vertical) parabola with vertex (h, k) and focal length $|p|$ is

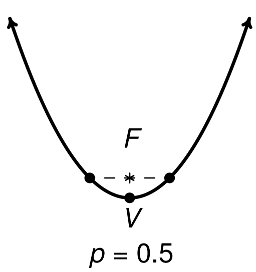
$$(x-h)^2 = 4p(y-k)$$

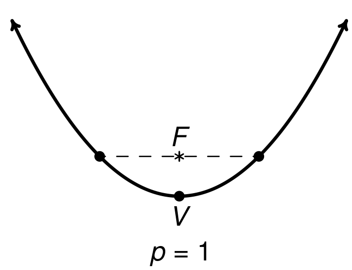
If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards.

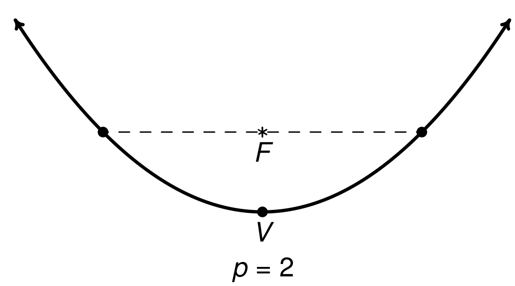
[a]That is, a parabola which opens either upwards or downwards.

Notice that in the standard equation of the parabola above, only one of the variables, x, is squared. As we'll see in the coming sections, this is a quick way to distinguish the equation of a parabola from equations representing the other conic sections.

Before embarking on an example, we take a moment to better illustrate the affect of the focal length $|p|$ on the graph of a parabola. Below we sketch three parabolas with focal length 0.5, 1, and 2. In each case, the focus is denoted by an '$*$.' In general, as the focal length $|p|$ increases, the parabola becomes wider.







The dashed line segment in each of the illustrations above is called the *latus rectum* of the parabola. More specifically, the **latus rectum** of a parabola is the line segment with endpoints on the parabola which contains the focus and is parallel to the directrix.[2]

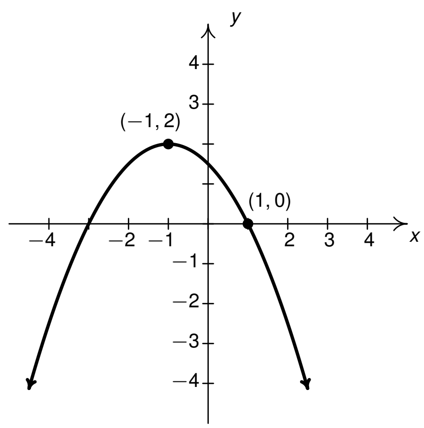
We leave it to the reader to show that the length of the latus rectum, called the **focal diameter** of the parabola is $|4p| = 4|p|$, which appears ever so conveniently in the standard form as stated in Equation 8.1. Knowing the focus and focal diameter allows to plot two points on the parabola in addition to the vertex, thus producing a more accurate graph.

Example 8.2.1.

1. Graph $(x+1)^2 = -8(y-3)$ in the xy-plane. Find the vertex, focus, and directrix. State the focal length, focal diameter, and find the endpoints of the latus rectum.

2. Find the standard form of the equation of the parabola with focus $(2, 1)$ and directrix $y = -4$.

3. Find the standard form of the equation of the parabola sketched below:



Solution.

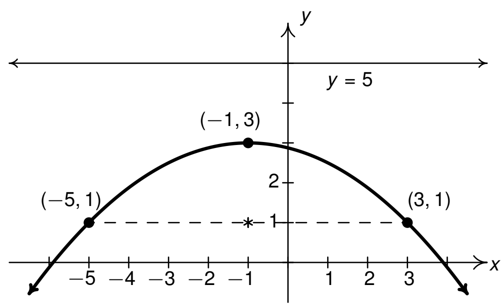
1. Rewriting $(x+1)^2 = -8(y-3)$ as $(x-(-1))^2 = -8(y-3)$, we identify $h = -1$ and $k = 3$ in Equation 8.1, so the vertex is $(-1, 3)$. Additionally, we have $4p = -8$ so $p = -2$. Since $p < 0$, the focus is *below* the vertex so the parabola opens *downwards*.

   The focal length is $|p| = 2$, which means the focus is 2 units below the vertex. From $(-1, 3)$, we move down 2 units and find the focus at $(-1, 3-2) = (-1, 1)$. Likewise the directrix is 2 units above the vertex, or the horizontal line $y = 3 + 2 = 5$.

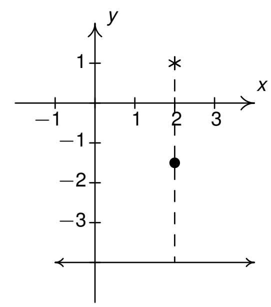
   The focal diameter is $|4p| = |-8| = 8$, which means the parabola is 8 units wide at the focus. Hence, the endpoints of the latus rectum are 4 units to the left and right of the focus. Starting at $(-1, 1)$ and moving to the left 4 units, we arrive at $(-1-4, 1) = (-5, 1)$. Starting at $(-1, 1)$ and moving to the right 4 units we arrive at $(-1+4, 1) = (3, 1)$. The final graph appears below on the left.

[2] Hence, the endpoints of the latus rectum are two points on 'opposite' sides of the parabola.

2. We begin by sketching the data given to us below on the right. Since the focus is $(2,1)$, we know the the vertex lies on the vertical line $x = 2$. Moreover, since the vertex is halfway between the focus and directrix, we know the vertex is exactly $\frac{5}{2}$ units *below* the focus at $\left(2, 1 - \frac{5}{2}\right) = \left(2, -\frac{3}{2}\right)$. This gives $h = 2$ and $k = -\frac{3}{2}$. Since the focus of the parabola is $\frac{5}{2}$ units *above* the vertex we know $p = +\frac{5}{2}$. Using Equation 8.1, we get our final answer: $(x-2)^2 = 4\left(\frac{5}{2}\right)\left(y - \left(-\frac{3}{2}\right)\right)^2$ or $(x-2)^2 = 10\left(y + \frac{3}{2}\right)^2$.



The graph of $(x+1)^2 = -8(y-3)$



3. From the graph, we assume the point labeled $(-1,2)$ is the vertex, which means in the context of Equation 8.1, $h = -1$ and $k = 2$. Hence, at this point, we know the equation is $(x-(-1))^2 = 4p(y-2)$, or, more simply $(x+1)^2 = 4p(y-2)$.

   To determine the value of p, we see $(1,0)$ is on the graph so when $x = 1$, $y = 0$. Substituting these values into our equation gives $(1+1)^2 = 4p(0-2)$ so $4 = -8p$ or $p = -\frac{1}{2}$. (The fact $p < 0$ tracks with the parabola opening downwards.) Hence, $4p = 4\left(-\frac{1}{2}\right) = -2$ so the equation of the parabola is $(x+1)^2 = -2(y-2)$. We leave it to the reader to check our answer analytically and graphically. $\square$

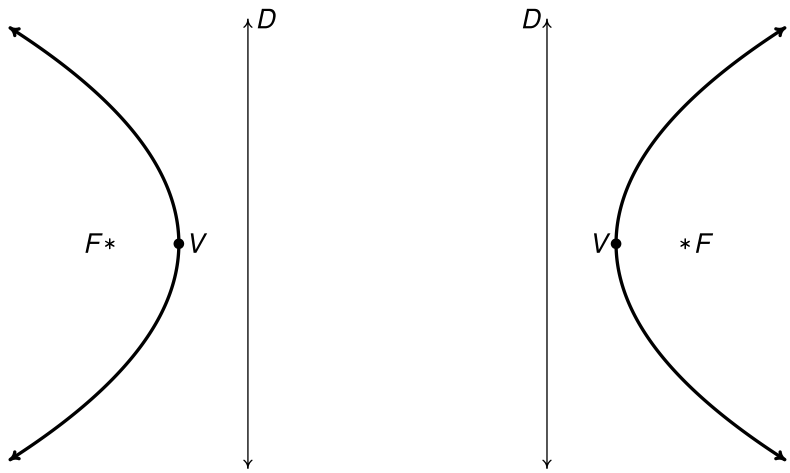
We can produce 'horizontal' parabolas in the xy-plane by reflecting our so-called 'vertical' parabolas about the line $y = x$. As you may recall from Section 5.6, we accomplish this algebraically by interchanging the variables x and y. Such parabolas necessarily open to the left or to the right, which means that unlike the vertical parabolas, these parabolas do not represent y as a function of x. As we shall see, however, they can *implicitly* describe y as a function of x, provided certain restrictions are in place.

Equation 8.2. The Standard Equation of a Horizontal Parabola:
The equation of a (horizontal) parabola with vertex (h,k) and focal length $|p|$ is

$$(y-k)^2 = 4p(x-h)$$

If $p > 0$, the parabola opens to the right; if $p < 0$, it opens to the left.

As we saw in Section 5.6, when we reflect a horizontal line across the line $y = x$, we obtain a vertical line, and, as a result, the directrix of a *horizontal* parabola is a *vertical* line. Moreover, the focus of a horizontal parabola is either to the *left* or right of the directrix. Schematically:



$p < 0$ $\qquad$ $p > 0$

Example 8.2.2.

1. For each of the equations below:

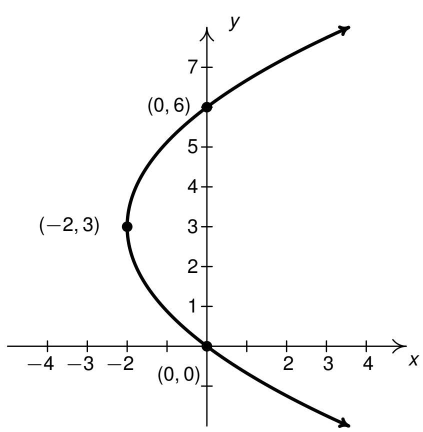
   - Graph the equation in the xy-plane.
   - Find the vertex, focus, and directrix. State the focal length, focal diameter, and find the endpoints of the latus rectum.

   (a) $(y-2)^2 = 12(x+1)$. $\qquad$ (b) $y^2 + 4y + 8x = 4$

2. Represent each of the parabolas in number 1 as the graphs of two or more explicit functions of x.

3. Find the standard form of the parabola satisfying the following characteristics:

   (a) The focus is $(-4, 2)$ and the directrix is the y-axis.
   (b) The parabola whose graph is sketched below:



Solution.

1. (a) Rewriting $(y-2)^2 = 12(x+1)$ as $(y-2)^2 = 12(x-(-1))$, we identify $h = -1$ and $k = 2$ so per Equation 8.2, the vertex is $(-1, 2)$. We also see that $4p = 12$ so $p = 3$. Since $p > 0$, this means the focus is to the *right* of the vertex so the parabola opens to the *right*.

   The focal length is $|p| = 3$, which means the focus is 3 units to the right of the vertex. From $(-1, 2)$, we move 3 units to the right and find the focus at $(-1+3, 2) = (2, 2)$. Likewise the directrix is 3 units to the left of the vertex, the vertical line $x = -1-3 = -4$.

   The focal diameter is $|4p| = |12| = 12$, which means the parabola is 12 units wide at the focus. Hence, the endpoints of the latus rectum are 6 units above and below the focus. Starting at $(2, 2)$ and moving down 6 units, we arrive at $(2, 2-6) = (2, -4)$. Starting at $(2, 2)$ and moving up 6 units we arrive at $(2, 2+6) = (2, 8)$. The final graph appears below on the left.

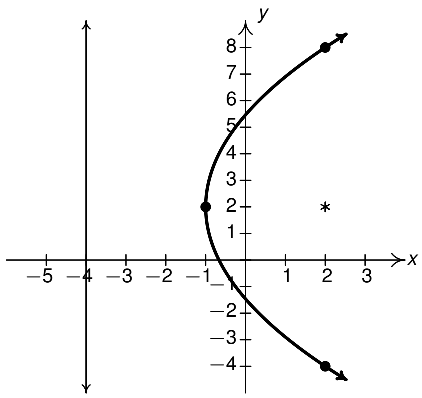
   (b) Unlike the previous example, the equation $y^2 + 4y + 8x = 4$ is not in the form described in Equation 8.2. In order to get an equivalent equation in the form prescribed by Equation 8.2, we need to complete the square in y on the left-hand side of the equation. Once that is done, we factor out the coefficient of x on the other side of the equation below.

$$\begin{array}{rcll} y^2+4y+8x & = & 4 & \\ y^2+4y & = & -8x+4 & \\ y^2+4y+4 & = & -8x+4+4 & \text{complete the square in } y. \\ (y+2)^2 & = & -8x+8 & \text{factor} \\ (y+2)^2 & = & -8(x-1) & \end{array}$$

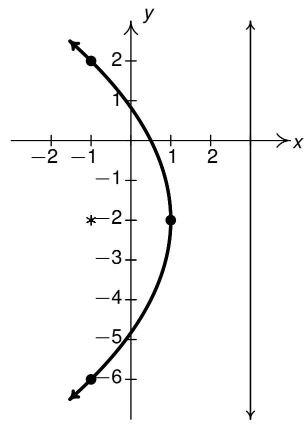
   The equation $(y+2)^2 = -8(x-1)$, rewritten as $(y-(-2))^2 = -8(x-1)$ is in the form given in Equation 8.2. Identifying $h = 1$ and $k = -2$, we get the vertex is $(1, -2)$. Moreover, we see $4p = -8$ so that $p = -2$. The fact that $p < 0$, means the focus will be the *left* of the vertex so the parabola will open to the *left*.

   Since the focal length is $|p| = 2$, the focus is 2 units to the left of the vertex. From $(1, -2)$ and move left 2 units and arrive at the focus $(1-2, -2) = (-1, -2)$. Similarly, the directrix is 2 units to the right of the vertex, the vertical line $x = 1+2 = 3$.

   Since the focal diameter is $|4p|$ is 8, the parabola is 8 units wide at the focus. Starting at the focus $(-1, -2)$ we move down 4 units and get $(-1, -2-4) = (-1, -6)$. Moving up 4 units from the focus we get $(-1, -2+4) = (-1, 2)$. Hence, $(-1, -6)$ and $(-1, 2)$ are the endpoints of the latus rectum. The final graph appears below on the right.



The graph of $(y-2)^2 = 12(x+1)$



The graph of $y^2 + 4y + 8x = 4$

2. To describe these parabolas as graphs of functions of x, we solve each equation for y in terms of x.

   (a) Starting with $(y-2)^2 = 12(x+1)$, we extract square roots to get $y - 2 = \pm\sqrt{12(x+1)}$. Isolating y, we get $y = 2 \pm \sqrt{12(x+1)}$ which simplifies to $y = 2 \pm 2\sqrt{3x+3}$.

   We let $f(x) = 2 + \sqrt{3x+3}$ and $g(x) = 2 - \sqrt{3x+3}$. Note that since $\sqrt{3x+3} \geq 0$ by definition, $f(x) = 2 + \sqrt{3x+3} \geq 2$ which means the graph of f describes the *upper* half of the parabola. Similarly, the graph of $g(x) = 2 - \sqrt{3x+3}$ describes the *lower* half of the parabola.

   (b) We can solve $y^2 + 4y + 8x = 4$ for y by completing the square or using the quadratic formula. We leave the former to the reader,[3] and proceed with the latter for the sake of practice.

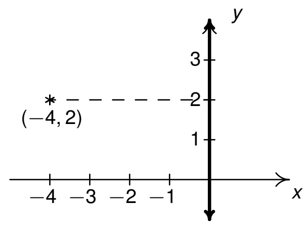
   To use the quadratic formula, we need to set the equation to 0: $y^2 + 4y + 8x - 4 = 0$. Since we are solving for y, we identify $a = 1$, $b = 4$ and $c = 8x - 4$. We find the discriminant $b^2 - 4ac = (4)^2 - 4(1)(8x-4) = 32 - 32x$ so

$$y = \frac{-4 \pm \sqrt{32-32x}}{2} = \frac{-4 \pm \sqrt{32(1-x)}}{2} = \frac{-4 \pm 4\sqrt{2(1-x)}}{2} = -2 \pm 2\sqrt{2-2x}.$$

   We identify $f(x) = -2 + 2\sqrt{2-2x}$ and $g(x) = -2 - 2\sqrt{2-2x}$. Since $\sqrt{2-2x} \geq 0$, we see the graph of f traces out the *upper* half of the parabola while the graph of g traces out the *lower* half of the parabola.

3. (a) We sketch the data below. Since the focus is $(-4, 2)$ and the directrix is a vertical line, we know the vertex must lie on the horizontal line $y = 2$. Moreover, we know the vertex must lie midway between the focus and directrix which in this case is $(-2, 2)$. This gives $h = -2$ and $k = 2$. Since the focus is 2 units to the *left* of the vertex, we know $p = -2$. Using Equation 8.2, we get our answer as $(y-2)^2 = 4(-2)(x-(-2))$ or $(y-2)^2 = -8(x+2)$.

[3]The standard form of the parabola will make an appearance using this route.



(b) From the graph, we may infer the vertex of the parabola is $(-2, 3)$ so $h = -2$ and $k = 3$. Per Equation 8.2, we have $(y - 3)^2 = 4p(x - (-2))$ or $(y - 3)^2 = 4p(x + 2)$. Since the graph contains $(0, 6)$, we substitute $x = 0$ and $y = 6$ to get $(6 - 3)^2 = 4p(0 + 2)$. We find $p = \frac{9}{8}$ which gives our final answer $(y - 3)^2 = 4\left(\frac{9}{8}\right)(x + 2)$ or $(y - 3)^2 = \frac{9}{2}(x + 2)$. □

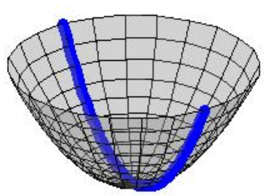
As we have seen, not all equations which describe parabolas will immediately match Equation 8.1 or Equation 8.2. Indeed, completing the square as we did with the equation in number 1b in Example 8.2.2 will be a necessary skill not only in this section, but in the rest of this chapter.

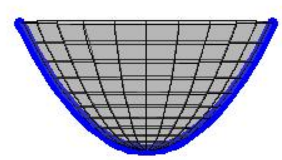
For parabolas, we summarize the procedure for putting an equation of a parabola into standard form below. Of key importance is that in the equation for a parabola, one, and only one, of the variables are squared.

To Write the Equation of a Parabola in Standard Form

1. Group the variable which is squared on one side of the equation and position the non-squared variable and the constant on the other side.
2. Complete the square if necessary and divide by the coefficient of the perfect square.
3. Factor out the coefficient of the non-squared variable from it and the constant.

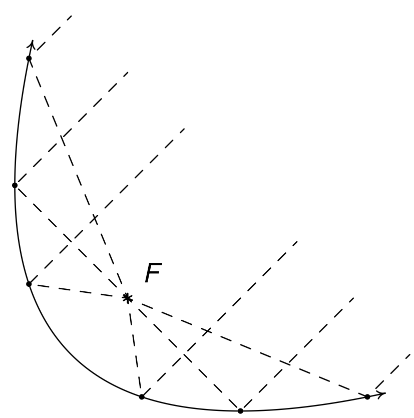
In studying quadratic functions, we have seen parabolas used to model physical phenomena such as the trajectories of projectiles. Other applications of the parabola concern its 'reflective property' which necessitates knowing about the focus of a parabola. For example, many satellite dishes are formed in the shape of a **paraboloid of revolution** as depicted below.





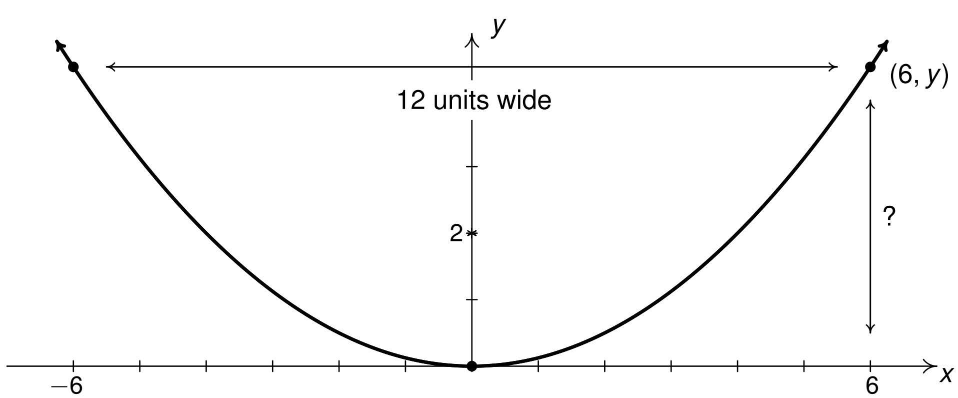
Every cross section through the vertex of the paraboloid is a parabola with the same focus. To see why this is important, imagine the dashed lines below as electromagnetic waves heading towards a parabolic dish. It turns out that the waves reflect off the parabola and concentrate at the focus which then becomes the optimal place for the receiver.

If, on the other hand, we imagine the dashed lines as emanating from the focus, we see that the waves are reflected off the parabola in a coherent fashion as in the case in a flashlight. Here, the bulb is placed at the focus and the light rays are reflected off a parabolic mirror to give directional light.



Example 8.2.3. A satellite dish is to be constructed in the shape of a paraboloid of revolution. If the receiver placed at the focus is located 2 ft above the vertex of the dish, and the dish is to be 12 feet wide, how deep will the dish be?

Solution. One way to approach this problem is to determine the equation of the parabola suggested to us by this data. For simplicity, we'll assume the vertex is $(0, 0)$ and the parabola opens upwards. Our standard form for such a parabola is $x^2 = 4py$. Since the focus is 2 units above the vertex, we know $p = 2$, so we have $x^2 = 8y$. Visually,



Since the parabola is 12 feet wide, we know the edge is 6 feet from the vertex. To find the depth, we are looking for the y value when $x = 6$. Substituting $x = 6$ into the equation of the parabola yields $6^2 = 8y$ or $y = \frac{36}{8} = \frac{9}{2} = 4.5$. Hence, the dish will be 4.5 feet deep. □

8.2.1 Exercises

In Exercises 1 - 8, graph of the given equations in the *xy*-plane. Find the vertex, focus and directrix. Include the endpoints of the latus rectum in your sketch.

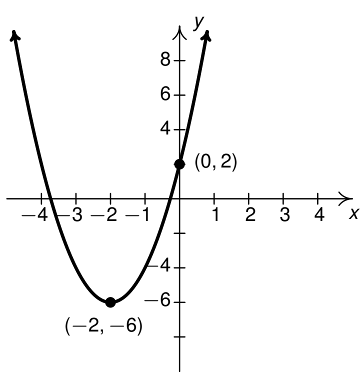
1. $(x-3)^2 = -16y$
2. $\left(x+\frac{7}{3}\right)^2 = 2\left(y+\frac{5}{2}\right)$
3. $(y-2)^2 = -12(x+3)$
4. $(y+4)^2 = 4x$
5. $(x-1)^2 = 4(y+3)$
6. $(x+2)^2 = -20(y-5)$
7. $(y-4)^2 = 18(x-2)$
8. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$

In Exercises 9 - 14, put the equation into standard form. Find the vertex, focus and directrix.[4]

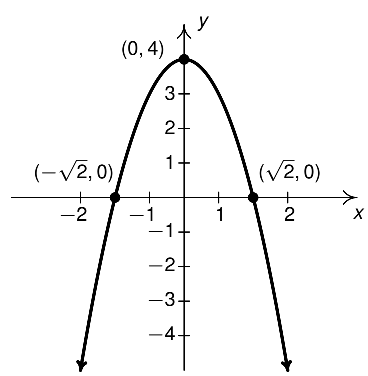
9. $y^2 - 10y - 27x + 133 = 0$
10. $25x^2 + 20x + 5y - 1 = 0$
11. $x^2 + 2x - 8y + 49 = 0$
12. $2y^2 + 4y + x - 8 = 0$
13. $x^2 - 10x + 12y + 1 = 0$
14. $3y^2 - 27y + 4x + \frac{211}{4} = 0$
15. For each of the equations given in Exercises 1 - 14 that do **not** describe y as a function of x, find two or more explicit functions of x represented by each of the equations. (See Example 8.2.2.)

In Exercises 16 - 19, find an equation for the parabola whose graph is given.

16.

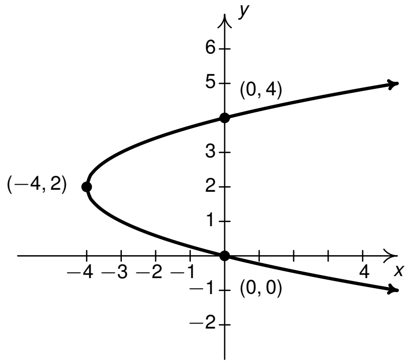


17.

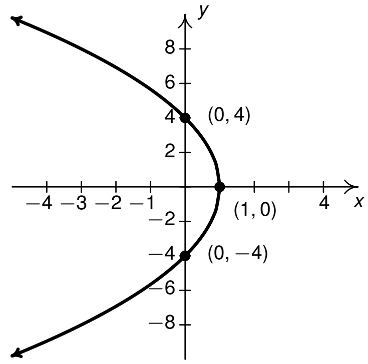


[4]. . . assuming the equation were graphed in the *xy*-plane.

18.



19.



In Exercises 20 - 23, find an equation for the parabola which fits the given criteria.

20. Vertex $(7, 0)$, focus $(0, 0)$.

21. Focus $(10, 1)$, directrix $x = 5$.

22. Vertex $(-8, -9)$; $(0, 0)$ and $(-16, 0)$ are points on the curve.

23. The endpoints of latus rectum are $(-2, -7)$ and $(4, -7)$.

24. The mirror in Carl's flashlight is a paraboloid of revolution. If the mirror is 5 centimeters in diameter and 2.5 centimeters deep, where should the light bulb be placed so it is at the focus of the mirror?

25. A parabolic Wi-Fi antenna is constructed by taking a flat sheet of metal and bending it into a parabolic shape.[5] If the cross section of the antenna is a parabola which is 45 centimeters wide and 25 centimeters deep, where should the receiver be placed to maximize reception?

26. A parabolic arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch.

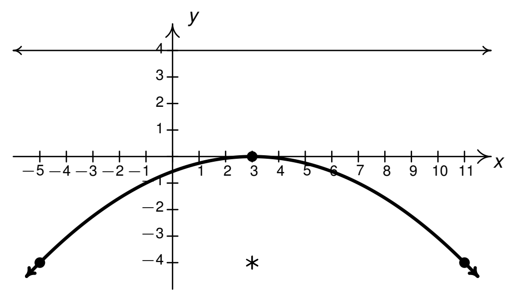
27. A popular novelty item is the 'mirage bowl.' Follow this link to see another startling application of the reflective property of the parabola.

28. With the help of your classmates, research spinning liquid mirrors. To get you started, here.

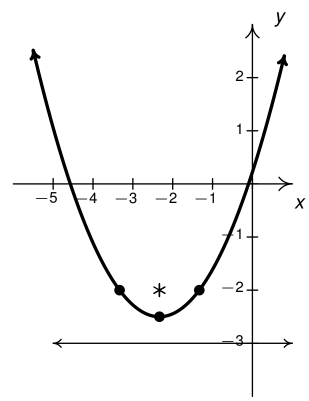
[5]This shape is called a 'parabolic cylinder.'

8.2.2 Answers

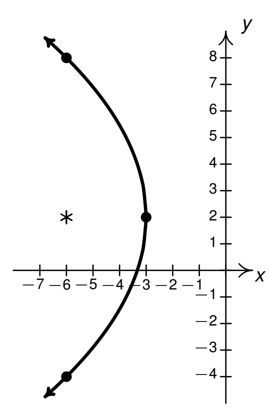
1. $(x-3)^2 = -16y$
   Vertex $(3, 0)$
   Focus $(3, -4)$
   Directrix $y = 4$
   Endpoints of latus rectum $(-5, -4)$, $(11, -4)$



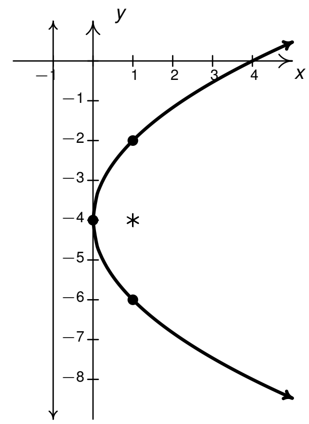
2. $\left(x+\frac{7}{3}\right)^2 = 2\left(y+\frac{5}{2}\right)$
   Vertex $\left(-\frac{7}{3}, -\frac{5}{2}\right)$
   Focus $\left(-\frac{7}{3}, -2\right)$
   Directrix $y = -3$
   Endpoints of latus rectum $\left(-\frac{10}{3}, -2\right)$, $\left(-\frac{4}{3}, -2\right)$



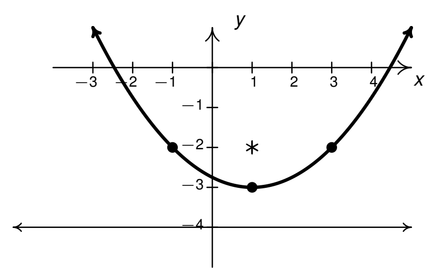
3. $(y-2)^2 = -12(x+3)$
   Vertex $(-3, 2)$
   Focus $(-6, 2)$
   Directrix $x = 0$
   Endpoints of latus rectum $(-6, 8)$, $(-6, -4)$



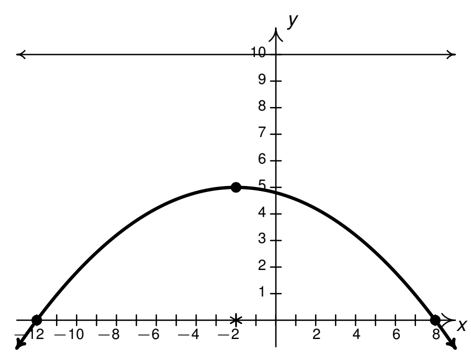
4. $(y+4)^2 = 4x$
Vertex $(0,-4)$
Focus $(1,-4)$
Directrix $x=-1$
Endpoints of latus rectum $(1,-2)$, $(1,-6)$



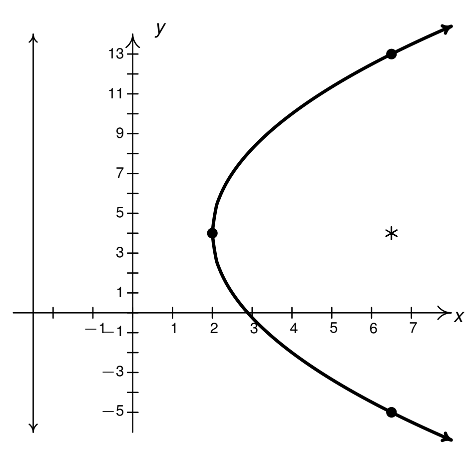
5. $(x-1)^2 = 4(y+3)$
Vertex $(1,-3)$
Focus $(1,-2)$
Directrix $y=-4$
Endpoints of latus rectum $(3,-2)$, $(-1,-2)$



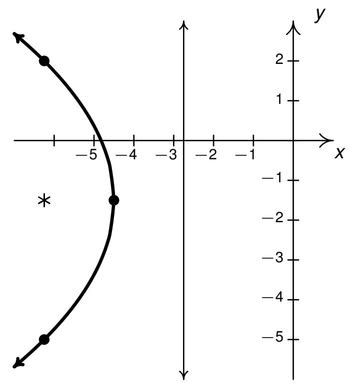
6. $(x+2)^2 = -20(y-5)$
Vertex $(-2,5)$
Focus $(-2,0)$
Directrix $y=10$
Endpoints of latus rectum $(-12,0)$, $(8,0)$



7. $(y-4)^2 = 18(x-2)$
Vertex $(2,4)$
Focus $\left(\frac{13}{2},4\right)$
Directrix $x=-\frac{5}{2}$
Endpoints of latus rectum $\left(\frac{13}{2},-5\right)$, $\left(\frac{13}{2},13\right)$



8. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$
Vertex $\left(-\frac{9}{2}, -\frac{3}{2}\right)$
Focus $\left(-\frac{25}{4}, -\frac{3}{2}\right)$
Directrix $x = -\frac{11}{4}$
Endpoints of latus rectum $\left(-\frac{25}{4}, 2\right)$, $\left(-\frac{25}{4}, -5\right)$



9. $(y-5)^2 = 27(x-4)$
Vertex $(4, 5)$
Focus $\left(\frac{43}{4}, 5\right)$
Directrix $x = -\frac{11}{4}$

10. $\left(x+\frac{2}{5}\right)^2 = -\frac{1}{5}(y-1)$
Vertex $\left(-\frac{2}{5}, 1\right)$
Focus $\left(-\frac{2}{5}, \frac{19}{20}\right)$
Directrix $y = \frac{21}{20}$

11. $(x+1)^2 = 8(y-6)$
Vertex $(-1, 6)$
Focus $(-1, 8)$
Directrix $y = 4$

12. $(y+1)^2 = -\frac{1}{2}(x-10)$
Vertex $(10, -1)$
Focus $\left(\frac{79}{8}, -1\right)$
Directrix $x = \frac{81}{8}$

13. $(x-5)^2 = -12(y-2)$
Vertex $(5, 2)$
Focus $(5, -1)$
Directrix $y = 5$

14. $\left(y-\frac{9}{2}\right)^2 = -\frac{4}{3}(x-2)$
Vertex $\left(2, \frac{9}{2}\right)$
Focus $\left(\frac{5}{3}, \frac{9}{2}\right)$
Directrix $x = \frac{7}{3}$

15. The equations which do not represent y as a function of x are: 3, 4, 7, 8, 9, 12, 14.

For number 3:

- $f(x) = 2 + 2\sqrt{-3x-9}$ represents the upper half of the parabola.
- $g(x) = 2 - 2\sqrt{-3x-9}$ represents the lower half of the parabola.

For number 4:

- $f(x) = -4 + 2\sqrt{x}$ represents the upper half of the parabola.
- $g(x) = -4 - 2\sqrt{x}$ represents the lower half of the parabola.

For number 7:

- $f(x) = 4 + 3\sqrt{2x-4}$ represents the upper half of the parabola.
- $g(x) = 4 - 3\sqrt{2x-4}$ represents the lower half of the parabola.

For number 8:

- $f(x) = -\frac{3}{2} + \frac{1}{2}\sqrt{-28x - 126}$ represents the upper half of the parabola.
- $g(x) = -\frac{3}{2} - \frac{1}{2}\sqrt{-28x - 126}$ represents the lower half of the parabola.

For number 9:

- $f(x) = 5 + 3\sqrt{3x - 12}$ represents the upper half of the parabola.
- $g(x) = 5 - 3\sqrt{3x - 12}$ represents the lower half of the parabola.

For number 12:

- $f(x) = -1 + \frac{1}{2}\sqrt{-2x + 20}$ represents the upper half of the parabola.
- $g(x) = -1 - \frac{1}{2}\sqrt{-2x + 20}$ represents the lower half of the parabola.

For number 14:

- $f(x) = \frac{9}{2} + \frac{2}{3}\sqrt{-3x + 6}$ represents the upper half of the parabola.
- $f(x) = \frac{9}{2} - \frac{2}{3}\sqrt{-3x + 6}$ represents the lower half of the parabola.

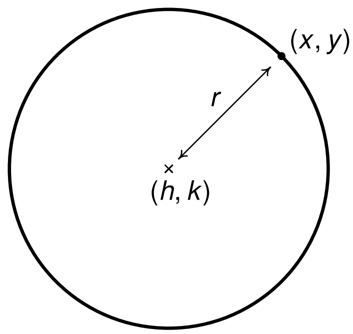
16. $(x + 2)^2 = \frac{1}{2}(y + 6)$
17. $x^2 = -\frac{1}{2}(y - 4)$
18. $(y - 2)^2 = x + 4$
19. $y^2 = -16(x - 1)$
20. $y^2 = -28(x - 7)$
21. $(y - 1)^2 = 10\left(x - \frac{15}{2}\right)$
22. $(x + 8)^2 = \frac{64}{9}(y + 9)$
23. $(x - 1)^2 = 6\left(y + \frac{17}{2}\right)$ or $(x - 1)^2 = -6\left(y + \frac{11}{2}\right)$
24. The bulb should be placed 0.625 centimeters above the vertex of the mirror.[6]
25. The receiver should be placed 5.0625 centimeters from the vertex of the cross section of the antenna.
26. The arch can be modeled by $x^2 = -(y - 9)$ or $y = 9 - x^2$. One foot in from the base of the arch corresponds to either $x = \pm 2$, so the height is $y = 9 - (\pm 2)^2 = 5$ feet.

[6] As verified by Carl himself!

8.3 Circles

Our next entry in the conic sections menagerie is the circle. Recall from Geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

Definition 8.2. A **circle** with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.



From the diagram, we see that a point (x, y) is on the circle if and only if its distance to (h, k) is r. We express this relationship algebraically using the Distance Formula, Equation A.1, as

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

Equation 8.3. The Standard Equation of a Circle:
The equation of a circle with center (h, k) and radius $r > 0$ is $(x-h)^2 + (y-k)^2 = r^2$.

Note in the standard equation of a circle, *both* of the variables squared. This is a quick way to distinguish the equation of a circle from that of a parabola in which only *one* of the variables is squared.

We put Equation 8.3 to good use in the following example.

Example 8.3.1.

1. For each of the equations below:

   - Graph the equation in the xy-plane.
   - Find the center and radius.

   (a) $(x+2)^2 + (y-1)^2 = 4$

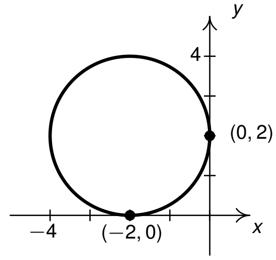
   (b) $3x^2 - 6x + 3y^2 + 4y - 4 = 0$

2. Graph $f(x) = -\sqrt{4x - x^2}$.

3. Find the standard form of the circle satisfying the following characteristics:

   (a) The points $(-1, 3)$ and $(2, 4)$ are the endpoints of a diameter.

   (b) The circle whose graph is below.



Solution.

1. (a) Rewriting $(x+2)^2 + (y-1)^2 = 4$ as $(x-(-2))^2 + (y-1)^2 = (2)^2$, we identify $h = -2$, $k = 1$ and $r = 2$. Thus we have a circle centered at $(-2, 1)$ with a radius of 2.

   To help us create a detailed graph, we start from the center $(-2, 1)$ and move two units to the left and two units up and down to the right to identify four points on the graph.[1] We get $(-2-2, 1) = (-4, 1)$, $(-2+2, 1) = (0, 1)$, $(-2, 1+2) = (-2, 3)$ and $(-2, 1-2) = (-2, -1)$. Our graph is below on the left.

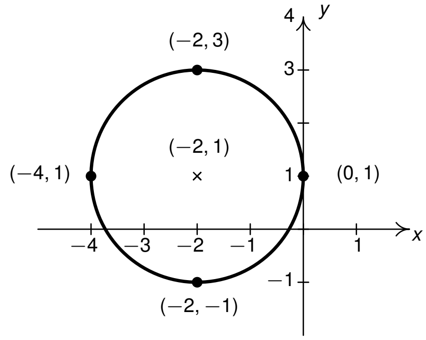
   (b) In order to make use of Equation 8.3, we need to put $3x^2 - 6x + 3y^2 + 4y - 4 = 0$ into standard form. To that end, we complete the square on both the x and y terms and collect the constants to the other side of the equation as demonstrated below.

$$
\begin{array}{rcll}
3x^2 - 6x + 3y^2 + 4y - 4 & = & 0 & \\
3x^2 - 6x + 3y^2 + 4y & = & 4 & \text{add 4 to both sides} \\
3\left(x^2 - 2x\right) + 3\left(y^2 + \frac{4}{3}y\right) & = & 4 & \text{factor out leading coefficients} \\
3\left(x^2 - 2x + \underline{1}\right) + 3\left(y^2 + \frac{4}{3}y + \underline{\underline{\frac{4}{9}}}\right) & = & 4 + 3\underline{(1)} + 3\underline{\underline{\left(\frac{4}{9}\right)}} & \text{complete the square in } x, y \\
3(x-1)^2 + 3\left(y + \frac{2}{3}\right)^2 & = & \frac{25}{3} & \text{factor} \\
(x-1)^2 + \left(y + \frac{2}{3}\right)^2 & = & \frac{25}{9} & \text{divide both sides by 3} \\
(x-1)^2 + \left(y - \left(-\frac{2}{3}\right)\right)^2 & = & \left(\frac{5}{3}\right)^2 & \text{rewrite in the form of Equation 8.3.}
\end{array}
$$

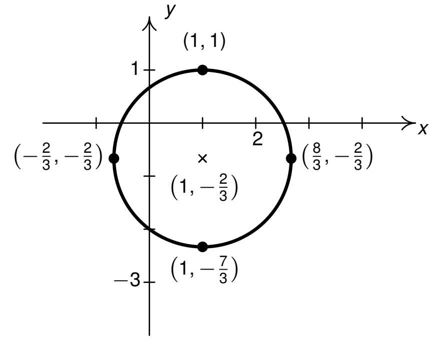
[1]Note the center of the circle is *not* on the graph of the circle!

From Equation 8.3, we identify $h = 1$, $k = -\frac{2}{3}$, and $r = \frac{5}{3}$. Hence, we have a circle with center $\left(1, -\frac{2}{3}\right)$ and radius $\frac{5}{3}$.

As above, we find four points on the circle by starting at the center $\left(1, -\frac{2}{3}\right)$ and moving up, down, to the left, and to the right $\frac{5}{3}$ units. Doing so produces the following points: $\left(-\frac{2}{3}, -\frac{2}{3}\right)$, $\left(\frac{8}{3}, -\frac{2}{3}\right)$, $(1, 1)$, and $\left(1, -\frac{7}{3}\right)$. Our graph is below on the right.



The graph of $(x + 2)^2 + (y - 1)^2 = 4$



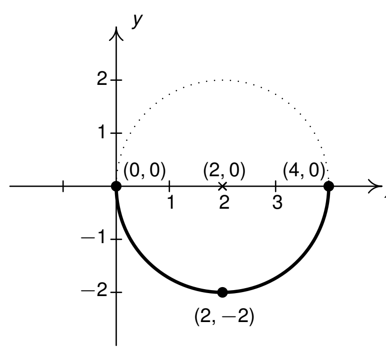
The graph of $3x^2 - 6x + 3y^2 + 4y - 4 = 0$

2. We are asked to graph $f(x) = -\sqrt{4x - x^2}$, which, at first glance, seems out of place in this section. However, to graph means we graph the equation $y = -\sqrt{4x - x^2}$.

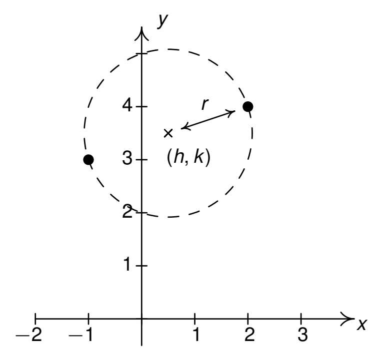
   Squaring both sides, we get $y^2 = (-\sqrt{4x - x^2})^2$ or $y^2 = 4x - x^2$. Rearranging this equation gives $x^2 - 4x + y^2 = 0$. Completing the square, we obtain $(x - 2)^2 + y^2 = 4$ which, when rewritten as $(x - 2)^2 + (y - 0)^2 = (2)^2$ is precisely the standard form of a circle as written in Equation 8.3.

   With $h = 2$, $k = 0$ and $r = 2$, we know the graph of $(x - 2)^2 + y^2 = 4$ is a circle of radius 2 centered at $(2, 0)$. However, the graph we want isn't the *entire* circle.[2] Indeed, we want the graph of $y = -\sqrt{4x - x^2}$. Because of the '$-$', we want the *lower* semicircle, graphed below on the left.

3. (a) We recall that a diameter of a circle is a line segment containing the center and two points on the circle. We plot the data given to us below on the right.



The graph of $f(x) = -\sqrt{4x - x^2}$.



[2]For one thing, the graph of a circle fails the Vertical Line Test so it does not represent y as a function of x in this case.

Since the given points are endpoints of a diameter, we know their midpoint (h, k) is the center of the circle. Likewise, the diameter of the circle is the distance between the given points, so we can find the radius of the circle by taking half of this distance. Using Equations A.2 and A.1, respectively, we get:

$$\begin{aligned}(h,k) &= \left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right) \\ &= \left(\frac{-1+2}{2}, \frac{3+4}{2}\right) \\ &= \left(\frac{1}{2}, \frac{7}{2}\right)\end{aligned}$$

$$\begin{aligned} r &= \frac{1}{2}\sqrt{(x_1-x_0)^2+(y_1-y_0)^2} \\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2} \\ &= \frac{1}{2}\sqrt{3^2+1^2} \\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2 = \frac{10}{4} = \frac{5}{2}$, our answer becomes $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{5}{2}$

(b) From the graph given to us, we are safe to assume the center of the circle is $(-2, 2)$ since the circle appears to be *tangent* to the coordinate axes at $(-2, 0)$ and $(0, 2)$.[3] Moreover, since the distance from $(-2, 2)$ to either of $(-2, 0)$ or $(0, 2)$ is 2, the radius of the circle is 2. Per Equation 8.3, our answer is $(x - (-2))^2 + (y - 2)^2 = (2)^2$ or $(x + 2)^2 + (y - 2)^2 = 4$. □

In number 1b above, we needed to transform a given equation into the standard form as stated in Equation 8.3. We record these steps below. Note that given an equation that represents a circle, *both* variables need to be squared and the squared terms must have the *same* coefficients.

To Write the Equation of a Circle in Standard Form

1. Group common variables together on one side of the equation and put the constant on the other.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

It is possible to obtain equations like $(x - 3)^2 + (y + 1)^2 = 0$ or $(x - 3)^2 + (y + 1)^2 = -1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations.

We close this section with a brief discussion of the so-called *Unit Circle*.[4]

Definition 8.3. The **Unit Circle** is the circle centered at $(0, 0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2 + y^2 = 1$.

In some ways, we may think of the Unit Circle as the progenitor of all circles. Indeed, if we divide both sides of Equation 8.3 by r^2, we obtain the alternate standard form of a circle below.

[3] Recall that for every point P on the circle, the tangent line at P is perpendicular to the radial line containing the center and P. Since the circle is tangent to the x-axis at $(-2, 0)$, the center must lie on a line perpendicular to the x-axis which contains $(-2, 0)$ or $x = -2$. Likewise, the circle is tangent to the y-axis at $(0, 2)$, the center must lie on $y = 2$. Hence the center is $(-2, 2)$.

[4] Widely regarded as the most important circle in all of mathematics.

Equation 8.4. The Alternate Standard Equation of a Circle: The equation of a circle with center (h, k) and radius $r > 0$ is

$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$

Taking this one step further, we may rewrite Equation 8.4 as

$$\left(\frac{x-h}{r}\right)^2 + \left(\frac{y-k}{r}\right)^2 = 1.$$

Hence, every circle can be obtained from the Unit Circle via the transformations discussed in Section 5.4.[5] Our last example has us find some important points on the the Unit Circle.

Example 8.3.2. Find the points on the unit circle with y-coordinate $\frac{\sqrt{3}}{2}$.

Solution. Note that all points (x, y) on the Unit Circle satisfy the equation $x^2 + y^2 = 1$. Hence, our first step is to replace y with $\frac{\sqrt{3}}{2}$ and solve for x.

$$\begin{aligned}
x^2 + y^2 &= 1 \\
x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\
\frac{3}{4} + x^2 &= 1 \\
x^2 &= \frac{1}{4} \\
x &= \pm\sqrt{\frac{1}{4}} \quad \text{extract square roots} \\
x &= \pm\frac{1}{2}
\end{aligned}$$

We find $x = \pm\frac{1}{2}$ so our final answers are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. $\square$

[5]See Exercise 28.

8.3.1 Exercises

In Exercises 1 - 6, graph the circle in the xy-plane. Find the center and radius.

1. $(x+1)^2 + (y+5)^2 = 100$
2. $(x-4)^2 + (y+2)^2 = 9$
3. $(x+3)^2 + \left(y - \frac{7}{13}\right)^2 = \frac{1}{4}$
4. $(x-5)^2 + (y+9)^2 = (\ln(8))^2$
5. $(x+e)^2 + \left(y - \sqrt{2}\right)^2 = \pi^2$
6. $(x-\pi)^2 + \left(y - e^2\right)^2 = 91^{\frac{2}{3}}$

In Exercises 7 - 12, complete the square in order to put the equation into standard form. Identify the center and the radius or explain why the equation does not represent a circle.[6]

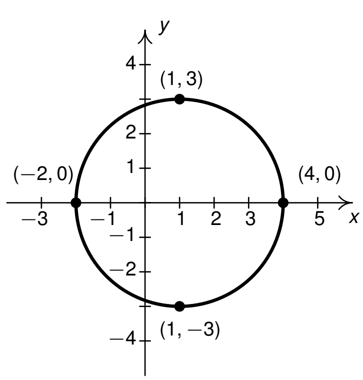
7. $x^2 - 4x + y^2 + 10y = -25$
8. $-2x^2 - 36x - 2y^2 - 112 = 0$
9. $3x^2 + 3y^2 + 24x - 30y - 3 = 0$
10. $x^2 + y^2 + 5x - y - 1 = 0$
11. $x^2 + x + y^2 - \frac{6}{5}y = 1$
12. $4x^2 + 4y^2 - 24y + 36 = 0$
13. For each of the odd numbered equations given in Exercises 1 - 11, find two or more explicit functions of x represented by each of the equations. (See Example 8.2.2 in Section 8.2.)

In Exercises 14 - 17, graph each function by recognizing it as a semicircle.

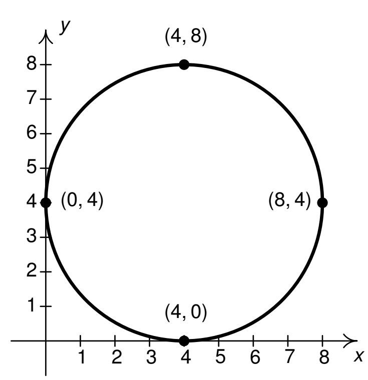
14. $f(x) = \sqrt{4 - x^2}$
15. $g(x) = -\sqrt{6x - x^2}$
16. $f(x) = -\sqrt{3 - 2x - x^2}$
17. $g(x) = -2 + \sqrt{9 - x^2}$

In Exercises 18 - 21, find an equation for the circle or semicircle whose graph is given.

18.

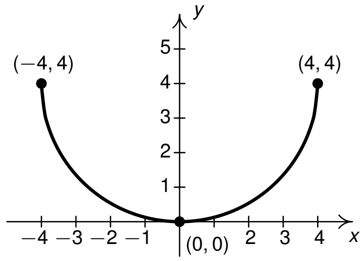


19.

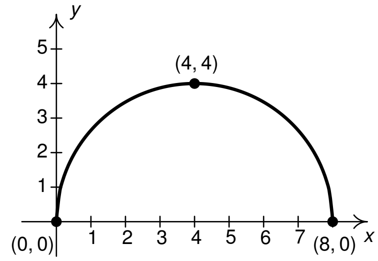


[6]. . . assuming the equation were graphed in the xy-plane.

20.



21.



In Exercises 22 - 25, find the standard equation of the circle which satisfies the given criteria.

22. center $(3, 5)$, passes through $(-1, -2)$

23. center $(3, 6)$, passes through $(-1, 4)$

24. endpoints of a diameter: $(3, 6)$ and $(-1, 4)$

25. endpoints of a diameter: $\left(\frac{1}{2}, 4\right)$, $\left(\frac{3}{2}, -1\right)$

26. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height is 136 feet.[7] Find an equation for the wheel assuming that its center lies on the y-axis and that the ground is the x-axis.

27. Verify that the following points lie on the Unit Circle:
$(\pm 1, 0)$, $(0, \pm 1)$, $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$, $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ and $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$

28. Discuss with your classmates how to obtain the alternate standard equation of a circle, Equation 8.4, from the equation of the Unit Circle, $x^2 + y^2 = 1$ using the transformations discussed in Section 5.4. (Thus every circle is just a few transformations away from the Unit Circle.)

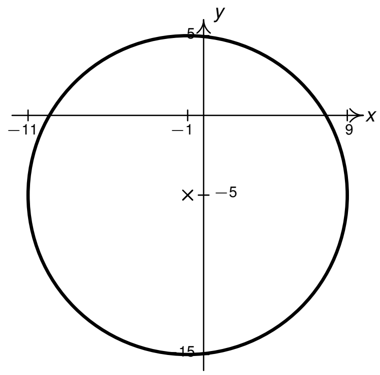
29. Find a one-to-one function whose graph is half of a circle.

HINT: Think piecewise . . .

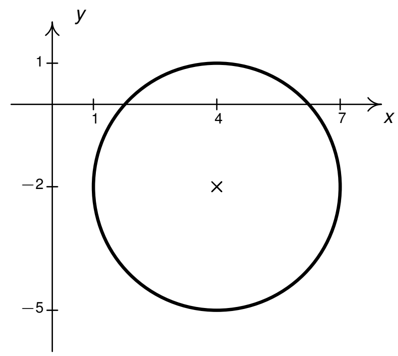
[7]Source: Cedar Point's webpage.

8.3.2 Answers

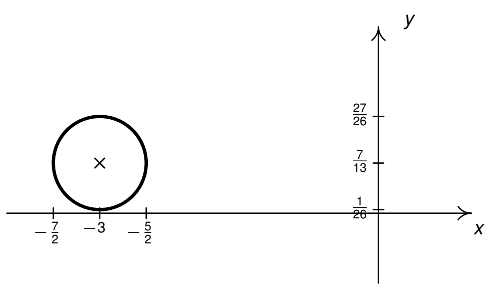
1. Center $(-1,-5)$, radius 10



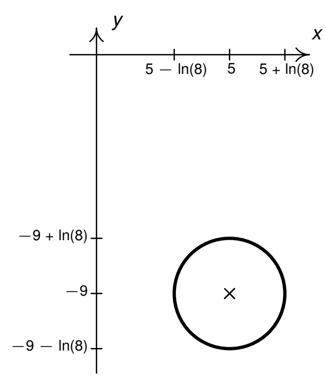
2. Center $(4,-2)$, radius 3



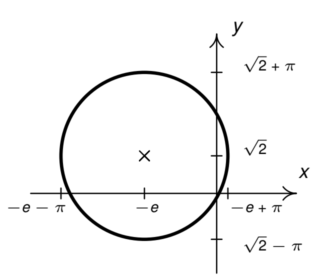
3. Center $\left(-3, \frac{7}{13}\right)$, radius $\frac{1}{2}$



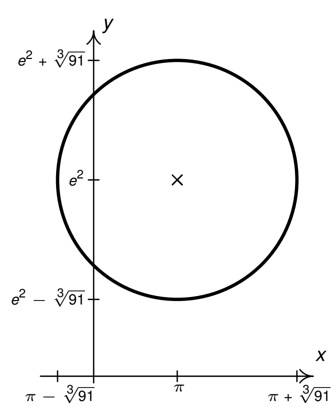
4. Center $(5,-9)$, radius $\ln(8)$



5. Center $\left(-e, \sqrt{2}\right)$, radius π



6. Center (π, e^2), radius $\sqrt[3]{91}$



7. $(x-2)^2+(y+5)^2=4$
Center $(2,-5)$, radius $r=2$

8. $(x+9)^2+y^2=25$
Center $(-9,0)$, radius $r=5$

9. $(x+4)^2+(y-5)^2=42$
Center $(-4,5)$, radius $r=\sqrt{42}$

10. $\left(x+\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{30}{4}$
Center $\left(-\frac{5}{2},\frac{1}{2}\right)$, radius $r=\frac{\sqrt{30}}{2}$

11. $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{5}\right)^2=\frac{161}{100}$
Center $\left(-\frac{1}{2},\frac{3}{5}\right)$, radius $r=\frac{\sqrt{161}}{10}$

12. $x^2+(y-3)^2=0$
This is not a circle.

13.

For number 1:

- $f(x)=-5+\sqrt{99-2x-x^2}$ represents the upper semicircle.
- $g(x)=-5-\sqrt{99-2x-x^2}$ represents the lower semicircle.

For number 3:

- $f(x)=\frac{7}{13}+\frac{1}{2}\sqrt{-4x^2-24x-35}$ represents the upper semicircle.
- $g(x)=\frac{7}{13}-\frac{1}{2}\sqrt{-4x^2-24x-35}$ represents the lower semicircle.

For number 5:

- $f(x)=\sqrt{2}+\sqrt{\pi^2-e^2-2ex-x^2}$ represents the upper semicircle.
- $g(x)=\sqrt{2}-\sqrt{\pi^2-e^2-2ex-x^2}$ represents the lower semicircle.

For number 7:

- $f(x)=-5+\sqrt{4x-x^2}$ represents the upper semicircle.
- $g(x)=-5-\sqrt{4x-x^2}$ represents the lower semicircle.

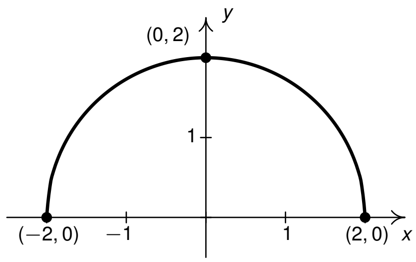
For number 9:

- $f(x)=5+\sqrt{26-8x-x^2}$ represents the upper semicircle.
- $g(x)=5-\sqrt{26-8x-x^2}$ represents the lower semicircle.

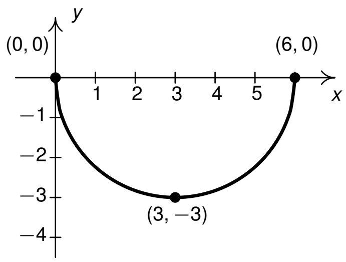
For number 11:

- $f(x)=\frac{3}{5}+\frac{1}{5}\sqrt{34-25x-25x^2}$ represents the upper semicircle.
- $g(x)=\frac{3}{5}-\frac{1}{5}\sqrt{34-25x-25x^2}$ represents the lower semicircle.

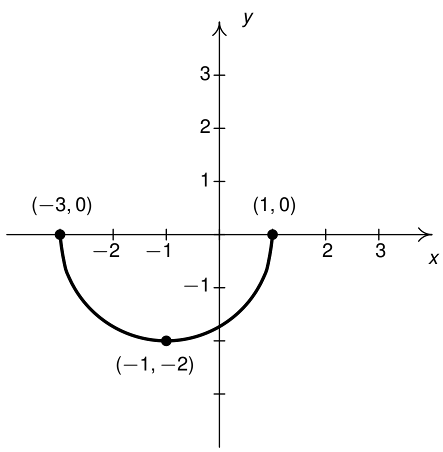
14. $f(x) = \sqrt{4 - x^2}$



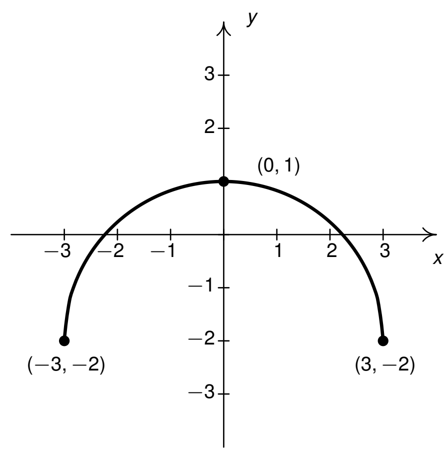
15. $g(x) = -\sqrt{6x - x^2}$



16. $f(x) = -\sqrt{3 - 2x - x^2}$



17. $g(x) = -2 + \sqrt{9 - x^2}$



18. $(x - 1)^2 + y^2 = 9$

19. $(x - 4)^2 + (y - 4)^2 = 16$

20. $y = 4 - \sqrt{16 - x^2}$

21. $y = \sqrt{8x - x^2}$

22. $(x - 3)^2 + (y - 5)^2 = 65$

23. $(x - 3)^2 + (y - 6)^2 = 20$

24. $(x - 1)^2 + (y - 5)^2 = 5$

25. $(x - 1)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{13}{2}$

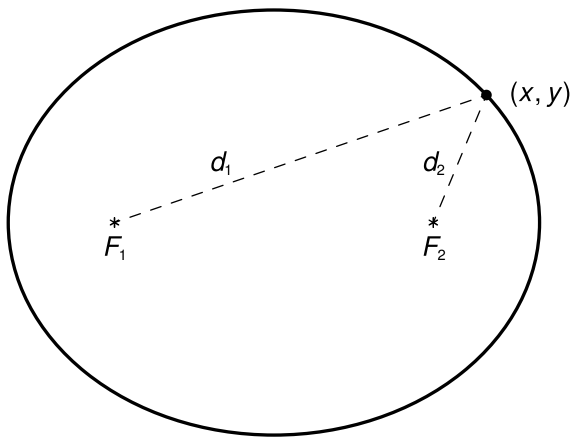
26. $x^2 + (y - 72)^2 = 4096$

8.4 Ellipses

In the definition of a circle, Definition 8.2, we fixed a point called the **center** and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance d to use in our definition.

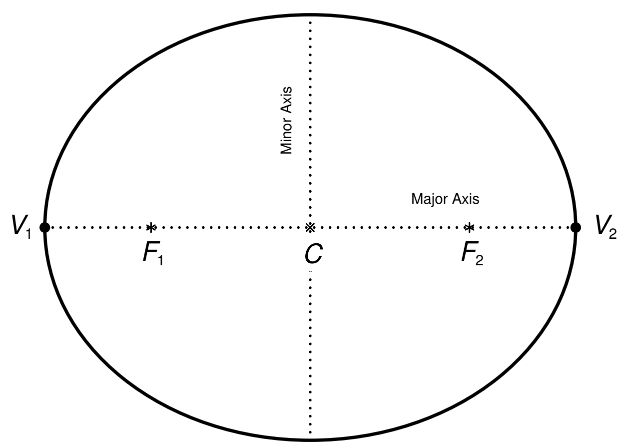
Definition 8.4. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci**[a] of the ellipse.

[a] the plural of 'focus'



$d_1 + d_2 = d$ for all (x, y) on the ellipse

We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse. Each ellipse has an assortment of parameters associated with it which we sketch below.

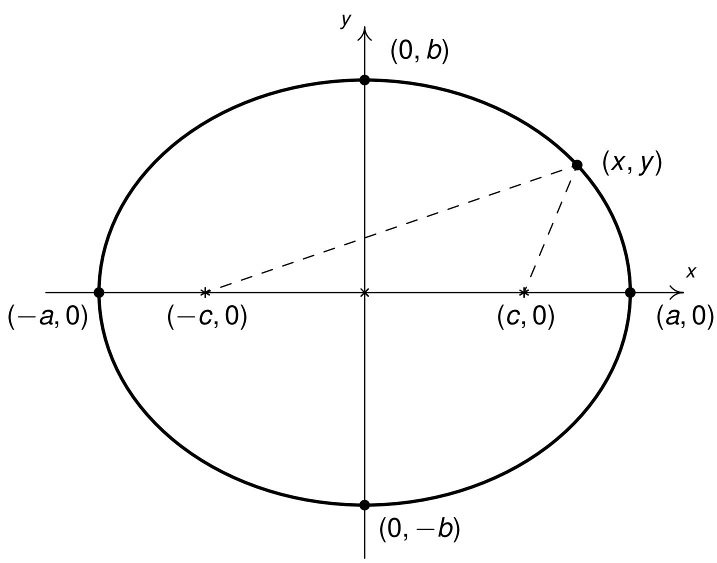


An ellipse with center C; foci F_1, F_2; and vertices V_1, V_2

As depicted above, the **center** of the ellipse is the midpoint of the line segment connecting the two foci. The **major axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The **minor axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The **vertices** of an ellipse are the points of the ellipse which lie on the major axis.

Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. Also note that the major axis is the longer of the two axes through the center, hence the moniker 'major.' Likewise, the minor axis is the shorter of the two, whence the adjective 'minor.'

In order to derive the standard equation of an ellipse, we assume that the ellipse has its center at $(0,0)$, its major axis along the x-axis, and has foci $(c,0)$ and $(-c,0)$ and vertices $(-a,0)$ and $(a,0)$. We will label the y-intercepts of the ellipse as $(0,b)$ and $(0,-b)$ (We assume a, b, and c are all positive numbers.)



Note that since $(a,0)$ is on the ellipse, it must satisfy the conditions of Definition 8.4. That is, the distance from $(-c,0)$ to $(a,0)$ plus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all of these points lie on the x-axis, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) + \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a+c)+(a-c) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance d mentioned in the definition of the ellipse is none other than the length of the major axis. We now use that fact $(0,b)$ is on the ellipse, along with the fact that $d=2a$ to get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (0,b) + \text{distance from } (c,0) \text{ to } (0,b) &= 2a \\
\sqrt{(0-(-c))^2+(b-0)^2} + \sqrt{(0-c)^2+(b-0)^2} &= 2a \\
\sqrt{b^2+c^2} + \sqrt{b^2+c^2} &= 2a \\
2\sqrt{b^2+c^2} &= 2a \\
\sqrt{b^2+c^2} &= a
\end{aligned}$$

From this, we get $a^2 = b^2 + c^2$, or $b^2 = a^2 - c^2$, which will prove useful later. Now consider a point (x, y) on the ellipse. Applying Definition 8.4, we get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (x,y) + \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\
\sqrt{(x-(-c))^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} &= 2a \\
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a
\end{aligned}$$

In order to make sense of this situation, we need to make good use of Intermediate Algebra.

$$\begin{aligned}
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a \\
\sqrt{(x+c)^2+y^2} &= 2a - \sqrt{(x-c)^2+y^2} \\
\left(\sqrt{(x+c)^2+y^2}\right)^2 &= \left(2a - \sqrt{(x-c)^2+y^2}\right)^2 \\
(x+c)^2+y^2 &= 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2 + (x-c)^2 - (x+c)^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2 - 4cx \\
a\sqrt{(x-c)^2+y^2} &= a^2 - cx \\
\left(a\sqrt{(x-c)^2+y^2}\right)^2 &= \left(a^2 - cx\right)^2 \\
a^2\left((x-c)^2+y^2\right) &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - c^2x^2 + a^2y^2 &= a^4 - a^2c^2 \\
\left(a^2-c^2\right)x^2 + a^2y^2 &= a^2\left(a^2-c^2\right)
\end{aligned}$$

We are nearly finished. Recall that $b^2 = a^2 - c^2$ so that

$$\begin{aligned}
\left(a^2-c^2\right)x^2 + a^2y^2 &= a^2\left(a^2-c^2\right) \\
b^2x^2 + a^2y^2 &= a^2b^2 \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1
\end{aligned}$$

This equation is for an ellipse centered at the origin. To get the formula for the ellipse centered at (h, k), we could use the transformations from Section 5.4 or re-derive the equation using Definition 8.4 and the distance formula to obtain the formula below.

Equation 8.5. The Standard Equation of an Ellipse:
For positive unequal numbers a and b, the equation of an ellipse with center (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Some remarks about Equation 8.5 are in order. First note that the values a and b determine how far in the x and y directions, respectively, one counts from the center to arrive at points on the ellipse.

Also note that if $a > b$, then we have an ellipse whose major axis is horizontal, and hence, the foci lie to the left and right of the center. In this case, as we've seen in the derivation, the distance from the center to the focus, c, can be found by $c = \sqrt{a^2 - b^2}$.

If $b > a$, the roles of the major and minor axes are reversed, and the foci lie above and below the center. In this case, $c = \sqrt{b^2 - a^2}$. In either case, it's best to just remember that c is the distance from the center to each focus, and, formulaically, $c = \sqrt{\text{bigger denominator} - \text{smaller denominator}}$.

Finally, it is worth mentioning that if we compare Equation 8.5 with the alternate standard equation of the circle, Equation 8.4, the only difference between the forms is that with a circle, the denominators are the same, and with an ellipse, they are different.

If we take a transformational approach, we can consider both Equations 8.5 and 8.4 as shifts and stretches of the Unit Circle $x^2 + y^2 = 1$ in Definition 8.3. Replacing x with $(x - h)$ and y with $(y - k)$ causes the usual horizontal and vertical shifts. Replacing x with $\frac{x}{a}$ and y with $\frac{y}{b}$ causes the usual vertical and horizontal stretches.

In other words, it is perfectly fine to think of an ellipse as the deformation of a circle in which the circle is stretched farther in one direction than the other.

Example 8.4.1.

1. Graph each of the following equations below in the xy-plane. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.

   (a) $25(x+1)^2 + 9(y-2)^2 = 225$.

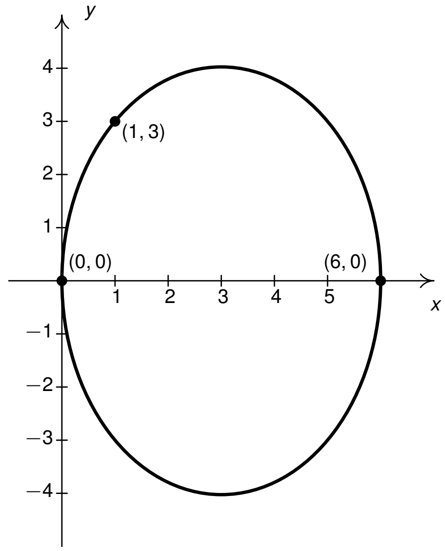
   (b) $x^2 + 4y^2 - 2x + 24y + 33 = 0$.

2. Graph $f(x) = 1 + 2\sqrt{-x^2 - 4x - 3}$

3. Find the standard form of the equation of an ellipse which satisfies the following characteristics:

   (a) the foci are at $(2, 1)$ and $(4, 1)$ and vertex $(0, 1)$.

(b) the ellipse graphed below:



Solution.

1. (a) To put $25(x+1)^2 + 9(y-2)^2 = 225$ in the form prescribed by Equation 8.5, we rewrite the quantity $(x+1)$ as $(x-(-1))$ and divide through by 225 to obtain an expression equal to 1:

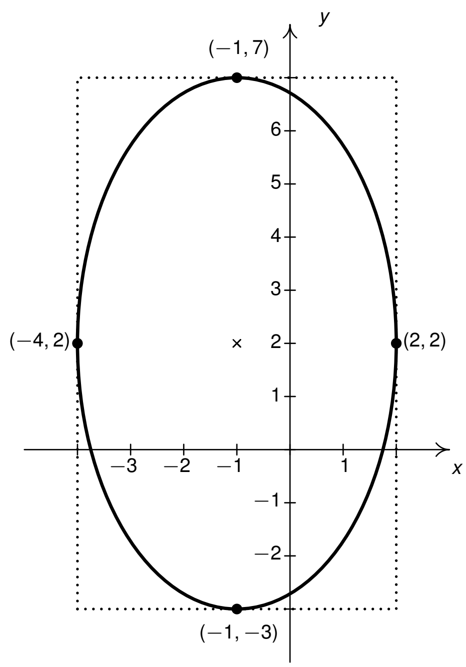
$$25(x+1)^2 + 9(y-2)^2 = 225 \rightarrow \frac{(x-(-1))^2}{9} + \frac{(y-2)^2}{25} = 1 \leftrightarrow \frac{(x-(-1))^2}{(3)^2} + \frac{(y-2)^2}{(5)^2} = 1.$$

We identify $h = -1$ and $k = 2$, so the center of the ellipse is $(-1, 2)$. We have $a = 3$ so we move 3 units left and right from the center to obtain two points on the ellipse: $(-1-3, 2) = (-4, 2)$ and $(-1+3, 2) = (2, 2)$. Likewise, since $b = 5$, we move up and down 5 units from the center to find two more points on the ellipse: $(-1, 2+5) = (-1, 7)$ and $(-1, 2-5) = (-1, -3)$.

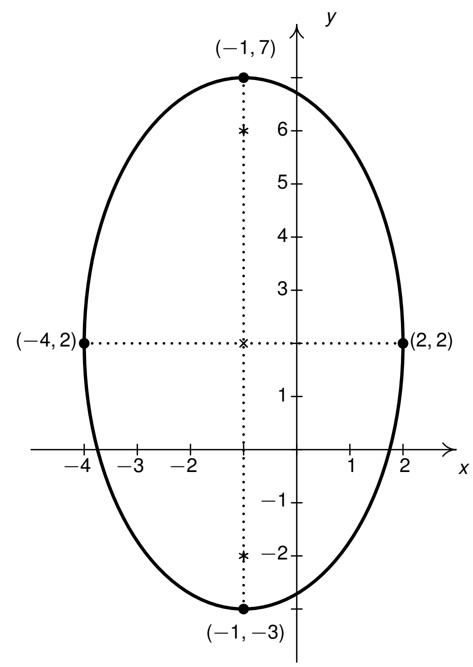
As an aid to sketching, we draw a rectangle matching this description, called a **guide rectangle**, and sketch the ellipse inside this rectangle as seen below on the left.

Since we moved farther from the center in the y direction than in the x direction, the major axis will lie along the vertical line $x = -1$, while the minor axis lies along the horizontal line $y = 2$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1, 7)$ and $(-1, -3)$, and the endpoints of the minor axis are $(-4, 2)$ and $(2, 2)$. (Notice these points are the four points we used to draw the guide rectangle.)

To find the foci, we find $c = \sqrt{25-9} = \sqrt{16} = 4$, which means the foci lie 4 units from the center. Since the major axis is vertical, the foci lie 4 units above and below the center, at $(-1, 2-4) = (-1, -2)$ and $(-1, 2+4) = (-1, 6)$. Our final graph appears below on the right.



The graph of $25(x+1)^2 + 9(y-2)^2 = 225$.



The graph of $25(x+1)^2 + 9(y-2)^2 = 225$.

(b) In the equation $x^2 + 4y^2 - 2x + 24y + 33 = 0$ we have a sum of two squares with unequal coefficients, it's a good bet we have an ellipse on our hands.[1]

In order to put this equation into the form stated in Equation 8.5, we need to complete both squares and then divide, if necessary, to get the right-hand side equal to 1:

$$\begin{aligned}
x^2 + 4y^2 - 2x + 24y + 33 &= 0 && \\
x^2 - 2x + 4y^2 + 24y &= -33 && \text{subtract 33 from both sides} \\
x^2 - 2x + 4\left(y^2 + 6y\right) &= -33 && \text{factor out leading coefficients} \\
\left(x^2 - 2x + \underline{1}\right) + 4\left(y^2 + 6y + \underline{\underline{9}}\right) &= -33 + \underline{1} + 4(\underline{\underline{9}}) && \text{complete the squares} \\
(x-1)^2 + 4(y+3)^2 &= 4 && \text{factor} \\
\frac{(x-1)^2 + 4(y+3)^2}{4} &= \frac{4}{4} && \text{divide through by 4} \\
\frac{(x-1)^2}{4} + (y+3)^2 &= 1 && \\
\frac{(x-1)^2}{(2)^2} + \frac{(y-(-3))^2}{(1)^2} &= 1 && \text{rewrite in the form of Equation 8.5}
\end{aligned}$$

[1] Recall the equation of a parabola has exactly *one* squared variable and the equation of a circle has two squared variables, but with *identical* coefficients.

Now that this equation is in the standard form of Equation 8.5, identify $h = 1$ and $k = -3$ so our ellipse is centered at $(1, -3)$. With $a = 2$, we move 2 units left and right from the center to get two points on the ellipse: $(1 - 2, -3) = (-1, -3)$ and $(1 + 2, -3) = (3, -3)$. Since $b = 1$, we move 1 unit up and down from the center to obtain two additional points on the ellipse: $(1, -3 + 1) = (1, -2)$ and $(1, -3 - 1) = (1, -4)$.

Since we moved farther from the center in the x direction than in the y direction, the major axis will lie along the horizontal line $y = -3$ so the minor axis lies along the vertical line $x = 1$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1, -3)$ and $(3, -3)$, and the endpoints of the minor axis are $(1, -2)$ and $(1, -4)$.

To find the foci, we find $c = \sqrt{4 - 1} = \sqrt{3}$, which means the foci lie $\sqrt{3}$ units from the center. Since the major axis is horizontal, the foci lie $\sqrt{3}$ units to the left and right of the center, at $(1 - \sqrt{3}, -3)$ and $(1 + \sqrt{3}, -3)$. Plotting all of this information gives the graph below on the left.

2. At first glance, it doesn't seem as if the function $f(x) = 1 + 2\sqrt{-x^2 - 4x - 3}$ will have *any* graph owing to the presence of the '$-$' signs which decorate *all* of the terms beneath the radical. However, since x is a *variable*, $-x^2 - 4x - 3$ is not necessarily negative.[2]

   Recall to graph the function $f(x) = 1+2\sqrt{-x^2 - 4x - 3}$, we graph the equation $y = 1+2\sqrt{-x^2 - 4x - 3}$. To make sense of this equation in the context of this chapter, we first isolate, then eliminate, the square root in order to obtain a quadratic equation in one or more variables.

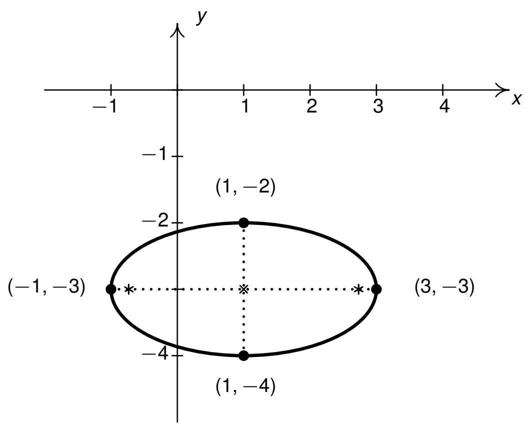
   From $y = 1+2\sqrt{-x^2 - 4x - 3}$, we get $y-1 = 2\sqrt{-x^2 - 4x - 3}$ so that $(y-1)^2 = (2\sqrt{-x^2 - 4x - 3})^2$. Hence, $(y - 1)^2 = 4(-x^2 - 4x - 3)$ which is equivalent to $4x^2 + 16x + (y - 1)^2 = -12$. At this point, we see we have the sum of two squared variables with unequal coefficients present which indicates an ellipse, so we complete the square on x and rewrite the equation so it fits Equation 8.5:

$$\begin{aligned}
4x^2 + 16x + (y-1)^2 &= -12 && \\
4\left(x^2 + 4x\right) + (y-1)^2 &= -12 && \text{factor out leading coefficient of } x^2 \\
4\left(x^2 + 4x + \underline{4}\right) + (y-1)^2 &= -12 + 4(\,\underline{4}\,) && \text{complete the square in } x \\
4(x+2)^2 + (y-1)^2 &= 4 && \text{factor} \\
\frac{4(x+2)^2 + (y-1)^2}{4} &= \frac{4}{4} && \text{divide through by 4} \\
(x+2)^2 + \frac{(y-1)^2}{4} &= 1 && \\
\frac{(x-(-2))^2}{(1)^2} + \frac{(y-1)^2}{(2)^2} &= 1 && \text{rewrite in the form of Equation 8.5}
\end{aligned}$$

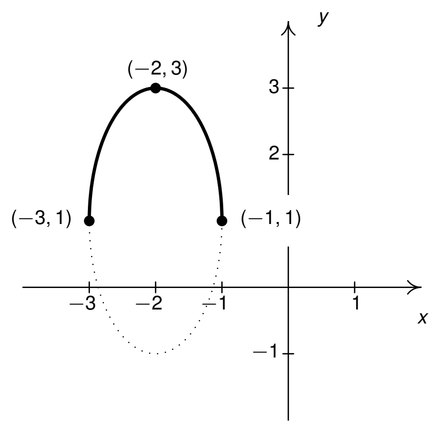
   We identify $h = -2$ and $k = 1$ so the ellipse is centered at $(-2, 1)$. With $a = 1$, we move 1 unit to the left and to the right an obtain the points $(-2 - 1, 1) = (-3, 1)$ and $(-2 + 1, 1) = (-1, 1)$. With $b = 2$, we move 2 units up and down from the center to obtain two more points on the graph of the ellipse: $(-2, 1 + 2) = (-2, 3)$ and $(-2, 1 - 2) = (-2, -1)$.

[2]Indeed, we leave it to the reader to show $-x^2 - 4x - 3 \geq 0$ on $[-3, -1]$.

However, the graph of $f(x) = 1 + 2\sqrt{-x^2 - 4x - 3}$ cannot be the *entire* ellipse, else it would violate the vertical line test. This means the graph of f must be a *portion* of the ellipse. Since, by definition $\sqrt{-x^2 - 4x - 3} \geq 0$, we know $f(x) = 1 + 2\sqrt{-x^2 - 4x - 3} \geq 1$. Hence, the graph of f must be the *upper* half of the ellipse, as denoted below on the right.

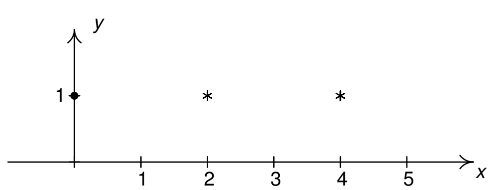


The graph of $x^2 + 4y^2 - 2x + 24y + 33 = 0$.



The graph of $f(x) = 1 + 2\sqrt{-x^2 - 4x - 3}$.

3. (a) We plot the information given to us below and notice immediately that the major axis is horizontal, which means $a > b$. Since the center is the midpoint of the foci, we know the center of the ellipse is $(3, 1)$ which means $h = 3$ and $k = 1$. Since one vertex is at $(0, 1)$, which is 3 units from the center, we have that $a = 3$, so $a^2 = 9$. At this point, all that remains is to find b^2.



Since the foci, $(2, 1)$ and $(4, 1)$, are 1 unit away from the center, we have $c = 1$. Putting this together with the fact that $a > b$, we get $c = \sqrt{a^2 - b^2}$, or $1 = \sqrt{9 - b^2}$. Squaring both sides gives $1 = 9 - b^2$ or $b^2 = 8$. Feeding all of this data into Equation 8.5, we get our final answer:

$$\frac{(x-3)^2}{9} + \frac{(y-1)^2}{8} = 1.$$

(b) From the diagram, we infer the ellipse is taller than it is wide. More specifically, the labeled points $(0, 0)$ and $(6, 0)$ are the endpoints of the minor axis. This gives the center is $(3, 0)$, so $h = 3$ and $k = 0$. Moreover, we have $a = 3$.

While it certainly *appears* that the vertices are $(3, \pm 4)$, in which case we'd have $b = 4$, these points aren't labeled. Instead, we use the labeled point $(1, 3)$ to calculate b^2. At this stage, we know the equation of the ellipse is

$$\frac{(x-3)^2}{9} + \frac{y^2}{b^2} = 1,$$

so upon substituting $x = 1$ and $y = 3$, we obtain $\frac{4}{9} + \frac{9}{b^2} = 1$. Solving this equation, we get $b^2 = \frac{81}{5}$. Hence, our final answer is:

$$\frac{(x-3)^2}{9} + \frac{5(y-1)^2}{81} = 1.$$

Note the vertices of the ellipse are $\left(3, \pm\sqrt{\frac{81}{5}}\right) \approx (3, \pm 4)$ but they are not *exactly* $(3, \pm 4)$. □

As seen in Example 8.4.1 above, it is often necessary to algebraically manipulate a given equation into the standard form of Equation 8.5 in order to graph. We summarize one approach below.

To Write the Equation of an Ellipse in Standard Form

1. Group common variables together on one side of the equation and put the constant on the other.
2. Complete the square on both variables as needed.
3. Divide both sides, if needed, to obtain 1 on one side of the equation.

If we think of a circle as being 'perfectly round,' then ellipses, being deformed circles, have varying degrees of 'roundness.' We quantify this idea with the notion of *eccentricity* defined formally below.

Definition 8.5. The **eccentricity** of an ellipse, denoted e, is the following ratio:

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

In an ellipse, the foci are closer to the center than the vertices, so $0 < e < 1$. The ellipse below on left has eccentricity $e \approx 0.66$; for the ellipse below on the right, $e \approx 0.98$. In general, the closer the eccentricity is to 0, the less 'eccentric' or more 'circular' the ellipse appears. On the other hand, the closer the eccentricity is to 1, the more 'eccentric' the ellipse is and it appears less 'circular.'

$e \approx 0.66$ $\qquad$ $e \approx 0.98$

According to Kepler's Laws of Planetary Motion, each planet orbits the Sun in an elliptical path with the Sun at one focus. The eccentricity is therefore an important orbital parameter. We investigate the orbit of Mercury in the following example.

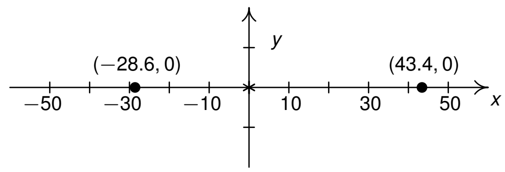
Example 8.4.2. According to NASA, Mercury orbits the Sun in an elliptical orbit. If at perihelion,[3] Mercury is 28.6 megamiles from the Sun and at aphelion,[4] Mercury is 43.4 megamiles from the Sun, find the eccentricity of Mercury's orbit, rounded to three decimal places.

[3]the closest Mercury is to the Sun

[4]the farthest Mercury is from the Sun

Solution. Per Kepler's Laws, the orbit of Mercury is an ellipse with the Sun at one focus. Since we are told to assume the Sun is positioned at $(0,0)$, we are free to choose if the major axis of the ellipse lies on the x- or y-axis. We choose the former.

With the information given, we know one vertex is 28.6 units away from the focus and the other is 43.4 units. Again, we are free to chose which direction is which, so we decide to put one vertex at $(-28.6, 0)$ and the other is $(43.4, 0)$. Schematically, we have:

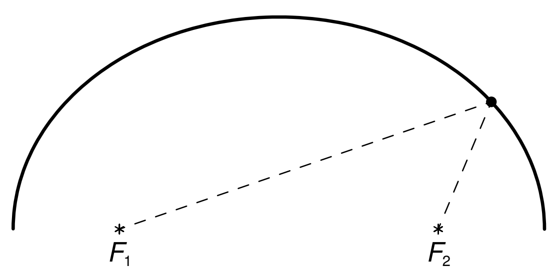


Since the center is the midpoint of the vertices, we find the center to be $(7.4, 0)$. This means the focus $(0,0)$ is 7.4 units from the center and each vertex is 36 units from the center. This is precisely what we need to determine the eccentricity of Mercury's orbit:

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}} = \frac{7.4}{36} \approx 0.205.$$

To our delight, we find answer agrees with NASA. With an eccentricity of 0.205, we expect Mercury's orbit to be fairly 'round.' In Exercise 33, we invite the reader to find the equation of Mercury's orbit to confirm this conclusion graphically. □

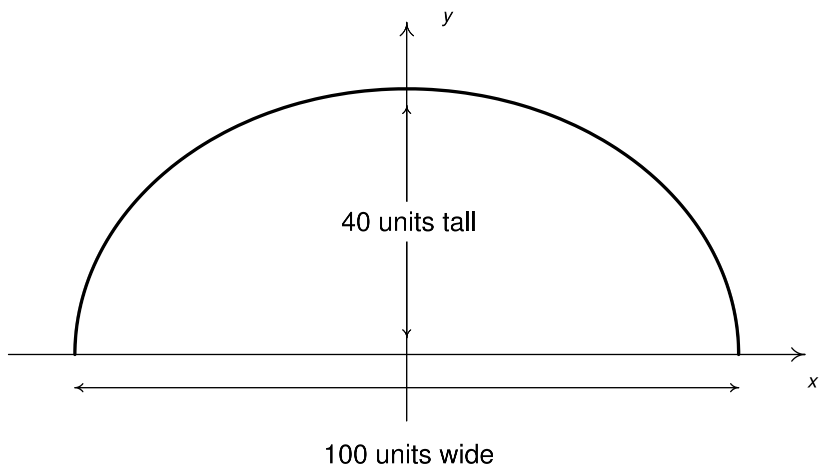
As with parabolas, ellipses have a reflective property. If we imagine the dashed lines below representing sound waves, then it can be shown that the waves emanating from one focus reflect off the top of the ellipse and head towards the other focus.



Such geometry is exploited in the construction of so-called 'Whispering Galleries'. If a person whispers at one focus, a person standing at the other focus will hear the first person as if they were standing right next to them. We explore the Whispering Galleries in our last example.

Example 8.4.3. Jamie and Jason want to exchange secrets (terrible secrets) from across a crowded whispering gallery. Recall that a whispering gallery is a room which, in cross section, is half of an ellipse. If the room is 40 feet high at the center and 100 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

Solution. Ultimately, we are looking for information about the relative position of foci and the vertices, so we assume the whispering gallery can be represented by the upper half of an ellipse centered at the origin with its major axis along the x-axis. Graphing the data yields the schematic below.



Since the ellipse is 100 units wide and 40 units tall, we get $a = 50$ and $b = 40$, respectively. From this, we get $c = \sqrt{50^2 - 40^2} = \sqrt{900} = 30$, which means the foci are 30 units from the center. Hence, the foci are $50 - 30 = 20$ units from the vertices so Jason and Jamie should stand 20 feet from opposite ends of the gallery to exchange their secrets in what amounts to very public privacy. $\square$

8.4.1 Exercises

In Exercises 1 - 8, graph the ellipse in the xy-plane. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.

1. $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
2. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$
3. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
4. $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
5. $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
6. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
7. $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
8. $\dfrac{(x-4)^2}{8} + \dfrac{(y-2)^2}{18} = 1$

In Exercises 9 - 14, put the equation in standard form. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.[5]

9. $9x^2 + 25y^2 - 54x - 50y - 119 = 0$
10. $12x^2 + 3y^2 - 30y + 39 = 0$
11. $5x^2 + 18y^2 - 30x + 72y + 27 = 0$
12. $x^2 - 2x + 2y^2 - 12y + 3 = 0$
13. $9x^2 + 4y^2 - 4y - 8 = 0$
14. $6x^2 + 5y^2 - 24x + 20y + 14 = 0$
15. For each of the odd numbered equations given in Exercises 1 - 13, find two or more explicit functions of x represented by each of the equations. (See Example 8.2.2 in Section 8.2.)

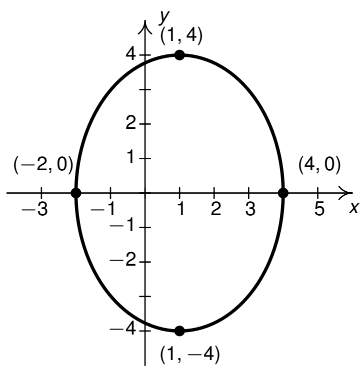
In Exercises 16 - 19, graph each function by recognizing it as a semi ellipse.

16. $f(x) = \sqrt{16 - 4x^2}$
17. $g(x) = -\frac{1}{2}\sqrt{6x - x^2}$
18. $f(x) = -2\sqrt{3 - 2x - x^2}$
19. $g(x) = -2 + 2\sqrt{9 - x^2}$

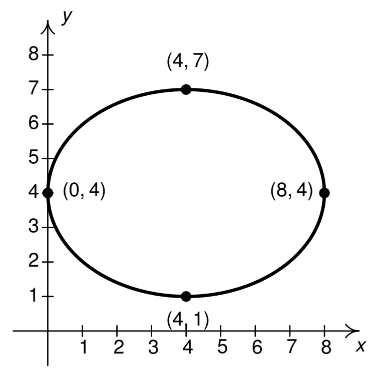
[5]. . . assuming the equation were graphed in the xy-plane.

In Exercises 20 - 23, find an equation for the ellipse or semi ellipse whose graph is given.

20.



21.



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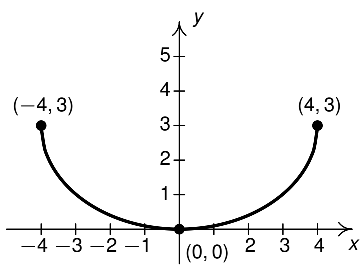
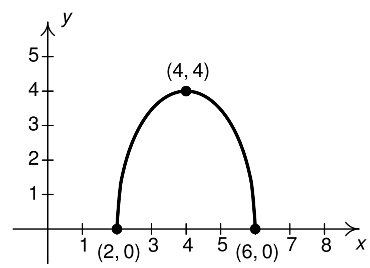


23.



In Exercises 24 - 29, find the standard form of the equation of the ellipse which has the given properties.

24. Center $(3, 7)$, Vertex $(3, 2)$, Focus $(3, 3)$
25. Foci $(0, \pm 5)$, Vertices $(0, \pm 8)$.
26. Foci $(\pm 3, 0)$, length of the Minor Axis 10
27. Vertices $(3, 2)$, $(13, 2)$; Endpoints of the Minor Axis $(8, 4)$, $(8, 0)$
28. Center $(5, 2)$, Vertex $(0, 2)$, eccentricity $\frac{1}{2}$
29. All points on the ellipse are in Quadrant IV except $(0, -9)$ and $(8, 0)$. (One might also say that the ellipse is "tangent to the axes" at those two points.)
30. Repeat Example 8.4.3 for a whispering gallery 200 feet wide and 75 feet tall.
31. An elliptical arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch. Compare your result with your answer to Exercise 26 in Section 8.2.

32. The Earth's orbit around the sun is an ellipse with the sun at one focus and eccentricity $e \approx 0.0167$. The length of the semimajor axis (that is, half of the major axis) is defined to be 1 astronomical unit (AU). The vertices of the elliptical orbit are given special names: 'aphelion' is the vertex farthest from the sun, and 'perihelion' is the vertex closest to the sun. Find the distance in AU between the sun and aphelion and the distance in AU between the sun and perihelion.

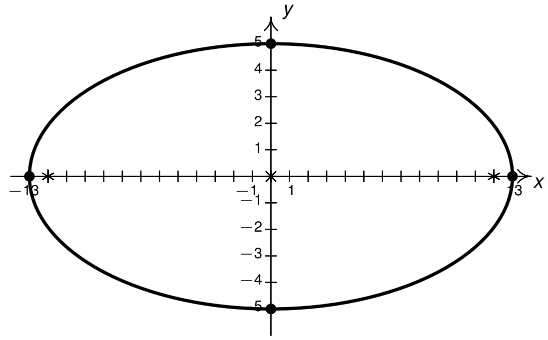
33. This exercise is a follow-up to Example 8.4.2. Find the equation of the ellipse which models the orbit of Mercury. Graph the ellipse using a graphing utility, and comment on the 'roundness' of the orbit.

34. Some famous examples of whispering galleries include St. Paul's Cathedral in London, England, National Statuary Hall in Washington, D.C., and The Cincinnati Museum Center. With the help of your classmates, research these whispering galleries. How does the whispering effect compare and contrast with the scenario in Example 8.4.3?

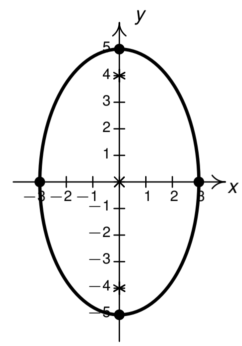
35. With the help of your classmates, research "extracorporeal shock-wave lithotripsy". It uses the reflective property of the ellipsoid to dissolve kidney stones.

8.4.2 Answers

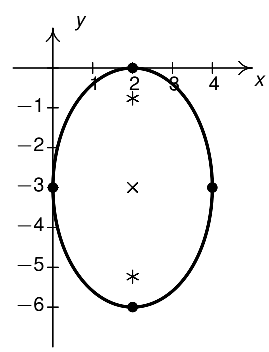
1. $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
   Center $(0, 0)$
   Major axis along $y = 0$
   Minor axis along $x = 0$
   Vertices $(13, 0)$, $(-13, 0)$
   Endpoints of Minor Axis $(0, -5)$, $(0, 5)$
   Foci $(12, 0)$, $(-12, 0)$
   $e = \frac{12}{13}$



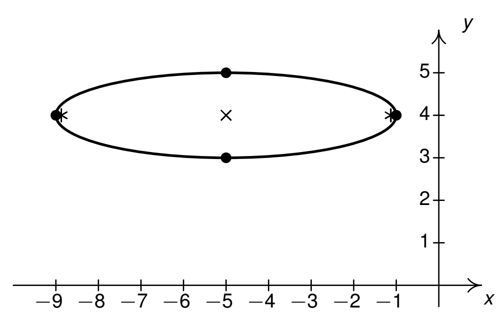
2. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$
   Center $(0, 0)$
   Major axis along $x = 0$
   Minor axis along $y = 0$
   Vertices $(0, 5)$, $(0, -5)$
   Endpoints of Minor Axis $(-3, 0)$, $(3, 0)$
   Foci $(0, -4)$, $(0, 4)$
   $e = \frac{4}{5}$



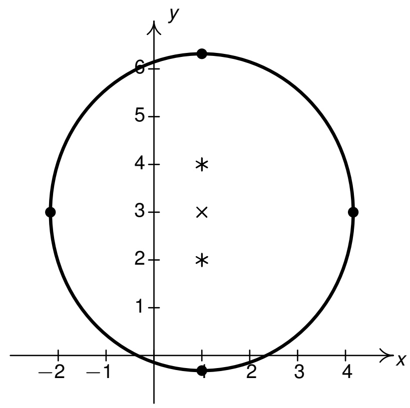
3. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
   Center $(2, -3)$
   Major axis along $x = 2$
   Minor axis along $y = -3$
   Vertices $(2, 0)$, $(2, -6)$
   Endpoints of Minor Axis $(0, -3)$, $(4, -3)$
   Foci $(2, -3 + \sqrt{5})$, $(2, -3 - \sqrt{5})$
   $e = \frac{\sqrt{5}}{3}$



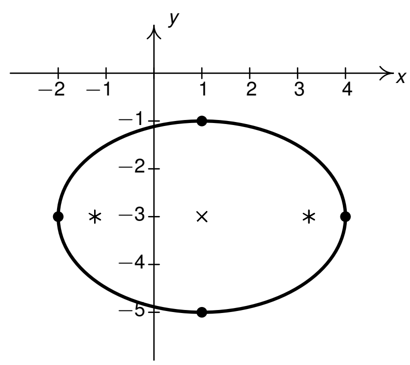
4. $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
   Center $(-5, 4)$
   Major axis along $y = 4$
   Minor axis along $x = -5$
   Vertices $(-9, 4)$, $(-1, 4)$
   Endpoints of Minor Axis $(-5, 3)$, $(-5, 5)$
   Foci $(-5 + \sqrt{15}, 4)$, $(-5 - \sqrt{15}, 4)$
   $e = \frac{\sqrt{15}}{4}$



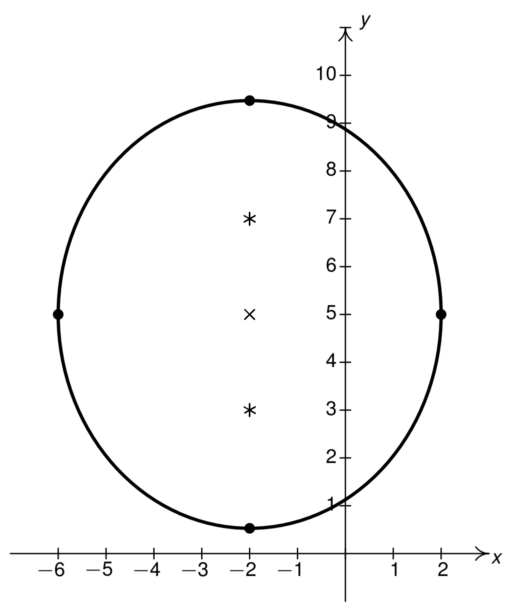
5. $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
Center $(1,3)$
Major axis along $x = 1$
Minor axis along $y = 3$
Vertices $(1, 3+\sqrt{11})$, $(1, 3-\sqrt{11})$
Endpoints of the Minor Axis
$(1-\sqrt{10}, 3)$, $(1+\sqrt{10}, 3)$
Foci $(1,2)$, $(1,4)$
$e = \frac{\sqrt{11}}{11}$



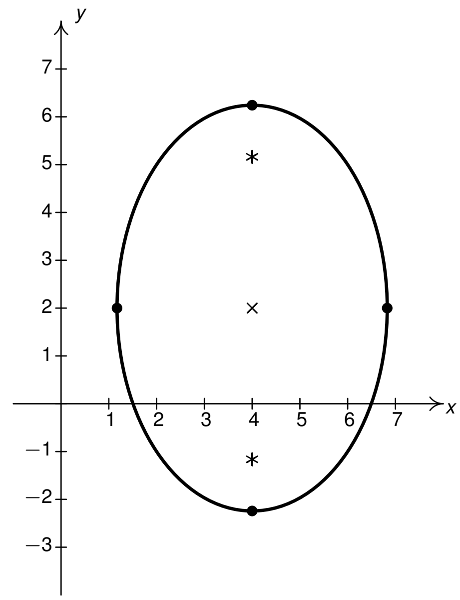
6. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
Center $(1,-3)$
Major axis along $y = -3$
Minor axis along $x = 1$
Vertices $(4,-3)$, $(-2,-3)$
Endpoints of the Minor Axis $(1,-1)$, $(1,-5)$
Foci $(1+\sqrt{5}, -3)$, $(1-\sqrt{5}, -3)$
$e = \frac{\sqrt{5}}{3}$



7. $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
Center $(-2,5)$
Major axis along $x = -2$
Minor axis along $y = 5$
Vertices $(-2, 5+2\sqrt{5})$, $(-2, 5-2\sqrt{5})$
Endpoints of the Minor Axis $(-6,5)$, $(2,5)$ Foci
$(-2,7)$, $(-2,3)$
$e = \frac{\sqrt{5}}{5}$



8. $\dfrac{(x-4)^2}{8}+\dfrac{(y-2)^2}{18}=1$
Center $(4,2)$
Major axis along $x=4$
Minor axis along $y=2$
Vertices $(4,2+3\sqrt{2})$, $(4,2-3\sqrt{2})$
Endpoints of the Minor Axis
$(4-2\sqrt{2},2)$, $(4+2\sqrt{2},2)$
Foci $(4,2+\sqrt{10})$, $(4,2-\sqrt{10})$
$e=\frac{\sqrt{5}}{3}$



9. $\dfrac{(x-3)^2}{25}+\dfrac{(y-1)^2}{9}=1$
Center $(3,1)$
Major Axis along $y=1$
Minor Axis along $x=3$
Vertices $(8,1)$, $(-2,1)$
Endpoints of Minor Axis $(3,4)$, $(3,-2)$
Foci $(7,1)$, $(-1,1)$
$e=\frac{4}{5}$

10. $\dfrac{x^2}{3}+\dfrac{(y-5)^2}{12}=1$
Center $(0,5)$
Major axis along $x=0$
Minor axis along $y=5$
Vertices $(0,5-2\sqrt{3})$, $(0,5+2\sqrt{3})$
Endpoints of Minor Axis $(-\sqrt{3},5)$, $(\sqrt{3},5)$
Foci $(0,2)$, $(0,8)$
$e=\frac{\sqrt{3}}{2}$

11. $\dfrac{(x-3)^2}{18}+\dfrac{(y+2)^2}{5}=1$
Center $(3,-2)$
Major axis along $y=-2$
Minor axis along $x=3$
Vertices $(3-3\sqrt{2},-2)$, $(3+3\sqrt{2},-2)$
Endpoints of Minor Axis $(3,-2+\sqrt{5})$, $(3,-2-\sqrt{5})$
Foci $(3-\sqrt{13},-2)$, $(3+\sqrt{13},-2)$
$e=\frac{\sqrt{26}}{6}$

12. $\dfrac{(x-1)^2}{16}+\dfrac{(y-3)^2}{8}=1$
Center $(1,3)$
Major Axis along $y=3$
Minor Axis along $x=1$
Vertices $(5,3)$, $(-3,3)$
Endpoints of Minor Axis $(1,3+2\sqrt{2})$, $(1,3-2\sqrt{2})$
Foci $(1+2\sqrt{2},3)$, $(1-2\sqrt{2},3)$
$e=\frac{\sqrt{2}}{2}$

13. $\dfrac{x^2}{1}+\dfrac{4\left(y-\frac{1}{2}\right)^2}{9}=1$
Center $\left(0,\frac{1}{2}\right)$
Major Axis along $x=0$ (the y-axis)
Minor Axis along $y=\frac{1}{2}$
Vertices $(0,2)$, $(0,-1)$
Endpoints of Minor Axis $\left(-1,\frac{1}{2}\right)$, $\left(1,\frac{1}{2}\right)$
Foci $\left(0,\frac{1+\sqrt{5}}{2}\right)$, $\left(0,\frac{1-\sqrt{5}}{2}\right)$
$e=\frac{\sqrt{5}}{3}$

14. $\dfrac{(x-2)^2}{5}+\dfrac{(y+2)^2}{6}=1$
Center $(2,-2)$
Major Axis along $x=2$
Minor Axis along $y=-2$
Vertices $\left(2,-2+\sqrt{6}\right)$, $(2,-2-\sqrt{6})$
Endpoints of Minor Axis $\left(2-\sqrt{5},-2\right)$, $\left(2+\sqrt{5},-2\right)$
Foci $(2,-1)$, $(2,-3)$
$e=\frac{\sqrt{6}}{6}$

15. For number 1:

- $f(x)=\frac{5}{13}\sqrt{169-x^2}$ represents the upper half of the ellipse.
- $g(x)=-\frac{5}{13}\sqrt{169-x^2}$ represents the lower half of the ellipse.

For number 3:

- $f(x)=-3+\frac{3}{2}\sqrt{4x-x^2}$ represents the upper half of the ellipse.
- $g(x)=-3-\frac{3}{2}\sqrt{4x-x^2}$ represents the lower half of the ellipse.

For number 5:

- $f(x)=3+\frac{1}{10}\sqrt{990+220x-110x^2}$ represents the upper half of the ellipse.
- $g(x)=3-\frac{1}{10}\sqrt{990+220x-110x^2}$ represents the lower half of the ellipse.

For number 7:

- $f(x)=5+\frac{1}{2}\sqrt{60-20x-5x^2}$ represents the upper half of the ellipse.
- $g(x)=5-\frac{1}{2}\sqrt{60-20x-5x^2}$ represents the lower half of the ellipse.

For number 9:

- $f(x)=1+\frac{3}{5}\sqrt{16+6x-x^2}$ represents the upper half of the ellipse.
- $g(x)=1-\frac{3}{5}\sqrt{16+6x-x^2}$ represents the lower half of the ellipse.

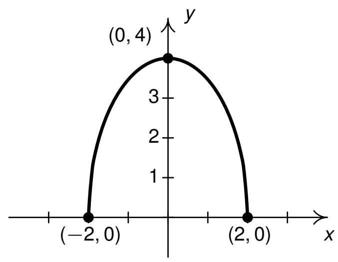
For number 11:

- $f(x)=-2+\frac{1}{6}\sqrt{90+60x-10x^2}$ represents the upper half of the ellipse.
- $g(x)=-2-\frac{1}{6}\sqrt{90+60x-10x^2}$ represents the lower half of the ellipse.

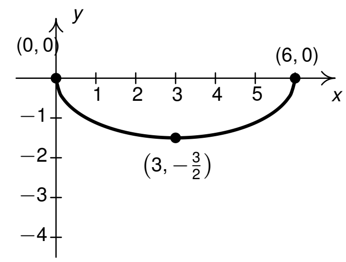
For number 13:

- $f(x) = \frac{1}{2} + \frac{3}{2}\sqrt{1 - x^2}$ represents the upper half of the ellipse.
- $g(x) = \frac{1}{2} - \frac{3}{2}\sqrt{1 - x^2}$ represents the lower half of the ellipse.

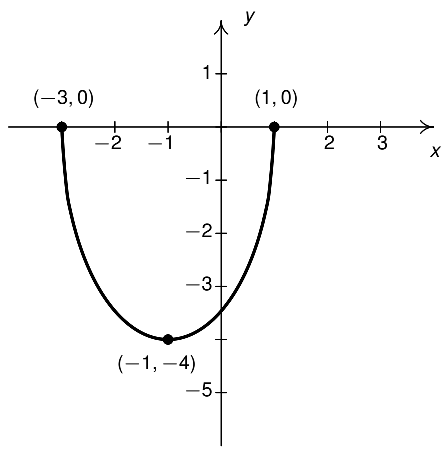
16. $f(x) = \sqrt{16 - 4x^2}$



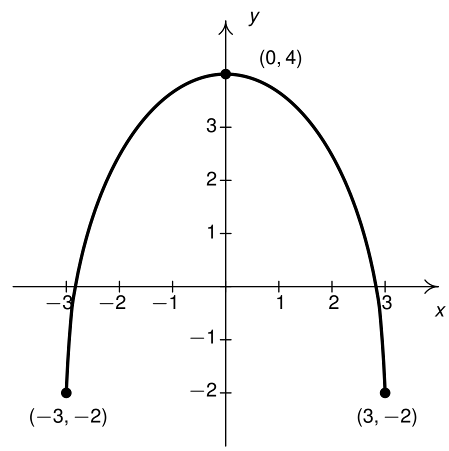
17. $g(x) = -\frac{1}{2}\sqrt{6x - x^2}$



18. $f(x) = -2\sqrt{3 - 2x - x^2}$



19. $g(x) = -2 + 2\sqrt{9 - x^2}$



20. $\dfrac{(x-1)^2}{9} + \dfrac{y^2}{16} = 1$

21. $\dfrac{(x-4)^2}{16} + \dfrac{(y-4)^2}{9} = 1$

22. $y = 3 - \frac{3}{4}\sqrt{16 - x^2}$

23. $y = 2\sqrt{8x - x^2 - 12}$

24. $\dfrac{(x-3)^2}{9} + \dfrac{(y-7)^2}{25} = 1$

25. $\dfrac{x^2}{39} + \dfrac{y^2}{64} = 1$

26. $\dfrac{x^2}{34} + \dfrac{y^2}{25} = 1$

27. $\dfrac{(x-8)^2}{25} + \dfrac{(y-2)^2}{4} = 1$

28. $\dfrac{(x-5)^2}{25} + \dfrac{4(y-2)^2}{75} = 1$

29. $\dfrac{(x-8)^2}{64} + \dfrac{(y+9)^2}{81} = 1$

30. Jamie and Jason should stand $100 - 25\sqrt{7} \approx 33.86$ feet from opposite ends of the gallery.

31. The arch can be modeled by the upper half of $\frac{x^2}{9} + \frac{y^2}{81} = 1$. One foot in from the base of the arch corresponds to either $x = \pm 2$. Plugging in $x = \pm 2$ gives $y = \pm 3\sqrt{5}$ and since y represents a height, we choose $y = 3\sqrt{5} \approx 6.71$ feet.

32. Distance from the sun to aphelion ≈ 1.0167 AU.
Distance from the sun to perihelion ≈ 0.9833 AU.

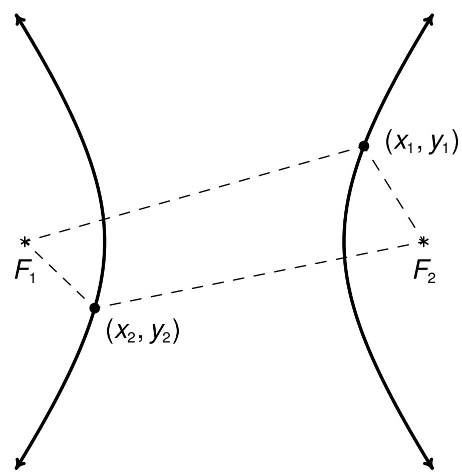
33. $\dfrac{(x - 7.4)^2}{1296} + \dfrac{y^2}{1241.24} = 1$. Graphing this equation[6] reveals a very 'round' orbit.

[6]. . . using the 'Zoom Square' setting . . .

8.5 Hyperbolas

In the definition of an ellipse, Definition 8.4, we fixed two points called foci and looked at points whose distances to the foci always **added** to a constant distance d. Those prone to syntactical tinkering may wonder what, if any, curve we'd generate if we replaced **added** with **subtracted**. The answer is a hyperbola.

Definition 8.6. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, a **hyperbola** is the set of all points (x, y) in the plane such that the absolute value of the difference of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** of the hyperbola.



In the figure above:

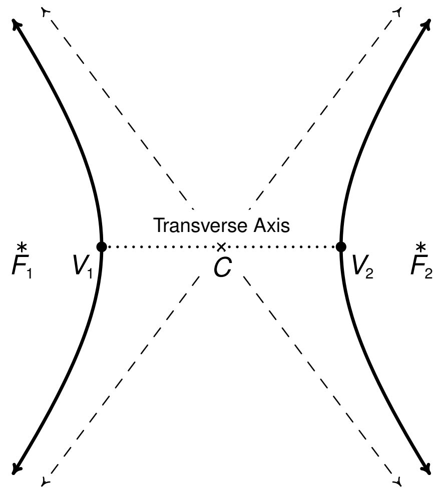
$$\text{the distance from } F_1 \text{ to } (x_1, y_1) - \text{the distance from } F_2 \text{ to } (x_1, y_1) \quad = \quad d$$

and

$$\text{the distance from } F_2 \text{ to } (x_2, y_2) - \text{the distance from } F_1 \text{ to } (x_2, y_2) \quad = \quad d$$

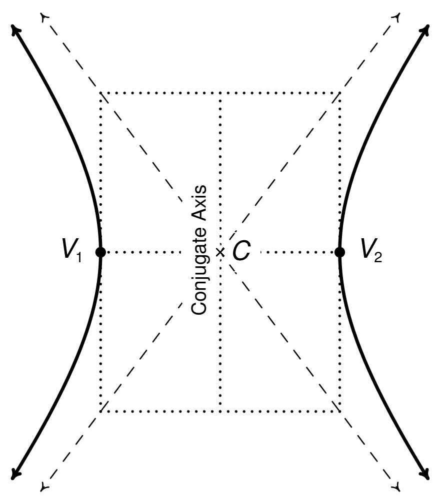
Note that the hyperbola has two parts, called **branches**. The **center** of the hyperbola is the midpoint of the line segment connecting the two foci. The **transverse axis** of the hyperbola is the line segment connecting two opposite ends of the hyperbola which also contains the center and foci. The **vertices** of a hyperbola are the points of the hyperbola which lie on the transverse axis.

In addition, we will show momentarily that the hyperbola has a pair of **asymptotes** which the branches of the hyperbola approach for large x and y values. They serve as guides to the graph. Schematically:



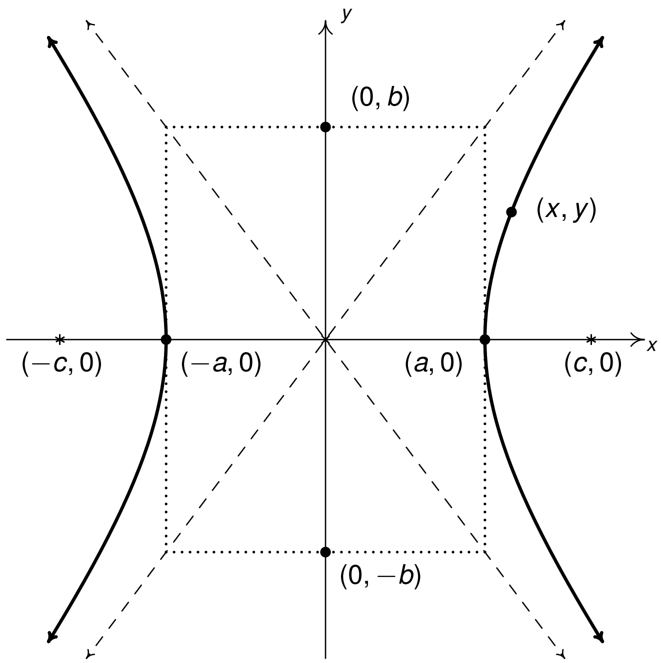
A hyperbola with center C; foci F_1, F_2; and vertices V_1, V_2 and asymptotes (dashed)

Before we derive the standard equation of the hyperbola, we need to discuss one further parameter, the **conjugate axis** of the hyperbola. The conjugate axis of a hyperbola is the line segment through the center which is perpendicular to the transverse axis and has the same length as the line segment through a vertex which connects the asymptotes. Schematically:



Note that in the diagram, we can construct a rectangle using line segments with lengths equal to the lengths of the transverse and conjugate axes whose center is the center of the hyperbola and whose diagonals are contained in the asymptotes. This **guide rectangle**, much akin to the one we saw Section 8.4 to help us graph ellipses, will aid us in graphing hyperbolas.

Suppose we wish to derive the equation of a hyperbola. For simplicity, we shall assume that the center is $(0, 0)$, the vertices are $(a, 0)$ and $(-a, 0)$ and the foci are $(c, 0)$ and $(-c, 0)$. We label the endpoints of the conjugate axis $(0, b)$ and $(0, -b)$. (Although b does not enter into our derivation, we will have to justify this choice as you shall see later.) As before, we assume a, b, and c are all positive numbers. Schematically:



Since $(a, 0)$ is on the hyperbola, it must satisfy the conditions of Definition 8.6. That is, the distance from $(-c, 0)$ to $(a, 0)$ minus the distance from $(c, 0)$ to $(a, 0)$ must equal the fixed distance d. Since all these points lie on the x-axis, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) - \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a+c) - (c-a) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance d from the definition of the hyperbola is actually the length of the transverse axis! (Where have we seen that type of coincidence before?) Now consider a point (x, y) on the hyperbola. Applying Definition 8.6, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (x,y) - \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\ \sqrt{(x-(-c))^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} &= 2a \\ \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= 2a \end{aligned}$$

Using the same arsenal of Intermediate Algebra weaponry we used in deriving the standard formula of an ellipse, Equation 8.5, we arrive at the following.[1]

[1] It is a good exercise to actually work this out.

$$\left(a^2 - c^2\right) x^2 + a^2 y^2 \quad = \quad a^2 \left(a^2 - c^2\right)$$

What remains is to determine the relationship between a, b and c. To that end, we note that since a and c are both positive numbers with $a < c$, we get $a^2 < c^2$ so that $a^2 - c^2$ is a negative number. Hence, $c^2 - a^2$ is a positive number. For reasons which will become clear soon, we solve the equation for $\frac{y^2}{x^2}$:

$$\begin{array}{rcl} \left(a^2 - c^2\right) x^2 + a^2 y^2 & = & a^2 \left(a^2 - c^2\right) \\ -\left(c^2 - a^2\right) x^2 + a^2 y^2 & = & -a^2 \left(c^2 - a^2\right) \\ a^2 y^2 & = & \left(c^2 - a^2\right) x^2 - a^2 \left(c^2 - a^2\right) \\ \dfrac{y^2}{x^2} & = & \dfrac{\left(c^2 - a^2\right)}{a^2} - \dfrac{\left(c^2 - a^2\right)}{x^2} \end{array}$$

As $|x| \to \infty$,[2] the quantity $\frac{(c^2-a^2)}{x^2} \to 0$ so that $\frac{y^2}{x^2} \approx \frac{(c^2-a^2)}{a^2}$. By setting $b^2 = c^2 - a^2$ we get $\frac{y^2}{x^2} \approx \frac{b^2}{a^2}$. This shows that $y \approx \pm\frac{b}{a}x$, so that $y = \pm\frac{b}{a}x$ are the asymptotes to the graph as predicted and our choice of labels for the endpoints of the conjugate axis is justified. In our equation of the hyperbola we can substitute $a^2 - c^2 = -b^2$ which yields

$$\begin{array}{rcl} \left(a^2 - c^2\right) x^2 + a^2 y^2 & = & a^2 \left(a^2 - c^2\right) \\ -b^2 x^2 + a^2 y^2 & = & -a^2 b^2 \\ \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} & = & 1 \end{array}$$

The equation above is for a hyperbola whose center is the origin and which opens to the left and right. If the hyperbola were centered at a point (h, k), we would get the following.

Equation 8.6. The Standard Equation of a Horizontal[a] Hyperbola
For positive numbers a and b, the equation of a horizontal hyperbola with center (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

[a]That is, a hyperbola whose branches open to the left and right

If the roles of x and y were interchanged, then the hyperbola's branches would open upwards and downwards and we would get a 'vertical' hyperbola.

Equation 8.7. The Standard Equation of a Vertical Hyperbola
For positive numbers a and b, the equation of a vertical hyperbola with center (h, k) is:

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

The values of a and b determine how far in the x and y directions, respectively, one counts from the center to determine the guide rectangle. In both cases, the distance from the center to the foci, c, as seen in

[2]Recall this means we are analyzing the behavior as $x \to \infty$ and as $x \to -\infty$.

the derivation, can be found by the formula $c = \sqrt{a^2 + b^2}$. Lastly, note that we can quickly distinguish the equation of a hyperbola from that of a circle or ellipse because the hyperbola formula involves a *difference* of squares where the circle and ellipse formulas both involve the *sum* of squares.

Example 8.5.1.

1. Graph each of the following equations below in the xy-plane. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

   (a) $25(x-2)^2 - 4y^2 = 100$.

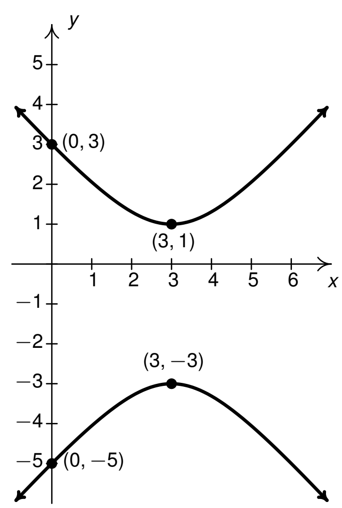
   (b) $9y^2 - x^2 - 6x = 10$.

2. Graph $f(x) = \sqrt{x^2 - 2x - 3}$.

3. Find the standard form of the equation of a hyperbola which satisfies the following characteristics:

   (a) the asymptotes are $y = \pm 2x$ and the vertices are $(\pm 5, 0)$.

   (b) the hyperbola graphed below:



Solution.

1. (a) Owing to the difference of squares in $25(x-2)^2 - 4y^2 = 100$, we work towards putting this equation into the form of Equation 8.6 or Equation 8.7. To that end, we rewrite y^2 as $(y-0)^2$ and divide through by 100:

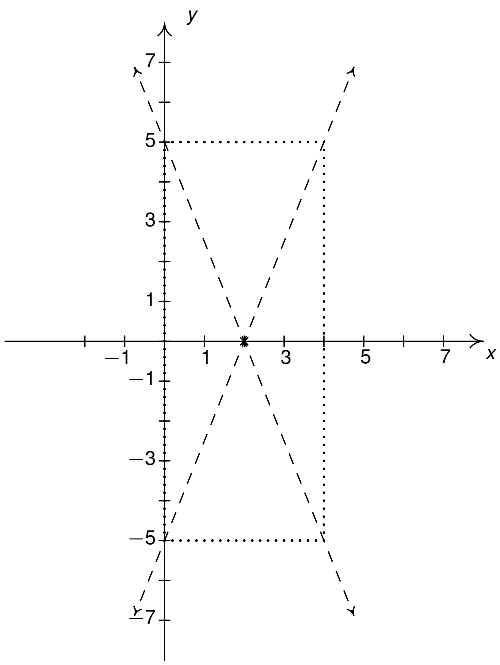
$$25(x-2)^2 - 4y^2 = 100 \rightarrow \frac{(x-2)^2}{4} - \frac{(y-0)^2}{25} = 1 \leftrightarrow \frac{(x-2)^2}{(2)^2} - \frac{(y-0)^2}{(5)^2} = 1.$$

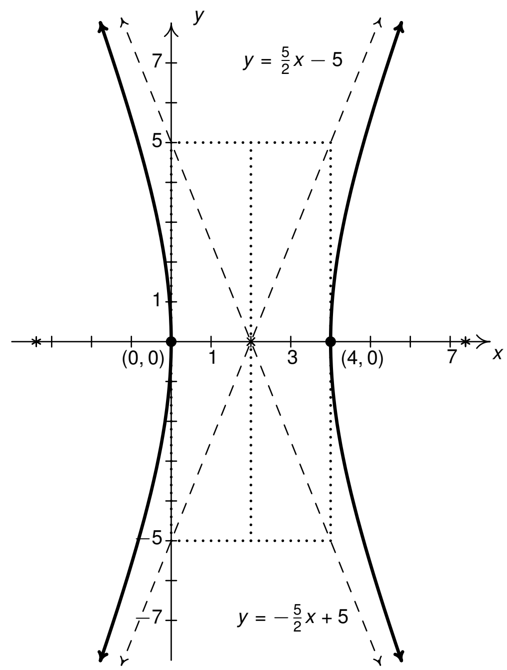
We identify $h = 2$ and $k = 0$, so the hyperbola is centered at $(2, 0)$. We also see $a = 2$, and $b = 5$, which means we move 2 units to the left and to the right of the center and 5 units up and down from the center to arrive at points on the guide rectangle: $(2-2,0) = (0,0)$, $(2+2,0) = (4,0)$, $(2,0+5) = (2,5)$, and $(2,0-5) = (2,-5)$. Since slant asymptotes pass through the center of the hyperbola as well as the corners of the rectangle, we get the set-up as drawn below on the left.

Since the y^2 term is being subtracted from the x^2 term, we are in the situation of Equation 8.6. Hence, the branches of the hyperbola open to the left and right so the transverse axis lies along the x-axis and the conjugate axis lies along the vertical line $x = 2$.

Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are $(0,0)$ and $(4,0)$. To find the foci, we need $c = \sqrt{a^2+b^2} = \sqrt{4+25} = \sqrt{29}$. Since the foci lie on the transverse axis, we move $\sqrt{29}$ units to the left and right of $(2,0)$ to arrive at $(2-\sqrt{29},0)$ (approximately $(-3.39,0)$) and $(2+\sqrt{29},0)$ (approximately $(7.39,0)$).

Lastly, to determine the equations of the asymptotes, recall that the asymptotes pass through the center of the hyperbola, $(2,0)$, as well as the corners of guide rectangle. As such, they have slopes of $\pm\frac{b}{a} = \pm\frac{5}{2}$. Feeding this information into the point-slope equation of a line, Equation A.5, we get $y - 0 = \pm\frac{5}{2}(x-2)$, so the asymptotes are $y = \frac{5}{2}x - 5$ and $y = -\frac{5}{2}x + 5$. Putting it all together, we get our final graph below on the right.





The graph of $25(x-2)^2 - 4y^2 = 100$.

(b) Since we have a difference of squares in $9y^2 - x^2 - 6x = 10$, we aim to transform our given equation into Equation 8.6 or Equation 8.7. As we've seen with the other conic sections, we begin with completing the square.

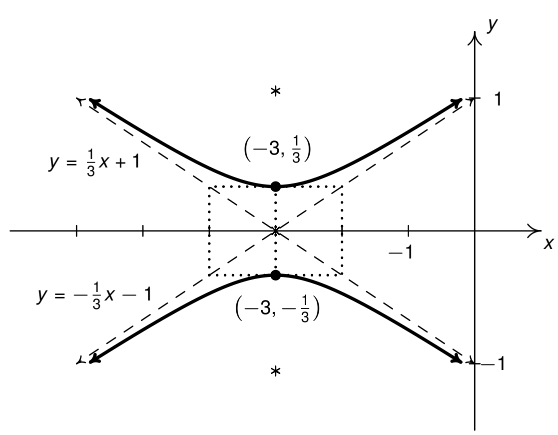
$$\begin{array}{rcll}
9y^2 - x^2 - 6x & = & 10 & \\
9y^2 - 1\left(x^2 + 6x\right) & = & 10 & \text{factor out leading coefficient of } x^2 \\
9y^2 - 1\left(x^2 + 6x + \underline{9}\right) & = & 10 + (-1)\,(\underline{9}) & \text{complete the square in } x \\
9y^2 - (x+3)^2 & = & 1 & \text{factor} \\
\dfrac{y^2}{\frac{1}{9}} - \dfrac{(x+3)^2}{1} & = & 1 & \text{rewrite} \\
\dfrac{(y-0)^2}{\left(\frac{1}{3}\right)^2} - \dfrac{(x-(-3))^2}{(1)^2} & = & 1 & \text{write in the form of Equation 8.7.}
\end{array}$$

Now that this equation is in the standard form of Equation 8.7, we identify $h = -3$ and $k = 0$ so the center is $(-3, 0)$. We also see so $a = 1$, and $b = \frac{1}{3}$ which means that we move 1 unit to the left and to the right of the center and $\frac{1}{3}$ units up and down from the center to arrive at points on the guide rectangle: $(-3-1, 0) = (-4, 0)$, $(-3+1, 0) = (-2, 0)$, $\left(-3, 0 + \frac{1}{3}\right) = \left(-3, \frac{1}{3}\right)$ and $\left(-3, 0 - \frac{1}{3}\right) = \left(-3, -\frac{1}{3}\right)$.

Since the x^2 term is being subtracted from the y^2 term, we know the branches of the hyperbola open upwards and downwards. This means the transverse axis lies along the vertical line $x = -3$ and the conjugate axis lies along the x-axis. As a result, we get the vertices are $\left(-3, \frac{1}{3}\right)$ and $\left(-3, -\frac{1}{3}\right)$.

To find the foci, we use $c = \sqrt{a^2 + b^2} = \sqrt{\frac{1}{9} + 1} = \frac{\sqrt{10}}{3}$. Since the foci lie on the transverse axis, we move $\frac{\sqrt{10}}{3}$ units above and below $(-3, 0)$ to arrive at $\left(-3, \frac{\sqrt{10}}{3}\right)$ and $\left(-3, -\frac{\sqrt{10}}{3}\right)$.

To determine the asymptotes, we use the fact the asymptotes pass through the center of the hyperbola, $(-3, 0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{1}{3}$. Once again we use the point-slope equation of a line, Equation A.5, to get the two asymptotes $y = \frac{1}{3}x + 1$ and $y = -\frac{1}{3}x - 1$. Our final graph is below.



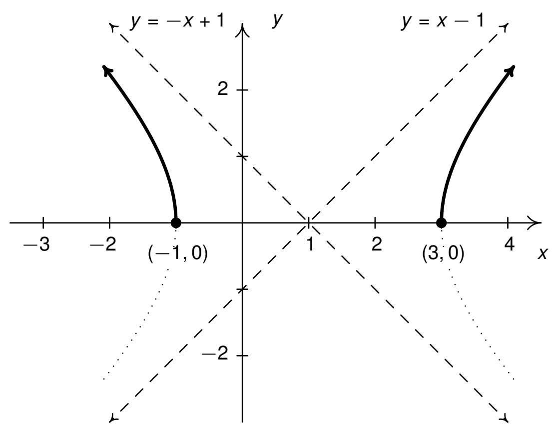
The graph of $9y^2 - x^2 - 6x = 10$.

2. Graphing $f(x) = \sqrt{x^2 - 2x - 5}$ amounts to graphing the equation $y = \sqrt{x^2 - 2x - 5}$. In order to use the tools we've learned in this chapter, we first square both sides to get a quadratic equation in two variables: $y^2 = (\sqrt{x^2 - 2x - 3})^2$. We get $y^2 = x^2 - 2x - 3$ or $y^2 - x^2 + 2x = -3$. We now set about transforming this equation into the form stated in Equation 8.6 or Equation 8.7.

$$\begin{array}{rcll}
y^2 - x^2 + 2x & = & -3 & \\
y^2 - 1\left(x^2 - 2x\right) & = & -3 & \text{factor out leading coefficient of } x^2 \\
y^2 - 1\left(x^2 - 2x + \underline{1}\right) & = & -3 + (-1)\left(\underline{1}\right) & \text{complete the square in } x \\
y^2 - (x-1)^2 & = & -4 & \text{factor} \\
-\dfrac{y^2}{4} + \dfrac{(x-1)^2}{4} & = & 1 & \text{divide through by } -4 \\
\dfrac{(x-1)^2}{(2)^2} - \dfrac{(y-0)^2}{(2)^2} & = & 1 & \text{write in the form of Equation 8.6.}
\end{array}$$

We get the equation into the form of Equation 8.6 and identify $h = 1$ and $k = 0$ so the center is $(1, 0)$. We have $a = b = 2$, which means we move 2 units to the left, to the right, up and down from the center to find points on the guide rectangle: $(1-2, 0) = (-1, 0)$, $(1+2, 0) = (3, 0)$, $(1, 0-2) = (1, -2)$ and $(1, 0+2) = (1, 2)$. Of these four points, the vertices are $(-1, 0)$ and $(3, 0)$ since the hyperbola opens to the left and to the right. As usual, we the guide rectangle helps us sketch the hyperbola along with its slant asymptotes, which we find are $y - 0 = \pm(x - 1)$ or $y = x - 1$ and $y = -x + 1$.

We know since f is a function, the graph of f cannot be the *entire* hyperbola, otherwise the graph would fail the vertical line test. Since, by definition, $\sqrt{x^2 - 2x - 5} \geq 0$, we know $f(x) \geq 0$. Hence the graph of f is the *upper* half of the hyperbola, as shown below.

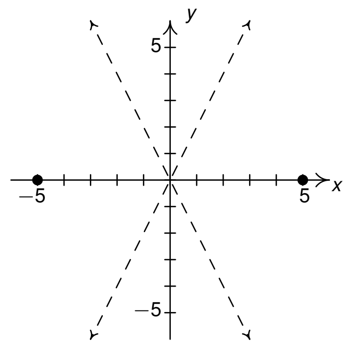


The graph of $f(x) = \sqrt{x^2 - 2x - 5}$.

3. (a) Plotting the data given to us below, we know the branches of the hyperbola open to the left and to the right. This means the our answer will take the form of Equation 8.6.

Since the center is the midpoint of the vertices, we see the center is $(0, 0)$, so $h = k = 0$. Moreover, since the vertices are exactly 5 units from the center, we know $a = 5$ so $a^2 = 25$. All that remains to find is the value of b^2.

Recall that the slopes of the asymptotes are $\pm\frac{b}{a}$. Since $a = 5$ and the slope of the line $y = 2x$ is 2, we have that $\frac{b}{5} = 2$, so $b = 10$. Hence, $b^2 = 100$. Our final answer is $\frac{x^2}{25} - \frac{y^2}{100} = 1$.



(b) From what we are given on the graph, the equation of the hyperbola takes the form of Equation 8.7. The vertices appear to be $(3, 1)$ and $(3, -3)$ whose midpoint gives us the center as $(3, -1)$. Hence, $h = 3$ and $k = -1$. Moreover, since the vertices are 2 units above and below the center, we know $b = 2$ so $b^2 = 4$. All that remains is for us to find the value of a^2.

Since we are given two additional points, $(0, 3)$ and $(0, -5)$, we choose one of them, $(0, 3)$ to find a^2 and use the other, $(0, -5)$ to partially check our answer.

At this stage, we know the equation of the hyperbola is

$$\frac{(y+1)^2}{4} - \frac{(x-3)^2}{a^2} = 1,$$

so substituting $x = 0$ and $y = 3$ into this equation, we get $\frac{16}{4} - \frac{9}{a^2} = 1$ so $a^2 = 3$. Hence, our final answer is

$$\frac{(y+1)^2}{4} - \frac{(x-3)^2}{3} = 1.$$

We leave it to the reader to check. □

As seen in Example 8.5.1, it is often the case we need to transform a given equation into the form specified by Equations 8.6 or 8.7. We summarize one method below.

To Write the Equation of a Hyperbola in Standard Form

1. Group common variables together on one side of the equation and put the constant on the other.
2. Complete the square on both variables as needed.
3. Divide both sides, if needed, to obtain 1 on one side of the equation.

Hyperbolas can be used in so-called 'trilateration,' or 'positioning' problems. The procedure outlined in the next example is the basis of the (now defunct) LOng Range Aid to Navigation (LORAN for short) system.[3]

Example 8.5.2.

1. Jeff is stationed 10 miles due west of Carl in an otherwise empty forest in an attempt to locate an elusive Sasquatch. At the stroke of midnight, Jeff records a Sasquatch call 9 seconds earlier than Carl. If the speed of sound that night is 760 miles per hour, determine a hyperbolic path along which Sasquatch must be located.

2. By a stroke of luck, Kai is also camping in the woods at this time. He is 6 miles due north of Jeff and heard the Sasquatch call 18 seconds after Jeff did. Use this added information to locate Sasquatch.

Solution.

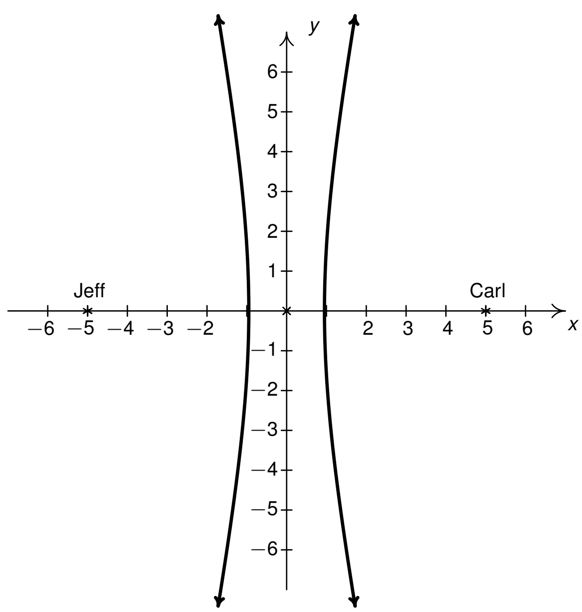
1. Since Jeff hears Sasquatch sooner, it is closer to Jeff than it is to Carl. Since the speed of sound is 760 miles per hour, we can determine how much closer Sasquatch is to Jeff by multiplying

$$760\,\frac{\text{miles}}{\text{hour}} \times \frac{1\text{ hour}}{3600\text{ seconds}} \times 9\text{ seconds} = 1.9\text{ miles}$$

   This means that Sasquatch is 1.9 miles closer to Jeff than it is to Carl. In other words, Sasquatch must lie on a path where

$$(\text{the distance to Carl}) - (\text{the distance to Jeff}) = 1.9$$

   This is exactly the situation in the definition of a hyperbola, Definition 8.6. In this case, Jeff and Carl are located at the foci,[4] and our fixed distance d is 1.9. For simplicity, we assume the hyperbola is centered at $(0,0)$ with its foci at $(-5,0)$ and $(5,0)$. Schematically:



[3] GPS now rules the positioning kingdom. Is there still a place for LORAN? Do satellites ever malfunction?

[4] We usually like to be the *center* of attention, but being the *focus* of attention works equally well.

We are seeking a curve of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in which the distance from the center to each focus is $c = 5$. As we saw in the derivation of the standard equation of the hyperbola, Equation 8.6, $d = 2a$, so that $2a = 1.9$, or $a = 0.95$ and $a^2 = 0.9025$. All that remains is to find b^2. To that end, we recall that $a^2 + b^2 = c^2$ so $b^2 = c^2 - a^2 = 25 - 0.9025 = 24.0975$. Since Sasquatch is closer to Jeff than it is to Carl, it must be on the western (left hand) branch of

$$\frac{x^2}{0.9025} - \frac{y^2}{24.0975} = 1.$$

2. Kai and Jeff are at the foci of a second hyperbola where the fixed distance d is:

$$760\,\frac{\text{miles}}{\text{hour}} \times \frac{1\text{ hour}}{3600\text{ seconds}} \times 18\text{ seconds} = 3.8\text{ miles}$$

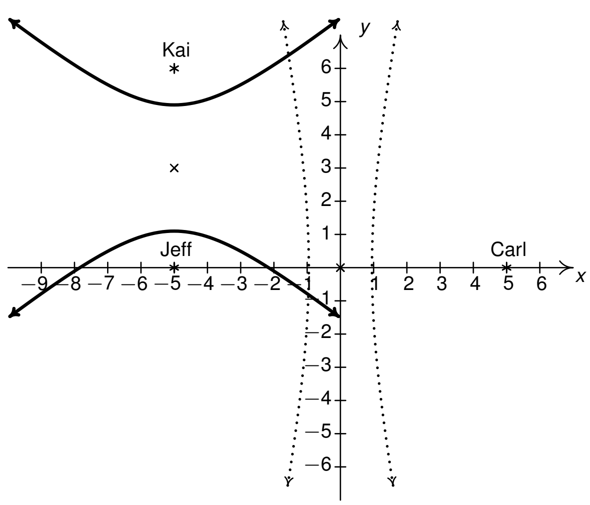
Since Jeff is positioned at $(-5, 0)$, we place Kai at $(-5, 6)$. This puts the center of the new hyperbola at $(-5, 3)$. Plotting Kai's position and the new center gives us the diagram below on the left.

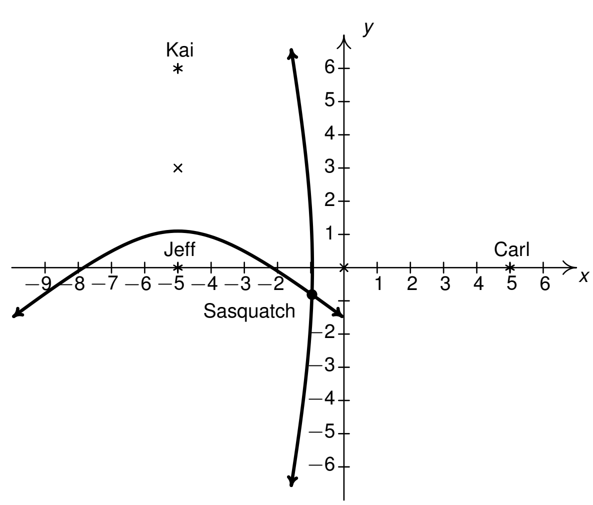
The second hyperbola is vertical, so it must be of the form $\frac{(y-3)^2}{b^2} - \frac{(x+5)^2}{a^2} = 1$. As before, the distance d is the length of the major axis, which in this case is $2b$. We get $2b = 3.8$ so that $b = 1.9$ and $b^2 = 3.61$. With Kai 6 miles due North of Jeff, we have that the distance from the center to the focus is $c = 3$. Since $a^2 + b^2 = c^2$, we get $a^2 = c^2 - b^2 = 9 - 3.61 = 5.39$.

Kai heard the Sasquatch call after Jeff, so Kai is farther from Sasquatch than Jeff. Thus Sasquatch must lie on the southern branch of the hyperbola

$$\frac{(y-3)^2}{3.61} - \frac{(x+5)^2}{5.39} = 1.$$

Looking at the western branch of the hyperbola determined by Jeff and Carl along with the southern branch of the hyperbola determined by Kai and Jeff, we see that there is exactly one point in common, and this is where Sasquatch must have been when it called.





To determine the coordinates of this point of intersection exactly, we would need techniques for solving systems of non-linear equations (which we won't see until Section 9.7), so we use a graphing utility. Doing so, we get Sasquatch is approximately at $(-0.9629, -0.8113)$. □

Each of the conic sections we have studied in this chapter result from graphing equations of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ for different choices of A, C, D, E, and[5] F. While we've seen examples demonstrate *how* to convert an equation from this general form to one of the standard forms, we close this chapter with some advice about *which* standard form to choose.[6]

Strategies for Identifying Conic Sections

Suppose the graph of equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is a non-degenerate conic section.[a]

- If just *one* variable is squared, the graph is a parabola. Rewrite the equation in the standard form given in Equation 8.1 (if x is squared) or Equation 8.2 (if y is squared).

If *both* variables are squared, look at the coefficients of x^2 and y^2, A and C.

- If $A = C$, the graph is a circle. Rewrite the equation in the standard form given in Equation 8.3.
- If $A \neq C$ but A and C have the *same* sign, the graph is an ellipse. Rewrite the equation in the standard form given in Equation 8.5.
- If A and C have the *different signs*, the graph is a hyperbola. Rewrite the equation in the standard form given in either Equation 8.6 or Equation 8.7.

[a]That is, a parabola, circle, ellipse, or hyperbola – see Section 8.1.

[5]See Section 14.4 to see why we skip B.

[6]We formalize this in Exercise 40.

8.5.1 Exercises

In Exercises 1 - 8, graph the hyperbola in the xy-plane. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

1. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$
2. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$
3. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$
4. $\dfrac{(y-3)^2}{11} - \dfrac{(x-1)^2}{10} = 1$
5. $\dfrac{(x+4)^2}{16} - (y-4)^2 = 1$
6. $\dfrac{(x+1)^2}{9} - \dfrac{(y-3)^2}{4} = 1$
7. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$
8. $\dfrac{(x-4)^2}{8} - \dfrac{(y-2)^2}{18} = 1$

In Exercises 9 - 12, put the equation in standard form. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.[7]

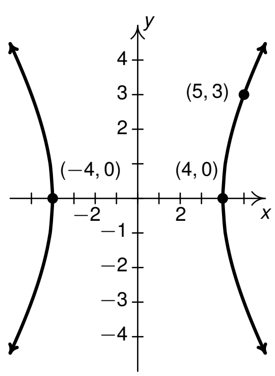
9. $12x^2 - 3y^2 + 30y - 111 = 0$
10. $18y^2 - 5x^2 + 72y + 30x - 63 = 0$
11. $9x^2 - 25y^2 - 54x - 50y - 169 = 0$
12. $-6x^2 + 5y^2 - 24x + 40y + 26 = 0$
13. For each of the odd numbered equations given in Exercises 1 - 11, find two or more explicit functions of x represented by each of the equations. (See Example 8.2.2 in Section 8.2.)

In Exercises 14 - 17, graph each function by recognizing it as a portion of a hyperbola.

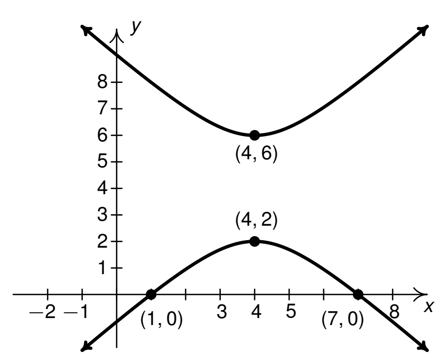
14. $f(x) = \sqrt{x^2 - 4}$
15. $g(x) = -\sqrt{x^2 - 4x}$
16. $f(x) = -2\sqrt{x^2 + 2x - 3}$
17. $g(x) = -2 + 2\sqrt{x^2 - 9}$

In Exercises 18 - 19, find an equation for the hyperbola whose graph is given.

18.



19.



[7]. . . assuming the equation were graphed in the xy-plane . . .

In Exercises 20 - 25, find the standard form of the equation of the hyperbola which has the given properties.

20. Center $(3,7)$, Vertex $(3,3)$, Focus $(3,2)$

21. Vertex $(0,1)$, Vertex $(8,1)$, Focus $(-3,1)$

22. Foci $(0,\pm 8)$, Vertices $(0,\pm 5)$.

23. Foci $(\pm 5,0)$, length of the Conjugate Axis 6

24. Vertices $(3,2)$, $(13,2)$; Endpoints of the Conjugate Axis $(8,4)$, $(8,0)$

25. Vertex $(-10,5)$, Asymptotes $y=\pm\frac{1}{2}(x-6)+5$

In Exercises 26 - 35, find the standard form of the equation using the guidelines on page 732 and then graph the conic section.

26. $x^2-2x-4y-11=0$

27. $x^2+y^2-8x+4y+11=0$

28. $9x^2+4y^2-36x+24y+36=0$

29. $9x^2-4y^2-36x-24y-36=0$

30. $y^2+8y-4x+16=0$

31. $4x^2+y^2-8x+4=0$

32. $4x^2+9y^2-8x+54y+49=0$

33. $x^2+y^2-6x+4y+14=0$

34. $2x^2+4y^2+12x-8y+25=0$

35. $4x^2-5y^2-40x-20y+160=0$

36. The location of an earthquake's epicenter – the point on the surface of the Earth directly above where the earthquake actually occurred – can be determined by a process similar to how we located Sasquatch in Example 8.5.2. (As we said back in Exercise 82 in Section 7.2, earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a course in Geology or the U.S. Geological Survey's Earthquake Hazards Program found here.) Our technique works only for relatively small distances because we need to assume that the Earth is flat in order to use hyperbolas in the plane. The P-waves ("P" stands for Primary) of an earthquake in Sasquatchia travel at 6 kilometers per second.[8] Station A records the waves first. Then Station B, which is 100 kilometers due north of Station A, records the waves 2 seconds later. Station C, which is 150 kilometers due west of Station A records the waves 3 seconds after that (a total of 5 seconds after Station A). Where is the epicenter?

37. The notion of eccentricity introduced for ellipses in Definition 8.5 in Section 8.4 is the same for hyperbolas in that we can define the eccentricity e of a hyperbola as

$$e=\frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

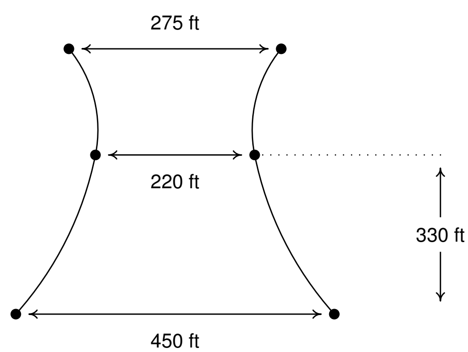
[8]Depending on the composition of the crust at a specific location, P-waves can travel between 5 kps and 8 kps.

(a) With the help of your classmates, explain why $e > 1$ for any hyperbola.

(b) Find the equation of the hyperbola with vertices $(\pm 3, 0)$ and eccentricity $e = 2$.

(c) With the help of your classmates, find the eccentricity of each of the hyperbolas in Exercises 1 - 8. What role does eccentricity play in the shape of the graphs?

38. On page 683 in Section 8.2, we discussed paraboloids of revolution when studying the design of satellite dishes and parabolic mirrors. In much the same way, 'natural draft' cooling towers are often shaped as **hyperboloids of revolution**. Each vertical cross section of these towers is a hyperbola. Suppose the a natural draft cooling tower has the cross section below. Suppose the tower is 450 feet wide at the base, 275 feet wide at the top, and 220 feet at its narrowest point (which occurs 330 feet above the ground.) Determine the height of the tower to the nearest foot.



39. With the help of your classmates, research the Cassegrain Telescope. It uses the reflective property of the hyperbola as well as that of the parabola to make an ingenious telescope.

40. With the help of your classmates show that if $Ax^2 + Cy^2 + Dx + Ey + F = 0$ determines a non-degenerate conic[9] then

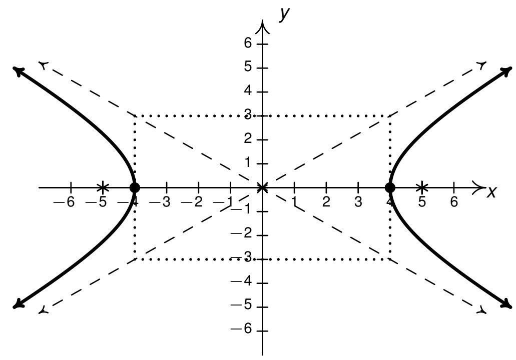
- $AC < 0$ means that the graph is a hyperbola
- $AC = 0$ means that the graph is a parabola
- $AC > 0$ means that the graph is an ellipse or circle

NOTE: This result will be generalized in Theorem 14.9 in Section 14.4.1.

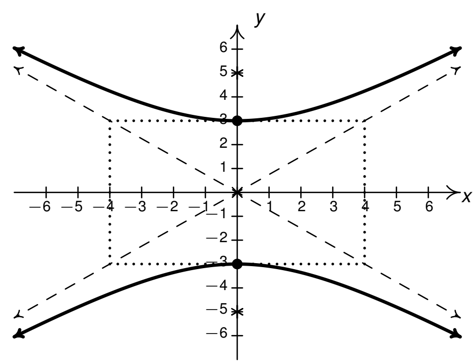
[9]Recall that this means its graph is either a circle, parabola, ellipse or hyperbola.

8.5.2 Answers

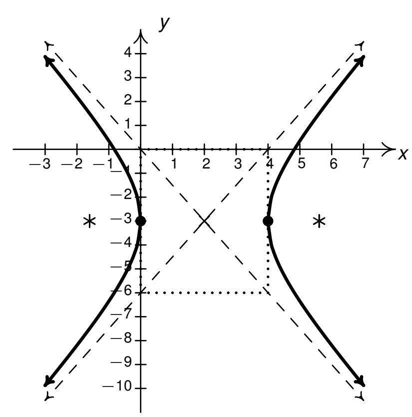
1. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$
   Center $(0,0)$
   Transverse axis on $y=0$
   Conjugate axis on $x=0$
   Vertices $(4,0), (-4,0)$
   Foci $(5,0), (-5,0)$
   Asymptotes $y = \pm\frac{3}{4}x$



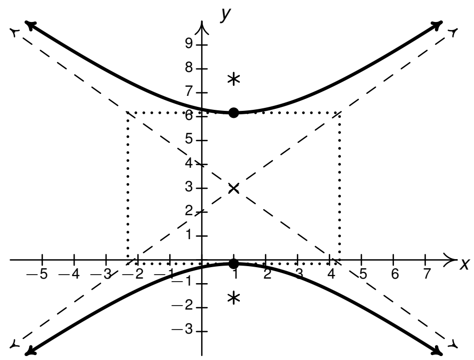
2. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$
   Center $(0,0)$
   Transverse axis on $x=0$
   Conjugate axis on $y=0$
   Vertices $(0,3), (0,-3)$
   Foci $(0,5), (0,-5)$
   Asymptotes $y = \pm\frac{3}{4}x$



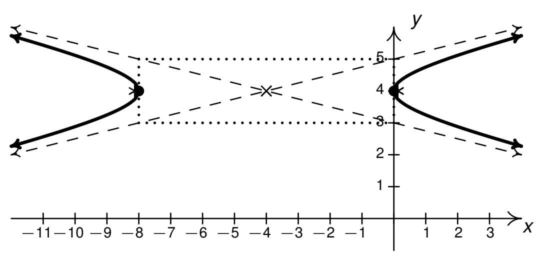
3. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$
   Center $(2,-3)$
   Transverse axis on $y=-3$
   Conjugate axis on $x=2$
   Vertices $(0,-3), (4,-3)$
   Foci $(2+\sqrt{13},-3), (2-\sqrt{13},-3)$
   Asymptotes $y = \pm\frac{3}{2}(x-2) - 3$



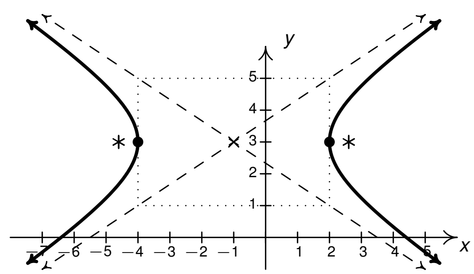
4. $\dfrac{(y-3)^2}{11} - \dfrac{(x-1)^2}{10} = 1$
Center $(1,3)$
Transverse axis on $x = 1$
Conjugate axis on $y = 3$
Vertices $(1, 3+\sqrt{11}), (1, 3-\sqrt{11})$
Foci $(1, 3+\sqrt{21}), (1, 3-\sqrt{21})$
Asymptotes $y = \pm\frac{\sqrt{110}}{10}(x-1)+3$



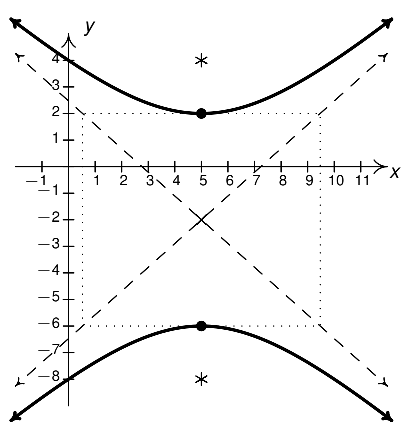
5. $\dfrac{(x+4)^2}{16} - \dfrac{(y-4)^2}{1} = 1$
Center $(-4,4)$
Transverse axis on $y = 4$
Conjugate axis on $x = -4$
Vertices $(-8,4), (0,4)$
Foci $(-4+\sqrt{17}, 4), (-4-\sqrt{17}, 4)$
Asymptotes $y = \pm\frac{1}{4}(x+4)+4$



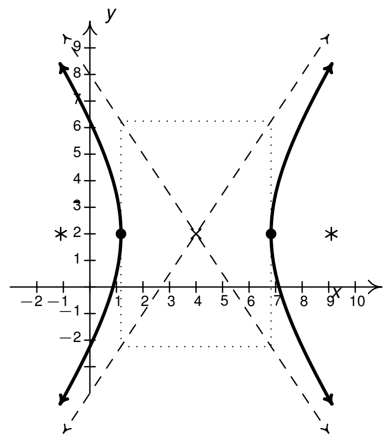
6. $\dfrac{(x+1)^2}{9} - \dfrac{(y-3)^2}{4} = 1$
Center $(-1,3)$
Transverse axis on $y = 3$
Conjugate axis on $x = -1$
Vertices $(2,3), (-4,3)$
Foci $\left(-1+\sqrt{13}, 3\right), \left(-1-\sqrt{13}, 3\right)$
Asymptotes $y = \pm\frac{2}{3}(x+1)+3$



7. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$
Center $(5,-2)$
Transverse axis on $x = 5$
Conjugate axis on $y = -2$
Vertices $(5,2), (5,-6)$
Foci $(5,4), (5,-8)$
Asymptotes $y = \pm\frac{2\sqrt{5}}{5}(x-5)-2$



8. $\dfrac{(x-4)^2}{8} - \dfrac{(y-2)^2}{18} = 1$
Center $(4, 2)$
Transverse axis on $y = 2$
Conjugate axis on $x = 4$
Vertices $\left(4 + 2\sqrt{2}, 2\right), \left(4 - 2\sqrt{2}, 2\right)$
Foci $\left(4 + \sqrt{26}, 2\right), \left(4 - \sqrt{26}, 2\right)$
Asymptotes $y = \pm\frac{3}{2}(x-4) + 2$



9. $\dfrac{x^2}{3} - \dfrac{(y-5)^2}{12} = 1$
Center $(0, 5)$
Transverse axis on $y = 5$
Conjugate axis on $x = 0$
Vertices $(\sqrt{3}, 5), (-\sqrt{3}, 5)$
Foci $(\sqrt{15}, 5), (-\sqrt{15}, 5)$
Asymptotes $y = \pm 2x + 5$

10. $\dfrac{(y+2)^2}{5} - \dfrac{(x-3)^2}{18} = 1$
Center $(3, -2)$
Transverse axis on $x = 3$
Conjugate axis on $y = -2$
Vertices $(3, -2 + \sqrt{5}), (3, -2 - \sqrt{5})$
Foci $(3, -2 + \sqrt{23}), (3, -2 - \sqrt{23})$
Asymptotes $y = \pm\frac{\sqrt{10}}{6}(x-3) - 2$

11. $\dfrac{(x-3)^2}{25} - \dfrac{(y+1)^2}{9} = 1$
Center $(3, -1)$
Transverse axis on $y = -1$
Conjugate axis on $x = 3$
Vertices $(8, -1), (-2, -1)$
Foci $\left(3 + \sqrt{34}, -1\right), \left(3 - \sqrt{34}, -1\right)$
Asymptotes $y = \pm\frac{3}{5}(x-3) - 1$

12. $\dfrac{(y+4)^2}{6} - \dfrac{(x+2)^2}{5} = 1$
Center $(-2, -4)$
Transverse axis on $x = -2$
Conjugate axis on $y = -4$
Vertices $\left(-2, -4 + \sqrt{6}\right), \left(-2, -4 - \sqrt{6}\right)$
Foci $\left(-2, -4 + \sqrt{11}\right), \left(-2, -4 - \sqrt{11}\right)$
Asymptotes $y = \pm\frac{\sqrt{30}}{5}(x+2) - 4$

13.

For number 1:

- $f(x) = \frac{3}{4}\sqrt{x^2 - 16}$ represents the upper half of the hyperbola.
- $g(x) = -\frac{3}{4}\sqrt{x^2 - 16}$ represents the lower half of the hyperbola.

For number 3:

- $f(x) = -3 + \frac{3}{2}\sqrt{x^2 - 4x}$ represents the upper half of the hyperbola.
- $g(x) = -3 - \frac{3}{2}\sqrt{x^2 - 4x}$ represents the lower half of the hyperbola.

For number 5:

- $f(x) = 4 + \frac{1}{4}\sqrt{x^2 + 8x}$ represents the upper half of the hyperbola.
- $g(x) = 4 - \frac{1}{4}\sqrt{x^2 + 8x}$ represents the lower half of the hyperbola.

For number 7:

- $f(x) = -2 + \frac{2}{5}\sqrt{5x^2 - 50x + 225}$ represents the upper half of the hyperbola.
- $g(x) = -2 - \frac{2}{5}\sqrt{5x^2 - 50x + 225}$ represents the lower half of the hyperbola.

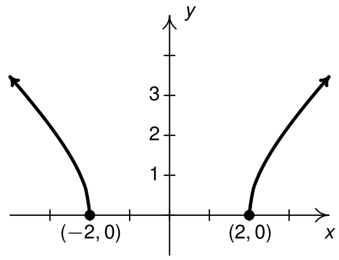
For number 9:

- $f(x) = 5 + 2\sqrt{x^2 - 3}$ represents the upper half of the hyperbola.
- $g(x) = 5 - 2\sqrt{x^2 - 3}$ represents the lower half of the hyperbola.

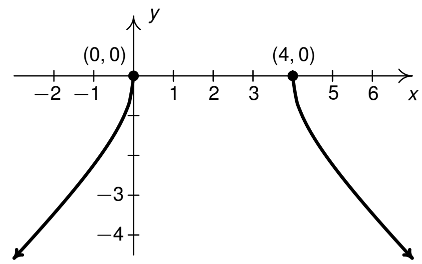
For number 11:

- $f(x) = -1 + \frac{3}{5}\sqrt{x^2 - 6x - 16}$ represents the upper half of the hyperbola.
- $g(x) = -1 - \frac{3}{5}\sqrt{x^2 - 6x - 16}$ represents the lower half of the hyperbola.

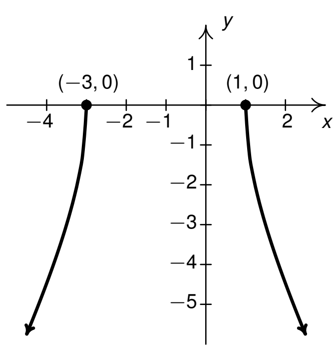
14. $f(x) = \sqrt{x^2 - 4}$



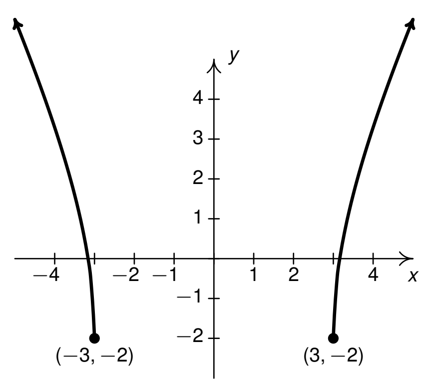
15. $g(x) = -\sqrt{x^2 - 4x}$



16. $f(x) = -2\sqrt{x^2 + 2x - 3}$



17. $g(x) = -2 + 2\sqrt{x^2 - 9}$



18. $\dfrac{x^2}{16} - \dfrac{y^2}{16} = 1$

19. $\dfrac{(y-4)^2}{4} - \dfrac{(x-4)^2}{3} = 1$

20. $\dfrac{(y-7)^2}{16} - \dfrac{(x-3)^2}{9} = 1$

21. $\dfrac{(x-4)^2}{16} - \dfrac{(y-1)^2}{33} = 1$

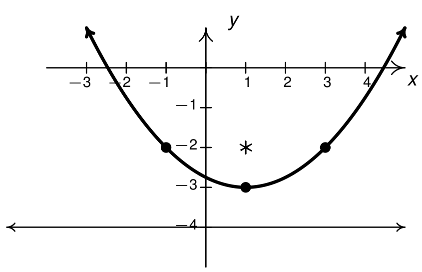
22. $\dfrac{y^2}{25} - \dfrac{x^2}{39} = 1$

23. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

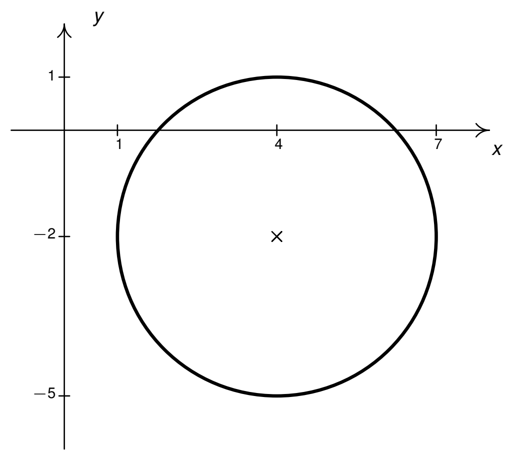
24. $\dfrac{(x-8)^2}{25} - \dfrac{(y-2)^2}{4} = 1$

25. $\dfrac{(x-6)^2}{256} - \dfrac{(y-5)^2}{64} = 1$

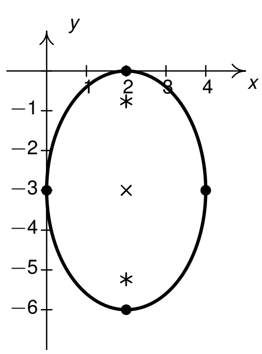
26. $(x-1)^2 = 4(y+3)$



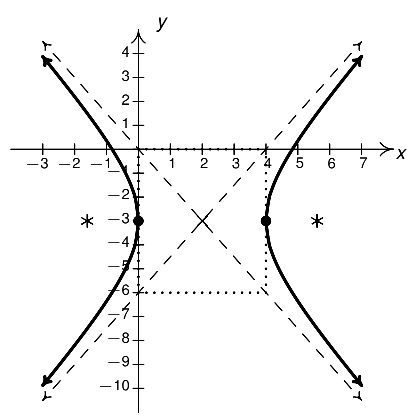
27. $(x-4)^2 + (y+2)^2 = 9$



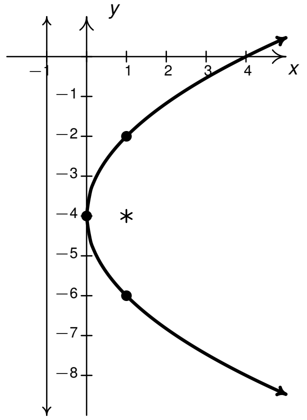
28. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$



29. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$

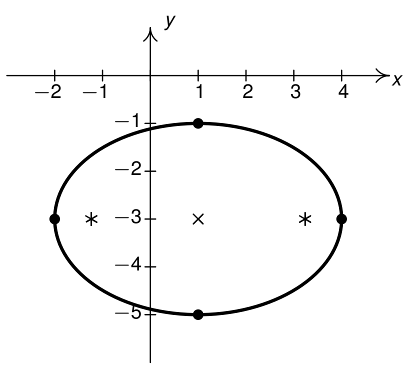


30. $(y+4)^2 = 4x$



31. $\dfrac{(x-1)^2}{1} + \dfrac{y^2}{4} = 0$
The graph is the point $(1,0)$ only.

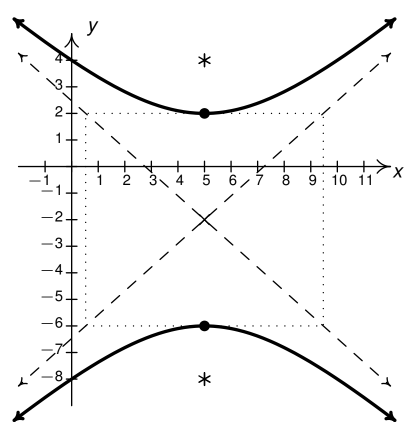
32. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$



33. $(x-3)^2 + (y+2)^2 = -1$
There is no graph.

34. $\dfrac{(x+3)^2}{2} + \dfrac{(y-1)^2}{1} = -\dfrac{3}{4}$
There is no graph.

35. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$



37. By placing Station A at $(0,-50)$ and Station B at $(0,50)$, the two second time difference yields the hyperbola $\frac{y^2}{36}-\frac{x^2}{2464}=1$ with foci A and B and center $(0,0)$. Placing Station C at $(-150,-50)$ and using foci A and C gives us a center of $(-75,-50)$ and the hyperbola $\frac{(x+75)^2}{225}-\frac{(y+50)^2}{5400}=1$. The point of intersection of these two hyperbolas which is closer to A than B and closer to A than C is $(-57.8444, -9.21336)$ so that is the epicenter.

38. (b) $\dfrac{x^2}{9}-\dfrac{y^2}{27}=1$.

39. The tower may be modeled (approximately)[10] by $\frac{x^2}{12100}-\frac{(y-330)^2}{34203}=1$. To find the height, we plug in $x=137.5$ which yields $y\approx 191$ or $y\approx 469$. Since the top of the tower is above the narrowest point, we get the tower is approximately 469 feet tall.

[10]The exact value underneath $(y-330)^2$ is $\frac{52707600}{1541}$ in case you need more precision.

Chapter 9

Systems of Equations and Matrices

9.1 Systems of Linear Equations: Gaussian Elimination

Up until now, when we concerned ourselves with solving different types of equations there was only one equation to solve at a time. Given an equation $f(x) = g(x)$, we could check our solutions geometrically by finding where the graphs of $y = f(x)$ and $y = g(x)$ intersect. The x-coordinates of these intersection points correspond to the solutions to the equation $f(x) = g(x)$, and the y-coordinates were largely ignored. If we modify the problem and ask for the intersection points of the graphs of $y = f(x)$ and $y = g(x)$, where both the solution to x and y are of interest, we have what is known as a *system of equations*, written as

$$\left\{ \begin{array}{rcl} y & = & f(x) \\ y & = & g(x) \end{array} \right.$$

The 'curly bracket' notation means we are to find all *pairs* of points (x, y) which satisfy *both* equations.
We assume the reader has some experience with systems of equations from high school algebra - specifically systems of linear equations comprised of two equations and two unknowns. We encourage the reader to read through Section A.6 before proceeding if for no other reason than to refresh themselves on the basic mechanics and vocabulary involved. In order to move this section beyond a review of high school algebra, we define what is meant by a linear equation in n variables.

Definition 9.1. A **linear equation in *n* variables**, $x_1, x_2, \ldots, x_n$, is an equation of the form:

$$a_1x_1 + a_2x_2 + ... + a_nx_n = c,$$

where $a_1, a_2, \ldots a_n$ and c are real numbers and at least one of $a_1, a_2, \ldots, a_n$ is nonzero.

Instead of using more familiar variables like x, y, and even z and/or w in Definition 9.1, we use subscripts to distinguish the different variables. We have no idea how many variables may be involved, so we use numbers to distinguish them instead of letters. (There is an endless supply of distinct numbers.)
As an example, the linear equation $3x_1 - x_2 = 4$ represents the same relationship between the variables x_1 and x_2 as the equation $3x - y = 4$ does between the variables x and y. And, just as we cannot combine the terms in the expression $3x - y$, we cannot combine the terms in the expression $3x_1 - x_2$.

Coupling more than one linear equation in n variables results in a **system of linear equations in n variables**. When solving these systems, it becomes increasingly important to keep track of what operations are performed to which equations and to develop a strategy based on the kind of manipulations (substitution and elimination) taught in high school. To this end, we first remind ourselves of the maneuvers which can be applied to a system of linear equations that result in an equivalent system.[1]

Theorem 9.1. Given a system of equations, the following moves will result in an equivalent system:

- Interchange the position of any two equations.
- Replace an equation with a nonzero multiple of itself.[a]
- Replace an equation with itself plus a nonzero multiple of another equation.

[a]That is, an equation which results from multiplying both sides of the equation by the same nonzero number.

The first move, while it obviously admits an equivalent system, seems silly to state, but our perception will change as we consider more equations and more variables in this, and later sections.

Consider the system of equations

$$\left\{ \begin{array}{rcr} x - \frac{1}{3}y + \frac{1}{2}z & = & 1 \\ y - \frac{1}{2}z & = & 4 \\ z & = & -1 \end{array} \right.$$

We have $z = -1$, so we substitute this into the second equation $y - \frac{1}{2}(-1) = 4$ to obtain $y = \frac{7}{2}$. Substituting $y = \frac{7}{2}$ and $z = -1$ into the first equation we get $x - \frac{1}{3}\left(\frac{7}{2}\right) + \frac{1}{2}(-1) = 1$. This gives $x = \frac{8}{3}$. The reader can verify that these values of x, y and z satisfy all three original equations.

It is tempting for us to write the solution to this system by extending the usual (x, y) notation to (x, y, z) and list our solution as $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$. The question quickly becomes what does an 'ordered triple' like $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$ represent? Just as ordered pairs are used to locate points on the two-dimensional plane, ordered triples can be used to locate points in space.[2]

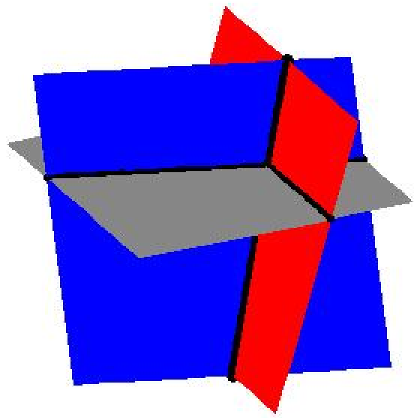
Moreover, just as equations involving the variables x and y describe graphs of one-dimensional lines and curves in the two-dimensional plane, equations involving variables x, y, and z describe objects called *surfaces* in three-dimensional space. Each of the equations in the above system can be visualized as a plane situated in three-space. Geometrically, the system is trying to find the intersection, or common point, of all three planes. If you imagine three sheets of notebook paper each representing a portion of these planes, you will start to see the complexities involved in how three such planes can intersect.

Below is a sketch of the three planes. It turns out that any two of these planes intersect in a line,[3] so our intersection point is where all three of these lines meet.

[1]That is, a system with the same solution set.

[2]You were asked to think about this in Exercise 21 in Section A.3.

[3]These lines are described by 'parametric solutions' to the systems formed by taking any two of these equations by themselves. (Again, see Section A.6.) We'll see an example of this sort of solution in this section shortly.



Since the geometry for equations involving more than two variables is complicated, we will focus our efforts on the algebra. Returning to the system

$$\left\{ \begin{array}{rcr} x - \frac{1}{3}y + \frac{1}{2}z & = & 1 \\ y - \frac{1}{2}z & = & 4 \\ z & = & -1 \end{array} \right.$$

we note the reason it was so easy to solve is because of its structure. The third equation is solved for z and the second equation involves only y and z. Since the coefficient of y is 1, it makes it easy to solve for y using our known value for z. Lastly, the coefficient of x in the first equation is 1 making it easy to substitute the known values of y and z and then solve for x.

We formalize this pattern below for the most general systems of linear equations. Again, we use subscripted variables to describe the general case. The variable with the smallest subscript in a given equation is typically called the *leading variable* of that equation.

Definition 9.2. A system of linear equations with variables $x_1, x_2, \ldots x_n$ is said to be in **triangular form** provided all of the following conditions hold:

1. The subscripts of the variables in each equation are always increasing from left to right.
2. The leading variable in each equation has coefficient 1.
3. The subscript on the leading variable in a given equation is greater than the subscript on the leading variable in the equation above it.
4. Any equation without variables[a] cannot be placed above an equation with variables.

[a] necessarily an identity or contradiction

In our previous system, if we make the obvious choices $x = x_1$, $y = x_2$, and $z = x_3$, we see that the system is in triangular form.[4] An example of a more complicated system in triangular form is

$$\left\{ \begin{array}{rcr} x_1 - 4x_3 + x_4 - x_6 & = & 6 \\ x_2 + 2x_3 & = & 1 \\ x_4 + 3x_5 - x_6 & = & 8 \\ x_5 + 9x_6 & = & 10 \end{array} \right.$$

Our goal henceforth will be to transform a given system of linear equations into triangular form using the moves in Theorem 9.1.

Example 9.1.1. Use Theorem 9.1 to put the following systems into triangular form and then solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.[5]

1. $\left\{ \begin{array}{rcr} 3x - y + z & = & 3 \\ 2x - 4y + 3z & = & 16 \\ x - y + z & = & 5 \end{array} \right.$
2. $\left\{ \begin{array}{rcr} 2x + 3y - z & = & 1 \\ 10x - z & = & 2 \\ 4x - 9y + 2z & = & 5 \end{array} \right.$
3. $\left\{ \begin{array}{rcr} 3x_1 + x_2 + x_4 & = & 6 \\ 2x_1 + x_2 - x_3 & = & 4 \\ x_2 - 3x_3 - 2x_4 & = & 0 \end{array} \right.$

Solution. For definitiveness, we label the topmost equation in each system $E1$, the equation beneath that $E2$, and so forth.

1. We put the system in triangular form using an algorithm known as *Gaussian Elimination*. Starting with x, we transform the system so that conditions 2 and 3 in Definition 9.2 are satisfied. Then we move on to the next variable, in this case y, and repeat.

   Since the variables in all of the equations have a consistent ordering from left to right, our first move is to get an x in $E1$'s spot with a coefficient of 1. While there are many ways to do this, the easiest is to apply the first move listed in Theorem 9.1 and interchange $E1$ and $E3$.

$$\left\{ \begin{array}{lrcr} (E1) & 3x - y + z & = & 3 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & x - y + z & = & 5 \end{array} \right. \xrightarrow{\text{Switch } E1 \text{ and } E3} \left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & 3x - y + z & = & 3 \end{array} \right.$$

   To satisfy Definition 9.2, we need to eliminate the x's from $E2$ and $E3$. We accomplish this by replacing each of them with a sum of themselves and a multiple of $E1$. To eliminate the x from $E2$, we need to multiply $E1$ by -2 then add; to eliminate the x from $E3$, we need to multiply $E1$ by -3 then add. Applying the third move listed in Theorem 9.1 twice, we get

$$\left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & 3x - y + z & = & 3 \end{array} \right. \xrightarrow[\text{Replace } E3 \text{ with } -3E1 + E3]{\text{Replace } E2 \text{ with } -2E1 + E2} \left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & -2y + z & = & 6 \\ (E3) & 2y - 2z & = & -12 \end{array} \right.$$

[4] If letters are used instead of subscripted variables, Definition 9.2 can be suitably modified using alphabetical order of the variables instead of numerical order on the subscripts of the variables.

[5] See Section A.6 for a review of these terms.

Now we enforce the conditions stated in Definition 9.2 for the variable y. To that end we need to get the coefficient of y in $E2$ equal to 1. We apply the second move listed in Theorem 9.1 and replace $E2$ with itself times $-\frac{1}{2}$.

$$\left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & -2y + z & = & 6 \\ (E3) & 2y - 2z & = & -12 \end{array} \right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{2}E2} \left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & 2y - 2z & = & -12 \end{array} \right.$$

To eliminate the y in $E3$, we add $-2E2$ to it.

$$\left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & 2y - 2z & = & -12 \end{array} \right. \xrightarrow{\text{Replace } E3 \text{ with } -2E2 + E3} \left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & -z & = & -6 \end{array} \right.$$

Finally, we apply the second move from Theorem 9.1 one last time and multiply $E3$ by -1 to satisfy the conditions of Definition 9.2 for the variable z.

$$\left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & -z & = & -6 \end{array} \right. \xrightarrow{\text{Replace } E3 \text{ with } -1E3} \left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & z & = & 6 \end{array} \right.$$

Substituting $z = 6$ into $E2$ gives $y - 3 = -3$ so that $y = 0$. With $y = 0$ and $z = 6$, $E1$ becomes $x - 0 + 6 = 5$, or $x = -1$. Hence, our solution is $(-1, 0, 6)$. We leave it to the reader to check that substituting the respective values for x, y, and z into the original system results in three identities.

Since there is a solution to the system, the system is classified as consistent. Since there are no free variables,[6] the system is classified as independent.

2. Proceeding as above, our first step is to get an equation with x in the $E1$ position with 1 as its coefficient. Since there is no easy fix, we multiply $E1$ by $\frac{1}{2}$.

$$\left\{ \begin{array}{lrcr} (E1) & 2x + 3y - z & = & 1 \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{array} \right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{2}E1} \left\{ \begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{array} \right.$$

Now it's time to take care of the x's in $E2$ and $E3$.

$$\left\{ \begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{array} \right. \xrightarrow[\text{Replace } E3 \text{ with } -4E1 + E3]{\text{Replace } E2 \text{ with } -10E1 + E2} \left\{ \begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y + 4z & = & -3 \\ (E3) & -15y + 4z & = & 3 \end{array} \right.$$

Our next step is to get the coefficient of y in $E2$ equal to 1. To that end, we have

[6] Again, see Section A.6 for a review of this concept, if needed.

$$\left\{\begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y + 4z & = & -3 \\ (E3) & -15y + 4z & = & 3 \end{array}\right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{15}E2} \left\{\begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & y - \frac{4}{15}z & = & \frac{1}{5} \\ (E3) & -15y + 4z & = & 3 \end{array}\right.$$

Finally, we rid $E3$ of y.

$$\left\{\begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & y - \frac{4}{15}z & = & \frac{1}{5} \\ (E3) & -15y + 4z & = & 3 \end{array}\right. \xrightarrow{\text{Replace } E3 \text{ with } 15E2 + E3} \left\{\begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & 0 & = & 6 \end{array}\right.$$

The last equation, $0 = 6$, is a contradiction so the system has no solution. According to Theorem 9.1, since this system has no solutions, neither does the original, thus we have an inconsistent system.

3. For our last system, we begin by multiplying $E1$ by $\frac{1}{3}$ to get a coefficient of 1 on x_1.

$$\left\{\begin{array}{lrcr} (E1) & 3x_1 + x_2 + x_4 & = & 6 \\ (E2) & 2x_1 + x_2 - x_3 & = & 4 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{3}E1} \left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1 + x_2 - x_3 & = & 4 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right.$$

Next we eliminate x_1 from $E2$

$$\left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1 + x_2 - x_3 & = & 4 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right. \xrightarrow[\text{with } -2E1 + E2]{\text{Replace } E2} \left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right.$$

We switch $E2$ and $E3$ to get a coefficient of 1 for x_2.

$$\left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right. \xrightarrow{\text{Switch } E2 \text{ and } E3} \left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & x_2 - 3x_3 - 2x_4 & = & 0 \\ (E3) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \end{array}\right.$$

Finally, we eliminate x_2 in $E3$.

$$\left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & x_2 - 3x_3 - 2x_4 & = & 0 \\ (E3) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \end{array}\right. \xrightarrow[\text{with } -\frac{1}{3}E2 + E3]{\text{Replace } E3} \left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & x_2 - 3x_3 - 2x_4 & = & 0 \\ (E3) & 0 & = & 0 \end{array}\right.$$

Equation $E3$ reduces to $0 = 0$,which is always true. Since we have no equations with x_3 or x_4 as leading variables, they are both 'free' variables so we have a consistent dependent system.

We 'parametrize' the solution set by letting $x_3 = s$ and $x_4 = t$. From $E2$, we get $x_2 = 3s + 2t$. Substituting this and $x_4 = t$ into $E1$, we have $x_1 + \frac{1}{3}(3s + 2t) + \frac{1}{3}t = 2$ which gives $x_1 = 2 - s - t$.

Our solution is the set $\{(2 - s - t, 2s + 3t, s, t) \,|\, -\infty < s, t < \infty\}$.[7] We leave it to the reader to verify that the substitutions $x_1 = 2 - s - t$, $x_2 = 3s + 2t$, $x_3 = s$ and $x_4 = t$ satisfy the equations in the original system regardless of the choices made for the parameters s and t. $\square$

Like all algorithms, Gaussian Elimination has the advantage of always producing what we need, but it can also be inefficient at times. For example, when solving the second system in Example 9.1.1, it is clear after we eliminated the x's in the second step to get the system

$$\left\{ \begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y + 4z & = & -3 \\ (E3) & -15y + 4z & = & 3 \end{array} \right.$$

that equations $E2$ and $E3$, taken together, produce a contradiction. (We have identical left hand sides and different right hand sides.) However, the algorithm takes an additional two steps to reach this conclusion. We also note that substitution in Gaussian Elimination is delayed until all the elimination is done, whence the name *back-substitution*. This may also be inefficient in many cases.

Lastly, we note that the last system in Example 9.1.1 is underdetermined,[8] and as it is consistent, we necessarily have free variables in our answer. We close this section with a standard 'mixture' type application of systems of linear equations which features an application of a consistent dependent system.

Example 9.1.2. Lucas needs to create a 500 milliliters (mL) of a 40% acid solution. He has stock solutions of 30% and 90% acid as well as all of the distilled water he wants. Set-up and solve a system of linear equations which determines all of the possible combinations of the stock solutions and water which would produce the required solution.

Solution. We are after three unknowns, the amount (in mL) of the 30% stock solution (which we'll call x), the amount (in mL) of the 90% stock solution (which we'll call y) and the amount (in mL) of water (which we'll call w). We now need to determine some relationships between these variables.

Our goal is to produce 500 milliliters of a 40% acid solution. This product has two defining characteristics. First, it must be 500 mL; second, it must be 40% acid. We take each of these qualities in turn.

First, the total volume of 500 mL must be the sum of the volumes of the two stock solutions and the water:

$$\text{amount of 30\% stock solution} + \text{amount of 90\% stock solution} + \text{amount of water} = 500\,\text{mL}$$

Using our defined variables, this reduces to $x + y + w = 500$.

Next, we need to make sure the final solution is 40% acid. Since water contains no acid, the acid will come from the stock solutions only. We find 40% of 500 mL to be 200 mL which means the final solution must contain 200 mL of acid. We have

$$\text{amount of acid in 30\% stock solution} + \text{amount of acid 90\% stock solution} = 200\,\text{mL}$$

The amount of acid in x mL of 30% stock is $0.30x$ and the amount of acid in y mL of 90% solution is $0.90y$. We have $0.30x + 0.90y = 200$. Converting to fractions,[9] our system of equations becomes

[7] Here, any choice of s and t determines a point in 4-dimensional space. Yeah, we have trouble visualizing that, too.

[8] Recall this means we have fewer equations than unknowns.

[9] We do this only because we believe students can use all of the practice with fractions they can get!

$$\left\{ \begin{array}{rcl} x + y + w & = & 500 \\ \frac{3}{10}x + \frac{9}{10}y & = & 200 \end{array} \right.$$

We first eliminate the x from the second equation

$$\left\{ \begin{array}{lrcl} (E1) & x + y + w & = & 500 \\ (E2) & \frac{3}{10}x + \frac{9}{10}y & = & 200 \end{array} \right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{3}{10}E1 + E2} \left\{ \begin{array}{lrcl} (E1) & x + y + w & = & 500 \\ (E2) & \frac{3}{5}y - \frac{3}{10}w & = & 50 \end{array} \right.$$

Next, we get a coefficient of 1 on the leading variable in $E2$

$$\left\{ \begin{array}{lrcl} (E1) & x + y + w & = & 500 \\ (E2) & \frac{3}{5}y - \frac{3}{10}w & = & 50 \end{array} \right. \xrightarrow{\text{Replace } E2 \text{ with } \frac{5}{3}E2} \left\{ \begin{array}{lrcl} (E1) & x + y + w & = & 500 \\ (E2) & y - \frac{1}{2}w & = & \frac{250}{3} \end{array} \right.$$

Notice that we have no equation to determine w, and as such, w is free. Setting $w = t$ in $E2$, we get $y = \frac{1}{2}t + \frac{250}{3}$. Substituting for w and y in $E1$ gives $x + \left(\frac{1}{2}t + \frac{250}{3}\right) + t = 500$ so that $x = -\frac{3}{2}t + \frac{1250}{3}$.
This system is consistent, dependent and its solution set is $\left\{\left(-\frac{3}{2}t + \frac{1250}{3}, \frac{1}{2}t + \frac{250}{3}, t\right) \mid -\infty < t < \infty\right\}$.
While this answer checks algebraically, we have neglected to take into account that x, y and w, being amounts of acid and water, need to be nonnegative. That is, $x \geq 0$, $y \geq 0$ and $w \geq 0$.
The constraint $x \geq 0$ gives us $-\frac{3}{2}t + \frac{1250}{3} \geq 0$, or $t \leq \frac{2500}{9}$. From $y \geq 0$, we get $\frac{1}{2}t + \frac{250}{3} \geq 0$ or $t \geq -\frac{500}{3}$. The condition $z \geq 0$ yields $t \geq 0$, and we see that when we take the set theoretic intersection of these intervals, we get $0 \leq t \leq \frac{2500}{9}$. This gives our final answer is $\left\{\left(-\frac{3}{2}t + \frac{1250}{3}, \frac{1}{2}t + \frac{250}{3}, t\right) \mid 0 \leq t \leq \frac{2500}{9}\right\}$.
Of what practical use is our answer? Suppose there is only 100 mL of the 90% solution remaining and it is due to expire. Can we use all of it to make our required solution? We would have $y = 100$ so that $\frac{1}{2}t + \frac{250}{3} = 100$, and we get $t = \frac{100}{3}$. This means the amount of 30% solution required is $x = -\frac{3}{2}t + \frac{1250}{3} = -\frac{3}{2}\left(\frac{100}{3}\right) + \frac{1250}{3} = \frac{1100}{3}$ mL, and for the water, $w = t = \frac{100}{3}$ mL. The reader is invited to check that mixing these three amounts of our constituent solutions produces the required 40% acid mix. $\Box$

9.1.1 Exercises

In Exercises 1 - 18, put each system of linear equations into triangular form and solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

1. $\left\{ \begin{array}{rcr} -5x + y & = & 17 \\ x + y & = & 5 \end{array} \right.$

2. $\left\{ \begin{array}{rcr} x + y + z & = & 3 \\ 2x - y + z & = & 0 \\ -3x + 5y + 7z & = & 7 \end{array} \right.$

3. $\left\{ \begin{array}{rcr} 4x - y + z & = & 5 \\ 2y + 6z & = & 30 \\ x + z & = & 5 \end{array} \right.$

4. $\left\{ \begin{array}{rcr} 4x - y + z & = & 5 \\ 2y + 6z & = & 30 \\ x + z & = & 6 \end{array} \right.$

5. $\left\{ \begin{array}{rcr} x + y + z & = & -17 \\ y - 3z & = & 0 \end{array} \right.$

6. $\left\{ \begin{array}{rcr} x - 2y + 3z & = & 7 \\ -3x + y + 2z & = & -5 \\ 2x + 2y + z & = & 3 \end{array} \right.$

7. $\left\{ \begin{array}{rcr} 3x - 2y + z & = & -5 \\ x + 3y - z & = & 12 \\ x + y + 2z & = & 0 \end{array} \right.$

8. $\left\{ \begin{array}{rcr} 2x - y + z & = & -1 \\ 4x + 3y + 5z & = & 1 \\ 5y + 3z & = & 4 \end{array} \right.$

9. $\left\{ \begin{array}{rcr} x - y + z & = & -4 \\ -3x + 2y + 4z & = & -5 \\ x - 5y + 2z & = & -18 \end{array} \right.$

10. $\left\{ \begin{array}{rcr} 2x - 4y + z & = & -7 \\ x - 2y + 2z & = & -2 \\ -x + 4y - 2z & = & 3 \end{array} \right.$

11. $\left\{ \begin{array}{rcr} 2x - y + z & = & 1 \\ 2x + 2y - z & = & 1 \\ 3x + 6y + 4z & = & 9 \end{array} \right.$

12. $\left\{ \begin{array}{rcr} x - 3y - 4z & = & 3 \\ 3x + 4y - z & = & 13 \\ 2x - 19y - 19z & = & 2 \end{array} \right.$

13. $\left\{ \begin{array}{rcr} x + y + z & = & 4 \\ 2x - 4y - z & = & -1 \\ x - y & = & 2 \end{array} \right.$

14. $\left\{ \begin{array}{rcr} x - y + z & = & 8 \\ 3x + 3y - 9z & = & -6 \\ 7x - 2y + 5z & = & 39 \end{array} \right.$

15. $\left\{ \begin{array}{rcr} 2x - 3y + z & = & -1 \\ 4x - 4y + 4z & = & -13 \\ 6x - 5y + 7z & = & -25 \end{array} \right.$

16. $\left\{ \begin{array}{rcr} 2x_1 + x_2 - 12x_3 - x_4 & = & 16 \\ -x_1 + x_2 + 12x_3 - 4x_4 & = & -5 \\ 3x_1 + 2x_2 - 16x_3 - 3x_4 & = & 25 \\ x_1 + 2x_2 - 5x_4 & = & 11 \end{array} \right.$

17. $\left\{ \begin{array}{rcr} x_1 - x_3 & = & -2 \\ 2x_2 - x_4 & = & 0 \\ x_1 - 2x_2 + x_3 & = & 0 \\ -x_3 + x_4 & = & 1 \end{array} \right.$

18. $\left\{ \begin{array}{rcr} x_1 - x_2 - 5x_3 + 3x_4 & = & -1 \\ x_1 + x_2 + 5x_3 - 3x_4 & = & 0 \\ x_2 + 5x_3 - 3x_4 & = & 1 \\ x_1 - 2x_2 - 10x_3 + 6x_4 & = & -1 \end{array} \right.$

19. Find two other forms of the parametric solution to Exercise 3 above by reorganizing the equations so that x or y can be the free variable.

20. At The Crispy Critter's Head Shop and Patchouli Emporium along with their dried up weeds, sunflower seeds and astrological postcards they sell an herbal tea blend. By weight, Type I herbal tea is 30% peppermint, 40% rose hips and 30% chamomile, Type II has percents 40%, 20% and 40%, respectively, and Type III has percents 35%, 30% and 35%, respectively. How much of each Type of tea is needed to make 2 pounds of a new blend of tea that is equal parts peppermint, rose hips and chamomile?

21. Discuss with your classmates how you would approach Exercise 20 above if they needed to use up a pound of Type I tea to make room on the shelf for a new canister.

22. If you were to try to make 100 mL of a 60% acid solution using stock solutions at 20% and 40%, respectively, what would the triangular form of the resulting system look like? Explain.

9.1.2 Answers

Because triangular form is not unique, we give only one possible answer to that part of the question. Yours may be different and still be correct.

1. $\left\{\begin{array}{rcl} x+y &=& 5 \\ y &=& 7 \end{array}\right.$

   Consistent independent
   Solution $(-2,7)$

2. $\left\{\begin{array}{rcr} x-\frac{5}{3}y-\frac{7}{3}z &=& -\frac{7}{3} \\ y+\frac{5}{4}z &=& 2 \\ z &=& 0 \end{array}\right.$

   Consistent independent
   Solution $(1,2,0)$

3. $\left\{\begin{array}{rcr} x-\frac{1}{4}y+\frac{1}{4}z &=& \frac{5}{4} \\ y+3z &=& 15 \\ 0 &=& 0 \end{array}\right.$

   Consistent dependent
   Solution $(-t+5,-3t+15,t)$
   for all real numbers t

4. $\left\{\begin{array}{rcr} x-\frac{1}{4}y+\frac{1}{4}z &=& \frac{5}{4} \\ y+3z &=& 15 \\ 0 &=& 1 \end{array}\right.$

   Inconsistent
   No solution

5. $\left\{\begin{array}{rcr} x+y+z &=& -17 \\ y-3z &=& 0 \end{array}\right.$

   Consistent dependent
   Solution $(-4t-17,3t,t)$
   for all real numbers t

6. $\left\{\begin{array}{rcr} x-2y+3z &=& 7 \\ y-\frac{11}{5}z &=& -\frac{16}{5} \\ z &=& 1 \end{array}\right.$

   Consistent independent
   Solution $(2,-1,1)$

7. $\left\{\begin{array}{rcr} x+y+2z &=& 0 \\ y-\frac{3}{2}z &=& 6 \\ z &=& -2 \end{array}\right.$

   Consistent independent
   Solution $(1,3,-2)$

8. $\left\{\begin{array}{rcr} x-\frac{1}{2}y+\frac{1}{2}z &=& -\frac{1}{2} \\ y+\frac{3}{5}z &=& \frac{3}{5} \\ 0 &=& 1 \end{array}\right.$

   Inconsistent
   no solution

9. $\left\{\begin{array}{rcr} x-y+z &=& -4 \\ y-7z &=& 17 \\ z &=& -2 \end{array}\right.$

   Consistent independent
   Solution $(1,3,-2)$

10. $\left\{\begin{array}{rcr} x-2y+2z &=& -2 \\ y &=& \frac{1}{2} \\ z &=& 1 \end{array}\right.$

    Consistent independent
    Solution $\left(-3,\frac{1}{2},1\right)$

11. $\begin{cases} x - \frac{1}{2}y + \frac{1}{2}z = \frac{1}{2} \\ y - \frac{2}{3}z = 0 \\ z = 1 \end{cases}$

Consistent independent
Solution $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$

12. $\begin{cases} x - 3y - 4z = 3 \\ y + \frac{11}{13}z = \frac{4}{13} \\ 0 = 0 \end{cases}$

Consistent dependent
Solution $\left(\frac{19}{13}t + \frac{51}{13}, -\frac{11}{13}t + \frac{4}{13}, t\right)$
for all real numbers t

13. $\begin{cases} x + y + z = 4 \\ y + \frac{1}{2}z = \frac{3}{2} \\ 0 = 1 \end{cases}$

Inconsistent
no solution

14. $\begin{cases} x - y + z = 8 \\ y - 2z = -5 \\ z = 1 \end{cases}$

Consistent independent
Solution $(4, -3, 1)$

15. $\begin{cases} x - \frac{3}{2}y + \frac{1}{2}z = -\frac{1}{2} \\ y + z = -\frac{11}{2} \\ 0 = 0 \end{cases}$

Consistent dependent
Solution $\left(-2t - \frac{35}{4}, -t - \frac{11}{2}, t\right)$
for all real numbers t

16. $\begin{cases} x_1 + \frac{2}{3}x_2 - \frac{16}{3}x_3 - x_4 = \frac{25}{3} \\ x_2 + 4x_3 - 3x_4 = 2 \\ 0 = 0 \\ 0 = 0 \end{cases}$

Consistent dependent
Solution $(8s - t + 7, -4s + 3t + 2, s, t)$
for all real numbers s and t

17. $\begin{cases} x_1 - x_3 = -2 \\ x_2 - \frac{1}{2}x_4 = 0 \\ x_3 - \frac{1}{2}x_4 = 1 \\ x_4 = 4 \end{cases}$

Consistent independent
Solution $(1, 2, 3, 4)$

18. $\begin{cases} x_1 - x_2 - 5x_3 + 3x_4 = -1 \\ x_2 + 5x_3 - 3x_4 = \frac{1}{2} \\ 0 = 1 \\ 0 = 0 \end{cases}$

Inconsistent
No solution

19. If x is the free variable then the solution is $(t, 3t, -t + 5)$ and if y is the free variable then the solution is $\left(\frac{1}{3}t, t, -\frac{1}{3}t + 5\right)$.

20. $\frac{4}{3} - \frac{1}{2}t$ pounds of Type I, $\frac{2}{3} - \frac{1}{2}t$ pounds of Type II and t pounds of Type III where $0 \leq t \leq \frac{4}{3}$.

9.2 Systems of Linear Equations: Augmented Matrices

In Section 9.1 we introduced Gaussian Elimination as a means of transforming a system of linear equations into triangular form with the ultimate goal of producing an equivalent system of linear equations which is easier to solve. If we take a step back and study the process, we see that all of our moves are determined entirely by the *coefficients* of the variables involved, and not the variables themselves. Much the same thing happened when we studied long division in Section 2.2. Just as we developed synthetic division to streamline that process, in this section, we introduce a similar bookkeeping device to help us solve systems of linear equations, the *matrix*.

A *matrix* as a rectangular array of real numbers. We typically enclose matrices with 'square brackets' of the likes of '[' and ']', and we size matrices by the number of rows and columns they have. For example, the *size* (sometimes called the *dimension*) of

$$\left[\begin{array}{rrr} 3 & 0 & -1 \\ 2 & -5 & 10 \end{array}\right]$$

is 2×3 because it has 2 rows and 3 columns. The individual numbers in a matrix are called its *entries* and are usually labeled with double subscripts: the first tells which row the element is in and the second tells which column it is in. The rows are numbered from top to bottom and the columns are numbered from left to right. Matrices themselves are usually denoted by uppercase letters (A, B, C, etc.) while their entries are usually denoted by the corresponding letter. So, for instance, if we have

$$A = \left[\begin{array}{rrr} 3 & 0 & -1 \\ 2 & -5 & 10 \end{array}\right]$$

then $a_{11} = 3$, $a_{12} = 0$, $a_{13} = -1$, $a_{21} = 2$, $a_{22} = -5$, and $a_{23} = 10$. We shall explore matrices as mathematical objects with their own algebra in Section 9.3 and introduce them here solely as a bookkeeping device. Consider the system of linear equations from number 2 in Example 9.1.1

$$\left\{ \begin{array}{lrcl} (E1) & 2x + 3y - z & = & 1 \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{array} \right.$$

We encode this system into a matrix by assigning each equation to a corresponding row. Within that row, each variable and the constant gets its own column, and to separate the variables on the left hand side of the equation from the constants on the right hand side, we use a vertical bar, $|$. Note that in $E2$, since y is not present, we record its coefficient as 0. The matrix associated with this system is

$$\begin{array}{c} \\ (E1) \rightarrow \\ (E2) \rightarrow \\ (E3) \rightarrow \end{array} \begin{array}{c} \begin{array}{rrrr} x \quad & y \quad & z \quad & c \end{array} \\ \left[\begin{array}{rrr|r} 2 & 3 & -1 & 1 \\ 10 & 0 & -1 & 2 \\ 4 & -9 & 2 & 5 \end{array}\right] \end{array}$$

This matrix is called an *augmented matrix* because the column containing the constants is appended to the matrix containing the coefficients.[1] To solve this system, we can use the same kind operations on

[1] We shall study the coefficient and constant matrices separately in Section 9.3.

the *rows* of the matrix that we performed on the *equations* of the system. More specifically, we have the following analog of Theorem 9.1 below.

Theorem 9.2. Row Operations: Given an augmented matrix for a system of linear equations, the following row operations produce an augmented matrix which corresponds to an equivalent system of linear equations.

- Interchange any two rows.
- Replace a row with a nonzero multiple of itself.[a]
- Replace a row with itself plus a nonzero multiple of another row.[b]

[a] That is, the row obtained by multiplying each entry in the row by the same nonzero number.
[b] Where we add entries in corresponding columns.

As a demonstration of the moves in Theorem 9.2, we revisit some of the steps that were used in solving the systems of linear equations in Example 9.1.1 of Section 9.1. The reader is encouraged to perform the indicated operations on the rows of the augmented matrix to see that the machinations are identical to what is done to the coefficients of the variables in the equations. We first see a demonstration of switching two rows using the first step of part 1 in Example 9.1.1.

$$\left\{\begin{array}{lrcr} (E1) & 3x-y+z & = & 3 \\ (E2) & 2x-4y+3z & = & 16 \\ (E3) & x-y+z & = & 5 \end{array}\right. \xrightarrow{\text{Switch } E1 \text{ and } E3} \left\{\begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & 2x-4y+3z & = & 16 \\ (E3) & 3x-y+z & = & 3 \end{array}\right.$$

$$\left[\begin{array}{rrr|r} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right] \xrightarrow{\text{Switch } R1 \text{ and } R3} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 5 \\ 2 & -4 & 3 & 16 \\ 3 & -1 & 1 & 3 \end{array}\right]$$

Next, we have a demonstration of replacing a row with a nonzero multiple of itself using the first step of part 3 in Example 9.1.1.

$$\left\{\begin{array}{lrcr} (E1) & 3x_1+x_2+x_4 & = & 6 \\ (E2) & 2x_1+x_2-x_3 & = & 4 \\ (E3) & x_2-3x_3-2x_4 & = & 0 \end{array}\right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{3}E1} \left\{\begin{array}{lrcr} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1+x_2-x_3 & = & 4 \\ (E3) & x_2-3x_3-2x_4 & = & 0 \end{array}\right.$$

$$\left[\begin{array}{rrrr|r} 3 & 1 & 0 & 1 & 6 \\ 2 & 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -2 & 0 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } \frac{1}{3}R1} \left[\begin{array}{rrrr|r} 1 & \frac{1}{3} & 0 & \frac{1}{3} & 2 \\ 2 & 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -2 & 0 \end{array}\right]$$

Finally, we have an example of replacing a row with itself plus a multiple of another row using the second step from part 2 in Example 9.1.1.

$$\left\{\begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x-z & = & 2 \\ (E3) & 4x-9y+2z & = & 5 \end{array}\right. \xrightarrow[\text{Replace } E3 \text{ with } -4E1+E3]{\text{Replace } E2 \text{ with } -10E1+E2} \left\{\begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y+4z & = & -3 \\ (E3) & -15y+4z & = & 3 \end{array}\right.$$

$$\left[\begin{array}{rrr|r} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 10 & 0 & -1 & 2 \\ 4 & -9 & 2 & 5 \end{array}\right] \xrightarrow[\text{Replace } R3 \text{ with } -4R1+R3]{\text{Replace } R2 \text{ with } -10R1+R2} \left[\begin{array}{rrr|r} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -15 & 4 & -3 \\ 0 & -15 & 4 & 3 \end{array}\right]$$

The matrix equivalent of 'triangular form' is *row echelon form*. The reader is encouraged to refer to Definition 9.2 for comparison. Note that the analog of 'leading variable' of an equation is 'leading entry' of a row. Specifically, the first nonzero entry (if it exists) in a row is called the *leading entry* of that row.

Definition 9.3. A matrix is said to be in **row echelon form** provided all of the following conditions hold:

1. The first nonzero entry in each row is 1.
2. The leading 1 of a given row must be to the right of the leading 1 of the row above it.
3. Any row of all zeros cannot be placed above a row with nonzero entries.

To solve a system of a linear equations using an augmented matrix, we encode the system into an augmented matrix and apply Gaussian Elimination to the rows to get the matrix into row-echelon form. We then decode the matrix and back substitute. The next example illustrates this nicely.

Example 9.2.1. Use an augmented matrix to transform the following system of linear equations into triangular form. Solve the system.

$$\left\{\begin{array}{rcl} 3x - y + z & = & 8 \\ x + 2y - z & = & 4 \\ 2x + 3y - 4z & = & 10 \end{array}\right.$$

Solution. We first encode the system into an augmented matrix.

$$\left\{\begin{array}{rcl} 3x - y + z & = & 8 \\ x + 2y - z & = & 4 \\ 2x + 3y - 4z & = & 10 \end{array}\right. \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{rrr|r} 3 & -1 & 1 & 8 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & -4 & 10 \end{array}\right]$$

Thinking back to Gaussian Elimination at an equations level, our first order of business is to get x in $E1$ with a coefficient of 1. At the matrix level, this means getting a leading 1 in $R1$. This is in accordance with the first criteria in Definition 9.3. To that end, we interchange $R1$ and $R2$.

$$\left[\begin{array}{rrr|r} 3 & -1 & 1 & 8 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & -4 & 10 \end{array}\right] \xrightarrow{\text{Switch } R1 \text{ and } R2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 3 & -1 & 1 & 8 \\ 2 & 3 & -4 & 10 \end{array}\right]$$

Our next step is to eliminate the x's from $E2$ and $E3$. From a matrix standpoint, this means we need 0's below the leading 1 in $R1$. This guarantees the leading 1 in $R2$ will be to the right of the leading 1 in $R1$ in accordance with the second requirement of Definition 9.3.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 3 & -1 & 1 & 8 \\ 2 & 3 & -4 & 10 \end{array}\right] \xrightarrow[\text{Replace } R3 \text{ with } -2R1+R3]{\text{Replace } R2 \text{ with } -3R1+R2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & -7 & 4 & -4 \\ 0 & -1 & -2 & 2 \end{array}\right]$$

Now we repeat the above process for the variable y. That is, we need the leading entry in $R2$ to be 1.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & -7 & 4 & -4 \\ 0 & -1 & -2 & 2 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -\frac{1}{7}R2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & -1 & -2 & 2 \end{array}\right]$$

To ensure the leading 1 in $R3$ is to the right of the leading 1 in $R2$, we get a 0 in the second column of $R3$.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & -1 & -2 & 2 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } R2 + R3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & -\frac{18}{7} & \frac{18}{7} \end{array}\right]$$

Finally, we get the leading entry in $R3$ to be 1.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & -\frac{18}{7} & \frac{18}{7} \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -\frac{7}{18}R3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Decoding from the matrix gives a system in triangular form

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{\begin{array}{rcr} x + 2y - z & = & 4 \\ y - \frac{4}{7}z & = & \frac{4}{7} \\ z & = & -1 \end{array}\right.$$

We get $z = -1$, $y = \frac{4}{7}z + \frac{4}{7} = \frac{4}{7}(-1) + \frac{4}{7} = 0$ and $x = -2y + z + 4 = -2(0) + (-1) + 4 = 3$ for a final answer of $(3, 0, -1)$. We leave it to the reader to check. □

As part of Gaussian Elimination, we used row operations to obtain 0's beneath each leading 1 to put the matrix into row echelon form. If we also require that 0's are the only numbers above a leading 1, we have what is known as the *reduced row echelon form* of the matrix.

Definition 9.4. A matrix is said to be in **reduced row echelon form** provided both of the following conditions hold:

1. The matrix is in row echelon form.

2. The leading 1s are the only nonzero entry in their respective columns.

Of what significance is the reduced row echelon form of a matrix? To illustrate, let's take the row echelon form from Example 9.2.1 and perform the necessary steps to put into reduced row echelon form. We start by using the leading 1 in $R3$ to zero out the numbers in the rows above it.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[\text{Replace } R2 \text{ with } \frac{4}{7}R3 + R2]{\text{Replace } R1 \text{ with } R3 + R1} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Finally, we take care of the 2 in $R1$ above the leading 1 in $R2$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } -2R2+R1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

To our surprise and delight, when we decode this matrix, we obtain the solution instantly without having to deal with any back-substitution at all.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{\begin{array}{rcr} x & = & 3 \\ y & = & 0 \\ z & = & -1 \end{array}\right.$$

Note that in the previous discussion, we could have started with $R2$ and used it to get a zero above its leading 1 and then done the same for the leading 1 in $R3$. By starting with $R3$, however, we get more zeros first, and the more zeros there are, the faster the remaining calculations will be.[2] It is also worth noting that while a matrix has several[3] row echelon forms, it has only one reduced row echelon form. The process by which we have put a matrix into reduced row echelon form is called *Gauss-Jordan Elimination*.

Example 9.2.2. Solve the following system using an augmented matrix. Use Gauss-Jordan Elimination to put the augmented matrix into reduced row echelon form.

$$\left\{\begin{array}{rcl} x_2 - 3x_1 + x_4 & = & 2 \\ 2x_1 + 4x_3 & = & 5 \\ 4x_2 - x_4 & = & 3 \end{array}\right.$$

Solution. We first encode the system into a matrix. (Pay attention to the subscripts!)

$$\left\{\begin{array}{rcl} x_2 - 3x_1 + x_4 & = & 2 \\ 2x_1 + 4x_3 & = & 5 \\ 4x_2 - x_4 & = & 3 \end{array}\right. \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{rrrr|r} -3 & 1 & 0 & 1 & 2 \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

Next, we get a leading 1 in the first column of $R1$.

$$\left[\begin{array}{rrrr|r} -3 & 1 & 0 & 1 & 2 \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } -\frac{1}{3}R1} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

Now we eliminate the nonzero entry below our leading 1.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -2R1+R2} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & 4 & \frac{2}{3} & \frac{19}{3} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

We proceed to get a leading 1 in $R2$.

[2] Carl also finds starting with $R3$ to be more symmetric, in a purely poetic way.
[3] infinite, in fact

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & 4 & \frac{2}{3} & \frac{19}{3} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } \frac{3}{2}R2} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

We now zero out the entry below the leading 1 in $R2$.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -4R2+R3} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & -24 & -5 & -35 \end{array}\right]$$

Next, it's time for a leading 1 in $R3$.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & -24 & -5 & -35 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -\frac{1}{24}R3} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

The matrix is now in row echelon form. To get the reduced row echelon form, we start with the last leading 1 we produced and work to get 0's above it.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -6R3+R2} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

Lastly, we get a 0 above the leading 1 of $R2$.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } \frac{1}{3}R2+R1} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{5}{12} & -\frac{5}{12} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

At last, we decode to get

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{5}{12} & -\frac{5}{12} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{\begin{array}{rcr} x_1 - \frac{5}{12}x_4 & = & -\frac{5}{12} \\ x_2 - \frac{1}{4}x_4 & = & \frac{3}{4} \\ x_3 + \frac{5}{24}x_4 & = & \frac{35}{24} \end{array}\right.$$

We see x_4 is free and assign it the parameter t. We obtain $x_3 = -\frac{5}{24}t + \frac{35}{24}$, $x_2 = \frac{1}{4}t + \frac{3}{4}$, and $x_1 = \frac{5}{12}t - \frac{5}{12}$. Our solution is $\left\{ \left(\frac{5}{12}t - \frac{5}{12}, \frac{1}{4}t + \frac{3}{4}, -\frac{5}{24}t + \frac{35}{24}, t \right) : -\infty < t < \infty \right\}$ which we leave to the reader to check. $\square$

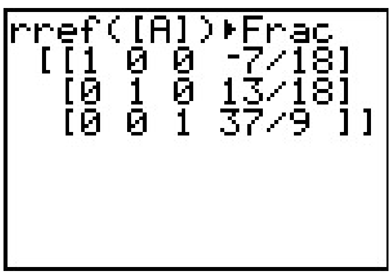
Like all good algorithms, putting a matrix in row echelon or reduced row echelon form can easily be programmed into a calculator, and, doubtless, your graphing calculator has such a feature. We make use of thus feature in next example to avoid tedious arithmetic.

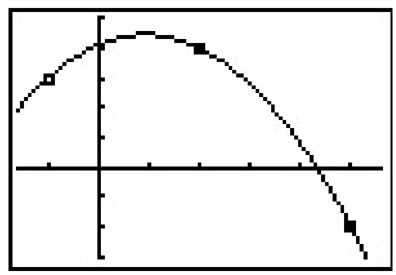
Example 9.2.3. Find the quadratic function passing through the points $(-1, 3)$, $(2, 4)$, $(5, -2)$.

Solution. According to Definition 1.10, a quadratic function has the form $f(x) = ax^2 + bx + c$ where $a \neq 0$. Our goal is to find a, b and c so that the three given points are on the graph of f.
Since $(-1, 3)$ is on the graph of f, we now $f(-1) = 3$. This gives $a(-1)^2 + b(-1) + c = 3$, or $a - b + c = 3$. This last equation is a linear equation with the variables a, b and c. Similarly, if the point $(2, 4)$ is on the graph of f, then $f(2) = 4$, so $4a + 2b + c = 4$. Lastly, the point $(5, -2)$ is on the graph of f gives us $25a + 5b + c = -2$. Putting these together, we obtain a system of three linear equations, which we encode into the matrix below.

$$\left\{ \begin{array}{rcr} a - b + c & = & 3 \\ 4a + 2b + c & = & 4 \\ 25a + 5b + c & = & -2 \end{array} \right. \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 4 & 2 & 1 & 4 \\ 25 & 5 & 1 & -2 \end{array} \right]$$

Using a calculator,[4] we find $a = -\frac{7}{18}$, $b = \frac{13}{18}$ and $c = \frac{37}{9}$. Hence, the one and only quadratic which fits the bill is $f(x) = -\frac{7}{18}x^2 + \frac{13}{18}x + \frac{37}{9}$. To verify this analytically, we see that $f(-1) = 3$, $f(2) = 4$, and $f(5) = -2$. We can use the calculator to check our solution as well by plotting the three data points and the function f.





The graph of $f(x) = -\frac{7}{18}x^2 + \frac{13}{18}x + \frac{37}{9}$
with the points $(-1, 3)$, $(2, 4)$ and $(5, -2)$

$\square$

[4]We've tortured you enough already with fractions in this exposition!

9.2.1 Exercises

In Exercises 1 - 6, state whether the given matrix is in reduced row echelon form, row echelon form only or in neither of those forms.

1. $\left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right]$

2. $\left[\begin{array}{rrr|r} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right]$

3. $\left[\begin{array}{rrr|r} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right]$

4. $\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

5. $\left[\begin{array}{rrrr|r} 1 & 0 & 4 & 3 & 0 \\ 0 & 1 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

6. $\left[\begin{array}{rrr|r} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \end{array}\right]$

In Exercises 7 - 12, the following matrices are in reduced row echelon form. Determine the solution of the corresponding system of linear equations or state that the system is inconsistent.

7. $\left[\begin{array}{rr|r} 1 & 0 & -2 \\ 0 & 1 & 7 \end{array}\right]$

8. $\left[\begin{array}{rrr|r} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 19 \end{array}\right]$

9. $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 6 & -6 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right]$

10. $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

11. $\left[\begin{array}{rrrr|r} 1 & 0 & -8 & 1 & 7 \\ 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

12. $\left[\begin{array}{rrr|r} 1 & 0 & 9 & -3 \\ 0 & 1 & -4 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$

In Exercises 13 - 26, solve the following systems of linear equations using the techniques discussed in this section. Compare and contrast these techniques with those you used to solve the systems in the Exercises in Section 9.1.

13. $\left\{\begin{array}{rcr} -5x + y & = & 17 \\ x + y & = & 5 \end{array}\right.$

14. $\left\{\begin{array}{rcr} x + y + z & = & 3 \\ 2x - y + z & = & 0 \\ -3x + 5y + 7z & = & 7 \end{array}\right.$

15. $\left\{\begin{array}{rcr} 4x - y + z & = & 5 \\ 2y + 6z & = & 30 \\ x + z & = & 5 \end{array}\right.$

16. $\left\{\begin{array}{rcr} x - 2y + 3z & = & 7 \\ -3x + y + 2z & = & -5 \\ 2x + 2y + z & = & 3 \end{array}\right.$

17. $\left\{\begin{array}{rcr} 3x - 2y + z & = & -5 \\ x + 3y - z & = & 12 \\ x + y + 2z & = & 0 \end{array}\right.$

18. $\left\{\begin{array}{rcr} 2x - y + z & = & -1 \\ 4x + 3y + 5z & = & 1 \\ 5y + 3z & = & 4 \end{array}\right.$

19. $\left\{\begin{array}{rcr} x - y + z & = & -4 \\ -3x + 2y + 4z & = & -5 \\ x - 5y + 2z & = & -18 \end{array}\right.$

20. $\left\{\begin{array}{rcr} 2x - 4y + z & = & -7 \\ x - 2y + 2z & = & -2 \\ -x + 4y - 2z & = & 3 \end{array}\right.$

21. $\left\{\begin{array}{rcl} 2x - y + z & = & 1 \\ 2x + 2y - z & = & 1 \\ 3x + 6y + 4z & = & 9 \end{array}\right.$

22. $\left\{\begin{array}{rcr} x - 3y - 4z & = & 3 \\ 3x + 4y - z & = & 13 \\ 2x - 19y - 19z & = & 2 \end{array}\right.$

23. $\left\{\begin{array}{rcr} x + y + z & = & 4 \\ 2x - 4y - z & = & -1 \\ x - y & = & 2 \end{array}\right.$

24. $\left\{\begin{array}{rcr} x - y + z & = & 8 \\ 3x + 3y - 9z & = & -6 \\ 7x - 2y + 5z & = & 39 \end{array}\right.$

25. $\left\{\begin{array}{rcr} 2x - 3y + z & = & -1 \\ 4x - 4y + 4z & = & -13 \\ 6x - 5y + 7z & = & -25 \end{array}\right.$

26. $\left\{\begin{array}{rcr} x_1 - x_3 & = & -2 \\ 2x_2 - x_4 & = & 0 \\ x_1 - 2x_2 + x_3 & = & 0 \\ -x_3 + x_4 & = & 1 \end{array}\right.$

27. It's time for another meal at our local buffet. This time, 22 diners (5 of whom were children) feasted for \$162.25, before taxes. If the kids buffet is \$4.50, the basic buffet is \$7.50, and the deluxe buffet (with crab legs) is \$9.25, find out how many diners chose the deluxe buffet.

28. Carl wants to make a party mix consisting of almonds (which cost \$7 per pound), cashews (which cost \$5 per pound), and peanuts (which cost \$2 per pound.) If he wants to make a 10 pound mix with a budget of \$35, what are the possible combinations almonds, cashews, and peanuts? (You may find it helpful to review Example 9.1.2 in Section 9.1.)

29. Using Example 9.2.3 as a guide, determine the values of coefficients a, b, and c so the graph of the given function below contains the points $(-2, 1)$, $(1, 4)$, $(3, -2)$:

    (a) a quadratic function: $f(x) = ax^2 + bx + c$

    (b) a function of the form: $g(x) = ax^3 + bx + c$

    (c) a function of the form: $h(x) = ax^{-1} + bx^2 + c$

30. At 9 PM, the temperature was 60°F; at midnight, the temperature was 50°F; and at 6 AM, the temperature was 70°F . Use the technique in Example 9.2.3 to fit a quadratic function to these data with the temperature, T, measured in degrees Fahrenheit, as the dependent variable, and the number of hours after 9 PM, t, measured in hours, as the independent variable. What was the coldest temperature of the night? When did it occur?

31. The price for admission into the Stitz-Zeager Sasquatch Museum and Research Station is \$15 for adults and \$8 for kids 13 years old and younger. When the Zahlenreich family visits the museum their bill is \$38 and when the Nullsatz family visits their bill is \$39. One day both families went together and took an adult babysitter along to watch the kids and the total admission charge was \$92. Later that summer, the adults from both families went without the kids and the bill was \$45.

    Is that enough information to determine how many adults and children are in each family? If not, state whether the resulting system is inconsistent or consistent dependent. In the latter case, give at least two plausible solutions.

32. Use the technique in Example 9.2.3 to find the line between the points $(-3,4)$ and $(6,1)$. How does your answer compare to the slope-intercept form of the line in Equation A.6?

33. With the help of your classmates, find at least two different row echelon forms for the matrix

$$\left[\begin{array}{rr|r} 1 & 2 & 3 \\ 4 & 12 & 8 \end{array}\right]$$

9.2.2 Answers

1. Reduced row echelon form
2. Neither
3. Row echelon form only
4. Reduced row echelon form
5. Reduced row echelon form
6. Row echelon form only
7. $(-2, 7)$
8. $(-3, 20, 19)$
9. $(-3t + 4, -6t - 6, 2, t)$
   for all real numbers t
10. Inconsistent
11. $(8s - t + 7, -4s + 3t + 2, s, t)$
    for all real numbers s and t
12. $(-9t - 3, 4t + 20, t)$
    for all real numbers t
13. $(-2, 7)$
14. $(1, 2, 0)$
15. $(-t + 5, -3t + 15, t)$
    for all real numbers t
16. $(2, -1, 1)$
17. $(1, 3, -2)$
18. Inconsistent
19. $(1, 3, -2)$
20. $\left(-3, \frac{1}{2}, 1\right)$
21. $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$
22. $\left(\frac{19}{13}t + \frac{51}{13}, -\frac{11}{13}t + \frac{4}{13}, t\right)$
    for all real numbers t
23. Inconsistent
24. $(4, -3, 1)$
25. $\left(-2t - \frac{35}{4}, -t - \frac{11}{2}, t\right)$
    for all real numbers t
26. $(1, 2, 3, 4)$
27. This time, 7 diners chose the deluxe buffet.
28. If t represents the amount (in pounds) of peanuts, then we need $1.5t - 7.5$ pounds of almonds and $17.5 - 2.5t$ pounds of cashews. Since we can't have a negative amount of nuts, $5 \leq t \leq 7$.
29. (a) $f(x) = -\frac{4}{5}x^2 + \frac{1}{5}x + \frac{23}{5}$ (b) $g(x) = -0.4x^3 + 2.2x + 2.2$ (c) $h(x) = 0.6x^{-1} - 0.7x^2 + 4.1$
30. $T(t) = \frac{20}{27}t^2 - \frac{50}{9}t + 60$. Lowest temperature of the evening $\frac{595}{12} \approx 49.58^{\circ}$F at 12:45 AM.

31. Let x_1 and x_2 be the numbers of adults and children, respectively, in the Zahlenreich family and let x_3 and x_4 be the numbers of adults and children, respectively, in the Nullsatz family. The system of equations determined by the given information is

$$\left\{ \begin{array}{rcl} 15x_1 + 8x_2 & = & 38 \\ 15x_3 + 8x_4 & = & 39 \\ 15x_1 + 8x_2 + 15x_3 + 8x_4 & = & 77 \\ 15x_1 + 15x_3 & = & 45 \end{array} \right.$$

We subtracted the cost of the babysitter in E3 so the constant is 77, not 92. This system is consistent dependent and its solution is $\left(\frac{8}{15}t + \frac{2}{5}, -t + 4, -\frac{8}{15}t + \frac{13}{5}, t\right)$. Our variables represent numbers of adults and children so they must be whole numbers. Running through the values $t = 0, 1, 2, 3, 4$ yields only one solution where all four variables are whole numbers; $t = 3$ gives us $(2, 1, 1, 3)$. Thus there are 2 adults and 1 child in the Zahlenreichs and 1 adult and 3 kids in the Nullsatzs.

9.3 Matrix Arithmetic

In Section 9.2, we used a special class of matrices, the augmented matrices, to assist us in solving systems of linear equations. In this section, we study matrices as mathematical objects of their own accord, temporarily divorced from systems of linear equations. To do so conveniently requires some more notation. When we write $A = \left[a_{ij}\right]_{m \times n}$, we mean A is an m by n matrix[1] and a_{ij} is the entry found in the ith row and jth column. Schematically, we have

$$A \quad = \quad \begin{array}{cc} \begin{array}{c} \text{j counts columns} \\ \text{from left to right} \\ \longrightarrow \end{array} & \\ \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right] & \Bigg\downarrow \quad \begin{array}{c} \text{i counts rows} \\ \text{from top to bottom} \end{array} \end{array}$$

With this new notation we can define what it means for two matrices to be equal.

Definition 9.5. Matrix Equality: Two matrices are said to be **equal** if they are the same size and their corresponding entries are equal. More specifically, if $A = \left[a_{ij}\right]_{m \times n}$ and $B = \left[b_{ij}\right]_{p \times r}$, we write $A = B$ provided

1. $m = p$ and $n = r$
2. $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and all $1 \leq j \leq n$.

Essentially, two matrices are equal if they are the same size and they have the same numbers in the same spots. For example, the two 2×3 matrices below are, despite appearances, equal.

$$\left[\begin{array}{rrr} 0 & -2 & 9 \\ 25 & 117 & -3 \end{array} \right] = \left[\begin{array}{ccc} \ln(1) & \sqrt[3]{-8} & e^{2\ln(3)} \\ 125^{2/3} & 3^2 \cdot 13 & \log(0.001) \end{array} \right]$$

Now that we have an agreed upon understanding of what it means for two matrices to equal each other, we may begin defining arithmetic operations on matrices. Our first operation is addition.

Definition 9.6. Matrix Addition: Given two matrices of the same size, the matrix obtained by adding the corresponding entries of the two matrices is called the **sum** of the two matrices. More specifically, if $A = \left[a_{ij}\right]_{m \times n}$ and $B = \left[b_{ij}\right]_{m \times n}$, we define

$$A + B = \left[a_{ij}\right]_{m \times n} + \left[b_{ij}\right]_{m \times n} = \left[a_{ij} + b_{ij}\right]_{m \times n}$$

As an example, consider the sum below.

$$\left[\begin{array}{rr} 2 & 3 \\ 4 & -1 \\ 0 & -7 \end{array} \right] + \left[\begin{array}{rr} -1 & 4 \\ -5 & -3 \\ 8 & 1 \end{array} \right] = \left[\begin{array}{rr} 2 + (-1) & 3 + 4 \\ 4 + (-5) & (-1) + (-3) \\ 0 + 8 & (-7) + 1 \end{array} \right] = \left[\begin{array}{rr} 1 & 7 \\ -1 & -4 \\ 8 & -6 \end{array} \right]$$

[1] Recall that means A has m rows and n columns.

It is worth the reader's time to think what would have happened had we reversed the order of the summands above. As we would expect, we arrive at the same answer. In general, $A + B = B + A$ for matrices A and B, provided they are the same size so that the sum is defined in the first place. This is the *commutative property* of matrix addition. To see why this is true in general, we appeal to the definition of matrix addition. Given $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{m\times n}$,

$$A + B = \left[a_{ij}\right]_{m\times n} + \left[b_{ij}\right]_{m\times n} = \left[a_{ij} + b_{ij}\right]_{m\times n} = \left[b_{ij} + a_{ij}\right]_{m\times n} = \left[b_{ij}\right]_{m\times n} + \left[a_{ij}\right]_{m\times n} = B + A$$

where the second equality is the definition of $A + B$, the third equality holds by the commutative law of real number addition, and the fourth equality is the definition of $B + A$. In other words, matrix addition is commutative because real number addition is.

A similar argument shows the *associative property* of matrix addition also holds, inherited in turn from the associative law of real number addition. Specifically, for matrices A, B, and C of the same size, $(A + B) + C = A + (B + C)$. In other words, when adding more than two matrices, it doesn't matter how they are grouped. This means that we can write $A + B + C$ without parentheses and there is no ambiguity as to what this means.[2] These properties and more are summarized in the following theorem.

Theorem 9.3. Properties of Matrix Addition

- **Commutative Property:** For all $m \times n$ matrices, $A + B = B + A$
- **Associative Property:** For all $m \times n$ matrices, $(A + B) + C = A + (B + C)$
- **Additive Identity:** If $0_{m\times n}$ is the $m \times n$ matrix whose entries are all 0, for all $m \times n$ matrices A

  $$A + 0_{m\times n} = 0_{m\times n} + A = A$$

  That is, the additive identity for a matrix is the matrix of the additive identity for each of its entries.
- **Additive Inverse:** For every given $m \times n$ matrix $A = \left[a_{ij}\right]_{m\times n}$, the matrix $B = \left[-a_{ij}\right]_{m\times n}$ satisfies

  $$A + B = B + A = 0_{m\times n}$$

  That is, the additive inverse of a matrix is the matrix of the additive inverses of each of its entries.

The identity property is easily verified by resorting to the definition of matrix addition; just as the number 0 is the additive identity for real numbers, the matrix comprised of all 0's does the same job for matrices.

To establish the inverse property, we note that per the definition of matrix addition,

$$A + B = \left[a_{ij}\right]_{m\times n} + \left[-a_{ij}\right]_{m\times n} = \left[a_{ij} - a_{ij}\right]_{m\times n} = [0]_{m\times n} = 0_{m\times n}.$$

The fact that $B + A = 0_{m\times n}$ as well comes from the commutative property of matrix addition.

More about the additive inverse is true. If a matrix $C = \left[c_{ij}\right]_{m\times n}$ satisfies $A + C = 0_{m\times n}$, then once again by the definition of matrix addition, we must have $a_{ij} + c_{ij} = 0$, or $c_{ij} = -a_{ij}$ for all i and j. This shows the

[2] We have seen this idea before in Sections 5.2 and 5.3.

matrix C must be the matrix B as described in Theorem 9.3 which shows the additive inverse of a matrix is *unique*. In general, we denote the additive inverse of a matrix A using the (suggestive) symbol $-A$.

With the concept of additive inverse well in hand, we may now discuss what is meant by subtracting matrices. You may remember from arithmetic that $a - b = a + (-b)$; that is, subtraction is defined as 'adding the opposite (inverse).' We extend this concept to matrices. For two matrices A and B of the same size, we define $A - B = A + (-B)$. At the level of entries, this amounts to

$$A - B = A + (-B) = \left[a_{ij}\right]_{m\times n} + \left[-b_{ij}\right]_{m\times n} = \left[a_{ij} + \left(-b_{ij}\right)\right]_{m\times n} = \left[a_{ij} - b_{ij}\right]_{m\times n}$$

Thus to subtract two matrices of equal size, we subtract their corresponding entries. Surprised?

Our next task is to define what it means to multiply a matrix by a real number. Thinking back to arithmetic, you may recall that multiplication, at least by a natural number, can be thought of as 'rapid addition.' For example, $2 + 2 + 2 = 3 \cdot 2$. We know from algebra[3] that $3x = x + x + x$, so it seems natural that given a matrix A, we define $3A = A + A + A$. If $A = \left[a_{ij}\right]_{m\times n}$, we have

$$3A = A + A + A = \left[a_{ij}\right]_{m\times n} + \left[a_{ij}\right]_{m\times n} + \left[a_{ij}\right]_{m\times n} = \left[a_{ij} + a_{ij} + a_{ij}\right]_{m\times n} = \left[3a_{ij}\right]_{m\times n}$$

In other words, multiplying the *matrix* in this fashion by 3 is the same as multiplying *each entry* by 3. This leads us to the following definition.

Definition 9.7. Scalar[a] Multiplication: We define the product of a real number and a matrix to be the matrix obtained by multiplying each of its entries by said real number. More specifically, if k is a real number and $A = \left[a_{ij}\right]_{m\times n}$, we define

$$kA = k\left[a_{ij}\right]_{m\times n} = \left[ka_{ij}\right]_{m\times n}$$

[a]The word 'scalar' here refers to real numbers. 'Scalar multiplication' in this context means we are multiplying a matrix by a real number (a scalar). We will discuss this term momentarily.

The word 'scalar' means 'scaling factor' as we explain below. Every point $P(x, y)$ in the plane can be represented by its position matrix, P:

$$(x, y) \leftrightarrow P = \begin{bmatrix} x \\ y \end{bmatrix}$$

Suppose we take the point $(-2, 1)$ and multiply its position matrix by 3. We have

$$3P = 3\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(-2) \\ 3(1) \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}.$$

This new matrix corresponds to the point $(-6, 3)$ which is the result of scaling both the horizontal and vertical directions by a factor of 3.

As did matrix addition, scalar multiplication inherits many properties from real number arithmetic. Below we summarize these properties.

[3]The Distributive Property, in particular.

Theorem 9.4. Properties of Scalar Multiplication

- **Associative Property:** For every $m \times n$ matrix A and scalars k and r, $(kr)A = k(rA)$.
- **Identity Property:** For all $m \times n$ matrices A, $1A = A$.
- **Additive Inverse Property:** For all $m \times n$ matrices A, $-A = (-1)A$.
- **Distributive Property of Scalar Multiplication over Scalar Addition:**
  For every $m \times n$ matrix A and scalars k and r,
  $$(k+r)A = kA + rA$$
- **Distributive Property of Scalar Multiplication over Matrix Addition:**
  For all $m \times n$ matrices A and B scalars k,
  $$k(A+B) = kA + kB$$
- **Zero Product Property:** If A is an $m \times n$ matrix and k is a scalar, then
  $$kA = 0_{m\times n} \quad \text{if and only if} \quad k = 0 \quad \text{or} \quad A = 0_{m\times n}$$

As with the other results in this section, Theorem 9.4 can be proved using the definitions of scalar multiplication and matrix addition. For example, to prove that $k(A+B) = kA+kB$ for a scalar k and $m \times n$ matrices A and B, we start by adding A and B, then multiplying by k and seeing how that compares with the sum of kA and kB.

$$k(A+B) = k\left(\left[a_{ij}\right]_{m\times n} + \left[b_{ij}\right]_{m\times n}\right) = k\left[a_{ij}+b_{ij}\right]_{m\times n} = \left[k\left(a_{ij}+b_{ij}\right)\right]_{m\times n} = \left[ka_{ij}+kb_{ij}\right]_{m\times n}$$

As for $kA + kB$, we have

$$kA + kB = k\left[a_{ij}\right]_{m\times n} + k\left[b_{ij}\right]_{m\times n} = \left[ka_{ij}\right]_{m\times n} + \left[kb_{ij}\right]_{m\times n} = \left[ka_{ij}+kb_{ij}\right]_{m\times n} \checkmark$$

which establishes the property. The remaining proofs are similar and are left to the reader.
The properties in Theorems 9.3 and 9.4 establish an algebraic system that lets us treat matrices and scalars more or less as we would real numbers and variables. In the following example, we challenge the reader to justify each and every step of the calculations using either properties of matrix arithmetic.

Example 9.3.1. Solve for the matrix A:

$$3A - \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A\right) = \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 9 & 12 \\ -3 & 39 \end{bmatrix}$$

Solution.

$$
\begin{aligned}
3A - \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) &= \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 9 & 12 \\ -3 & 39 \end{bmatrix} \\
3A + \left\{ - \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) \right\} &= \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{3}\right)(9) & \left(\frac{1}{3}\right)(12) \\ \left(\frac{1}{3}\right)(-3) & \left(\frac{1}{3}\right)(39) \end{bmatrix} \\
3A + (-1) \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) &= \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -1 & 13 \end{bmatrix} \\
3A + \left\{ (-1) \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + (-1)(5A) \right\} &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + (-1) \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + (-1)(5A) &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + \begin{bmatrix} (-1)(2) & (-1)(-1) \\ (-1)(3) & (-1)(5) \end{bmatrix} + ((-1)(5))A &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} + (-5)A &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + (-5)A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
(3 + (-5))A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} + \left(- \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} \right) &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} + \left(- \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} \right) \\
(-2)A + 0_{2\times 2} &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} \\
(-2)A &= \begin{bmatrix} -1 - (-2) & 6 - 1 \\ 5 - (-3) & 11 - (-5) \end{bmatrix} \\
(-2)A &= \begin{bmatrix} 1 & 5 \\ 8 & 16 \end{bmatrix} \\
\left(-\frac{1}{2}\right)((-2)A) &= -\frac{1}{2} \begin{bmatrix} 1 & 5 \\ 8 & 16 \end{bmatrix} \\
\left(\left(-\frac{1}{2}\right)(-2)\right) A &= \begin{bmatrix} \left(-\frac{1}{2}\right)(1) & \left(-\frac{1}{2}\right)(5) \\ \left(-\frac{1}{2}\right)(8) & \left(-\frac{1}{2}\right)(16) \end{bmatrix} \\
1A &= \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \\ -4 & -\frac{16}{2} \end{bmatrix} \\
A &= \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \\ -4 & -8 \end{bmatrix}
\end{aligned}
$$

The reader is encouraged to check our answer in the original equation. □

While the solution to the previous example is written in excruciating detail, in practice many of the steps above are omitted. The reader is encouraged to solve the equation in Example 9.3.1 as they would any other linear equation, for example: $3a - (2 + 5a) = -4 + \frac{1}{3}(9)$.

We now turn our attention to *matrix multiplication* - that is, multiplying a matrix by another matrix. Based on the 'no surprises' trend so far in the section, you may expect that in order to multiply two matrices, they must be of the same size and you find the product by multiplying the corresponding entries. While this kind of product is used in other areas of mathematics,[4] we define matrix multiplication to serve us in solving systems of linear equations.

To that end, we begin by defining the product of a row and a column. We motivate the general definition with an example. Consider the two matrices A and B below.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{bmatrix}$$

Let $R1$ denote the first row of A and $C1$ denote the first column of B. To find the 'product' of $R1$ with $C1$, denoted $R1 \cdot C1$, we first find the product of the first entry in $R1$ and the first entry in $C1$. Next, we add to that the product of the second entry in $R1$ and the second entry in $C1$, and so on until we reach the last entry in $R1$ and the last entry in $C1$.

Using entry notation, $R1 \cdot C1 = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = (2)(3) + (0)(4) + (-1)(5) = 6 + 0 + (-5) = 1$. We can visualize this schematically as follows

$$\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{bmatrix}$$

$$\underbrace{\begin{matrix} \boxed{2} & 0 & -1 \end{matrix} \quad \begin{matrix} \boxed{3} \\ 4 \\ 5 \end{matrix}}_{\substack{a_{11}b_{11} \\ (2)(3)}} \; + \; \underbrace{\begin{matrix} 2 & \boxed{0} & -1 \end{matrix} \quad \begin{matrix} 3 \\ \boxed{4} \\ 5 \end{matrix}}_{\substack{a_{12}b_{21} \\ (0)(4)}} \; + \; \underbrace{\begin{matrix} 2 & 0 & \boxed{-1} \end{matrix} \quad \begin{matrix} 3 \\ 4 \\ \boxed{5} \end{matrix}}_{\substack{a_{13}b_{31} \\ (-1)(5)}}$$

To find $R2 \cdot C3$ where $R2$ denotes the second row of A and $C3$ denotes the third column of B, we proceed similarly. We start with finding the product of the first entry of $R2$ with the first entry in $C3$ then add to it the product of the second entry in $R2$ with the second entry in $C3$, and so forth. Using entry notation, we have $R2 \cdot C3 = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = (-10)(2) + (3)(-5) + (5)(-2) = -45$. Schematically,

$$\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{bmatrix}$$

[4] See this article on the Hadamard Product.

$$\underbrace{\begin{array}{ccccc} & & & & \boxed{2} \\ \boxed{-10} & 3 & 5 & & -5 \\ & & & & -2 \end{array}}_{a_{21}b_{13} = (-10)(2) = -20} + \underbrace{\begin{array}{ccccc} & & & & 2 \\ -10 & \boxed{3} & 5 & & \boxed{-5} \\ & & & & -2 \end{array}}_{a_{22}b_{23} = (3)(-5) = -15} + \underbrace{\begin{array}{ccccc} & & & & 2 \\ -10 & 3 & \boxed{5} & & -5 \\ & & & & \boxed{-2} \end{array}}_{a_{23}b_{33} = (5)(-2) = -10}$$

Generalizing this process, we have the following definition.

Definition 9.8. Product of a Row and a Column: Suppose $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times r}$. Let Ri denote the ith row of A and let Cj denote the jth column of B. The **product of R_i and C_j, denoted $R_i \cdot C_j$** is the real number defined by

$$Ri \cdot Cj = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots a_{in}b_{nj}$$

Note that in order to multiply a row by a column, the number of entries in the row must match the number of entries in the column. We are now in the position to define matrix multiplication.

Definition 9.9. Matrix Multiplication: Suppose $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times r}$. Let Ri denote the ith row of A and let Cj denote the jth column of B. The **product of A and B**, denoted AB, is the matrix

$$AB = [Ri \cdot Cj]_{m\times r}$$

that is

$$AB = \left[\begin{array}{cccc} R1 \cdot C1 & R1 \cdot C2 & \ldots & R1 \cdot Cr \\ R2 \cdot C1 & R2 \cdot C2 & \ldots & R2 \cdot Cr \\ \vdots & \vdots & & \vdots \\ Rm \cdot C1 & Rm \cdot C2 & \ldots & Rm \cdot Cr \end{array}\right]$$

There are a number of subtleties in Definition 9.9 which warrant closer inspection. First and foremost, Definition 9.9 tells us that the ij-entry of a matrix product AB is the ith row of A times the jth column of B. In order for this to be defined, the number of entries in the rows of A must match the number of entries in the columns of B. This means that the number of columns of A must match[5] the number of rows of B. In other words, to multiply A times B, the second dimension of A must match the first dimension of B, which is why in Definition 9.9, $A_{m\times \underline{n}}$ is being multiplied by a matrix $B_{\underline{n}\times r}$.
Furthermore, the product matrix AB has as many rows as A and as many columns of B. As a result, when multiplying a matrix $A_{\underline{m}\times n}$ by a matrix $B_{n\times \underline{r}}$, the result is the matrix $AB_{\underline{m}\times \underline{r}}$.
Returning to our example matrices below, we see that A is a $2 \times \underline{3}$ matrix and B is a $\underline{3} \times 4$ matrix. This means that the product matrix AB is defined and will be a 2×4 matrix.

$$A = \left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array}\right] \quad B = \left[\begin{array}{rrrr} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{array}\right]$$

[5] The reader is encouraged to think this through carefully.

Using Ri to denote the ith row of A and Cj to denote the jth column of B, we form AB per to Definition 9.9:

$$AB = \left[\begin{array}{cccc} R1\cdot C1 & R1\cdot C2 & R1\cdot C3 & R1\cdot C4 \\ R2\cdot C1 & R2\cdot C2 & R2\cdot C3 & R2\cdot C4 \end{array}\right] = \left[\begin{array}{rrrr} 1 & 2 & 6 & -4 \\ 7 & 14 & -45 & 47 \end{array}\right]$$

Note that the product BA is not defined, since B is a $3 \times \underline{4}$ matrix while A is a $\underline{2} \times 3$ matrix; B has more columns than A has rows, and so it is not possible to multiply a row of B by a column of A.

Even when the dimensions of A and B are compatible such that AB and BA are both defined, the product AB and BA aren't necessarily equal.[6] In other words, AB may not equal BA which means matrix multiplication is not, in general, commutative. That being said, several other real number properties are inherited by matrix multiplication, as illustrated in our next theorem.

Theorem 9.5. Properties of Matrix Multiplication Let A, B and C be matrices such that all of the matrix products below are defined and let k be a real number.

- **Associative Property of Matrix Multiplication:** $(AB)C = A(BC)$
- **Associative Property with Scalar Multiplication:** $k(AB) = (kA)B = A(kB)$
- **Identity Property:**

  For a natural number k, the $\boldsymbol{k \times k}$ **identity matrix**, denoted I_k, is defined by $I_k = \left[d_{ij}\right]_{k\times k}$ where

  $$d_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

  For all $m \times n$ matrices, $I_m A = A I_n = A$.
- **Distributive Property of Matrix Multiplication over Matrix Addition:**

  $$A(B \pm C) = AB \pm AC \text{ and } (A \pm B)C = AC \pm BC$$

The one property in Theorem 9.5 which begs further investigation is, without doubt, the multiplicative identity. The entries in a matrix where $i = j$ comprise what is called the *main diagonal* of the matrix. The identity matrix has 1's along its main diagonal and 0's everywhere else. A few examples of the matrix I_k mentioned in Theorem 9.5 are given below. The reader is encouraged to see how they match the definition of the identity matrix presented there.

$$\underbrace{[1]}_{I_1} \qquad \underbrace{\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]}_{I_2} \qquad \underbrace{\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}_{I_3} \qquad \underbrace{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{I_4}$$

The identity matrix is an example of what is called a *square matrix* as it has the same number of rows as columns. Note that to in order to verify that the identity matrix acts as a multiplicative identity, some care

[6]And may not even have the same dimensions. For example, if A is a 2×3 matrix and B is a 3×2 matrix, then AB is defined and is a 2×2 matrix while BA is also defined... but is a 3×3 matrix!

must be taken depending on the order of the multiplication. For example, take the matrix 2×3 matrix A:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

In order for the product I_kA to be defined, $k = 2$; similarly, for AI_k to be defined, $k = 3$. We leave it to the reader to show $I_2A = A$ and $AI_3 = A$. In other words,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

While the proofs of the properties in Theorem 9.5 are computational in nature, the notation becomes quite involved very quickly, so they are left to a course in Linear Algebra. The following example provides some practice with matrix multiplication and its properties. As usual, some valuable lessons are to be learned.

Example 9.3.2.

1. Find AB for $A = \begin{bmatrix} -23 & -1 & 17 \\ 46 & 2 & -34 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix}$.

2. Find $C^2 - 5C + 10I_2$ for $C = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$.

3. Suppose M is a 4×4 matrix. Use Theorem 9.5 to expand $(M - 2I_4)(M + 3I_4)$.

Solution.

1. We have $AB = \begin{bmatrix} -23 & -1 & 17 \\ 46 & 2 & -34 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

2. Just as x^2 means x times itself, C^2 denotes the matrix C times itself. We get

$$\begin{aligned} C^2 - 5C + 10I_2 &= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 10 \\ -15 & -20 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \\ &= \begin{bmatrix} -5 & -10 \\ 15 & 10 \end{bmatrix} + \begin{bmatrix} 5 & 10 \\ -15 & -10 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

3. We expand $(M-2I_4)(M+3I_4)$ with the same pedantic zeal we showed in Example 9.3.1. The reader is encouraged to determine which property of matrix arithmetic justifies each step.

$$\begin{array}{rcl}(M-2I_4)(M+3I_4) & = & (M-2I_4)M+(M-2I_4)(3I_4) \\ & = & MM-(2I_4)M+M(3I_4)-(2I_4)(3I_4) \\ & = & M^2-2(I_4M)+3(MI_4)-2(I_4(3I_4)) \\ & = & M^2-2M+3M-2(3(I_4I_4)) \\ & = & M^2+M-6I_4\end{array}$$

□

Example 9.3.2 illustrates some interesting features of matrix multiplication. First note that in the first problem, neither A nor B is the zero matrix, yet the product AB is the zero matrix. Hence, the the zero product property enjoyed by real numbers and scalar multiplication does not hold for matrix multiplication.
The second and third problems introduce us to polynomials involving matrices. The reader is encouraged to step back and compare our expansion of the matrix product $(M-2I_4)(M+3I_4)$ in third probem with the product $(x-2)(x+3)$ from real number algebra. The exercises explore this kind of parallel further.

As we mentioned earlier, a point $P(x,y)$ in the xy-plane can be represented as a 2×1 position matrix. We now show that matrix multiplication can be used to rotate these points, and hence graphs of equations.

Example 9.3.3. Let $R = \left[\begin{array}{rr}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{array}\right]$.

1. Plot $P(2,-2)$, $Q(4,0)$, $S(0,3)$, and $T(-3,-3)$ in the plane as well as the points RP, RQ, RS, and RT. Plot the lines $y=x$ and $y=-x$ as guides. What does R appear to be doing to these points?

2. If a point P is on the hyperbola $x^2-y^2=4$, show that the point RP is on the curve $y=\frac{2}{x}$.

Solution.

1. For $P(2,-2)$, the position matrix is $P=\left[\begin{array}{r}2\\-2\end{array}\right]$, and

$$\begin{array}{rcl}RP & = & \left[\begin{array}{rr}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{array}\right]\left[\begin{array}{r}2\\-2\end{array}\right] \\ & = & \left[\begin{array}{r}2\sqrt{2}\\0\end{array}\right]\end{array}$$

We have that R takes $(2,-2)$ to $(2\sqrt{2},0)$. Similarly, we find $(4,0)$ is moved to $(2\sqrt{2},2\sqrt{2})$, $(0,3)$ is moved to $\left(-\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right)$, and $(-3,-3)$ is moved to $(0,-3\sqrt{2})$. We plot these points below on the left along with the lines $y=x$ and $y=-x$. We see that the matrix R is rotating these points counterclockwise by 45°.

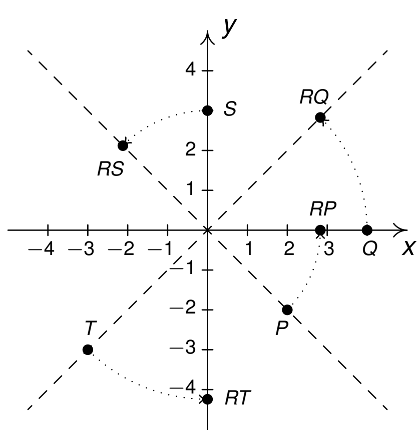
2. For a generic point $P(x, y)$ on the hyperbola $x^2 - y^2 = 4$, we have

$$\begin{aligned} RP &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \\ \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{bmatrix} \end{aligned}$$

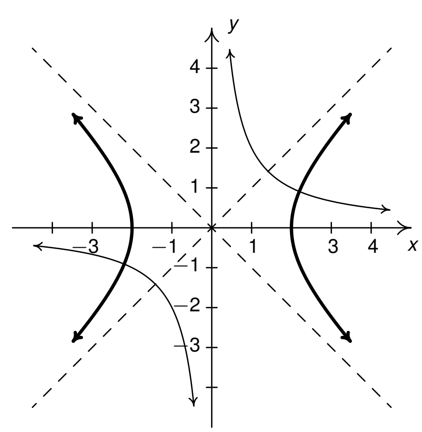
which means R takes (x, y) to $\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)$. To show that this point is on the curve $y = \frac{2}{x}$, we replace x with $\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and y with $\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$ and simplify.

$$\begin{aligned} y &= \frac{2}{x} \\ \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y &\stackrel{?}{=} \frac{2}{\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y} \\ \left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right) &\stackrel{?}{=} \left(\frac{2}{\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y}\right)\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right) \\ \left(\frac{\sqrt{2}}{2}x\right)^2 - \left(\frac{\sqrt{2}}{2}y\right)^2 &\stackrel{?}{=} 2 \\ \frac{x^2}{2} - \frac{y^2}{2} &\stackrel{?}{=} 2 \\ x^2 - y^2 &\stackrel{\checkmark}{=} 4 \end{aligned}$$

Since (x, y) is on the hyperbola $x^2 - y^2 = 4$, we know that this last equation is true. Since all of our steps are reversible, this last equation is equivalent to our original equation, showing the graph of $y = \frac{2}{x}$ is none other than the hyperbola $x^2 - y^2 = 4$ when rotated counterclockwise by 45°. Below on the right are the graphs of $x^2 - y^2 = 4$ (thicker line) and $y = \frac{2}{x}$ for comparison.



Plotting P and RP.



Graphing $x^2 - y^2 = 4$ and $\boldsymbol{y = \frac{2}{x}}$.

□

When we started this section, we mentioned that we would temporarily consider matrices as their own entities, but that the algebra developed here would ultimately allow us to solve systems of linear equations. To that end, consider the system

$$\left\{ \begin{array}{rcl} 3x - y + z & = & 8 \\ x + 2y - z & = & 4 \\ 2x + 3y - 4z & = & 10 \end{array} \right.$$

In Section 9.2, we encoded this system into the augmented matrix

$$\left[\begin{array}{rrr|r} 3 & -1 & 1 & 8 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & -4 & 10 \end{array} \right]$$

Recall that the entries to the left of the vertical line come from the coefficients of the variables in the system, while those on the right comprise the associated constants. For that reason, we may form the *coefficient matrix* A, the *unknowns matrix* X and the *constant matrix* B as below

$$A = \left[\begin{array}{rrr} 3 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & -4 \end{array} \right] \quad X = \left[\begin{array}{c} x \\ y \\ z \end{array} \right] \quad B = \left[\begin{array}{r} 8 \\ 4 \\ 10 \end{array} \right]$$

We now consider the matrix equation $AX = B$.

$$\begin{array}{rcl} AX & = & B \\ \left[\begin{array}{rrr} 3 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & -4 \end{array} \right] \left[\begin{array}{c} x \\ y \\ z \end{array} \right] & = & \left[\begin{array}{r} 8 \\ 4 \\ 10 \end{array} \right] \\ \left[\begin{array}{r} 3x - y + z \\ x + 2y - z \\ 2x + 3y - 4z \end{array} \right] & = & \left[\begin{array}{r} 8 \\ 4 \\ 10 \end{array} \right] \end{array}$$

We see that finding a solution (x, y, z) to the original system corresponds to finding a solution X for the matrix equation $AX = B$. If we think about solving the real number equation $ax = b$, we would simply 'divide' both sides by a. Is it possible to 'divide' both sides of the matrix equation $AX = B$ by the matrix A? This is the central topic of Section 9.4.

9.3.1 Exercises

For each pair of matrices A and B in Exercises 1 - 7, find the following, if defined

- $3A$
- $-B$
- A^2
- $A - 2B$
- AB
- BA

1. $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -2 \\ 4 & 8 \end{bmatrix}$

2. $A = \begin{bmatrix} -1 & 5 \\ -3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 10 \\ -7 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 0 & 8 \\ -3 & 1 & 4 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 & -5 \\ 7 & -9 & 11 \end{bmatrix}$

5. $A = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 1 & 8 \end{bmatrix}$

7. $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix}$

In Exercises 8 - 21, use the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -3 \\ -5 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 10 & -\frac{11}{2} & 0 \\ \frac{3}{5} & 5 & 9 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & -13 \\ -\frac{4}{3} & 0 \\ 6 & 8 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -9 \\ 0 & 0 & -5 \end{bmatrix}$$

to compute the following or state that the indicated operation is undefined.

8. $7B - 4A$
9. AB
10. BA
11. $E + D$
12. ED
13. $CD + 2I_2A$
14. $A - 4I_2$
15. $A^2 - B^2$
16. $(A + B)(A - B)$
17. $A^2 - 5A - 2I_2$
18. $E^2 + 5E - 36I_3$
19. EDC
20. CDE
21. $ABCEDI_2$
22. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $E_2 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

    Compute E_1A, E_2A and E_3A. What effect did each of the E_i matrices have on the rows of A? Create E_4 so that its effect on A is to multiply the bottom row by -6. How would you extend this idea to matrices with more than two rows?

In Exercises 23 - 29, consider the following scenario. In the small village of Pedimaxus in the country of Sasquatchia, all 150 residents get one of the two local newspapers. Market research has shown that in any given week, 90% of those who subscribe to the Pedimaxus Tribune want to keep getting it, but 10% want to switch to the Sasquatchia Picayune. Of those who receive the Picayune, 80% want to continue with it and 20% want switch to the Tribune. We can express this situation using matrices. Specifically, let X be the 'state matrix' given by

$$X = \begin{bmatrix} T \\ P \end{bmatrix}$$

where T is the number of people who get the Tribune and P is the number of people who get the Picayune in a given week. Let Q be the 'transition matrix' given by

$$Q = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix}$$

such that QX will be the state matrix for the next week.

23. Let's assume that when Pedimaxus was founded, all 150 residents got the Tribune. (Let's call this Week 0.) This would mean

$$X = \begin{bmatrix} 150 \\ 0 \end{bmatrix}$$

Since 10% of that 150 want to switch to the Picayune, we should have that for Week 1, 135 people get the Tribune and 15 people get the Picayune. Show that QX in this situation is indeed

$$QX = \begin{bmatrix} 135 \\ 15 \end{bmatrix}$$

24. Assuming that the percentages stay the same, we can get to the subscription numbers for Week 2 by computing Q^2X. How many people get each paper in Week 2?

25. Explain why the transition matrix does what we want it to do.

26. If the conditions do not change from week to week, then Q remains the same and we have what's known as a **Stochastic Process**[7] because Week n's numbers are found by computing Q^nX. Choose a few values of n and, with the help of your classmates and calculator, find out how many people get each paper for that week. You should start to see a pattern as $n \to \infty$.

27. If you didn't see the pattern, we'll help you out. Let

$$X_s = \begin{bmatrix} 100 \\ 50 \end{bmatrix}.$$

Show that $QX_s = X_s$ This is called the **steady state** because the number of people who get each paper didn't change for the next week. Show that $Q^nX \to X_s$ as $n \to \infty$.

[7]More specifically, we have a Markov Chain, which is a special type of stochastic process.

28. Now let

$$S = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Show that $Q^n \to S$ as $n \to \infty$.

29. Show that $SY = X_s$ for any matrix Y of the form

$$Y = \begin{bmatrix} y \\ 150 - y \end{bmatrix}$$

This means that no matter how the distribution starts in Pedimaxus, if Q is applied often enough, we always end up with 100 people getting the Tribune and 50 people getting the Picayune.

30. Let $z = a + bi$ and $w = c + di$ be arbitrary complex numbers. Associate z and w with the matrices

$$Z = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \text{ and } W = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

Show that complex number addition, subtraction and multiplication are mirrored by the associated *matrix* arithmetic. That is, show that $Z + W$, $Z - W$ and ZW produce matrices which can be associated with the complex numbers $z + w$, $z - w$ and zw, respectively.

31. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -3 \\ -5 & 2 \end{bmatrix}$$

Compare $(A + B)^2$ to $A^2 + 2AB + B^2$. Discuss with your classmates what constraints must be placed on two arbitrary matrices A and B so that both $(A + B)^2$ and $A^2 + 2AB + B^2$ exist. When will $(A + B)^2 = A^2 + 2AB + B^2$? In general, what is the correct formula for $(A + B)^2$?

In Exercises 32 - 36, consider the following definitions. A square matrix is said to be an **upper triangular matrix** if all of its entries below the main diagonal are zero and it is said to be a **lower triangular matrix** if all of its entries above the main diagonal are zero. For example,

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -9 \\ 0 & 0 & -5 \end{bmatrix}$$

from Exercises 8 - 21 above is an upper triangular matrix whereas

$$F = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

is a lower triangular matrix. (Zeros are allowed on the main diagonal.) Discuss the following questions with your classmates.

32. Give an example of a matrix which is neither upper triangular nor lower triangular.

33. Is the product of two $n \times n$ upper triangular matrices always upper triangular?

34. Is the product of two $n \times n$ lower triangular matrices always lower triangular?

35. Given the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

write A as LU where L is a lower triangular matrix and U is an upper triangular matrix?

36. Are there any matrices which are simultaneously upper and lower triangular?

9.3.2 Answers

1. For $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ 4 & 8 \end{bmatrix}$

   - $3A = \begin{bmatrix} 6 & -9 \\ 3 & 12 \end{bmatrix}$
   - $-B = \begin{bmatrix} -5 & 2 \\ -4 & -8 \end{bmatrix}$
   - $A^2 = \begin{bmatrix} 1 & -18 \\ 6 & 13 \end{bmatrix}$
   - $A - 2B = \begin{bmatrix} -8 & 1 \\ -7 & -12 \end{bmatrix}$
   - $AB = \begin{bmatrix} -2 & -28 \\ 21 & 30 \end{bmatrix}$
   - $BA = \begin{bmatrix} 8 & -23 \\ 16 & 20 \end{bmatrix}$

2. For $A = \begin{bmatrix} -1 & 5 \\ -3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 10 \\ -7 & 1 \end{bmatrix}$

   - $3A = \begin{bmatrix} -3 & 15 \\ -9 & 18 \end{bmatrix}$
   - $-B = \begin{bmatrix} -2 & -10 \\ 7 & -1 \end{bmatrix}$
   - $A^2 = \begin{bmatrix} -14 & 25 \\ -15 & 21 \end{bmatrix}$
   - $A - 2B = \begin{bmatrix} -5 & -15 \\ 11 & 4 \end{bmatrix}$
   - $AB = \begin{bmatrix} -37 & -5 \\ -48 & -24 \end{bmatrix}$
   - $BA = \begin{bmatrix} -32 & 70 \\ 4 & -29 \end{bmatrix}$

3. For $A = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 0 & 8 \\ -3 & 1 & 4 \end{bmatrix}$

   - $3A = \begin{bmatrix} -3 & 9 \\ 15 & 6 \end{bmatrix}$
   - $-B = \begin{bmatrix} -7 & 0 & -8 \\ 3 & -1 & -4 \end{bmatrix}$
   - $A^2 = \begin{bmatrix} 16 & 3 \\ 5 & 19 \end{bmatrix}$
   - $A - 2B$ is not defined
   - $AB = \begin{bmatrix} -16 & 3 & 4 \\ 29 & 2 & 48 \end{bmatrix}$
   - BA is not defined

4. For $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 & -5 \\ 7 & -9 & 11 \end{bmatrix}$

   - $3A = \begin{bmatrix} 6 & 12 \\ 18 & 24 \end{bmatrix}$
   - $-B = \begin{bmatrix} 1 & -3 & 5 \\ -7 & 9 & -11 \end{bmatrix}$
   - $A^2 = \begin{bmatrix} 28 & 40 \\ 60 & 88 \end{bmatrix}$
   - $A - 2B$ is not defined
   - $AB = \begin{bmatrix} 26 & -30 & 34 \\ 50 & -54 & 58 \end{bmatrix}$
   - BA is not defined

5. For $A = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

- $3A = \begin{bmatrix} 21 \\ 24 \\ 27 \end{bmatrix}$
- $-B = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix}$
- A^2 is not defined
- $A - 2B$ is not defined
- $AB = \begin{bmatrix} 7 & 14 & 21 \\ 8 & 16 & 24 \\ 9 & 18 & 27 \end{bmatrix}$
- $BA = [50]$

6. For $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 1 & 8 \end{bmatrix}$

- $3A = \begin{bmatrix} 3 & -6 \\ -9 & 12 \\ 15 & -18 \end{bmatrix}$
- $-B = \begin{bmatrix} 5 & -1 & -8 \end{bmatrix}$
- A^2 is not defined
- $A - 2B$ is not defined
- AB is not defined
- $BA = \begin{bmatrix} 32 & -34 \end{bmatrix}$

7. For $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix}$

- $3A = \begin{bmatrix} 6 & -9 & 15 \\ 9 & 3 & -6 \\ -21 & 3 & -3 \end{bmatrix}$
- $-B = \begin{bmatrix} -1 & -2 & -1 \\ -17 & -33 & -19 \\ -10 & -19 & -11 \end{bmatrix}$
- $A^2 = \begin{bmatrix} -40 & -4 & 11 \\ 23 & -10 & 15 \\ -4 & 21 & -36 \end{bmatrix}$
- $A - 2B = \begin{bmatrix} 0 & -7 & 3 \\ -31 & -65 & -40 \\ -27 & -37 & -23 \end{bmatrix}$
- $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. $7B - 4A = \begin{bmatrix} -4 & -29 \\ -47 & -2 \end{bmatrix}$

9. $AB = \begin{bmatrix} -10 & 1 \\ -20 & -1 \end{bmatrix}$

10. $BA = \begin{bmatrix} -9 & -12 \\ 1 & -2 \end{bmatrix}$

11. $E + D$ is undefined

12. $ED = \begin{bmatrix} \frac{67}{3} & 11 \\ -\frac{178}{3} & -72 \\ -30 & -40 \end{bmatrix}$

13. $CD + 2I_2A = \begin{bmatrix} \frac{238}{3} & -126 \\ \frac{863}{15} & \frac{361}{5} \end{bmatrix}$

14. $A - 4I_2 = \begin{bmatrix} -3 & 2 \\ 3 & 0 \end{bmatrix}$

15. $A^2 - B^2 = \begin{bmatrix} -8 & 16 \\ 25 & 3 \end{bmatrix}$

16. $(A + B)(A - B) = \begin{bmatrix} -7 & 3 \\ 46 & 2 \end{bmatrix}$

17. $A^2 - 5A - 2I_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

18. $E^2 + 5E - 36I_3 = \begin{bmatrix} -30 & 20 & -15 \\ 0 & 0 & -36 \\ 0 & 0 & -36 \end{bmatrix}$

19. $EDC = \begin{bmatrix} \frac{3449}{15} & -\frac{407}{6} & 99 \\ -\frac{9548}{15} & -\frac{101}{3} & -648 \\ -324 & -35 & -360 \end{bmatrix}$

20. CDE is undefined

21. $ABCEDI_2 = \begin{bmatrix} -\frac{90749}{15} & -\frac{28867}{5} \\ -\frac{156601}{15} & -\frac{47033}{5} \end{bmatrix}$

22. $E_1A = \begin{bmatrix} d & e & f \\ a & b & c \end{bmatrix}$ E_1 interchanged $R1$ and $R2$ of A.

$E_2A = \begin{bmatrix} 5a & 5b & 5c \\ d & e & f \end{bmatrix}$ E_2 multiplied $R1$ of A by 5.

$E_3A = \begin{bmatrix} a - 2d & b - 2e & c - 2f \\ d & e & f \end{bmatrix}$ E_3 replaced $R1$ in A with $R1 - 2R2$.

$E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -6 \end{bmatrix}$

9.4 Systems of Linear Equations: Matrix Inverses

We concluded Section 9.3 by showing how we can rewrite a system of linear equations as the matrix equation $AX = B$ where A and B are known matrices and the solution matrix X of the equation corresponds to the solution of the system. In this section, we develop the method for solving such an equation. To that end, consider the system

$$\left\{ \begin{array}{rcr} 2x - 3y & = & 16 \\ 3x + 4y & = & 7 \end{array} \right.$$

To write this as a matrix equation, we follow the procedure outlined on page 778. We find the coefficient matrix A, the unknowns matrix X and constant matrix B to be

$$A = \left[\begin{array}{rr} 2 & -3 \\ 3 & 4 \end{array} \right] \quad X = \left[\begin{array}{c} x \\ y \end{array} \right] \quad B = \left[\begin{array}{r} 16 \\ 7 \end{array} \right]$$

In order to motivate how we solve a matrix equation like $AX = B$, we revisit solving a similar equation involving real numbers. For instance, to solve $3x = 5$, we divide both sides by 3 and obtain $x = \frac{5}{3}$. How can we go about defining an analogous process for matrices? To answer this question, we solve $3x = 5$ again, but this time, we pay attention to the properties of real numbers being used at each step. Recall that dividing by 3 is the same as multiplying by $\frac{1}{3} = 3^{-1}$, the so-called *multiplicative inverse*[1] of 3.

$$\begin{array}{rclr} 3x & = & 5 & \\ 3^{-1}(3x) & = & 3^{-1}(5) & \text{Multiply by the (multiplicative) inverse of 3} \\ \left(3^{-1} \cdot 3\right) x & = & 3^{-1}(5) & \text{Associative property of multiplication} \\ 1 \cdot x & = & 3^{-1}(5) & \text{Inverse property} \\ x & = & 3^{-1}(5) & \text{Multiplicative Identity} \end{array}$$

If we wish to check our answer, we substitute $x = 3^{-1}(5)$ into the original equation

$$\begin{array}{rclr} 3x & \stackrel{?}{=} & 5 & \\ 3\left(3^{-1}(5)\right) & \stackrel{?}{=} & 5 & \\ \left(3 \cdot 3^{-1}\right)(5) & \stackrel{?}{=} & 5 & \text{Associative property of multiplication} \\ 1 \cdot 5 & \stackrel{?}{=} & 5 & \text{Inverse property} \\ 5 & \stackrel{\checkmark}{=} & 5 & \text{Multiplicative Identity} \end{array}$$

Thinking back to Theorem 9.5, we know that matrix multiplication enjoys both an associative property and a multiplicative identity. What's missing from the mix is a multiplicative inverse for the coefficient matrix A. Assuming we can find such a beast, we can mimic our solution (and check) to $3x = 5$ as follows

[1] Every nonzero real number a has a multiplicative inverse, denoted a^{-1}, such that $a^{-1} \cdot a = a \cdot a^{-1} = 1$.

Solving $AX = B$ | Checking our answer

$$\begin{array}{rcl} AX & = & B \\ A^{-1}(AX) & = & A^{-1}B \\ \left(A^{-1}A\right)X & = & A^{-1}B \\ I_2X & = & A^{-1}B \\ X & = & A^{-1}B \end{array} \qquad \begin{array}{rcl} AX & \stackrel{?}{=} & B \\ A\left(A^{-1}B\right) & \stackrel{?}{=} & B \\ \left(AA^{-1}\right)B & \stackrel{?}{=} & B \\ I_2B & \stackrel{?}{=} & B \\ B & \stackrel{\checkmark}{=} & B \end{array}$$

The matrix A^{-1} is read 'A-inverse' and we will define it formally later in the section. At this stage, we have no idea if such a matrix A^{-1} exists, but that won't deter us from trying to find it.[2]

We want A^{-1} to satisfy two equations, $A^{-1}A = I_2$ and $AA^{-1} = I_2$, making A^{-1} necessarily a 2×2 matrix.[3] Hence, A^{-1} has the form

$$A^{-1} = \left[\begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array}\right]$$

for real numbers x_1, x_2, x_3 and x_4. For reasons which will become clear later, we focus our attention on the equation $AA^{-1} = I_2$. We have

$$\begin{array}{rcl} AA^{-1} & = & I_2 \\ \left[\begin{array}{rr} 2 & -3 \\ 3 & 4 \end{array}\right]\left[\begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array}\right] & = & \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \\ \left[\begin{array}{cc} 2x_1 - 3x_3 & 2x_2 - 3x_4 \\ 3x_1 + 4x_3 & 3x_2 + 4x_4 \end{array}\right] & = & \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \end{array}$$

This gives rise to *two* more systems of equations

$$\left\{\begin{array}{rcl} 2x_1 - 3x_3 & = & 1 \\ 3x_1 + 4x_3 & = & 0 \end{array}\right. \qquad \left\{\begin{array}{rcl} 2x_2 - 3x_4 & = & 0 \\ 3x_2 + 4x_4 & = & 1 \end{array}\right.$$

At this point, it may seem absurd to continue with this venture. After all, the intent was to solve *one* system of equations, and in doing so, we have produced *two* more to solve. Remember, the objective of this discussion is to develop a general *method* which, when used in the correct scenarios, allows us to do far more than just solve a system of equations.

If we set about solving these systems using augmented matrices using the techniques in Section 9.2, we see that not only do both systems have the same coefficient matrix, this coefficient matrix is none other than the matrix A itself! (We will come back to this observation in a moment.)

---

[2]Much like Carl's quest to find Sasquatch.

[3]Since matrix multiplication isn't necessarily commutative, at this stage, these are two different equations.

$$\left\{ \begin{array}{rcl} 2x_1 - 3x_3 & = & 1 \\ 3x_1 + 4x_3 & = & 0 \end{array} \right. \xrightarrow{\text{Encode into a matrix}} \left[\begin{array}{rr|r} 2 & -3 & 1 \\ 3 & 4 & 0 \end{array} \right]$$

$$\left\{ \begin{array}{rcl} 2x_2 - 3x_4 & = & 0 \\ 3x_2 + 4x_4 & = & 1 \end{array} \right. \xrightarrow{\text{Encode into a matrix}} \left[\begin{array}{rr|r} 2 & -3 & 0 \\ 3 & 4 & 1 \end{array} \right]$$

To solve these two systems, we use Gauss-Jordan Elimination to put the augmented matrices into reduced row echelon form. (We leave the details to the reader.) For the first system, we get

$$\left[\begin{array}{rr|r} 2 & -3 & 1 \\ 3 & 4 & 0 \end{array} \right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{rr|r} 1 & 0 & \frac{4}{17} \\ 0 & 1 & -\frac{3}{17} \end{array} \right]$$

which gives $x_1 = \frac{4}{17}$ and $x_3 = -\frac{3}{17}$. To solve the second system, we use the exact same row operations, in the same order, and we obtain

$$\left[\begin{array}{rr|r} 2 & -3 & 0 \\ 3 & 4 & 1 \end{array} \right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{rr|r} 1 & 0 & \frac{3}{17} \\ 0 & 1 & \frac{2}{17} \end{array} \right]$$

which means $x_2 = \frac{3}{17}$ and $x_4 = \frac{2}{17}$. Hence,

$$A^{-1} = \left[\begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \right] = \left[\begin{array}{rr} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right]$$

We can check to see that A^{-1} behaves as it should by computing AA^{-1}

$$AA^{-1} = \left[\begin{array}{rr} 2 & -3 \\ 3 & 4 \end{array} \right] \left[\begin{array}{rr} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right] = \left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \right] = I_2 \checkmark$$

As an added bonus,

$$A^{-1}A = \left[\begin{array}{rr} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right] \left[\begin{array}{rr} 2 & -3 \\ 3 & 4 \end{array} \right] = \left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \right] = I_2 \checkmark$$

We can now return to the problem at hand. From our discussion on page 787, we know

$$X = A^{-1}B = \left[\begin{array}{rr} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right] \left[\begin{array}{r} 16 \\ 7 \end{array} \right] = \left[\begin{array}{r} 5 \\ -2 \end{array} \right]$$

so that our final solution to the system is $(x, y) = (5, -2)$.

As mentioned, the point of this exercise was not just to solve the system of linear equations, but to develop a general method for finding A^{-1}. To that end, we analyze the foregoing discussion in a more general context. To find A^{-1}, we used two augmented matrices, both containing the entries as A:

$$\left[\begin{array}{rr|r} 2 & -3 & 1 \\ 3 & 4 & 0 \end{array}\right] = \left[\begin{array}{c|c} A & \begin{array}{c} 1 \\ 0 \end{array} \end{array}\right]$$

$$\left[\begin{array}{rr|r} 2 & -3 & 0 \\ 3 & 4 & 1 \end{array}\right] = \left[\begin{array}{c|c} A & \begin{array}{c} 0 \\ 1 \end{array} \end{array}\right]$$

We also note that the reduced row echelon forms of these augmented matrices can be written as

$$\left[\begin{array}{rr|r} 1 & 0 & \frac{4}{17} \\ 0 & 1 & -\frac{3}{17} \end{array}\right] = \left[\begin{array}{c|c} I_2 & \begin{array}{c} x_1 \\ x_3 \end{array} \end{array}\right]$$

$$\left[\begin{array}{rr|r} 1 & 0 & \frac{3}{17} \\ 0 & 1 & \frac{2}{17} \end{array}\right] = \left[\begin{array}{c|c} I_2 & \begin{array}{c} x_2 \\ x_4 \end{array} \end{array}\right]$$

where we have identified the entries to the left of the vertical bar as the identity I_2 and the entries to the right of the vertical bar as the solutions to our systems. The long and short of the solution process can be summarized as

$$\left[\begin{array}{c|c} A & \begin{array}{c} 1 \\ 0 \end{array} \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_2 & \begin{array}{c} x_1 \\ x_3 \end{array} \end{array}\right]$$

$$\left[\begin{array}{c|c} A & \begin{array}{c} 0 \\ 1 \end{array} \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_2 & \begin{array}{c} x_2 \\ x_4 \end{array} \end{array}\right]$$

Since the row operations for both processes are the same, all of the arithmetic on the left hand side of the vertical bar is identical in both problems. The only difference between the two processes is what happens to the constants to the right of the vertical bar. As long as we keep these separated into columns, we can combine our efforts into one 'super-sized' augmented matrix and describe the above process as

$$\left[\begin{array}{c|cc} A & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_2 & \begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \end{array}\right]$$

We have the identity matrix I_2 appearing as the right hand side of the first super-sized augmented matrix and the left hand side of the second super-sized augmented matrix. To our surprise and delight, the elements on the right hand side of the second super-sized augmented matrix are none other than those which comprise A^{-1}. Hence, we have

$$\left[\begin{array}{c|c} A & I_2 \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_2 & A^{-1} \end{array}\right]$$

In other words, the process of finding A^{-1} for a matrix A can be viewed as performing a series of row operations which transform A into the identity matrix of the same dimension. We can view this process as follows. In trying to find A^{-1}, we are trying to 'undo' multiplication by the matrix A. The identity matrix in the super-sized augmented matrix $[A|I]$ keeps a running memory of all of the moves required to 'undo' A. This results in exactly what we want, A^{-1}. We are now ready to formalize and generalize the foregoing discussion. We begin with the formal definition of an invertible matrix.[4]

[4]If this rhetoric sounds familiar, it should. See section 5.6.

Definition 9.10. An $n \times n$ matrix A is said to be **invertible** if there exists a matrix A^{-1}, read 'A inverse', such that $A^{-1}A = AA^{-1} = I_n$.

Note that, as a consequence of our definition, invertible matrices are square, and as such, the conditions in Definition 9.10 force the matrix A^{-1} to be same dimensions as A, that is, $n \times n$. Since not all matrices are square, not all matrices are invertible. However, just because a matrix is square doesn't guarantee it is invertible. (See the exercises.)

Our first result summarizes some of the important characteristics of invertible matrices and their inverses.

Theorem 9.6. Suppose A is an $n \times n$ matrix.

1. If A is invertible then A^{-1} is unique.
2. A is invertible if and only if $AX = B$ has a unique solution for every $n \times r$ matrix B.

The proofs of the properties in Theorem 9.6 rely on a healthy mix of definition and matrix arithmetic.

To establish the first property, we assume that A is invertible and suppose the matrices B and C act as inverses for A. That is, $BA = AB = I_n$ and $CA = AC = I_n$. We need to show that B and C are, in fact, the same matrix. To see this, we note that $B = I_nB = (CA)B = C(AB) = CI_n = C$. Hence, any two matrices that act like A^{-1} are, in fact, the same matrix.[5]

To prove the second property of Theorem 9.6, we note that if A is invertible then the discussion on page 787 shows the solution to $AX = B$ to be $X = A^{-1}B$, and since A^{-1} is unique, so is $A^{-1}B$.

Conversely, if $AX = B$ has a unique solution for every $n \times r$ matrix B, then, in particular, there is a unique solution X_0 to the equation $AX = I_n$. The solution matrix X_0 is our candidate for A^{-1}. We have $AX_0 = I_n$ by definition, but we need to also show $X_0A = I_n$. To that end, we note that $A(X_0A) = (AX_0)A = I_nA = A$. In other words, the matrix X_0A is a solution to the equation $AX = A$. Clearly, $X = I_n$ is also a solution to the equation $AX = A$, and since we are assuming every such equation as a *unique* solution, we must have $X_0A = I_n$. Hence, we have $X_0A = AX_0 = I_n$, so that $X_0 = A^{-1}$ and A is invertible.

The foregoing discussion justifies our quest to find A^{-1} using our super-sized augmented matrix approach

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$

We are, in essence, trying to find the unique solution to the equation $AX = I_n$ using row operations.

What does all of this mean for a system of linear equations? Theorem 9.6 tells us that if we write the system in the form $AX = B$, then if the coefficient matrix A is invertible, there is only one solution to the system $-$ that is, if A is invertible, the system is consistent and independent.[6]

We also know that the process by which we find A^{-1} is determined completely by A, and not by the constants in B. This answers the question as to why we would bother doing row operations on a super-sized augmented matrix to find A^{-1} instead of an ordinary augmented matrix to solve a system; by finding

[5]If this proof sounds familiar, it should. See the discussion following Theorem 5.13.

[6]It can be shown that a matrix is invertible if and only if when it serves as a coefficient matrix for a system of equations, the system is always consistent independent. It amounts to the second property in Theorem 9.6 where the matrices B are restricted to being $n \times 1$ matrices. We note that, owing to how matrix multiplication is defined, being able to find unique solutions to $AX = B$ for $n \times 1$ matrices B gives you the same statement about solving such equations for $n \times r$ matrices $-$ since we can find a unique solution to them one column at a time.

A^{-1} we have done all of the row operations we ever need to do, once and for all, since we can quickly solve *any* equation $AX = B$ using *one* multiplication, $A^{-1}B$.

Example 9.4.1. Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$

1. Use row operations to find A^{-1}. Check your answer by finding $A^{-1}A$ and AA^{-1}.
2. Use A^{-1} to solve the following systems of equations

$$\text{(a)} \left\{ \begin{array}{rcl} 3x+y+2z & = & 26 \\ -y+5z & = & 39 \\ 2x+y+4z & = & 117 \end{array} \right. \qquad \text{(b)} \left\{ \begin{array}{rcl} 3x+y+2z & = & 4 \\ -y+5z & = & 2 \\ 2x+y+4z & = & 5 \end{array} \right. \qquad \text{(c)} \left\{ \begin{array}{rcl} 3x+y+2z & = & 1 \\ -y+5z & = & 0 \\ 2x+y+4z & = & 0 \end{array} \right.$$

Solution.

1. We begin with a super-sized augmented matrix and proceed with Gauss-Jordan elimination.

$$\left[\begin{array}{rrr|rrr} 3 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{with } \frac{1}{3}R1]{\text{Replace } R1} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R1+R3]{\text{Replace } R3 \text{ with}} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right] \xrightarrow[\text{with } (-1)R2]{\text{Replace } R2} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right] \xrightarrow[-\frac{1}{3}R2+R3]{\text{Replace } R3 \text{ with}} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & \frac{13}{3} & -\frac{2}{3} & \frac{1}{3} & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & \frac{13}{3} & -\frac{2}{3} & \frac{1}{3} & 1 \end{array}\right] \xrightarrow[\text{with } \frac{3}{13}R3]{\text{Replace } R3} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right] \xrightarrow[\substack{\text{Replace } R2 \text{ with} \\ 5R3+R2}]{\substack{\text{Replace } R1 \text{ with} \\ -\frac{2}{3}R3+R1}} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & 0 & \frac{17}{39} & -\frac{2}{39} & -\frac{2}{13} \\ 0 & 1 & 0 & -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & 0 & \frac{17}{39} & -\frac{2}{39} & -\frac{2}{13} \\ 0 & 1 & 0 & -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right] \xrightarrow[-\frac{1}{3}R2+R1]{\text{Replace } R1 \text{ with}} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ 0 & 1 & 0 & -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right]$$

We find $A^{-1} = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix}$. To check our answer, we compute

$$A^{-1}A = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

and

$$AA^{-1} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

2. Each of the systems in this part has A as its coefficient matrix. The only difference between the systems is the constants which is the matrix B in the associated matrix equation $AX = B$. We solve each of them using the formula $X = A^{-1}B$.

(a) $X = A^{-1}B = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix} \begin{bmatrix} 26 \\ 39 \\ 117 \end{bmatrix} = \begin{bmatrix} -39 \\ 91 \\ 26 \end{bmatrix}$. Our solution is $(-39, 91, 26)$.

(b) $X = A^{-1}B = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{5}{13} \\ \frac{19}{13} \\ \frac{9}{13} \end{bmatrix}$. We get $\left(\frac{5}{13}, \frac{19}{13}, \frac{9}{13}\right)$.

(c) $X = A^{-1}B = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{9}{13} \\ -\frac{10}{13} \\ -\frac{2}{13} \end{bmatrix}$. We find $\left(\frac{9}{13}, -\frac{10}{13}, -\frac{2}{13}\right)$.[7]

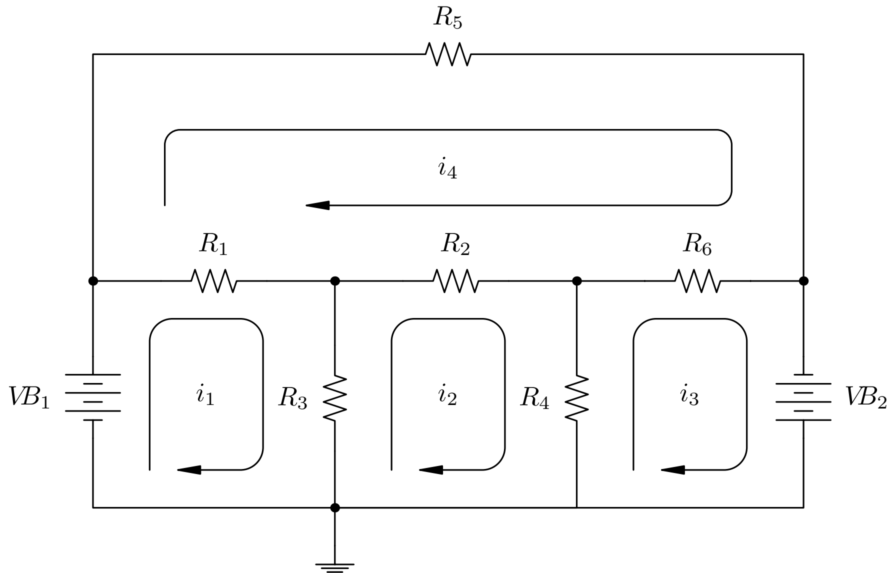
$\square$

In Example 9.4.1, we see that finding one inverse matrix can enable us to solve an entire family of systems of linear equations. There are many examples of where this comes in handy 'in the wild', and we chose our example for this section from the field of electronics. We also take this opportunity to introduce the student to how we can compute inverse matrices using the calculator.

Example 9.4.2. Consider the circuit diagram below.[8] We have two batteries with source voltages VB_1 and VB_2, measured in volts V, along with six resistors with resistances R_1 through R_6, measured in kiloohms, $k\Omega$. Using Ohm's Law and Kirchhoff's Voltage Law, we can relate the voltage supplied to the circuit by the two batteries to the voltage drops across the six resistors in order to find the four 'mesh' currents: i_1, i_2, i_3 and i_4, measured in milliamps, mA. If we think of electrons flowing through the circuit, we can think of the

[7] Note that the solution is the first column of the A^{-1}. The reader is encouraged to meditate on this 'coincidence'.
[8] The authors wish to thank Don Anthan of Lakeland Community College for the design of this example.

voltage sources as providing the ‘push’ which makes the electrons move, the resistors as obstacles for the electrons to overcome, and the mesh current as a net rate of flow of electrons around the indicated loops.



The system of linear equations associated with this circuit is

$$\left\{ \begin{array}{rcl} (R_1 + R_3)\, i_1 - R_3 i_2 - R_1 i_4 & = & VB_1 \\ -R_3 i_1 + (R_2 + R_3 + R_4)\, i_2 - R_4 i_3 - R_2 i_4 & = & 0 \\ -R_4 i_2 + (R_4 + R_6)\, i_3 - R_6 i_4 & = & -VB_2 \\ -R_1 i_1 - R_2 i_2 - R_6 i_3 + (R_1 + R_2 + R_5 + R_6)\, i_4 & = & 0 \end{array} \right.$$

Assuming the resistances are all $1k\Omega$, find the mesh currents if the battery voltages are

- $VB_1 = 10V$ and $VB_2 = 5V$
- $VB_1 = 10V$ and $VB_2 = 0V$
- $VB_1 = 0V$ and $VB_2 = 10V$
- $VB_1 = 10V$ and $VB_2 = 10V$

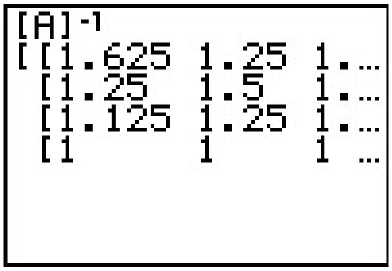
Solution.
Substituting the resistance values into our system of equations, we get

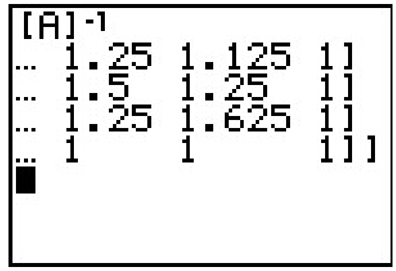
$$\left\{ \begin{array}{rcl} 2i_1 - i_2 - i_4 & = & VB_1 \\ -i_1 + 3i_2 - i_3 - i_4 & = & 0 \\ -i_2 + 2i_3 - i_4 & = & -VB_2 \\ -i_1 - i_2 - i_3 + 4i_4 & = & 0 \end{array} \right.$$

This corresponds to the matrix equation $AX = B$ where

$$A = \left[\begin{array}{rrrr} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 4 \end{array} \right] \quad X = \left[\begin{array}{c} i_1 \\ i_2 \\ i_3 \\ i_4 \end{array} \right] \quad B = \left[\begin{array}{r} VB_1 \\ 0 \\ -VB_2 \\ 0 \end{array} \right]$$

When we input the matrix A into the calculator, we find



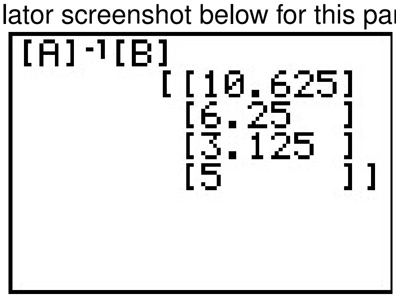


from which we have $A^{-1} = \left[\begin{array}{rrrr} 1.625 & 1.25 & 1.125 & 1 \\ 1.25 & 1.5 & 1.25 & 1 \\ 1.125 & 1.25 & 1.625 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right]$.

To solve the four systems given to us, we find $X = A^{-1}B$ where the value of B is determined by the given values of VB_1 and VB_2

$$B = \left[\begin{array}{r} 10 \\ 0 \\ -5 \\ 0 \end{array}\right], \quad B = \left[\begin{array}{r} 10 \\ 0 \\ 0 \\ 0 \end{array}\right], \quad B = \left[\begin{array}{r} 0 \\ 0 \\ -10 \\ 0 \end{array}\right], \quad B = \left[\begin{array}{r} 10 \\ 0 \\ 10 \\ 0 \end{array}\right]$$

- For $VB_1 = 10V$ and $VB_2 = 5V$, the calculator gives $i_1 = 10.625\ mA$, $i_2 = 6.25\ mA$, $i_3 = 3.125\ mA$, and $i_4 = 5\ mA$. We include a calculator screenshot below for this part (and this part only!) for reference.



- By keeping $VB_1 = 10V$ and setting $VB_2 = 0V$, we are removing the effect of the second battery. We get $i_1 = 16.25\ mA$, $i_2 = 12.5\ mA$, $i_3 = 11.25\ mA$, and $i_4 = 10\ mA$.

- Here, we are zeroing out VB_1 and making $VB_2 = 10$, essentially removing the effect of the first battery. We find $i_1 = -11.25\ mA$, $i_2 = -12.5\ mA$, $i_3 = -16.25\ mA$, and $i_4 = -10\ mA$, where the negatives indicate that the current is flowing in the opposite direction as is indicated on the diagram. The reader is encouraged to study the symmetry here, and if need be, hold up a mirror to the diagram to literally 'see' what is happening.

- For $VB_1 = 10V$ and $VB_2 = 10V$, we get $i_1 = 5\ mA$, $i_2 = 0\ mA$, $i_3 = -5\ mA$, and $i_4 = 0\ mA$. The mesh currents i_2 and i_4 being zero is a consequence of both batteries 'pushing' in equal but opposite directions, causing the net flow of electrons in these two regions to cancel out. $\square$

9.4.1 Exercises

In Exercises 1 - 8, find the inverse of the matrix or state that the matrix is not invertible.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

3. $C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

4. $D = \begin{bmatrix} 2 & -1 \\ 16 & -9 \end{bmatrix}$

5. $E = \begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$

6. $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

7. $G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

8. $H = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

In Exercises 9 - 11, use one matrix inverse to solve the following systems of linear equations.

9. $\left\{ \begin{array}{rcl} 3x + 7y & = & 26 \\ 5x + 12y & = & 39 \end{array} \right.$

10. $\left\{ \begin{array}{rcl} 3x + 7y & = & 0 \\ 5x + 12y & = & -1 \end{array} \right.$

11. $\left\{ \begin{array}{rcl} 3x + 7y & = & -7 \\ 5x + 12y & = & 5 \end{array} \right.$

In Exercises 12 - 14, use the inverse of E from Exercise 5 above to solve the following systems of linear equations.

12. $\left\{ \begin{array}{rcl} 3x + 4z & = & 1 \\ 2x - y + 3z & = & 0 \\ -3x + 2y - 5z & = & 0 \end{array} \right.$

13. $\left\{ \begin{array}{rcl} 3x + 4z & = & 0 \\ 2x - y + 3z & = & 1 \\ -3x + 2y - 5z & = & 0 \end{array} \right.$

14. $\left\{ \begin{array}{rcl} 3x + 4z & = & 0 \\ 2x - y + 3z & = & 0 \\ -3x + 2y - 5z & = & 1 \end{array} \right.$

15. This exercise is a continuation of Example 9.3.3 in Section 9.3 and gives another application of matrix inverses. Recall that given the position matrix P for a point in the plane, the matrix RP corresponds to a point rotated 45° counterclockwise from P where

$$R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(a) Find R^{-1}.

(b) If RP rotates a point counterclockwise 45°, what should $R^{-1}P$ do? Check your answer by finding $R^{-1}P$ for various points on the coordinate axes and the lines $y = \pm x$.

(c) Find $R^{-1}P$ where P corresponds to a generic point $P(x, y)$. Verify that this takes points on the curve $y = \frac{2}{x}$ to points on the curve $x^2 - y^2 = 4$.

16. A Sasquatch's diet consists of three primary foods: Ippizuti Fish, Misty Mushrooms, and Sun Berries. Each serving of Ippizuti Fish is 500 calories, contains 40 grams of protein, and has no Vitamin X. Each serving of Misty Mushrooms is 50 calories, contains 1 gram of protein, and 5 milligrams of Vitamin X. Finally, each serving of Sun Berries is 80 calories, contains no protein, but has 15 milligrams of Vitamin X.[9]

    (a) If an adult male Sasquatch requires 3200 calories, 130 grams of protein, and 275 milligrams of Vitamin X daily, use a matrix inverse to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries he needs to eat each day.

    (b) An adult female Sasquatch requires 3100 calories, 120 grams of protein, and 300 milligrams of Vitamin X daily. Use the matrix inverse you found in part (a) to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries she needs to eat each day.

    (c) An adolescent Sasquatch requires 5000 calories, 400 grams of protein daily, but no Vitamin X daily.[10] Use the matrix inverse you found in part (a) to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries she needs to eat each day.

17. Matrices can be used in cryptography. Suppose we wish to encode the message 'BIGFOOT LIVES'. We start by assigning a number to each letter of the alphabet, say $A = 1$, $B = 2$ and so on. We reserve 0 to act as a space. Hence, our message 'BIGFOOT LIVES' corresponds to the string of numbers '2, 9, 7, 6, 15, 15, 20, 0, 12, 9, 22, 5, 19.' To encode this message, we use an invertible matrix. Any invertible matrix will do, but for this exercise, we choose

    $$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$$

    Since A is 3×3 matrix, we encode our message string into a matrix M with 3 rows. To do this, we take the first three numbers, 2 9 7, and make them our first column, the next three numbers, 6 15 15, and make them our second column, and so on. We put 0's to round out the matrix.

    $$M = \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix}$$

    To encode the message, we find the product AM

    $$AM = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 42 & 100 & -23 & 38 \\ 1 & 3 & 36 & 39 & 57 \\ -12 & -42 & -152 & -46 & -133 \end{bmatrix}$$

[9] Misty Mushrooms and Sun Berries are the only known fictional sources of Vitamin X.

[10] Vitamin X is needed to sustain Sasquatch longevity only.

So our coded message is '12, 1, -12, 42, 3, -42, 100, 36, -152, -23, 39, -46, 38, 57, -133.' To decode this message, we start with this string of numbers, construct a message matrix as we did earlier (we should get the matrix AM again) and then multiply by A^{-1}.

(a) Find A^{-1}.

(b) Use A^{-1} to decode the message and check this method actually works.

(c) Decode the message '14, 37, -76, 128, 21, -151, 31, 65, -140'

(d) Choose another invertible matrix and encode and decode your own messages.

18. Using the matrices A from Exercise 1, B from Exercise 2 and D from Exercise 4, show $AB = D$ and $D^{-1} = B^{-1}A^{-1}$. That is, show that $(AB)^{-1} = B^{-1}A^{-1}$.

19. Let M and N be invertible $n \times n$ matrices. Show that $(MN)^{-1} = N^{-1}M^{-1}$ and compare your work to Exercise 40 in Section 5.6.

9.4.2 Answers

1. $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

2. $B^{-1} = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$

3. C is not invertible

4. $D^{-1} = \begin{bmatrix} \frac{9}{2} & -\frac{1}{2} \\ 8 & -1 \end{bmatrix}$

5. $E^{-1} = \begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix}$

6. $F^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{7}{4} & -\frac{9}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

7. G is not invertible

8. $H^{-1} = \begin{bmatrix} 16 & 0 & 3 & 0 \\ -90 & -\frac{1}{2} & -\frac{35}{2} & \frac{7}{2} \\ 5 & 0 & 1 & 0 \\ -36 & 0 & -7 & 1 \end{bmatrix}$

The coefficient matrix is B^{-1} from Exercise 2 above so the inverse we need is $(B^{-1})^{-1} = B$.

9. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 26 \\ 39 \end{bmatrix} = \begin{bmatrix} 39 \\ -13 \end{bmatrix}$ So $x = 39$ and $y = -13$.

10. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ So $x = 7$ and $y = -3$.

11. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} -7 \\ 5 \end{bmatrix} = \begin{bmatrix} -119 \\ 50 \end{bmatrix}$ So $x = -119$ and $y = 50$.

The coefficient matrix is $E = \begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$ from Exercise 5, so $E^{-1} = \begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix}$

12. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ So $x = -1$, $y = 1$ and $z = 1$.

13. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ -6 \end{bmatrix}$ So $x = 8$, $y = -3$ and $z = -6$.

14. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$ So $x = 4$, $y = -1$ and $z = -3$.

16. (a) The adult male Sasquatch needs: 3 servings of Ippizuti Fish, 10 servings of Misty Mushrooms, and 15 servings of Sun Berries daily.

    (b) The adult female Sasquatch needs: 3 servings of Ippizuti Fish and 20 servings of Sun Berries daily. (No Misty Mushrooms are needed!)

    (c) The adolescent Sasquatch requires 10 servings of Ippizuti Fish daily. (No Misty Mushrooms or Sun Berries are needed!)

17. (a) $A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix}$

    (b) $\begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix} \begin{bmatrix} 12 & 42 & 100 & -23 & 38 \\ 1 & 3 & 36 & 39 & 57 \\ -12 & -42 & -152 & -46 & -133 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix}$ ✓

    (c) ‘LOGS RULE’

9.5 Determinants and Cramer's Rule

9.5.1 Definition and Properties of the Determinant

In this section we assign to each square matrix A a real number, called the *determinant* of A, which will eventually lead us to yet another technique for solving consistent independent systems of linear equations. The determinant is defined recursively, that is, we define it for 1×1 matrices and give a rule by which we can reduce determinants of $n \times n$ matrices to a sum of determinants of $(n-1) \times (n-1)$ matrices.[1] This means we will be able to evaluate the determinant of a 2×2 matrix as a sum of the determinants of 1×1 matrices; the determinant of a 3×3 matrix as a sum of the determinants of 2×2 matrices, and so forth.

To explain how we will take an $n \times n$ matrix and distill an $(n-1) \times (n-1)$ matrix from it, we use introduce the following notation.

Definition 9.11. Given an $n \times n$ matrix A where $n > 1$, the matrix A_{ij} is the $(n-1) \times (n-1)$ matrix formed by deleting the ith row of A and the jth column of A.

For the matrix A below, we obtain A_{23} by deleting the second row and third column of A:

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix} \xrightarrow{\text{Delete } R2 \text{ and } C3} A_{23} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

We are now in the position to define the determinant of a matrix.

Definition 9.12. Given an $n \times n$ matrix A the **determinant of A**, denoted $\det(A)$, is defined as follows

- If $n = 1$, then $A = [a_{11}]$ and $\det(A) = \det([a_{11}]) = a_{11}$.
- If $n > 1$, then $A = \left[a_{ij}\right]_{n \times n}$ and

$$\det(A) = \det\left(\left[a_{ij}\right]_{n \times n}\right) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + - \ldots + (-1)^{1+n}a_{1n}\det(A_{1n})$$

There are two commonly used notations for the determinant of a matrix A: '$\det(A)$' and '$|A|$' We have chosen to use the notation $\det(A)$ as opposed to $|A|$ because we find that the latter is often confused with absolute value, especially in the context of a 1×1 matrix.

In the expansion $a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + - \ldots + (-1)^{1+n}a_{1n}\det(A_{1n})$, the notation '$+ - \ldots + (-1)^{1+n}a_{1n}$' means that the signs alternate and the final sign is dictated by the sign of the quantity $(-1)^{1+n}$.

Since the entries a_{11}, a_{12} and so forth up through a_{1n} comprise the first row of A, we say we are finding the determinant of A by 'expanding along the first row'. Later in the section, we will develop a formula for $\det(A)$ which allows us to find it by expanding along any row.

Applying Definition 9.12 to the matrix $A = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$ we get

[1] We will talk more about the term 'recursively' in Section 10.1.

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}\right) \\
&= 4\det(A_{11}) - (-3)\det(A_{12}) \\
&= 4\det([1]) + 3\det([2]) \\
&= 4(1) + 3(2) \\
&= 10
\end{aligned}$$

For a generic 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we get

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\
&= a\det(A_{11}) - b\det(A_{12}) \\
&= a\det([d]) - b\det([c]) \\
&= ad - bc
\end{aligned}$$

This formula is worth remembering

Equation 9.1. For a 2×2 matrix,

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Applying Definition 9.12 to the 3×3 matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$ we obtain

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}\right) \\
&= 3\det(A_{11}) - 1\det(A_{12}) + 2\det(A_{13}) \\
&= 3\det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 0 & 5 \\ 2 & 4 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}\right) \\
&= 3((-1)(4) - (5)(1)) - ((0)(4) - (5)(2)) + 2((0)(1) - (-1)(2)) \\
&= 3(-9) - (-10) + 2(2) \\
&= -13
\end{aligned}$$

To evaluate the determinant of a 4×4 matrix, we would have to evaluate the determinants of *four* 3×3 matrices, each of which involves the finding the determinants of *three* 2×2 matrices. As you can see, our method of evaluating determinants quickly gets out of hand and many of you may be reaching for the calculator. There is some mathematical machinery which can assist us in calculating determinants and we present that here. Before we state the theorem, we need some more terminology.

Definition 9.13. Let A be an $n \times n$ matrix and A_{ij} be defined as in Definition 9.11.

- The ***ij* minor** of A, denoted M_{ij} is defined by $M_{ij} = \det\left(A_{ij}\right)$.
- The ***ij* cofactor** of A, denoted C_{ij} is defined by $C_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j}\det\left(A_{ij}\right)$.

We note that in Definition 9.12, the sum

$$a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + - \ldots + (-1)^{1+n} a_{1n}\det(A_{1n})$$

can be rewritten as

$$a_{11}(-1)^{1+1}\det(A_{11}) + a_{12}(-1)^{1+2}\det(A_{12}) + \ldots + a_{1n}(-1)^{1+n}\det(A_{1n})$$

which, in the language of cofactors is

$$a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$$

We are now ready to state our main theorem concerning determinants.

Theorem 9.7. Properties of the Determinant: Let $A = \left[a_{ij}\right]_{n \times n}$.

- We may find the determinant by expanding along any row. That is, for any $1 \leq k \leq n$,

$$\det(A) = a_{k1}C_{k1} + a_{k2}C_{k2} + \ldots + a_{kn}C_{kn}$$

- If A' is the matrix obtained from A by:
  - interchanging any two rows, then $\det(A') = -\det(A)$.
  - replacing a row with a nonzero multiple (say c) of itself, then $\det(A') = c\det(A)$
  - replacing a row with itself plus a multiple of another row, then $\det(A') = \det(A)$
- If A has two identical rows, or a row consisting of all 0's, then $\det(A) = 0$.
- If A is upper or lower triangular,[a] then $\det(A)$ is the product of the entries on the main diagonal.[b]
- If B is an $n \times n$ matrix, then $\det(AB) = \det(A)\det(B)$.
- $\det(A^n) = \det(A)^n$ for all natural numbers n.
- A is invertible if and only if $\det(A) \neq 0$. In this case, $\det\left(A^{-1}\right) = \dfrac{1}{\det(A)}$.

[a]See Exercise 9.3.1 in 9.3.

[b]See page 774 in Section 9.3.

Unfortunately, while we can easily *demonstrate* the results in Theorem 9.7, the proofs of most of these properties are beyond the scope of this text. We could prove these properties for generic 2×2 or even

3×3 matrices by brute force computation, but this manner of proof belies the elegance and symmetry of the determinant. We will prove what few properties we can after we have developed some more tools such as the Principle of Mathematical Induction in Section 10.3.[2]

For the moment, let us demonstrate some of the properties listed in Theorem 9.7 on the matrix A below. (Others will be discussed in the Exercises.)

$$A = \left[\begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{array}\right]$$

We found $\det(A) = -13$ by expanding along the first row. To take advantage of the 0 in the second row, we use Theorem 9.7 to find $\det(A) = -13$ by expanding along that row.

$$\begin{aligned}
\det\left(\left[\begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{array}\right]\right) &= 0C_{21} + (-1)C_{22} + 5C_{23} \\
&= (-1)(-1)^{2+2}\det(A_{22}) + 5(-1)^{2+3}\det(A_{23}) \\
&= -\det\left(\left[\begin{array}{rr} 3 & 2 \\ 2 & 4 \end{array}\right]\right) - 5\det\left(\left[\begin{array}{rr} 3 & 1 \\ 2 & 1 \end{array}\right]\right) \\
&= -((3)(4) - (2)(2)) - 5((3)(1) - (2)(1)) \\
&= -8 - 5 \\
&= -13 \checkmark
\end{aligned}$$

In general, the sign of $(-1)^{i+j}$ in front of the minor in the expansion of the determinant follows an alternating pattern. Below is the pattern for 2×2, 3×3 and 4×4 matrices, and it extends naturally to higher dimensions.

$$\left[\begin{array}{cc} + & - \\ - & + \end{array}\right] \qquad \left[\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}\right] \qquad \left[\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array}\right]$$

The reader is cautioned, however, against reading too much into these sign patterns. In the example above, we expanded the 3×3 matrix A by its second row and the term which corresponds to the second entry ended up being negative even though the sign attached to the minor is $(+)$. These signs represent only the signs of the $(-1)^{i+j}$ in the formula; the sign of the corresponding entry as well as the minor itself determine the ultimate sign of the term in the expansion of the determinant.

To illustrate some of the other properties in Theorem 9.7, we use row operations to transform our 3×3 matrix A into an upper triangular matrix, keeping track of the row operations, and labeling each successive matrix. In what follows we follow the Gauss Jordan algorithm without worrying about getting leading 1's.

[2]For a very elegant treatment, take a course in Linear Algebra. There, you will most likely see the treatment of determinants logically reversed than what is presented here. Specifically, the determinant is defined as a function which takes a square matrix to a real number and satisfies some of the properties in Theorem 9.7. From that function, a formula for the determinant is developed.

$$\underbrace{\left[\begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{array}\right]}_{A} \xrightarrow[\text{with } -\frac{2}{3}R1 + R3]{\text{Replace } R3} \underbrace{\left[\begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & \frac{1}{3} & \frac{8}{3} \end{array}\right]}_{B} \xrightarrow[\frac{1}{3}R2 + R3]{\text{Replace } R3 \text{ with}} \underbrace{\left[\begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & \frac{13}{3} \end{array}\right]}_{C}$$

Theorem 9.7 guarantees us that $\det(A) = \det(B) = \det(C)$ since we are replacing a row with itself plus a multiple of another row moving from one matrix to the next. Furthermore, since C is upper triangular, $\det(C)$ is the product of the entries on the main diagonal, in this case $\det(C) = (3)(-1)\left(\frac{13}{3}\right) = -13$. This demonstrates the utility of using row operations to assist in calculating determinants.

This also sheds some light on the connection between a determinant and invertibility. Recall from Section 9.4 that in order to find A^{-1}, we attempt to transform A to I_n using row operations

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$

As we apply our allowable row operations on A to put it into reduced row echelon form, the determinant of the intermediate matrices can vary from the determinant of A by at most a *nonzero* multiple. This means that if $\det(A) \neq 0$, then the determinant of A's reduced row echelon form must also be nonzero. Per Definition 9.3, this means that all the main diagonal entries on A's reduced row echelon form must be 1. Hence, A's reduced row echelon form is I_n so A is invertible.

Conversely, if A is invertible, then A can be transformed into I_n using row operations. Since $\det(I_n) = 1 \neq 0$, our same logic implies $\det(A) \neq 0$. Basically, we have established that the determinant *determines* whether or not the matrix A is invertible.[3]

It is worth noting that when we first introduced the notion of a matrix inverse, it was in the context of solving a linear matrix equation. In effect, we were trying to 'divide' both sides of the matrix equation $AX = B$ by the matrix A. Just like we cannot divide a real number by 0, Theorem 9.7 tells us we cannot 'divide' by a matrix whose *determinant* is 0. We also know that if the coefficient matrix of a system of linear equations is invertible, then system is consistent and independent. It follows, then, that if the determinant of said coefficient is not zero, the system is consistent and independent.

[3]In Section 9.5.2, we learn determinants (specifically cofactors) are deeply connected with the inverse of a matrix.

9.5.2 Cramer's Rule and Matrix Adjoints

In this section, we introduce a theorem which enables us to solve a system of linear equations by means of determinants only. As usual, the theorem is stated in full generality, using numbered unknowns x_1, x_2, etc., instead of the more familiar letters x, y, z, etc. As with many results in this chapter, the proof of the general case is best left to a course in Linear Algebra.

Theorem 9.8. Cramer's Rule: Suppose $AX = B$ is the matrix form of a system of n linear equations in n unknowns where A is the coefficient matrix, X is the unknowns matrix, and B is the constant matrix. If $\det(A) \neq 0$, then the corresponding system is consistent and independent and the solution for unknowns $x_1, x_2, \ldots x_n$ is given by:

$$x_j = \frac{\det\left(A_j\right)}{\det(A)},$$

where A_j is the matrix A whose jth column has been replaced by the constants in B.

In words, Cramer's Rule tells us we can solve for each unknown, one at a time, by finding the ratio of the determinant of A_j to that of the determinant of the coefficient matrix. The matrix A_j is found by replacing the column in the coefficient matrix which holds the coefficients of x_j with the constants of the system. The following example fleshes out this method.

Example 9.5.1. Use Cramer's Rule to solve for the indicated unknowns.

1. Solve $\left\{ \begin{array}{rcr} 2x_1 - 3x_2 & = & 4 \\ 5x_1 + x_2 & = & -2 \end{array} \right.$ for x_1 and x_2

2. Solve $\left\{ \begin{array}{rcr} 2x - 3y + z & = & -1 \\ x - y + z & = & 1 \\ 3x - 4z & = & 0 \end{array} \right.$ for z.

Solution.

1. Writing this system in matrix form, we find

$$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

To find the matrix A_1, we remove the column of the coefficient matrix A which holds the coefficients of x_1 and replace it with the corresponding entries in B. Likewise, we replace the column of A which corresponds to the coefficients of x_2 with the constants to form the matrix A_2. This yields

$$A_1 = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 4 \\ 5 & -2 \end{bmatrix}$$

Computing determinants, we get $\det(A) = 17$, $\det\left(A_1\right) = -2$ and $\det\left(A_2\right) = -24$, so that

$$x_1 = \frac{\det(A_1)}{\det(A)} = -\frac{2}{17} \qquad x_2 = \frac{\det(A_2)}{\det(A)} = -\frac{24}{17}$$

The reader can check that the solution to the system is $\left(-\frac{2}{17}, -\frac{24}{17}\right)$.

2. To use Cramer's Rule to find z, we identify x_3 as z. We have

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & -4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad A_3 = A_z = \begin{bmatrix} 2 & -3 & -1 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

Expanding both $\det(A)$ and $\det(A_z)$ along the third rows (to take advantage of the 0's) gives

$$z = \frac{\det(A_z)}{\det(A)} = \frac{-12}{-10} = \frac{6}{5}$$

The reader is encouraged to solve this system for x and y similarly and check the answer. $\Box$

It is worth noting that finding determinants is very 'computationally expensive' meaning they take quite a bit of time and effort to compute for humans and machines alike. For that reason, determinants, and, in particular, Cramer's Rule are used mostly in theoretical contexts as convenient ways to describe and discuss solutions to systems of linear equations.[4]

Our last application of determinants is to develop an alternative method for finding the inverse of a matrix. As with the discussion in Section 9.4 when we developed the first algorithm to find matrix inverses, we ask that you indulge us as we proceed to work to describe a *general* method here via *example*.
Let us consider the 3×3 matrix A which we so extensively studied in Section 9.5.1

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

We found through a variety of methods that $\det(A) = -13$. To our surprise and delight, its inverse below has a remarkable number of 13's in the denominators of its entries. This is no coincidence.

$$A^{-1} = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix}$$

Recall that to find A^{-1}, we are essentially solving the matrix equation $AX = I_3$, where $X = \left[x_{ij}\right]_{3\times 3}$ is a 3×3 matrix. Because of how matrix multiplication is defined, the first column of I_3 is the product of A with the first column of X, the second column of I_3 is the product of A with the second column of X and the third column of I_3 is the product of A with the third column of X. In other words, we are solving three equations[5]

[4]A classic case of this is the Wronskian from Differential Equations.

[5]The reader is encouraged to stop and think this through.

$$A\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad A\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad A\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can solve each of these systems using Cramer's Rule. Focusing on the first system, we have

$$A_1 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 1 & 4 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 5 \\ 2 & 0 & 4 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

If we expand $\det(A_1)$ along the first row, we get

$$\begin{aligned} \det(A_1) &= \det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 0 & 5 \\ 0 & 4 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right) \end{aligned}$$

Amazingly, this is none other than the C_{11} cofactor of A. The reader is invited to check this, as well as the claims that $\det(A_2) = C_{12}$ and $\det(A_3) = C_{13}$.[6] (To see this, though it seems unnatural to do so, expand along the first row.) Cramer's Rule tells us

$$x_{11} = \frac{\det(A_1)}{\det(A)} = \frac{C_{11}}{\det(A)}, \quad x_{21} = \frac{\det(A_2)}{\det(A)} = \frac{C_{12}}{\det(A)}, \quad x_{31} = \frac{\det(A_3)}{\det(A)} = \frac{C_{13}}{\det(A)}$$

So the first column of the inverse matrix X is:

$$\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} \dfrac{C_{11}}{\det(A)} \\ \dfrac{C_{12}}{\det(A)} \\ \dfrac{C_{13}}{\det(A)} \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$$

Notice the reversal of the subscripts going from the unknown to the corresponding cofactor of A. This trend continues and we get

$$\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} C_{21} \\ C_{22} \\ C_{23} \end{bmatrix} \qquad \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix}$$

Putting all of these together, we have obtained a new and surprising formula for A^{-1}, namely

[6] In a solid Linear Algebra course you will learn that the properties in Theorem 9.7 hold equally well if the word 'row' is replaced by the word 'column'. We're not going to get into column operations in this text, but they do make some of what we're trying to say easier to follow.

$$A^{-1} = \frac{1}{\det(A)} \left[\begin{array}{ccc} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{array} \right]$$

To see that this does indeed yield A^{-1}, we find all of the cofactors of A

$$\begin{array}{rclrclrcl} C_{11} & = & -9, & C_{21} & = & -2, & C_{31} & = & 7 \\ C_{12} & = & 10, & C_{22} & = & 8, & C_{32} & = & -15 \\ C_{13} & = & 2, & C_{23} & = & -1, & C_{33} & = & -3 \end{array}$$

And, as promised,

$$A^{-1} = \frac{1}{\det(A)} \left[\begin{array}{ccc} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{array} \right] = -\frac{1}{13} \left[\begin{array}{rrr} -9 & -2 & 7 \\ 10 & 8 & -15 \\ 2 & -1 & -3 \end{array} \right] = \left[\begin{array}{rrr} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array} \right]$$

To generalize this to invertible $n \times n$ matrices, we need another definition and a theorem. Our definition gives a special name to the cofactor matrix, and the theorem gives us a recipe for the matrix inverse.

Definition 9.14. Let A be an $n \times n$ matrix, and C_{ij} denote the ij cofactor of A. The **adjoint** of A, denoted adj(A) is the matrix whose ij-entry is the ji cofactor of A, C_{ji}. That is

$$\text{adj}(A) = \left[\begin{array}{cccc} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{array} \right]$$

This new notation greatly shortens the statement of the formula for the inverse of a matrix.

Theorem 9.9. Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

For 2×2 matrices, Theorem 9.9 reduces to a fairly simple formula.

Equation 9.2. For an invertible 2×2 matrix,

$$\left[\begin{array}{cc} a & b \\ c & d \end{array} \right]^{-1} = \frac{1}{ad - bc} \left[\begin{array}{rr} d & -b \\ -c & a \end{array} \right]$$

The proof of Theorem 9.9 is, like so many of the results in this section, best left to a course in Linear Algebra. In such a course, not only do you gain some more sophisticated proof techniques, you also gain a larger perspective. Within the scope of this text, we will prove a few results involving determinants in Section 10.3 using the Principle of Mathematical Induction. Until then, make sure you have a handle on the *mechanics* of matrices and the theory will come eventually.

9.5.3 Exercises

Exercise ideas: revisit fitting curves to three points - function condition means det not zero (?)
Determinant formula for line
Geometry of determinant - wait to vectors (?)
Follow up in inverse determinant section: What about $ax^m + bx^n + cx^p$ for these three points?
In Exercises 1 - 8, compute the determinant of the given matrix. (Some of these matrices appeared in Exercises 1 - 8 in Section 9.4.)

1. $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

2. $C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

3. $Q = \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}$

4. $L = \begin{bmatrix} \dfrac{1}{x^3} & \dfrac{\ln(x)}{x^3} \\ -\dfrac{3}{x^4} & \dfrac{1 - 3\ln(x)}{x^4} \end{bmatrix}$

5. $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

6. $G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

7. $V = \begin{bmatrix} i & j & k \\ -1 & 0 & 5 \\ 9 & -4 & -2 \end{bmatrix}$

8. $H = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

In Exercises 9 - 14, use Cramer's Rule to solve the system of linear equations.

9. $\left\{ \begin{array}{rcl} 3x + 7y & = & 26 \\ 5x + 12y & = & 39 \end{array} \right.$

10. $\left\{ \begin{array}{rcr} 2x - 4y & = & 5 \\ 10x + 13y & = & -6 \end{array} \right.$

11. $\left\{ \begin{array}{rcr} x + y & = & 8000 \\ 0.03x + 0.05y & = & 250 \end{array} \right.$

12. $\left\{ \begin{array}{rcl} \frac{1}{2}x - \frac{1}{5}y & = & 1 \\ 6x + 7y & = & 3 \end{array} \right.$

13. $\left\{ \begin{array}{rcl} x + y + z & = & 3 \\ 2x - y + z & = & 0 \\ -3x + 5y + 7z & = & 7 \end{array} \right.$

14. $\left\{ \begin{array}{rcr} 3x + y - 2z & = & 10 \\ 4x - y + z & = & 5 \\ x - 3y - 4z & = & -1 \end{array} \right.$

In Exercises 15 - 16, use Cramer's Rule to solve for x_4.

15. $\left\{ \begin{array}{rcr} x_1 - x_3 & = & -2 \\ 2x_2 - x_4 & = & 0 \\ x_1 - 2x_2 + x_3 & = & 0 \\ -x_3 + x_4 & = & 1 \end{array} \right.$

16. $\left\{ \begin{array}{rcr} 4x_1 + x_2 & = & 4 \\ x_2 - 3x_3 & = & 1 \\ 10x_1 + x_3 + x_4 & = & 0 \\ -x_2 + x_3 & = & -3 \end{array} \right.$

In Exercises 17 - 18, find the inverse of the given matrix using their determinants and adjoints.

17. $B = \left[\begin{array}{rr} 12 & -7 \\ -5 & 3 \end{array} \right]$

18. $F = \left[\begin{array}{rrr} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{array} \right]$

19. Carl's Sasquatch Attack! Game Card Collection is a mixture of common and rare cards. Each common card is worth \$0.25 while each rare card is worth \$0.75. If his entire 117 card collection is worth \$48.75, how many of each kind of card does he own?

20. How much of a 5 gallon 40% salt solution should be replaced with pure water to obtain 5 gallons of a 15% solution?

21. How much of a 10 liter 30% acid solution must be replaced with pure acid to obtain 10 liters of a 50% solution?

22. Daniel's Exotic Animal Rescue houses snakes, tarantulas and scorpions. When asked how many animals of each kind he boards, Daniel answered: 'We board 49 total animals, and I am responsible for each of their 272 legs and 28 tails.' How many of each animal does the Rescue board? (Recall: tarantulas have 8 legs and no tails, scorpions have 8 legs and one tail, and snakes have no legs and one tail.)

23. This exercise is a continuation of Exercise 16 in Section 9.4. Just because a system is consistent independent doesn't mean it will admit a solution that makes sense in an applied setting. Using the nutrient values given for Ippizuti Fish, Misty Mushrooms, and Sun Berries, use Cramer's Rule to determine the number of servings of Ippizuti Fish needed to meet the needs of a daily diet which requires 2500 calories, 1000 grams of protein, and 400 milligrams of Vitamin X. Now use Cramer's Rule to find the number of servings of Misty Mushrooms required. Does a solution to this diet problem exist?

24. Let $R = \left[\begin{array}{rr} -7 & 3 \\ 11 & 2 \end{array} \right]$, $S = \left[\begin{array}{rr} 1 & -5 \\ 6 & 9 \end{array} \right]$ $T = \left[\begin{array}{rr} 11 & 2 \\ -7 & 3 \end{array} \right]$, and $U = \left[\begin{array}{rr} -3 & 15 \\ 6 & 9 \end{array} \right]$

    (a) Show that $\det(RS) = \det(R)\det(S)$

    (b) Show that $\det(T) = -\det(R)$

    (c) Show that $\det(U) = -3\det(S)$

25. For M, N, and P below, show that $\det(M) = 0$, $\det(N) = 0$ and $\det(P) = 0$.

$$M = \left[\begin{array}{rrr} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{array} \right], \quad N = \left[\begin{array}{rrr} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right], \quad P = \left[\begin{array}{rrr} 1 & 2 & 3 \\ -2 & -4 & -6 \\ 7 & 8 & 9 \end{array} \right]$$

26. This exercise is a follow-up to Exercise 29 in Section 9.2. Suppose you wish to determine coefficients a, b, and c so the the graph of $f(x) = ax^m + bx^n + cx^p$ contains the points $(-2, 1)$, $(1, 4)$, $(3, -2)$. With help from your classmates, discuss if there a unique solution for every selection of m, n, and p? If not, under what conditions is there a unique solution?

27. Let A be an arbitrary invertible 3×3 matrix.

   (a) Show that $\det(I_3) = 1$. (See footnote[7] below.)

   (b) Using the facts that $AA^{-1} = I_3$ and $\det(AA^{-1}) = \det(A)\det(A^{-1})$, show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

The purpose of Exercises 28 - 31 is to introduce you to the eigenvalues and eigenvectors of a matrix.[8] We begin with an example using a 2×2 matrix and then guide you through some exercises using a 3×3 matrix. Consider the matrix

$$C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$$

from Exercise 2. We know that $\det(C) = 0$ which means that $CX = 0_{2\times 2}$ does not have a unique solution. So there is a nonzero matrix Y with $CY = 0_{2\times 2}$. In fact, every matrix of the form

$$Y = \begin{bmatrix} -\frac{5}{2}t \\ t \end{bmatrix}$$

is a solution to $CX = 0_{2\times 2}$, so there are infinitely many matrices such that $CX = 0_{2\times 2}$. But consider the matrix

$$X_{41} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

It is NOT a solution to $CX = 0_{2\times 2}$, but rather,

$$CX_{41} = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 123 \\ 287 \end{bmatrix} = 41 \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

In fact, if Z is of the form

$$Z = \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix}$$

then

$$CZ = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix} \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{123}{7}t \\ 41t \end{bmatrix} = 41 \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix} = 41Z$$

for all t. The big question is "How did we know to use 41?"

We need a number λ such that $CX = \lambda X$ has nonzero solutions. We have demonstrated that $\lambda = 0$ and $\lambda = 41$ both worked. Are there others? If we look at the matrix equation more closely, what we *really*

[7] If you think about it for just a moment, you'll see that $\det(I_n) = 1$ for any natural number n. The formal proof of this fact requires the Principle of Mathematical Induction (Section 10.3) so we'll stick with $n = 3$ for the time being.

[8] This material is usually given its own chapter in a Linear Algebra book so clearly we're not able to tell you everything you need to know about eigenvalues and eigenvectors. They are a nice application of determinants, though, so we're going to give you enough background so that you can start playing around with them.

wanted was a nonzero solution to $(C - \lambda I_2)X = 0_{2\times 2}$ which we know exists if and only if the determinant of $C - \lambda I_2$ is zero.[9] So we computed

$$\det(C - \lambda I_2) = \det\left(\begin{bmatrix} 6-\lambda & 15 \\ 14 & 35-\lambda \end{bmatrix}\right) = (6-\lambda)(35-\lambda) - 14 \cdot 15 = \lambda^2 - 41\lambda$$

This is called the **characteristic polynomial** of the matrix C and it has two zeros: $\lambda = 0$ and $\lambda = 41$. That's how we knew to use 41 in our work above. The fact that $\lambda = 0$ showed up as one of the zeros of the characteristic polynomial just means that C itself had determinant zero which we already knew. Those two numbers are called the **eigenvalues** of C. The corresponding matrix solutions to $CX = \lambda X$ are called the **eigenvectors** of C and the 'vector' portion of the name will make more sense after you've studied vectors.

Now it's your turn. In the following exercises, you'll be using the matrix G from Exercise 6.

$$G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$$

28. Show that the characteristic polynomial of G is $p(\lambda) = -\lambda(\lambda - 1)(\lambda - 22)$. That is, compute $\det(G - \lambda I_3)$.

29. Let $G_0 = G$. Find the parametric description of the solution to the system of linear equations given by $GX = 0_{3\times 3}$.

30. Let $G_1 = G - I_3$. Find the parametric description of the solution to the system of linear equations given by $G_1X = 0_{3\times 3}$. Show that any solution to $G_1X = 0_{3\times 3}$ also has the property that $GX = 1X$.

31. Let $G_{22} = G - 22I_3$. Find the parametric description of the solution to the system of linear equations given by $G_{22}X = 0_{3\times 3}$. Show that any solution to $G_{22}X = 0_{3\times 3}$ also has the property that $GX = 22X$.

[9]Think about this.

9.5.4 Answers

1. $\det(B) = 1$
2. $\det(C) = 0$
3. $\det(Q) = x^2$
4. $\det(L) = \dfrac{1}{x^7}$
5. $\det(F) = -12$
6. $\det(G) = 0$
7. $\det(V) = 20i + 43j + 4k$
8. $\det(H) = -2$
9. $x = 39,\ y = -13$
10. $x = \frac{41}{66},\ y = -\frac{31}{33}$
11. $x = 7500,\ y = 500$
12. $x = \frac{76}{47},\ y = -\frac{45}{47}$
13. $x = 1,\ y = 2,\ z = 0$
14. $x = \frac{121}{60},\ y = \frac{131}{60},\ z = -\frac{53}{60}$
15. $x_4 = 4$
16. $x_4 = -1$
17. $B^{-1} = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$
18. $F^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{7}{4} & -\frac{9}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$
19. Carl owns 78 common cards and 39 rare cards.
20. 3.125 gallons.
21. $\frac{20}{7} \approx 2.85$ liters.
22. The rescue houses 15 snakes, 21 tarantulas and 13 scorpions.
23. Using Cramer's Rule, we find we need 53 servings of Ippizuti Fish to satisfy the dietary requirements. The number of servings of Misty Mushrooms required, however, is -1120. Since it's impossible to have a negative number of servings, there is no solution to the applied problem, despite there being a solution to the mathematical problem. A cautionary tale about using Cramer's Rule: just because you are guaranteed a mathematical answer for each variable doesn't mean the solution will make sense in the 'real' world.

9.6 Partial Fraction Decomposition

This section uses systems of linear equations to rewrite rational functions in a form more palatable to Calculus students. In College Algebra, the function

$$f(x) = \frac{x^2 - x - 6}{x^4 + x^2} \tag{1}$$

is written in the best form possible to construct a sign diagram and to find zeros and asymptotes, but certain applications in Calculus require us to rewrite $f(x)$ as

$$f(x) = \frac{x + 7}{x^2 + 1} - \frac{1}{x} - \frac{6}{x^2} \tag{2}$$

If we are given the form of $f(x)$ in (2), it is a matter of Intermediate Algebra to determine a common denominator to obtain the form of $f(x)$ given in (1). The focus of this section is to develop a method by which we start with $f(x)$ in the form of (1) and 'resolve it into *partial fractions*' to obtain the form in (2). Essentially, we need to reverse the least common denominator process.

Starting with the form of $f(x)$ in (1), we begin by factoring the denominator

$$\frac{x^2 - x - 6}{x^4 + x^2} = \frac{x^2 - x - 6}{x^2\left(x^2 + 1\right)}$$

We now think about which individual denominators could contribute to obtain $x^2\left(x^2 + 1\right)$ as the least common denominator. Certainly x^2 and x^2+1, but are there any other factors? Since x^2+1 is an irreducible quadratic[1] there are no factors of it that have real coefficients which can contribute to the denominator.

The factor x^2, however, is not irreducible, since we can think of it as $x^2 = xx = (x - 0)(x - 0)$, a so-called 'repeated' linear factor.[2] This means it's possible that a term with a denominator of just x contributed to the expression as well. What about something like $x\left(x^2 + 1\right)$? This, too, could contribute, but we would then wish to break down that denominator into x and $\left(x^2 + 1\right)$, so we leave out a term of that form.

At this stage, we have guessed

$$\frac{x^2 - x - 6}{x^4 + x^2} = \frac{x^2 - x - 6}{x^2\left(x^2 + 1\right)} = \frac{?}{x} + \frac{?}{x^2} + \frac{?}{x^2 + 1}$$

Our next task is to determine what form the unknown numerators take. It stands to reason that since the expression $\frac{x^2-x-6}{x^4+x^2}$ is 'proper' in the sense that the degree of the numerator is less than the degree of the denominator, we are safe to make the ansatz that all of the partial fraction resolvents are also. This means that the numerator of the fraction with x as its denominator is just a constant and the numerators on the terms involving the denominators x^2 and $x^2 + 1$ are at most linear polynomials.

In other words, we guess that there are real numbers A, B, C, D and E so that

$$\frac{x^2 - x - 6}{x^4 + x^2} = \frac{x^2 - x - 6}{x^2\left(x^2 + 1\right)} = \frac{A}{x} + \frac{Bx + C}{x^2} + \frac{Dx + E}{x^2 + 1}$$

[1] Recall this means it has no real zeros; see Section 2.4.

[2] Recall this means $x = 0$ is a zero of multiplicity 2.

However, if we look more closely at the term $\frac{Bx+C}{x^2}$, we see that $\frac{Bx+C}{x^2} = \frac{Bx}{x^2} + \frac{C}{x^2} = \frac{B}{x} + \frac{C}{x^2}$. The term $\frac{B}{x}$ has the same form as the term $\frac{A}{x}$ which means it contributes nothing new to our expansion. Hence, we drop it and, after re-labeling, we find ourselves with our new guess:

$$\frac{x^2-x-6}{x^4+x^2} = \frac{x^2-x-6}{x^2\left(x^2+1\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

Our next task is to determine the values of our unknowns. Clearing denominators gives

$$x^2-x-6 = Ax\left(x^2+1\right) + B\left(x^2+1\right) + (Cx+D)x^2$$

Gathering the like powers of x we have

$$x^2-x-6 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

In order for this to hold for all values of x in the domain of f, we equate the coefficients of corresponding powers of x on each side of the equation[3] and obtain the system of linear equations

$$\left\{\begin{array}{rrcrl} (E1) & A+C & = & 0 & \text{From equating coefficients of } x^3 \\ (E2) & B+D & = & 1 & \text{From equating coefficients of } x^2 \\ (E3) & A & = & -1 & \text{From equating coefficients of } x \\ (E4) & B & = & -6 & \text{From equating the constant terms} \end{array}\right.$$

To solve this system of equations, we could use any of the methods presented in Sections 9.1 through 9.5, but none of these methods are as efficient as the good old-fashioned substitution from High School algebra. From $E3$, we have $A = -1$ and we substitute this into $E1$ to get $C = 1$. Similarly, since $E4$ gives us $B = -6$, we have from $E2$ that $D = 7$. We get

$$\frac{x^2-x-6}{x^4+x^2} = \frac{x^2-x-6}{x^2\left(x^2+1\right)} = -\frac{1}{x} - \frac{6}{x^2} + \frac{x+7}{x^2+1}$$

which matches the formula given in (2).

As we have seen in this opening example, resolving a rational function into partial fractions takes two steps: first, we need to determine the *form* of the decomposition, and then we need to determine the unknown coefficients which appear in said form.

Theorem 2.18 guarantees that any polynomial with real coefficients can be factored over the real numbers as a product of linear factors and irreducible quadratic factors. Once we have this factorization of the denominator of a rational function, the next theorem tells us the form the decomposition takes. The reader is encouraged to review the Factor Theorem (Theorem 2.8) and its connection to the role of multiplicity to fully appreciate the statement of the following theorem.

[3]We will justify this shortly.

Theorem 9.10. Suppose $R(x) = \dfrac{N(x)}{D(x)}$ is a rational function where the degree of $N(x)$ less than the degree of $D(x)$ and $N(x)$ and $D(x)$ have no common factors.[a]

- If α is a real zero of D of multiplicity m which corresponds to the linear factor $ax + b$, the partial fraction decomposition includes

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_m}{(ax+b)^m}$$

  for real numbers $A_1, A_2, \ldots A_m$.

- If α is a non-real zero of D of multiplicity m which corresponds to the irreducible quadratic $ax^2 + bx + c$, the partial fraction decomposition includes

$$\frac{B_1 x + C_1}{ax^2+bx+c} + \frac{B_2 x + C_2}{\left(ax^2+bx+c\right)^2} + \ldots + \frac{B_m x + C_m}{\left(ax^2+bx+c\right)^m}$$

  for real numbers $B_1, B_2, \ldots B_m$ and $C_1, C_2, \ldots C_m$.

[a] In other words, $R(x)$ is a proper rational function which has been fully reduced.

The proof of Theorem 9.10 is best left to a course in Abstract Algebra. Notice that the theorem provides for the general case, so we need to use subscripts, A_1, A_2, etc., to denote different unknown coefficients as opposed to the usual convention of A, B, etc.. The stress on multiplicities is to help us correctly group factors in the denominator. For example, consider the rational function

$$\frac{3x-1}{\left(x^2-1\right)\left(2-x-x^2\right)}$$

Factoring the denominator to find the zeros, we get $(x+1)(x-1)(1-x)(2+x)$. We find $x = -1$ and $x = -2$ are zeros of multiplicity one but that $x = 1$ is a zero of multiplicity two due to the two different factors $(x-1)$ and $(1-x)$. One way to handle this is to note that $(1-x) = -(x-1)$ so

$$\frac{3x-1}{(x+1)(x-1)(1-x)(2+x)} = \frac{3x-1}{-(x-1)^2(x+1)(x+2)} = \frac{1-3x}{(x-1)^2(x+1)(x+2)}$$

from which we proceed with the partial fraction decomposition

$$\frac{1-3x}{(x-1)^2(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{x+2}$$

Turning our attention to non-real zeros, we note that the tool of choice to determine the irreducibility of a quadratic $ax^2 + bx + c$ is the discriminant, $b^2 - 4ac$. If $b^2 - 4ac < 0$, the quadratic admits a *pair* of non-real complex conjugate zeros. Even though *one* irreducible quadratic gives *two* distinct non-real zeros, we list the terms with denominators involving a given irreducible quadratic only once to avoid duplication in the form of the decomposition. The trick, of course, is factoring the denominator or otherwise finding the zeros and their multiplicities in order to apply Theorem 9.10. We recommend that the reader review the techniques set forth in Sections 2.3 and 2.4.

Next, we state a theorem that if two polynomials are equal, the corresponding coefficients of the like powers of x are equal. This is the principal by which we shall determine the unknown coefficients in our partial fraction decomposition.

Theorem 9.11. Suppose

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = b_m x^m + m_{m-1}x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0$$

for all x in an open interval I. Then $n = m$ and $a_i = b_i$ for all $i = 1 \ldots n$.

Believe it or not, the proof of Theorem 9.11 is a consequence of Theorem 2.16. Define $p(x)$ to be the difference of the left hand side of the equation in Theorem 9.11 and the right hand side. Then $p(x) = 0$ for all x in the open interval I. If $p(x)$ were a nonzero polynomial of degree k, then, by Theorem 2.16, p could have at most k zeros in I, k being a *finite* number. Since $p(x) = 0$ for *all* real numbers x in I, p has infinitely many zeros, and hence, p is the zero polynomial. This means there can be no nonzero terms in $p(x)$ and the theorem follows. Arguably, the best way to make sense of either of the two preceding theorems is to work some examples.

Example 9.6.1. Resolve the following rational functions into partial fractions.

1. $R(x) = \dfrac{x+5}{2x^2-x-1}$
2. $f(z) = \dfrac{3}{z^3-2z^2+z}$
3. $F(s) = \dfrac{3}{s^3-s^2+s}$
4. $r(x) = \dfrac{4x^3}{x^2-2}$
5. $G(z) = \dfrac{z^3+5z-1}{z^4+6z^2+9}$
6. $H(s) = \dfrac{8s^2}{s^4+16}$

Solution.

1. We begin by factoring the denominator to find $2x^2 - x - 1 = (2x+1)(x-1)$. We get $x = -\frac{1}{2}$ and $x = 1$ are both zeros of multiplicity one and thus we know

$$\frac{x+5}{2x^2-x-1} = \frac{x+5}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

Clearing denominators, we get $x+5 = A(x-1)+B(2x+1)$ so that $x+5 = (A+2B)x+B-A$. Equating coefficients, we get the system

$$\left\{ \begin{array}{rcl} A+2B & = & 1 \\ -A+B & = & 5 \end{array} \right.$$

This system is readily handled using the Addition Method from Section A.6, and after adding both equations, we get $3B = 6$ so $B = 2$. Using back substitution, we find $A = -3$. Our answer is easily checked by getting a common denominator and adding the fractions.

$$\frac{x+5}{2x^2-x-1} = \frac{2}{x-1} - \frac{3}{2x+1}$$

2. Factoring the denominator gives $z^3 - 2z^2 + z = z\left(z^2 - 2z + 1\right) = z(z-1)^2$ which gives $z = 0$ as a zero of multiplicity one and $z = 1$ as a zero of multiplicity two. We have

$$\frac{3}{z^3 - 2z^2 + z} = \frac{3}{z(z-1)^2} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

Clearing denominators, we get $3 = A(z-1)^2 + Bz(z-1) + Cz$, which, after gathering up the like terms becomes $3 = (A+B)z^2 + (-2A - B + C)z + A$. Our system is

$$\left\{ \begin{array}{rcl} A + B & = & 0 \\ -2A - B + C & = & 0 \\ A & = & 3 \end{array} \right.$$

Substituting $A = 3$ into $A + B = 0$ gives $B = -3$, and substituting both for A and B in $-2A - B + C = 0$ gives $C = 3$. Our final answer is

$$\frac{3}{z^3 - 2z^2 + z} = \frac{3}{z} - \frac{3}{z-1} + \frac{3}{(z-1)^2}$$

3. The denominator factors as $s\left(s^2 - s + 1\right)$. We see immediately that $s = 0$ is a zero of multiplicity one, but the zeros of $s^2 - s + 1$ aren't as easy to discern. The quadratic doesn't factor easily, so we check the discriminant and find it to be $(-1)^2 - 4(1)(1) = -3 < 0$. We find its zeros are not real so it is an irreducible quadratic. The form of the partial fraction decomposition is then

$$\frac{3}{s^3 - s^2 + s} = \frac{3}{s\left(s^2 - s + 1\right)} = \frac{A}{s} + \frac{Bs + C}{s^2 - s + 1}$$

Clearing denominators gives $3 = A\left(s^2 - s + 1\right) + (Bs + C)s$ or $3 = (A+B)s^2 + (-A + C)s + A$, hence

$$\left\{ \begin{array}{rcl} A + B & = & 0 \\ -A + C & = & 0 \\ A & = & 3 \end{array} \right.$$

From $A = 3$ and $A + B = 0$, we get $B = -3$. From $-A + C = 0$, we get $C = A = 3$. We get

$$\frac{3}{s^3 - s^2 + s} = \frac{3}{s} + \frac{3 - 3s}{s^2 - s + 1}$$

4. Since $\frac{4x^3}{x^2-2}$ isn't proper, we first use long division and obtain a quotient of $4x$ with a remainder of $8x$. Rewriting, $\frac{4x^3}{x^2-2} = 4x + \frac{8x}{x^2-2}$ so we focus on resolving $\frac{8x}{x^2-2}$ into partial fractions. The quadratic $x^2 - 2$, though it doesn't factor nicely, is, nevertheless, reducible. Solving $x^2 - 2 = 0$ gives us $x = \pm\sqrt{2}$, so using Theorem 2.16,[4] we have $x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$. Hence,

[4]Alternatively, we can recognize $x^2 - 2 = x^2 - (\sqrt{2})^2$ and use the Difference of Squares formula on page 1425.

$$\frac{8x}{x^2-2} = \frac{8x}{\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)} = \frac{A}{x-\sqrt{2}} + \frac{B}{x+\sqrt{2}}$$

Clearing fractions, we get $8x = A\left(x+\sqrt{2}\right) + B\left(x-\sqrt{2}\right)$ or $8x = (A+B)x + (A-B)\sqrt{2}$ which gives

$$\left\{\begin{array}{rcl} A+B & = & 8 \\ (A-B)\sqrt{2} & = & 0 \end{array}\right.$$

From $(A-B)\sqrt{2} = 0$, we get $A = B$, which, when substituted into $A + B = 8$ gives $B = 4$. Hence, $A = B = 4$ and we get

$$\frac{4x^3}{x^2-2} = 4x + \frac{8x}{x^2-2} = 4x + \frac{4}{x+\sqrt{2}} + \frac{4}{x-\sqrt{2}}$$

5. At first glance, the denominator $D(z) = z^4 + 6z^2 + 9$ appears irreducible. However, $D(z)$ has three terms, and the exponent on the first term is exactly twice that of the second. Rewriting $D(z) = \left(z^2\right)^2 + 6z^2 + 9$, we see it is a quadratic in disguise and factor $D(z) = \left(z^2+3\right)^2$. Since $z^2 + 3$ clearly has no real zeros, it is irreducible and the form of the decomposition is

$$\frac{z^3+5z-1}{z^4+6z^2+9} = \frac{z^3+5z-1}{\left(z^2+3\right)^2} = \frac{Az+B}{z^2+3} + \frac{Cz+D}{\left(z^2+3\right)^2}$$

After the usual clearing of denominators, we have $z^3 + 5z - 1 = (Az + B)\left(z^2+3\right) + Cz + D$ which gives $z^3 + 5z - 1 = Az^3 + Bz^2 + (3A + C)z + 3B + D$. Our system is

$$\left\{\begin{array}{rcr} A & = & 1 \\ B & = & 0 \\ 3A+C & = & 5 \\ 3B+D & = & -1 \end{array}\right.$$

We have $A = 1$ and $B = 0$ from which we get $C = 2$ and $D = -1$. Our final answer is

$$\frac{z^3+5z-1}{z^4+6z^2+9} = \frac{z}{z^2+3} + \frac{2z-1}{\left(z^2+3\right)^2}$$

6. Once again, the difficulty in our last example is factoring the denominator. In an attempt to get a quadratic in disguise, we write

$$s^4 + 16 = \left(s^2\right)^2 + 4^2 = \left(s^2\right)^2 + 8s^2 + 4^2 - 8s^2 = \left(s^2+4\right)^2 - 8s^2$$

and obtain a difference of two squares: $\left(s^2+4\right)^2$ and $8s^2 = \left(2s\sqrt{2}\right)^2$. Hence,

$$s^4+16=\left(s^2+4-2s\sqrt{2}\right)\left(s^2+4+2s\sqrt{2}\right)=\left(s^2-2s\sqrt{2}+4\right)\left(s^2+2s\sqrt{2}+4\right)$$

The discriminant of both of these quadratics works out to be $-8<0$, which means they are irreducible. We leave it to the reader to verify that, despite having the same discriminant, these quadratics have different zeros. The partial fraction decomposition takes the form

$$\frac{8s^2}{s^4+16}=\frac{8s^2}{\left(s^2-2s\sqrt{2}+4\right)\left(s^2+2s\sqrt{2}+4\right)}=\frac{As+B}{s^2-2s\sqrt{2}+4}+\frac{Cs+D}{s^2+2s\sqrt{2}+4}$$

We get $8s^2=(As+B)\left(s^2+2s\sqrt{2}+4\right)+(Cs+D)\left(s^2-2s\sqrt{2}+4\right)$ or

$$8s^2=(A+C)s^3+(2A\sqrt{2}+B-2C\sqrt{2}+D)s^2+(4A+2B\sqrt{2}+4C-2D\sqrt{2})s+4B+4D$$

which gives the system

$$\left\{\begin{array}{rcl} A+C & = & 0 \\ 2A\sqrt{2}+B-2C\sqrt{2}+D & = & 8 \\ 4A+2B\sqrt{2}+4C-2D\sqrt{2} & = & 0 \\ 4B+4D & = & 0 \end{array}\right.$$

From $A+C=0$, we get $A=-C$. Likewise, from $4B+4D=0$, we get $B=-D$. Substituting these into the remaining two equations gives

$$\left\{\begin{array}{rcl} -2C\sqrt{2}-D-2C\sqrt{2}+D & = & 8 \\ -4C-2D\sqrt{2}+4C-2D\sqrt{2} & = & 0 \end{array}\right.$$

or

$$\left\{\begin{array}{rcl} -4C\sqrt{2} & = & 8 \\ -4D\sqrt{2} & = & 0 \end{array}\right.$$

We get $C=-\sqrt{2}$ so that $A=-C=\sqrt{2}$ and $D=0$ which means $B=-D=0$. We get

$$\frac{8s^2}{s^4+16}=\frac{s\sqrt{2}}{s^2-2s\sqrt{2}+4}-\frac{s\sqrt{2}}{s^2+2s\sqrt{2}+4}$$

$\square$

9.6.1 Exercises

In Exercises 1 - 6, find only the *form* needed to begin the process of partial fraction decomposition. Do not create the system of linear equations or attempt to find the actual decomposition.

1. $\dfrac{7}{(x-3)(x+5)}$
2. $\dfrac{5x+4}{x(x-2)(2-x)}$
3. $\dfrac{m}{(7x-6)(x^2+9)}$
4. $\dfrac{ax^2+bx+c}{x^3(5x+9)(3x^2+7x+9)}$
5. $\dfrac{\text{A polynomial of degree } < 9}{(x+4)^5(x^2+1)^2}$
6. $\dfrac{\text{A polynomial of degree } < 7}{x(4x-1)^2(x^2+5)(9x^2+16)}$

In Exercises 7 - 18, find the partial fraction decomposition of the following rational expressions.

7. $\dfrac{2x}{x^2-1}$
8. $\dfrac{-7x+43}{3x^2+19x-14}$
9. $\dfrac{11z^2-5z-10}{5z^3-5z^2}$
10. $\dfrac{-2z^2+20z-68}{z^3+4z^2+4z+16}$
11. $\dfrac{-s^2+15}{4s^4+40s^2+36}$
12. $\dfrac{-21s^2+s-16}{3s^3+4s^2-3s+2}$
13. $\dfrac{5x^4-34x^3+70x^2-33x-19}{(x-3)^2}$
14. $\dfrac{x^6+5x^5+16x^4+80x^3-2x^2+6x-43}{x^3+5x^2+16x+80}$
15. $\dfrac{-7z^2-76z-208}{z^3+18z^2+108z+216}$
16. $\dfrac{-10z^4+z^3-19z^2+z-10}{z^5+2z^3+z}$
17. $\dfrac{4s^3-9s^2+12s+12}{s^4-4s^3+8s^2-16s+16}$
18. $\dfrac{2s^2+3s+14}{(s^2+2s+9)(s^2+s+5)}$
19. Find a partial fraction decomposition of $R(z) = \dfrac{4}{z^4-1}$ over the *complex* numbers.
20. In light of Theorem 2.16, we know all polynomial functions can be reduced to a product of *linear* factors - if we use complex numbers. It turns out that Theorem 9.10 holds true with complex numbers as well (though when using complex numbers, there are no irreducible quadratics.) Discuss with your classmates how this ultimately means every rational function is a sum of shifted Laurent Monomials.[5]
21. One of the most common algebraic error the authors encounter when teaching Calculus is along the lines of:

$$\frac{8}{x^2-9} \neq \frac{8}{x^2} - \frac{8}{9}$$

Think about why if the above were true, this section would have no need to exist.

[5]See Section 3.1.

9.6.2 Answers

1. $\dfrac{A}{x-3}+\dfrac{B}{x+5}$

2. $\dfrac{A}{x}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$

3. $\dfrac{A}{7x-6}+\dfrac{Bx+C}{x^2+9}$

4. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{5x+9}+\dfrac{Ex+F}{3x^2+7x+9}$

5. $\dfrac{A}{x+4}+\dfrac{B}{(x+4)^2}+\dfrac{C}{(x+4)^3}+\dfrac{D}{(x+4)^4}+\dfrac{E}{(x+4)^5}+\dfrac{Fx+G}{x^2+1}+\dfrac{Hx+I}{(x^2+1)^2}$

6. $\dfrac{A}{x}+\dfrac{B}{4x-1}+\dfrac{C}{(4x-1)^2}+\dfrac{Dx+E}{x^2+5}+\dfrac{Fx+G}{9x^2+16}$

7. $\dfrac{2x}{x^2-1}=\dfrac{1}{x+1}+\dfrac{1}{x-1}$

8. $\dfrac{-7x+43}{3x^2+19x-14}=\dfrac{5}{3x-2}-\dfrac{4}{x+7}$

9. $\dfrac{11z^2-5z-10}{5z^3-5z^2}=\dfrac{3}{z}+\dfrac{2}{z^2}-\dfrac{4}{5(z-1)}$

10. $\dfrac{-2z^2+20z-68}{z^3+4z^2+4z+16}=-\dfrac{9}{z+4}+\dfrac{7z-8}{z^2+4}$

11. $\dfrac{-s^2+15}{4s^4+40s^2+36}=\dfrac{1}{2(s^2+1)}-\dfrac{3}{4(s^2+9)}$

12. $\dfrac{-21s^2+s-16}{3s^3+4s^2-3s+2}=-\dfrac{6}{s+2}-\dfrac{3s+5}{3s^2-2s+1}$

13. $\dfrac{5x^4-34x^3+70x^2-33x-19}{(x-3)^2}=5x^2-4x+1+\dfrac{9}{x-3}-\dfrac{1}{(x-3)^2}$

14. $\dfrac{x^6+5x^5+16x^4+80x^3-2x^2+6x-43}{x^3+5x^2+16x+80}=x^3+\dfrac{x+1}{x^2+16}-\dfrac{3}{x+5}$

15. $\dfrac{-7z^2-76z-208}{z^3+18z^2+108z+216}=-\dfrac{7}{z+6}+\dfrac{8}{(z+6)^2}-\dfrac{4}{(z+6)^3}$

16. $\dfrac{-10z^4+z^3-19z^2+z-10}{z^5+2z^3+z}=-\dfrac{10}{z}+\dfrac{1}{z^2+1}+\dfrac{z}{(z^2+1)^2}$

17. $\dfrac{4s^3-9s^2+12s+12}{s^4-4s^3+8s^2-16s+16}=\dfrac{1}{s-2}+\dfrac{4}{(s-2)^2}+\dfrac{3s+1}{s^2+4}$

18. $\dfrac{2s^2+3s+14}{(s^2+2s+9)(s^2+s+5)}=\dfrac{1}{s^2+2s+9}+\dfrac{1}{s^2+s+5}$

19. $R(z)=\dfrac{4}{z^4-1}=\dfrac{1}{z-1}-\dfrac{1}{z+1}+\dfrac{i}{z-i}-\dfrac{i}{z+i}$

9.7 Systems of Non-Linear Equations

In this section, we study systems of non-linear equations. In non-linear equations, we can have variables to powers other than 1, we can have different variables multiplied together, or variable can occur as arguments of exponential and logarithmic functions.

Unlike the systems of linear equations for which we have developed several algorithmic solution techniques, there is no general algorithm to solve systems of non-linear equations. Moreover, all of the usual hazards of non-linear equations like extraneous solutions and domain restrictions are once again present.

Along with the tried and true techniques of substitution and elimination, we shall often need equal parts tenacity and ingenuity to see a problem through to the end. You may find it necessary to review topics throughout the text which pertain to solving equations involving the various functions we have studied thus far. To get the section rolling we begin with a fairly routine example.

Example 9.7.1. Solve the following systems of equations. Verify answers algebraically and graphically.

1. $\left\{ \begin{array}{rcr} x^2+y^2 & = & 4 \\ 4x^2+9y^2 & = & 36 \end{array} \right.$
2. $\left\{ \begin{array}{rcr} x^2+y^2 & = & 4 \\ 4x^2-9y^2 & = & 36 \end{array} \right.$
3. $\left\{ \begin{array}{rcr} x^2+y^2 & = & 4 \\ y-2x & = & 0 \end{array} \right.$
4. $\left\{ \begin{array}{rcr} x^2+y^2 & = & 4 \\ y-x^2 & = & 0 \end{array} \right.$

SOLUTION:

1. Since both equations contain x^2 and y^2 only, we can use elimination as seen in Section A.6:

$$\left\{ \begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & 4x^2+9y^2 & = & 36 \end{array} \right. \xrightarrow[-4E1+E2]{\text{Replace } E2 \text{ with}} \left\{ \begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & 5y^2 & = & 20 \end{array} \right.$$

From $5y^2 = 20$, we get $y^2 = 4$ or $y = \pm 2$. To find the associated x values, we substitute each value of y into one of the equations to find the resulting value of x.

Choosing $x^2 + y^2 = 4$, we find that for both $y = -2$ and $y = 2$, we get $x = 0$. Our solution is thus $\{(0, 2), (0, -2)\}$. To verify these answers algebraically, we would need to show that the pair $(x, y) = (0, 2)$ and $(x, y) = (0, -2)$ each satisfy *both* equations. We leave this to the reader.

To check our answer graphically, we sketch both equations and look for their points of intersection.

The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2. To graph $4x^2 + 9y^2 = 36$, we convert to standard form $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and recognize it as an ellipse centered at $(0, 0)$ with a major axis along the x-axis of length 6 and a minor axis along the y-axis of length 4.

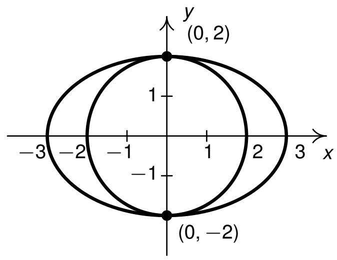
We see from the graph that the two curves intersect at their y-intercepts only, $(0, \pm 2)$.

2. We proceed as before to eliminate one of the variables

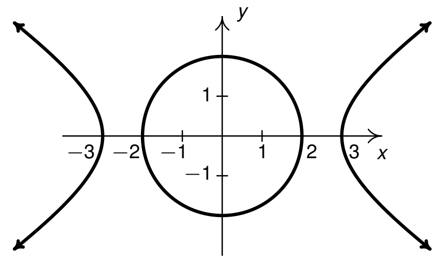
$$\left\{ \begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & 4x^2-9y^2 & = & 36 \end{array} \right. \xrightarrow[-4E1+E2]{\text{Replace } E2 \text{ with}} \left\{ \begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & -13y^2 & = & 20 \end{array} \right.$$

Since the equation $-13y^2 = 20$ admits no real solution, the system is inconsistent. To verify this graphically, we note that $x^2 + y^2 = 4$ is the same circle as before, but when writing the second equation in standard form, $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we find a hyperbola centered at $(0, 0)$ opening to the left and right with a transverse axis of length 6 and a conjugate axis of length 4.

We see that the circle and the hyperbola have no points in common, hence, there are no solutions.



Graphs for $\begin{cases} x^2 + y^2 = 4 \\ 4x^2 + 9y^2 = 36 \end{cases}$



Graphs for $\begin{cases} x^2 + y^2 = 4 \\ 4x^2 - 9y^2 = 36 \end{cases}$

3. Since there are no like terms among the two equations, elimination won't work here. Instead, we proceed using substitution.

   From the equation $y - 2x = 0$, we get $y = 2x$. Substituting this into $x^2 + y^2 = 4$ gives $x^2 + (2x)^2 = 4$. Solving, we find $5x^2 = 4$ or $x = \pm\frac{2\sqrt{5}}{5}$.

   Returning to the equation we used for the substitution, $y = 2x$, we find $y = \frac{4\sqrt{5}}{5}$ when $x = \frac{2\sqrt{5}}{5}$, so one solution is $\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right)$ and the other is $\left(-\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5}\right)$. Hence, our final answer is $\left\{\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right), \left(-\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5}\right)\right\}$. As before, we leave the algebraic check to the reader.

   The graph of $x^2 + y^2 = 4$ is our circle from before and the graph of $y - 2x = 0$, or $y = 2x$ is a line through the origin with slope 2. Even though we cannot easily verify the numerical values of the points of intersection from our sketch, we can be sure there are just two solutions: one in Quadrant I and one in Quadrant III. This observation, combined with our (your) algebraic check gives us confidence our solution is correct.[1]

4. While it may be tempting to solve $y - x^2 = 0$ as $y = x^2$ and substitute, we note that this system is set up for elimination.[2]

$$\begin{cases} (E1) & x^2 + y^2 = 4 \\ (E2) & y - x^2 = 0 \end{cases} \xrightarrow[E1 + E2]{\text{Replace } E2 \text{ with}} \begin{cases} (E1) & x^2 + y^2 = 4 \\ (E2) & y^2 + y = 4 \end{cases}$$

   From $y^2 + y = 4$ we get $y^2 + y - 4 = 0$ which gives $y = \frac{-1\pm\sqrt{17}}{2}$. Due to the complicated nature of these answers, it is worth our time to make a quick sketch of both equations first to head off any extraneous solutions we may encounter.

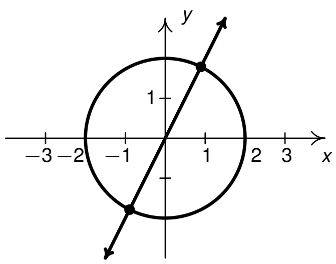
[1] Of course, we could check our answers more accurately using a graphing utility.

[2] We encourage the reader to solve the system using substitution to see that you get the same solution.

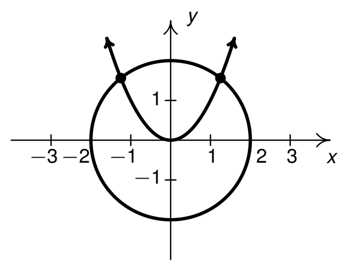
We see that the circle $x^2 + y^2 = 4$ intersects the parabola $y = x^2$ exactly *twice*, and both of these points have a positive y value.

Of the two solutions for y, only $y = \frac{-1+\sqrt{17}}{2}$ is positive, so to get our solution, we substitute this into $y - x^2 = 0$ and solve for x. We get $x = \pm\sqrt{\frac{-1+\sqrt{17}}{2}} = \pm\frac{\sqrt{-2+2\sqrt{17}}}{2}$.

Our final answer is $\left\{\left(\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right), \left(-\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right)\right\}$. Checking these answers algebraically amounts to a true test of anyone's algebraic mettle and as such is left to the reader.



Graphs for $\left\{\begin{array}{rcl} x^2 + y^2 & = & 4 \\ y - 2x & = & 0 \end{array}\right.$



Graphs for $\left\{\begin{array}{rcl} x^2 + y^2 & = & 4 \\ y - x^2 & = & 36 \end{array}\right.$

$\square$

A couple of remarks about Example 9.7.1 are in order. First note that, unlike systems of linear equations, it is possible for a system of non-linear equations to have more than one solution without having infinitely many solutions. In fact, while we characterize systems of nonlinear equations as being 'consistent' or 'inconsistent,' we generally don't use the labels 'dependent' or 'independent'.

Secondly, as we saw with the last problem, sometimes making a quick sketch of the problem situation can save a lot of time and effort. While in general the curves in a system of non-linear equations may not be easily visualized, it pays to take advantage when they are. Our next example provides some considerable review of many of the topics introduced in this text.

Example 9.7.2. Solve the following systems of equations. Verify answers algebraically and graphically.

1. $\left\{\begin{array}{rcl} x^2 + 2xy - 16 & = & 0 \\ y^2 + 2xy - 16 & = & 0 \end{array}\right.$
2. $\left\{\begin{array}{rcl} y + 4e^{2x} & = & 1 \\ y^2 + 2e^x & = & 1 \end{array}\right.$
3. $\left\{\begin{array}{rcl} z(x-2) & = & x \\ yz & = & y \\ (x-2)^2 + y^2 & = & 1 \end{array}\right.$

Solution.

1. At first glance, it doesn't appear as though elimination will do us any good since it's clear that we cannot completely eliminate one of the variables. The alternative, however, namely solving one of the equations for one variable and substituting it into the other, is very intimidating.

   Returning to elimination, we note that it is possible to eliminate the troublesome xy term, and the constant term as well. Doing so we get a more tractable relationship between x and y:

$$\left\{ \begin{array}{rrcl} (E1) & x^2+2xy-16 & = & 0 \\ (E2) & y^2+2xy-16 & = & 0 \end{array} \right. \xrightarrow[-E1+E2]{\text{Replace } E2 \text{ with}} \left\{ \begin{array}{rrcl} (E1) & x^2+2xy-16 & = & 0 \\ (E2) & y^2-x^2 & = & 0 \end{array} \right.$$

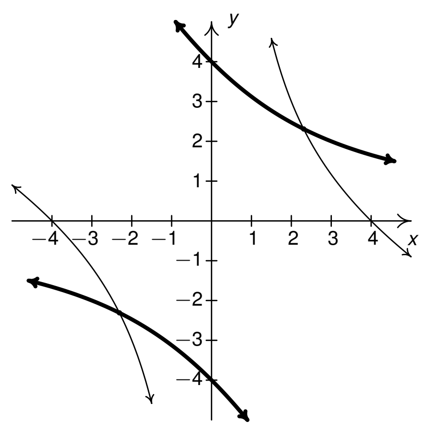
We get $y^2 - x^2 = 0$ or $y = \pm x$. Substituting $y = x$ into $E1$ we get $x^2 + 2x^2 - 16 = 0$ so that $x^2 = \frac{16}{3}$ or $x = \pm\frac{4\sqrt{3}}{3}$. On the other hand, when we substitute $y = -x$ into $E1$, we get $x^2 - 2x^2 - 16 = 0$ or $x^2 = -16$ which gives no real solutions.

Substituting each of $x = \pm\frac{4\sqrt{3}}{3}$ into $y = x$ yields the solution $\left\{\left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}\right), \left(-\frac{4\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3}\right)\right\}$. As usual, we leave it to the reader to verifying this solution algebraically.

To verify this equation graphically, we solve $x^2 + 2xy - 16 = 0$ for y to obtain $y = \frac{16-x^2}{2x}$. We produce the graph of this equation using the techniques described in Section 3.2.

Solving the second equation, $y^2 + 2xy - 16 = 0$, for y, however, is more complicated. The quadratic formula gives $y = -x \pm \sqrt{x^2+16}$ which requires Calculus or a graphing utility to graph.

As it happens, however, we don't need either because the equation $y^2+2xy-16 = 0$ can be obtained from the equation $x^2 + 2xy - 16 = 0$ by interchanging 'y' and 'x.' Thinking back to Section 5.6, this means we can obtain the graph of $y^2 + 2xy - 16 = 0$ by reflecting the graph of $x^2 + 2xy - 16 = 0$ across the line $y = x$. Doing so confirms that the two graphs intersect twice: once in Quadrant I, and once in Quadrant III as required.[3]



The graphs of $x^2 + 2xy - 16 = 0$ and $\mathbf{y^2 + 2xy - 16 = 0}$

2. Unlike the previous problem, there seems to be no avoiding substitution and a bit of algebraic unpleasantness. Solving $y + 4e^{2x} = 1$ for y, we get $y = 1 - 4e^{2x}$ which, when substituted into the second equation, yields $\left(1 - 4e^{2x}\right)^2 + 2e^x = 1$.

   After expanding and gathering like terms, we get $16e^{4x} - 8e^{2x} + 2e^x = 0$. Factoring gives us $2e^x\left(8e^{3x} - 4e^x + 1\right) = 0$, and since $2e^x \neq 0$ for any real x, we are left with solving $8e^{3x} - 4e^x + 1 = 0$.

[3] An amazing coincidence. Of course, we could use a graphing utility to verify our solutions if this didn't happen to be the case.

We have three terms, and even though this is not a 'quadratic in disguise', we can benefit from the substitution $u = e^x$. The equation becomes $8u^3 - 4u + 1 = 0$. Using the techniques set forth in Section 2.3, we find $u = \frac{1}{2}$ is a zero and factor $8u^3 - 4u + 1$. We find $8u^3 - 4u + 1 = 0$ is equivalent to $\left(u - \frac{1}{2}\right)\left(8u^2 + 4u - 2\right) = 0$. So in addition to $u = \frac{1}{2}$, we need to solve $8u^2 + 4u - 2 = 0$.

We use the quadratic formula to solve $8u^2 + 4u - 2 = 0$ and find $u = \frac{-1 \pm \sqrt{5}}{4}$. Since $u = e^x$, we now must solve $e^x = \frac{1}{2}$ and $e^x = \frac{-1 \pm \sqrt{5}}{4}$. From $e^x = \frac{1}{2}$, we get $x = \ln\left(\frac{1}{2}\right) = -\ln(2)$.

As for $e^x = \frac{-1 \pm \sqrt{5}}{4}$, we first note that $\frac{-1-\sqrt{5}}{4} < 0$, so $e^x = \frac{-1-\sqrt{5}}{4}$ has no real solutions. We are left with $e^x = \frac{-1+\sqrt{5}}{4}$, so that $x = \ln\left(\frac{-1+\sqrt{5}}{4}\right)$.

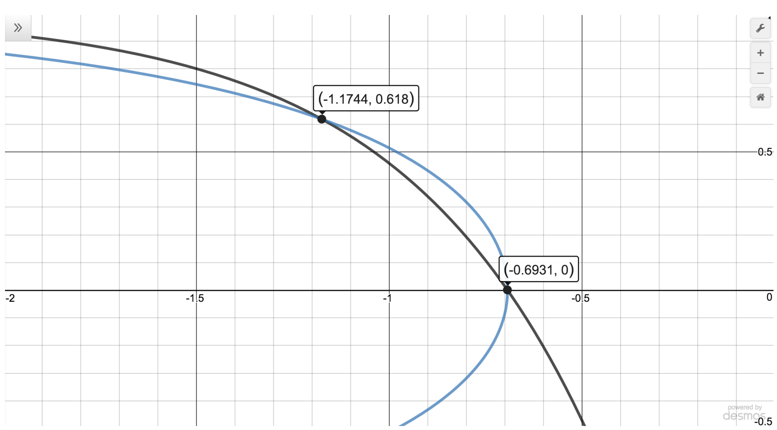
We now return to $y = 1 - 4e^{2x}$ to find the accompanying y values for each of our solutions for x:

$$\begin{array}{rcl} x & = & -\ln(2) : \\ y & = & 1 - 4e^{2x} \\ & = & 1 - 4e^{-2\ln(2)} \\ & = & 1 - 4e^{\ln\left(\frac{1}{4}\right)} \\ & = & 1 - 4\left(\frac{1}{4}\right) \\ & = & 0 \end{array}$$

$$\begin{array}{rcl} x & = & \ln\left(\frac{-1+\sqrt{5}}{4}\right) : \\ y & = & 1 - 4e^{2x} \\ & = & 1 - 4e^{2\ln\left(\frac{-1+\sqrt{5}}{4}\right)} \\ & = & 1 - 4e^{\ln\left(\frac{-1+\sqrt{5}}{4}\right)^2} \\ & = & 1 - 4\left(\frac{-1+\sqrt{5}}{4}\right)^2 \\ & = & 1 - 4\left(\frac{3-\sqrt{5}}{8}\right) \\ & = & \frac{-1+\sqrt{5}}{2} \end{array}$$

We get two solutions, $\left\{(-\ln(2), 0), \left(\ln\left(\frac{-1+\sqrt{5}}{4}\right), \frac{-1+\sqrt{5}}{2}\right)\right\}$. It is a good review of the properties of logarithms to verify both solutions algebraically, so we leave that to the reader.

While we are able to sketch $y = 1 - 4e^{2x}$ using the techniques in Section 7.1, the second equation is more difficult and we resort to using a graphing utility. We see the two graphs intersect at $(-0.6931, 0) \approx (-\ln(2), 0)$ and $(-1.1744, 0.618) \approx \left(\ln\left(\frac{-1+\sqrt{5}}{4}\right), \frac{-1+\sqrt{5}}{2}\right)$.



Graphs for $\left\{\begin{array}{rcl} y + 4e^{2x} & = & 1 \\ y^2 + 2e^x & = & 1 \end{array}\right.$

3. Our last system involves three variables and provides an opportunity to gain some insight on how to keep such systems organized. Labeling the equations as before, we have

$$\left\{ \begin{array}{lrcl} E1 & z(x-2) & = & x \\ E2 & yz & = & y \\ E3 & (x-2)^2+y^2 & = & 1 \end{array} \right.$$

The easiest equation to start with appears to be $E2$. While it may be tempting to divide both sides of $E2$ by y, we caution against this practice because it presupposes $y \neq 0$. Instead, we take $E2$ and rewrite it as $yz - y = 0$ so $y(z-1) = 0$. From this, we get two cases: $y = 0$ or $z = 1$.

Case 1: $y = 0$. Substituting $y = 0$ into $E1$ and $E3$, we get

$$\left\{ \begin{array}{lrcl} E1 & z(x-2) & = & x \\ E3 & (x-2)^2 & = & 1 \end{array} \right.$$

Solving $E3$ for x gives $x = 1$ or $x = 3$. Substituting these values into $E1$ gives $z = -1$ when $x = 1$ and $z = 3$ when $x = 3$. We obtain two solutions, $(1, 0, -1)$ and $(3, 0, 3)$.

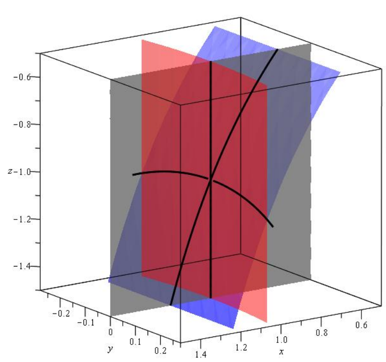
Case 2: $z = 1$. Substituting $z = 1$ into $E1$ and $E3$ gives us

$$\left\{ \begin{array}{lrcl} E1 & (1)(x-2) & = & x \\ E3 & (x-2)^2+y^2 & = & 1 \end{array} \right.$$

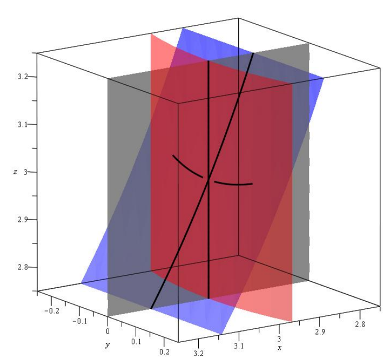
Equation $E1$ gives us $x - 2 = x$ or $-2 = 0$, which is a contradiction. This means we have no solution to the system in this case, even though $E3$ is satisfied by infinitely many pairs of points (x, y).

Hence, our final answer is $\{(1, 0, -1), (3, 0, 3)\}$. These points are easy enough to check algebraically in our three original equations, so that is left to the reader.

As for verifying these solutions graphically, they require plotting surfaces in three dimensions and looking for intersection points. While this is beyond the scope of this book, we provide a snapshot of the graphs of our three equations near one of the solution points, $(1, 0, -1)$ and $(3, 0, 3)$. □



Near $(-1, 0, 1)$



Near $(3, 0, 3)$

Example 9.7.2 showcases some of the ingenuity and tenacity mentioned at the beginning of the section. Sometimes you just have to look at a system the right way to find the most efficient method to solve it. Sometimes you just have to try something.

Next we explore some common application problems which give rise to systems of nonlinear equations.

Example 9.7.3. Carl decides to explore the Meander River, the location of several recent Sasquatch sightings. From camp, he canoes downstream five miles to check out a purported Sasquatch nest. Finding nothing, he immediately turns around, retraces his route (this time traveling upstream), and returns to camp 3 hours after he left. If Carl canoes at a rate of 6 miles per hour in still water, how fast was the Meander River flowing on that day?

Solution. We are given information about distances, rates (speeds) and times. The basic principle relating these quantities is:

$$\text{distance} = \text{rate} \cdot \text{time}$$

The first observation to make, however, is that the distance, rate and time given to us aren't 'compatible': the distance given is the distance for only *part* of the trip, the rate given is the speed Carl can canoe in still water, not in a flowing river, and the time given is the duration of the *entire* trip. Ultimately, we are after the speed of the river, so let's call that R measured in miles per hour to be consistent with the other rate given to us. To get started, let's divide the trip into its two parts: the initial trip downstream and the return trip upstream. For the downstream trip, all we know is that the distance traveled is 5 miles.

$$\begin{array}{rcl} \text{distance downstream} & = & \text{rate traveling downstream} \cdot \text{time traveling downstream} \\ 5\,\text{miles} & = & \text{rate traveling downstream} \cdot \text{time traveling downstream} \end{array}$$

Since the return trip upstream followed the same route as the trip downstream, we know that the distance traveled upstream is also 5 miles.

$$\begin{array}{rcl} \text{distance upstream} & = & \text{rate traveling upstream} \cdot \text{time traveling upstream} \\ 5\,\text{miles} & = & \text{rate traveling upstream} \cdot \text{time traveling upstream} \end{array}$$

We are told Carl can canoe at a rate of 6 miles per hour in still water. How does this figure into the rates traveling upstream and downstream? The speed the canoe travels in the river is a combination of the speed at which Carl can propel the canoe in still water, 6 miles per hour, and the speed of the river, which we're calling R. When traveling downstream, the river is helping Carl along, so we *add* these two speeds:

$$\begin{array}{rcl} \text{rate traveling downstream} & = & \text{rate Carl propels the canoe} + \text{speed of the river} \\ & = & 6\,\frac{\text{miles}}{\text{hour}} + R\,\frac{\text{miles}}{\text{hour}} \end{array}$$

So our downstream speed is $(6 + R)\frac{\text{miles}}{\text{hour}}$. Substituting this into our 'distance-rate-time' equation for the downstream part of the trip, we get:

$$\begin{array}{rcl} 5\,\text{miles} & = & \text{rate traveling downstream} \cdot \text{time traveling downstream} \\ 5\,\text{miles} & = & (6 + R)\,\frac{\text{miles}}{\text{hour}} \cdot \text{time traveling downstream} \end{array}$$

When traveling upstream, Carl works *against* the current. Since Carl must move the canoe faster than the river's speed to move upstream, we *subtract* the river's speed *from* Carl's canoeing speed to get:

$$\begin{aligned} \text{rate traveling upstream} &= \text{rate Carl propels the canoe} - \text{river speed} \\ &= 6\,\frac{\text{miles}}{\text{hour}} - R\,\frac{\text{miles}}{\text{hour}} \end{aligned}$$

Proceeding as before, we get

$$\begin{aligned} 5\,\text{miles} &= \text{rate traveling upstream} \cdot \text{time traveling upstream} \\ 5\,\text{miles} &= (6 - R)\,\frac{\text{miles}}{\text{hour}} \cdot \text{time traveling upstream} \end{aligned}$$

The last piece of information given to us is that the total trip lasted 3 hours. If we let t_{down} denote the time of the downstream trip and t_{up} the time of the upstream trip, we have: $t_{\text{down}} + t_{\text{up}} = 3$ hours. Substituting t_{down} and t_{up} into the 'distance-rate-time' equations, we get a system of three equations in three unknowns below. Note that since the variables in equations $E1$ and $E2$ are multiplied together, these two equations are nonlinear.

$$\left\{ \begin{array}{lrcl} E1 & (6+R)\,t_{\text{down}} & = & 5 \\ E2 & (6-R)\,t_{\text{up}} & = & 5 \\ E3 & t_{\text{down}} + t_{\text{up}} & = & 3 \end{array} \right.$$

Since we are ultimately after R, we need to use these three equations to get at least one equation involving *only* R. We start with equation $E1$. We know that both $(6+R) \neq 0$ and $t_{\text{down}} \neq 0$ since the product of these two quantities is 5 and is nonzero.[4] Hence, we may solve $E1$ for t_{down} by dividing both sides by the quantity $(6+R)$ to get $t_{\text{down}} = \frac{5}{6+R}$. Similarly, we use $E2$ to get $t_{\text{up}} = \frac{5}{6-R}$. Substituting these into $E3$, we get:[5]

$$\frac{5}{6+R} + \frac{5}{6-R} = 3.$$

Clearing denominators, we get $5(6-R) + 5(6+R) = 3(6+R)(6-R)$ which reduces to $R^2 = 16$. We find $R = \pm 4$, and since R represents the speed of the river, we choose $R = 4$. On the day in question, the Meander River is flowing at a rate of 4 miles per hour. □

One of the important lessons to learn from Example 9.7.3 is that speeds, and more generally, rates, are additive. As we see in our next example, the concept of rate and its associated principles can be applied to a wide variety of problems - not just 'distance-rate-time' scenarios.

Example 9.7.4. Working alone, Taylor can weed the garden in 4 hours. If Carl helps, they can weed the garden in 3 hours. How long would it take for Carl to weed the garden on his own?

Solution. The key relationship between work and time which we use in this problem is:

$$\text{amount of work done} = \text{rate of work} \cdot \text{time spent working}$$

We are told that, working alone, Taylor can weed the garden in 4 hours. In Taylor's case then:

$$\begin{aligned} \text{amount of work Taylor does} &= \text{rate of Taylor working} \cdot \text{time Taylor spent working} \\ 1\,\text{garden} &= (\text{rate of Taylor working}) \cdot (4\,\text{hours}) \end{aligned}$$

[4] This is a restatement of the Zero Product Property from Section A.2.
[5] The reader is encouraged to verify that the units in this equation are consistent. For starters, the units on the '3' is 'hours.'

So we have that the rate Taylor works is $\frac{1 \text{ garden}}{4 \text{ hours}} = \frac{1}{4} \frac{\text{garden}}{\text{hour}}$. We are also told that when working together, Taylor and Carl can weed the garden in just 3 hours. We have:

$$\begin{aligned}\text{amount of work done together} &= \text{rate of working together} \cdot \text{time spent working together} \\ 1 \text{ garden} &= (\text{rate of working together}) \cdot (3 \text{ hours})\end{aligned}$$

From this, we find that the rate of Taylor and Carl working together is $\frac{1 \text{ garden}}{3 \text{ hours}} = \frac{1}{3} \frac{\text{garden}}{\text{hour}}$. We are asked to find out how long it would take for Carl to weed the garden on his own. Let us call this unknown t, measured in hours to be consistent with the other times given to us in the problem. Then:

$$\begin{aligned}\text{amount of work Carl does} &= \text{rate of Carl working} \cdot \text{time Carl spent working} \\ 1 \text{ garden} &= (\text{rate of Carl working}) \cdot (t \text{ hours})\end{aligned}$$

In order to find t, we need to find the rate of Carl working, so let's call this quantity R, with units $\frac{\text{garden}}{\text{hour}}$. Using the fact that rates are additive, we have:

$$\begin{aligned}\text{rate working together} &= \text{rate of Taylor working} + \text{rate of Carl working} \\ \frac{1}{3} \frac{\text{garden}}{\text{hour}} &= \frac{1}{4} \frac{\text{garden}}{\text{hour}} + R \frac{\text{garden}}{\text{hour}}\end{aligned}$$

so that $R = \frac{1}{12} \frac{\text{garden}}{\text{hour}}$. Substituting this into our 'work-rate-time' equation for Carl, we get:

$$\begin{aligned}1 \text{ garden} &= (\text{rate of Carl working}) \cdot (t \text{ hours}) \\ 1 \text{ garden} &= \left(\frac{1}{12} \frac{\text{garden}}{\text{hour}}\right) \cdot (t \text{ hours})\end{aligned}$$

Solving $1 = \frac{1}{12} t$, we get $t = 12$, so it takes Carl 12 hours to weed the garden on his own.[6] □

As is common with 'word problems' like Examples 9.7.3 and 9.7.4, there is no short-cut to the answer. Note that in Examples 9.7.3, we formalized the system of non-linear equations before solving whereas in Example 9.7.4, the system remained much in the background. We encourage the reader to carefully think through and apply the basic principles of rate to each (potentially different!) situation. It is time well spent. We also encourage the tracking of units, especially in the early stages of the problem. Not only does this promote uniformity in the units, it also serves as a quick means to check if an equation makes sense.[7]

[6] Carl would much rather spend his time writing open-source Mathematics texts than gardening anyway.

[7] In other words, make sure you don't try to add apples to oranges!

9.7.1 Exercises

Exercise idea: follow up on last example and have students formalize the system there using given variables.

In Exercises 1 - 6, solve the given system of nonlinear equations. Sketch the graph of both equations on the same set of axes to verify the solution set.

1. $\left\{ \begin{array}{rcl} x^2 - y & = & 4 \\ x^2 + y^2 & = & 4 \end{array} \right.$

2. $\left\{ \begin{array}{rcl} x^2 + y^2 & = & 4 \\ x^2 - y & = & 5 \end{array} \right.$

3. $\left\{ \begin{array}{rcl} x^2 + y^2 & = & 16 \\ 16x^2 + 4y^2 & = & 64 \end{array} \right.$

4. $\left\{ \begin{array}{rcl} x^2 + y^2 & = & 16 \\ 9x^2 - 16y^2 & = & 144 \end{array} \right.$

5. $\left\{ \begin{array}{rcl} x^2 + y^2 & = & 16 \\ \frac{1}{9}y^2 - \frac{1}{16}x^2 & = & 1 \end{array} \right.$

6. $\left\{ \begin{array}{rcl} x^2 + y^2 & = & 16 \\ x - y & = & 2 \end{array} \right.$

In Exercises 7 - 15, solve the given system of nonlinear equations. Use a graph to help you avoid any potential extraneous solutions.

7. $\left\{ \begin{array}{rcl} x^2 - y^2 & = & 1 \\ x^2 + 4y^2 & = & 4 \end{array} \right.$

8. $\left\{ \begin{array}{rcl} \sqrt{x+1} - y & = & 0 \\ x^2 + 4y^2 & = & 4 \end{array} \right.$

9. $\left\{ \begin{array}{rcl} x + 2y^2 & = & 2 \\ x^2 + 4y^2 & = & 4 \end{array} \right.$

10. $\left\{ \begin{array}{rcl} (x-2)^2 + y^2 & = & 1 \\ x^2 + 4y^2 & = & 4 \end{array} \right.$

11. $\left\{ \begin{array}{rcl} x^2 + y^2 & = & 25 \\ y - x & = & 1 \end{array} \right.$

12. $\left\{ \begin{array}{rcl} x^2 + y^2 & = & 25 \\ x^2 + (y-3)^2 & = & 10 \end{array} \right.$

13. $\left\{ \begin{array}{rcl} y & = & x^3 + 8 \\ y & = & 10x - x^2 \end{array} \right.$

14. $\left\{ \begin{array}{rcl} x^2 - xy & = & 8 \\ y^2 - xy & = & 8 \end{array} \right.$

15. $\left\{ \begin{array}{rcl} x^2 + y^2 & = & 25 \\ 4x^2 - 9y & = & 0 \\ 3y^2 - 16x & = & 0 \end{array} \right.$

16. A certain bacteria culture follows the Law of Uninbited Growth, Equation 7.4. After 10 minutes, there are 10,000 bacteria. Five minutes later, there are 14,000 bacteria. How many bacteria were present initially? How long before there are 50,000 bacteria?

Consider the system of nonlinear equations below

$$\left\{ \begin{array}{rcl} \dfrac{4}{x} + \dfrac{3}{y} & = & 1 \\ \dfrac{3}{x} + \dfrac{2}{y} & = & -1 \end{array} \right.$$

If we let $u = \frac{1}{x}$ and $v = \frac{1}{y}$ then the system becomes

$$\left\{ \begin{array}{rcl} 4u + 3v & = & 1 \\ 3u + 2v & = & -1 \end{array} \right.$$

This associated system of linear equations can then be solved using any of the techniques presented earlier in the chapter to find that $u = -5$ and $v = 7$. Thus $x = \frac{1}{u} = -\frac{1}{5}$ and $y = \frac{1}{v} = \frac{1}{7}$.

We say that the original system is **linear in form** because its equations are not linear but a few substitutions reveal a structure that we can treat like a system of linear equations. Each system in Exercises 17 - 19 is linear in form. Make the appropriate substitutions and solve for x and y.

17. $\begin{cases} 4x^3 + 3\sqrt{y} & = & 1 \\ 3x^3 + 2\sqrt{y} & = & -1 \end{cases}$

18. $\begin{cases} 4e^x + 3e^{-y} & = & 1 \\ 3e^x + 2e^{-y} & = & -1 \end{cases}$

19. $\begin{cases} 4\ln(x) + 3y^2 & = & 1 \\ 3\ln(x) + 2y^2 & = & -1 \end{cases}$

20. Solve the following system

$$\begin{cases} x^2 + \sqrt{y} + \log_2(z) & = & 6 \\ 3x^2 - 2\sqrt{y} + 2\log_2(z) & = & 5 \\ -5x^2 + 3\sqrt{y} + 4\log_2(z) & = & 13 \end{cases}$$

21. Systems of nonlinear equations show up in third semester Calculus in the midst of some really cool problems. The system below came from a problem in which we were asked to find the dimensions of a rectangular box with a volume of 1000 cubic inches that has minimal surface area. The variables x, y and z are the dimensions of the box and λ is called a Lagrange multiplier. With the help of your classmates, solve the system.[8]

$$\begin{cases} 2y + 2z & = & \lambda yz \\ 2x + 2z & = & \lambda xz \\ 2y + 2x & = & \lambda xy \\ xyz & = & 1000 \end{cases}$$

22. According to Theorem 2.18 in Section 2.4, the polynomial $p(x) = x^4 + 4$ can be factored into the product linear and irreducible quadratic factors. In this exercise, we present a method for obtaining that factorization.

    (a) Show that p has no real zeros.

    (b) Because p has no real zeros, its factorization must be of the form $(x^2 + ax + b)(x^2 + cx + d)$ where each factor is an irreducible quadratic. Expand this quantity and gather like terms together.

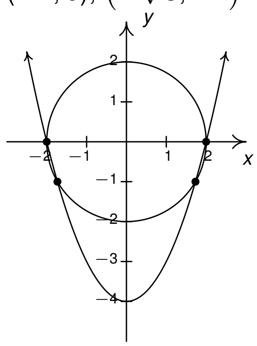
    (c) Create and solve the system of nonlinear equations which results from equating the coefficients of the expansion found above with those of $x^4 + 4$. You should get four equations in the four unknowns a, b, c and d. Write $p(x)$ in factored form.

23. Factor $q(x) = x^4 + 6x^2 - 5x + 6$.

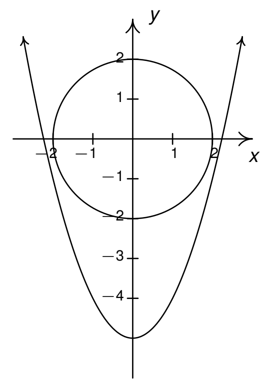
[8] If using λ bothers you, change it to w when you solve the system.

9.7.2 Answers

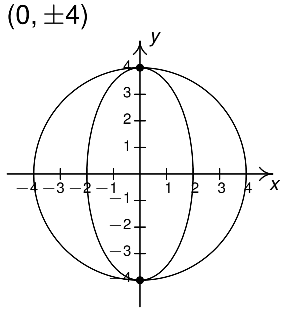
1. $(\pm 2, 0)$, $\left(\pm\sqrt{3}, -1\right)$



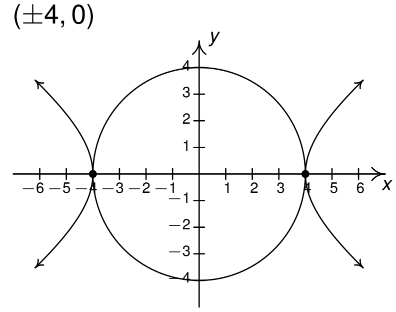
2. No solution



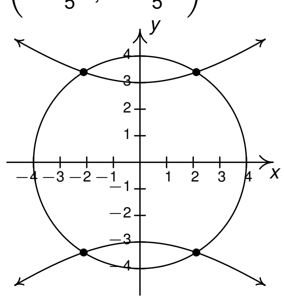
3. $(0, \pm 4)$



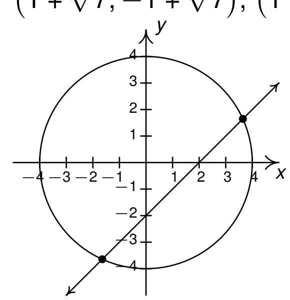
4. $(\pm 4, 0)$



5. $\left(\pm\frac{4\sqrt{7}}{5}, \pm\frac{12\sqrt{2}}{5}\right)$



6. $\left(1+\sqrt{7}, -1+\sqrt{7}\right)$, $\left(1-\sqrt{7}, -1-\sqrt{7}\right)$



7. $\left(\pm\frac{2\sqrt{10}}{5}, \pm\frac{\sqrt{15}}{5}\right)$
8. $(0, 1)$
9. $(0, \pm 1)$, $(2, 0)$
10. $\left(\frac{4}{3}, \pm\frac{\sqrt{5}}{3}\right)$
11. $(3, 4)$, $(-4, -3)$
12. $(\pm 3, 4)$
13. $(-4, -56)$, $(1, 9)$, $(2, 16)$
14. $(-2, 2)$, $(2, -2)$
15. $(3, 4)$
16. Initially, there are $\frac{250000}{49} \approx 5102$ bacteria. It will take $\frac{5\ln(49/5)}{\ln(7/5)} \approx 33.92$ minutes for the colony to grow to 50,000 bacteria.

17. $\left(-\sqrt[3]{5}, 49\right)$

18. No solution

19. $\left(e^{-5}, \pm\sqrt{7}\right)$

20. $(1, 4, 8)$, $(-1, 4, 8)$

21. $x = 10$, $y = 10$, $z = 10$, $\lambda = \frac{2}{5}$

22. (c) $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$

23. $x^4 + 6x^2 - 5x + 6 = (x^2 - x + 1)(x^2 + x + 6)$

9.8 Inequalities and Regions in the Plane

With few exceptions, we have spent our time in this course graphing equations relating two variables. In this section, we explore graphing inequalities relating two variables which usually a two-dimensional *region* in the plane instead of a one dimensional line or curve.[1] In our first example, we restrict our attention to looking at regions in the xy-plane bounded by equations which describe y as a function of x.

Example 9.8.1.

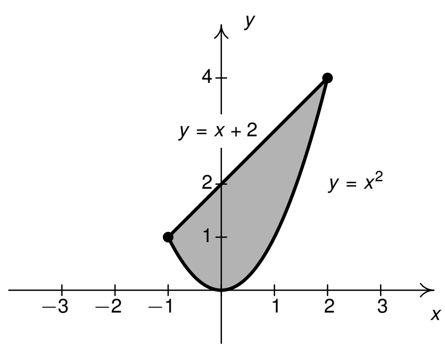
1. Sketch the following sets of points in the xy-plane.

   (a) $R = \{(x, y) \,|\, y > |x|\}$ (b) $S = \{(x, y) \,|\, y \leq 6 - x^2\}$

   (c) $T = \{(x, y) \,|\, |x| < y \leq 6 - x^2\}$

2. Find a set builder description for each of the following regions described below:

   (a) The region graphed below:



The region R

   (b) The region S between the graphs[2] of $f(x) = x^3 - 3x$ and $g(x) = x$.

Solution.

1. (a) The set R consists of all points (x, y) whose y-coordinate is greater than $|x|$. If we graph $y = |x|$, we get the familiar '$\vee$' shape with vertex at $(0, 0)$. Hence, in order for an ordered pair (x, y) to satisfy $y > |x|$, the point (x, y) must be *above* the graph of $y = |x|$.

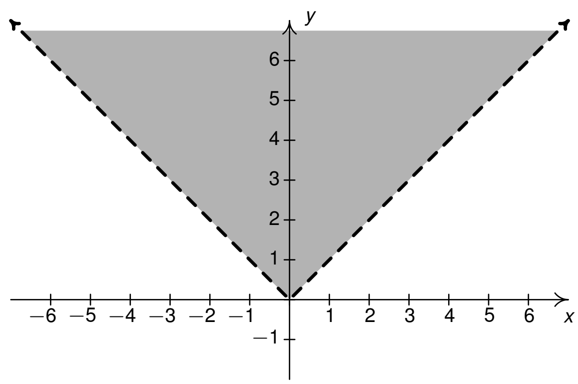
   Since the inequality here, $y > |x|$ is strict, we use a dashed line with which to indicate the graph of $y = |x|$ and shade the region above, or 'inside,' the '$\vee$' as shown below on the left.

   Note that one way to check our answer is to choose points both inside and outside the shaded region to verify the inequality either holds or it doesn't, respectively.

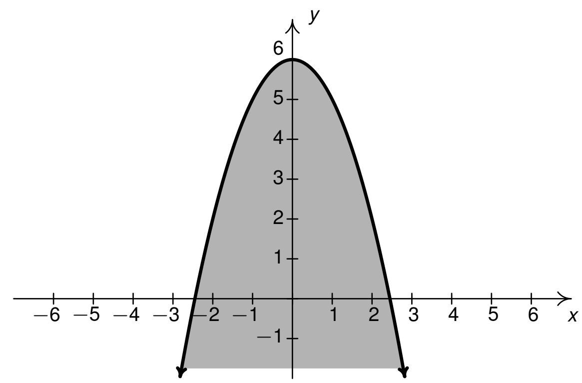
[1] We have *some* experience describing simple regions in the plane using inequalities from our work in Section 5.5.

[2] sometimes written as 'the region bounded by the graphs' . . .

(b) Using the same reasoning as above we note a point (x, y) is in S if its y-coordinate is less than or equal to the y-coordinate on the parabola $y = 6 - x^2$. Hence, we graph the points *below* the parabola ($y < 6 - x^2$) along with the points *on* the parabola ($y = 6 - x^2$) below on the right.



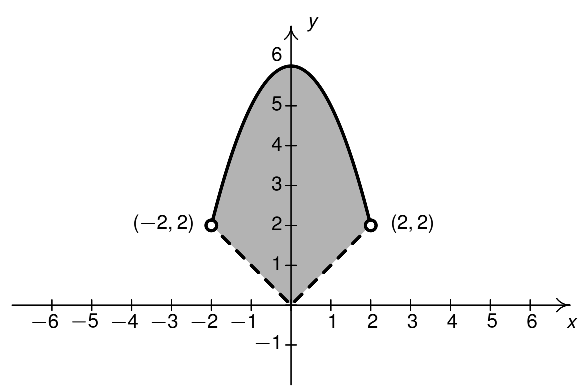
$R = \{(x, y) : y > |x|\}$



$S = \{(x, y) : y \leq 2 - x^2\}$

(c) For a point (x, y) to be in T, the y-coordinate must satisfy $|x| < y \leq 6 - x^2$ which means it must belong to both R and S.[3] Thus we shade the region *between* $y = |x|$ and $y = 6 - x^2$, keeping those points on the parabola, but not the points on $y = |x|$.

To get an accurate graph, we need to find where these two graphs intersect, so we set $|x| = 6 - x^2$. Using the techniques discussed in Sections 1.3 and 1.4, we find $x = -2, 2$. To find the associated y-coordinates of the intersection points, we substitute $x = \pm 2$ into either $y = |x|$ or $y = 6 - x^2$ (or both, to check!) In both cases, we find the associated y-coordinate to be 2, so the intersection points are $(-2, 2)$ and $(2, 2)$. On our graph, however, these are the location of 'holes' owing to the strictness of the inequality $y > |x|$.



$T = \{(x, y) : |x| < y \leq 6 - x^2\}$

2. (a) Verbally, we may describe R as the region *between* the graphs of $y = x^2$ and $y = x + 2$. More specifically, R is the set of points *above* (or on) the parabola $y = x^2$ but *below* (or on) the line $y = x + 2$. That is, for the points (x, y) in R, we have $x^2 \leq y \leq x + 2$.

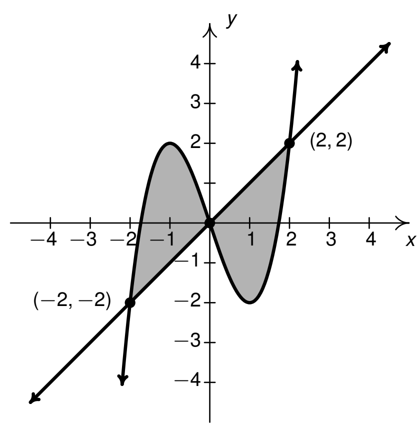
[3] Said differently, T is the set-theoretic intersection of R and S, $T = R \cap S$ as described in A.1.

To find the restrictions on x, we find the intersection of the graphs. Solving $x^2 = x + 2$ gives $x = -1$ and $x = 2$. Hence, to be in R, points (x, y) need to satisfy $-1 \leq x \leq 2$. Hence, one way to describe R is $R = \{(x, y) \,|\, -1 \leq x \leq 2,\ x^2 \leq y \leq x + 2\}$.

While this answer is correct, the fact that the line and the parabola intersect only twice producing only *one* region where the inequality $x^2 \leq y \leq x + 2$ is true, we may shorten our description of R to just $R = \{(x, y) \,|\, x^2 \leq y \leq x + 2\}$.

(b) Since we are given a verbal description of S, we first sketch the graph to get a sense of the geometric situation. To graph $f(x) = x^3 - 3x$, we use what we learned in Section 2.1: end behavior, behavior near the x-intercepts, and symmetry. The graph of $g(x) = x$ is a line with slope 1 through the origin.

Graphing both of these functions on the same pair of axes, we see three intersection points. Solving $f(x) = g(x)$, that is $x^3 - 3x = x$ gives $x = 0$ and $x = \pm 2$. These solutions correspond to the intersection points $(-2, -2)$, $(0, 0)$, and $(2, 2)$. S is described as the region '*between*' these graphs, so we shade between the graphs accordingly.



Graphing $f(x) = x^3 - 3x$ and $g(x) = x$

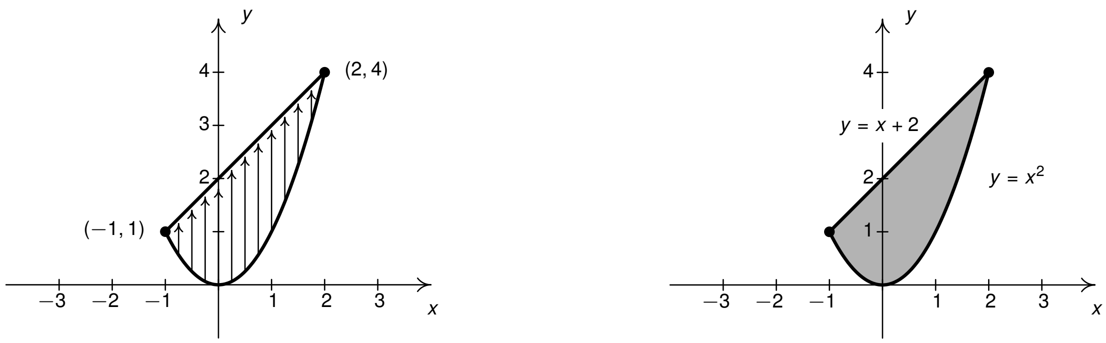
From the graph, we see that for $-2 < x < 0$, the graph of f is above the graph of g. Algebraically, this means that for all x with $-2 \leq x \leq 0$, $f(x) \geq g(x)$ or $x^3 - 3x \geq x$. At $x = 0$, the situation changes and we have $f(x) \leq g(x)$ or $x^3 - 3x \leq x$ for $0 \leq x \leq 2$. One to describe S is to describe each of these pieces and take their set theoretic union.

We first consider the piece for $-2 \leq x \leq 0$. Unlike the previous problem, we cannot completely describe this region in terms of the y-values since there are *two* regions where $x \leq y \leq x^3 - 3x$ is true: one where $-2 \leq x \leq 0$ (the one we want) and one here $x \geq 2$ (which we don't want.) Hence, we describe this first piece as $\{(x, y) \,|\, -2 \leq x \leq 0,\ x \leq y \leq x^3 - 3x\}$. Following this methodology for the second piece, we obtain the complete description for S below:

$$S = \{(x, y) \,|\, -2 \leq x \leq 0,\ x \leq y \leq x^3 - 3x\} \cup \{(x, y) \,|\, 0 \leq x \leq 2,\ x^3 - 3x \leq y \leq x\}.$$

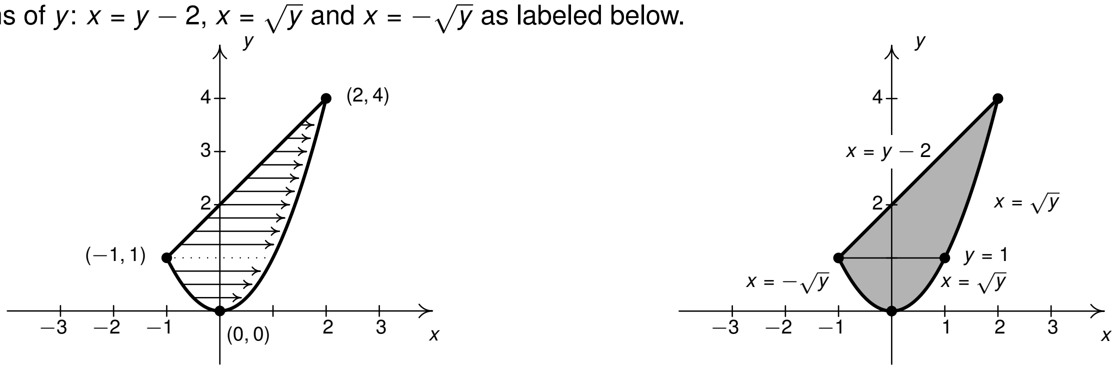
$\square$

A few remarks about Example 9.8.1 are in order. First note that each of the regions presented here can be views as graphs of relations as described in Section 5.5. Moreover, there are many ways to describe a region so our answers to number 2 above are by no means unique.
In particular, our solution $R = \{(x, y) \,|\, -1 \leq x \leq 2,\ x^2 \leq y \leq x + 2\}$ to number 2a can be visualized as 'filling up' the region R from the bottom curve, $y = x^2$ to the top curve, $y = x + 2$ as x runs from the leftmost extent of the region at $x = -1$ to the rightmost extent at $x = 2$ as indicated below. The notion that each x determines where to start and stop filling the region is a consequence of us viewing the bounding curves as functions of x. That is, for each x, we can determine the lower boundary of the region, $y = x^2$ and the upper boundary of the region, $y = x + 2$.



'Filling up' R from bottom to top. $\longrightarrow$ The region R.

There are times in Calculus where it may be convenient to describe the region R as filling up left-to-right as y varies from the bottom most extent, $y = 0$ to the top most extent, $y = 4$. In this case, we need to describe the bounding curves as functions of y. To that end, we solve $y = x + 2$ and $y = x^2$ for x and get three functions of y: $x = y - 2$, $x = \sqrt{y}$ and $x = -\sqrt{y}$ as labeled below.



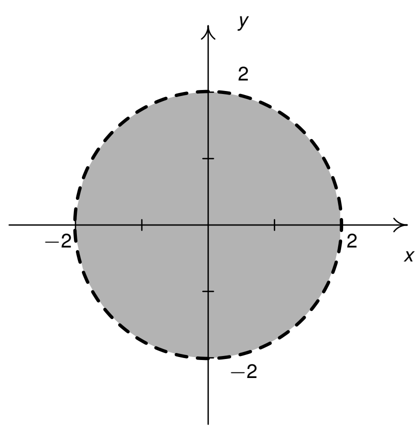
'Filling up' R from left to right. $\longrightarrow$ The region R

Based on the diagram, we see we need to describe R as *two* pieces. The first piece being bounded on the left by $x = -\sqrt{y}$ and on the right $x = \sqrt{y}$ from $y = 0$ to $y = 1$, and the second piece bounded on the left by $x = y - 2$ and on the right by $x = \sqrt{y}$ from $y = 1$ to $y = 4$. That is,

$$R = \{(x, y) \,|\, 0 \leq y \leq 1,\ -\sqrt{y} \leq x \leq \sqrt{y}\} \cup \{(x, y) \,|\, 1 \leq y \leq 4,\ y - 2 \leq x \leq \sqrt{y}\}.$$

Not all regions in the plane are best described using functions of x or y.[4] Suppose, for instance, we wish to sketch the region $\{(x, y) \,|\, x^2 < 4 - y^2.\}$ Algebraically, we wish to plot all points (x, y) for which the inequality $x^2 < 4 - y^2$ is true. One way to proceed is to mimic the 'sign diagram' routine we use for solving nonlinear inequalities in one variable: rewrite the inequality so as to obtain 0 on one side of the inequality, find the zeros of the non-zero side, choose test values determined by the zeros, and record our solution. First, we gather all of the terms on one side and leave a 0 on the other: $x^2 + y^2 - 4 < 0$. Next, we find the zeros of the left hand side, that is, where is $x^2 + y^2 - 4 = 0$. Rewriting, we get $x^2 + y^2 = 4$ which describes the circle of radius 2 centered at the origin. In other words, instead of obtaining a few *numbers* which divide the real number *line* into *intervals*, we get an equation of a *curve*, in this case, a circle, which divides the *plane* into two *regions* - the 'inside' and 'outside' of the circle.

Just like we used test *values* to determine whether or not an interval belongs to the solution of the inequality, we use test *points* in the each of the regions to see which of these belong to our solution set.[5] We choose $(0, 0)$ to represent the region inside the circle and $(0, 3)$ to represent the points outside of the circle. When we substitute $(0, 0)$ into $x^2 + y^2 - 4 < 0$, we get $-4 < 4$ which is true. This means $(0, 0)$ and all the other points inside the circle are part of the solution. On the other hand, when we substitute $(0, 3)$ into the same inequality, we get $5 < 0$ which is false. This means $(0, 3)$ along with all other points outside the circle are not part of the solution. What about points on the circle itself? Choosing a point on the circle, say $(0, 2)$, we get $0 < 0$, which means the circle itself does not satisfy the inequality.[6] As a result, we leave the circle dashed in the final diagram.



The solution to $x^2 < 4 - y^2$

We put this technique to good use in the following example.

[4]See Section 14.2 for instance!

[5]The theory behind why all this works is, surprisingly, the same theory which guarantees that sign diagrams work the way they do - continuity and the Intermediate Value Theorem - but in this case, applied to functions of more than one variable.

[6]Another way to see this is that points on the circle satisfy $x^2 + y^2 - 4 = 0$, so they do not satisfy $x^2 + y^2 - 4 < 0$.

Example 9.8.2.

1. Sketch the following regions in the xy-plane:

   (a) $R = \{(x, y) \mid y^2 - 4 \leq x < y + 2\}$

   (b) The solution to: $\left\{ \begin{array}{rcl} x^2 + y^2 & \geq & 4 \\ x^2 - 2x + y^2 - 2y & \leq & 0 \end{array} \right.$

2. Find a set builder description for the region between the Unit Circle and the graph of $x^2 + y^2 = 16$.

Solution.

1. (a) The inequality $y^2 - 4 \leq x < y + 2$ is a compound inequality which translates as $y^2 - 4 \leq x$ *and* $x < y + 2$. Hence, our approach is to solve each inequality separately and take the set theoretic intersection to determine the region which satisfies both inequalities.

   Starting with $y^2 - 4 \leq x$, we rewrite this as $y^2 - x - 4 \leq 0$ and set about graphing $y^2 - x - 4 = 0$. Since only one variable is squared, we know this equation describes a parabola. Rewriting in standard form, we get $y^2 = x + 4$, so the vertex is $(-4, 0)$ and the parabola opens to the right.

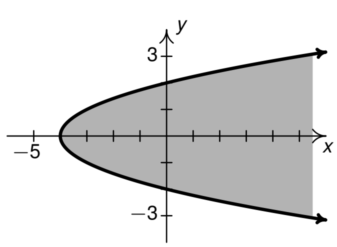
   Using the test points $(-5, 0)$ and $(0, 0)$, we find that the solution to the inequality includes the region to the *right* of, or 'inside', the parabola. The points on the parabola itself are also part of the solution, owing to the 'equality' in the 'inequality.' (We could also check a point on the parabola such as $(-4, 0)$ satisfies the inequality.)

   Turning our attention to $x < y + 2$, we first rewrite this as $x - y - 2 < 0$ and focus our attention on $x - y - 2 = 0$. Rewriting, we have the line $y = x - 2$ and using the test points $(0, 0)$ and $(0, -4)$, we find points in the region *above* the line satisfy the inequality. (Owing to the strictness of the inequality, the points on the line itself do not.)[7]

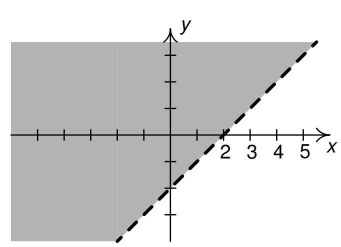
   We see that these two regions do overlap but to make the graph more precise, we seek the intersection of these two curves. That is, we need to solve the system of nonlinear equations

$$\left\{ \begin{array}{lrcl} (E1) & y^2 & = & x + 4 \\ (E2) & y & = & x - 2 \end{array} \right.$$

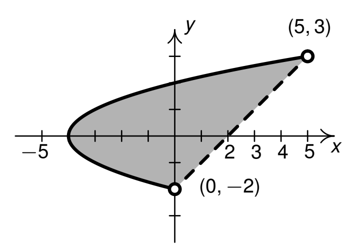
   Solving $E1$ for x, we get $x = y^2 - 4$. Substituting this into $E2$ gives $y = y^2 - 4 - 2$, or $y^2 - y - 6 = 0$. We find $y = -2$ and $y = 3$ and since $x = y^2 - 4$, we get that the graphs intersect at $(0, -2)$ and $(5, 3)$. Putting all of this together, we get our final answer below.



$y^2 - 4 \leq x$



$x < y + 2$



$y^2 - 4 \leq x < y + 2$

[7]We could also have rewritten $x < y + 2$ as $y > x - 2$ directly and sketched the region as explained in Example 9.8.1.

(b) Like any system, our solution to this problem requires us to graph the points (x, y) which satisfy *both* inequalities. To do this, we solve each inequality separately and take the set theoretic intersection of the solution sets.

We begin with the inequality $x^2 + y^2 \geq 4$ which we rewrite as $x^2 + y^2 - 4 \geq 0$. The points which satisfy $x^2 + y^2 - 4 = 0$ form the circle $x^2 + y^2 = 4$. Using test points $(0, 0)$, $(0, 2)$, and $(0, 3)$ we find that our solution comprises the region *outside* the circle along with the circle itself.

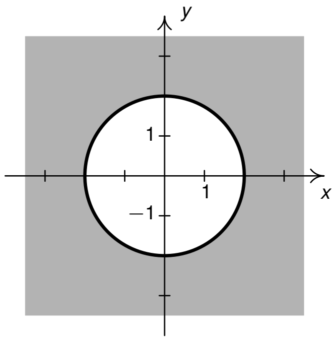
Moving to $x^2 - 2x + y^2 - 2y \leq 0$, we start with $x^2 - 2x + y^2 - 2y = 0$. Completing the squares, we obtain $(x - 1)^2 + (y - 1)^2 = 2$, which is a circle centered at $(1, 1)$ with a radius of $\sqrt{2}$.

Choosing $(1, 1)$ to represent the inside of the circle, $(0, 0)$ as a point on the circle, and $(1, 3)$ as a point outside of the circle, we find that the solution to the inequality is the inside of the circle, including the circle itself.

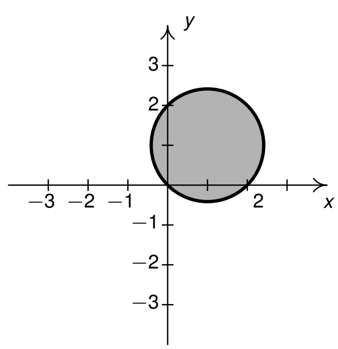
Our final answer, then, consists of the points on or outside of the circle $x^2 + y^2 = 4$ which lie on or inside the circle $(x - 1)^2 + (y - 1)^2 = 2$. To produce the most accurate graph, we need to find where these circles intersect. To that end, we solve the system

$$\begin{cases} (E1) & x^2 + y^2 = 4 \\ (E2) & x^2 - 2x + y^2 - 2y = 0 \end{cases}$$

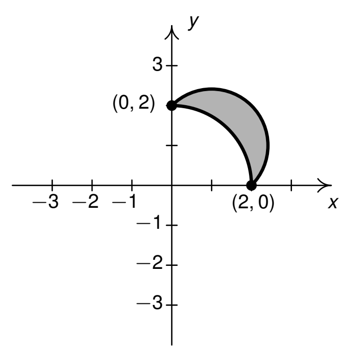
We can eliminate both the x^2 and y^2 by replacing $E2$ with $-E1 + E2$. Doing so produces $-2x - 2y = -4$. Solving for y gives $y = 2 - x$. Substituting this into $E1$ gives $x^2 + (2 - x)^2 = 4$ which simplifies to $x^2 + 4 - 4x + x^2 = 4$ or $2x^2 - 4x = 0$. Factoring yields $2x(x - 2)$ which gives $x = 0$ or $x = 2$. Substituting these values into $y = 2 - x$ gives the points $(0, 2)$ and $(2, 0)$. The intermediate graphs and final solution are below.



$x^2 + y^2 \geq 4$



$x^2 - 2x + y^2 - 2y \leq 0$

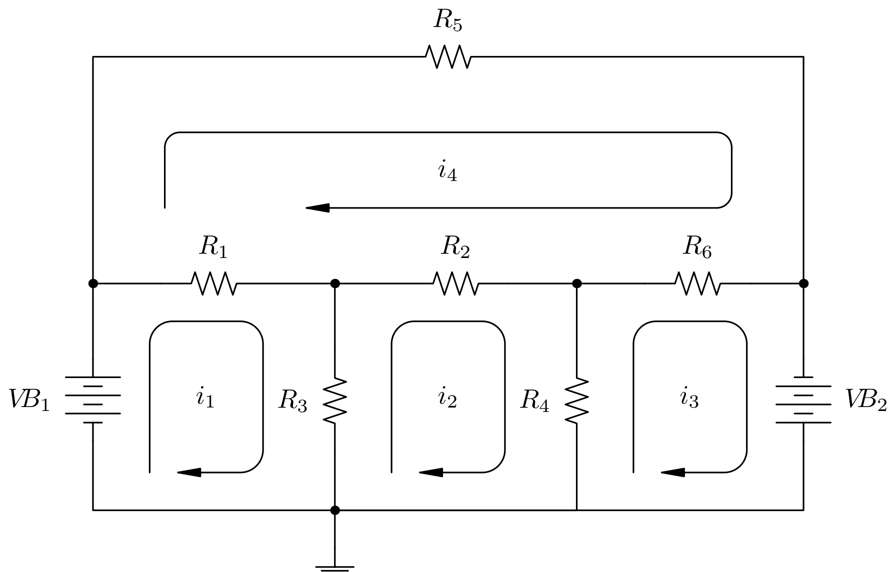


Solution to the system.

2. We first recall the Unit Circle is the circle centered at the origin with radius 1 and hence is described algebraically by the equation $x^2 + y^2 = 1$. The equation $x^2 + y^2 = 16$ describes a circle centered at the origin with radius 4. Hence, the 'region between' these two circles indicates we want is *outside* the Unit Circle but *inside* the circle $x^2 + y^2 = 16$.

   Based on our experience from the earlier problems, we know points outside the Unit Circle satisfy $x^2 + y^2 > 1$ whereas the points inside the circle $x^2 + y^2 = 16$ satisfy $x^2 + y^2 < 16$. Hence, the points we seek satisfy *both* inequalities, so our solution is $\{(x, y) \mid 1 < x^2 + y^2 < 16\}$. □

We close this section with a follow-up to Example 9.4.2 in Section 9.4. Recall in the circuit diagrammed below, we have two batteries with source voltages VB_1 and VB_2, measured in volts V, along with six resistors with resistances R_1 through R_6, measured in kiloohms, $k\Omega$.Recall if we think of electrons flowing through the circuit, we can think of the voltage sources as providing the 'push' which makes the electrons move, the resistors as obstacles for the electrons to overcome, and the mesh current as a net rate of flow of electrons around the indicated loops.



Using Ohm's Law and Kirchhoff's Voltage Law, we can relate the voltage supplied to the circuit by the two batteries to the voltage drops across the six resistors in order to find the four 'mesh' currents: i_1, i_2, i_3 and i_4, measured in milliamps, mA. This gives rise to the following system of linear equations:

$$\left\{ \begin{array}{rcl} (R_1 + R_3)\, i_1 - R_3 i_2 - R_1 i_4 & = & VB_1 \\ -R_3 i_1 + (R_2 + R_3 + R_4)\, i_2 - R_4 i_3 - R_2 i_4 & = & 0 \\ -R_4 i_2 + (R_4 + R_6)\, i_3 - R_6 i_4 & = & -VB_2 \\ -R_1 i_1 - R_2 i_2 - R_6 i_3 + (R_1 + R_2 + R_5 + R_6)\, i_4 & = & 0 \end{array} \right.$$

In Example 9.4.2, we found that under the assumptions $VB_1 = 10V$, $VB_2 = 5V$, and all the resistances are all $1k\Omega$, the mesh currents worked out to be $i_1 = 10.625\ mA$, $i_2 = 6.25\ mA$, $i_3 = 3.125\ mA$, and $i_4 = 5\ mA$. In our final example, we assume $VB_1 = 10V$ and $VB_2 = 5V$ and work to find what combination of resistances would combine to produce these mesh currents.

Example 9.8.3. For the circuit described above, if $VB_1 = 10V$ and $VB_2 = 5V$, find the possible combinations of resistances which yield the currents $i_1 = 10.625\ mA$, $i_2 = 6.25\ mA$, $i_3 = 3.125\ mA$, and $i_4 = 5\ mA$.

Solution.
We begin by substituting the known values of the currents into the system of equations. We get:

$$\left\{ \begin{array}{rcr} 5.625R_1 + 4.375R_3 & = & 10 \\ 1.25R_2 - 4.375R_3 + 3.125R_4 & = & 0 \\ -3.125R_4 - 1.875R_6 & = & -5 \\ -5.625R_1 - 1.25R_2 + 5R_5 + 1.875R_6 & = & 0 \end{array} \right.$$

The coefficient matrix for this system is 4×6 (4 equations with 6 unknowns) and is therefore not invertible. We do know, however, this system is consistent, since setting all the resistance values equal to 1 corresponds to our situation Example 9.4.2. This means we have an underdetermined consistent system which is necessarily dependent. To solve this system, we encode it into an augmented matrix

$$\left[\begin{array}{rrrrrr|r} 5.25 & 0 & 4.375 & 0 & 0 & 0 & 10 \\ 0 & 1.25 & -4.375 & 3.125 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.125 & 0 & -1.875 & -5 \\ -5.625 & -1.25 & 0 & 0 & 5 & 1.875 & 0 \end{array} \right]$$

A graphing utility gives the reduced-row echelon form of the matrix as:

$$\left[\begin{array}{rrrrrr|r} 1 & 0 & 0.\overline{7} & 0 & 0 & 0 & 1.\overline{7} \\ 0 & 1 & -3.5 & 0 & 0 & -1.5 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0.6 & 1.6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right]$$

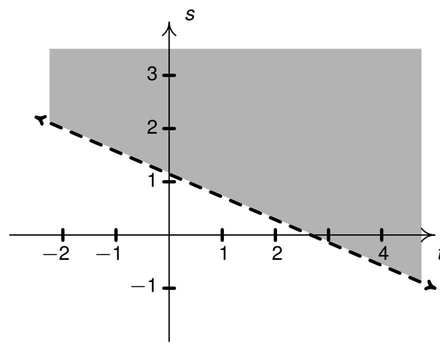
from which we obtain the system:

$$\left\{ \begin{array}{rcr} R_1 + 0.\overline{7}R_3 & = & 1.\overline{7} \\ R_2 - 3.5R_3 - 1.5R_6 & = & -4 \\ R_4 + 0.6R_6 & = & 1.6 \\ R_5 & = & 1 \end{array} \right.$$

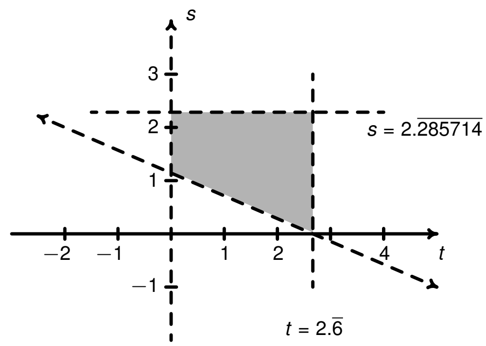
We can solve for R_1, R_2, R_4 and R_5 leaving R_3 and R_6 as free variables. Labeling $R_3 = s$ and $R_6 = t$, we have $R_1 = -0.\overline{7}s + 1.\overline{7}$, $R_2 = 3.5s + 1.5t - 4$, $R_4 = -0.6t + 1.6$ and $R_5 = 1$.

Since resistance values are always positive, we need to restrict our values of s and t. We know $R_3 = s > 0$ and when we combine that with $R_1 = -0.\overline{7}s + 1.\overline{7} > 0$, we get $0 < s < 2.\overline{285714}$. Similarly, $R_6 = t > 0$ and with $R_4 = -0.6t + 1.6 > 0$, we find $0 < t < 2.\overline{6}$.

In order visualize the inequality $R_2 = 3.5s + 1.5t - 4 > 0$, we graph the line $3.5s + 1.5t - 4 = 0$ on the ts-plane below on the left and shade accordingly. Imposing the additional conditions $0 < s < 2.\overline{285714}$ and $0 < t < 2.\overline{6}$, we find our values of s and t restricted to the region R graphed below on the right.



The region where $3.5s + 1.5t - 4 > 0$



The region R for our parameters s and t.

Hence, our final answer is:

$$\left\{ \begin{array}{rcr} R_1 & = & -0.\overline{7}s + 1.\overline{7} \\ R_2 & = & 3.5s + 1.5t - 4 \\ R_4 & = & -0.6t + 1.6 \\ R_5 & = & 1, \end{array} \right.$$

where s and t are pulled from the region $R = \left\{ (s,t) \,|\, 0 < s < 2.\overline{285714},\ 0 < t < 2.\overline{6},\ 3.5s + 1.5t - 4 > 0 \right\}$. The reader is encouraged to check that the solution presented in Example 9.4.2, namely all resistance values equal to 1, corresponds to a pair (s,t) in this region. □

9.8.1 Exercises

In Exercises 1 - 6, sketch the solution to each system of nonlinear inequalities in the plane.

1. $\left\{ \begin{array}{rcl} x^2 - y^2 & \leq & 1 \\ x^2 + 4y^2 & \geq & 4 \end{array} \right.$

2. $\left\{ \begin{array}{rcl} x^2 + y^2 & < & 25 \\ x^2 + (y-3)^2 & \geq & 10 \end{array} \right.$

3. $\left\{ \begin{array}{rcl} (x-2)^2 + y^2 & < & 1 \\ x^2 + 4y^2 & < & 4 \end{array} \right.$

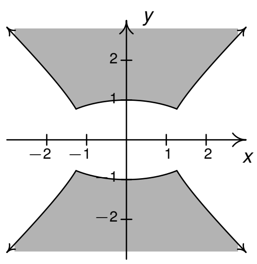
4. $\left\{ \begin{array}{rcl} y & > & 10x - x^2 \\ y & < & x^3 + 8 \end{array} \right.$

5. $\left\{ \begin{array}{rcl} x + 2y^2 & > & 2 \\ x^2 + 4y^2 & \leq & 4 \end{array} \right.$

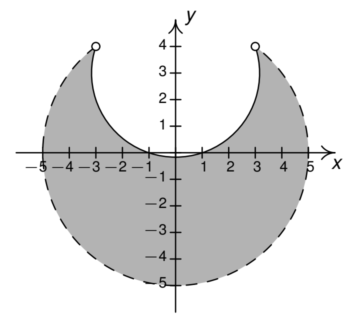
6. $\left\{ \begin{array}{rcl} x^2 + y^2 & \geq & 25 \\ y - x & \leq & 1 \end{array} \right.$

9.8.2 Answers

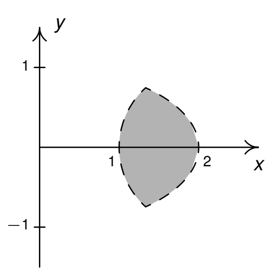
1. $\begin{cases} x^2 - y^2 \leq 1 \\ x^2 + 4y^2 \geq 4 \end{cases}$



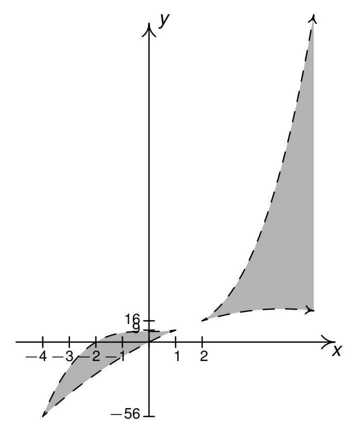
2. $\begin{cases} x^2 + y^2 < 25 \\ x^2 + (y-3)^2 \geq 10 \end{cases}$



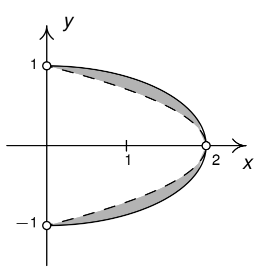
3. $\begin{cases} (x-2)^2 + y^2 < 1 \\ x^2 + 4y^2 < 4 \end{cases}$



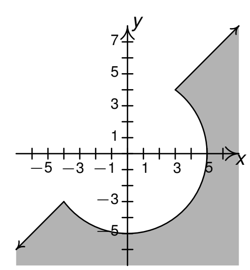
4. $\begin{cases} y > 10x - x^2 \\ y < x^3 + 8 \end{cases}$



5. $\begin{cases} x + 2y^2 > 2 \\ x^2 + 4y^2 \leq 4 \end{cases}$



6. $\begin{cases} x^2 + y^2 \geq 25 \\ y - x \leq 1 \end{cases}$



Chapter 10

Sequences and the Binomial Theorem

10.1 Sequences

In this section, we introduce *sequences* which are an important class of functions whose domains are, more or less, the set of natural numbers.[1] Before we get to far ahead of ourselves, let's look at what the term 'sequence' means mathematically. Informally, we can think of a sequence as an infinite list of numbers. For example, consider the sequence

$$\frac{1}{2}, -\frac{3}{4}, \frac{9}{8}, -\frac{27}{16}, \ldots \qquad (1)$$

As usual, the periods of ellipsis, ..., indicate that the proposed pattern continues forever. Each of the numbers in the list is called a *term*, and we call $\frac{1}{2}$ the 'first term', $-\frac{3}{4}$ the 'second term', $\frac{9}{8}$ the 'third term' and so forth. In numbering them this way, we are setting up a function, which we'll call 'a' per tradition, between the natural numbers and the terms in the sequence.

n	$a(n)$
1	$\frac{1}{2}$
2	$-\frac{3}{4}$
3	$\frac{9}{8}$
4	$-\frac{27}{16}$
$\vdots$	$\vdots$

In other words, $a(n)$ is the n^{th} term in the sequence. We formalize these ideas in our definition of a sequence and introduce some accompanying notation.

Definition 10.1. A **sequence** is a function a whose domain is the natural numbers. The value $a(n)$ is often written as a_n and is called the $\boldsymbol{n}^{\textbf{th}}$ **term** of the sequence. The sequence itself is usually denoted using the notation: a_n, $n \geq 1$ or the notation: $\{a_n\}_{n=1}^{\infty}$.

[1] Recall that this is the set $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Applying the notation provided in Definition 10.1 to the sequence given (1), we have $a_1 = \frac{1}{2}$, $a_2 = -\frac{3}{4}$, $a_3 = \frac{9}{8}$.

Suppose we wanted to know a_{117}, that is, the 117^{th} term in the sequence. While the pattern of the sequence is apparent, it would benefit us greatly to have an explicit formula for a_n. Unfortunately, there is no general algorithm that will produce a formula for every sequence, so any formulas we do develop will come from that greatest of teachers, experience. In other words, it is time for an example.

Example 10.1.1. Write the first four terms of the following sequences.

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$
2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$
3. $\{2n-1\}_{n=1}^{\infty}$
4. $\left\{\dfrac{1+(-1)^i}{i}\right\}_{i=2}^{\infty}$
5. $a_1 = 7$, $a_{n+1} = 2 - a_n$, $n \geq 1$
6. $f_0 = 1$, $f_n = n \cdot f_{n-1}$, $n \geq 1$

Solution.

1. Since we are given $n \geq 1$, the first four terms of the sequence are a_1, a_2, a_3 and a_4. Since the notation a_1 means the same thing as $a(1)$, we obtain our first term by replacing every occurrence of n in the formula for a_n with $n = 1$ to get $a_1 = \frac{5^{1-1}}{3^1} = \frac{1}{3}$. Proceeding similarly, we get $a_2 = \frac{5^{2-1}}{3^2} = \frac{5}{9}$, $a_3 = \frac{5^{3-1}}{3^3} = \frac{25}{27}$ and $a_4 = \frac{5^{4-1}}{3^4} = \frac{125}{81}$.

2. For this sequence we have $k \geq 0$, so the first four terms are b_0, b_1, b_2 and b_3. Proceeding as before, replacing in this case the variable k with the appropriate whole number, beginning with 0, we get $b_0 = \frac{(-1)^0}{2(0)+1} = 1$, $b_1 = \frac{(-1)^1}{2(1)+1} = -\frac{1}{3}$, $b_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{5}$ and $b_3 = \frac{(-1)^3}{2(3)+1} = -\frac{1}{7}$. As a side-note, this sequence is called an *alternating* sequence since the signs alternate between '+' and '$-$.' The reader is encouraged to think what component of the formula is producing this effect.

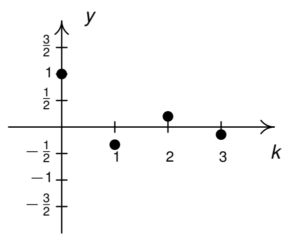
3. The notation $\{2n-1\}_{n=1}^{\infty}$ means $a_n = 2n - 1$, $n \geq 1$. We get $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and $a_4 = 7$. In other words, we get the first four odd natural numbers. The reader is encouraged to examine whether or not this pattern continues indefinitely.

4. Here, we are using the letter i as a counter, not as the imaginary unit we saw in Section 2.4. Proceeding as before, we set $a_i = \frac{1+(-1)^i}{i}$, $i \geq 2$. We find $a_2 = 1$, $a_3 = 0$, $a_4 = \frac{1}{2}$ and $a_5 = 0$.

5. To obtain the terms of this sequence, we start with $a_1 = 7$ and use the equation $a_{n+1} = 2 - a_n$ for $n \geq 1$ to generate successive terms. When $n = 1$, this equation becomes $a_{1+1} = 2 - a_1$ which simplifies to $a_2 = 2 - a_1 = 2 - 7 = -5$. When $n = 2$, the equation becomes $a_{2+1} = 2 - a_2$ so we get $a_3 = 2 - a_2 = 2 - (-5) = 7$. Finally, when $n = 3$, we get $a_{3+1} = 2 - a_3$ so $a_4 = 2 - a_3 = 2 - 7 = -5$.

6. As with the problem above, we are given a place to start with $f_0 = 1$ and given a formula to build other terms of the sequence. Substituting $n = 1$ into the equation $f_n = n \cdot f_{n-1}$, we get $f_1 = 1 \cdot f_0 = 1 \cdot 1 = 1$. Advancing to $n = 2$, we get $f_2 = 2 \cdot f_1 = 2 \cdot 1 = 2$. Finally, $f_3 = 3 \cdot f_2 = 3 \cdot 2 = 6$. $\square$

Some remarks about Example 10.1.1 are in order. We first note that since sequences are functions, we can graph them in the same way we graph functions. For example, if we wish to graph the sequence $\{b_k\}_{k=0}^{\infty}$ from Example 10.1.1, we graph the equation $y = b(k)$ for the values $k \geq 0$. That is, we plot the points $(k, b(k))$ for the values of k in the domain, $k = 0, 1, 2, \ldots$. The resulting collection of points is the graph of the sequence. Note that we do not connect the dots in a pleasing fashion as we are used to doing, because the domain is just the whole numbers in this case, not a collection of intervals of real numbers.[2]



Graphing $y = b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$

Speaking of $\{b_k\}_{k=0}^{\infty}$, the astute and mathematically minded reader will correctly note that this technically isn't a sequence, since according to Definition 10.1, sequences are functions whose domains are the *natural* numbers, not the *whole* numbers, as is the case with $\{b_k\}_{k=0}^{\infty}$. In other words, to satisfy Definition 10.1, we need to shift the variable k so it starts at $k = 1$ instead of $k = 0$.

To see how we can do this, it helps to think of the problem graphically. What we want is to shift the graph of $y = b(k)$ to the right one unit, and thinking back to Section 5.4, we can accomplish this by replacing k with $k - 1$ in the definition of $\{b_k\}_{k=0}^{\infty}$.

Specifically, let $c_k = b_{k-1}$ where $k - 1 \geq 0$. We get $c_k = \frac{(-1)^{k-1}}{2(k-1)+1} = \frac{(-1)^{k-1}}{2k-1}$, where now $k \geq 1$. We leave to the reader to verify that $\{c_k\}_{k=1}^{\infty}$ generates the same list of numbers as does $\{b_k\}_{k=0}^{\infty}$, but the former satisfies Definition 10.1, while the latter does not.

Like so many things in this text, we acknowledge that this point is pedantic and join the vast majority of authors who adopt a more relaxed view of Definition 10.1 to include any function which generates a list of numbers which can then be matched up with the natural numbers.[3]

One last note about Example 10.1.1 concerns the manner in which the sequences in numbers 5 and 6 are defined. We say these two sequences are described '*recursively*.' In each instance, an initial value of the sequence is given which is then followed by a *recursion equation* – a formula which enables us to use known terms of the sequence to determine other terms.

The terms of the sequence from number 6 is given notation and name: $f_n = n!$ is called *n-factorial*. Using the '!' notation, we can describe the factorial sequence as: $0! = 1$ and $n! = n(n-1)!$ for $n \geq 1$.

After $0! = 1$ the next four terms, written out in detail, are $1! = 1 \cdot 0! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. From this, we see a more informal way of computing $n!$, which is $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ with $0! = 1$ as a special case. (We will study factorials in greater detail in Section 10.4.)[4]

[2] If you feel a sense of nostalgia, you should see Section 1.1.

[3] We're basically talking about the 'countably infinite' subsets of the real number line when we do this.

[4] Another famous sequence, the Fibonacci Numbers are defined also recursively and are explored in the exercises.

While none of the sequences in Example 10.1.1 worked out to be the sequence in (1), they do give us some insight into what kinds of patterns to look for. Two patterns in particular are given in the next definition.

Definition 10.2. Arithmetic and Geometric Sequences: Suppose $\{a_n\}_{n=k}^{\infty}$ is a sequence[a]

- If there is a number d so that $a_{n+1} = a_n + d$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called an **arithmetic sequence**. The number d is called the **common difference**.
- If there is a number r so that $a_{n+1} = ra_n$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called a **geometric sequence**. The number r is called the **common ratio**.

[a]Note that we have adjusted for the fact that not all 'sequences' begin at $n = 1$.

In English, an arithmetic sequence is one in which we proceed from one term to the next by always *adding* the fixed number d. If this sort of 'constant change' idea sounds familiar, it should. Indeed, arithmetic sequences are merely *linear* functions, something we will explore in more detail shortly. Note the name 'common difference' comes from a slight rewrite of the recursion equation from $a_{n+1} = a_n + d$ to $a_{n+1} - a_n = d$. That is, every pair of successive terms has the *same* or *common* difference, d.

Analogously, a geometric sequence is one in which we proceed from one term to the next by always *multiplying* by the same fixed number r. If this notion sounds familiar, it is because geometric sequences are, in fact, *exponential* functions. Again, we will explore this connection in more detail later. We note that if $a_n \neq 0$, we can rearrange the recursion equation to get $\frac{a_{n+1}}{a_n} = r$. Hence, every pair of successive terms has the *same* or *common* ratio, r.

Some sequences are arithmetic, some are geometric and some are neither as the next example illustrates.[5]

Example 10.1.2. Determine if the following sequences are arithmetic, geometric or neither. If arithmetic, find the common difference d; if geometric, find the common ratio r.

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$
2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$
3. $\{2n - 1\}_{n=1}^{\infty}$
4. $\dfrac{1}{2}, -\dfrac{3}{4}, \dfrac{9}{8}, -\dfrac{27}{16}, \ldots$

Solution. A good rule of thumb to keep in mind when working with sequences is "When in doubt, write it out!" Writing out the first several terms can help you identify the pattern of the sequence should one exist.

1. From Example 10.1.1, we know that the first four terms of this sequence are $\frac{1}{3}$, $\frac{5}{9}$, $\frac{25}{27}$ and $\frac{125}{81}$. To see if this is an arithmetic sequence, we look at the successive differences of terms. We find that $a_2 - a_1 = \frac{5}{9} - \frac{1}{3} = \frac{2}{9}$ and $a_3 - a_2 = \frac{25}{27} - \frac{5}{9} = \frac{10}{27}$. Since we get different numbers, there is no 'common difference' and we have established that the sequence is *not* arithmetic.

   To see if the sequence is geometric, we compute the ratios of successive terms. The first three ratios *suggest* the sequence is geometric:

$$\frac{a_2}{a_1} = \frac{\frac{5}{9}}{\frac{1}{3}} = \frac{5}{3}, \quad \frac{a_3}{a_2} = \frac{\frac{25}{27}}{\frac{5}{9}} = \frac{5}{3} \quad \text{and} \quad \frac{a_4}{a_3} = \frac{\frac{125}{81}}{\frac{25}{27}} = \frac{5}{3}$$

[5]Can a sequence be both arithmetic *and* geometric? See Exercise 37.

To *prove* the sequence is geometric, however, we must show that $\frac{a_{n+1}}{a_n} = r$ for all n:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{5^{(n+1)-1}}{3^{n+1}}}{\frac{5^{n-1}}{3^n}} = \frac{5^n}{3^{n+1}} \cdot \frac{3^n}{5^{n-1}} = \frac{5}{3}$$

Hence, the sequence is geometric with common ratio $r = \frac{5}{3}$.

2. Again, we have Example 10.1.1 to thank for providing the first four terms of this sequence: 1, $-\frac{1}{3}$, $\frac{1}{5}$ and $-\frac{1}{7}$. We find $b_1 - b_0 = -\frac{4}{3}$ and $b_2 - b_1 = \frac{8}{15}$. Hence, the sequence is not arithmetic. To see if it is geometric, we compute $\frac{b_1}{b_0} = -\frac{1}{3}$ and $\frac{b_2}{b_1} = -\frac{3}{5}$. Since there is no 'common ratio,' we conclude the sequence is not geometric, either.

3. As we saw in Example 10.1.1, the sequence $\{2n-1\}_{n=1}^{\infty}$ generates the odd numbers: $1, 3, 5, 7, \ldots$. Computing the first few differences, we find $a_2 - a_1 = 2$, $a_3 - a_2 = 2$, and $a_4 - a_3 = 2$. This suggests that the sequence is arithmetic. To prove this is the case, we find

$$a_{n+1} - a_n = (2(n+1) - 1) - (2n - 1) = 2n + 2 - 1 - 2n + 1 = 2$$

This establishes that the sequence is arithmetic with common difference $d = 2$. To see if it is geometric, we compute $\frac{a_2}{a_1} = 3$ and $\frac{a_3}{a_2} = \frac{5}{3}$. Since these ratios are different, we conclude the sequence is not geometric.

4. We met our last sequence at the beginning of the section. Given that $a_2 - a_1 = -\frac{5}{4}$ and $a_3 - a_2 = \frac{15}{8}$, the sequence is not arithmetic. Computing the first few ratios, however, gives us $\frac{a_2}{a_1} = -\frac{3}{2}$, $\frac{a_3}{a_2} = -\frac{3}{2}$ and $\frac{a_4}{a_3} = -\frac{3}{2}$. Since these are the only terms given to us, we *assume* that the pattern of ratios continue in this fashion and conclude that the sequence is geometric. □

We are now one step away from determining an explicit formula for the sequence given in (1). We know that it is a geometric sequence and our next result gives us the explicit formula we require.

Equation 10.1. Formulas for Arithmetic and Geometric Sequences:

- An arithmetic sequence with first term $a_1 = a$ and common difference d is given by

$$a_n = a + (n-1)d, \quad n \geq 1$$

- A geometric sequence with first term $a_1 = a$ and common ratio $r \neq 0$ is given by

$$a_n = ar^{n-1}, \quad n \geq 1$$

An intuitive way to arrive at Equation 10.1 appeals to Definition 10.2 directly. Given an arithmetic sequence with first term a and common difference d, the way we get from one term to the next is by adding d. Hence, the terms of the sequence are: a, $a+d$, $a+2d$, $a+3d$, ... We see that to reach the nth term, we add d to a exactly $(n-1)$ times, which is exactly what the formula says.[6]

Note if we rewrite the formula $a_n = a_1 + (n-1)d$ using traditional function notation as $a(n) = a(1) + d(n-1)$ we can see arithmetic sequences are linear functions.[7] Indeed, relabeling the function a as 'f' and the independent variable n as 'x,' we can make the identifications $x_0 = 1$, and $m = d$ so as to put the equation $a(n) = a(1) + d(n-1)$ into the form of Equation 1.1:

$$\begin{array}{rcrcl} a(n) & = & a(1) & + & d(n-1) \\ f(x) & = & f(1) & + & m(x-1) \end{array}$$

Hence, arithmetic sequences are linear functions with slope d whose domains are the natural numbers.

The derivation of the formula for geometric series follows similarly. Here, we start with the first term a and go from one term to the next by multiplying by r. We get a, ar, ar^2, ar^3 and so forth. The nth term results from multiplying a by r exactly $(n-1)$ times.[8]

In the same way arithmetic sequences are linear functions, geometric sequences are exponential functions. Writing $a_n = a_1 r^{n-1}$ as $a(n) = a(1)r^{n-1}$, we can relabel a as f and n as x and make the identifications $x_0 = 1$ and $b = r$ to put the equation into the form described in Definition 7.2:

$$\begin{array}{rcl} a(n) & = & a(1)r^{n-1} \\ f(x) & = & f(1)b^{x-1} \end{array}$$

So, geometric sequences are exponential functions with base r whose domains are the natural numbers.

With Equation 10.1 in place, we finally have the tools required to find an explicit formula for the nth term of the sequence given in (1). We know from Example 10.1.2 that it is geometric with common ratio $r = -\frac{3}{2}$. The first term is $a = \frac{1}{2}$ so by Equation 10.1 we get $a_n = ar^{n-1} = \frac{1}{2}\left(-\frac{3}{2}\right)^{n-1}$ for $n \geq 1$. After a touch of simplifying, we get $a_n = \frac{(-3)^{n-1}}{2^n}$ for $n \geq 1$. Note that we can easily check our answer by substituting in values of n and seeing that the formula generates the sequence given in (1). We leave this to the reader. In particular, the 117th term in the sequence is $a_{117} = \frac{1}{2}\left(-\frac{3}{2}\right)^{117-1} = \frac{3^{116}}{2^{117}}$.

Our next example gives us more practice finding patterns.

Example 10.1.3. Find an explicit formula for the n^{th} term of the following sequences.

1. $0.9, 0.09, 0.009, 0.0009, \ldots$
2. $\dfrac{2}{5}, 2, -\dfrac{2}{3}, -\dfrac{2}{7}, \ldots$
3. $1, -\dfrac{2}{7}, \dfrac{4}{13}, -\dfrac{8}{19}, \ldots$

Solution.

1. Although this sequence may seem strange, the reader can verify it is actually a geometric sequence with common ratio $r = 0.1 = \frac{1}{10}$. With $a = 0.9 = \frac{9}{10}$, we get $a_n = \frac{9}{10}\left(\frac{1}{10}\right)^{n-1}$ for $n \geq 0$. Simplifying, we get $a_n = \frac{9}{10^n}$, $n \geq 1$. There is more to this sequence than meets the eye and we shall return to this example in the next section.

[6] We formalize this argument in Section 10.3.

[7] Note here a is a *function*, so the expressions $a(n)$ and $a(1)$ here represent the *outputs* from a. On the other hand, the expression $d(n-1)$ indicates *multiplication* of the real numbers d and $(n-1)$.

[8] We note here that the reason $r = 0$ is excluded from Equation 10.1 is to avoid an instance of 0^0 which is an indeterminate form. (See the remarks following Definition 2.1 in Section 2.1.)

2. As the reader can verify, this sequence is neither arithmetic nor geometric. In an attempt to find a pattern, we rewrite the second term with a denominator to make all the terms appear as fractions and associate the '$-$' with the denominators so we have a constant numerator:

$$\frac{2}{5}, \frac{2}{1}, \frac{2}{-3}, -\frac{2}{-7}, \ldots$$

This tells us that we can tentatively sketch out the formula for the sequence as $a_n = \frac{2}{D_n}$ where D_n is the sequence of denominators.

The sequence of the denominators: $5, 1, -3, -7, \ldots$ is seen to be an arithmetic sequence with a common difference of -4. Using Equation 10.1 with $a = 5$ and $d = -4$, we get the nth denominator by the formula $D_n = 5 + (n-1)(-4) = 9 - 4n$ for $n \geq 1$. Hence, our final answer is $a_n = \frac{2}{9-4n}$, $n \geq 1$.

3. The sequence as given is neither arithmetic nor geometric, so we proceed as in the last problem to try to get patterns individually for the numerator and denominator. Letting C_n and D_n denote the sequence of numerators and denominators, respectively, so that $a_n = \frac{C_n}{D_n}$.

After some experimentation,[9] we choose to write the first term as a fraction and associate the negatives '$-$' with the numerators. This yields

$$\frac{1}{1}, \frac{-2}{7}, \frac{4}{13}, \frac{-8}{19}, \ldots$$

The numerators form the sequence $1, -2, 4, -8, \ldots$ which is geometric with $a = 1$ and $r = -2$, so we get $C_n = (-2)^{n-1}$, for $n \geq 1$.

The denominators $1, 7, 13, 19, \ldots$ form an arithmetic sequence with $a = 1$ and $d = 6$. Hence, we get $D_n = 1 + 6(n-1) = 6n - 5$, for $n \geq 1$.

Putting these two formulas together, we obtain our formula for $a_n = \frac{C_n}{D_n} = \frac{(-2)^{n-1}}{6n-5}$, for $n \geq 1$. We leave it to the reader to show that this checks out. $\square$

While the last problem in Example 10.1.3 was neither geometric nor arithmetic, it did resolve into a combination of these two kinds of sequences. If handed the sequence $2, 5, 10, 17, \ldots$, we would be hard-pressed to find a formula for a_n if we restrict our attention to these two archetypes. We said before that there is no general algorithm for finding the explicit formula for the nth term of a given sequence, and it is only through experience gained from evaluating sequences from explicit formulas that we learn to begin to recognize number patterns.

The pattern $1, 4, 9, 16, \ldots$ is rather recognizable as the squares, so the formula $a_n = n^2$, $n \geq 1$ may not be too hard to determine. With this in mind, it's possible to see $2, 5, 10, 17, \ldots$ as the sequence $1 + 1, 4 + 1, 9 + 1, 16 + 1, \ldots$, so that $a_n = n^2 + 1$, $n \geq 1$.

Of course, since we are given only a small *sample* of the sequence, we shouldn't be too disappointed to find out this isn't the *only* formula which generates this sequence. For example, consider the sequence defined by $b_n = -\frac{1}{4}n^4 + \frac{5}{2}n^3 - \frac{31}{4}n^2 + \frac{25}{2}n - 5$, $n \geq 1$. The reader is encouraged to verify that it also

[9] Here we take 'experimentation' to mean a frustrating guess-and-check session.

produces the terms $2, 5, 10, 17$. In fact, it can be shown that given any finite sample of a sequence, there are infinitely many explicit formulas all of which generate those same finite points. This means that there will be infinitely many correct answers to some of the exercises in this section.[10] Just because your answer doesn't match ours doesn't mean it's wrong. As always, when in doubt, write your answer out. As long as it produces the same terms in the same order as what the problem wants, your answer is correct.

Sequences play a major role in the Mathematics of Finance, as we have already seen with Equation 7.2 in Section 7.6. Recall that if we invest P dollars at an annual percentage rate r and compound the interest n times per year, the formula for A_k, the amount in the account after k compounding periods, is $A_k = P\left(1+\frac{r}{n}\right)^k = \left[P\left(1+\frac{r}{n}\right)\right]\left(1+\frac{r}{n}\right)^{k-1}$, $k \geq 1$. We leave it to the reader to show this is a geometric sequence with first term $P\left(1+\frac{r}{n}\right)$ and common ratio $\left(1+\frac{r}{n}\right)$.

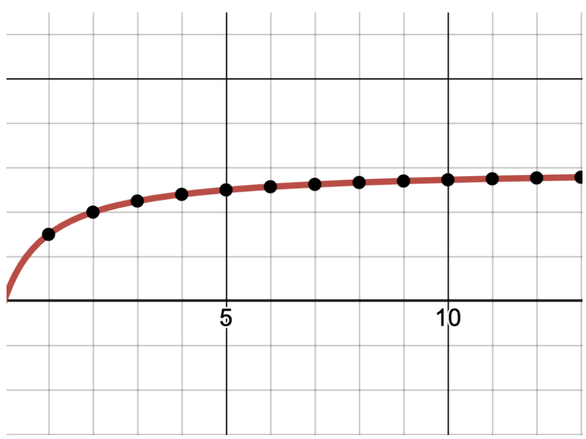
In section 7.6, we showed $\displaystyle\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = Pe^{rt}$, where we noted, at the time, that the limit here was taken on a discrete, rather than continuous variable. That didn't stop us from using the limit properties listed in Theorem 6.2. We talk more on the limits of sequences next.

10.1.1 Limits of Sequences

Consider the sequence $a_n = \frac{3n}{n+1}$, $n \geq 1$. Suppose we wished to find $\displaystyle\lim_{n\to\infty} a_n$.

We should first note that even though the domain of the sequence a_n is discrete, we can nonetheless discuss what happens as $n \to \infty$ since for any real number $M > 0$, we can find a natural number $n > M$. (We could, for instance take $n = \lfloor M \rfloor + 1$.)

Next, note that we can visualize the sequence $a_n = \frac{3n}{n+1}$, $n \geq 1$ as being points on the graph of $f(x) = \frac{3x}{x+1}$:



Comparing leading terms of numerator and denominator, as $x \to \infty$, $\frac{3x}{x+1} \approx \frac{3x}{x} = 3$. Hence, $\displaystyle\lim_{x\to\infty} \frac{3x}{x+1} = 3$, which means **all** of the y-values on the graph of $y = f(x)$, including the y-values of the graph of the sequence, approach 3 as $x \to \infty$. It stands to reason, then that $\displaystyle\lim_{n\to\infty} \frac{3n}{n+1} = 3$.

[10]For more on this, see When Every Answer is Correct: Why Sequences and Number Patterns Fail the Test.

The long and short of the above argument is that since $\lim_{x\to\infty} \frac{3x}{x+1}$ exists, $\lim_{n\to\infty} \frac{3n}{n+1} = \lim_{x\to\infty} \frac{3x}{x+1}$. While the syntax of '$\lim_{n\to\infty} \frac{3n}{n+1} = \lim_{x\to\infty} \frac{3x}{x+1}$' appears to be just a switch in a dummy variable,[11] the switch from 'n' to 'x' indicates switching from a **discrete** variable to a **continuous** one. This sort of maneuver is called **passing to a continuous variable** and is one of the primary ways we can use what we've already studied to analyze limits of sequences.

Theorem 10.1. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for all $n \geq k$ for some natural number k, then $\lim_{n\to\infty} a_n = L$.

Note the phrase '$f(n) = a_n$ for all $n \geq k$ for some natural number k' indicates the focus on what is happening as $n \to \infty$. In other words, the first finitely many sequence values do not impact the value of $\lim_{n\to\infty} a_n$. We are just concerned with the long-run or end behavior here. The reason Theorem 10.1 works, indeed why all of the limit properties discussed in Section 6.1 work with sequences as well as functions of continuous variables is because the formal definitions associated with limits of both sequences and functions share the same mathematical 'bones.' (See a Calculus instructor for more details.) We'll explore these connections more deeply in the Exercises.

We have special words to describe sequences which have limits and those which do not.

Definition 10.3. If $\{a_n\}$ is a sequence and $\lim_{n\to\infty} a_n = L$, we say the sequence $\{a_n\}$ **converges** to L. If $\lim_{n\to\infty} a_n$ does not exist, we say the sequence **diverges**.

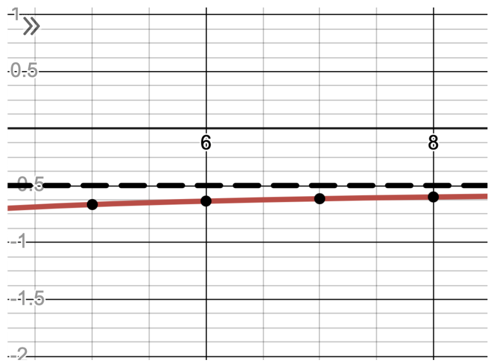
Example 10.1.4.

1. Determine the following limits by passing to a continuous variable. Check your answers graphically.
   (a) $\lim_{n\to\infty} \frac{1-n^2}{2n^2-3n+1}$.
   (b) $\lim_{n\to\infty} ne^{-2n}$.
2. What difficulties do you encounter when trying to pass the limit $\lim_{n\to\infty} \frac{(-1)^n}{n^2+1}$ to a continuous variable? What appears to be the limit?

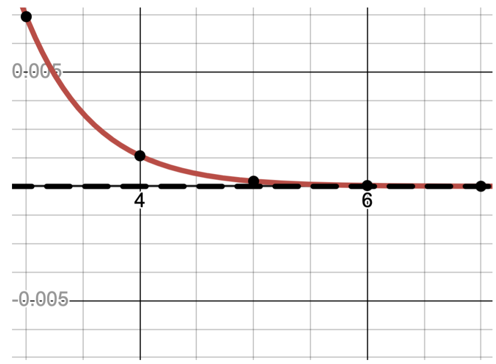
Solution.

1. (a) Passing $\lim_{n\to\infty} \frac{1-n^2}{2n^2-3n+1}$ to a continuous variable gives $\lim_{x\to\infty} \frac{1-x^2}{2x^2-3x+1}$. Comparing leading terms, we have that as $x \to \infty$, $\frac{1-x^2}{2x^2-3x+1} \approx \frac{-x^2}{2x^2} = -\frac{1}{2}$, so $\lim_{x\to\infty} \frac{1-x^2}{2x^2-3x+1} = -\frac{1}{2}$. By Theorem 10.1, $\lim_{n\to\infty} \frac{1-n^2}{2n^2-3n+1} = -\frac{1}{2}$. This tracks given the graph below on the left.
   (b) Passing $\lim_{n\to\infty} ne^{-2n}$ to a continuous variable gives $\lim_{x\to\infty} xe^{-2x}$. Here we do not have leading terms to compare - but we do have an indeterminate form '$\infty \cdot 0$', That being said, as discussed following Example 7.4.5 in Section 7.4, the factor e^{-2x} will dominate the factor x as $x \to \infty$, so we have $\lim_{n\to\infty} ne^{-2n} = \lim_{x\to\infty} xe^{-2x} = 0$. The graph below on the right confirms this.

[11] which it would be without context



$a_n = \frac{1-n^2}{2n^2-3n+1}$ along with $y = \frac{1-x^2}{2x^2-3x+1}$

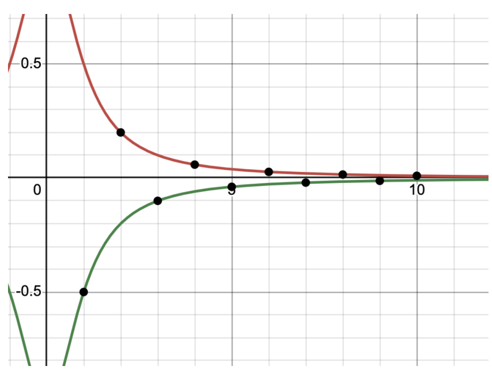


$a_n = ne^{-2n}$ along with $y = xe^{-2x}$

2. When passing $\lim\limits_{n\to\infty} \frac{(-1)^n}{n^2+1}$ to a continuous variable, we get the factor $(-1)^x$ which is non-real for several[12] real numbers such as $x = \frac{1}{2}$, $x = 117.23$, or $x = 3000\pi$. That being said, we note that the factor $(-1)^n$ alternates the sign of each term: $(+)1$ if n is even and -1 if n is odd: $-\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \frac{1}{17}, \ldots$.

   This means that we can bound the sequence we're interested in between two other sequences we know something about: $-\frac{1}{n^2+1} \leq \frac{(-1)^n}{n^2+1} \leq \frac{1}{n^2+1}$.

   Passing to a continuous variable, $\lim\limits_{n\to\infty} -\frac{1}{n^2+1} = \lim\limits_{x\to\infty} -\frac{1}{x^2+1} = 0$ and $\lim\limits_{n\to\infty} \frac{1}{n^2+1} = \lim\limits_{x\to\infty} \frac{1}{x^2+1} = 0$, so it stands to reason that $\lim\limits_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$ as well. We sketch the situation out below.



$\square$

The sequence in number 2 in Example 10.1.4 is an example of an **alternating sequence**, so-named because the terms alternate in sign. Alternating sequences play a large role in the study of infinite series (whatever those are) in Calculus,[13] so it is worth pointing them out here.

[12] Actually, 'uncountably infinite' ...

[13] We'll touch on these in the next section, too.

Next, the reasoning we used to determine $\lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$ is sound and is codified in the following theorem. We state the result for both sequences (discrete functions) and (continuous) functions.

Theorem 10.2. The Squeeze Theorem:

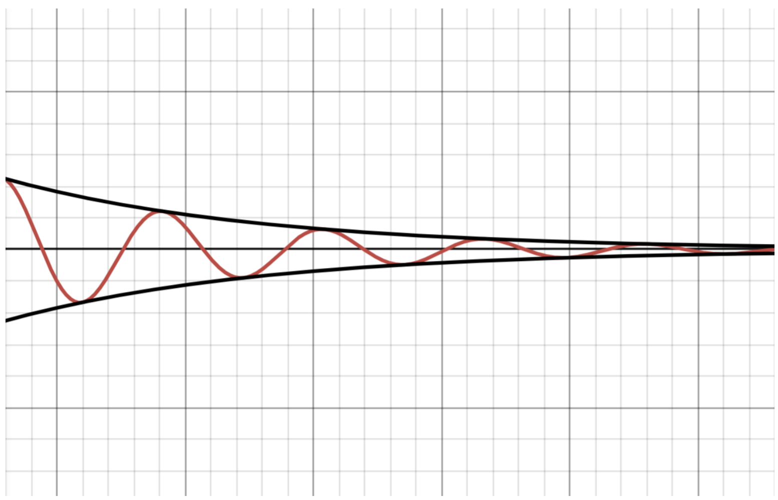
- Suppose for some natural number k, $b_n \leq a_n \leq c_n$ for all $n \geq k$.

  If $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} a_n = L$.

- Suppose for some real number M, $g(x) \leq f(x) \leq h(x)$ for all $x \geq M$.

  If $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$, then $\lim_{x\to\infty} f(x) = L$.

The Squeeze Theorem is so-named because the two sequences (functions) which bound the middle sequence (function) 'squeeze' the middle function to the common limit, L.



The graph of $y = f(x)$ being 'squeezed' to a common limit by the graphs of $y = g(x)$ and $y = h(x)$.

Passing to a continuous variable in conjunction with the Squeeze Theorem can be used to prove certain classes of Geometric Sequences converge. We have the following:

Theorem 10.3. Limits of Geometric Sequences: Given a geometric sequence with common ratio r:

1. If $-1 < r < 1$, the sequence converges to 0.

2. If $r = 1$, the sequence converges to the first term, a.

3. If $r > 1$ or $r \leq -1$, the sequence diverges.

We encourage the reader to think through each of the cases stated in Theorem 10.3 to make sure the conclusions seem reasonable. We'll have occasion to cite Theorem 10.3 in the next section. For now, it's time for some Exercises.

10.1.2 Exercises

In Exercises 1 - 13, write out the first four terms of the given sequence.

1. $a_n = 2^n - 1$, $n \geq 0$
2. $d_j = (-1)^{\frac{j(j+1)}{2}}$, $j \geq 1$
3. $\{5k - 2\}_{k=1}^{\infty}$
4. $\left\{ \dfrac{n^2+1}{n+1} \right\}_{n=0}^{\infty}$
5. $\left\{ \dfrac{x^n}{n^2} \right\}_{n=1}^{\infty}$
6. $\left\{ \dfrac{\ln(n)}{n} \right\}_{n=1}^{\infty}$
7. $a_1 = 3$, $a_{n+1} = a_n - 1$, $n \geq 1$
8. $d_0 = 12$, $d_m = \dfrac{d_{m-1}}{100}$, $m \geq 1$
9. $b_1 = 2$, $b_{k+1} = 3b_k + 1$, $k \geq 1$
10. $c_0 = -2$, $c_j = \dfrac{c_{j-1}}{(j+1)(j+2)}$, $j \geq 1$
11. $a_1 = 117$, $a_{n+1} = \dfrac{1}{a_n}$, $n \geq 1$
12. $s_0 = 1$, $s_{n+1} = x^{n+1} + s_n$, $n \geq 0$
13. $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ (This is the famous Fibonacci Sequence)

In Exercises 14 - 21 determine if the given sequence is arithmetic, geometric or neither. If it is arithmetic, find the common difference d; if it is geometric, find the common ratio r.

14. $\{3n - 5\}_{n=1}^{\infty}$
15. $a_n = n^2 + 3n + 2$, $n \geq 1$
16. $\dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{12}, \dfrac{1}{24}, \ldots$
17. $\left\{ 3\left(\dfrac{1}{5}\right)^{n-1} \right\}_{n=1}^{\infty}$
18. $17, 5, -7, -19, \ldots$
19. $2, 22, 222, 2222, \ldots$
20. $0.9, 9, 90, 900, \ldots$
21. $a_n = \dfrac{n!}{2}$, $n \geq 0$.

In Exercises 22 - 30, find an explicit formula for the n^{th} term of the given sequence.[14]

22. $3, 5, 7, 9, \ldots$
23. $1, -\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \ldots$
24. $1, \dfrac{2}{3}, \dfrac{4}{5}, \dfrac{8}{7}, \ldots$
25. $1, \dfrac{2}{3}, \dfrac{1}{3}, \dfrac{4}{27}, \ldots$
26. $1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \ldots$
27. $x, -\dfrac{x^3}{3}, \dfrac{x^5}{5}, -\dfrac{x^7}{7}, \ldots$
28. $0.9, 0.99, 0.999, 0.9999, \ldots$
29. $27, 64, 125, 216, \ldots$
30. $1, 0, 1, 0, \ldots$

[14] Use the formulas in Equation 10.1 as needed.

In Exercises 31 - 33, find the indicated limit by using Theorem 10.1 and passing to a continuous variable.[15]

31. $\displaystyle\lim_{n\to\infty} \frac{2n^2 - 3n + 1}{4 - n^2}$

32. $\displaystyle\lim_{k\to\infty} \frac{k^2 + 7k - 3}{3k - k^3}$

33. $\displaystyle\lim_{m\to\infty} \frac{117m^{42} + 3m + 1}{e^{2m} + 6}$

In Exercises 34 - 36, use the Squeeze Theorem, Theorem 10.2 to help you determine the limit.[16]

34. $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{3n + 1}$

35. $\displaystyle\lim_{k\to\infty} 1 - \left(-\frac{2}{3}\right)^k$

36. $\displaystyle\lim_{m\to\infty} \frac{(-1)^{\frac{m^2-m}{2}}}{m!}$

37. Find a sequence which is both arithmetic and geometric. (Hint: Start with $a_n = c$ for all n.)

38. Show that a geometric sequence can be transformed into an arithmetic sequence by taking the natural logarithm of the terms.

39. Thomas Robert Malthus is credited with saying, "The power of population is indefinitely greater than the power in the earth to produce subsistence for man. Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will show the immensity of the first power in comparison with the second." (See this webpage for more information.) Discuss this quote with your classmates from a sequences point of view.

40. This classic problem involving sequences shows the power of geometric sequences. Suppose that a wealthy benefactor agrees to give you one penny today and then double the amount she gives you each day for 30 days. So, for example, you get two pennies on the second day and four pennies on the third day. How many pennies do you get on the 30$^{\text{th}}$ day? What is the total dollar value of the gift you have received?

41. Research the terms 'arithmetic mean' and 'geometric mean.' With the help of your classmates, show that a given term of a arithmetic sequence a_k, $k \geq 2$ is the arithmetic mean of the term immediately preceding, a_{k-1} it and immediately following it, a_{k+1}. State and prove an analogous result for geometric sequences.

42. Discuss with your classmates how the results of this section might change if we were to examine sequences of other mathematical things like complex numbers or matrices. Find an explicit formula for the n^{th} term of the sequence $i, -1, -i, 1, i, \ldots$. List out the first four terms of the matrix sequences we discussed in Exercise 9.3.1 in Section 9.3.

[15] See Example 10.1.4.

[16] See part 2 of Example 10.1.4.

10.1.3 Answers

1. $0, 1, 3, 7$
2. $-1, -1, 1, 1$
3. $3, 8, 13, 18$
4. $1, 1, \frac{5}{3}, \frac{5}{2}$
5. $x, \frac{x^2}{4}, \frac{x^3}{9}, \frac{x^4}{16}$
6. $0, \frac{\ln(2)}{2}, \frac{\ln(3)}{3}, \frac{\ln(4)}{4}$
7. $3, 2, 1, 0$
8. $12, 0.12, 0.0012, 0.000012$
9. $2, 7, 22, 67$
10. $-2, -\frac{1}{3}, -\frac{1}{36}, -\frac{1}{720}$
11. $117, \frac{1}{117}, 117, \frac{1}{117}$
12. $1, x+1, x^2+x+1, x^3+x^2+x+1$
13. $1, 1, 2, 3$
14. arithmetic, $d = 3$
15. neither
16. geometric, $r = \frac{1}{2}$
17. geometric, $r = \frac{1}{5}$
18. arithmetic, $d = -12$
19. neither
20. geometric, $r = 10$
21. neither
22. $a_n = 1 + 2n,\ n \geq 1$
23. $a_n = \left(-\frac{1}{2}\right)^{n-1},\ n \geq 1$
24. $a_n = \frac{2^{n-1}}{2n-1},\ n \geq 1$
25. $a_n = \frac{n}{3^{n-1}},\ n \geq 1$
26. $a_n = \frac{1}{n^2},\ n \geq 1$
27. $\frac{(-1)^{n-1}x^{2n-1}}{2n-1},\ n \geq 1$
28. $a_n = \frac{10^n-1}{10^n},\ n \geq 1$
29. $a_n = (n+2)^3,\ n \geq 1$
30. $a_n = \frac{1+(-1)^{n-1}}{2},\ n \geq 1$
31. $\lim\limits_{n\to\infty} \frac{2n^2-3n+1}{4-n^2} = -2$
32. $\lim\limits_{k\to\infty} \frac{k^2+7k-3}{3k-k^3} = 0$
33. $\lim\limits_{m\to\infty} \frac{117m^{42}+3m+1}{e^{2m}+6} = 0$
34. $\lim\limits_{n\to\infty} \frac{(-1)^n}{3n+1} = 0$
35. $\lim\limits_{k\to\infty} 1 - \left(-\frac{2}{3}\right)^k = 1$
36. $\lim\limits_{m\to\infty} \frac{(-1)^{\frac{m^2-m}{2}}}{m!} = 0$

10.2 Summation Notation

In Section 10.1, we showed how the formula for compound interest is a geometric sequence. In retirement planning, it is seldom the case that an investor deposits a set amount of money into an account and waits for it to grow. Usually, additional payments of principal are made at regular intervals and the value of the investment grows accordingly. This kind of investment is called an *annuity* and will be discussed in later in this section once we have developed more mathematical machinery that enables us to *add* sequences.

In the previous section, we introduced sequences. Each of the numbers in the sequence is called a 'term' which implies these numbers are meant to be added. To that end, we introduce the following notation which is used to describe the sum of (some of the) terms of a sequence.

Definition 10.4. Summation Notation: Given a sequence $\{a_n\}_{n=k}^{\infty}$ and numbers m and p satisfying $k \leq m \leq p$, the summation from m to p of the sequence $\{a_n\}$ is written

$$\sum_{n=m}^{p} a_n = a_m + a_{m+1} + \ldots + a_p$$

The variable n is called the **index of summation**. The number m is called the **lower limit of summation** while the number p is called the **upper limit of summation**.

In English, Definition 10.4 is simply defining a short-hand notation for adding up the terms of the sequence $\{a_n\}_{n=k}^{\infty}$ from a_m through a_p. The symbol Σ is the capital Greek letter sigma and is shorthand for 'sum'. The lower and upper limits of the summation tells us which term to start with and which term to end with, respectively. For example, using the sequence $a_n = 2n - 1$ for $n \geq 1$, we can write $a_3 + a_4 + a_5 + a_6$ as

$$\begin{aligned}
\sum_{n=3}^{6}(2n-1) &= (2(3)-1) + (2(4)-1) + (2(5)-1) + (2(6)-1) \\
&= 5 + 7 + 9 + 11 \\
&= 32
\end{aligned}$$

The index variable is considered a 'dummy variable' in the sense that it may be changed to any letter without affecting the value of the summation. For instance,

$$\sum_{n=3}^{6}(2n-1) = \sum_{k=3}^{6}(2k-1) = \sum_{j=3}^{6}(2j-1)$$

One place you may encounter summation notation is in mathematical definitions. For example, summation notation allows us to define polynomials as functions of the form

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

for real numbers a_k, $k = 0, 1, \ldots n$. The reader is invited to compare this with what is given in Definition 2.4. Summation notation is particularly useful when talking about matrix operations. For example, we can write the product of the ith row R_i of a matrix $A = [a_{ij}]_{m \times n}$ and the j^{th} column C_j of a matrix $B = [b_{ij}]_{n \times r}$ as

$$Ri \cdot Cj = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Again, the reader is encouraged to write out the sum and compare it to Definition 9.8. Our next example gives us practice with this new notation.

Example 10.2.1.

1. Find the following sums.

   (a) $\displaystyle\sum_{k=1}^{4} \frac{13}{100^k}$ (b) $\displaystyle\sum_{n=0}^{4} \frac{n!}{2}$ (c) $\displaystyle\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n$

2. Write the following sums using summation notation.

   (a) $1 + 3 + 5 + \ldots + 117$

   (b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots + \frac{1}{117}$

   (c) $0.9 + 0.09 + 0.009 + \ldots 0.\underbrace{0 \cdots 0}_{n-1 \text{ zeros}}9$

Solution.

1. (a) We substitute $k = 1$ into the formula $\frac{13}{100^k}$ and add successive terms until we reach $k = 4$.

$$\begin{aligned} \sum_{k=1}^{4} \frac{13}{100^k} &= \frac{13}{100^1} + \frac{13}{100^2} + \frac{13}{100^3} + \frac{13}{100^4} \\ &= 0.13 + 0.0013 + 0.000013 + 0.00000013 \\ &= 0.13131313 \end{aligned}$$

   (b) Proceeding as in (a), we replace every occurrence of n with the values 0 through 4. We recall the factorials, $n!$ as defined in number Example 10.1.1, number 6 and get:

$$\begin{aligned} \sum_{n=0}^{4} \frac{n!}{2} &= \frac{0!}{2} + \frac{1!}{2} + \frac{2!}{2} + \frac{3!}{2} = \frac{4!}{2} \\ &= \frac{1}{2} + \frac{1}{2} + \frac{2 \cdot 1}{2} + \frac{3 \cdot 2 \cdot 1}{2} + \frac{4 \cdot 3 \cdot 2 \cdot 1}{2} \\ &= \frac{1}{2} + \frac{1}{2} + 1 + 3 + 12 \\ &= 17 \end{aligned}$$

(c) We proceed as before, replacing the index n, but *not* the variable x, with the values 1 through 5 and adding the resulting terms.

$$\begin{aligned}\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n &= \frac{(-1)^{1+1}}{1}(x-1)^1 + \frac{(-1)^{2+1}}{2}(x-1)^2 + \frac{(-1)^{3+1}}{3}(x-1)^3 \\ &\quad + \frac{(-1)^{1+4}}{4}(x-1)^4 + \frac{(-1)^{1+5}}{5}(x-1)^5 \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}\end{aligned}$$

2. The key to writing these sums with summation notation is to find the pattern of the terms. To that end, we make good use of the techniques presented in Section 10.1.

(a) The terms of the sum 1, 3, 5, etc., form an arithmetic sequence with first term $a = 1$ and common difference $d = 2$. Using Equation 10.1, we get $a_n = 1 + (n-1)2 = 2n - 1$, $n \geq 1$.

At this stage, we have the formula for the terms, namely $2n - 1$, and the lower limit of the summation, $n = 1$. To finish the problem, we need to determine the upper limit of the summation. In other words, we need to determine which value of n produces the term 117. Setting $a_n = 117$, we get $2n - 1 = 117$ or $n = 59$. Our final answer is

$$1 + 3 + 5 + \ldots + 117 = \sum_{n=1}^{59} (2n-1)$$

(b) We rewrite all of the terms as fractions, the subtraction as addition, and associate the negatives '$-$' with the numerators to get

$$\frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \ldots + \frac{1}{117}$$

The numerators, 1, -1, etc. can be described by the geometric sequence[1] $C_n = (-1)^{n-1}$ for $n \geq 1$, while the denominators are given by the arithmetic sequence[2] $D_n = n$ for $n \geq 1$. Hence, we get the formula $a_n = \frac{(-1)^{n-1}}{n}$ for our terms, and we find the lower and upper limits of summation to be $n = 1$ and $n = 117$, respectively. Thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots + \frac{1}{117} = \sum_{n=1}^{117} \frac{(-1)^{n-1}}{n}$$

(c) Thanks to Example 10.1.3, we know that one formula for the n^{th} term is $a_n = \frac{9}{10^n}$ for $n \geq 1$. This gives us a formula for the summation as well as a lower limit of summation.

[1] This is indeed a geometric sequence with first term $a = 1$ and common ratio $r = -1$.

[2] It is an arithmetic sequence with first term $a = 1$ and common difference $d = 1$.

To determine the upper limit of summation, we note that to produce the $n-1$ zeros to the right of the decimal point before the 9, we need a denominator of 10^n. Hence, n is the upper limit of summation.

Since n is used in the limits of the summation, we need to choose a different letter for the index of summation.[3] We choose k and get

$$0.9 + 0.09 + 0.009 + \ldots 0.\underbrace{0\cdots 0}_{n-1 \text{ zeros}}9 \quad = \quad \sum_{k=1}^{n} \frac{9}{10^k}$$

□

The following theorem presents some general properties of summation notation.

Theorem 10.4. Properties of Summation Notation: Suppose $\{a_n\}$ and $\{b_n\}$ are sequences so that the following sums are defined.

- **Sum and Difference Property:** $\displaystyle\sum_{n=m}^{p} (a_n \pm b_n) = \sum_{n=m}^{p} a_n \pm \sum_{n=m}^{p} b_n$
- **Distributive Property:** $\displaystyle\sum_{n=m}^{p} c\, a_n = c \sum_{n=m}^{p} a_n$, for any real number c.
- **Additive Index Property:** $\displaystyle\sum_{n=m}^{j} a_n + \sum_{n=j+1}^{p} a_n = \sum_{n=m}^{p} a_n$, for any natural number $m \leq j < j+1 \leq p$.
- **Re-indexing:** $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m+r}^{p+r} a_{n-r}$, for any integer r.

There is much to be learned by thinking about why the properties hold, so we leave the proof of these properties to the reader.[4]

Example 10.2.2.

1. If $\displaystyle\sum_{n=2}^{50} (a_n - 3b_n) = 17$ and $\displaystyle\sum_{n=2}^{50} a_n = 10$, find $\displaystyle\sum_{n=2}^{50} b_n$.
2. If $\displaystyle\sum_{n=1}^{20} a_n = -3$ and $\displaystyle\sum_{n=1}^{21} a_n = 7$, find a_{21}.
3. Rewrite the sum so the index starts at 0: $\displaystyle\sum_{n=2}^{437} n(n-1)x^{n-2}$

[3] To see why, try writing the summation using 'n' as the index.
[4] To get started, remember the mantra "When in doubt, write it out!"

Solution.

1. Using the Sum and Difference Property along with the Distributive Property of Theorem 10.4, we get:

$$\sum_{n=2}^{50}(a_n - 3b_n) = \sum_{n=2}^{50} a_n - \sum_{n=2}^{50} 3b_n = \sum_{n=2}^{50} a_n - 3\sum_{n=2}^{50} b_n$$

   Hence, $\displaystyle\sum_{n=2}^{50} a_n - 3\sum_{n=2}^{50} b_n = 17$. If $\displaystyle\sum_{n=2}^{50} a_n = 10$, then $\displaystyle 10 - 3\sum_{n=2}^{50} b_n = 17$ so $\displaystyle\sum_{n=2}^{50} b_n = -\frac{7}{3}$.

2. There are at least two ways to approach this problem. By definition, $\displaystyle\sum_{n=1}^{21} a_n = a_1 + a_2 + ... + a_{21}$. That is, we add up the first 21 terms of the sequence a_n. Similarly, $\displaystyle\sum_{n=1}^{20} a_n = a_1 + a_2 + ... + a_{20}$ means we add up the first 20 terms of the sequence. Hence, $\displaystyle a_{21} = \sum_{n=1}^{21} a_n - \sum_{n=1}^{20} a_n = 7 - (-3) = 10$.

   Alternatively, we can use the Additive Index Property:

$$\sum_{n=1}^{21} a_n = \sum_{n=1}^{20} a_n + \sum_{n=21}^{21} a_n = \sum_{n=1}^{20} a_n + a_{21},$$

   which gives $\displaystyle a_{21} = \sum_{n=1}^{21} a_n - \sum_{n=1}^{20} a_n = 7 - (-3) = 10$ as well.

3. To re-index $\displaystyle\sum_{n=2}^{437} n(n-1)x^{n-2}$ so n starts at 0, we follow the formula in Theorem 10.2.2 with $r = -2$:

$$\sum_{n=2}^{437} n(n-1)x^{n-2} = \sum_{n=2+(-2)}^{437+(-2)} (n-(-2))(n-(-2)-1)x^{n-(-2)-2} = \sum_{n=0}^{435}(n+2)(n+1)x^n.$$

   We leave it to the reader to check by writing out the first few, and last few, terms.

   Alternatively, to better see *why* the re-indexing works in this way, we can introduce a new counter, k. We want this new counter to start at $k = 0$ whereas the current counter starts at $n = 2$, so we want $k = n - 2$. When $n = 2$, $k = 0$, as required, and when $n = 437$, $k = 435$.

   Moreover, $n = k + 2$, so substituting this into the sum, we get

$$\sum_{n=2}^{437} n(n-1)x^{n-2} = \sum_{k=0}^{435}(k+2)((k+2)-1)x^{(k+2)-2} = \sum_{k=0}^{435}(k+2)(k+1)x^k,$$

which is the same sum we had before, just with a different dummy variable. □

We now turn our attention to the sums involving arithmetic and geometric sequences. Given an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$, we let S denote the sum of the first n terms. To derive a formula for S, we write it out in two different ways

$$\begin{array}{rcccccccccc} S & = & a & + & (a+d) & + & \ldots & + & (a+(n-2)d) & + & (a+(n-1)d) \\ S & = & (a+(n-1)d) & + & (a+(n-2)d) & + & \ldots & + & (a+d) & + & a \end{array}$$

If we add these two equations and combine the terms which are aligned vertically, we get

$$2S = (2a + (n-1)d) + (2a + (n-1)d) + \ldots + (2a + (n-1)d) + (2a + (n-1)d)$$

The right hand side of this equation contains n terms, all of which are equal to $(2a + (n-1)d)$ so we get $2S = n(2a + (n-1)d)$. Dividing both sides of this equation by 2, we obtain the formula

$$S = \frac{n}{2}(2a + (n-1)d)$$

If we rewrite the quantity $2a + (n-1)d$ as $a + (a + (n-1)d) = a_1 + a_n$, we get the formula

$$S = n\left(\frac{a_1 + a_n}{2}\right)$$

A helpful way to remember this last formula is to recognize that we have expressed the sum as the product of the number of terms n and the *average* of the first and n^{th} terms.

To derive the formula for the geometric sum, we start with a geometric sequence $a_k = ar^{k-1}$, $k \geq 1$, and let S once again denote the sum of the first n terms. Comparing S and rS, we get

$$\begin{array}{rcccccccccccc} S & = & a & + & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & & \\ rS & = & & & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & + & ar^n \end{array}$$

Subtracting the second equation from the first forces all of the terms except a and ar^n to cancel out and we get $S - rS = a - ar^n$. Factoring, we get $S(1-r) = a(1-r^n)$. Assuming $r \neq 1$, we can divide both sides by the quantity $(1-r)$ to obtain

$$S = a\left(\frac{1-r^n}{1-r}\right)$$

If we distribute a through the numerator, we get $a - ar^n = a_1 - a_{n+1}$ which yields the formula

$$S = \frac{a_1 - a_{n+1}}{1-r}$$

In the case when $r = 1$, we get the formula

$$S = \underbrace{a + a + \ldots + a}_{n \text{ times}} = n\,a$$

Our results are summarized below.[5]

Equation 10.2. Sums of Arithmetic and Geometric Sequences:

- The sum S of the first n terms of an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$ is

$$S = \sum_{k=1}^{n} a_k = n\left(\frac{a_1 + a_n}{2}\right) = \frac{n}{2}(2a + (n-1)d)$$

- The sum S of the first n terms of a geometric sequence $a_k = ar^{k-1}$ for $k \geq 1$ is

1. $S = \displaystyle\sum_{k=1}^{n} a_k = \frac{a_1 - a_{n+1}}{1-r} = a\left(\frac{1-r^n}{1-r}\right)$, if $r \neq 1$.
2. $S = \displaystyle\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a = na$, if $r = 1$.

While we have made an honest effort to derive the formulas in Equation 10.2, formal proofs require the machinery in Section 10.3.

Example 10.2.3.

1. (a) Find the sum: $1 + 3 + 5 + \ldots + 117$

   (b) Find a formula for the sum $\displaystyle\sum_{k=1}^{n} k$.

2. The classic wheat and chessboard problem asks the following question. Given a chessboard with its squares numbered 1 to 64, suppose on the first square was placed one grain of wheat, the second square, two grains, the third square, four grains, and so on, each square receiving twice the number of grains as its predecessor. How many total grains of wheat would end up on the chessboard?

Solution.

1. (a) Recognizing the terms of $1 + 3 + 5 + \ldots + 117$ as 1, 3, 5, and so on, we see we have an arithmetic sequence with $a = 1$ and $d = 2$. Using Equation 10.1, we get a formula for the terms $a_n = 1 + 2(n-1) = 2n - 1$ for $n \geq 1$. In order to use the formula in Equation 10.2, we need to determine the number of terms being added, n. Setting $2n - 1 = 117$, we find $n = 59$. Feeding in all of our data into Equation 10.2, we get $1 + 3 + 5 + \ldots + 117 = 59\left(\frac{1+117}{2}\right) = 3481$.

   (b) Applying the adage 'when in doubt, write it out,' we have $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + \ldots + n$. We see the terms here form an arithmetic sequence with $a = d = 1$. Moreover, we are adding exactly n terms, so Equation 10.2 gives $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$.

[5] Alternatively, we can use Exercise 40 in Section 2.2.

As a side note, the special case: $1+2+3+\ldots+100$ was allegedly given to Carl Friedrich Gauss while he was in elementary school. Instead of computing the sum in a brute force method, he arrived at the answer by grouping $1+99=100$, $2+98=100$, etc. so that he had 50 groups of 100 with 50 left over for a total of 5050. This is the exact same methodology we used to prove the sum of the arithmetic sequence formula in Equation 10.2.

2. Since we are *doubling* the number of grains of wheat as we move from one square to the next, a geometric sequence with $r=2$ describes the number of grains on each individual square.

   Since we start with one grain on the first square, the number of grains on the kth square is $a_k = (1)(2)^{k-1} = 2^{k-1}$ for $k \geq 1$.

   Adding up the number of grains on each square gives:

$$1+2+\ldots+2^{64-1} = 1+2+\ldots 2^{63} = \frac{1-2^{64}}{1-2} = 2^{64}-1 \approx 1.8 \times 10^{19},$$

   in accordance with Equation 10.2. (The weight of these grains would total approximately 2.6×10^{15} pounds which is approximately 15 times the entire biomass of the planet.) □

An important application of the geometric sum formula is the investment plan called an *annuity*. Annuities differ from the kind of investments we studied in Section 7.6 in that payments are deposited into the account on an on-going basis, and this complicates the mathematics a little.[6]

Suppose you have an account with annual interest rate r which is compounded n times per year. We let $i = \frac{r}{n}$ denote the interest rate per period. Suppose we wish to make ongoing deposits of P dollars at the *end* of each compounding period. Let A_k denote the amount in the account after k compounding periods. Then $A_1 = P$, because we have made our first deposit at the *end* of the first compounding period and no interest has been earned. During the second compounding period, we earn interest on A_1 so that our initial investment has grown to $A_1(1+i) = P(1+i)$ in accordance with Equation 7.1. Adding our second payment at the end of the second period, we get

$$A_2 = A_1(1+i)+P = P(1+i)+P = P(1+i)\left(1+\frac{1}{1+i}\right)$$

The reason for factoring out the $P(1+i)$ will become apparent in short order. During the third compounding period, we earn interest on A_2 which then grows to $A_2(1+i)$. We add our third payment at the end of the third compounding period to obtain

$$A_3 = A_2(1+i)+P = P(1+i)\left(1+\frac{1}{1+i}\right)(1+i)+P = P(1+i)^2\left(1+\frac{1}{1+i}+\frac{1}{(1+i)^2}\right)$$

During the fourth compounding period, A_3 grows to $A_3(1+i)$, and when we add the fourth payment, we factor out $P(1+i)^3$ to get

$$A_4 = P(1+i)^3\left(1+\frac{1}{1+i}+\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}\right)$$

[6] The reader may wish to re-read the discussion on compound interest in Section 7.6 before proceeding.

This pattern continues so that at the end of the kth compounding, we get

$$A_k = P(1+i)^{k-1}\left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{k-1}}\right)$$

The sum in the parentheses above is the sum of the first k terms of a geometric sequence with $a = 1$ and $r = \frac{1}{1+i}$. Using Equation 10.2, we get

$$1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{k-1}} = 1\left(\frac{1-\dfrac{1}{(1+i)^k}}{1-\dfrac{1}{1+i}}\right) = \frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}$$

Hence, we get

$$A_k = P(1+i)^{k-1}\left(\frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}\right) = \frac{P\left((1+i)^k - 1\right)}{i}$$

If we let t be the number of years this investment strategy is followed, then $k = nt$, and we get the formula for the future value of an *ordinary annuity*.

Equation 10.3. Future Value of an Ordinary Annuity: Suppose an annuity offers an annual interest rate r compounded n times per year. Let $i = \frac{r}{n}$ be the interest rate per compounding period. If a deposit P is made at the end of each compounding period, the amount A in the account after t years is given by

$$A = \frac{P\left((1+i)^{nt} - 1\right)}{i}$$

The reader is encouraged to substitute $i = \frac{r}{n}$ into Equation 10.3 and simplify. Some familiar equations arise which are cause for pause and meditation. One last note: if the deposit P is made a the *beginning* of the compounding period instead of at the end, the annuity is called an *annuity-due*. We leave the derivation of the formula for the future value of an annuity-due as an exercise for the reader.

Example 10.2.4. An ordinary annuity offers a 6% annual interest rate, compounded monthly.

1. If monthly payments of \$50 are made, find the value of the annuity in 30 years.
2. How many years will it take for the annuity to grow to \$100,000?

Solution.

1. We have $r = 0.06$ and $n = 12$ so that $i = \frac{r}{n} = \frac{0.06}{12} = 0.005$. With $P = 50$ and $t = 30$,

$$A = \frac{50\left((1+0.005)^{(12)(30)} - 1\right)}{0.005} \approx 50225.75$$

Our final answer is \$50,225.75.

2. To find how long it will take for the annuity to grow to \$100,000, we set $A = 100000$ and solve for t. We isolate the exponential and take natural logs of both sides of the equation.

$$\begin{array}{rcl} 100000 & = & \dfrac{50\left((1+0.005)^{12t}-1\right)}{0.005} \\ 10 & = & (1.005)^{12t}-1 \\ (1.005)^{12t} & = & 11 \\ \ln\left((1.005)^{12t}\right) & = & \ln(11) \\ 12t\ln(1.005) & = & \ln(11) \\ t & = & \frac{\ln(11)}{12\ln(1.005)} \approx 40.06 \end{array}$$

This means that it takes just over 40 years for the investment to grow to \$100,000. Comparing this with our answer to part 1, we see that in just 10 additional years, the value of the annuity nearly doubles. This is a lesson worth remembering. □

10.2.1 Geometric Series

As defined in Section 10.1, sequences are an *infinite* list of numbers. So far in this section, we have concerned ourselves with adding only *finitely* many terms. In Calculus, *infinite* sums, called **series** are studied at great length. While we do not have the mathematical machinery to embark upon an exhaustive study here, we can nevertheless focus our attention on what is arguably one of the most prevalent and useful types of series, **geometric series**.

As a motivating example, consider the number $0.\overline{9}$. We can write this number as

$$0.\overline{9} = 0.9999... = 0.9 + 0.09 + 0.009 + 0.0009 + ...$$

From Example 10.2.1, we know we can write the sum of the first n of these terms as

$$0.\underbrace{9\cdots9}_{n \text{ nines}} = .9 + 0.09 + 0.009 + \ldots 0.\underbrace{0\cdots0}_{n-1 \text{ zeros}}9 = \sum_{k=1}^{n} \frac{9}{10^k}$$

Using Equation 10.2, we have

$$\sum_{k=1}^{n} \frac{9}{10^k} = \sum_{k=1}^{n} \frac{9}{10}\left(\frac{1}{10^{k-1}}\right) = \sum_{k=1}^{n} \frac{9}{10}\left(\frac{1}{10}\right)^{k-1} = \frac{9}{10}\left(\frac{1-\dfrac{1}{10^n}}{1-\dfrac{1}{10}}\right) = 1 - \frac{1}{10^n}$$

It stands to reason that we should define $0.\overline{9} = \lim\limits_{n\to\infty}\left(1-\frac{1}{10^n}\right)$. Passing to a continuous variable along with our knowledge of exponential functions gives $\lim\limits_{n\to\infty}\left(1-\frac{1}{10^n}\right) = \lim\limits_{x\to\infty}\left(1-\frac{1}{10^x}\right) = 1 - 0 = 1$.

We have just argued that $0.\overline{9} = 1$, which may shock some readers.[7]

[7] To make this more palatable, it is usually accepted that $0.\overline{3} = \frac{1}{3}$ so that $0.\overline{9} = 3\left(0.\overline{3}\right) = 3\left(\frac{1}{3}\right) = 1$.

Note that in this manner, any non-terminating decimal can be thought of as an infinite sum whose denominators are the powers of 10, so the phenomenon of adding up infinitely many terms and arriving at a finite number is not as foreign of a concept as it may appear. We have the following theorem.

Theorem 10.5. Geometric Series: Given the sequence $a_k = ar^{k-1}$ for $k \geq 1$, where $|r| < 1$,

$$a + ar + ar^2 + \ldots = \sum_{k=1}^{\infty} ar^{k-1} = \lim_{n \to \infty} \sum_{k=1}^{n} ar^{k-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the sum $a + ar + ar^2 + \ldots$ does not exist.

The justification of the result in Theorem 10.5 comes from taking the formula in Equation 10.2 for the sum of the first n terms of a geometric sequence and taking the limit as $n \to \infty$.
Assuming $|r| < 1$ means $-1 < r < 1$, so per Theorem 10.3, $\lim\limits_{n \to \infty} r^n = 0$. Using this fact along with the Limit Properties listed in Theorem 6.2:

$$\lim_{n \to \infty} \sum_{k=1}^{n} ar^{k-1} = \lim_{n \to \infty} a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r}$$

We'll explore what goes wrong when $|r| \geq 1$ in some of the Exercises. For now, we put this theorem to good use in the following example.

Example 10.2.5.

1. Find the sum: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$.
2. Represent $4.2\overline{17}$ as a fraction in lowest terms.

Solution.

1. We recognize $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ as a geometric series with $a = r = \frac{1}{2}$. Using Theorem 10.5, we get

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

   The interested reader is invited to research this sum as it relates to Zeno's Dichotomy Paradox.

2. To use Theorem 10.5 as it applies to the repeating decimal $4.2\overline{17}$, we first need to rewrite this decimal in terms of a geometric series. Expanding $4.2\overline{17} = 4.2 + 0.017 + 0.00017 + 0.0000017 + \ldots$, we see the series $0.017 + 0.00017 + 0.0000017 + \ldots$ is geometric with $a = 0.017$ and $r = 0.01$. Hence, we can apply Theorem 10.5 to that part of the decimal to get:

$$0.017 + 0.00017 + 0.0000017 + \ldots = \frac{0.017}{1 - 0.01} = \frac{\frac{17}{1000}}{\frac{99}{100}} = \frac{17}{990}$$

   Hence, $4.2\overline{17} = 4.2 + \frac{17}{990} = \frac{42}{10} + \frac{17}{990} = \frac{835}{198}$. □

We note that another popular method for converting repeating decimals to fractions goes something like this: let $x = 4.2\overline{17}$. Then, $100x = 421.7\overline{17}$. Hence, $99x = 100x - x = 421.7\overline{17} - 4.2\overline{17} = 417.5$. Hence, $x = \frac{417.5}{99} = \frac{835}{198}$. While this procedure results in the same (correct!) answer, the manipulations involved (such as the multiplication and subtraction) are actually using some of the properties listed in Theorem 10.4 extended to infinite sums via Theorem 6.2.

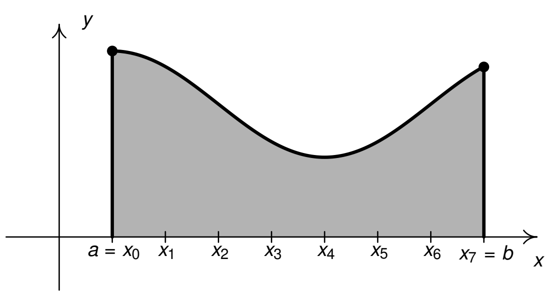
10.2.2 Area

One of the (two) major geometric problems studied in Calculus is finding the area under a curve[8] (more specifically, the area between the graph of a function and the x-axis.)[9] In this section, we explore how summation notation is used to help better formulate this problem, and, as with our study of Geometric Series, sneak a peak into Calculus itself.

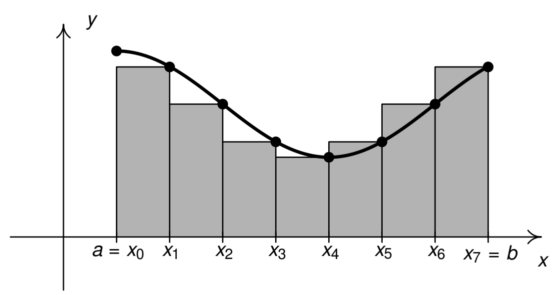
Suppose we wish to determine the area between the graph of a continuous function $y = f(x)$ over the interval $[a, b]$ and the x-axis as shown below on the left. Since we don't know any area formulas for arbitrary regions, we stick to what we know - rectangles.

To keep things simple, we divide $[a, b]$ into n equal pieces (subintervals), and use the right-endpoints of each piece to determine the height of the rectangles.[10] We let x_k represent the right endpoint of the kth subinterval, so the height of the kth rectangle is $f(x_k)$.

The width of the kth rectangle is the length of the kth subinterval. Since the interval itself is $b - a$ units long and we are dividing the interval into n *equal* pieces, each piece is $\frac{b-a}{n}$ units long. For brevity, we'll call this length 'Δx.' Below on the right is a depiction of RS_7, a 'right endpoint sum' using 7 (equally spaced) subintervals.[11]



Area under the graph of $y = f(x)$



Visualizing RS_7, a 'right endpoint sum.'

The idea here is to approximate the area of the shaded region by the sum of the areas of the rectangles. In symbols:

$$\text{Area} \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_7)\Delta x = \sum_{k=1}^{7} f(x_k)\Delta x$$

[8]The other is the concept of tangent lines which we touch on (awesome pun!) in Section 6.2.

[9]The area can actually represent a wide variety of things such as displacement, probability, or, as odd as it sounds, volume.

[10]In Calculus, you'll also use left endpoints and midpoints . . . .

[11]On intervals over which the function is *increasing*, we find the area of rectangles *overestimates* the area we want; on intervals over which the function is *decreasing*, we find the area of the rectangles *underestimates* the area we want.

Our ultimate goal is to find a formula for the area approximation as described above as a function of the number of rectangles n and look to see what happens as $n \to \infty$.
We first note that the right endpoints x_k, are terms in an arithmetic sequence: the first right endpoint, x_1 is Δx to the right of $a = x_0$, so $x_1 = x_0 + \Delta x$; the second right endpoint, x_2 is Δx units to the right of x_1, so $x_2 = x_1 + \Delta x$; the third right endpoint $x_3 = x_2 + \Delta x$ and so on. In general, $x_k = x_{k-1} + \Delta x$, proving the x_k are terms of an arithmetic sequence with common difference $d = \Delta x$. It follows that x_k, the kth right endpoint is $k\Delta x$ units to the right of $x_0 = a$, so that $x_k = a + k\Delta x$. We summarize the notation and formulas for right endpoint sums below.

Summary of Formulas for Right Endpoint Sums, RS_n

- Number of rectangles: n
- Width of each rectangle: $\Delta x = \dfrac{b-a}{n}$
- Right endpoint: $x_k = a + k\Delta x$
- Height of kth rectangle: $f(x_k)$
- Area $\approx RS_n$ = the sum of the area of the rectangles $= \displaystyle\sum_{k=1}^{n} f(x_k)\Delta x_k$

Below we summarize some common summation formulas we'll need when actually computing these sums. Formal proofs of these require the machinery of Section 10.3 and are found there.

Summation Formulas

- $\displaystyle\sum_{k=1}^{n} c = cn$
- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

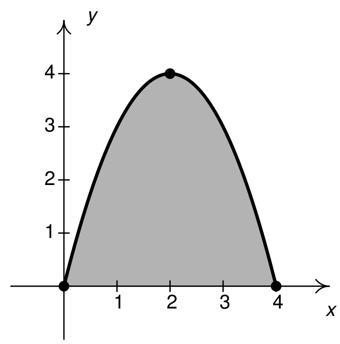
It is high time for an example.

Example 10.2.6. Consider $f(x) = 4x - x^2$ over the interval $[0, 4]$.

1. Graph f over this interval and shade the area between the graph of f and the x-axis.
2. Compute RS_n for $n = 4$ and $n = 8$. Interpret your results graphically.
3. Find a formula for RS_n in terms of n and determine $\displaystyle\lim_{n\to\infty} RS_n$.

Solution.

1. The graph of $f(x) = 4x - x^2$ is a parabola with intercepts $(0,0)$ and $(4,0)$ with a vertex at $(2,4)$.

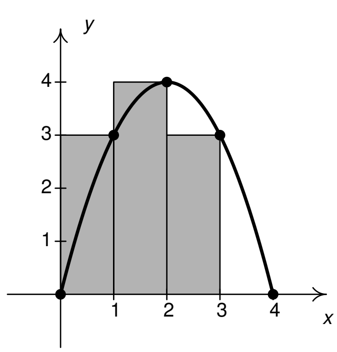


Area under the graph of $y = f(x)$

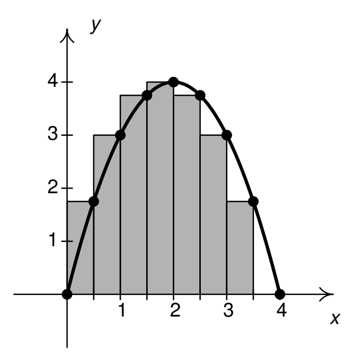
2. To find RS_4, we begin by chopping up the interval $[0,4]$ into 4 equal pieces so each subinterval has length $\Delta x = \frac{4}{4} = 1$ unit. Our right endpoints are: $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$. We find $f(1) = 3$, $f(2) = 4$, $f(3) = 3$, and $f(4) = 0$. Hence,

$$RS_4 = \sum_{k=1}^{4} f(x_k)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x = (3)(1) + (4)(1) + 3(1) + (0)(1) = 10.$$

Geometrically we have approximated the area under the graph of f to be 10 square units by the adding the areas of the shaded rectangles shaded below on the left. (Note that since $f(x_4) = f(4) = 0$, the fourth 'rectangle' has 0 height.)



Visualizing RS_4



Visualizing RS_8

To find RS_8, we divide the interval $[0,4]$ into 8 equal pieces, so each has length $\Delta x = \frac{4}{8} = 0.5$ units. This produces the right endpoints: $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$, $x_5 = 2.5$, $x_6 = 3$, $x_7 = 3.5$, $x_8 = 4$. In addition to the function values we used to compute RS_4, we need $f(0.5) = 1.75$, $f(1.5) = 3.75$, $f(2.5) = 3.75$, and $f(3.5) = 1.75$. Hence,

$$
\begin{aligned}
RS_8 &= \sum_{k=1}^{8} f(x_k)\Delta x \\
&= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\
&\quad + f(x_5)\Delta x + f(x_6)\Delta x + f(x_7)\Delta x + f(x_8)\Delta x \\
&= (1.75)(0.5) + (3)(0.5) + (3.75)(0.5) + (4)(0.5) \\
&\quad + (3.75)(0.5) + (3)(0.5) + (1.75)(0.5) + (0)(0.5) \\
&= 10.5
\end{aligned}
$$

Hence, the area under the graph f is approximately 10.5 square units as approximated by the sum of the rectangles above on the right. (Again, since $f(x_8) = f(4) = 0$, the eighth 'rectangle' has 0 height.)

3. To find a formula for RS_n, we imagine dividing the interval $[0, 4]$ into n equal pieces each of length $\Delta x = \frac{4}{n}$. We have n right endpoints, x_1, x_2, $\ldots x_n$ where $x_k = 0 + k\Delta x = \frac{4k}{n}$. Since $f(x) = 4x - x^2$,

$$f(x_k) = 4x_k - x_k^2 = 4\left(\frac{4k}{n}\right) - \left(\frac{4k}{n}\right)^2 = \frac{16k}{n} - \frac{16k^2}{n^2}.$$

Hence,

$$
\begin{aligned}
RS_n &= \sum_{k=1}^{n} f(x_k)\Delta x \\
&= \sum_{k=1}^{n} \left[\frac{16k}{n} - \frac{16k^2}{n^2}\right]\left(\frac{4}{n}\right) \\
&= \sum_{k=1}^{n} \left[\frac{64k}{n^2} - \frac{64k^2}{n^3}\right] && \text{Distribute the } \frac{4}{n}. \\
&= \sum_{k=1}^{n} \frac{64k}{n^2} - \sum_{k=1}^{n} \frac{64k^2}{n^3} && \text{Sum and Difference Property} \\
&= \frac{64}{n^2}\sum_{k=1}^{n} k - \frac{64}{n^3}\sum_{k=1}^{n} k^2 && \text{Distributive Property[12]} \\
&= \frac{64}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) && \text{Summation Formulas} \\
&= \frac{32(n+1)}{n} - \frac{32(n+1)(2n+1)}{3n^2} = \frac{32n^2 - 32}{3n^2}
\end{aligned}
$$

[12]Note: the counter here is 'k,' not 'n,' so as far as k is concerned, 'n' is a constant so we can factor it out of the summation.

Note we can partially check our answer at thus point by substituting $n = 4$ and $n = 8$ to our formula to RS_n to see if we recover our answers from above. We get $RS_4 = \frac{32(4)^2-32}{3(4^2)} = \frac{480}{48} = 10$ and $RS_8 = \frac{32(8)^2-32}{3(8)^2} = \frac{2016}{192} = 10.5$, as required.

To find $\lim\limits_{n\to\infty} RS_n = \lim\limits_{n\to\infty} \frac{32n^2-32}{3n^2}$, we compare the leading term of the numerator and denominator. As $n \to \infty$, $\frac{32n^2-32}{3n^2} \approx \frac{32n^2}{3n^2} = \frac{32}{3}$. Hence, $\lim\limits_{n\to\infty} RS_n = \frac{32}{3}$. Hence, as we use more and more rectangles,[13] the sum total of the area of those rectangles approaches $\frac{32}{3}$ square units. In Calculus, we more or less **define** the area under f to be $\frac{32}{3}$ square units. □

It is worth noting that, as with other examples in the text, Example 10.2.6 is more or less lifted straight out of a Calculus lecture. That being said, the vast majority of the mechanics here involve precalculus notions.[14] In general, the machinations in Calculus amount to applying the limit concept to the mechanics of precalculus.

[13] even though they become skinnier and skinner and hence, *individually* have smaller and smaller areas . . .

[14] The only Calculus bit is the limit concept which I guess is precalculus now . . .

10.2.3 Exercises

In Exercises 1 - 8, find the value of each sum using Definition 10.4.

1. $\displaystyle\sum_{g=4}^{9} (5g+3)$
2. $\displaystyle\sum_{k=3}^{8} \frac{1}{k}$
3. $\displaystyle\sum_{j=0}^{5} 2^j$
4. $\displaystyle\sum_{k=0}^{2} (3k-5)x^k$
5. $\displaystyle\sum_{i=1}^{4} \frac{1}{4}(i^2+1)$
6. $\displaystyle\sum_{n=1}^{100} (-1)^n$
7. $\displaystyle\sum_{n=1}^{5} \frac{(n+1)!}{n!}$
8. $\displaystyle\sum_{j=1}^{3} \frac{5!}{j!\,(5-j)!}$

In Exercises 9 - 16, rewrite the sum using summation notation.

9. $8+11+14+17+20$
10. $1-2+3-4+5-6+7-8$
11. $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7}$
12. $1+2+4+\cdots+2^{29}$
13. $2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5}$
14. $-\ln(3) + \ln(4) - \ln(5) + \cdots + \ln(20)$
15. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$
16. $\frac{1}{2}(x-5) + \frac{1}{4}(x-5)^2 + \frac{1}{6}(x-5)^3 + \frac{1}{8}(x-5)^4$

In Exercises 17 - 28, use the formulas in Equation 10.2 to find the sum.

17. $\displaystyle\sum_{n=1}^{10} 5n+3$
18. $\displaystyle\sum_{n=1}^{20} 2n-1$
19. $\displaystyle\sum_{k=0}^{15} 3-k$
20. $\displaystyle\sum_{n=1}^{10} \left(\frac{1}{2}\right)^n$
21. $\displaystyle\sum_{n=1}^{5} \left(\frac{3}{2}\right)^n$
22. $\displaystyle\sum_{k=0}^{5} 2\left(\frac{1}{4}\right)^k$
23. $1+4+7+\ldots+295$
24. $4+2+0-2-\ldots-146$
25. $1+3+9+\ldots+2187$
26. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{256}$
27. $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + - \cdots + \frac{3}{256}$
28. $\displaystyle\sum_{n=1}^{10} -2n + \left(\frac{5}{3}\right)^n$

In Exercises 29 - 32, use Theorem 10.5 to find the sum of the given geometric series.[15]

29. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$
30. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, 3^{n-1}}{4^n}$
31. $\displaystyle\sum_{m=2}^{\infty} \frac{3}{2^{m-1}}$
32. $\displaystyle\sum_{k=0}^{\infty} x^k$, $|x| < 1$.

[15] Remember, when in doubt . . .

In Exercises 33 - 36, use Theorem 10.5 to express each repeating decimal as a fraction of integers.

33. $0.\overline{7}$
34. $0.\overline{13}$
35. $10.\overline{159}$
36. $-5.8\overline{67}$

In Exercises 37 - 42, use Equation 10.3 to compute the future value of the annuity with the given terms. In all cases, assume the payment is made monthly, the interest rate given is the annual rate, and interest is compounded monthly.

37. payments are \$300, interest rate is 2.5%, term is 17 years.
38. payments are \$50, interest rate is 1.0%, term is 30 years.
39. payments are \$100, interest rate is 2.0%, term is 20 years
40. payments are \$100, interest rate is 2.0%, term is 25 years
41. payments are \$100, interest rate is 2.0%, term is 30 years
42. payments are \$100, interest rate is 2.0%, term is 35 years
43. Suppose an ordinary annuity offers an annual interest rate of 2%, compounded monthly, for 30 years. What should the monthly payment be to have \$100,000 at the end of the term?
44. In this exercise, we Theorem 10.5 to represent $f(x) = \dfrac{1}{x^2+4}$ as a series.
    (a) Show that $f(x) = \dfrac{\frac{1}{4}}{1 - \left(-\frac{x^2}{4}\right)}$.
    (b) Use the formula in Theorem 10.5: $\dfrac{a}{1-r} = \displaystyle\sum_{k=1}^{\infty} ar^{k-1}$ to write $f(x)$ as an infinite series.
    (c) Graph $y = f(x)$ along with some partial sums of the series. What do you notice?
45. Using Example 10.2.6 as a guide, find and simplify formula for the right endpoint sum, RS_n, for each of the functions below on the specified interval. Find $\displaystyle\lim_{n\to\infty} RS_n$ to find the area between the graph of f and the x-axis.
    (a) $f(x) = 4 - x$ over the interval $[0, 4]$.
    (b) $f(x) = 3x^2$ over the interval $[1, 3]$.
    (c) $f(x) = 12 - x - x^2$ over the interval $[0, 3]$.
46. Prove the properties listed in Theorem 10.4.
47. Show that the formula for the future value of an annuity due is

$$A = P(1+i)\left[\frac{(1+i)^{nt} - 1}{i}\right]$$

48. Discuss with your classmates what goes wrong when trying to find the following sums.[16]

(a) $\displaystyle\sum_{k=1}^{\infty} 2^{k-1}$ (b) $\displaystyle\sum_{k=1}^{\infty} (1.0001)^{k-1}$ (c) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1}$

49. In this exercise, we walk through the proof of Cauchy's Bound, Theorem 2.10 in Section 2.3.

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial of degree n and let Z be the largest zero of f in absolute value and let M be the largest of the numbers: $\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}$.

(a) Since $P(Z) = 0$, solve for Z^n: $Z^n = \frac{a_{n-1}}{a_n} Z^{n-1} + \ldots + \frac{a_1}{a_n} Z + \frac{a_0}{a_n}$.

(b) If $-1 \leq Z \leq 1$, then Cauchy's Bound is immediately satisfied since Z would automatically lie in the interval $[-(M+1), M+1]$. So we assume $|Z| > 1$.

Under the assumption $|Z| > 1$. explain why[17]

$$\begin{aligned} |Z|^n &= \left| \frac{a_{n-1}}{a_n} Z^{n-1} + \ldots + \frac{a_1}{a_n} Z + \frac{a_0}{a_n} \right| \\ &\leq \frac{|a_{n-1}|}{|a_n|} |Z|^{n-1} + \ldots + \frac{|a_1|}{|a_n|} |Z| + \frac{|a_0|}{|a_n|} \end{aligned}$$

(c) Use the definition of M along with the Geometric Sum Formula, Equation 10.2 to show:

$$|Z|^n \leq M\left(|Z|^{n-1} + \ldots + |Z| + 1\right) = M\,\frac{1 - |Z|^n}{1 - |Z|} = M\,\frac{|Z|^n - 1}{|Z| - 1}$$

(d) Now use the fact that $|Z| > 1$ to rearrange the above inequality to get:

$$|Z| - 1 \leq M\,\frac{|Z|^n - 1}{|Z|^n} = M\left(1 - \frac{1}{|Z|^n}\right)$$

(e) Use the fact that $1 - \frac{1}{|Z|^n} < 1$ to get:

$$|Z| - 1 \leq M\left(1 - \frac{1}{|Z|^n}\right) < M(1) = M$$

(f) From $|Z| - 1 < M$, we get $|Z| < M + 1$. Hence, Z lies in the interval $[-(M+1), M+1]$.

[16] When in doubt . . .

[17] Feel free to use the Triangle Inequality, as needed. See Exercise 55 in Section 1.3.

10.2.4 Answers

1. 213
2. $\frac{341}{280}$
3. 63
4. $-5 - 2x + x^2$
5. $\frac{17}{2}$
6. 0
7. 20
8. 25
9. $\displaystyle\sum_{k=1}^{5}(3k+5)$
10. $\displaystyle\sum_{k=1}^{8}(-1)^{k-1}k$
11. $\displaystyle\sum_{k=1}^{4}(-1)^{k-1}\frac{x^{2k-1}}{2k-1}$
12. $\displaystyle\sum_{k=1}^{30}2^{k-1}$
13. $\displaystyle\sum_{k=1}^{5}\frac{k+1}{k}$
14. $\displaystyle\sum_{k=3}^{20}(-1)^{k}\ln(k)$
15. $\displaystyle\sum_{k=1}^{6}\frac{(-1)^{k-1}}{k^2}$
16. $\displaystyle\sum_{k=1}^{4}\frac{1}{2k}(x-5)^k$
17. 305
18. 400
19. -72
20. $\dfrac{1023}{1024}$
21. $\dfrac{633}{32}$
22. $\dfrac{1365}{512}$
23. 14652
24. -5396
25. 3280
26. $\dfrac{255}{256}$
27. $\dfrac{513}{256}$
28. $\dfrac{17771050}{59049}$
29. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=2$
30. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,3^{n-1}}{4^n}=\frac{4}{21}$
31. $\displaystyle\sum_{m=2}^{\infty}\frac{3}{2^{m-1}}=3$
32. $\displaystyle\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}$
33. $\dfrac{7}{9}$
34. $\dfrac{13}{99}$
35. $\dfrac{3383}{333}$
36. $-\dfrac{5809}{990}$
37. \$76,163.67
38. \$20,981.40
39. \$29,479.69
40. \$38,882.12
41. 49,272.55
42. 60,754.80
43. For \$100,000, the monthly payment is $\approx$ \$202.95.
44. (b) $f(x)=\dfrac{\frac{1}{4}}{1-\left(-\frac{x^2}{4}\right)}=\displaystyle\sum_{k=1}^{\infty}\frac{1}{4}\left(-\frac{x^2}{4}\right)^{k-1}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\,x^{2k-2}}{4^k}$

    (c) No matter how many terms are added, the graph of the series seems to only account for a portion of the graph of $y=f(x)$. This is due to the fact that geometric series converge only when the ratio $|r|<1$. In this case, $r=-\frac{x^2}{4}$ so $|r|<1$ corresponds to the interval $(-2,2)$.
45. Using Example 10.2.6 as a guide, find the area between the graph of each function below and the x-axis by evaluating the limit of a right endpoint sum.

    (a) $RS_n = 8-\frac{8}{n}$; Area is 8 units^2
    (b) $RS_n = 26+\frac{24}{n}+\frac{4}{n^2}$; Area is 26 units^2
    (c) $RS_n = \frac{45}{2}-\frac{18}{n}-\frac{9}{2n^2}$; Area is $\frac{45}{2}$ units^2

10.3 Mathematical Induction

The Chinese philosopher Confucius is credited with the saying, “A journey of a thousand miles begins with a single step.” In many ways, this is the central theme of this section. Here we introduce a method of proof, Mathematical Induction, which allows us to *prove* many of the formulas we have merely *motivated* in Sections 10.1 and 10.2 by starting with just a single step. A good example is the formula for arithmetic sequences we touted in Equation 10.1. Arithmetic sequences are defined recursively, starting with $a_1 = a$ and then $a_{n+1} = a_n + d$ for $n \geq 1$. This tells us that we start the sequence with a and we go from one term to the next by successively adding d. In symbols,

$$a, a+d, a+2d, a+3d, a+4d+\ldots$$

The pattern *suggested* here is that to reach the nth term, we start with a and add d to it exactly $n-1$ times, leading to the formula $a_n = a + (n-1)d$ for $n \geq 1$. In order to *prove* this is the case, we have:

The Principle of Mathematical Induction (PMI):
Suppose $P(n)$ is a sentence involving the natural number n.

IF

1. $P(1)$ is true **and**
2. whenever $P(k)$ is true, it follows that $P(k+1)$ is also true

THEN the sentence $P(n)$ is true for all natural numbers n.

The Principle of Mathematical Induction, or PMI for short, is exactly that - a principle.[1] It is a property of the natural numbers we either choose to accept or reject. The notation which is used here, ‘$P(n)$,’ acts just like function notation. For example, if $P(n)$ is the sentence (formula) ‘$n^2 + 1 = 3$’, then $P(1)$ would be ‘$1^2 + 1 = 3$’, which is false. In this case, the construction $P(k+1)$ would be ‘$(k+1)^2 + 1 = 3$’.

In English, the PMI says that if we want to prove that a formula works for all natural numbers n, we start by showing it is true for $n = 1$ (the ‘*base step*’) and then show that *if* it is true for a generic natural number k, *then* it must be true for the next natural number, $k + 1$ (the ‘*inductive step*’). In essence, by showing that $P(k+1)$ must always be true when $P(k)$ is true, we are showing that the formula $P(1)$ can be used to get the formula $P(2)$, which in turn can be used to derive the formula $P(3)$, which in turn can be used to establish the formula $P(4)$, and so on, for all natural numbers n.

One might liken Mathematical Induction to a repetitive process like climbing stairs.[2] If you are sure that (1) you can get on the stairs (the base case) and (2) you can climb from any one step to the next step (the inductive step), then presumably you can climb the entire staircase.[3] We get some more practice with induction in the following example.

[1]Another word for this you may have seen is ‘axiom.’
[2]Falling dominoes is the most widely used metaphor in the mainstream College Algebra books.
[3]This is how Carl climbed the stairs in the Cologne Cathedral. Well, that, and encouragement from Kai.

Example 10.3.1. Prove the following assertions using the Principle of Mathematical Induction.

1. If $a_1 = 4$ and $a_{n+1} = -\dfrac{a_n}{2}$ for $n \geq 1$, then prove $a_n = (-1)^{n-1}2^{3-n}$ for $n \geq 1$.
2. $1 + 3 + 5 + \ldots + (2n-1) = n^2$
3. $3^n > 100n$ for $n > 5$.

Solution.

1. To prove $a_n = (-1)^{n-1}2^{3-n}$ for $n \geq 1$ by induction, we first identify the sentence $P(n)$ as the equation $a_n = (-1)^{n-1}2^{3-n}$. The sentence $P(1)$ is the equation $a_1 = (-1)^{1-1}2^{3-1}$ or, after simplifying, $a_1 = 4$, which we are told is true.

   Next, we *assume* the sentence $P(k)$ is true, that is, $a_k = (-1)^{k-1}2^{3-k}$ (this is called the '*induction hypothesis*') and must use this to *deduce* $P(k+1)$ is true. That is, we need to use the fact that $a_k = (-1)^{k-1}2^{3-k}$ to show $a_{k+1} = (-1)^{(k+1)-1}2^{3-(k+1)}$ or, after simplifying, $a_{k+1} = (-1)^k 2^{2-k}$.

   We are told $a_{k+1} = -\frac{a_k}{2}$ and we are assuming $a_k = (-1)^{k-1}2^{3-k}$, so we put these together to get

   $$a_{k+1} = -\frac{a_k}{2} = -\frac{(-1)^{k-1}2^{3-k}}{2} = (-1)^1\frac{(-1)^{k-1}2^{3-k}}{2^1} = (-1)^{k-1+1}2^{3-k+1} = (-1)^k 2^{2-k},$$

   as required. Hence, by induction, $a_n = (-1)^{n-1}2^{3-n}$ for $n \geq 1$.

   We take a moment and recognize the sequence here, as described, is a geometric sequence with $a = 4$ and $r = -\frac{1}{2}$. Using Equation 10.1 we arrive at the explicit formula for $a_n = 4\left(-\frac{1}{2}\right)^{n-1}$ for $n \geq 1$ which we leave to the reader to show reduces to $a_n = (-1)^{n-1}2^{3-n}$. (Note: You'll be asked to prove Equation 10.1 in Exercise 8a.)

2. As above, our first step is to identify the sentence $P(n)$ which is the equation $1+3+5+\ldots+(2n-1) = n^2$ which is more precisely written using summation notation: $\displaystyle\sum_{j=1}^{n}(2j-1) = n^2$. (Note we use '$j$' as our dummy variable here since 'n' is already used and we usually reserve 'k' for the induction variable.)

   The sentence $P(1)$ is $\displaystyle\sum_{j=1}^{1}(2j-1) = 1^2$ which reduces to $2(1) - 1 = 1$ which is true. Next, we assume $P(k)$ is true, $\displaystyle\sum_{j=1}^{k}(2j-1) = k^2$, and use it to show $P(k+1)$ is true: $\displaystyle\sum_{j=1}^{k+1}(2j-1) = (k+1)^2$. We have:

   $$\underbrace{\sum_{j=1}^{k+1}(2j-1)}_{\text{adding } k+1 \text{ terms}} = \underbrace{\sum_{j=1}^{k}(2j-1)}_{\text{adding the first } k \text{ terms}} + \underbrace{(2(k+1)-1)}_{\text{adding the first } k+1 \text{ term}} = \underbrace{k^2}_{P(k)} \underbrace{+2k+1}_{\text{simplify}} = \underbrace{(k+1)^2}_{\text{factor}},$$

   as required. Hence, by induction, $1 + 3 + 5 + \ldots + (2n-1) = n^2$ for all natural numbers $n \geq 0$.

As with the first example, this problem, too, can be shown using a previous result. The sequence being added in the equation $1 + 3 + 5 + ... + (2n - 1) = n^2$ is arithmetic, so Equation 10.2 applies to give the sum as $\frac{n}{2}(1 + (2n - 1)) = n^2$. We'll prove Equation 10.2 for arithmetic sequences in the next example. We leave the case for geometric sequences to the reader in Exercise 8b.

3. The first wrinkle we encounter in this problem is that we are asked to prove this formula for $n > 5$ instead of $n \geq 1$. Since n is a natural number, this means our base step occurs at $n = 6$. We can still use the PMI in this case, but our conclusion will be that the formula is valid for all $n \geq 6$.

   We let $P(n)$ be the inequality $3^n > 100n$, and check that $P(6)$ is true. Comparing $3^6 = 729$ and $100(6) = 600$, we see $3^6 > 100(6)$ as required.

   Next, we assume that $P(k)$ is true, that is we assume $3^k > 100k$. We need to show that $P(k + 1)$ is true, that is, we need to show $3^{k+1} > 100(k + 1)$. Since $3^{k+1} = 3 \cdot 3^k$, the induction hypothesis gives $3^{k+1} = 3 \cdot 3^k > 3(100k) = 300k$.

   To complete the proof, we need to show $300k > 100(k + 1)$ for $k \geq 6$. Solving $300k > 100(k + 1)$ we get $k > \frac{1}{2}$. Since $k \geq 6$, we know this is true.

   Putting all of this together, we have $3^{k+1} = 3 \cdot 3^k > 3(100k) = 300k > 100(k+1)$, and hence $P(k+1)$ is true. By induction, $3^n > 100n$ for all $n \geq 6$. $\square$

One of the things that may seem troubling about proving statements by induction is the induction hypothesis: that is, assuming that $P(k)$ is true. After all, isn't that what we are trying to prove? When we assume $P(k)$ is true, we are doing so with the *express purpose* of showing that $P(k + 1)$ follows. That is, we are interested in showing *how* we go 'from one step to the next.'

As mentioned at the beginning of this section, induction is the formal way to prove many the formulas we've used in Sections 10.1 and 10.2. Indeed, now that we have some experience using the PMI to prove formulas, we return to proving the formula for an arithmetic sequence.

Recall we define an arithmetic sequence recursively as: $a_1 = a$ and $a_{n+1} = a_n + d$ for $n \geq 1$. We need to prove $a_n = a + (n - 1)d$ for $n \geq 1$. Identifying $P(n)$ as the formula $a_n = a + (n - 1)d$, we see $P(1)$ is $a_1 = a + (1 - 1)d = a$, which is true.

Next, we assume $P(k)$ is true, that is, $a_k = a+(k-1)d$ and use this to show $P(k+1)$, or $a_{k+1} = a+((k+1)-1)d$ or $a_{k+1} = a + kd$ is true. We know $a_{k+1} = a_k + d$ from the definition of arithmetic sequence, hence

$$a_{k+1} = a_k + d = a + (k - 1)d + d = a + kd,$$

as required. Hence, $a_n = a + (n - 1)d$, for all natural numbers $n \geq 1$.

We conclude this section with three more proofs by induction.

Example 10.3.2. Prove the following assertions using the Principle of Mathematical Induction.

1. The sum formula for arithmetic sequences: $\displaystyle\sum_{j=1}^{n}(a+(j-1)d) = \frac{n}{2}(2a+(n-1)d)$.
2. For a complex number z, $(\overline{z})^n = \overline{z^n}$ for $n \geq 1$.
3. Let A be an $n \times n$ matrix and let A' be the matrix obtained by replacing a row R of A with cR for some real number c. Use the definition of determinant to show $\det(A') = c\det(A)$.

Solution.

1. We set $P(n)$ to be the equation we are asked to prove, namely $\displaystyle\sum_{j=1}^{n}(a+(j-1)d) = \frac{n}{2}(2a+(n-1)d)$. The statement $P(1)$, $\displaystyle\sum_{j=1}^{1}(a+(j-1)d) = \frac{1}{2}(2a+(1-1)d)$, reduces to $a+(0)d = \frac{1}{2}(2a)$ or $a = a$, which is true. Next we assume $P(k)$ is true, that is, we assume $\displaystyle\sum_{j=1}^{k}(a+(j-1)d) = \frac{k}{2}(2a+(k-1)d)$ and use this to show $P(k+1)$ is true: $\displaystyle\sum_{j=1}^{k+1}(a+(j-1)d) = \frac{k+1}{2}(2a+(k+1-1)d) = \frac{k+1}{2}(2a+kd)$:

$$\underbrace{\sum_{j=1}^{k+1}(a+(j-1)d)}_{\text{adding } k+1 \text{ terms}} = \underbrace{\sum_{j=1}^{k}(a+(j-1)d)}_{\text{adding the first } k \text{ terms}} + \underbrace{(a+((k+1)-1)d)}_{\text{adding the first } k+1 \text{ term}} = \underbrace{\frac{k}{2}(2a+(k-1)d)}_{P(k)} + \underbrace{a+kd}_{\text{simplify}}.$$

   We leave it to the reader to show that, indeed,

$$\frac{k}{2}(2a+(k-1)d) + a + kd = \frac{k+1}{2}(2a+d),$$

   which completes the proof that $P(k+1)$ is true. By induction, $\displaystyle\sum_{j=1}^{n}(a+(j-1)d) = \frac{n}{2}(2a+(n-1)d)$ for all natural numbers n.

2. We let $P(n)$ be the equation $(\overline{z})^n = \overline{z^n}$. The base case $P(1)$ is $(\overline{z})^1 = \overline{z^1}$ reduces to $\overline{z} = \overline{z}$ which is true. We now assume $P(k)$ is true, that is, we assume $(\overline{z})^k = \overline{z^k}$ and use this to show that $P(k+1)$ is true, namely $(\overline{z})^{k+1} = \overline{z^{k+1}}$.

   Since $(\overline{z})^{k+1} = (\overline{z})^k\ \overline{z}$, we can use the induction hypothesis to write $(\overline{z})^k = \overline{z^k}$. Hence,

$$(\overline{z})^{k+1} = (\overline{z})^k\ \overline{z} = \overline{z^k}\,\overline{z} = \overline{z^k z} = \overline{z^{k+1}},$$

   where the second-to-last equality is courtesy of the product rule for conjugates[4] This shows $P(k+1)$ is true and hence, by induction, $(\overline{z})^n = \overline{z^n}$ for all natural numbers n.

[4]See Exercise 47 in Section 2.4: $\overline{z}\,\overline{w} = \overline{zw}$.

3. To prove this determinant property, we use induction on n, where we take $P(n)$ to be that the property we wish to prove is true for all $n \times n$ matrices. For the base case, we note that if A is a 1×1 matrix, then $A = [a]$ so $A' = [ca]$. By definition, $\det(A) = a$ and $\det(A') = ca$ so we have $\det(A') = c\det(A)$.

   Now suppose that the property we wish to prove is true for all $k \times k$ matrices. Let A be a $(k+1)\times(k+1)$ matrix. We have two cases, depending on if the row R being replaced is the first row of A.

   CASE 1: The row R being replaced is the first row of A. By definition,

$$\det(A') = \sum_{p=1}^{n} a'_{1p}C'_{1p}$$

   where the $1p$ cofactor of A' is $C'_{1p} = (-1)^{(1+p)}\det\left(A'_{1p}\right)$ and A'_{1p} is the $k \times k$ matrix obtained by deleting the 1st row and pth column of A'.[5]

   Since the first row of A' is c times the first row of A, we have $a'_{1p} = c\,a_{1p}$. In addition, since the remaining rows of A' are identical to those of A, $A'_{1p} = A_{1p}$. (To obtain these matrices, the first row of A' is removed.) Hence $\det\left(A'_{1p}\right) = \det\left(A_{1p}\right)$, so that $C'_{1p} = C_{1p}$. As a result, we get

$$\det(A') = \sum_{p=1}^{n} a'_{1p}C'_{1p} = \sum_{p=1}^{n} c\,a_{1p}C_{1p} = c\sum_{p=1}^{n} a_{1p}C_{1p} = c\det(A),$$

   as required. Hence, $P(k+1)$ is true in this case, which means the result is true in this case for all natural numbers $n \geq 1$. (You'll note that we did not use the induction hypothesis at all in this case. It is possible to restructure the proof so that induction is only used where it is needed. While mathematically more elegant, it is less intuitive.)

   CASE 2: The row R being replaced is the not the first row of A. By definition,

$$\det(A') = \sum_{p=1}^{n} a'_{1p}C'_{1p},$$

   where in this case, $a'_{1p} = a_{1p}$, since the first rows of A and A' are the same. The matrices A'_{1p} and A_{1p}, on the other hand, are different but in a very predictable way $-$ the row in A'_{1p} which corresponds to the row cR in A' is exactly c times the row in A_{1p} which corresponds to the row R in A.

   This means A'_{1p} and A_{1p} are $k \times k$ matrices which satisfy the induction hypothesis. Hence, we know $\det\left(A'_{1p}\right) = c\det\left(A_{1p}\right)$ and $C'_{1p} = c\,C_{1p}$. We get

$$\det(A') = \sum_{p=1}^{n} a'_{1p}C'_{1p} = \sum_{p=1}^{n} a_{1p}c\,C_{1p} = c\sum_{p=1}^{n} a_{1p}C_{1p} = c\det(A),$$

[5] See Section 9.5 for a review of this notation.

which establishes $P(k+1)$ to be true. Hence by induction, we have shown that the result holds in this case for $n \geq 1$ and we are done. $\square$

While we have used the Principle of Mathematical Induction to prove some of the formulas we have merely motivated in the text, our main use of this result comes in Section 10.4 to prove the celebrated Binomial Theorem. The ardent Mathematics student will no doubt see the PMI in many courses yet to come. Sometimes it is explicitly stated and sometimes it remains hidden in the background. If ever you see a property stated as being true 'for all natural numbers n', it's a solid bet that the formal proof requires the Principle of Mathematical Induction.

10.3.1 Exercises

In Exercises 1 - 7, prove each assertion using the Principle of Mathematical Induction.

1. $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

2. $\displaystyle\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$

3. $2^n > 500n$ for $n > 12$

4. $3^n \geq n^3$ for $n \geq 4$

5. Use the Product Rule for Absolute Value to show $|x^n| = |x|^n$ for all real numbers x and all natural numbers $n \geq 1$

6. Use the Product Rule for Logarithms to show $\log(x^n) = n\log(x)$ for all real numbers $x > 0$ and all natural numbers $n \geq 1$.

7. $\left[\begin{array}{cc} a & 0 \\ 0 & b \end{array}\right]^n = \left[\begin{array}{cc} a^n & 0 \\ 0 & b^n \end{array}\right]$ for $n \geq 1$.

8. Prove Equations 10.1 and 10.2 for the case of geometric sequences. That is:

   (a) For the sequence $a_1 = a$, $a_{n+1} = ra_n$, $n \geq 1$, prove $a_n = ar^{n-1}$, $n \geq 1$.

   (b) $\displaystyle\sum_{j=1}^{n} ar^{j-1} = a\left(\frac{1-r^n}{1-r}\right)$, if $r \neq 1$, $\displaystyle\sum_{j=1}^{n} ar^{j-1} = na$, if $r = 1$.

9. Prove that the determinant of a lower triangular matrix is the product of the entries on the main diagonal. (See Exercise 9.3.1 in Section 9.3.) Use this result to then show $\det(I_n) = 1$ where I_n is the $n \times n$ identity matrix.

10. Prove the Power Rule for Limits (see Theorem 6.2 in Section **??**): $\displaystyle\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n$, where n is any natural number.

11. Discuss the classic 'paradox' All Horses are the Same Color problem with your classmates.

10.3.2 Selected Answers

1. Let $P(n)$ be the sentence $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$. For the base case, $n = 1$, we get

$$\begin{aligned} \sum_{j=1}^{1} j^2 &\stackrel{?}{=} \frac{(1)(1+1)(2(1)+1)}{6} \\ 1^2 &= 1 \checkmark \end{aligned}$$

We now assume $P(k)$ is true and use it to show $P(k+1)$ is true. We have

$$\begin{aligned} \sum_{j=1}^{k+1} j^2 &\stackrel{?}{=} \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\ \sum_{j=1}^{k} j^2 + (k+1)^2 &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \underbrace{\frac{k(k+1)(2k+1)}{6}}_{\text{Using } P(k)} + (k+1)^2 &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)(k(2k+1) + 6(k+1))}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)\left(2k^2+7k+6\right)}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)(k+2)(2k+3)}{6} &= \frac{(k+1)(k+2)(2k+3)}{6} \checkmark \end{aligned}$$

By induction, $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all natural numbers $n \geq 1$.

4. Let $P(n)$ be the sentence $3^n > n^3$. Our base case is $n = 4$ and we check $3^4 = 81$ and $4^3 = 64$ so that $3^4 > 4^3$ as required. We now assume $P(k)$ is true, that is $3^k > k^3$, and try to show $P(k+1)$ is true. We note that $3^{k+1} = 3 \cdot 3^k > 3k^3$ and so we are done if we can show $3k^3 > (k+1)^3$ for $k \geq 4$. We can solve the inequality $3x^3 > (x+1)^3$ using the techniques of Section 4.1, and doing so gives us $x > \frac{1}{\sqrt[3]{3}-1} \approx 2.26$. Hence, for $k \geq 4$, $3^{k+1} = 3 \cdot 3^k > 3k^3 > (k+1)^3$ so that $3^{k+1} > (k+1)^3$. By induction, $3^n > n^3$ is true for all natural numbers $n \geq 4$.

6. Let $P(n)$ be the sentence $\log\left(x^n\right) = n\log(x)$. For the duration of this argument, we assume $x > 0$. The base case $P(1)$ amounts checking that $\log\left(x^1\right) = 1\log(x)$ which is clearly true. Next we assume $P(k)$ is true, that is $\log\left(x^k\right) = k\log(x)$ and try to show $P(k+1)$ is true. Using the Product Rule for Logarithms along with the induction hypothesis, we get

$$\log\left(x^{k+1}\right) = \log\left(x^k \cdot x\right) = \log\left(x^k\right) + \log(x) = k\log(x) + \log(x) = (k+1)\log(x)$$

Hence, $\log\left(x^{k+1}\right) = (k+1)\log(x)$. By induction $\log\left(x^n\right) = n\log(x)$ is true for all $x > 0$ and all natural numbers $n \geq 1$.

9. Let A be an $n \times n$ lower triangular matrix. We proceed to prove the $\det(A)$ is the product of the entries along the main diagonal by inducting on n. For $n = 1$, $A = [a]$ and $\det(A) = a$, so the result is (trivially) true. Next suppose the result is true for $k \times k$ lower triangular matrices. Let A be a $(k+1) \times (k+1)$ lower triangular matrix. Expanding $\det(A)$ along the first row, we have

$$\det(A) = \sum_{p=1}^{n} a_{1p}C_{1p}$$

Since $a_{1p} = 0$ for $2 \leq p \leq k+1$, this simplifies $\det(A) = a_{11}C_{11}$. By definition, we know that $C_{11} = (-1)^{1+1}\det\left(A_{11}\right) = \det\left(A_{11}\right)$ where A_{11} is $k \times k$ matrix obtained by deleting the first row and first column of A. Since A is lower triangular, so is A_{11} and, as such, the induction hypothesis applies to A_{11}. In other words, $\det\left(A_{11}\right)$ is the product of the entries along A_{11}'s main diagonal. Now, the entries on the main diagonal of A_{11} are the entries $a_{22}, a_{33}, \ldots, a_{(k+1)(k+1)}$ from the main diagonal of A. Hence,

$$\det(A) = a_{11}\det\left(A_{11}\right) = a_{11}\left(a_{22}a_{33}\cdots a_{(k+1)(k+1)}\right) = a_{11}a_{22}a_{33}\cdots a_{(k+1)(k+1)}$$

We have $\det(A)$ is the product of the entries along its main diagonal. This shows $P(k+1)$ is true, and, hence, by induction, the result holds for all $n \times n$ upper triangular matrices. The $n \times n$ identity matrix I_n is a lower triangular matrix whose main diagonal consists of all 1's. Hence, $\det\left(I_n\right) = 1$, as required.

10.4 The Binomial Theorem

In this section, we aim to prove the celebrated *Binomial Theorem*. Simply stated, the Binomial Theorem is a formula for the expansion of quantities $(a+b)^n$ for natural numbers n. In High School Algebra, you probably have seen specific instances of the formula, namely

$$\begin{array}{rcl} (a+b)^1 & = & a+b \\ (a+b)^2 & = & a^2+2ab+b^2 \\ (a+b)^3 & = & a^3+3a^2b+3ab^2+b^3 \end{array}$$

If we wanted the expansion for $(a+b)^4$ we would write $(a+b)^4 = (a+b)(a+b)^3$ and use the formula that we have for $(a+b)^3$ to get $(a+b)^4 = (a+b)\left(a^3+3a^2b+3ab^2+b^3\right) = a^4+4a^3b+6a^2b^2+4ab^3+b^4$. Generalizing this a bit, we see that if we have a formula for $(a+b)^k$, we can obtain a formula for $(a+b)^{k+1}$ by rewriting the latter as $(a+b)^{k+1} = (a+b)(a+b)^k$. Clearly this means Mathematical Induction plays a major role in the proof of the Binomial Theorem.[1] Before we can state the theorem we need to revisit the sequence of factorials which were introduced in Example 10.1.1 number 6 in Section 10.1.

Definition 10.5. Factorials:
For a whole number n, ***n* factorial**, denoted $n!$, is the term f_n of the sequence:

$$f_0 = 1,\, f_n = n \cdot f_{n-1}, \quad n \geq 1.$$

Recall this means $0! = 1$ and $n! = n(n-1)!$ for $n \geq 1$. Hence, $1! = 1 \cdot 0! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. Informally, $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ with $0! = 1$ as our 'base case.' Our first example familiarizes us with some of the basics of factorial computations.

Example 10.4.1.

1. Simplify the following expressions.

   (a) $\dfrac{3!\,2!}{0!}$ (b) $\dfrac{7!}{5!}$ (c) $\dfrac{1000!}{998!\,2!}$ (d) $\dfrac{(k+2)!}{(k-1)!}$, $k \geq 1$

2. Prove $n! > 3^n$ for all $n \geq 7$.

Solution.

1. We keep in mind the mantra, "When in doubt, write it out!" as we simplify the following.

   (a) Recall $0! = 1$, by definition, $3! = 3 \cdot 2 \cdot 1 = 6$ and $2! = 2 \cdot 1 = 2$. Hence, $\frac{3!\,2!}{0!} = \frac{(6)(2)}{1} = 12$.

   (b) We have $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ so $\frac{7!}{5!} = \frac{5040}{120} = 42$.

   While this is correct, we note that we could have saved ourselves some of time had we approached the problem as follows:

[1] It's pretty much the reason Section 10.3 is in the book.

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 = 42$$

In fact, should we want to fully exploit the recursive nature of the factorial, we can write

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5!}{5!} = \frac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42$$

(c) Keeping in mind the lesson we learned from the previous problem, we have

$$\frac{1000!}{998!\,2!} = \frac{1000 \cdot 999 \cdot 998!}{998! \cdot 2!} = \frac{1000 \cdot 999 \cdot \cancel{998!}}{\cancel{998!} \cdot 2!} = \frac{999000}{2} = 499500$$

(d) This problem continues the theme which we have seen in the previous two problems. We first note that since $k+2$ is larger than $k-1$, $(k+2)!$ contains all of the factors of $(k-1)!$ and as a result we can get the $(k-1)!$ to cancel from the denominator.

To see this, we begin by writing out $(k+2)!$ starting with $(k+2)$ and multiplying it by the numbers which precede it until we reach $(k-1)$: $(k+2)! = (k+2)(k+1)(k)(k-1)!$. As a result, we have

$$\frac{(k+2)!}{(k-1)!} = \frac{(k+2)(k+1)(k)(k-1)!}{(k-1)!} = \frac{(k+2)(k+1)(k)\cancel{(k-1)!}}{\cancel{(k-1)!}} = k(k+1)(k+2)$$

The stipulation $k \geq 1$ is there to ensure that all of the factorials involved are defined.

2. We proceed by induction and let $P(n)$ be the inequality $n! > 3^n$. The base case here is $n = 7$ and we see that $7! = 5040$ is larger than $3^7 = 2187$, so $P(7)$ is true.

   Next, we assume that $P(k)$ is true, that is, we assume $k! > 3^k$ and attempt to show $P(k+1)$ follows. Using the properties of the factorial, we have $(k+1)! = (k+1)k!$ and since $k! > 3^k$, we have $(k+1)! > (k+1)3^k$. Since $k \geq 7$, $k+1 \geq 8$, so $(k+1)3^k \geq 8 \cdot 3^k > 3 \cdot 3^k = 3^{k+1}$.

   Putting all of this together, we have $(k+1)! = (k+1)k! > (k+1)3^k > 3^{k+1}$ which shows $P(k+1)$ is true. By the Principle of Mathematical Induction, we have $n! > 3^n$ for all $n \geq 7$. $\square$

Of all of the mathematical functions we have discussed in the text, factorials grow most quickly. In Example 10.4.1 above, we proved that $n!$ overtakes 3^n at $n = 7$. 'Overtakes' may be too polite a word, since $n!$ thoroughly trounces 3^n for $n \geq 7$, as any reasonable set of data will show.

It can be shown that for any real number $x > 0$, not only does $n!$ eventually overtake x^n, but the ratio $\frac{x^n}{n!} \to 0$ as $n \to \infty$. (This is extremely important for Calculus.)

Applications of factorials in the wild often involve counting arrangements. For example, if you have fifty songs on your mp3 player and wish arrange these songs in a playlist in which the order of the songs matters, it turns out that there are 50! different possible playlists.

If you wish to select only ten of the songs to create a playlist, then there are $\frac{50!}{40!}$ such playlists. If, on the other hand, you just want to select ten song files out of the fifty to put on a flash memory card so that now the order no longer matters, there are $\frac{50!}{40!10!}$ ways to achieve this.[2]

While some of these ideas are explored in the Exercises, the authors encourage you to take courses such as Finite Mathematics, Discrete Mathematics and Statistics. We introduce these concepts here because this is how the factorials make their way into the Binomial Theorem, as our next definition indicates.

Definition 10.6. Binomial Coefficients: Given two whole numbers n and j with $n \geq j$, the binomial coefficient $\boldsymbol{\binom{n}{j}}$ (read, 'n choose j') is the whole number given by

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

The name 'binomial coefficient' will be justified shortly. For now, we can physically interpret $\binom{n}{j}$ as the number of ways to select j items from n items where the order of the items selected is unimportant. For example, suppose you won two free tickets to a special screening of the latest Hollywood blockbuster and have five good friends each of whom would love to accompany you to the movies. There are $\binom{5}{2}$ ways to choose who goes with you. Applying Definition 10.6, we get

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10$$

So there are 10 different ways to distribute those two tickets among five friends. (Some will see it as 10 ways to decide which three friends have to stay home.) The reader is encouraged to verify this by actually taking the time to list all of the possibilities.

We now state and prove a theorem which is crucial to the proof of the Binomial Theorem.

Theorem 10.6. For natural numbers n and j with $n \geq j$,

$$\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$$

The proof of Theorem 10.6 is purely computational and uses the definition of binomial coefficients, the recursive property of factorials and common denominators.

[2]For reference,

$$\begin{aligned} 50! &= 30414093201713378043612608166064768844377641568960512000000000000, \\ \frac{50!}{40!} &= 37276043023296000, \quad \text{and} \\ \frac{50!}{40!10!} &= 10272278170 \end{aligned}$$

$$
\begin{aligned}
\binom{n}{j-1} + \binom{n}{j} &= \frac{n!}{(j-1)!(n-(j-1))!} + \frac{n!}{j!(n-j)!} \\
&= \frac{n!}{(j-1)!(n-j+1)!} + \frac{n!}{j!(n-j)!} \\
&= \frac{n!}{(j-1)!(n-j+1)(n-j)!} + \frac{n!}{j(j-1)!(n-j)!} \\
&= \frac{n!\,j}{j(j-1)!(n-j+1)(n-j)!} + \frac{n!(n-j+1)}{j(j-1)!(n-j+1)(n-j)!} \\
&= \frac{n!\,j}{j!(n-j+1)!} + \frac{n!(n-j+1)}{j!(n-j+1)!} \\
&= \frac{n!\,j + n!(n-j+1)}{j!(n-j+1)!} \\
&= \frac{n!\,(j+(n-j+1))}{j!(n-j+1)!} \\
&= \frac{(n+1)n!}{j!(n+1-j))!} \\
&= \frac{(n+1)!}{j!((n+1)-j))!} \\
&= \binom{n+1}{j} \checkmark
\end{aligned}
$$

We are now in position to state and prove the Binomial Theorem where we see that binomial coefficients are just that - coefficients in the binomial expansion.

Theorem 10.7. Binomial Theorem: For nonzero real numbers a and b,

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$$

for all natural numbers n.

To get a feel of what this theorem is saying and how it really isn't as hard to remember as it may first appear, let's consider the specific case of $n = 4$. According to the theorem, we have

$$\begin{aligned}
(a+b)^4 &= \sum_{j=0}^{4} \binom{4}{j} a^{4-j} b^j \\
&= \binom{4}{0} a^{4-0} b^0 + \binom{4}{1} a^{4-1} b^1 + \binom{4}{2} a^{4-2} b^2 + \binom{4}{3} a^{4-3} b^3 + \binom{4}{4} a^{4-4} b^4 \\
&= \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} a b^3 + \binom{4}{4} b^4
\end{aligned}$$

We forgo the simplification of the coefficients in order to note the pattern in the expansion. First note that in each term, the total of the exponents is 4 which matched the exponent of the binomial $(a+b)^4$. The exponent on a begins at 4 and decreases by one as we move from one term to the next while the exponent on b starts at 0 and increases by one each time.

Also note that the binomial coefficients themselves have a pattern. The upper number, 4, matches the exponent on the binomial $(a+b)^4$ whereas the lower number changes from term to term and matches the exponent of b in that term.

This is no coincidence and corresponds to the kind of counting we discussed earlier. If we think of obtaining $(a+b)^4$ by multiplying $(a+b)(a+b)(a+b)(a+b)$, our answer is the sum of all possible products with exactly four factors - some a, some b. If we wish to count, for instance, the number of ways we obtain 1 factor of b out of a total of 4 possible factors, thereby forcing the remaining 3 factors to be a, the answer is $\binom{4}{1}$. Hence, the term $\binom{4}{1}a^3b$ is in the expansion. The other terms which appear cover the remaining cases.

While the foregoing discussion gives an indication as to *why* the theorem is true, a formal proof requires Mathematical Induction.[3]

To prove the Binomial Theorem, we let $P(n)$ be the expansion formula given in the statement of the theorem and we note that $P(1)$ is true since

$$\sum_{j=0}^{1} \binom{1}{j} a^{1-j} b^j = \binom{1}{0} a^{1-0} b^0 + \binom{1}{1} a^{1-1} b^1 = a + b = (a+b)^1.$$

Now we assume that $P(k)$ is true. That is, we assume that we can expand $(a+b)^k$ using the formula given in Theorem 10.7 and attempt to show that $P(k+1)$ is true.

[3]and a fair amount of tenacity and attention to detail.

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\
&= (a+b)\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\
&= a\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j + b\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\
&= \sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1}
\end{aligned}$$

Our goal is to combine as many of the terms as possible within the two summations.

As the counter j in the first summation runs from 0 through k, we get terms involving a^{k+1}, $a^k b$, $a^{k-1}b^2$, $\ldots$, ab^k. In the second summation, we get terms involving $a^k b$, $a^{k-1}b^2$, $\ldots$, ab^k, b^{k+1}. In other words, apart from the first term in the first summation and the last term in the second summation, we have terms common to both summations.

Our next move is to 'kick out' the terms which we cannot combine and rewrite the summations so that we can combine them. To that end, we note

$$\sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j = a^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j$$

and

$$\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1} = \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} + b^{k+1}$$

so that

$$(a+b)^{k+1} = a^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} + b^{k+1}$$

We now wish to write

$$\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1}$$

as a single summation. The wrinkle is that the first summation starts with $j = 1$, while the second starts with $j = 0$. Even though the sums produce terms with the same powers of a and b, they do so for different values of j. To resolve this, we need to shift the index on the second summation so that the index j starts at $j = 1$ instead of $j = 0$ and we make use of Theorem 10.4 in the process.

$$\begin{aligned}
\sum_{j=0}^{k-1} \binom{k}{j} a^{k-j} b^{j+1} &= \sum_{j=0+1}^{k-1+1} \binom{k}{j-1} a^{k-(j-1)} b^{(j-1)+1} \\
&= \sum_{j=1}^{k} \binom{k}{j-1} a^{k+1-j} b^{j}
\end{aligned}$$

We can now combine our two sums using Theorem 10.4 and simplify using Theorem 10.6

$$\begin{aligned}
\sum_{j=1}^{k} \binom{k}{j} a^{k+1-j} b^{j} + \sum_{j=0}^{k-1} \binom{k}{j} a^{k-j} b^{j+1} &= \sum_{j=1}^{k} \binom{k}{j} a^{k+1-j} b^{j} + \sum_{j=1}^{k} \binom{k}{j-1} a^{k+1-j} b^{j} \\
&= \sum_{j=1}^{k} \left[\binom{k}{j} + \binom{k}{j-1} \right] a^{k+1-j} b^{j} \\
&= \sum_{j=1}^{k} \binom{k+1}{j} a^{k+1-j} b^{j}
\end{aligned}$$

Using this and the fact that $\binom{k+1}{0} = 1$ and $\binom{k+1}{k+1} = 1$, we get

$$\begin{aligned}
(a+b)^{k+1} &= a^{k+1} + \sum_{j=1}^{k} \binom{k+1}{j} a^{k+1-j} b^{j} + b^{k+1} \\
&= \binom{k+1}{0} a^{k+1} b^{0} + \sum_{j=1}^{k} \binom{k+1}{j} a^{k+1-j} b^{j} + \binom{k+1}{k+1} a^{0} b^{k+1} \\
&= \sum_{j=0}^{k+1} \binom{k+1}{j} a^{(k+1)-j} b^{j}
\end{aligned}$$

which shows that $P(k+1)$ is true. Hence, by induction, we have established that the Binomial Theorem holds for all natural numbers n.

Example 10.4.2. Use the Binomial Theorem to find the following.

1. $(x-2)^4$
2. 2.1^3
3. The term containing x^3 in the expansion $(2x+y)^5$

Solution.

1. Since $(x-2)^4 = (x+(-2))^4$, we identify $a = x$, $b = -2$ and $n = 4$ and obtain

$$
\begin{aligned}
(x-2)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j}(-2)^j \\
&= \binom{4}{0} x^{4-0}(-2)^0 + \binom{4}{1} x^{4-1}(-2)^1 + \binom{4}{2} x^{4-2}(-2)^2 + \binom{4}{3} x^{4-3}(-2)^3 + \binom{4}{4} x^{4-4}(-2)^4 \\
&= x^4 - 8x^3 + 24x^2 - 32x + 16
\end{aligned}
$$

2. At first this problem seem misplaced, but we can write $2.1^3 = (2+0.1)^3$. Identifying $a = 2$, $b = 0.1$ and $n = 3$, we get

$$
\begin{aligned}
(2+0.1)^3 &= \sum_{j=0}^{3} \binom{3}{j} 2^{3-j}(0.1)^j \\
&= \binom{3}{0} 2^{3-0}(0.1)^0 + \binom{3}{1} 2^{3-1}(0.1)^1 + \binom{3}{2} 2^{3-2}(0.1)^2 + \binom{3}{3} 2^{3-3}(0.1)^3 \\
&= 8 + 1.2 + 0.06 + 0.001 \\
&= 9.261
\end{aligned}
$$

3. Identifying $a = 2x$, $b = y$ and $n = 5$, the Binomial Theorem gives

$$(2x+y)^5 = \sum_{j=0}^{5} \binom{5}{j} (2x)^{5-j} y^j$$

Since we are concerned with only the term containing x^3, there is no need to expand the entire sum. The exponents on each term must add to 5 and if the exponent on x is 3, the exponent on y must be 2. Plucking out the term $j = 2$, we get

$$\binom{5}{2} (2x)^{5-2} y^2 = 10(2x)^3 y^2 = 80x^3y^2$$

$\square$

An important application of binomial coefficients is computing probabilities using the eponymous *binomial distribution*. Suppose an experiment has a probability p of 'success' and a probability of $1-p$ of 'failure.'[4] For instance, suppose we roll a 'fair' six-sided die. Let us say a 'success' is rolling a four. Then the probability here of a success is $p = \frac{1}{6}$ while the probability of failure here, or *not* rolling a four, is $1 - \frac{1}{6} = \frac{5}{6}$. If we run this experiment n times, then the probability of *exactly* j successes is given by $\binom{n}{j} p^j (1-p)^{n-j}$.

[4]In other words, there are just two possible outcomes: success or failure, and the fact these probabilities add to 1 means one or the other, but not both, will happen. This situation is called a *Bernoulli Trial*.

Here, the binomial coefficient counts the number of ways we can produce j successes out of n trials. The 'bi' in 'binomial' comes from the fact that each trial produces one of two outcomes: a 'success' (with a probability of p) or 'failure' (with probability $1 - p$).

So, for instance, if we roll the fair die 5 times, the probability we get *exactly* 2 fours is:

$$\binom{5}{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{5-2} = \frac{625}{3888} \approx 16\%$$

Moreover, the probability $\binom{n}{j} p^j (1-p)^{n-j}$ is the jth term in the binomial expansion of $((1-p)+p)^n = 1^n = 1$. That is,

$$1 = 1^n = ((1-p)+p)^n = \sum_{j=0}^{n} \binom{n}{j} (1-p)^{n-j} p^j$$

The fact that the *sum* of the probabilities of all the possibilities (0 successful trials up through n successful trials) is 1 can be loosely translated as the probability *something* will happen is 100%.

Suppose we wanted to compute the probability of rolling *at least* 2 fours on 5 rolls. To achieve this, we add the probabilities of obtaining exactly 2 fours, 3 fours, 4 fours, and 5 fours. That is, we get a partial sum of the binomial expansion:

$$\begin{aligned}
\sum_{j=2}^{5} \binom{5}{j}\left(\frac{5}{6}\right)^{5-j}\left(\frac{5}{6}\right)^{j} &= \underbrace{\binom{5}{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{5-2}}_{\text{probability of 2 fours}} + \underbrace{\binom{5}{3}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{5-3}}_{\text{probability of 3 fours}} \\
&\quad + \underbrace{\binom{5}{4}\left(\frac{1}{6}\right)^{4}\left(\frac{5}{6}\right)^{5-4}}_{\text{probability of 4 fours}} + \underbrace{\binom{5}{5}\left(\frac{1}{6}\right)^{5}\left(\frac{5}{6}\right)^{5-5}}_{\text{probability of 5 fours}} \\
&= \tfrac{736}{3888} \approx 20\%
\end{aligned}$$

We summarize the properties of the binomial distribution below.

Theorem 10.8. Binomial Distribution: If an experiment has a probability of success of p then the probability of *exactly* j successes in n independent Bernoulli Trials is:

$$\binom{n}{j} p^j (1-p)^{n-j}$$

for $0 \leq j \leq n$.

The probability of *at least* k successes in n independent Bernoulli Trials is:

$$\sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

for $0 \leq k \leq n$.

We close this section with Pascal's Triangle, named in honor of the mathematician Blaise Pascal. Pascal's Triangle is obtained by arranging the binomial coefficients in the triangular fashion below.

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & & & \searrow\swarrow & & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & & & \searrow\swarrow & & \searrow\swarrow & & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
 & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \vdots & & & &
\end{array}$$

Since $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all whole numbers n, each row of Pascal's Triangle is bookended with 1.

To generate the numbers in the middle of the rows (from the third row onwards), we take advantage of the additive relationship expressed in Theorem 10.6. For instance,

$$\binom{1}{0} + \binom{1}{1} = \binom{2}{1}, \quad \binom{2}{0} + \binom{2}{1} = \binom{3}{1}, \quad \binom{2}{1} + \binom{2}{2} = \binom{3}{2}$$

and so forth. This relationship is indicated by the arrows in the array above.

With these two facts in hand, we can quickly generate Pascal's Triangle in the following way: we start with the first two rows, 1 and 1 1. Each successive row begins and ends with 1 and the middle numbers are generated using Theorem 10.6.

Below we attempt to demonstrate this building process to generate the first five rows of Pascal's Triangle.

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ & & \searrow\swarrow & & \\ \boxed{1} & & \underline{1+1} & & \boxed{1} \end{array} \quad \longrightarrow \quad \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ & & \searrow\swarrow & & \searrow\swarrow & & \\ \boxed{1} & & \underline{1+2} & & \underline{2+1} & & \boxed{1} \end{array} \quad \longrightarrow \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \\ \boxed{1} & & \underline{1+3} & & \underline{3+3} & & \underline{3+1} & & \boxed{1} \end{array} \quad \longrightarrow \quad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

To see how we can use Pascal's Triangle to expedite the Binomial Theorem, suppose we wish to expand $(3x-y)^4$. The coefficients we need are $\binom{4}{j}$ for $j = 0, 1, 2, 3, 4$ and are the numbers which form the fifth row of Pascal's Triangle.

Since we know that the exponent of $(3x)$ in the first term is 4 and then decreases by one as we go from left to right while the exponent of $(-y)$ starts at 0 in the first term and then increases by one as we move from left to right, we quickly obtain

$$\begin{aligned} (3x-y)^4 &= (1)(3x)^4 + (4)(3x)^3(-y) + (6)(3x)^2(-y)^2 + 4(3x)(-y)^3 + 1(-y)^4 \\ &= 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4 \end{aligned}$$

We would like to stress that Pascal's Triangle is a very quick method to expand an *entire* binomial. If only a term (or two or three) is required, then the Binomial Theorem is definitely the way to go.

10.4.1 Exercises

In Exercises 1 - 9, simplify the given expression.

1. $(3!)^2$
2. $\dfrac{10!}{7!}$
3. $\dfrac{7!}{2^3 3!}$
4. $\dfrac{9!}{4!3!2!}$
5. $\dfrac{(n+1)!}{n!}$, $n \geq 0$.
6. $\dfrac{(k-1)!}{(k+2)!}$, $k \geq 1$.
7. $\dbinom{8}{3}$
8. $\dbinom{117}{0}$
9. $\dbinom{n}{n-2}$, $n \geq 2$

In Exercises 10 - 13, use Pascal's Triangle to expand the given binomial.

10. $(x+2)^5$
11. $(2x-1)^4$
12. $\left(\frac{1}{3}x + y^2\right)^3$
13. $\left(x - x^{-1}\right)^4$

In Exercises 14 - 17, use Pascal's Triangle to simplify the given power of a complex number.

14. $(1+2i)^4$
15. $\left(-1 + i\sqrt{3}\right)^3$
16. $\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)^3$
17. $\left(\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i\right)^4$

In Exercises 18 - 22, use the Binomial Theorem to find the indicated term.

18. The term containing x^3 in the expansion $(2x-y)^5$
19. The term containing x^{117} in the expansion $(x+2)^{118}$
20. The term containing $x^{\frac{7}{2}}$ in the expansion $\left(\sqrt{x} - 3\right)^8$
21. The term containing x^{-7} in the expansion $\left(2x - x^{-3}\right)^5$
22. The constant term in the expansion $\left(x + x^{-1}\right)^8$
23. Use the Prinicple of Mathematical Induction to prove $n! > 2^n$ for $n \geq 4$.
24. Prove $\displaystyle\sum_{j=0}^{n} \binom{n}{j} = 2^n$ for all natural numbers n. (HINT: Use the Binomial Theorem!)
25. With the help of your classmates, research Patterns and Properties of Pascal's Triangle.
26. You've just won three tickets to see the new film, '$8.\overline{9}$.' Five of your friends, Albert, Beth, Chuck, Dan, and Eugene, are interested in seeing it with you. With the help of your classmates, list all the possible ways to distribute your two extra tickets among your five friends. Now suppose you've come down with the flu. List all the different ways you can distribute the three tickets among these five friends. How does this compare with the first list you made? What does this have to do with the fact that $\binom{5}{2} = \binom{5}{3}$?

10.4.2 Answers

1. 36
2. 720
3. 105
4. 1260
5. $n+1$
6. $\frac{1}{k(k+1)(k+2)}$
7. 56
8. 1
9. $\frac{n(n-1)}{2}$
10. $(x+2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$
11. $(2x-1)^4 = 16x^4 - 32x^3 + 24x^2 - 8x + 1$
12. $\left(\frac{1}{3}x + y^2\right)^3 = \frac{1}{27}x^3 + \frac{1}{3}x^2y^2 + xy^4 + y^6$
13. $\left(x - x^{-1}\right)^4 = x^4 - 4x^2 + 6 - 4x^{-2} + x^{-4}$
14. $-7 - 24i$
15. 8
16. i
17. -1
18. $80x^3y^2$
19. $236x^{117}$
20. $-24x^{\frac{7}{2}}$
21. $-40x^{-7}$
22. 70

Chapter 11

Foundations of Trigonometry

11.1 The Radian Measure of Angles

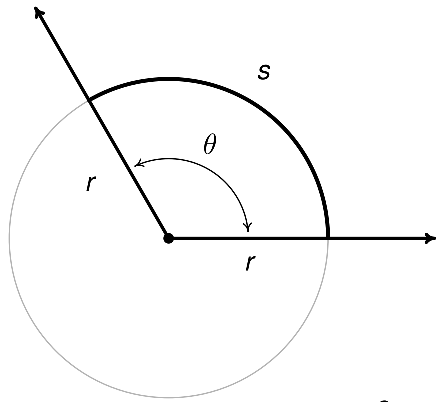
In Section B.1, we review the concept of (oriented) angles and degree measure. While degrees are the unit of choice for many applications of trigonometry, we introduce here the concept of the **radian measure** of an angle. As we will see, this concept naturally ties angles to real numbers. While the concept may seem foreign at first, we assure the reader that the utility of radian measure in modeling real-world phenomena is well worth the effort. We begin our development with a definition from Geometry.

Definition 11.1. The real number π is defined to be the ratio of a circle's circumference to its diameter. In symbols, given a circle of circumference C and diameter d,

$$\pi = \frac{C}{d}$$

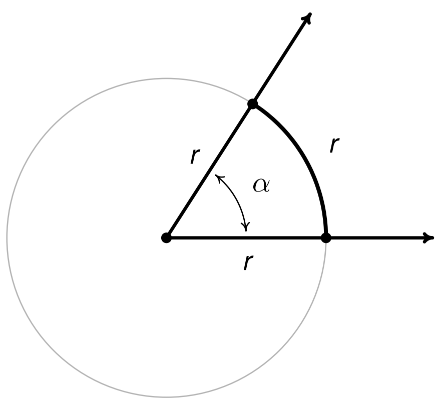
While Definition 11.1 is quite possibly the 'standard' definition of π, the authors would be remiss if we didn't mention that buried in this definition is actually a theorem. As the reader is probably aware, the number π is a mathematical constant - that is, it doesn't matter *which* circle is selected, the ratio of its circumference to its diameter will have the same value as any other circle. While this is indeed true, it is far from obvious and leads to a counterintuitive scenario which is explored in the Exercises. Since the diameter of a circle is twice its radius, we can quickly rearrange the equation in Definition 11.1 to get a formula more useful for our purposes, namely: $2\pi = \frac{C}{r}$. Hence, for any circle, the ratio of its circumference to its radius is 2π.

Suppose we take a *portion* of the circle as depicted below, and we compare some arc measuring s units in length to the radius. Let θ be the **central angle** subtended by this arc, that is, an angle whose vertex is the center of the circle and whose determining rays pass through the endpoints of the arc. Using proportionality (similarity) arguments, it stands to reason that the ratio $\frac{s}{r}$ should also be a constant among all circles. It is this ratio, $\frac{s}{r}$, which defines the **radian measure** of an angle.

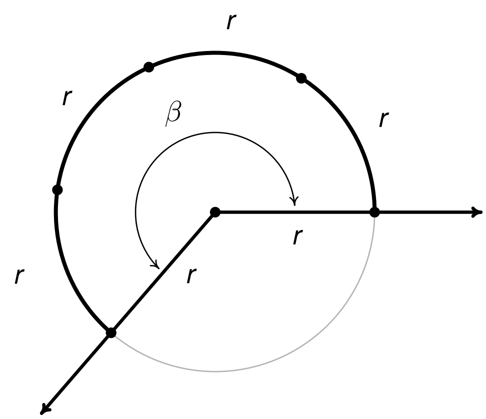


The radian measure of θ is $\dfrac{s}{r}$.

To get a better feel for radian measure, we note that an angle with radian measure 1 means the corresponding arc length s equals the radius of the circle r, that is, $s = r$. When the radian measure is 2, we have $s = 2r$; when the radian measure is 3, $s = 3r$, and so forth. Thus the radian measure of an angle θ tells us how many 'radius lengths' we need to sweep out along the circle to subtend the angle θ.



α has radian measure 1



β has radian measure 4

Since one revolution sweeps out the circumference $2\pi r$, one revolution has radian measure $\frac{2\pi r}{r} = 2\pi$. From this we can find the radian measure of other central angles using proportions, just like we did with degrees. For instance, half of a revolution has radian measure $\frac{1}{2}(2\pi) = \pi$, a quarter revolution has radian measure $\frac{1}{4}(2\pi) = \frac{\pi}{2}$, and so forth. Note that, by definition, the radian measure of an angle is a length divided by another length so that these measurements are actually dimensionless and are considered 'pure' numbers. For this reason, we do not use any symbols to denote radian measure, but we use the word 'radians' to denote these dimensionless units as needed. For instance, we say one revolution measures '2π radians,' half of a revolution measures 'π radians,' and so forth.

As with degree measure, the distinction between the angle itself and its measure is often blurred in practice, so when we write '$\theta = \frac{\pi}{2}$', we mean θ is an angle which measures $\frac{\pi}{2}$ radians.[1] We extend radian measure

[1]The authors are well aware that we are now identifying radians with real numbers. We will justify this shortly.

to oriented angles, just as we did with degrees beforehand, so that a positive measure indicates counter-clockwise rotation and a negative measure indicates clockwise rotation.[2] Much like before, two positive angles α and β are supplementary if $\alpha + \beta = \pi$ and complementary if $\alpha + \beta = \frac{\pi}{2}$. Finally, we leave it to the reader to show that when using radian measure, two angles α and β are coterminal if and only if $\beta = \alpha + 2\pi k$ for some integer k.

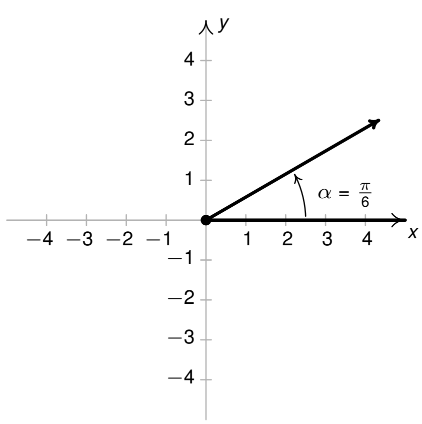
Example 11.1.1. Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. $\alpha = \dfrac{\pi}{6}$
2. $\beta = -\dfrac{4\pi}{3}$
3. $\gamma = \dfrac{9\pi}{4}$
4. $\phi = -\dfrac{5\pi}{2}$

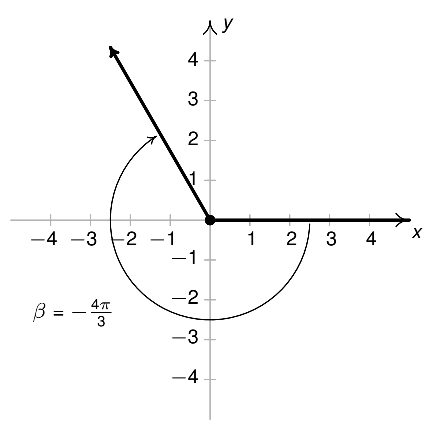
Solution.

1. The angle $\alpha = \frac{\pi}{6}$ is positive, so we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\frac{(\pi/6)}{2\pi} = \frac{1}{12}$ of a revolution. Thus α is a Quadrant I angle. Coterminal angles θ are of the form $\theta = \alpha + 2\pi \cdot k$, for some integer k. To make the arithmetic a bit easier, we note that $2\pi = \frac{12\pi}{6}$, thus when $k = 1$, we get $\theta = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$. Substituting $k = -1$ gives $\theta = \frac{\pi}{6} - \frac{12\pi}{6} = -\frac{11\pi}{6}$ and when we let $k = 2$, we get $\theta = \frac{\pi}{6} + \frac{24\pi}{6} = \frac{25\pi}{6}$.

2. Since $\beta = -\frac{4\pi}{3}$ is negative, we start at the positive x-axis and rotate clockwise $\frac{(4\pi/3)}{2\pi} = \frac{2}{3}$ of a revolution. We find β to be a Quadrant II angle. To find coterminal angles, we proceed as before using $2\pi = \frac{6\pi}{3}$, and compute $\theta = -\frac{4\pi}{3} + \frac{6\pi}{3} \cdot k$ for integer values of k. We obtain $\frac{2\pi}{3}$, $-\frac{10\pi}{3}$ and $\frac{8\pi}{3}$ as coterminal angles.



$\alpha = \frac{\pi}{6}$ in standard position.



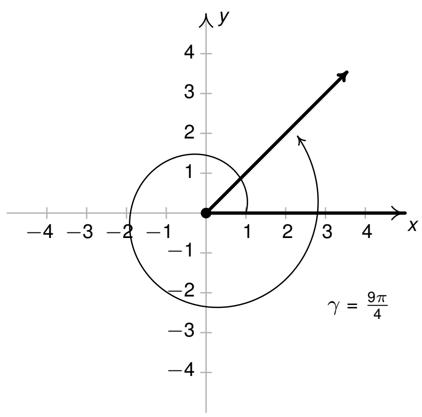
$\beta = -\frac{4\pi}{3}$ in standard position.

3. Since $\gamma = \frac{9\pi}{4}$ is positive, we rotate counter-clockwise from the positive x-axis. One full revolution accounts for $2\pi = \frac{8\pi}{4}$ of the radian measure with $\frac{\pi}{4}$ or $\frac{1}{8}$ of a revolution remaining. We have γ as a

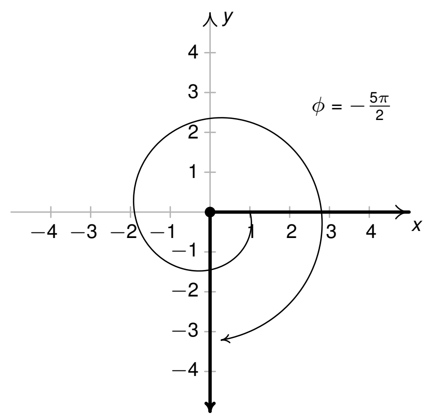
[2]This, in turn, endows the subtended arcs with an orientation as well. We address this in short order.

Quadrant I angle. All angles coterminal with γ are of the form $\theta = \frac{9\pi}{4} + \frac{8\pi}{4} \cdot k$, where k is an integer. Working through the arithmetic, we find: $\frac{\pi}{4}$, $-\frac{7\pi}{4}$ and $\frac{17\pi}{4}$.

4. To graph $\phi = -\frac{5\pi}{2}$, we begin our rotation clockwise from the positive x-axis. As $2\pi = \frac{4\pi}{2}$, after one full revolution clockwise, we have $\frac{\pi}{2}$ or $\frac{1}{4}$ of a revolution remaining. Since the terminal side of ϕ lies on the negative y-axis, ϕ is a quadrantal angle. To find coterminal angles, we compute $\theta = -\frac{5\pi}{2} + \frac{4\pi}{2} \cdot k$ for a few integers k and obtain $-\frac{\pi}{2}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$.



$\gamma = \frac{9\pi}{4}$ in standard position.



$\phi = -\frac{5\pi}{2}$ in standard position.

□

It is worth mentioning that we could have plotted the angles in Example 11.1.1 by first converting them to degree measure and following the procedure set forth in Example B.1.2. While converting back and forth from degrees and radians is certainly a good skill to have, it is best that you learn to 'think in radians' as well as you can 'think in degrees'. The authors would, however, be derelict in our duties if we ignored the basic conversion between these systems altogether. Since one revolution counter-clockwise measures 360° and the same angle measures 2π radians, we can use the proportion $\frac{2\pi \text{ radians}}{360^\circ}$, or its reduced equivalent, $\frac{\pi \text{ radians}}{180^\circ}$, as the conversion factor between the two systems. For example, to convert 60° to radians we find $60^\circ \left(\frac{\pi \text{ radians}}{180^\circ}\right) = \frac{\pi}{3}$ radians, or simply $\frac{\pi}{3}$. To convert from radian measure back to degrees, we multiply by the ratio $\frac{180^\circ}{\pi \text{ radian}}$. For example, $-\frac{5\pi}{6}$ radians is equal to $\left(-\frac{5\pi}{6} \text{ radians}\right)\left(\frac{180^\circ}{\pi \text{ radians}}\right) = -150^\circ$.[3] Hence, an angle which measures 1 in radian measure is equal to $\frac{180^\circ}{\pi} \approx 57.2958^\circ$. To summarize:

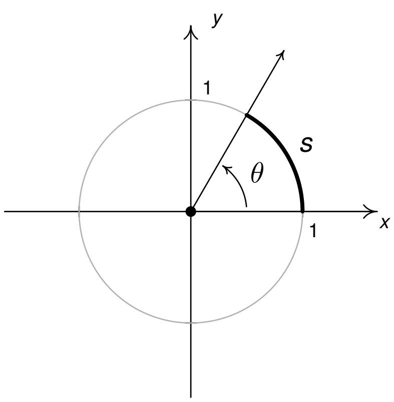
Equation 11.1. Degree - Radian Conversion:

- To convert degree measure to radian measure, multiply by $\dfrac{\pi \text{ radians}}{180^\circ}$
- To convert radian measure to degree measure, multiply by $\dfrac{180^\circ}{\pi \text{ radians}}$

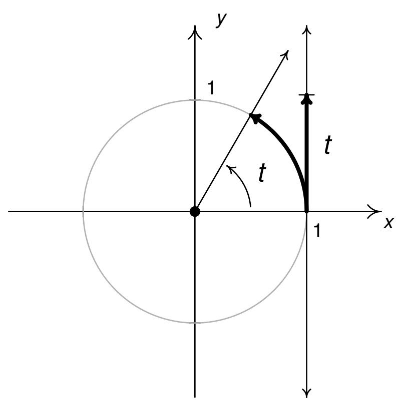
[3] Note that the negative sign indicates clockwise rotation in both systems, and so it is carried along accordingly.

In light of Example 11.1.1 and Equation 11.1, the reader may well wonder what the allure of radian measure is. The numbers involved are, admittedly, much more complicated than degree measure. The answer lies in how easily angles in radian measure can be identified with real numbers. Consider the Unit Circle, $x^2 + y^2 = 1$, as drawn below, the angle θ in standard position and the corresponding arc measuring s units in length. By definition, and the fact that the Unit Circle has radius 1, the radian measure of θ is $\frac{s}{r} = \frac{s}{1} = s$ so that, once again blurring the distinction between an angle and its measure, we have $\theta = s$. In order to identify real numbers with oriented angles, we essentially 'wrap' the real number line around the Unit Circle and associating to each real number t an *oriented* arc on the Unit Circle with initial point $(1, 0)$.

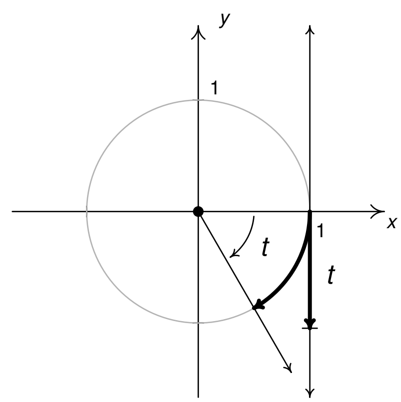
Viewing the vertical line $x = 1$ as another real number line demarcated like the y-axis, given a real number $t > 0$, we 'wrap' the (vertical) interval $[0, t]$ around the Unit Circle in a counter-clockwise fashion. The resulting arc has a length of t units and therefore the corresponding angle has radian measure equal to t. If $t < 0$, we wrap the interval $[t, 0]$ *clockwise* around the Unit Circle. Since we have defined clockwise rotation as having negative radian measure, the angle determined by this arc has radian measure equal to t. If $t = 0$, we are at the point $(1, 0)$ on the x-axis which corresponds to an angle with radian measure 0. In this way, we identify each real number t with the corresponding angle with radian measure t.



On the Unit Circle, $\theta = s$.



Identifying $t > 0$ with an angle.



Identifying $t < 0$ with an angle.

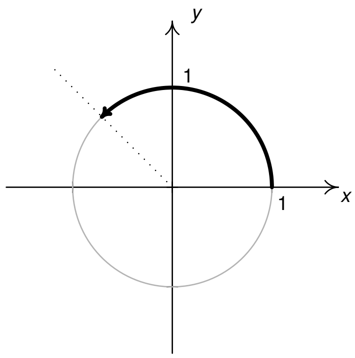
Example 11.1.2. Sketch the oriented arc on the Unit Circle corresponding to each of the following real numbers.

1. $t = \dfrac{3\pi}{4}$
2. $t = -2\pi$
3. $t = -2$
4. $t = 117$

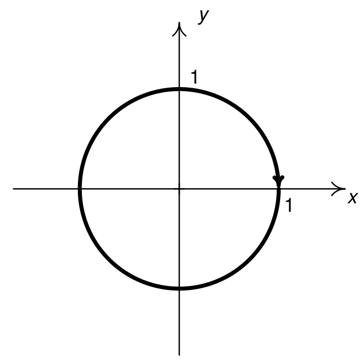
Solution.

1. The arc associated with $t = \frac{3\pi}{4}$ is the arc on the Unit Circle which subtends the angle $\frac{3\pi}{4}$ in radian measure. Since $\frac{3\pi}{4}$ is $\frac{3}{8}$ of a revolution, we have an arc which begins at the point $(1, 0)$ proceeds counter-clockwise up to midway through Quadrant II.

2. Since one revolution is 2π radians, and $t = -2\pi$ is negative, we graph the arc which begins at $(1, 0)$ and proceeds *clockwise* for one full revolution.



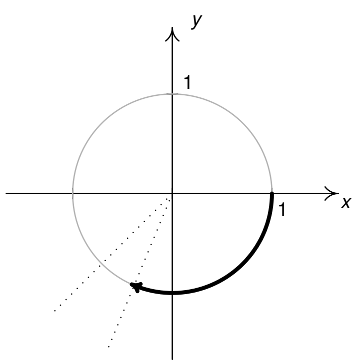
$t = \frac{3\pi}{4}$



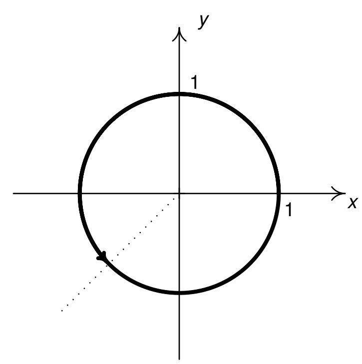
$t = -2\pi$

3. Like $t = -2\pi$, $t = -2$ is negative, so we begin our arc at $(1, 0)$ and proceed clockwise around the unit circle. Since $\pi \approx 3.14$ and $\frac{\pi}{2} \approx 1.57$, we find that rotating 2 radians clockwise from the point $(1, 0)$ lands us in Quadrant III. To more accurately place the endpoint, we proceed as we did in Example B.1.1, successively halving the angle measure until we find $\frac{5\pi}{8} \approx 1.96$ which tells us our arc extends just a bit beyond the quarter mark into Quadrant III.

4. Since 117 is positive, the arc corresponding to $t = 117$ begins at $(1, 0)$ and proceeds counter-clockwise. As 117 is much greater than 2π, we wrap around the Unit Circle several times before finally reaching our endpoint. We approximate $\frac{117}{2\pi}$ as 18.62 which tells us we complete 18 revolutions counter-clockwise with 0.62, or just shy of $\frac{5}{8}$ of a revolution to spare. In other words, the terminal side of the angle which measures 117 radians in standard position is just short of being midway through Quadrant III.



$t = -2$

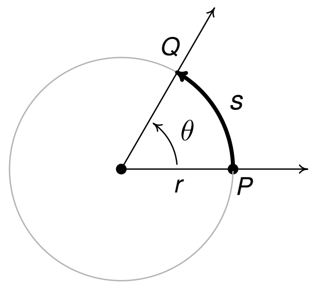


$t = 117$

□

11.1.1 Applications of Radian Measure: Circular Motion

Now that we have paired angles with real numbers via radian measure, a whole world of applications awaits us. Our first excursion into this realm comes by way of circular motion. Suppose an object is moving as pictured below along a circular path of radius r from the point P to the point Q in an amount of time t.



Here s represents a *displacement* so that $s > 0$ means the object is traveling in a counter-clockwise direction and $s < 0$ indicates movement in a clockwise direction. Note that with this convention the formula we used to define radian measure, namely $\theta = \frac{s}{r}$, still holds since a negative value of s incurred from a clockwise displacement matches the negative we assign to θ for a clockwise rotation. In Physics, the **average velocity** of the object, denoted $\overline{v}$ and read as 'v-bar', is defined as the average rate of change of the position of the object with respect to time.[4] As a result, we have $\overline{v} = \frac{\text{displacement}}{\text{time}} = \frac{s}{t}$. The quantity $\overline{v}$ has units of $\frac{\text{length}}{\text{time}}$ and conveys two ideas: the direction in which the object is moving and how fast the position of the object is changing. The contribution of direction in the quantity $\overline{v}$ is either to make it positive (in the case of counter-clockwise motion) or negative (in the case of clockwise motion), so that the quantity $|\overline{v}|$ quantifies how fast the object is moving - it is the **speed** of the object. Measuring θ in radians we have $\theta = \frac{s}{r}$ thus $s = r\theta$ and

$$\overline{v} = \frac{s}{t} = \frac{r\theta}{t} = r \cdot \frac{\theta}{t}$$

The quantity $\frac{\theta}{t}$ is called the **average angular velocity** of the object. It is denoted by $\overline{\omega}$ and is read 'omega-bar'. The quantity $\overline{\omega}$ is the average rate of change of the angle θ with respect to time and thus has units $\frac{\text{radians}}{\text{time}}$. If $\overline{\omega}$ is constant throughout the duration of the motion, then it can be shown[5] that the average velocities involved, namely $\overline{v}$ and $\overline{\omega}$, are the same as their instantaneous counterparts, v and ω, respectively. In this case, v is simply called the 'velocity' of the object and ω is called the 'angular velocity.'[6]

If the path of the object were 'uncurled' from a circle to form a line segment, then the velocity of the object on that line segment would be the same as the velocity on the circle. For this reason, the quantity v is often called the *linear* velocity of the object in order to distinguish it from the *angular* velocity, ω. Putting together the ideas of the previous paragraph, we get the following.

Equation 11.2. Velocity for Circular Motion: For an object moving on a circular path of radius r with constant angular velocity ω, the (linear) velocity of the object is given by $v = r\omega$.

We need to talk about units here. The units of v are $\frac{\text{length}}{\text{time}}$, the units of r are length only, and the units of ω are $\frac{\text{radians}}{\text{time}}$. Thus the left hand side of the equation $v = r\omega$ has units $\frac{\text{length}}{\text{time}}$, whereas the right hand side has units length $\cdot \frac{\text{radians}}{\text{time}} = \frac{\text{length} \cdot \text{radians}}{\text{time}}$. The supposed contradiction in units is resolved by remembering

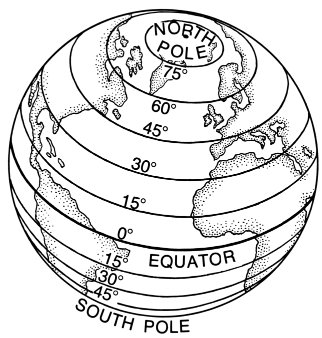
[4]See Definition 3.5 in Section 3.1 for a review of this concept.

[5]You guessed it, using Calculus . . .

[6]See Example 3.1.3 in Section 3.1 for more of a discussion on instantaneous velocity.

that radians are a dimensionless quantity and angles in radian measure are identified with real numbers so that the units $\frac{\text{length}\cdot\text{radians}}{\text{time}}$ reduce to the units $\frac{\text{length}}{\text{time}}$. We are long overdue for an example.

Example 11.1.3. Assuming that the surface of the Earth is a sphere, any point on the Earth can be thought of as an object traveling on a circle (this is the **parallel of latitude** of the point) as seen in the figure below.[7] Since it takes the Earth (approximately) 24 hours to rotate, the object takes 24 hours to complete one revolution along this circle. Lakeland Community College is at 41.628° north latitude, and it can be shown[8] that the radius of the earth at this Latitude is approximately 2960 miles. Find the linear velocity, in miles per hour, of Lakeland Community College as the world turns.



Solution. To use the formula $v = r\omega$, we first need to compute the angular velocity ω. The earth makes one revolution in 24 hours, and one revolution is 2π radians, so $\omega = \frac{2\pi \text{ radians}}{24 \text{ hours}} = \frac{\pi}{12 \text{ hours}}$. Note that once again, we are identifying angles in radian measure as real numbers so we can drop the 'radian' units as they are dimensionless. Also note that for simplicity's sake, we assume that we are viewing the rotation of the earth as counter-clockwise so $\omega > 0$. Hence, the linear velocity is

$$v = 2960 \text{ miles} \cdot \frac{\pi}{12 \text{ hours}} \approx 775 \, \frac{\text{miles}}{\text{hour}}$$

□

It is worth noting that the quantity $\frac{1 \text{ revolution}}{24 \text{ hours}}$ in Example 11.1.3 is called the **ordinary frequency** of the motion and is usually denoted by the variable f. The ordinary frequency is a measure of how often an object makes a complete cycle of the motion. The fact that $\omega = 2\pi f$ suggests that ω is also a frequency. Indeed, it is called the **angular frequency** of the motion. On a related note, the quantity $T = \frac{1}{f}$ is called the **period** of the motion and is the amount of time it takes for the object to complete one cycle of the motion. In the scenario of Example 11.1.3, the period of the motion is 24 hours, or one day.

The concepts of frequency and period help frame the equation $v = r\omega$ in a new light. That is, if ω is fixed, points which are farther from the center of rotation need to travel faster to maintain the same angular frequency since they have farther to travel to make one revolution in one period's time. The distance of

[7]Diagram credit: Pearson Scott Foresman [Public domain], via Wikimedia Commons.

[8]We will discuss how we arrived at this approximation in Example 11.2.5.

the object to the center of rotation is the radius of the circle, r, and is the 'magnification factor' which relates ω and v. We will have more to say about frequencies and periods in Section 11.3. While we have exhaustively discussed velocities associated with circular motion, we have yet to discuss a more natural question: if an object is moving on a circular path of radius r with a fixed angular velocity (frequency) ω, what is the position of the object at time t? The answer to this question is the very heart of Trigonometry and is answered in the next section.

11.1.2 Exercises

In Exercises 9 - 20, graph the oriented angle in standard position. Classify each angle according to where its terminal side lies and then give two coterminal angles, one of which is positive and the other negative.

1. $\frac{\pi}{3}$
2. $\frac{5\pi}{6}$
3. $-\frac{11\pi}{3}$
4. $\frac{5\pi}{4}$
5. $\frac{3\pi}{4}$
6. $-\frac{\pi}{3}$
7. $\frac{7\pi}{2}$
8. $\frac{\pi}{4}$
9. $-\frac{\pi}{2}$
10. $\frac{7\pi}{6}$
11. $-\frac{5\pi}{3}$
12. 3π
13. -2π
14. $-\frac{\pi}{4}$
15. $\frac{15\pi}{4}$
16. $-\frac{13\pi}{6}$

In Exercises 17 - 24, convert the angle from degree measure into radian measure, giving the exact value in terms of π.

17. 0°
18. 240°
19. 135°
20. -270°
21. -315°
22. 150°
23. 45°
24. -225°

In Exercises 25 - 32, convert the angle from radian measure into degree measure.

25. π
26. $-\frac{2\pi}{3}$
27. $\frac{7\pi}{6}$
28. $\frac{11\pi}{6}$
29. $\frac{\pi}{3}$
30. $\frac{5\pi}{3}$
31. $-\frac{\pi}{6}$
32. $\frac{\pi}{2}$

In Exercises 33 - 37, sketch the oriented arc on the Unit Circle which corresponds to the given real number.

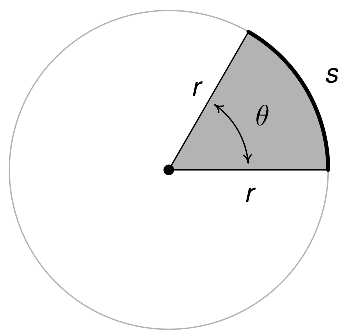
33. $t = \frac{5\pi}{6}$
34. $t = -\pi$
35. $t = 6$
36. $t = -2$
37. $t = 12$
38. A yo-yo which is 2.25 inches in diameter spins at a rate of 4500 revolutions per minute. How fast is the edge of the yo-yo spinning in miles per hour? Round your answer to two decimal places.
39. How many revolutions per minute would the yo-yo in exercise 38 have to complete if the edge of the yo-yo is to be spinning at a rate of 42 miles per hour? Round your answer to two decimal places.
40. In the yo-yo trick 'Around the World,' the performer throws the yo-yo so it sweeps out a vertical circle whose radius is the yo-yo string. If the yo-yo string is 28 inches long and the yo-yo takes 3 seconds to complete one revolution of the circle, compute the speed of the yo-yo in miles per hour. Round your answer to two decimal places.
41. A computer hard drive contains a circular disk with diameter 2.5 inches and spins at a rate of 7200 revolutions per minute. Find the linear speed of a point on the edge of the disk in miles per hour.

42. A rock got stuck in the tread of my tire and when I was driving 70 miles per hour, the rock came loose and hit the inside of the wheel well of the car. How fast, in miles per hour, was the rock traveling when it came out of the tread? (The tire has a diameter of 23 inches.)

43. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height is 136 feet. (Remember this from Exercise 26 in Section 8.3?) It completes two revolutions in 2 minutes and 7 seconds.[9] Assuming the riders are at the edge of the circle, how fast are they traveling in miles per hour?

44. Consider the circle of radius r pictured below with central angle θ, measured in radians, and subtended arc of length s. Prove that the area of the shaded sector is $A = \frac{1}{2}r^2\theta$.

(Hint: Use the proportion $\frac{A}{\text{area of the circle}} = \frac{s}{\text{circumference of the circle}}$.)



In Exercises 45 - 50, use the result of Exercise 44 to compute the areas of the circular sectors with the given central angles and radii.

45. $\theta = \dfrac{\pi}{6}$, $r = 12$

46. $\theta = \dfrac{5\pi}{4}$, $r = 100$

47. $\theta = 330^\circ$, $r = 9.3$

48. $\theta = \pi$, $r = 1$

49. $\theta = 240^\circ$, $r = 5$

50. $\theta = 1^\circ$, $r = 117$

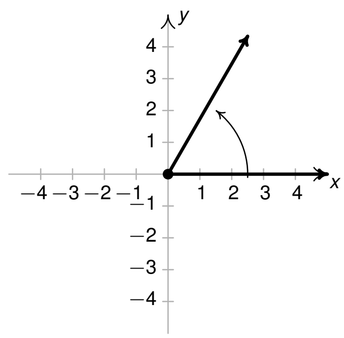
51. Imagine a rope tied around the Earth at the equator. Show that you need to add only 2π feet of length to the rope in order to lift it one foot above the ground around the entire equator. (You do NOT need to know the radius of the Earth to show this.)

52. With the help of your classmates, look for a proof that π is indeed a constant.

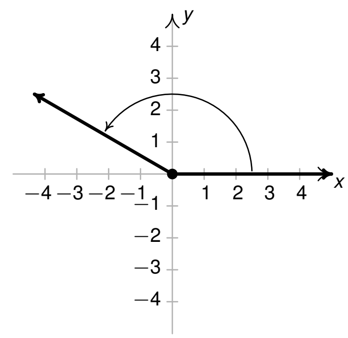
[9]Source: Cedar Point's webpage.

11.1.3 Answers

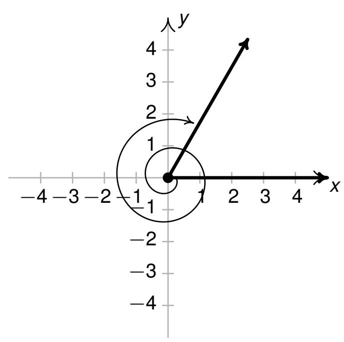
1. $\frac{\pi}{3}$ is a Quadrant I angle
coterminal with $\frac{7\pi}{3}$ and $-\frac{5\pi}{3}$



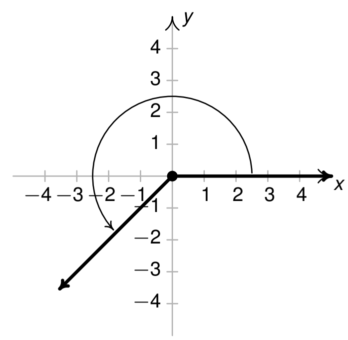
2. $\frac{5\pi}{6}$ is a Quadrant II angle
coterminal with $\frac{17\pi}{6}$ and $-\frac{7\pi}{6}$



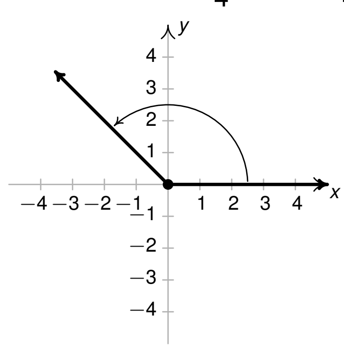
3. $-\frac{11\pi}{3}$ is a Quadrant I angle
coterminal with $\frac{\pi}{3}$ and $-\frac{5\pi}{3}$



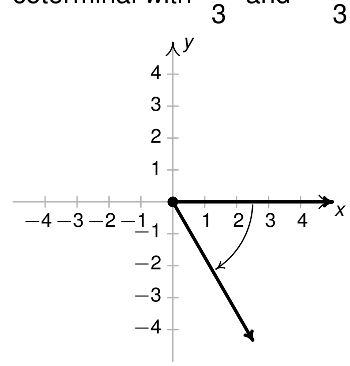
4. $\frac{5\pi}{4}$ is a Quadrant III angle
coterminal with $\frac{13\pi}{4}$ and $-\frac{3\pi}{4}$



5. $\frac{3\pi}{4}$ is a Quadrant II angle
coterminal with $\frac{11\pi}{4}$ and $-\frac{5\pi}{4}$

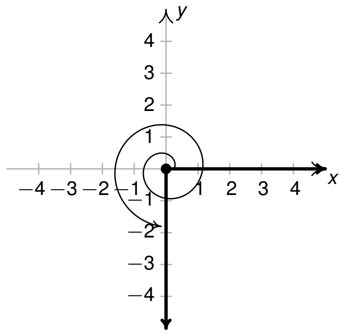


6. $-\frac{\pi}{3}$ is a Quadrant IV angle
coterminal with $\frac{5\pi}{3}$ and $-\frac{7\pi}{3}$



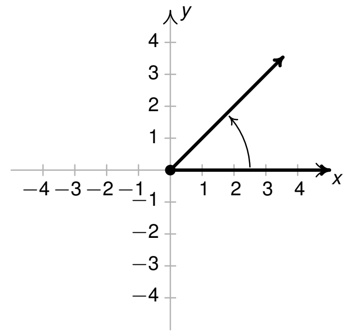
7. $\frac{7\pi}{2}$ lies on the negative y-axis

coterminal with $\frac{3\pi}{2}$ and $-\frac{\pi}{2}$



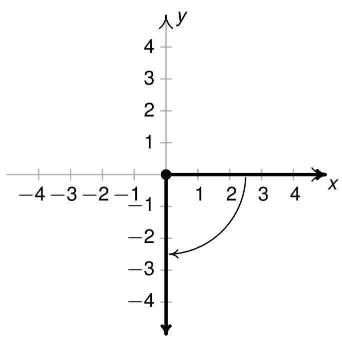
8. $\frac{\pi}{4}$ is a Quadrant I angle

coterminal with $\frac{9\pi}{4}$ and $-\frac{7\pi}{4}$



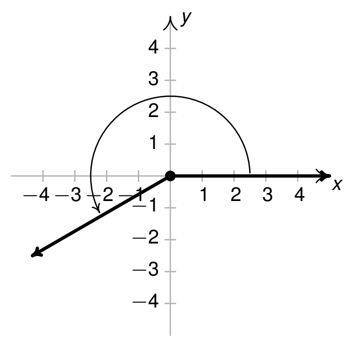
9. $-\frac{\pi}{2}$ lies on the negative y-axis

coterminal with $\frac{3\pi}{2}$ and $-\frac{5\pi}{2}$



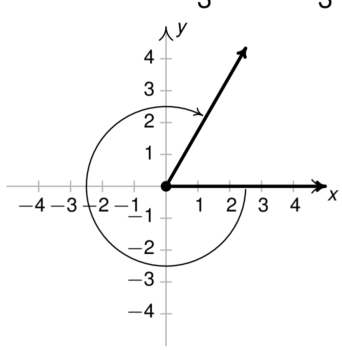
10. $\frac{7\pi}{6}$ is a Quadrant III angle

coterminal with $\frac{19\pi}{6}$ and $-\frac{5\pi}{6}$



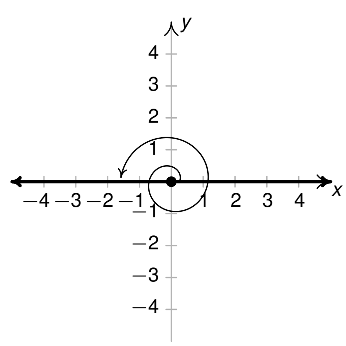
11. $-\frac{5\pi}{3}$ is a Quadrant I angle

coterminal with $\frac{\pi}{3}$ and $-\frac{11\pi}{3}$



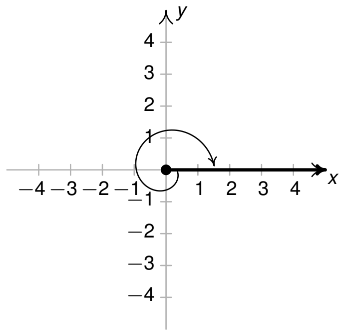
12. 3π lies on the negative x-axis

coterminal with π and $-\pi$



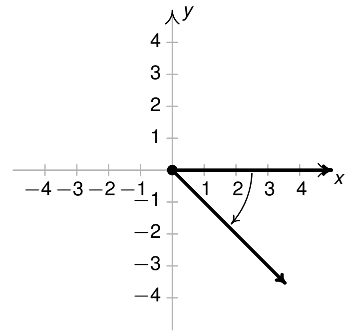
13. -2π lies on the positive x-axis

coterminal with 2π and -4π



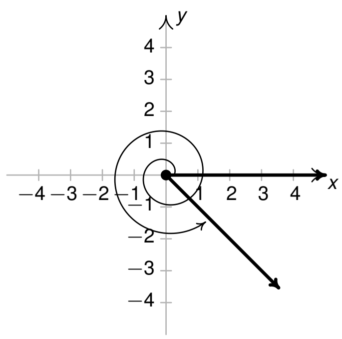
14. $-\dfrac{\pi}{4}$ is a Quadrant IV angle

coterminal with $\dfrac{7\pi}{4}$ and $-\dfrac{9\pi}{4}$



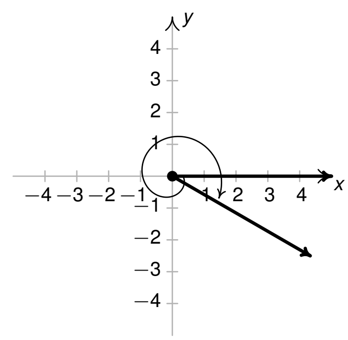
15. $\dfrac{15\pi}{4}$ is a Quadrant IV angle

coterminal with $\dfrac{7\pi}{4}$ and $-\dfrac{\pi}{4}$



16. $-\dfrac{13\pi}{6}$ is a Quadrant IV angle

coterminal with $\dfrac{11\pi}{6}$ and $-\dfrac{\pi}{6}$



17. 0

18. $\dfrac{4\pi}{3}$

19. $\dfrac{3\pi}{4}$

20. $-\dfrac{3\pi}{2}$

21. $-\dfrac{7\pi}{4}$

22. $\dfrac{5\pi}{6}$

23. $\dfrac{\pi}{4}$

24. $-\dfrac{5\pi}{4}$

25. 180°

26. -120°

27. 210°

28. 330°

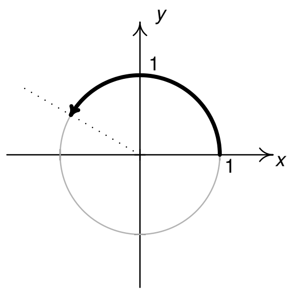
29. 60°

30. 300°

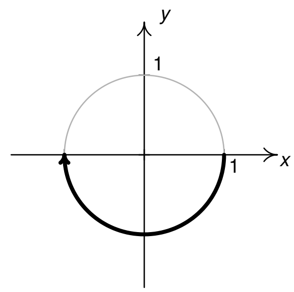
31. -30°

32. 90°

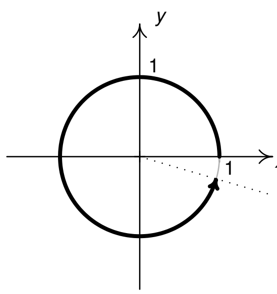
33. $t = \dfrac{5\pi}{6}$



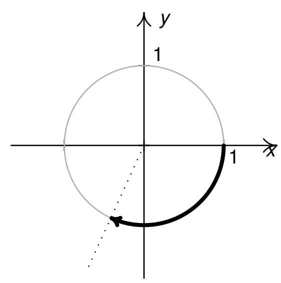
34. $t = -\pi$



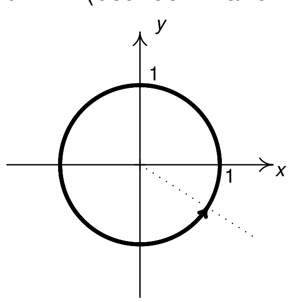
35. $t = 6$



36. $t = -2$



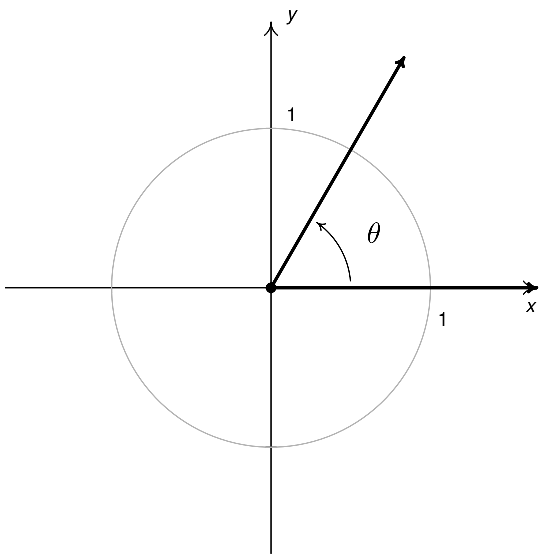
37. $t = 12$ (between 1 and 2 revolutions)

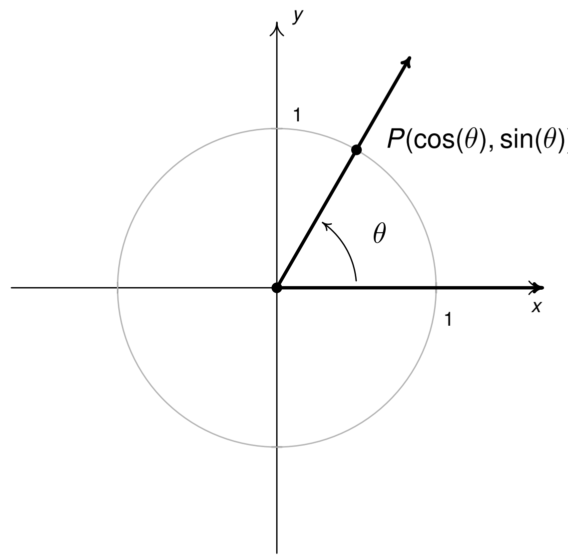


38. About 30.12 miles per hour
39. About 6274.52 revolutions per minute
40. About 3.33 miles per hour
41. About 53.55 miles per hour
42. 70 miles per hour
43. About 4.32 miles per hour
45. 12π square units
46. 6250π square units
47. $79.2825\pi \approx 249.07$ square units
48. $\dfrac{\pi}{2}$ square units
49. $\dfrac{50\pi}{3}$ square units
50. $38.025\pi \approx 119.46$ square units

11.2 The Circular Functions: Sine and Cosine

In Section 11.1.1, we introduced circular motion and derived a formula which describes the linear velocity of an object moving on a circular path at a constant angular velocity. One of the goals of this section is describe the *position* of such an object. To that end, consider an angle θ in standard position and let P denote the point where the terminal side of θ intersects the Unit Circle, as diagrammed below.





By associating the point P with the angle θ, we are assigning a *position* on the Unit Circle to the angle θ. Since for each angle θ, the terminal side of θ, when graphed in standard position, intersects The Unit Circle only once, the mapping of θ to P is a function.[1] Since there is only *one* way to describe a point using rectangular coordinates,[2] the mappings of θ to each of the x and y coordinates of P are also functions. We give these functions names in the following definition.

Definition 11.2. Suppose an angle θ is graphed in standard position. Let $P(x, y)$ be the point of intersection of the terminal side of P and the Unit Circle.

- The x-coordinate of P is called the **cosine** of θ, written $\cos(\theta)$.
- The y-coordinate of P is called the **sine** of θ, written $\sin(\theta)$.[a]

[a] The etymology of the name 'sine' is quite colorful, and the interested reader is invited to research it; the 'co' in 'cosine' is related to the concept of 'co'mplementary angles (see Sections B.1 and B.2) and is explained in detail in Section 12.2.

You may have already seen definitions for the sine and cosine of an (acute) angle in terms of ratios of sides of a right triangle.[3] While not incorrect, defining sine and cosine using right triangles limits the angles we can study to acute angles only. Definition 11.2, on the other hand, applies to *all* angles. Since these functions are defined in terms of points on the Unit *Circle*, they are called **circular** functions. Rest assured,

[1] See Definition 1.1 in Section 1.1.
[2] See page 1354 in Section A.3.
[3] For instance, Definition B.1 in Section B.2.

Definition 11.2 specializes to Definition B.1 when θ is an acute angle. We will see instances of this fact in the next example.

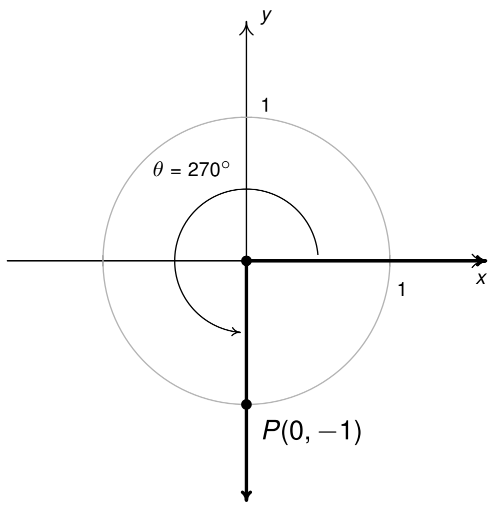
Example 11.2.1. Find the sine and cosine of the following angles.

1. $\theta = 270^\circ$
2. $\theta = -\pi$
3. $\theta = 45^\circ$
4. $\theta = \frac{\pi}{6}$
5. $\theta = \frac{5\pi}{6}$

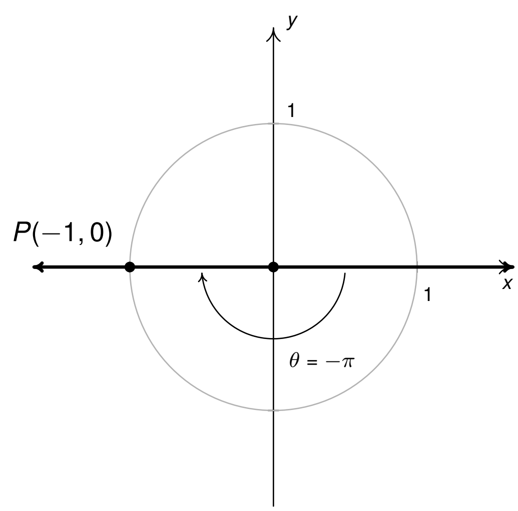
Solution.

1. To find $\cos(270^\circ)$ and $\sin(270^\circ)$, we plot the angle $\theta = 270^\circ$ in standard position and find the point on the terminal side of θ which lies on the Unit Circle. Since 270° represents $\frac{3}{4}$ of a counter-clockwise revolution, the terminal side of θ lies along the negative y-axis. Hence, the point we seek is $(0, -1)$ so that $\cos(270^\circ) = 0$ and $\sin(270^\circ) = -1$.

2. The angle $\theta = -\pi$ represents one half of a clockwise revolution so its terminal side lies on the negative x-axis. The point on the Unit Circle that lies on the negative x-axis is $(-1, 0)$ which means $\cos(-\pi) = -1$ and $\sin(-\pi) = 0$.



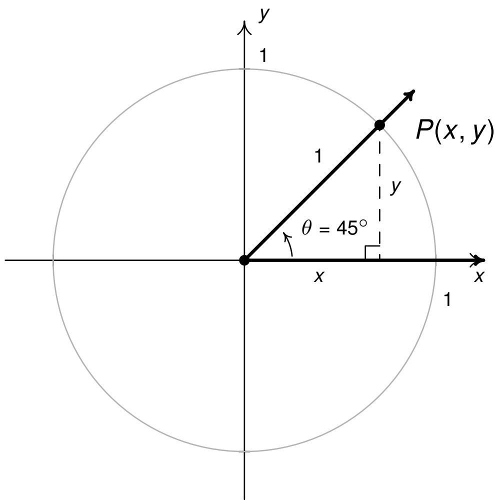
Finding $\cos(270^\circ)$ and $\sin(270^\circ)$

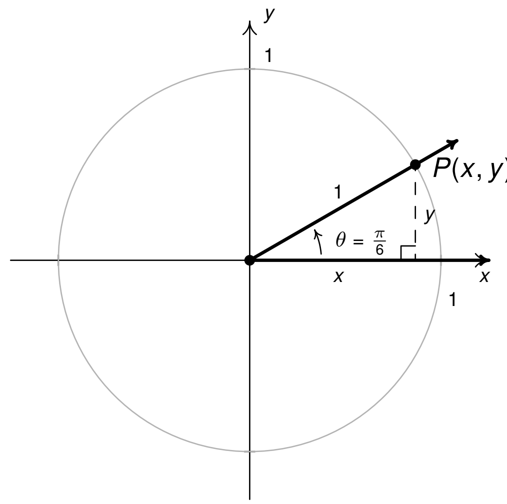


Finding $\cos(-\pi)$ and $\sin(-\pi)$

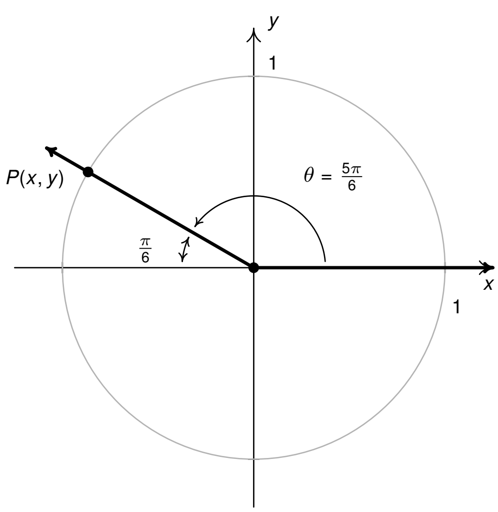
3. In Section B.2, we derived values for $\cos(45^\circ)$ and $\sin(45^\circ)$ using Definition B.1. In order to connect what we know from Section B.2 with what we are asked to find per Definition 11.2, we sketch $\theta = 45^\circ$ in standard position and let $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle. If we drop a perpendicular line segment from P to the x-axis as shown below on the left, we obtain a $45^\circ - 45^\circ - 90^\circ$ right triangle whose legs have lengths x and y units with hypotenuse 1. From our work in Section B.2, we obtain the (familiar) values $x = \cos(45^\circ) = \frac{\sqrt{2}}{2}$ and $y = \sin(45^\circ) = \frac{\sqrt{2}}{2}$.

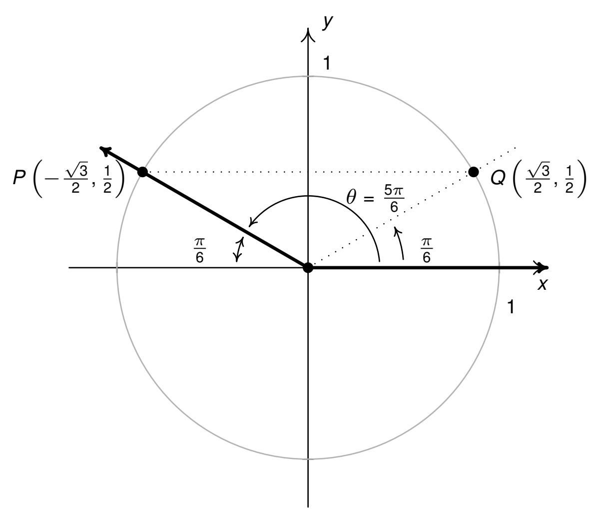
4. As before, the terminal side of $\theta = \frac{\pi}{6}$ does not lie on any of the coordinate axes, so we proceed using a triangle approach. Letting $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle, we drop a perpendicular line segment from P to the x-axis to form a $30^\circ - 60^\circ - 90^\circ$ right triangle. Re-using some of our work from Section B.2, we get $x = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.





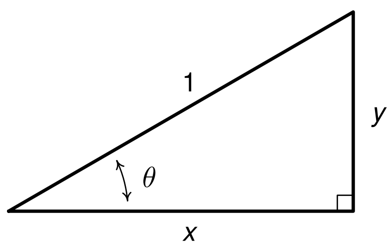
5. We plot $\theta = \frac{5\pi}{6}$ in standard position below on the left and, as usual, let $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle. In plotting θ, we find it lies $\frac{\pi}{6}$ radians short of one half revolution. Since we've just determined that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, we know the coordinates of the point Q below on the right are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Using symmetry, the coordinates of P are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, so $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

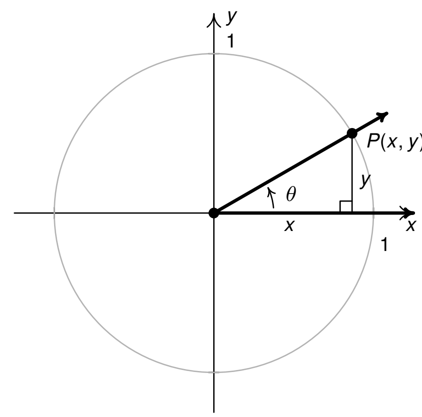




□

A few remarks are in order. First, after having re-used some of our work from Section B.2 in a few specific instances, we can reconcile Definition 11.2 with Definition B.1 in the case θ is an acute angle. We situate θ in a right triangle with hypotenuse length 1, adjacent side length 'x,' and the opposite side length 'y' as seen below on the left. Placing the vertex of θ at the origin and the adjacent side of θ along the x-axis as seen below on the right effectively puts θ in standard position with θ's adjacent side as the initial side of θ and the hypotenuse as the terminal side of θ. Since the hypotenuse of the triangle has length 1, we know the point $P(x, y)$ is on the Unit Circle.[4]

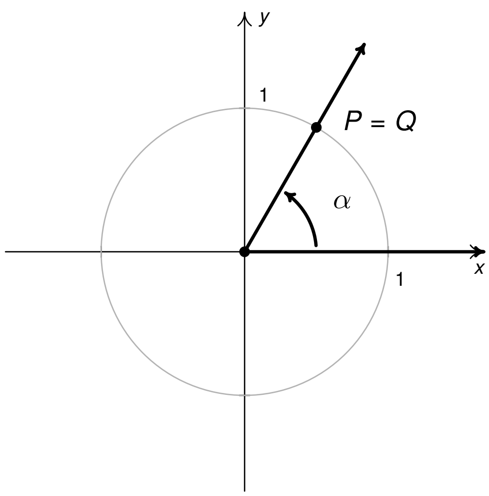




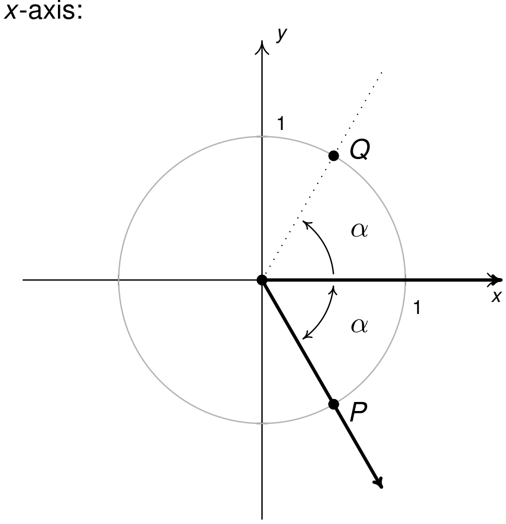
Definition B.1 gives $\cos(\theta) = \frac{x}{1} = x$ and $\sin(\theta) = \frac{y}{1} = y$ which exactly matches Definition 11.2. Hence, in the case of acute angles, the two definitions agree. In other words, the values of the *trigonometric ratios* of acute angles are the same as the corresponding *circular function* values.

A second important take-away from Example 11.2.1 is use of symmetry in number 5. Indeed, we found the sine and cosine of $\frac{5\pi}{6}$ using the (acute) angle $\frac{\pi}{6}$ 'for reference.' Since the Unit Circle is rife with symmetry, we would like to generalize this concept and exploit symmetry whenever possible. To that end, we introduce the notion of **reference angle**.

In general, for a non-quadrantal angle θ, the reference angle for θ (which we'll usually denote α) is the *acute* angle made between the terminal side of θ and the x-axis. If θ is a Quadrant I or IV angle, α is the angle between the terminal side of θ and the *positive* x-axis:



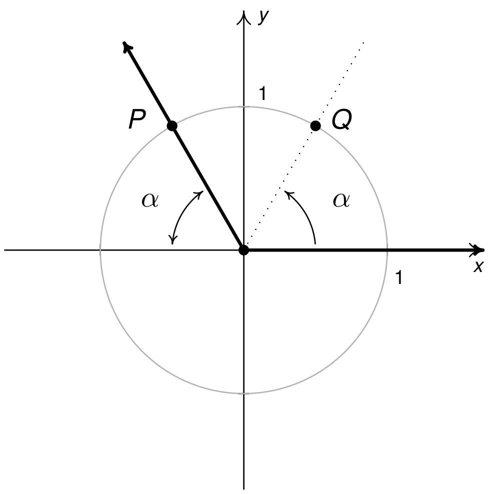
Reference angle α for a Quadrant I angle



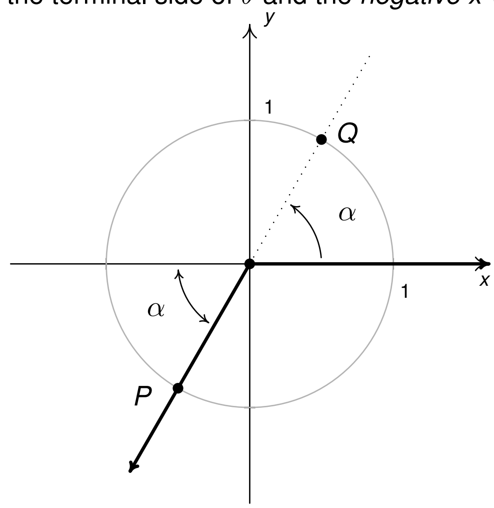
Reference angle α for a Quadrant IV angle

[4]Do you see why?

If θ is a Quadrant II or III angle, α is the angle between the terminal side of θ and the *negative* x-axis:



Reference angle α for a Quadrant II angle



Reference angle α for a Quadrant III angle

If we let P denote the point $(\cos(\theta), \sin(\theta))$, then P lies on the Unit Circle. Since the Unit Circle possesses symmetry with respect to the x-axis, y-axis and origin, regardless of where the terminal side of θ lies, there is a point Q symmetric with P which determines θ's reference angle, α. The only difference between the points P and Q are the signs of their coordinates, $\pm$. Hence, we have the following:

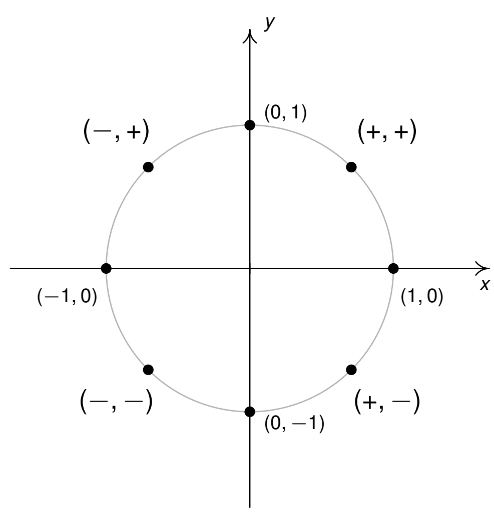
Theorem 11.1. Reference Angle Theorem. Suppose α is the reference angle for θ. Then:

$$\cos(\theta) = \pm\cos(\alpha) \text{ and } \sin(\theta) = \pm\sin(\alpha),$$

where the choice of the $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

In light of Theorem 11.1, it pays to know the sine and cosine values for certain common Quadrant I angles as well as to keep in mind the signs of the coordinates of points in the given quadrants.

θ(degrees)	θ(radians)	$\cos(\theta)$	$\sin(\theta)$
0°	0	1	0
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90°	$\frac{\pi}{2}$	0	1



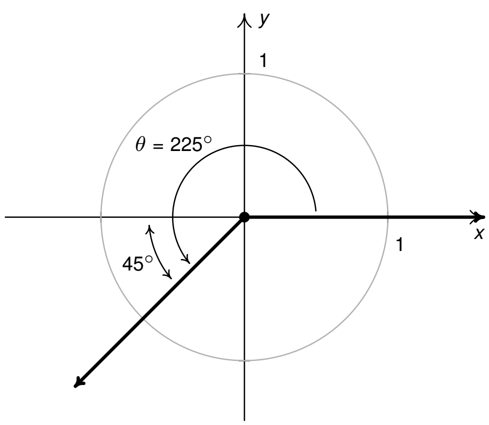
Example 11.2.2. Find the sine and cosine of the following angles.

1. $\theta = 225^\circ$
2. $\theta = \frac{11\pi}{6}$
3. $\theta = -\frac{5\pi}{4}$
4. $\theta = \frac{7\pi}{3}$

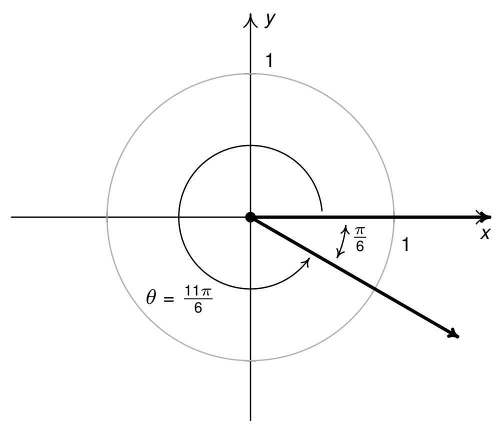
Solution.

1. We begin by plotting $\theta = 225^\circ$ in standard position and find its terminal side overshoots the negative x-axis to land in Quadrant III. Hence, we obtain θ's reference angle α by subtracting: $\alpha = \theta - 180^\circ = 225^\circ - 180^\circ = 45^\circ$. Since θ is a Quadrant III angle, both $\cos(\theta) < 0$ and $\sin(\theta) < 0$. The Reference Angle Theorem yields: $\cos(225^\circ) = -\cos(45^\circ) = -\frac{\sqrt{2}}{2}$ and $\sin(225^\circ) = -\sin(45^\circ) = -\frac{\sqrt{2}}{2}$.

2. The terminal side of $\theta = \frac{11\pi}{6}$, when plotted in standard position, lies in Quadrant IV, just shy of the positive x-axis. To find θ's reference angle α, we subtract: $\alpha = 2\pi - \theta = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$. Since θ is a Quadrant IV angle, $\cos(\theta) > 0$ and $\sin(\theta) < 0$, so the Reference Angle Theorem gives: $\cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{11\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$.



Finding $\cos(225^\circ)$ and $\sin(225^\circ)$

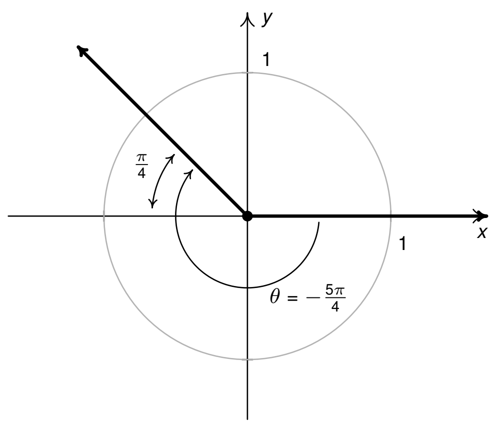


Finding $\cos\left(\frac{11\pi}{6}\right)$ and $\sin\left(\frac{11\pi}{6}\right)$

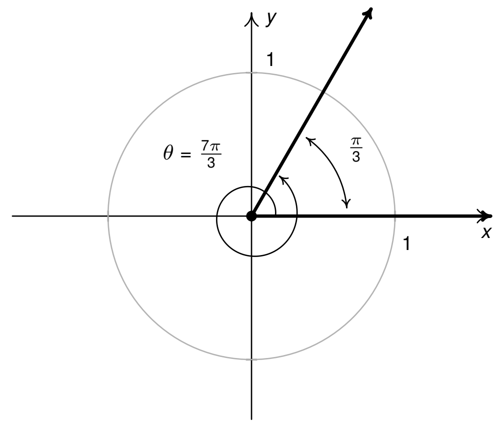
3. To plot $\theta = -\frac{5\pi}{4}$, we rotate *clockwise* an angle of $\frac{5\pi}{4}$ from the positive x-axis. The terminal side of θ, therefore, lies in Quadrant II making an angle of $\alpha = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$ radians with respect to the negative x-axis. Since θ is a Quadrant II angle, $\cos(\theta) < 0$ and $\sin(\theta) > 0$ so the Reference Angle Theorem gives: $\cos\left(-\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(-\frac{5\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

4. Since the angle $\theta = \frac{7\pi}{3}$ measures more than $2\pi = \frac{6\pi}{3}$, we find the terminal side of θ by rotating one full revolution followed by an additional $\alpha = \frac{7\pi}{3} - 2\pi = \frac{\pi}{3}$ radians. Hence, θ and α have the same terminal side,[5] and so $\cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

[5] Recall we say they are 'coterminal.'



Finding $\cos\left(-\frac{5\pi}{4}\right)$ and $\sin\left(-\frac{5\pi}{4}\right)$



Finding $\cos\left(\frac{7\pi}{3}\right)$ and $\sin\left(\frac{7\pi}{3}\right)$

$\square$

A couple of remarks are in order. First off, the reader may have noticed that when expressed in radian measure, the reference angle for a non-quadrantal angle is easy to spot. Reduced fraction multiples of π with a denominator of 6 have $\frac{\pi}{6}$ as a reference angle, those with a denominator of 4 have $\frac{\pi}{4}$ as their reference angle, and those with a denominator of 3 have $\frac{\pi}{3}$ as their reference angle.[6]

Also note in number 4 above, the angles $\frac{\pi}{3}$ and $\frac{7\pi}{3}$ are coterminal. As a result, have the same values for sine and cosine. It turns out that we can characterize coterminal angles in this manner, as stated below.

Theorem 11.2. Two angles α and β are coterminal if and only if:

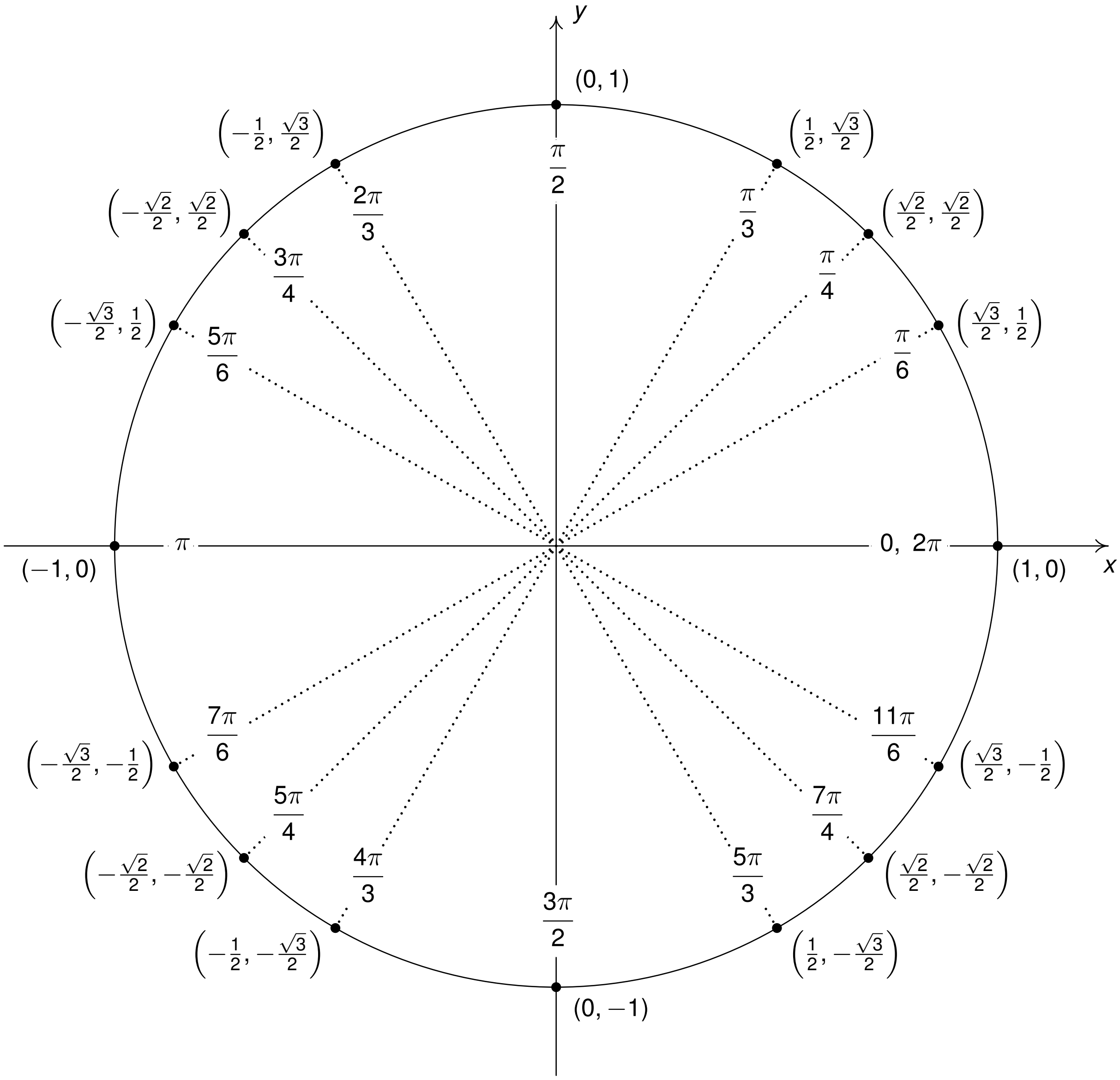
$$\cos(\alpha) = \cos(\beta) \text{ and } \sin(\alpha) = \sin(\beta).$$

Recall the phraseology 'if and only if' means there are two things to argue in Theorem 11.2: first, if α and β are co-terminal, then $\cos(\alpha) = \cos(\beta)$ and $\sin(\alpha) = \sin(\beta)$. This is immediate since coterminal share terminal sides, and, in particular, the (unique) point on the Unit Circle shared by said terminal side. Second, we need to argue that if $\cos(\alpha) = \cos(\beta)$ and $\sin(\alpha) = \sin(\beta)$, then α and β are coterminal.

To prove this second claim, note that when an angle is drawn in standard position, the terminal side of the angle is the ray that starts at the origin and is completely determined by *any* other point on the terminal side. If $\cos(\alpha) = \cos(\beta)$ and $\sin(\alpha) = \sin(\beta)$, then their terminal sides share a point on the Unit Circle, namely $(\cos(\alpha), \sin(\alpha)) = (\cos(\beta), \sin(\beta))$. Hence, α and β are coterminal.

The Reference Angle Theorem in conjunction with the table of sine and cosine values on Page 924 can be used to generate the figure on the next page. We recommend committing it to memory.

[6]For once, we have something convenient about using radian measure in contrast to the abstract theoretical nonsense about using them as a 'natural' way to match oriented angles with real numbers!



Important Points on the Unit Circle

Our next example uses The Reference Angle Theorem in a slightly more sophisticated context.

Example 11.2.3. Suppose α is an acute angle with $\cos(\alpha) = \frac{5}{13}$.

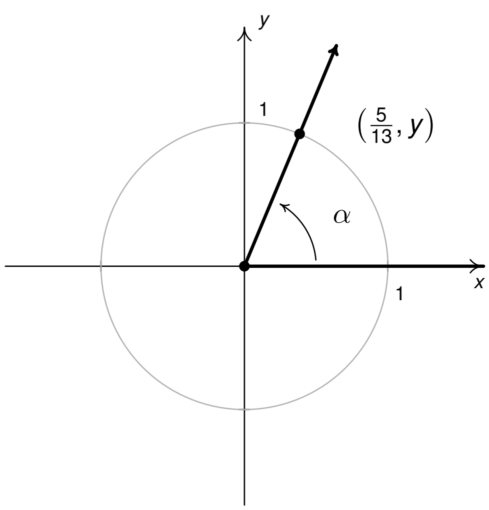
1. Find $\sin(\alpha)$ and use this to plot α in standard position.

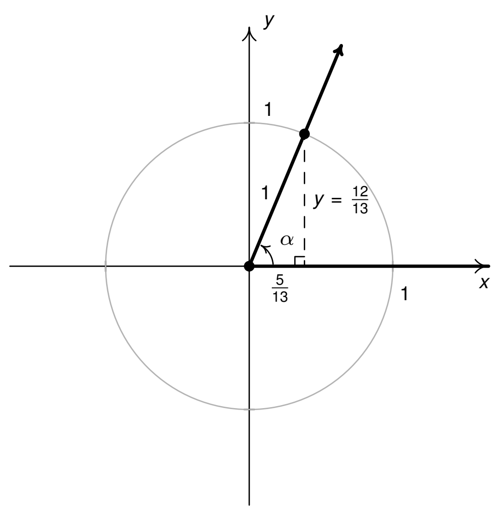
2. Find the sine and cosine of the following angles:

   (a) $\theta = \pi + \alpha$ (b) $\theta = 2\pi - \alpha$ (c) $\theta = 3\pi - \alpha$ (d) $\theta = \frac{\pi}{2} + \alpha$

Solution.

1. Since α is an acute angle, we know $0 < \alpha < \frac{\pi}{2}$ so the terminal side of α lies in Quadrant I. Moreover, since $\cos(\alpha) = \frac{5}{13}$, we know the x-coordinate of the intersection point of the terminal side of α and the Unit Circle is $\frac{5}{13}$. To find $\sin(\alpha)$, we need the y-coordinate. Taking a cue from Example 11.2.1, we drop a perpendicular from the terminal side of α to the x-axis as seen below on the right to form a right triangle with one leg measuring $\frac{5}{13}$ units and hypotenuse with a length of 1 unit.

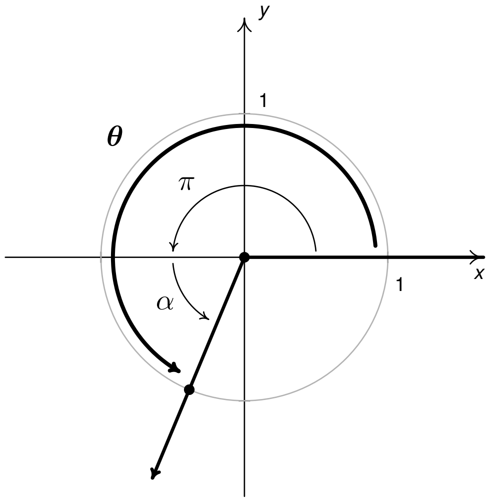




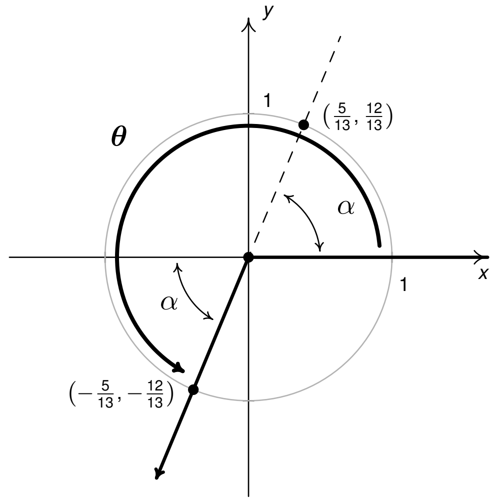
The Pythagorean Theorem gives $\left(\frac{5}{13}\right)^2 + y^2 = 1^2$ or $y = \frac{12}{13}$. Hence, $\sin(\alpha) = \frac{12}{13}$.

2. (a) To find the cosine and sine of $\theta = \pi + \alpha$, we first plot θ in standard position. We can imagine the sum of the angles $\pi + \alpha$ as a sequence of two rotations: a rotation of π radians followed by a rotation of α radians.[7] We see that α is the reference angle for θ. By The Reference Angle Theorem, $\cos(\theta) = \pm\cos(\alpha) = \pm\frac{5}{13}$ and $\sin(\theta) = \pm\sin(\alpha) = \pm\frac{12}{13}$. Since the terminal side of θ falls in Quadrant III, both $\cos(\theta)$ and $\sin(\theta)$ are negative, so $\cos(\theta) = -\frac{5}{13}$ and $\sin(\theta) = -\frac{12}{13}$.

[7] Since $\pi + \alpha = \alpha + \pi$, θ may be plotted by reversing the order of rotations given here. You should do this.

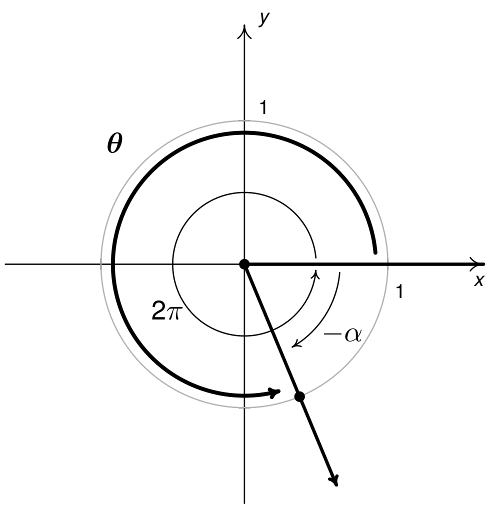


Visualizing $\boldsymbol{\theta = \pi + \alpha}$

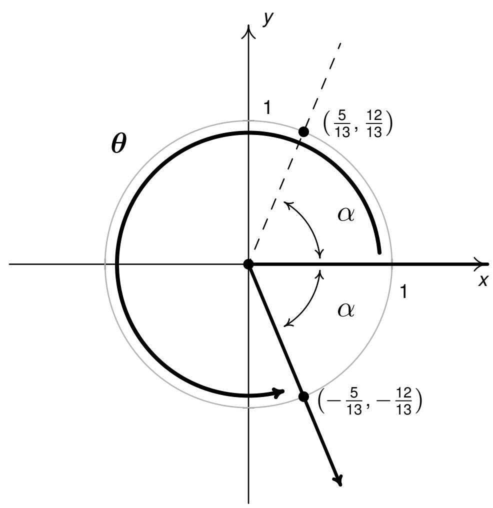


$\boldsymbol{\theta}$ has reference angle α

(b) Rewriting $\theta = 2\pi - \alpha$ as $\theta = 2\pi + (-\alpha)$, we can plot θ by visualizing one complete revolution counter-clockwise followed by a *clockwise* revolution, or 'backing up,' of α radians. Once again, we see that α is θ's reference angle. Since θ is a Quadrant IV angle, we choose the appropriate signs and get: $\cos(\theta) = \frac{5}{13}$ and $\sin(\theta) = -\frac{12}{13}$.

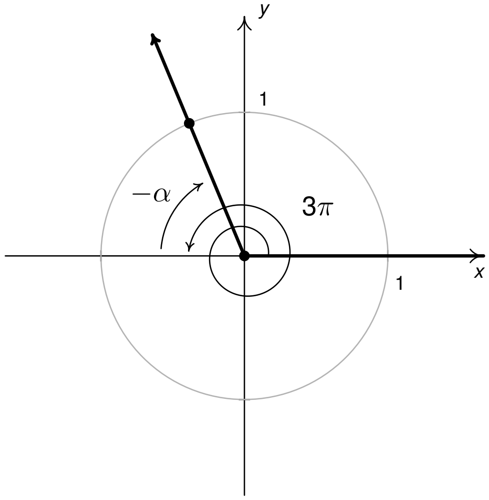


Visualizing $\boldsymbol{\theta = 2\pi - \alpha}$

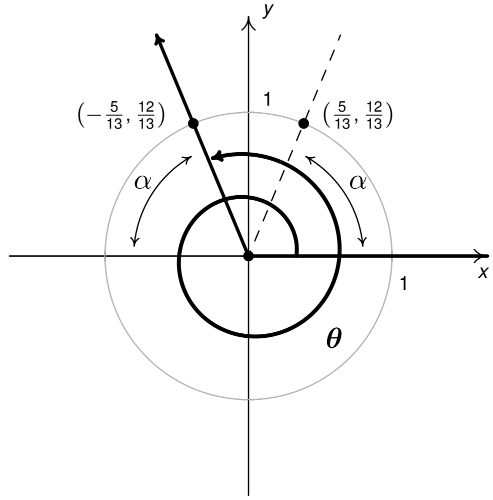


$\boldsymbol{\theta}$ has reference angle α

(c) Taking a cue from the previous problem, we rewrite $\theta = 3\pi - \alpha$ as $\theta = 3\pi + (-\alpha)$. The angle 3π represents one and a half revolutions counter-clockwise, so that when we 'back up' α radians, we end up in Quadrant II. Since α is the reference angle for θ, The Reference Angle Theorem gives $\cos(\theta) = -\frac{5}{13}$ and $\sin(\theta) = \frac{12}{13}$.

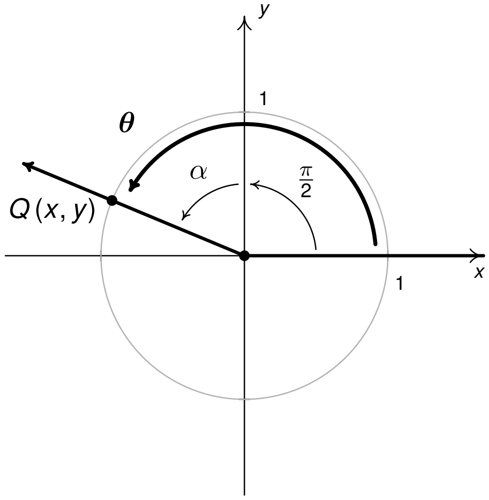


Visualizing $3\pi - \alpha$

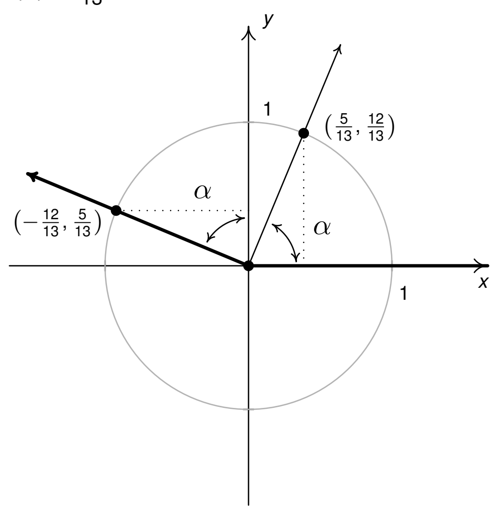


$\boldsymbol{\theta}$ has reference angle α

(d) To plot $\theta = \frac{\pi}{2} + \alpha$, we first rotate $\frac{\pi}{2}$ radians and follow up with α radians. The reference angle here is *not* α, so The Reference Angle Theorem is not immediately applicable. (It's important that you see why this is the case. Take a moment to think about this before reading on.) Let $Q(x, y)$ be the point on the terminal side of θ which lies on the Unit Circle so that $x = \cos(\theta)$ and $y = \sin(\theta)$. Once we graph α in standard position, we use the fact from Geometry that equal angles subtend equal chords to show that the dotted lines in the figure below are equal. Hence, $x = \cos(\theta) = -\frac{12}{13}$. Similarly, we find $y = \sin(\theta) = \frac{5}{13}$.



Visualizing $\boldsymbol{\theta = \frac{\pi}{2} + \alpha}$



Using symmetry to determine $Q(x, y)$

□

A couple of remarks about Example 11.2.3 are in order. First, we note the right triangle we used to find $\sin(\alpha)$ is a scaled 5-12-13 triangle. Recognizing this Pythagorean Triple[8] may have simplified our workflow.

[8]See Section B.2 for more examples of Pythagorean Triples.

Along the same lines, since, the Unit Circle, by definition, is described by the equation $x^2 + y^2 = 1$, we could substitute $x = \frac{5}{13}$ in order to find y. We leave it to the reader to show we get the exact same answer regardless of the approach used.

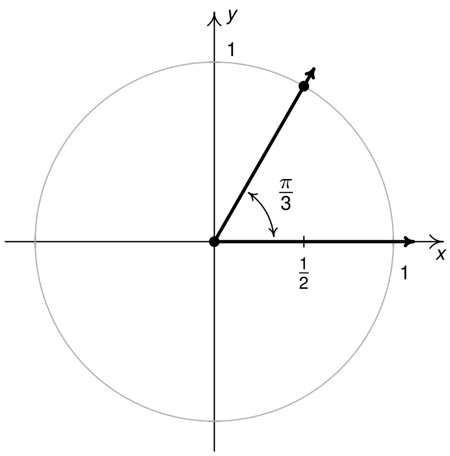
Our next example turns the tables and makes good use of the Unit Circle values given on 926 as well as Theorem 11.2 in a different way: instead of giving information about the angle and asking for sine or cosine values, we are given sine or cosine values and asked to produce the corresponding angles. In other words, we solve some rudimentary equations involving sine and cosine.[9]

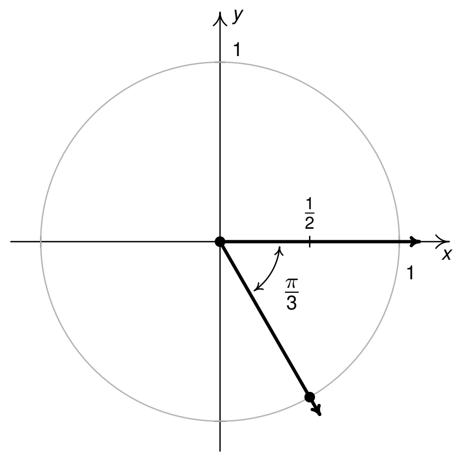
Example 11.2.4. Find all angles that satisfy the following equations. Express your answers in radians.[10]

1. $\cos(\theta) = \dfrac{1}{2}$
2. $\sin(\alpha) = -\dfrac{1}{2}$
3. $\cos(\beta) = 0.$
4. $\sin(\gamma) = \dfrac{3}{2}$

Solution.

1. If $\cos(\theta) = \frac{1}{2}$, then we know the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $x = \frac{1}{2}$. This means θ is a Quadrant I or IV angle. Since $\cos(\theta) = \frac{1}{2}$, we know from the values on page 926 that the reference angle is $\frac{\pi}{3}$.





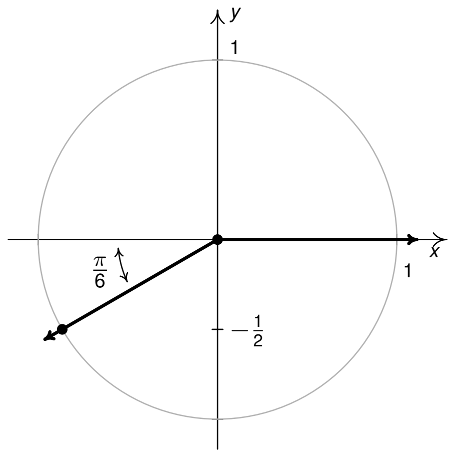
One solution in Quadrant I is $\theta = \frac{\pi}{3}$. Per Theorem 11.2, all other Quadrant I solutions must be coterminal with $\frac{\pi}{3}$. Recall from Section 11.1, two angles in radian measure are coterminal if and only if they differ by an integer multiple of 2π. Hence to describe all angles coterminal with a given angle, we add $2\pi k$ for integers $k = 0, \pm 1, \pm 2, \ldots$. Hence, we record our final answer as $\theta = \frac{\pi}{3} + 2\pi k$ for integers k. Proceeding similarly for the Quadrant IV case, we find the solution to $\cos(\theta) = \frac{1}{2}$ here is $\frac{5\pi}{3}$, so our answer in this Quadrant is $\theta = \frac{5\pi}{3} + 2\pi k$ for integers k.

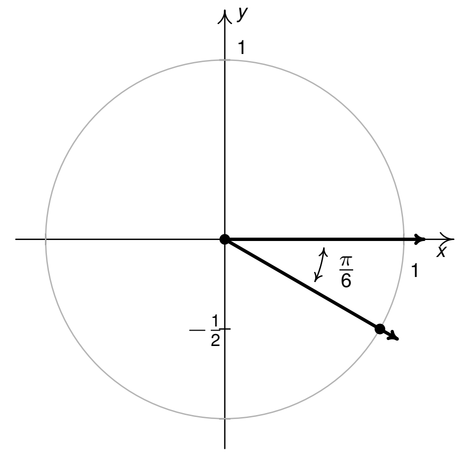
2. If $\sin(\alpha) = -\frac{1}{2}$, then when α is plotted in standard position, its terminal side intersects the Unit Circle at $y = -\frac{1}{2}$. From this, we determine α is a Quadrant III or Quadrant IV angle with reference angle $\frac{\pi}{6}$. In Quadrant III, one solution is $\frac{7\pi}{6}$, so we capture all Quadrant III solutions by adding integer

[9] We will study equations in more detail in Section 12.4.

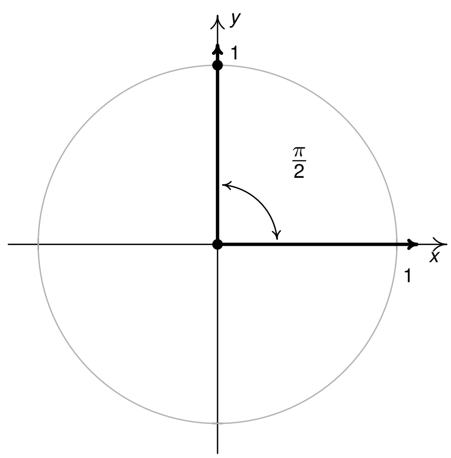
[10] This ensures we keep building the 'fluency with radians' which is so necessary for later work.

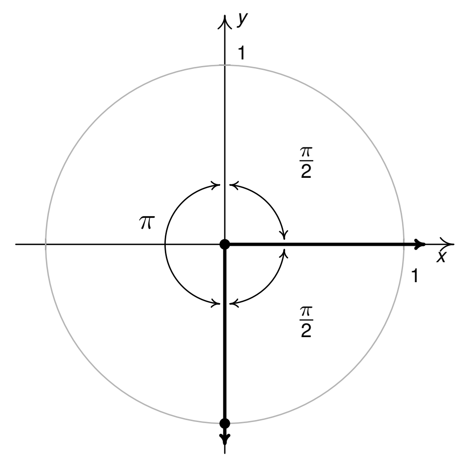
multiples of 2π: $\alpha = \frac{7\pi}{6} + 2\pi k$. In Quadrant IV, one solution is $\frac{11\pi}{6}$ so all the solutions here are of the form $\alpha = \frac{11\pi}{6} + 2\pi k$ for integers k.





3. If $\cos(\beta) = 0$, then the terminal side of β must lie on the line $x = 0$, also known as the y-axis.





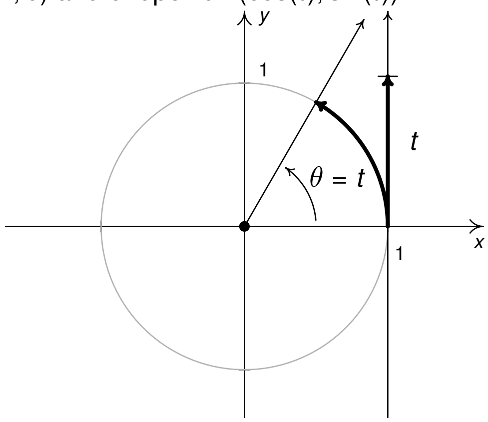
While, technically speaking, $\frac{\pi}{2}$ isn't a reference angle (it's not acute), we can nonetheless use it to find our answers. If we follow the procedure set forth in the previous examples, we find $\beta = \frac{\pi}{2} + 2\pi k$ and $\beta = \frac{3\pi}{2} + 2\pi k$ for integers, k. While this solution is correct, it can be shortened to $\beta = \frac{\pi}{2} + \pi k$ for integers k. The reader is encouraged to see the geometry using the diagram above on the left.

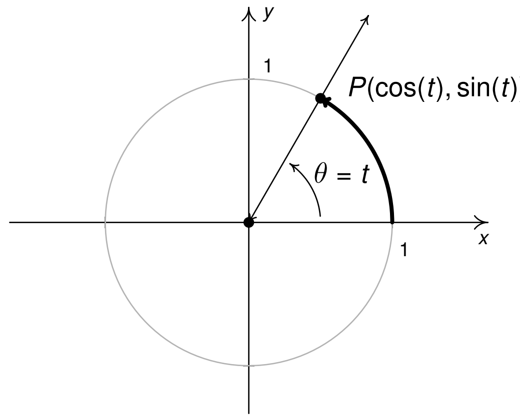
4. We are asked to solve $\sin(\gamma) = \frac{3}{2}$. Since sine values are y-coordinates on the Unit Circle, $\sin(\gamma)$ can't be any larger than 1. Hence, $\sin(t) = \frac{3}{2}$ has no solutions. □

One of the key items to take from Example 11.2.4 is that, in general, solutions to trigonometric equations consist of infinitely many answers. To get a feel for these answers, the reader is encouraged to follow our mantra from Chapter 10 - that is, 'When in doubt, write it out!' This is especially important when checking answers to the exercises.

For example, another Quadrant IV solution to $\sin(\theta) = -\frac{1}{2}$ is $\theta = -\frac{\pi}{6}$. Hence, the family of Quadrant IV answers to number 2 above could just have easily been written $\theta = -\frac{\pi}{6} + 2\pi k$ for integers k. While on the surface, this family may look different than the stated solution of $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k, we leave it to the reader to show they represent the same list of angles.

It is also worth noting that when asked to solve equations in algebra, we are usually looking for *real number* solutions. Thanks to the identifications made on page 909, we are able to regard the inputs to the sine and cosine functions as real numbers by identifying any real number t with an oriented angle θ measuring $\theta = t$ radians. That is, for each real number t, we associate an oriented arc t units in length with initial point $(1, 0)$ and endpoint $P(\cos(t), \sin(t))$.





In practice this means in expressions like '$\cos(\pi)$' and '$\sin(2)$,' the inputs can be thought of as either angles in radian measure or real numbers, whichever is more convenient.

Suppose, as in the Exercises, we are asked to find all *real number* solutions to the equation such as $\sin(t) = -\frac{1}{2}$. The discussion above allows us to find the *real number* solutions to this equation by *thinking* in angles. Indeed, we would solve this equation in the exact way we solved $\sin(\theta) = -\frac{1}{2}$ in Example 11.2.4 number 2. Our solution is only cosmetically different in that the variable used is t rather than θ: $t = \frac{7\pi}{6} + 2\pi k$ or $t = \frac{11\pi}{6} + 2\pi k$ for integers, k.

We will study the sine and cosine functions in greater detail in Section 11.3. Until then, keep in mind that any properties of the sine and cosine functions developed in the following sections which regard them as functions of *angles* in *radian* measure apply equally well if the inputs are regarded as *real numbers*.

11.2.1 Beyond the Unit Circle

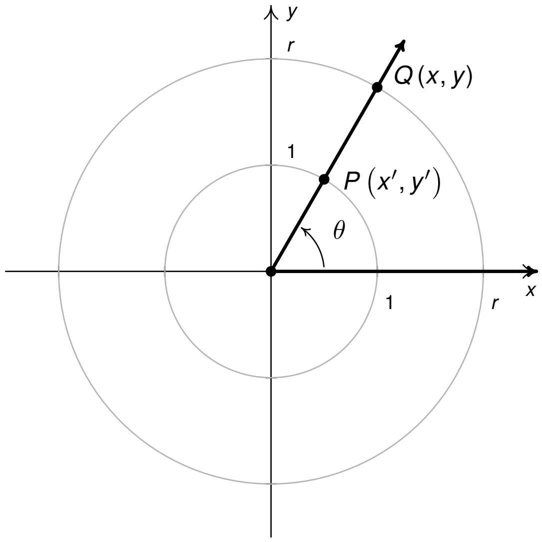
In Definition 11.2, we define the sine and cosine functions using the *Unit* Circle, $x^2 + y^2 = 1$. It turns out that we can use *any* circle centered at the origin to determine the sine and cosine values of angles. To show this, we essentially recycle the same similarity arguments used in Section B.2 to show the trigonometric ratios described in Definition B.1 are independent of the choice of right triangle used.[11]

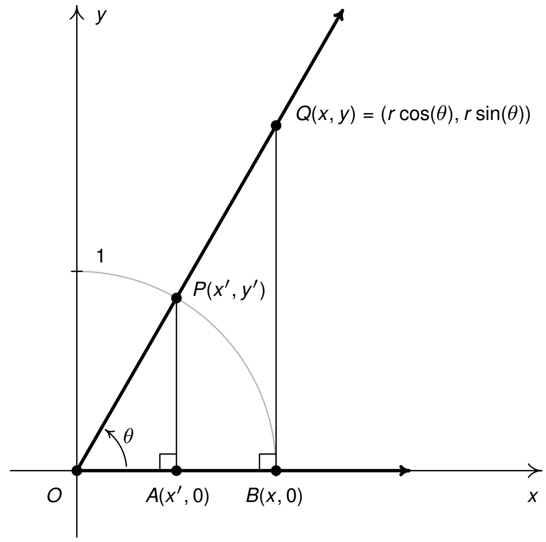
Consider for the moment the *acute* angle θ drawn below in standard position. Let $Q(x, y)$ be the point on the terminal side of θ which lies on the circle $x^2 + y^2 = r^2$, and let $P(x', y')$ be the point on the terminal side of θ which lies on the Unit Circle. Now consider dropping perpendiculars from P and Q to create two right triangles, $\triangle OPA$ and $\triangle OQB$. These triangles are similar,[12] thus it follows that $\frac{x}{x'} = \frac{r}{1} = r$, so $x = rx'$ and,

[11] Another approach uses transformations. See Exercise 71.

[12] Do you remember why?

similarly, we find $y = ry'$. Since, by definition, $x' = \cos(\theta)$ and $y' = \sin(\theta)$, we get the coordinates of Q to be $x = r\cos(\theta)$ and $y = r\sin(\theta)$. By reflecting these points through the x-axis, y-axis and origin, we obtain the result for all non-quadrantal angles θ, and we leave it to the reader to verify these formulas hold for the quadrantal angles as well.





Not only can we describe the coordinates of Q in terms of $\cos(\theta)$ and $\sin(\theta)$ but since the radius of the circle is $r = \sqrt{x^2 + y^2}$, we can also express $\cos(\theta)$ and $\sin(\theta)$ in terms of the coordinates of Q. These results are summarized in the following theorem.

Theorem 11.3. If $Q(x, y)$ is the point on the terminal side of an angle θ, plotted in standard position, which lies on the circle $x^2 + y^2 = r^2$ then $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Moreover,

$$\cos(\theta) = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \sin(\theta) = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

Note that in the case of the Unit Circle we have $r = \sqrt{x^2 + y^2} = 1$, so Theorem 11.3 reduces to our definitions of $\cos(\theta)$ and $\sin(\theta)$ in Definition 11.2. Our next example makes good use of Theorem 11.3.

Example 11.2.5.

1. Suppose that the terminal side of an angle θ, when plotted in standard position, contains the point $Q(4, -2)$. Find $\sin(\theta)$ and $\cos(\theta)$.

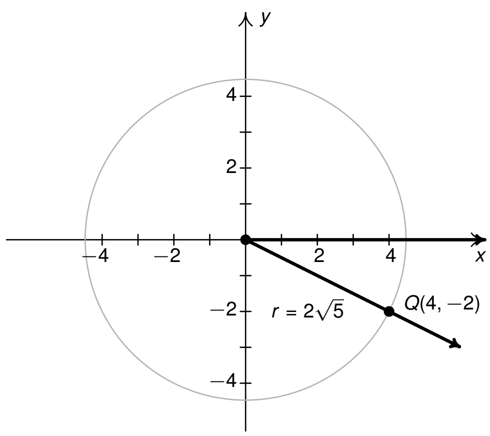
2. Suppose $\frac{\pi}{2} < \theta < \pi$ with $\sin(\theta) = \frac{8}{17}$. Find $\cos(\theta)$.

3. In Example 11.1.3 in Section 11.1, we approximated the radius of the earth at 41.628° north latitude to be 2960 miles. Justify this approximation if the spherical radius of the Earth is 3960 miles.

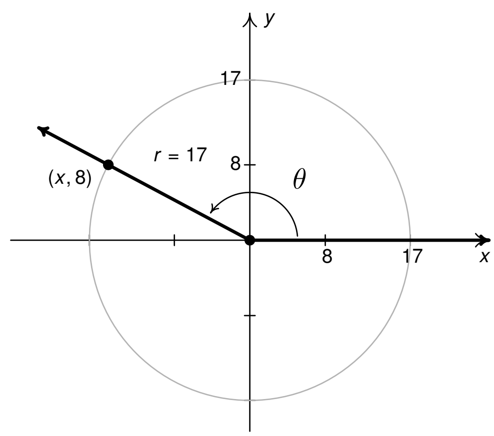
Solution.

1. Since we are given both the x and y coordinates of a point on the terminal side of this angle, we can use Theorem 11.3 directly. First, we find $r = \sqrt{x^2+y^2} = \sqrt{(-2)^2+4^2} = \sqrt{20} = 2\sqrt{5}$. This means the point Q lies on a circle of radius $2\sqrt{5}$ units as seen below on the left. Hence, $\cos(\theta) = \frac{x}{r} = \frac{4}{2\sqrt{5}} = \frac{2\sqrt{5}}{5}$ and $\sin(\theta) = \frac{y}{r} = \frac{-2}{2\sqrt{5}} = -\frac{\sqrt{5}}{5}$.

2. We are told $\frac{\pi}{2} < \theta < \pi$, so, in particular, θ is a Quadrant II angle. Per Theorem 11.3, $\sin(\theta) = \frac{8}{17} = \frac{y}{r}$ where y is the y-coordinate of the intersection point of the circle $x^2+y^2 = r^2$ and the terminal side of θ (when plotted in standard position, of course!) For convenience, we choose $r = 17$ so that $y = 8$, and we get the diagram below on the right. Since $x^2+y^2 = r^2$, we get $x^2+8^2 = 17^2$. We find $x = \pm 15$, and since θ is a Quadrant II angle, we get $x = -15$. Hence, $\cos(\theta) = -\frac{15}{17}$.

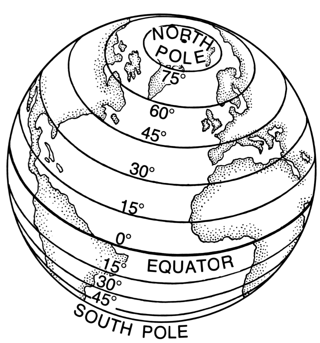


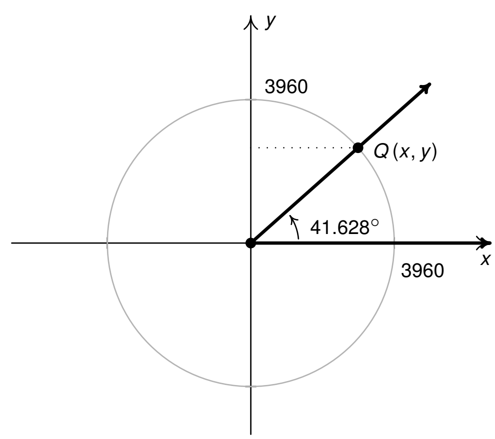
$Q(4,-2)$ lies on a circle of radius $2\sqrt{5}$ units



$0 < \theta < \frac{\pi}{2}$ with $\sin(\theta) = \frac{8}{17}$

3. Recall the diagram below on the left indicating the circles which are the parallels of latitude.[13]



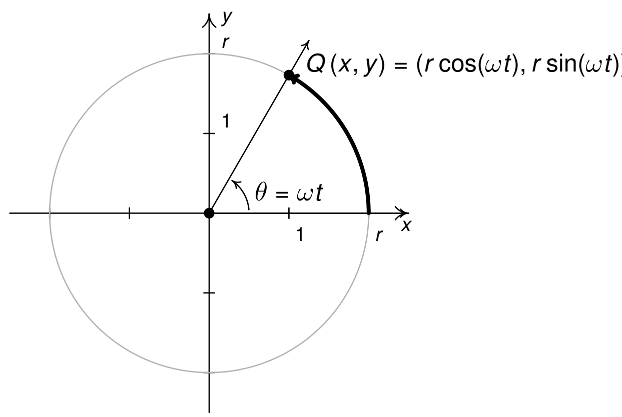


A point on the Earth at 41.628°N

[13]Diagram credit: Pearson Scott Foresman [Public domain], via Wikimedia Commons.

Assuming the Earth is a sphere of radius 3960 miles, a cross-section through the poles produces a circle of radius 3960 miles. Viewing the Equator as the x-axis, the value we seek is the x-coordinate of the point $Q(x, y)$ indicated in the figure above on the right. Using Theorem 11.3, we get $x = 3960\cos(41.628^\circ) \approx 2960$. Hence, the radius of the Earth at North Latitude 41.628° is approximately 2960 miles. $\square$

Theorem 11.3 gives us what we need to 'circle back' to the question posed at the the beginning of the section: how to describe the position of an object traveling in a circular path of radius r with constant angular velocity ω. Suppose that at time t, the object has swept out an angle measuring θ radians. If we assume that the object is at the point $(r, 0)$ when $t = 0$, the angle θ is in standard position. By definition, $\omega = \frac{\theta}{t}$ which we rewrite as $\theta = \omega t$. According to Theorem 11.3, the location of the object $Q(x, y)$ on the circle is found using the equations $x = r\cos(\theta) = r\cos(\omega t)$ and $y = r\sin(\theta) = r\sin(\omega t)$. Hence, at time t, the object is at the point $(r\cos(\omega t), r\sin(\omega t))$, as seen in the diagram below.



Equations for Circular Motion

We have just argued the following.

> **Equation 11.3.** Suppose an object is traveling in a circular path of radius r centered at the origin with constant angular velocity ω. If $t = 0$ corresponds to the point $(r, 0)$, then the x and y coordinates of the object are functions of t and are given by $x = r\cos(\omega t)$ and $y = r\sin(\omega t)$. Here, $\omega > 0$ indicates a counter-clockwise direction and $\omega < 0$ indicates a clockwise direction.

Example 11.2.6. Suppose we are in the situation of Example 11.1.3. Find the equations of motion of Lakeland Community College as the earth rotates.

Solution. From Example 11.1.3, we take $r = 2960$ miles and and $\omega = \frac{\pi}{12 \text{ hours}}$. Hence, the equations of motion are $x = r\cos(\omega t) = 2960\cos\left(\frac{\pi}{12}t\right)$ and $y = r\sin(\omega t) = 2960\sin\left(\frac{\pi}{12}t\right)$, where x and y are measured in miles and t is measured in hours.[14] $\square$

[14] We will revisit this concept of associating points with times in Section 14.5.

11.2.2 Exercises

In Exercises 1 - 20, find the exact value of the cosine and sine of the given angle.

1. $\theta = 0$
2. $\theta = \dfrac{\pi}{4}$
3. $\theta = \dfrac{\pi}{3}$
4. $\theta = \dfrac{\pi}{2}$
5. $\theta = \dfrac{2\pi}{3}$
6. $\theta = \dfrac{3\pi}{4}$
7. $\theta = \pi$
8. $\theta = \dfrac{7\pi}{6}$
9. $\theta = \dfrac{5\pi}{4}$
10. $\theta = \dfrac{4\pi}{3}$
11. $\theta = \dfrac{3\pi}{2}$
12. $\theta = \dfrac{5\pi}{3}$
13. $\theta = \dfrac{7\pi}{4}$
14. $\theta = \dfrac{23\pi}{6}$
15. $\theta = -\dfrac{13\pi}{2}$
16. $\theta = -\dfrac{43\pi}{6}$
17. $\theta = -\dfrac{3\pi}{4}$
18. $\theta = -\dfrac{\pi}{6}$
19. $\theta = \dfrac{10\pi}{3}$
20. $\theta = 117\pi$

In Exercises 21 - 29, find all of the angles which satisfy the given equation.

21. $\sin(\theta) = \dfrac{1}{2}$
22. $\cos(\theta) = -\dfrac{\sqrt{3}}{2}$
23. $\sin(\theta) = 0$
24. $\cos(\theta) = \dfrac{\sqrt{2}}{2}$
25. $\sin(\theta) = \dfrac{\sqrt{3}}{2}$
26. $\cos(\theta) = -1$
27. $\sin(\theta) = -1$
28. $\cos(\theta) = \dfrac{\sqrt{3}}{2}$
29. $\cos(\theta) = -1.001$

In Exercises 30 - 38, solve the equation for t. (See the remarks on page 11.2.)

30. $\cos(t) = 0$
31. $\sin(t) = -\dfrac{\sqrt{2}}{2}$
32. $\cos(t) = 3$
33. $\sin(t) = -\dfrac{1}{2}$
34. $\cos(t) = \dfrac{1}{2}$
35. $\sin(t) = -2$
36. $\cos(t) = 1$
37. $\sin(t) = 1$
38. $\cos(t) = -\dfrac{\sqrt{2}}{2}$

In Exercises 39 - 42, let θ be the angle in standard position whose terminal side contains the given point then compute $\cos(\theta)$ and $\sin(\theta)$.

39. $P(-7, 24)$
40. $Q(3, 4)$
41. $R(5, -9)$
42. $T(-2, -11)$

In Exercises 43 - 52, use the results developed throughout the section to find the requested value.

43. If $\sin(\theta) = -\dfrac{7}{25}$ with θ in Quadrant IV, what is $\cos(\theta)$?

44. If $\cos(\theta) = \dfrac{4}{9}$ with θ in Quadrant I, what is $\sin(\theta)$?

45. If $\sin(\theta) = \dfrac{5}{13}$ with θ in Quadrant II, what is $\cos(\theta)$?

46. If $\cos(\theta) = -\dfrac{2}{11}$ with θ in Quadrant III, what is $\sin(\theta)$?

47. If $\sin(\theta) = -\dfrac{2}{3}$ with θ in Quadrant III, what is $\cos(\theta)$?

48. If $\cos(\theta) = \dfrac{28}{53}$ with θ in Quadrant IV, what is $\sin(\theta)$?

49. If $\sin(\theta) = \dfrac{2\sqrt{5}}{5}$ and $\dfrac{\pi}{2} < \theta < \pi$, what is $\cos(\theta)$?

50. If $\cos(\theta) = \dfrac{\sqrt{10}}{10}$ and $2\pi < \theta < \dfrac{5\pi}{2}$, what is $\sin(\theta)$?

51. If $\sin(\theta) = -0.42$ and $\pi < \theta < \dfrac{3\pi}{2}$, what is $\cos(\theta)$?

52. If $\cos(\theta) = -0.98$ and $\dfrac{\pi}{2} < \theta < \pi$, what is $\sin(\theta)$?

In Exercises 53 - 58, use your calculator to approximate the given value to three decimal places. Make sure your calculator is in the proper angle measurement mode!

53. $\sin(78.95^\circ)$
54. $\cos(-2.01)$
55. $\sin(392.994)$
56. $\cos(207^\circ)$
57. $\sin(\pi^\circ)$
58. $\cos(e)$

In Exercises 59 - 63, write the given function as a nontrivial decomposition of functions as directed.

59. For $f(t) = 3t + \sin(2t)$, find functions g and h so that $f = g + h$.

60. For $f(\theta) = 3\cos(\theta) - \sin(4\theta)$, find functions g and h so that $f = g - h$.

61. For $f(t) = e^{-0.1t}\sin(3t)$, find functions g and h so that $f = gh$.

62. For $r(t) = \dfrac{\sin(t)}{t}$, find functions f and g so $r = \dfrac{f}{g}$.

63. For $r(\theta) = \sqrt{3\cos(\theta)}$, find functions f and g so $r = g \circ f$.

64. For each function $S(t)$ listed below, compute the average rate of change over the indicated interval.[15] What trends do you notice? Be sure your calculator is in radian mode!

$S(t)$	$[-0.1, 0.1]$	$[-0.01, 0.01]$	$[-0.001, 0.001]$
$\sin(t)$			
$\sin(2t)$			
$\sin(3t)$			
$\sin(4t)$			

In Exercises 65 - 68, find the equations of motion for the given scenario. Assume that the center of the motion is the origin, the motion is counter-clockwise and that $t = 0$ corresponds to a position along the positive x-axis. (See Equation 11.3 and Example 11.1.3.)

65. A point on the edge of the spinning yo-yo in Exercise 38 from Section 11.1.

    Recall: The diameter of the yo-yo is 2.25 inches and it spins at 4500 revolutions per minute.

66. The yo-yo in exercise 40 from Section 11.1.

    Recall: The radius of the circle is 28 inches and it completes one revolution in 3 seconds.

67. A point on the edge of the hard drive in Exercise 41 from Section 11.1.

    Recall: The diameter of the hard disk is 2.5 inches and it spins at 7200 revolutions per minute.

68. A passenger on the Big Wheel in Exercise 43 from Section 11.1.

    Recall: The diameter is 128 feet and completes 2 revolutions in 2 minutes, 7 seconds.

69. Consider the numbers: 0, 1, 2, 3, 4. Take the square root of each of these numbers, then divide each by 2. The resulting numbers should look hauntingly familiar. (See the values in the table on 924.)

70. On page 933, we see that the sine and cosine functions of *angles* can be considered functions of *real numbers*. With help from your classmates, discuss the domains and ranges of $f(t) = \sin(t)$ and $g(t) = \cos(t)$. Write your answers using interval notation.

71. Another way to establish Theorem 11.3 is to use transformations. Re-read the discussion following Theorem 8.4 in Chapter 8 and transform the Unit Circle, $x^2 + y^2 = 1$, to $x^2 + y^2 = r^2$ using horizontal and vertical stretches. Show if the coordinates on the Unit Circle are $(\cos(\theta), \sin(\theta))$, then the corresponding coordinates on $x^2 + y^2 = r^2$ are $(r\cos(\theta), r\sin(\theta))$.

72. In the scenario of Equation 11.3, we assumed that at $t = 0$, the object was at the point $(r, 0)$. If this is not the case, we can adjust the equations of motion by introducing a 'time delay.' If $t_0 > 0$ is the first time the object passes through the point $(r, 0)$, show, with the help of your classmates, the equations of motion are $x = r\cos(\omega(t - t_0))$ and $y = r\sin(\omega(t - t_0))$.

[15] See Definition 1.8 in Section 1.2.4 for a review of this concept, as needed.

11.2.3 Answers

1. $\cos(0) = 1,\ \sin(0) = 0$

2. $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

3. $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\ \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

4. $\cos\left(\frac{\pi}{2}\right) = 0,\ \sin\left(\frac{\pi}{2}\right) = 1$

5. $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2},\ \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

6. $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

7. $\cos(\pi) = -1,\ \sin(\pi) = 0$

8. $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2},\ \sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$

9. $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

10. $\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2},\ \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

11. $\cos\left(\frac{3\pi}{2}\right) = 0,\ \sin\left(\frac{3\pi}{2}\right) = -1$

12. $\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2},\ \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

13. $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

14. $\cos\left(\frac{23\pi}{6}\right) = \frac{\sqrt{3}}{2},\ \sin\left(\frac{23\pi}{6}\right) = -\frac{1}{2}$

15. $\cos\left(-\frac{13\pi}{2}\right) = 0,\ \sin\left(-\frac{13\pi}{2}\right) = -1$

16. $\cos\left(-\frac{43\pi}{6}\right) = -\frac{\sqrt{3}}{2},\ \sin\left(-\frac{43\pi}{6}\right) = \frac{1}{2}$

17. $\cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

18. $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},\ \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

19. $\cos\left(\frac{10\pi}{3}\right) = -\frac{1}{2},\ \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

20. $\cos(117\pi) = -1,\ \sin(117\pi) = 0$

21. $\sin(\theta) = \frac{1}{2}$ when $\theta = \frac{\pi}{6} + 2\pi k$ or $\theta = \frac{5\pi}{6} + 2\pi k$ for any integer k.

22. $\cos(\theta) = -\frac{\sqrt{3}}{2}$ when $\theta = \frac{5\pi}{6} + 2\pi k$ or $\theta = \frac{7\pi}{6} + 2\pi k$ for any integer k.

23. $\sin(\theta) = 0$ when $\theta = \pi k$ for any integer k.

24. $\cos(\theta) = \frac{\sqrt{2}}{2}$ when $\theta = \frac{\pi}{4} + 2\pi k$ or $\theta = \frac{7\pi}{4} + 2\pi k$ for any integer k.

25. $\sin(\theta) = \frac{\sqrt{3}}{2}$ when $\theta = \frac{\pi}{3} + 2\pi k$ or $\theta = \frac{2\pi}{3} + 2\pi k$ for any integer k.

26. $\cos(\theta) = -1$ when $\theta = (2k+1)\pi$ for any integer k.

27. $\sin(\theta) = -1$ when $\theta = \frac{3\pi}{2} + 2\pi k$ for any integer k.

28. $\cos(\theta) = \dfrac{\sqrt{3}}{2}$ when $\theta = \dfrac{\pi}{6} + 2\pi k$ or $\theta = \dfrac{11\pi}{6} + 2\pi k$ for any integer k.
29. $\cos(\theta) = -1.001$ never happens
30. $\cos(t) = 0$ when $t = \dfrac{\pi}{2} + \pi k$ for any integer k.
31. $\sin(t) = -\dfrac{\sqrt{2}}{2}$ when $t = \dfrac{5\pi}{4} + 2\pi k$ or $t = \dfrac{7\pi}{4} + 2\pi k$ for any integer k.
32. $\cos(t) = 3$ never happens.
33. $\sin(t) = -\dfrac{1}{2}$ when $t = \dfrac{7\pi}{6} + 2\pi k$ or $t = \dfrac{11\pi}{6} + 2\pi k$ for any integer k.
34. $\cos(t) = \dfrac{1}{2}$ when $t = \dfrac{\pi}{3} + 2\pi k$ or $t = \dfrac{5\pi}{3} + 2\pi k$ for any integer k.
35. $\sin(t) = -2$ never happens
36. $\cos(t) = 1$ when $t = 2\pi k$ for any integer k.
37. $\sin(t) = 1$ when $t = \dfrac{\pi}{2} + 2\pi k$ for any integer k.
38. $\cos(t) = -\dfrac{\sqrt{2}}{2}$ when $t = \dfrac{3\pi}{4} + 2\pi k$ or $t = \dfrac{5\pi}{4} + 2\pi k$ for any integer k.
39. $\cos(\theta) = -\dfrac{7}{25},\ \sin(\theta) = \dfrac{24}{25}$
40. $\cos(\theta) = \dfrac{3}{5},\ \sin(\theta) = \dfrac{4}{5}$
41. $\cos(\theta) = \dfrac{5\sqrt{106}}{106},\ \sin(\theta) = -\dfrac{9\sqrt{106}}{106}$
42. $\cos(\theta) = -\dfrac{2\sqrt{5}}{25},\ \sin(\theta) = -\dfrac{11\sqrt{5}}{25}$
43. If $\sin(\theta) = -\dfrac{7}{25}$ with θ in Quadrant IV, then $\cos(\theta) = \dfrac{24}{25}$.
44. If $\cos(\theta) = \dfrac{4}{9}$ with θ in Quadrant I, then $\sin(\theta) = \dfrac{\sqrt{65}}{9}$.
45. If $\sin(\theta) = \dfrac{5}{13}$ with θ in Quadrant II, then $\cos(\theta) = -\dfrac{12}{13}$.
46. If $\cos(\theta) = -\dfrac{2}{11}$ with θ in Quadrant III, then $\sin(\theta) = -\dfrac{\sqrt{117}}{11}$.

47. If $\sin(\theta) = -\frac{2}{3}$ with θ in Quadrant III, then $\cos(\theta) = -\frac{\sqrt{5}}{3}$.

48. If $\cos(\theta) = \frac{28}{53}$ with θ in Quadrant IV, then $\sin(\theta) = -\frac{45}{53}$.

49. If $\sin(\theta) = \frac{2\sqrt{5}}{5}$ and $\frac{\pi}{2} < \theta < \pi$, then $\cos(\theta) = -\frac{\sqrt{5}}{5}$.

50. If $\cos(\theta) = \frac{\sqrt{10}}{10}$ and $2\pi < \theta < \frac{5\pi}{2}$, then $\sin(\theta) = \frac{3\sqrt{10}}{10}$.

51. If $\sin(\theta) = -0.42$ and $\pi < \theta < \frac{3\pi}{2}$, then $\cos(\theta) = -\sqrt{0.8236} \approx -0.9075$.

52. If $\cos(\theta) = -0.98$ and $\frac{\pi}{2} < \theta < \pi$, then $\sin(\theta) = \sqrt{0.0396} \approx 0.1990$.

53. $\sin(78.95^\circ) \approx 0.981$

54. $\cos(-2.01) \approx -0.425$

55. $\sin(392.994) \approx -0.291$

56. $\cos(207^\circ) \approx -0.891$

57. $\sin(\pi^\circ) \approx 0.055$

58. $\cos(e) \approx -0.912$

59. One solution is $g(t) = 3t$ and $h(t) = \sin(2t)$.

60. One solution is $g(\theta) = 3\cos(\theta)$ and $h(\theta) = \sin(4\theta)$.

61. One solution is $g(t) = e^{-0.1t}$ and $h(t) = \sin(3t)$.

62. One solution is $f(t) = \sin(t)$ and $g(t) = t$.

63. One solution is $f(\theta) = 3\cos(\theta)$ and $g(\theta) = \sqrt{\theta}$.

64. As we zoom in towards 0, the average rate of change of $\sin(kt)$ approaches k.

$S(t)$	$[-0.1, 0.1]$	$[-0.01, 0.01]$	$[-0.001, 0.001]$
$\sin(t)$	≈ 0.9983	≈ 1	≈ 1
$\sin(2t)$	≈ 1.9867	≈ 1.9999	≈ 2
$\sin(3t)$	≈ 2.9552	≈ 2.9995	≈ 3
$\sin(4t)$	≈ 3.8942	≈ 3.9989	≈ 4

65. $r = 1.125$ inches, $\omega = 9000\pi \, \frac{\text{radians}}{\text{minute}}$, $x = 1.125\cos(9000\pi\, t)$, $y = 1.125\sin(9000\pi\, t)$. Here x and y are measured in inches and t is measured in minutes.

66. $r = 28$ inches, $\omega = \frac{2\pi}{3} \, \frac{\text{radians}}{\text{second}}$, $x = 28\cos\left(\frac{2\pi}{3}\, t\right)$, $y = 28\sin\left(\frac{2\pi}{3}\, t\right)$. Here x and y are measured in inches and t is measured in seconds.

67. $r = 1.25$ inches, $\omega = 14400\pi \, \frac{\text{radians}}{\text{minute}}$, $x = 1.25\cos(14400\pi\, t)$, $y = 1.25\sin(14400\pi\, t)$. Here x and y are measured in inches and t is measured in minutes.

68. $r = 64$ feet, $\omega = \frac{4\pi}{127} \, \frac{\text{radians}}{\text{second}}$, $x = 64\cos\left(\frac{4\pi}{127}\, t\right)$, $y = 64\sin\left(\frac{4\pi}{127}\, t\right)$. Here x and y are measured in feet and t is measured in seconds

11.3 Graphs of Sine and Cosine

On page 933, we discussed how to interpret the sine and cosine of real numbers. To review, we identify a real number t with an oriented angle θ measuring t radians[1] and define $\sin(t) = \sin(\theta)$ and $\cos(t) = \cos(\theta)$. Since every real number can be identified with one and only one angle θ this way, the domains of the functions $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are all real numbers, $(-\infty, \infty)$.

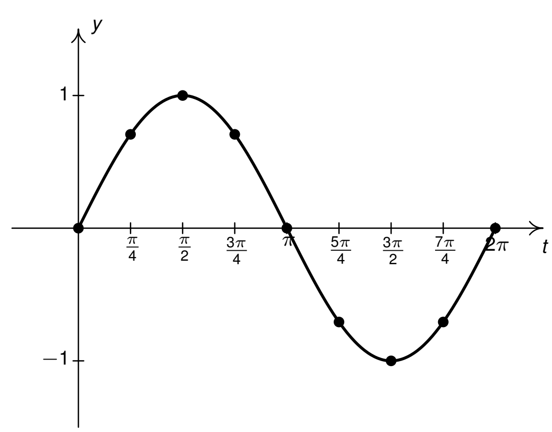
When it comes to range, recall that the sine and cosine of angles are coordinates of points on the Unit Circle and hence, each fall between -1 and 1 inclusive. Since the real number line,[2] when wrapped around the Unit Circle completely covers the circle, we can be assured that every point on the Unit Circle corresponds to at least one real number. Putting these two facts together, we conclude the range of $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are both $[-1, 1]$. We summarize these two important facts below.

Theorem 11.4. Domain and Range of the Cosine and Sine Functions:

- The function $f(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $g(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$

Our aim in this section is to become familiar with the graphs of $f(t) = \sin(t)$ and $g(t) = \cos(t)$. To that end, we begin by making a table and plotting points. We'll start by graphing $f(t) = \sin(t)$ by making a table of values and plotting the corresponding points. We'll keep the independent variable 't' for now and use the default 'y' as our dependent variable.[3] Note in the graph below, on the right, the scale of the horizontal and vertical axis is far from 1:1. (We will present a more accurately scaled graph shortly.)

t	$\sin(t)$	$(t, \sin(t))$
0	0	$(0, 0)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, \frac{\sqrt{2}}{2}\right)$
π	0	$(\pi, 0)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
2π	0	$(2\pi, 0)$



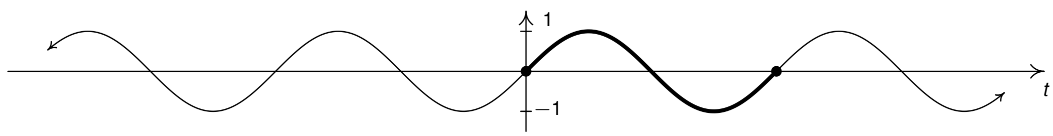
Graphing $y = \sin(t)$.

[1] which, you'll recall essentially 'wraps the real number line around the Unit Circle

[2] in particular the interval $[0, 2\pi)$

[3] Keep in mind that we're using 'y' here to denote the *output* from the sine function. It is a coincidence that the y-values on the graph of $y = \sin(t)$ correspond to the y-values on the Unit Circle.

If we plot additional points, we soon find that the graph repeats itself. This shouldn't come as too much of a surprise considering Theorem 11.2. In fact, in light of that theorem, we expect the function to repeat itself every 2π units. Below is a more accurately scaled graph highlighting the portion we had already graphed above. The graph is often described as having a 'wavelike' nature and is sometimes called a **sine wave** or, more technically, a **sinusoid**.



A more accurately scaled graph of $f(t) = \sin(t)$.

Note that by copying the highlighted portion of the graph and pasting it end-to-end, we obtain the entire graph of $f(t) = \sin(t)$. We give this 'repeating' property a name.

Definition 11.3. Periodic Functions: A function f is said to be **periodic** if there is a real number c so that $f(t + c) = f(t)$ for all real numbers t in the domain of f. The smallest positive number p for which $f(t + p) = f(t)$ for all real numbers t in the domain of f, if it exists, is called the **period** of f.

We have already seen a family of periodic functions in Section 1.2: the constant functions. However, despite being periodic a constant function has no period. (We'll leave that odd gem as an exercise for you.)

Returning to $f(t) = \sin(t)$, we see that by Definition 11.3, f is periodic since $\sin(t+2\pi) = \sin(t)$. To determine the period of f, we need to find the smallest real number p so that $f(t + p) = f(t)$ for all real numbers t or, said differently, the smallest positive real number p such that $\sin(t + p) = \sin(t)$ for all real numbers t.

We know that $\sin(t + 2\pi) = \sin(t)$ for all real numbers t but the question remains if any smaller real number will do the trick. Suppose $p > 0$ and $\sin(t + p) = \sin(t)$ for all real numbers t. Then, in particular, $\sin(0+p) = \sin(0) = 0$ so that $\sin(p) = 0$. From this we know p is a multiple of π. Since $\sin\left(\frac{\pi}{2}\right) \neq \sin\left(\frac{\pi}{2} + \pi\right)$, we know $p \neq \pi$. Hence, $p = 2\pi$ so the period of $f(t) = \sin(t)$ is 2π.

Having period 2π essentially means that we can completely understand everything about the function $f(t) = \sin(t)$ by studying *one* interval of length 2π, say $[0, 2\pi]$.[4] For this reason, when graphing sine (and cosine) functions, we typically restrict our attention to graphing these functions over the course of one period to produce one **cycle** of the graph.

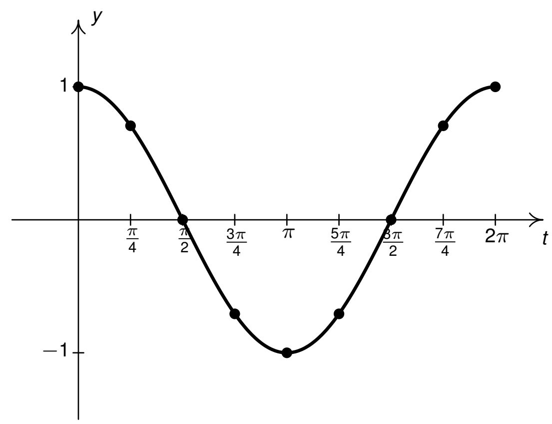
Not surprisingly, the graph of $g(t) = \cos(t)$ exhibits similar behavior as $f(t) = \sin(t)$ as seen below.[6]

[4]Technically, we should study the interval $[0, 2\pi)$,[5]since whatever happens at $t = 2\pi$ is the same as what happens at $t = 0$. As we will see shortly, $t = 2\pi$ gives us an extra 'check' when we go to graph these functions.

[5]In some advanced texts, the interval of choice is $[-\pi, \pi)$.

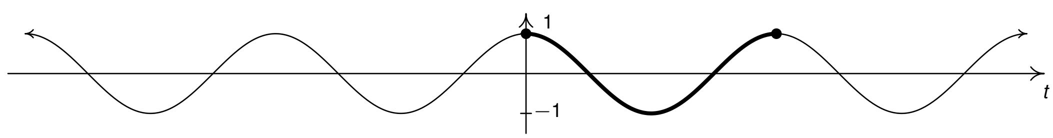
[6]Here note that the dependent variable 'y' represents the outputs from $g(t) = \cos(t)$ which are x-coordinates on the Unit CIrcle.

t	$\cos(t)$	$(t, \cos(t))$
0	1	$(0, 1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
π	-1	$(\pi, -1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, \frac{\sqrt{2}}{2}\right)$
2π	1	$(2\pi, 1)$



Graphing $y = \cos(t)$.

Like $f(t) = \sin(t)$, $g(t) = \cos(t)$ is a wavelike curve with period 2π. Moreover, the graphs of the sine and cosine functions have the same shape - differing only in what appears to be a horizontal shift. As we'll prove in Section 12.2, $\sin\left(t + \frac{\pi}{2}\right) = \cos(t)$, which means we can obtain the graph of $y = \cos(t)$ by shifting the graph of $y = \sin(t)$ to the left $\frac{\pi}{2}$ units.[7]



The graph of $g(t) = \cos(t)$.

While arguably the most important property shared by $f(t) = \sin(t)$ and $g(t) = \cos(t)$ is their periodic 'wavelike' nature,[8] their graphs suggest these functions are both continuous and smooth. Recall from Section 2.1 that, like polynomial functions, the graphs of the sine and cosine functions have no jumps, gaps, holes in the graph, vertical asymptotes, corners or cusps.

Note the graphs of both $f(t) = \sin(t)$ and $g(t) = \cos(t)$ meander as $t \to -\infty$ or as $\to \infty$. Said differently, none of the limits $\displaystyle\lim_{t \to -\infty} \sin(t)$, $\displaystyle\lim_{t \to \infty} \sin(t)$, $\displaystyle\lim_{t \to -\infty} \cos(t)$, or $\displaystyle\lim_{t \to \infty} \cos(t)$ exist. Even though these functions are 'trapped' (or **bounded**) between -1 and 1, neither graph has any horizontal asymptotes.

Lastly, the graphs of $f(t) = \sin(t)$ and $g(t) = \cos(t)$ suggest each enjoy one of the symmetries introduced in Section 2.1. The graph of $y = \sin(t)$ appears to be symmetric about the origin while the graph of $y = \cos(t)$ appears to be symmetric about the y-axis. Indeed, as we'll prove in Section 12.2, $f(t) = \sin(t)$ is, in fact, an odd function: [9] that is, $\sin(-t) = -\sin(t)$ and $g(t) = \cos(t)$ is an even function, so $\cos(-t) = \cos(t)$.

We summarize all of these properties in the following result.

[7]Hence, we can obtain the graph of $y = \sin(t)$ by shifting the graph of $y = \cos(t)$ to the *right* $\frac{\pi}{2}$ units: $\cos\left(t - \frac{\pi}{2}\right) = \sin(t)$.

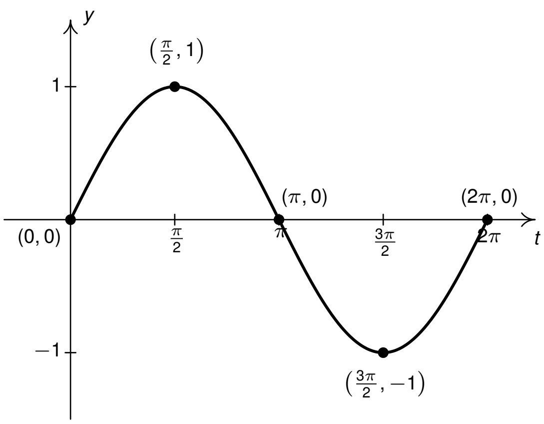
[8]this is the reason they are so useful in the Sciences and Engineering

[9]The reader may wish to review Definitions 2.2 and 2.3 as needed.

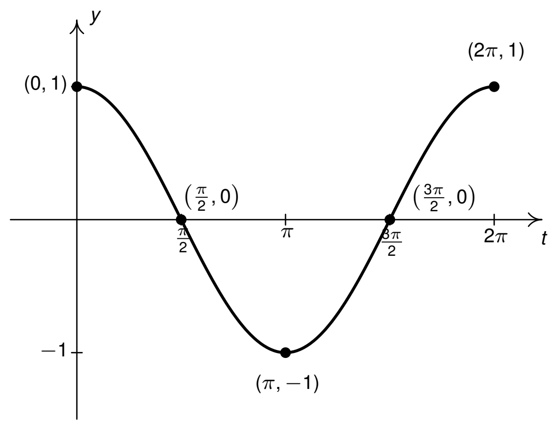
Theorem 11.5. Properties of the Cosine and Sine Functions

- The function $f(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is odd
  - has period 2π
- The function $g(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is even
  - has period 2π
- Conversion formulas: $\sin\left(t + \frac{\pi}{2}\right) = \cos(t)$ and $\cos\left(t - \frac{\pi}{2}\right) = \sin(t)$

Now that we know the basic shapes of the graphs of $y = \sin(t)$ and $y = \cos(t)$, we can use the results of Section 5.4 to graph more complicated functions using transformations. The fact that both of these functions are periodic means we only have to know what happens over the course of one period of the function in order to determine what happens to all points on the graph. To that end, we graph the '**fundamental cycle**' - the portion of each graph generated over the interval $[0, 2\pi]$ - for each sine and cosine:



The 'fundamental cycle' of $y = \sin(t)$.



The 'fundamental cycle' of $y = \cos(t)$.

In working through Section 5.4 , it was very helpful to track 'key points' through the transformations. The 'key points' we've indicated on the graphs above correspond to the quadrantal angles and generate the zeros and the extrema of functions. Since the quadrantal angles divide the interval $[0, 2\pi]$ into four equal pieces, we shall refer to these angles henceforth as the 'quarter marks.'

It is worth noting that because the transformations discussed in Section 5.4 are linear,[10] the *relative spacing* of the points before and after the transformations remains the same.[11] In particular, wherever the interval $[0, 2\pi]$ is mapped, the quarter marks of the new interval correspond to the quarter marks of $[0, 2\pi]$. (Can you see why?) We will exploit this fact in the following example.

[10] See the remarks at the beginning of Section 5.4.

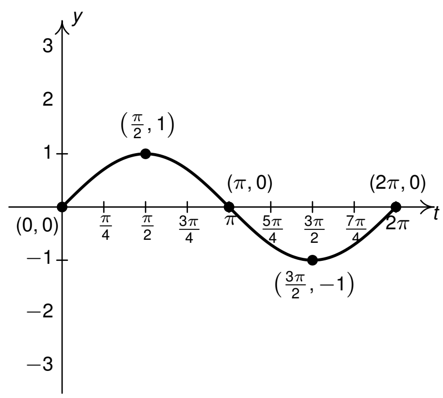
[11] If we use a linear function $f(t) = mt + b$ to transform the inputs, t, then $\Delta[f(t)] = m\Delta t$. That is, the change *after* the the transformation, $m\Delta t$, is just a multiple of the change *before* the transformation, Δt.

Example 11.3.1. Graph one cycle of the following functions. State the period of each.

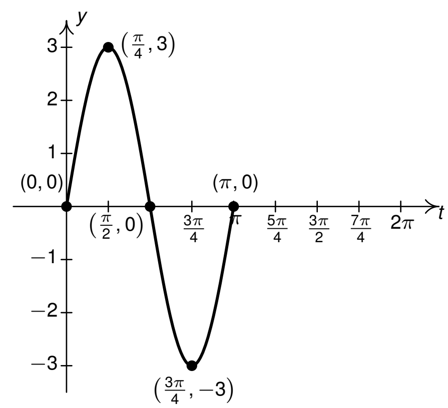
1. $f(t) = 3\sin(2t)$
2. $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$

Solution.

1. One way to proceed is to use Theorem 5.11 and follow the procedure outlined there. Starting with the fundamental cycle of $y = \sin(t)$, we divide each t-coordinate by 2 and multiply each y-coordinate by 3 to obtain one cycle of $y = 3\sin(2t)$.



The ‘fundamental cycle’ of $y = \sin(t)$.

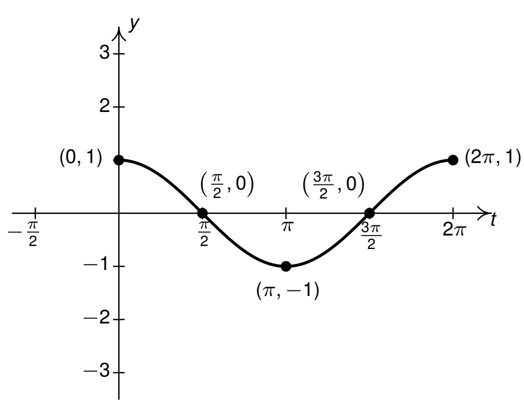


One cycle of $y = 3\sin(2t)$.

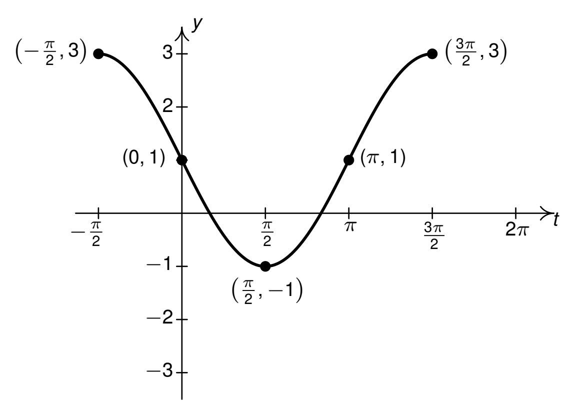
Since one cycle of $y = f(t)$ is completed over the interval $[0, \pi]$, the period of f is π.

2. Starting with the fundamental cycle of $y = \cos(t)$ and using Theorem 5.11, we subtract $\frac{\pi}{2}$ from each of the t-coordinates, then multiply each y-coordinate by 2, and add 1 to each y-coordinate.

We find one cycle of $y = g(t)$ is completed over the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, the period is $\frac{3\pi}{2} - \left(-\frac{\pi}{2}\right) = 2\pi$.



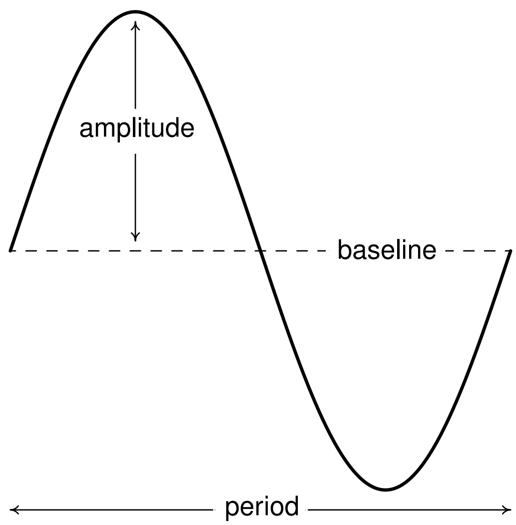
The ‘fundamental cycle’ of $y = \cos(t)$.



One cycle of $y = 2\cos\left(t + \frac{\pi}{2}\right)+1$

$\square$

As previously mentioned, the curves graphed in Example 11.3.1 are examples of sinusoids. A sinusoid is the result of taking the graph of $y = \sin(t)$ or $y = \cos(t)$ and performing any of the transformations mentioned in Section 5.4. We graph one cycle of a generic sinusoid below. Sinusoids can be characterized by four properties: period, phase shift, vertical shift (or 'baseline'), and amplitude.



We have already discussed the period of a sinusoid. If we think of t as measuring time, the period is how long it takes for the sinusoid to complete one cycle and is usually represented by the letter T. The standard period of both $\sin(t)$ and $\cos(t)$ is 2π, but horizontal scalings will change this.

In Example 11.3.1, for instance, the function $f(t) = 3\sin(2t)$ has period π instead of 2π because the graph is horizontally compressed by a factor of 2 as compared to the graph of $y = \sin(t)$. However, the period of $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$ is the same as the period of $\cos(t)$, 2π, since there are no horizontal scalings.

The **phase shift** of the sinusoid is the horizontal shift. Again, thinking of t as time, the phase shift of a sinusoid can be thought of as when the sinusoid 'starts' as compared to $t = 0$. Assuming there are no reflections across the y-axis, we can determine the phase shift of a sinusoid by finding where the value $t = 0$ on the graph of $y = \sin(t)$ or $y = \cos(t)$ is mapped to under the transformations.

For $f(t) = 3\sin(2t)$, the phase shift is '0' since the value $t = 0$ on the graph of $y = \sin(t)$ remains stationary under the transformations. Loosely speaking, this means both $y = \sin(t)$ and $y = 3\sin(2t)$ 'start' at the same time. The phase shift of $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$ is $-\frac{\pi}{2}$ or '$\frac{\pi}{2}$ to the *left*' since the value $t = 0$ on the graph of $y = \cos(t)$ is mapped to $t = -\frac{\pi}{2}$ on the graph of $y = 2\cos\left(t + \frac{\pi}{2}\right) + 1$. Again, loosely speaking, this means $y = 2\cos\left(t + \frac{\pi}{2}\right) + 1$ starts $\frac{\pi}{2}$ time units *earlier* than $y = \cos(t)$.

The vertical shift of a sinusoid is exactly the same as the vertical shifts in Section 5.4 and determines the new 'baseline' of the sinusoid. Thanks to symmetry, the vertical shift can always be found by averaging the maximum and minimum values of the sinusoid. For $f(t) = 3\sin(2t)$, the vertical shift is 0 whereas the vertical shift of $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$ is 1 or '1 up.'

The **amplitude** of the sinusoid is a measure of how 'tall' the wave is, as indicated in the figure below. Said differently, the amplitude measures how much the curve gets displaced from its 'baseline. ' The amplitude of the standard cosine and sine functions is 1, but vertical scalings can alter this.

In Example 11.3.1, the amplitude of $f(t) = 3\sin(2t)$ is 3, owing to the vertical stretch by a factor of 3 as compared with the graph of $y = \sin(t)$. In the case of $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$, the amplitude is 2 due to its vertical stretch as compared with the graph of $y = \cos(t)$. Note that the '+1' here does *not* affect the amplitude of the curve; it merely changes the 'baseline' from $y = 0$ to $y = 1$.

The following theorem shows how these four fundamental quantities relate to the parameters which describe a generic sinusoid. The proof follows from Theorem 5.11 and is left to the reader in Exercise 33.

Theorem 11.6. For $\omega > 0$, the graphs of

$$S(t) = A\sin(\omega t + \phi) + B \quad \text{and} \quad C(t) = A\cos(\omega t + \phi) + B$$

- have period $T = \dfrac{2\pi}{\omega}$
- have phase shift $-\dfrac{\phi}{\omega}$
- have amplitude $|A|$
- have vertical shift or 'baseline' B

The parameter ω mentioned above is called the **angular frequency**, or more simply, the **frequency** of the sinusoid and is the number of cycles the sinusoid completes over an interval of length 2π. That is, ω measures how 'frequently' the sinusoid repeats over an interval of length 2π. As we'll see in the next example, we can always ensure $\omega > 0$ using the even and odd properties of the cosine and sine functions, respectively. If t represents time, ω as represents how fast the sinusoid is being generated in terms of *radians* per unit time. In essence, it is the *angular speed* of the curve.

A quantity closely related to the angular frequency of the sinusoid is the **ordinary frequency** of the sinusoid, usually denoted f. The ordinary frequency of a sinusoid measures the number of cycles the sinusoid completes over an interval of length 1. Since the period, T represents the length of the interval required for a sinusoid to make one complete cycle, we have $f = \frac{1}{T}$. Once again, if t represents time, the ordinary frequency measures how fast the sinusoid is being generated in terms of *complete cycles* per unit time.[12]

Note that since $T = \frac{2\pi}{\omega}$, $f = \frac{1}{T} = \frac{\omega}{2\pi}$. Rewriting, we get $\omega = 2\pi f$. To understand this equation in terms of units, recall 1 complete cycle (revolution) around the Unit Circle counts for 2π radians. Hence, to get from f, measured in cycles per unit time, to ω, measured in radians per unit time, we need to multiply by 2π.

If the concepts of period and frequency seem familiar, they should. In Section 11.1, we discussed these very same ideas in the context of Example 11.1.3 and revisited them again in Section 11.2 in Equation 11.3. and Example 11.2.6. On the one hand, the notions presented here are more general, since they are not tied directly to circular motion. On the other hand, the stipulation in this section that $\omega > 0$ means we are restricting our attention to angular *speeds* instead of the more general angular *velocities*.[13]

Last, but not least, the quantity ϕ mentioned in Theorem 11.6 is called the **phase** or **phase angle** of the sinusoid. The phase of a sinusoid is the angle in the argument which corresponds to $t = 0$, and is important in describing waves in fields such as physics and electronics. When *graphing* sinusoids, however, we focus our attention on the horizontal shift induced by ϕ, $-\frac{\phi}{\omega}$.

We put Theorem 11.6 to good use in the next example.

[12]If t is time measured in seconds, then one cycle per second is 1 **Hertz**.

[13]Recall that velocity is speed with a direction. In 11.3, $\omega > 0$ indicated counter-clockwise motion while $\omega < 0$ indicated clockwise motion.

Example 11.3.2. Use Theorem 11.6 to determine the frequency, period, phase shift, amplitude, and vertical shift of each of the following functions and use this information to graph one cycle of each function.

1. $f(t) = 3\cos\left(\frac{\pi t - \pi}{2}\right) + 1$
2. $g(t) = \frac{1}{2}\sin(\pi - 2t) + \frac{3}{2}$

Solution.

1. To use Theorem 11.6, we first need to rewrite $f(t)$ in the form prescribed by Theorem 11.6. To that end, we rewrite: $f(t) = 3\cos\left(\frac{\pi t - \pi}{2}\right) + 1 = 3\cos\left(\frac{\pi}{2}t + \left(-\frac{\pi}{2}\right)\right) + 1$.

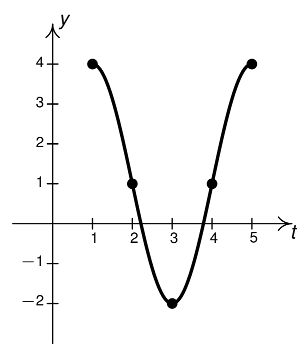
   From this, we identify $A = 3$, $\omega = \frac{\pi}{2}$, $\phi = -\frac{\pi}{2}$ and $B = 1$. According to Theorem 11.6, the frequency is $\omega = \frac{\pi}{2}$, the period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/2} = 4$, the phase shift is $-\frac{\phi}{\omega} = -\frac{-\pi/2}{\pi/2} = 1$ (indicating a shift to the *right* 1 unit), the amplitude is $|A| = |3| = 3$, and the vertical shift is $B = 1$ (indicating a shift *up* 1 unit.)

   To graph $y = f(t)$, we know one cycle begins at $t = 1$ (the phase shift.) Since the period is 4, we know the cycle ends 4 units later at $t = 1 + 4 = 5$. If we divide the interval $[1, 5]$ into four equal pieces, each piece has length $\frac{4}{4} = 1$. Hence, we to get our quarter marks, we start with $t = 1$ and add 1 unit until we reach the endpoint, $t = 5$. Our new quarter marks are: $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$.

   We now substitute these new quarter marks into $f(t)$ to obtain the corresponding y-values on the graph.[14] We connect the dots in a 'wavelike' fashion to produce the graph below on the right.

   Note that we can (partially) spot-check our answer by noting the average of the maximum and minimum is $\frac{4+(-2)}{2} = 1$ (our vertical shift) and the amplitude, $4 - 1 = 1 - (-2) = 3$ is indeed 3.

t	$f(t)$	$(t, f(t))$
1	4	$(1, 4)$
2	1	$(2, 1)$
3	-2	$(3, -2)$
4	1	$(4, 1)$
5	4	$(5, 4)$



One cycle of $y = f(t)$.

   Thought not asked for, this example provides a nice opportunity to interpret the ordinary frequency: $f = \frac{1}{T} = \frac{1}{4}$. Hence, $\frac{1}{4}$ of the sinusoid is traced out over an interval that is 1 unit long.

2. Turning our attention now to the function g, we first note that the coefficient of t is negative. In order to use Theorem 11.6, we need that coefficient to be positive. Hence, we first use the odd property of the sine function to rewrite $\sin(\pi - 2t)$ so that instead of a coefficient of -2, t has a coefficient of 2. We get $\sin(\pi - 2t) = \sin(-2t + \pi) = \sin(-(2t - \pi)) = -\sin(2t - \pi)$. Hence, $g(t) = -\frac{1}{2}\sin(2t + (-\pi)) + \frac{3}{2}$.

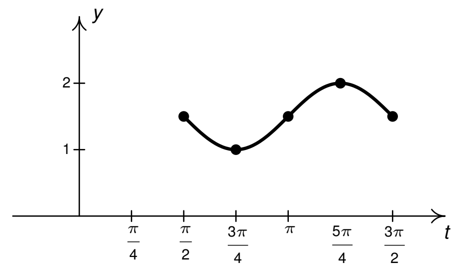
[14]Note when we substitute the quarter marks into $f(t)$, the argument of the cosine function simplifies to the quadrantal angles. That is, when we substitute $t = 1$, the argument of cosine simplifies to 0; when we substitute $t = 2$, the argument simplifies $\frac{\pi}{2}$ and so on. This provides a quick check of our calculations.

We idenitfy $A = -\frac{1}{2}$, $\omega = 2$, $\phi = -\pi$ and $B = \frac{3}{2}$. The frequency is $\omega = 2$, the period is $T = \frac{2\pi}{2} = \pi$, the phase shift is $-\frac{-\pi}{2} = \frac{\pi}{2}$ (indicating a shift *right* $\frac{\pi}{2}$ units), the amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$, and, finally, the vertical shift is *up* $\frac{3}{2}$.

Proceeding as before, we know one cycle of g starts at $t = \frac{\pi}{2}$ and ends at $t = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$. Dividing the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ into four equal pieces gives pieces of length $\frac{\pi}{4}$ units. Hence, to obtain our new quarter marks, we start at $t = \frac{\pi}{2}$ and add $\frac{\pi}{4}$ until we reach $t = \frac{3\pi}{2}$. Our new quarter marks are: $t = \frac{\pi}{2}$, $t = \frac{3\pi}{4}$, $t = \pi$, $t = \frac{5\pi}{4}$, $t = \frac{3\pi}{2}$. Substituting these values into g gives us the points to plot to produce the graph below on the right.

Again, we can quickly check the vertical shift by averaging the maximum and minimum values: $\frac{2+1}{2} = \frac{3}{2}$ and verify the amplitude: $2 - \frac{3}{2} = \frac{3}{2} - 1 = \frac{1}{2}$.

t	$g(t)$	$(t, g(t))$
$\frac{\pi}{2}$	$\frac{3}{2}$	$\left(\frac{\pi}{2}, \frac{3}{2}\right)$
$\frac{3\pi}{4}$	1	$\left(\frac{3\pi}{4}, 1\right)$
π	$\frac{3}{2}$	$\left(\pi, \frac{3}{2}\right)$
$\frac{5\pi}{4}$	2	$\left(\frac{5\pi}{4}, 2\right)$
$\frac{3\pi}{2}$	$\frac{3}{2}$	$\left(\frac{3\pi}{2}, \frac{3}{2}\right)$



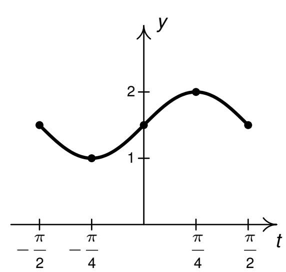
One cycle of $y = g(t)$.

□

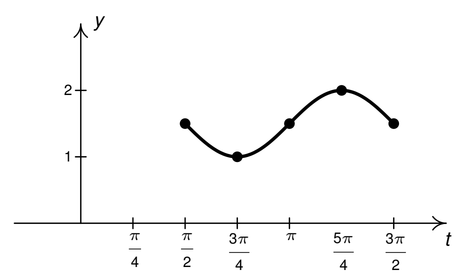
Note that in this section, we have discussed *two* ways to graph sinusoids: using Theorem 5.11 from Section 5.4 and using Theorem 11.6. Both methods will produce one cycle of the resulting sinusoid, but each method may produce a *different* cycle of the same sinusoid.

For example, if we graphed the function $g(t) = \frac{1}{2}\sin(\pi - 2t) + \frac{3}{2}$ from Example 11.3.2 using Theorem 5.11, we obtain the following:

t	$g(t)$	$(t, g(t))$
$\frac{\pi}{2}$	$\frac{3}{2}$	$\left(\frac{\pi}{2}, \frac{3}{2}\right)$
$\frac{\pi}{4}$	2	$\left(\frac{\pi}{4}, 2\right)$
0	$\frac{3}{2}$	$\left(0, \frac{3}{2}\right)$
$-\frac{\pi}{4}$	1	$\left(-\frac{\pi}{4}, 1\right)$
$-\frac{\pi}{2}$	$\frac{3}{2}$	$\left(-\frac{\pi}{2}, \frac{3}{2}\right)$



One cycle via Theorem 5.11.



One cycle via Theorem 11.6.

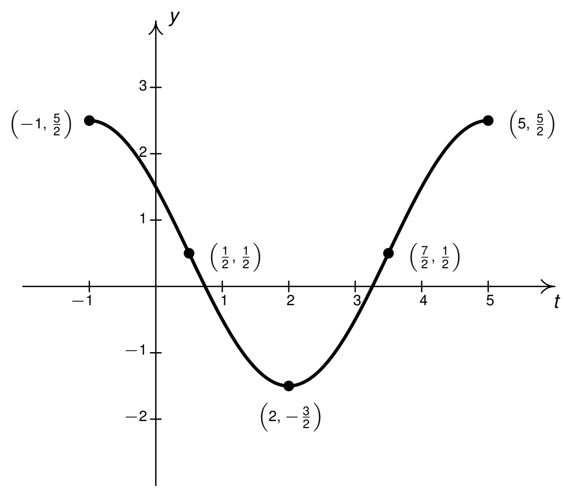
Comparing this result with the one obtained in Example 11.3.2 side by side, we see that one cycle ends right where the other starts. The cause of this discrepancy goes back to using the odd property of sine.

Essentially, the odd property of the sine function converts a reflection across the y-axis into a reflection across the t-axis. (Can you see why?) For this reason, whenever the coefficient of t is negative, Theorems 5.11 and 11.6 will produce different results.

In the Exercises, we assume the problems are worked using Theorem 11.6. If you choose to use Theorems 5.11 instead, your answer may look different than what is provided even though both your answer and the textbook's answer represent *one* cycle of the *same* function.

In the next example, we use Theorem 11.6 to determine the formula of a sinusoid given the graph of one cycle. Note that in some disciplines, sinusoids are written in terms of sines whereas in others, cosines functions are preferred. To cover all bases, we ask for both.

Example 11.3.3. Below is the graph of one complete cycle of a sinusoid $y = f(t)$.



One cycle of $y = f(t)$.

1. Write $f(t)$ in the form $C(t) = A\cos(\omega t + \phi) + B$, for $\omega > 0$.
2. Write $f(t)$ in the form $S(t) = A\sin(\omega t + \phi) + B$, for $\omega > 0$.

Solution.

1. Since one cycle is graphed over the interval $[-1, 5]$, its period is $T = 5 - (-1) = 6$. According to Theorem 11.6, $6 = T = \frac{2\pi}{\omega}$, so that $\omega = \frac{\pi}{3}$. Next, we see that the phase shift is -1, so we have $-\frac{\phi}{\omega} = -1$, or $\phi = \omega = \frac{\pi}{3}$.

   To find the baseline, we average the maximum and minimum values: $B = \frac{1}{2}\left[\frac{5}{2} + \left(-\frac{3}{2}\right)\right] = \frac{1}{2}(1) = \frac{1}{2}$. To find the amplitude, we subtract the maximum value from the baseline: $A = \frac{5}{2} - \frac{1}{2} = 2$.

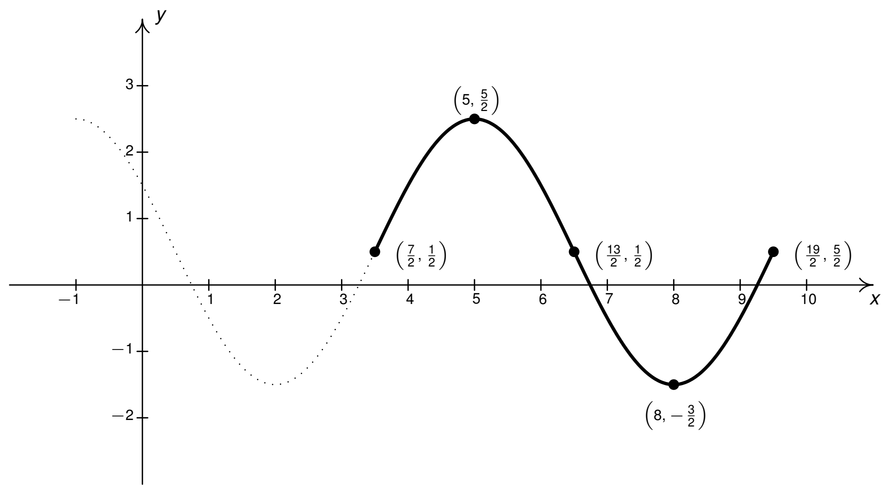
   Putting this altogether, we obtain our final answer is $f(t) = 2\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right) + \frac{1}{2}$.

2. Since we have written $f(t)$ in terms of cosines, we can use the conversion from sine to cosine as listed in Theorem 11.5. Since $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$, $\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right) = \sin\left(\left[\frac{\pi}{3}t + \frac{\pi}{3}\right] + \frac{\pi}{2}\right)$, so $\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}t + \frac{5\pi}{6}\right)$. Our final answer is $f(t) = 2\sin\left(\frac{\pi}{3}t + \frac{5\pi}{6}\right) + \frac{1}{2}$.

   However, for the sake of completeness, we provide another solution strategy which enables us to write $f(t)$ in terms of sines without starting with our answer from part 1.

Note that we obtain the period, amplitude, and vertical shift as before: $\omega = \frac{\pi}{3}$, $A = 2$ and $B = \frac{1}{2}$. The trickier part is finding the phase shift.

To that end, we imagine extending the graph of the given sinusoid as in the figure below so that we can identify a cycle beginning at $\left(\frac{7}{2}, \frac{1}{2}\right)$. Taking the phase shift to be $\frac{7}{2}$, we get $-\frac{\phi}{\omega} = \frac{7}{2}$, or $\phi = -\frac{7}{2}\omega = -\frac{7}{2}\left(\frac{\pi}{3}\right) = -\frac{7\pi}{6}$. Hence, our answer is $f(t) = 2\sin\left(\frac{\pi}{3}t - \frac{7\pi}{6}\right) + \frac{1}{2}$.



Extending the graph of $y = f(t)$.

□

Note that each of the answers given in Example 11.3.3 is one choice out of many possible answers. For example, when fitting a sine function to the data, we could have chosen to start at $\left(\frac{1}{2}, \frac{1}{2}\right)$ taking $A = -2$. In this case, the phase shift is $\frac{1}{2}$ so $\phi = -\frac{\pi}{6}$ for an answer of $f(t) = -2\sin\left(\frac{\pi}{3}t - \frac{\pi}{6}\right) + \frac{1}{2}$. The ultimate check of any solution is to graph the answer and check it matches the given data.

11.3.1 Applications of Sinusoids

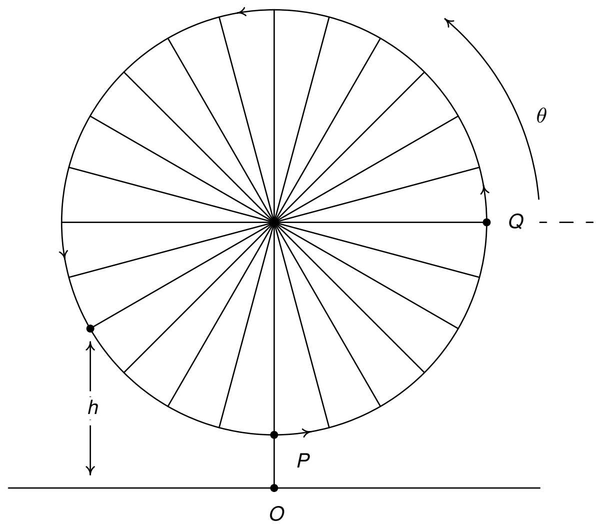
In the same way exponential functions can be used to model a wide variety of phenomena in nature,[15] the sine and cosine functions can be used to model their fair share of natural behaviors. Our first foray into sinusoidal motion revisits circular motion - in particular Equation 11.3 .

Example 11.3.4. Recall from Exercise 43 in Section 11.1 that The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height 136 feet. It completes two revolutions in 2 minutes and 7 seconds. Assuming that the riders are at the edge of the circle, find a sinusoid which describes the height of the passengers above the ground t seconds after they pass the point on the wheel closest to the ground.

Solution. We sketch the problem situation below and assume a counter-clockwise rotation.[16]

[15] See Section 7.6.

[16] Otherwise, we could just observe the motion of the wheel from the other side.



We know from the equations given on page 936 in Section 11.2.1 that the y-coordinate for counter-clockwise motion on a circle of radius r centered at the origin with constant angular velocity (frequency) ω is given by $y = r\sin(\omega t)$. Here, $t = 0$ corresponds to the point $(r, 0)$ so that θ, the angle measuring the amount of rotation, is in standard position.

In our case, the diameter of the wheel is 128 feet, so the radius is $r = 64$ feet. Since the wheel completes two revolutions in 2 minutes and 7 seconds (which is 127 seconds) the period $T = \frac{1}{2}(127) = \frac{127}{2}$ seconds. Hence, the angular frequency is $\omega = \frac{2\pi}{T} = \frac{4\pi}{127}$ radians per second.

Putting these two pieces of information together, we have that $y = 64\sin\left(\frac{4\pi}{127}t\right)$ describes the y-coordinate on the Giant Wheel after t seconds, assuming it is centered at $(0, 0)$ with $t = 0$ corresponding to point Q.

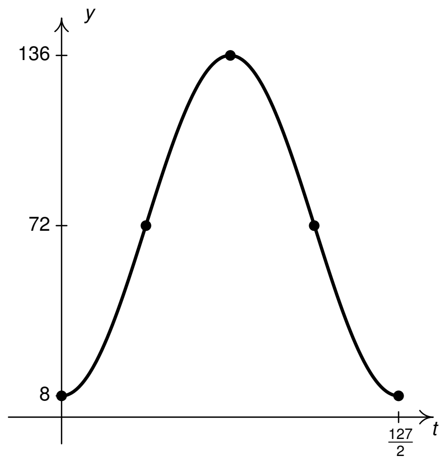
In order to find an expression for h, we take the point O in the figure as the origin. Since the base of the Giant Wheel ride is 8 feet above the ground and the Giant Wheel itself has a radius of 64 feet, its center is 72 feet above the ground. To account for this vertical shift upward,[17] we add 72 to our formula for y to obtain the new formula $h = y + 72 = 64\sin\left(\frac{4\pi}{127}t\right) + 72$.

Next, we need to adjust things so that $t = 0$ corresponds to the point P instead of the point Q. This is where the phase comes into play. Geometrically, we need to shift the angle θ in the figure back $\frac{\pi}{2}$ radians.

From the discussion on page 936, we know $\theta = \omega t = \frac{4\pi}{127}t$, so we (temporarily) write the height in terms of θ as $h = 64\sin(\theta) + 72$. Subtracting $\frac{\pi}{2}$ from θ gives $h(t) = 64\sin\left(\theta - \frac{\pi}{2}\right) + 72 = 64\sin\left(\frac{4\pi}{127}t - \frac{\pi}{2}\right) + 72$.

We can check the reasonableness of our answer by graphing $y = h(t)$ over the interval $\left[0, \frac{127}{2}\right]$ and visualizing the path of a person on the Big Wheel ride over the course of one rotation.

[17] We are readjusting our 'baseline' from $y = 0$ to $y = 72$.



□

A few remarks about Example 11.3.4 are in order. First, note that the amplitude of 64 in our answer corresponds to the radius of the Giant Wheel. This means that passengers on the Giant Wheel never stray more than 64 feet vertically from the center of the Wheel, which makes sense. Second, the phase shift of our answer works out to be $\frac{\pi/2}{4\pi/127} = \frac{127}{8} = 15.875$. This represents the 'time delay' (in seconds) we introduce by starting the motion at the point P as opposed to the point Q. Said differently, passengers which 'start' at P take 15.875 seconds to 'catch up' to the point Q.

Our next example revisits the daylight data first introduced in Section 1.4, Exercise 52b.

Example 11.3.5. According to the U.S. Naval Observatory website, the number of hours H of daylight that Fairbanks, Alaska received on the 21st day of the nth month of 2009 is given below. Here $t = 1$ represents January 21, 2009, $t = 2$ represents February 21, 2009, and so on.

Month Number	1	2	3	4	5	6	7	8	9	10	11	12
Hours of Daylight	5.8	9.3	12.4	15.9	19.4	21.8	19.4	15.6	12.4	9.1	5.6	3.3

1. Find a sinusoid which models these data and graph your answer along with the data.
2. Compare your answer to part 1 to one obtained using the regression feature of a graphing utility.

Solution.

1. To get a feel for the data, we plot it below on the left. At first glance, the data appear to be more like '$\wedge$'-shaped instead of sinusoidal, harkening back to Section 1.3. However, from experience, the hours of daylight is a cyclical process, which is why we attempt to fit this data to a sine function.

   Please note that when it comes down to it, fitting a sinusoid to data manually is not an exact science. We do our best to find the constants A, ω, ϕ and B so that the function $H(t) = A\sin(\omega t + \phi) + B$ closely matches the data. In this example, we first go after the vertical shift B to determine the baseline.

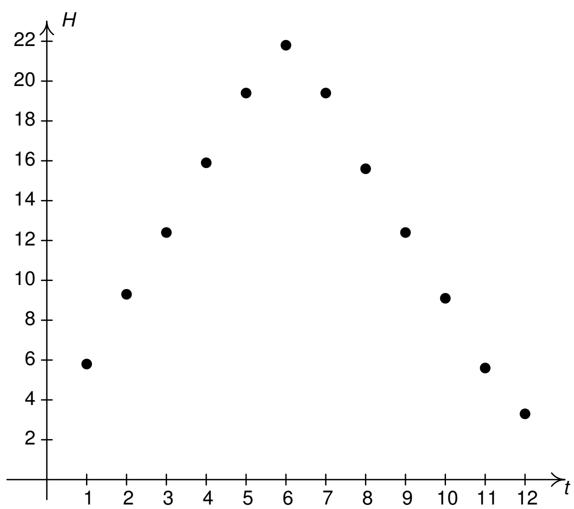
   In a typical sinusoid, the value of B is the average of the maximum and minimum values. So here we take $B = \frac{3.3+21.8}{2} = 12.55$.

Next is the amplitude A which is the displacement from the baseline to the maximum (and minimum) values. We find $A = 21.8 - 12.55 = 12.55 - 3.3 = 9.25$. At this stage, our sinusoid has the form: $H(t) = 9.25\sin(\omega t + \phi) + 12.55$.

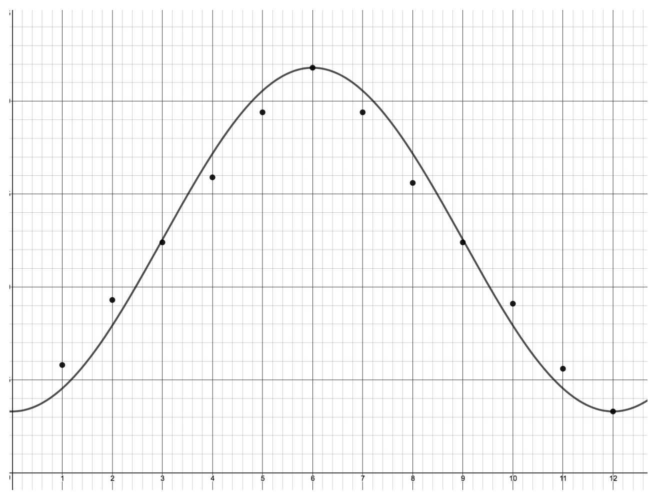
We proceed to find the angular frequency, ω. Since the data collected is over the span of a year (12 months), we take the period $T = 12$ months.[18] This means $\omega = \frac{2\pi}{T} = \frac{2\pi}{12} = \frac{\pi}{6}$.

The last quantity to find is the phase ϕ. Unlike the previous example, it is easier in this case to find the phase shift $-\frac{\phi}{\omega}$. Since we picked $A > 0$, the phase shift corresponds to the first value of t with $H(t) = 12.55$ (the baseline value).[19]

Here, we choose $t = 3$, since its corresponding H value of 12.4 is closer to 12.55 than the next value, 15.9, which corresponds to $t = 4$. Hence, $-\frac{\phi}{\omega} = 3$, so $\phi = -3\omega = -3\left(\frac{\pi}{6}\right) = -\frac{\pi}{2}$. We have $H(t) = 9.25\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 12.55$. Below on the right is a graph of our data with the curve $y = H(t)$.



Plotting the data.



Plotting the data long with $y = H(t)$

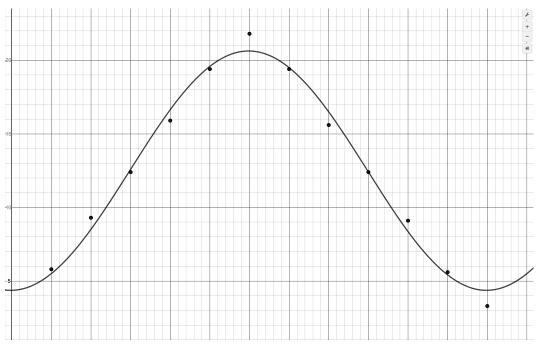
2. Using desmos to find a regression model produces $A \approx 8.36$ and $B \approx 12.06$, which closely match our answers from number 1. However, desmos calculates $\omega \approx -294.81$ and $\phi \approx -26.53$, which are far from our values $\omega = \frac{\pi}{6}$ and $\phi = -\frac{\pi}{2}$.

   Indeed, in Exercise 32, we invite the reader to graph $y = 8.36\sin(-294.81t - 26.53) + 12.06$ using desmos. At first glance, appears to be a solid gray rectangle. After zooming in, however, we see this rectangle is made up of literally hundreds of oscillations. Though desmos boasts this regression as having an R^2 value of 0.9886, meaning mathematically, this is a very good fit to the data, this curve doesn't model the real-world phenomenon of daylight hours in any sort of reasonable manner.

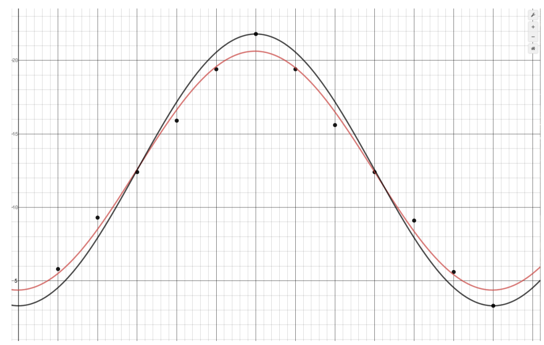
[18] Even though the data collected lies in the interval $[1, 12]$, which has a length of 11, we need to think of the data point at $t = 1$ as a representative sample of the amount of daylight for every day in January. That is, it represents $H(t)$ over the interval $[0, 1]$. Similarly, $t = 2$ is a sample of $H(t)$ over $[1, 2]$, and so forth.

[19] See the figure on page 948.

Since we know the period of the sinusoid is 12 months, we set $\omega = \frac{\pi}{6}$ and ask desmos to run a regression for the remaining three parameters. Desmos returns the values $A \approx -8.13$, $B \approx 12.5$, and $\phi \approx -4.70$, with an R^2 value of 0.987, so its regression curve is a much more reasonable $H(t) = -8.13 \sin\left(\frac{\pi}{6}t - 4.70\right) + 12.5$. Even though, at first glance, this curve appears much different from our solution to number 1, the we graph both functions below on the right and see they are very similar.[20] (Our solution to number 1 is the darker of the two curves.)



Plotting the desmos's regression curve.



Plotting both solutions.

□

The scenario described in Example 11.3.5 is a typical example of where the circular functions are useful outside the context of angles or circular motion. Indeed, sine and cosine functions are used extensively to model a wide range of periodic phenomena including signal analysis, wave physics, and even quantum mechanics. We close this section discussing limits involving sine and cosine.

11.3.2 Limits involving Sine and Cosine

We've already stated as fact (but not proven) that $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are continuous. Per Definition 6.4, this means that $\lim\limits_{t \to a} \sin(t) = \sin(a)$ and $\lim\limits_{t \to a} \cos(t) = \cos(a)$ for all real numbers, a. This means, for instance, we can compute $\lim\limits_{t \to \pi} \cos(t) = \cos(\pi) = -1$. Per the discussion following Definition 6.4, so long as we avoid the usual domain pitfalls, we can also compute:

$$\lim_{t \to \pi} \frac{2t\cos(t) - \sin(3t)}{\sqrt{\cos(2t)}} = \frac{2\pi\cos(\pi) - \sin(3\pi)}{\sqrt{\cos(2\pi)}} = \frac{-2\pi - 0}{\sqrt{1}} = -2\pi$$

In the next example we investigate a few (similar looking) limits of sinusoids.

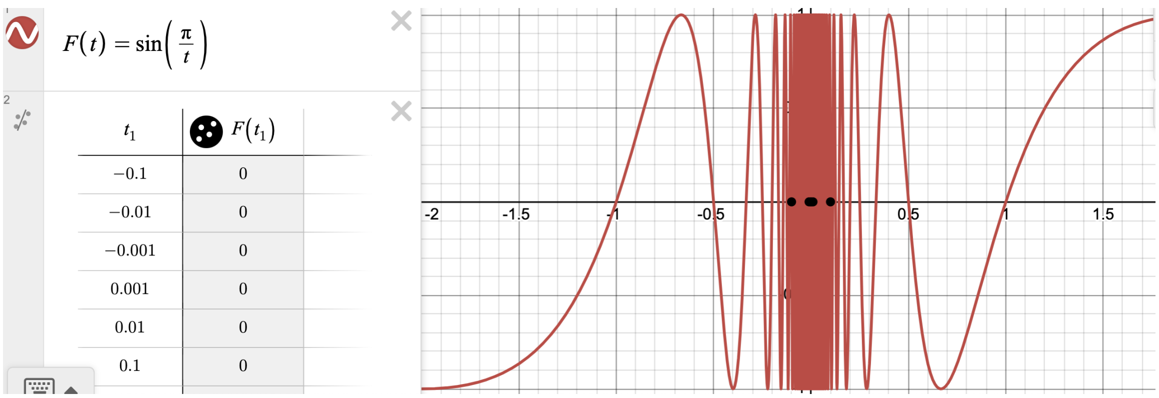
[20]In Section 12.2, we can show how similar these two functions are analytically. See Exercise 38.

Example 11.3.6. Investigate the following limits numerically, graphically, and analytically.

1. $\displaystyle\lim_{t\to 0} \sin\left(\frac{\pi}{t}\right)$
2. $\displaystyle\lim_{t\to 0} \sin\left(10^{23}\, t\right)$
3. $\displaystyle\lim_{t\to 0} \frac{\sin(t)}{t}$

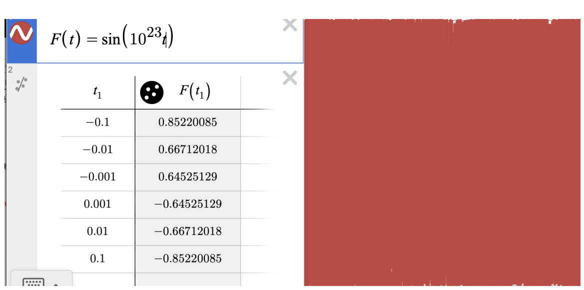
Solution.

1. To analyze $\displaystyle\lim_{t\to 0} \sin\left(\frac{\pi}{t}\right)$ numerically, we construct a table of values 'near' $t = 0$ for $F(t) = \sin\left(\frac{\pi}{t}\right)$. Choosing values such as $t = \pm 0.001$, $t = \pm 0.01$, etc., we get a list of '0's. This result shouldn't be too surprising since in each of these cases, $\frac{\pi}{t}$ results in an integer multiple of π. However, the graph near $t = 0$ (literally) paints a different picture. The graph appears to wildly oscillate near $t = 0$, suggesting that $\displaystyle\lim_{t\to 0} \sin\left(\frac{\pi}{t}\right)$ does not exist.

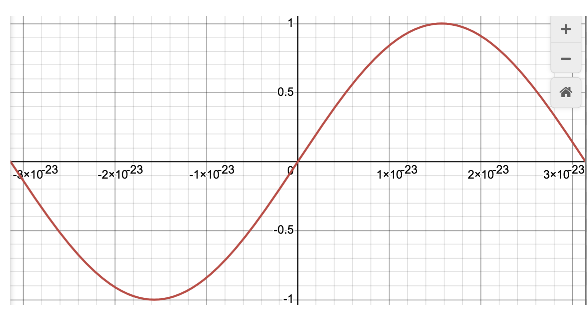


Indeed, as $t \to 0^-$, $\frac{\pi}{t} \to -\infty$ and as $t \to 0^+$, $\frac{\pi}{t} \to \infty$. In other words, all of the infinite oscillations experienced by the standard sine function, $f(t) = \sin(t)$ as $t \to -\infty$ and as $t \to \infty$ are being brought back to near $t = 0$ in $F(t) = \sin\left(\frac{\pi}{t}\right)$. Hence, $\displaystyle\lim_{t\to 0} \sin\left(\frac{\pi}{t}\right)$ does not exist.

2. We proceed as above to investigate $\displaystyle\lim_{t\to 0} \sin\left(10^{23}\, t\right)$. A table and a graph of $F(t) = \sin\left(10^{23}\, t\right)$ 'near' $t = 0$ produces numerical and graphical noise. Neither the table nor the graph suggest $\displaystyle\lim_{t\to 0} F(t)$ exists. However, we know $F(t) = \sin\left(10^{23}\, t\right)$ is a sinusoid with frequency $\omega = 10^{23}$. Zooming in by a factor of 10^{23}, we get one cycle of F which suggests the limit is, in fact, 0.



$F(t) = \sin\left(10^{23}\, t\right)$ 'near' $t = 0$



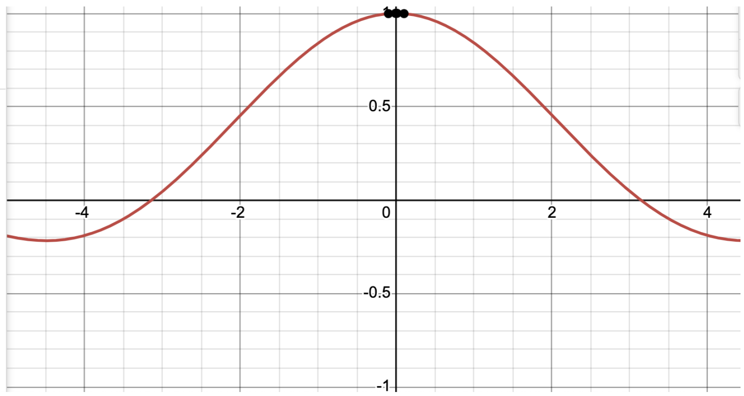
$F(t) = \sin\left(10^{23}\, t\right)$ 'nearer to ' $t = 0$

Indeed, we know $F(t) = \sin\left(10^{23}\,t\right)$ is continuous, so $\lim\limits_{t\to 0} \sin\left(10^{23}\,t\right) = \sin\left(10^{23}(0)\right) = \sin(0) = 0$.

3. Next we investigate $F(t) = \frac{\sin(t)}{t}$ near $t = 0$. The table and graph both suggest $\lim\limits_{t\to 0} \frac{\sin(t)}{t} = 1$.

$F(t) = \frac{\sin(t)}{t}$

t_1	$F(t_1)$
-0.1	0.99833417
-0.01	0.99998333
-0.001	0.99999983
0.001	0.99999983
0.01	0.99998333
0.1	0.99833417



Attempting to analyze this limit analytically, we see that $\lim\limits_{t\to 0} \frac{\sin(t)}{t}$ produces the indeterminate form '$\frac{0}{0}$'. At this stage, we do not have the tools to either prove or disprove our conjecture that $\lim\limits_{t\to 0} \frac{\sin(t)}{t} = 1$, but we will soon, [21] so stay tuned! $\square$

[21] See Exercises 80 through 82 in Section 11.4.

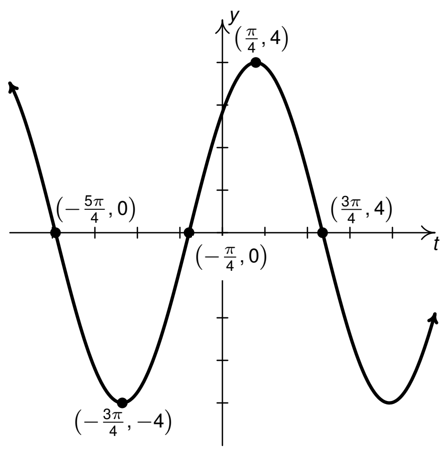
11.3.3 Exercises

In Exercises 1 - 12, graph one cycle of the given function. State the period, amplitude, phase shift and vertical shift of the function.

1. $f(t) = 3\sin(t)$
2. $g(t) = \sin(3t)$
3. $h(t) = -2\cos(t)$
4. $f(t) = \cos\left(t - \frac{\pi}{2}\right)$
5. $g(t) = -\sin\left(t + \frac{\pi}{3}\right)$
6. $h(t) = \sin(2t - \pi)$
7. $f(t) = -\frac{1}{3}\cos\left(\frac{1}{2}t + \frac{\pi}{3}\right)$
8. $g(t) = \cos(3t - 2\pi) + 4$
9. $h(t) = \sin\left(-t - \frac{\pi}{4}\right) - 2$
10. $f(t) = \frac{2}{3}\cos\left(\frac{\pi}{2} - 4t\right) + 1$
11. $g(t) = -\frac{3}{2}\cos\left(2t + \frac{\pi}{3}\right) - \frac{1}{2}$
12. $h(t) = 4\sin(-2\pi t + \pi)$

In Exercises 13 - 16, a sinusoid is graphed. Find a formula for the sinusoid in the form $S(t) = A\sin(\omega t + \phi) + B$ and $C(t) = A\cos(\omega t + \phi) + B$. Select ω so $\omega > 0$. Check your answer by graphing.

13.



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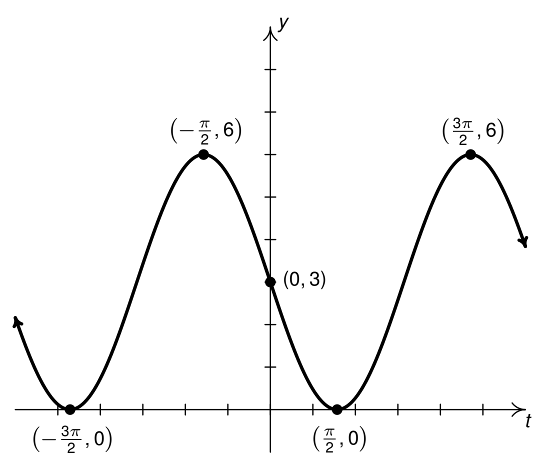

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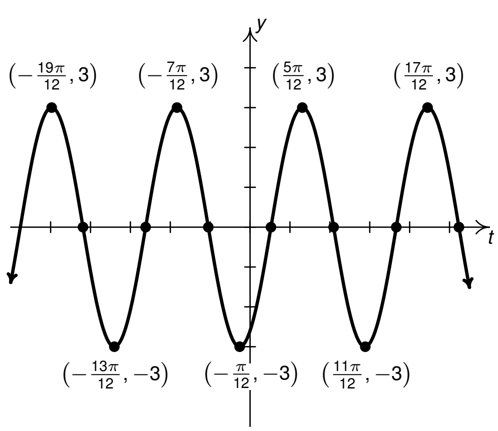

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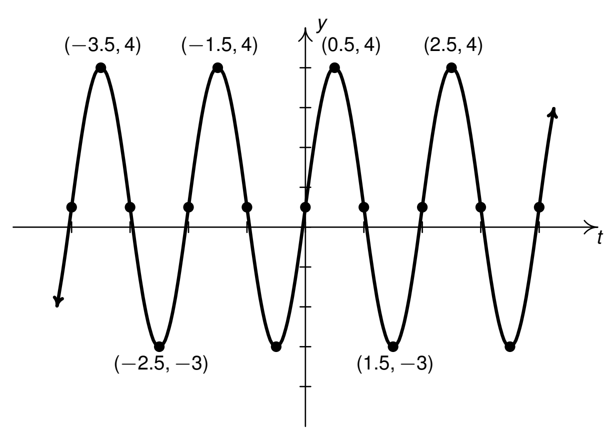


17. Use the graph of $S(t) = 4\sin(t)$ to graph each of the following functions. State the period of each.

    (a) $f(t) = |4\sin(t)|$

    (b) $g(t) = \sqrt{4\sin(t)}$

In Exercises 18 - 22, use a graphing utility to help you and your classmates discuss the given questions.

18. Graph $f(t) = \cos(3t) + \sin(t)$. Is this function periodic? If so, what is the period?

19. Graph $f(t) = t\sin(t)$ along with $y = \pm t$. What do you notice?

20. Graph $f(t) = \frac{\sin(t)}{t}$ along with $y = \pm\frac{1}{t}$. What do you notice?

    (a) What appears to be $\lim\limits_{t\to\infty} f(t)$? Interpret your answer graphically.

    (b) Use the Squeeze Theorem, Theorem 10.2 from Section 10.1 to prove your claim.

    HINT: Since $-1 \leq \sin(t) \leq 1$, for $t > 0$, $-\frac{1}{t} \leq \frac{\sin(t)}{t} \leq \frac{1}{t} \ldots$

21. Graph $f(t) = \cos\left(\frac{1}{t}\right)$.

    (a) Investigate $\lim\limits_{t\to 0} f(t)$. Does $\lim\limits_{t\to 0} f(t)$ exist? Why or why not?

    (b) Determine $\lim\limits_{t\to\infty} f(t)$ and interpret your answer graphically.

22. Graph $f(t) = e^{-0.1t}(\cos(2t) + \sin(2t))$ along with $y = \pm 2\,e^{-0.1t}$. What do you notice?

    (a) What appears to be $\lim\limits_{t\to\infty} f(t)$? Interpret your answer graphically.

    (b) Use the Squeeze Theorem, Theorem 10.2 from Section 10.1 to prove your claim.

    HINT: Since $-1 \leq \cos(2t) \leq 1$ and $-1 \leq \sin(2t) \leq 1$, $-2 \leq \cos(2t) + \sin(2t) \leq 2$.

    Hence, $-2e^{-0.1t} \leq e^{-0.1t}(\cos(2t) + \sin(2t)) \leq 2e^{-0.1t} \ldots$

23. Show every constant function f is periodic by explaining why $f(x + 117) = f(x)$ for all real numbers x. Then show that f has no period by showing that you cannot find a *smallest* number p such that $f(x + p) = f(x)$ for all real numbers x.

    Said differently, show that $f(x+p) = f(x)$ for all real numbers x for ALL values of $p > 0$, so no smallest value exists to satisfy the definition of 'period'.

24. The sounds we hear are made up of mechanical waves. The note 'A' above the note 'middle C' is a sound wave with ordinary frequency $f = 440$ Hertz $= 440\,\frac{\text{cycles}}{\text{second}}$. Find a sinusoid which models this note, assuming that the amplitude is 1 and the phase shift is 0.

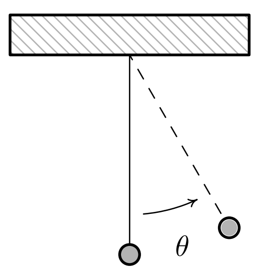
25. The voltage V in an alternating current source has amplitude $220\sqrt{2}$ and ordinary frequency $f = 60$ Hertz. Find a sinusoid which models this voltage. Assume that the phase is 0.

26. The London Eye is a popular tourist attraction in London, England and is one of the largest Ferris Wheels in the world. It has a diameter of 135 meters and makes one revolution (counter-clockwise) every 30 minutes. It is constructed so that the lowest part of the Eye reaches ground level, enabling passengers to simply walk on to, and off of, the ride. Find a sinsuoid which models the height h of the passenger above the ground in meters t minutes after they board the Eye at ground level.

27. On page 936 in Section 11.2.1, we found the x-coordinate of counter-clockwise motion on a circle of radius r with angular frequency ω to be $x = r\cos(\omega t)$, where $t = 0$ corresponds to the point $(r, 0)$. Suppose we are in the situation of Exercise 26 above. Find a sinsusoid which models the horizontal *displacement* x of the passenger from the center of the Eye in meters t minutes after they board the Eye. Here we take $x(t) > 0$ to mean the passenger is to the *right* of the center, while $x(t) < 0$ means the passenger is to the *left* of the center.

28. In Exercise 40 in Section 11.1, we introduced the yo-yo trick 'Around the World' in which a yo-yo is thrown so it sweeps out a vertical circle. As in that exercise, suppose the yo-yo string is 28 inches and it completes one revolution in 3 seconds. If the closest the yo-yo ever gets to the ground is 2 inches, find a sinsuoid which models the height h of the yo-yo above the ground in inches t seconds after it leaves its lowest point.

29. Consider the pendulum below. Ignoring air resistance, the angular displacement of the pendulum from the vertical position, θ, can be modeled as a sinusoid.[22]



The amplitude of the sinusoid is the same as the initial angular displacement, θ_0, of the pendulum and the period of the motion is given by

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

where ℓ is the length of the pendulum and g is the acceleration due to gravity.

[22] Provided θ is kept 'small.' Carl remembers the 'Rule of Thumb' as being 20° or less. Check with your friendly neighborhood physicist to make sure.

(a) Find a sinusoid which gives the angular displacement θ as a function of time, t. Arrange things so $\theta(0) = \theta_0$.

(b) In Exercise 24 section 4.1, you found the length of the pendulum needed in Jeff's antique Seth-Thomas clock to ensure the period of the pendulum is $\frac{1}{2}$ of a second. Assuming the initial displacement of the pendulum is 15°, find a sinusoid which models the displacement of the pendulum θ as a function of time, t, in seconds.

30. The table below lists the average temperature of Lake Erie as measured in Cleveland, Ohio on the first of the month for each month during the years 1971 – 2000.[23] For example, $t = 3$ represents the average of the temperatures recorded for Lake Erie on every March 1 for the years 1971 – 2000.

Month Number, t	1	2	3	4	5	6	7	8	9	10	11	12
Temperature ($^\circ$ F), T	36	33	34	38	47	57	67	74	73	67	56	46

(a) Using the techniques discussed in Example 11.3.5, fit a sinusoid to these data.

(b) Graph your model along with the data set to judge the reasonableness of the fit.

(c) Use the model from 30a to predict the average temperature recorded for Lake Erie on April 15$^{\text{th}}$ and September 15$^{\text{th}}$ during the years 1971–2000.[24]

(d) Compare your results to those obtained using a graphing utility.

31. The fraction of the moon illuminated at midnight Eastern Standard Time on the t^{th} day of June, 2009 is given in the table below.[25]

Day of June, t	3	6	9	12	15	18	21	24	27	30
Fraction Illuminated, F	0.81	0.98	0.98	0.83	0.57	0.27	0.04	0.03	0.26	0.58

(a) Using the techniques discussed in Example 11.3.5, fit a sinusoid to these data.[26]

(b) Graph your model along with the data set to judge the reasonableness of the fit.

(c) Use the model from 31a to predict the fraction of the moon illuminated on June 1, 2009. [27]

(d) Compare your results to those obtained using a graphing utility.

32. Use a graphing utility to graph $y = 8.36\sin(-294.81t - 26.53) + 12.06$. (This is the regression model produced by desmos in Example 11.3.5.) Zoom in, as needed, until you start to see the wave-like nature of the graph. Use Theorem 11.6 to determine a window which produces exactly one complete cycle of this sinusoid and check your answer graphically.

[23] See this website: http://www.erh.noaa.gov/cle/climate/cle/normals/laketempcle.html.

[24] The computed average is 41°F for April 15$^{\text{th}}$ and 71°F for September 15$^{\text{th}}$.

[25] See this website: http://www.usno.navy.mil/USNO/astronomical-applications/data-services/frac-moon-ill.

[26] You may want to plot the data before you find the phase shift.

[27] The listed fraction is 0.62.

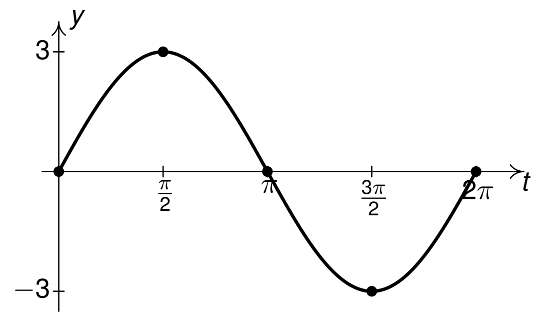
33. Use Theorem 5.11 to prove Theorem 11.6.

34. With the help of your classmates, research Amplitude Modulation and Frequency Modulation.

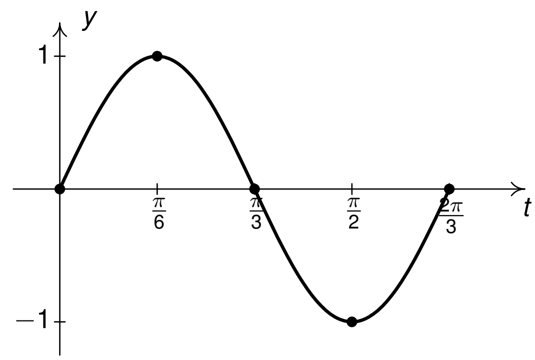
35. What other things in the world might be roughly sinusoidal? Look to see what models you can find for them and share your results with your class.

11.3.4 Answers

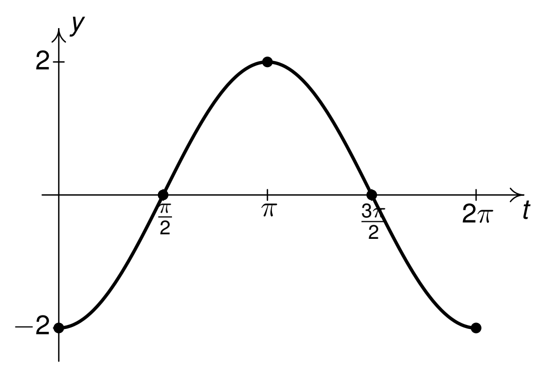
1. $f(t) = 3\sin(t)$
   Period: 2π
   Amplitude: 3
   Phase Shift: 0
   Vertical Shift: 0



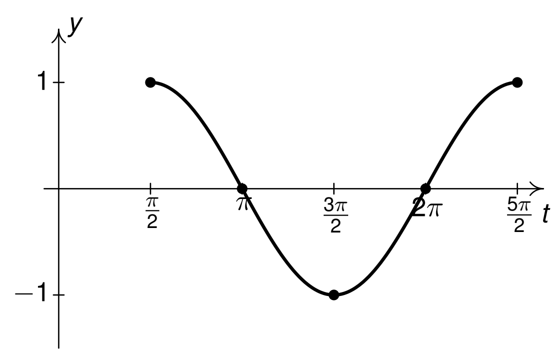
2. $g(t) = \sin(3t)$
   Period: $\frac{2\pi}{3}$
   Amplitude: 1
   Phase Shift: 0
   Vertical Shift: 0



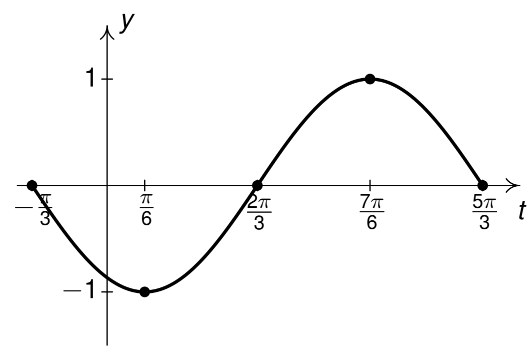
3. $h(t) = -2\cos(t)$
   Period: 2π
   Amplitude: 2
   Phase Shift: 0
   Vertical Shift: 0



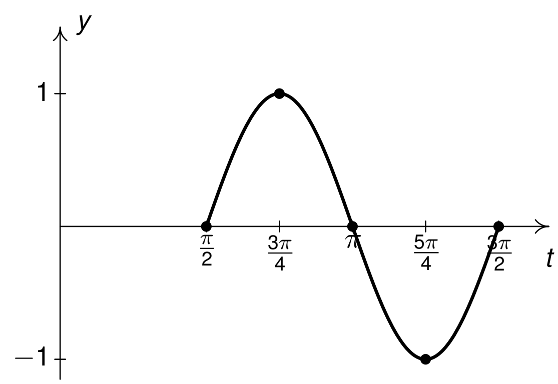
4. $f(t) = \cos\left(t - \frac{\pi}{2}\right)$
   Period: 2π
   Amplitude: 1
   Phase Shift: $\frac{\pi}{2}$
   Vertical Shift: 0



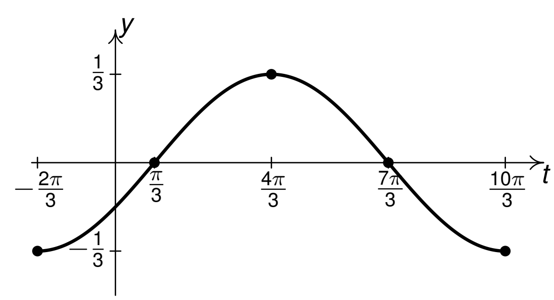
5. $g(t) = -\sin\left(t + \frac{\pi}{3}\right)$
Period: 2π
Amplitude: 1
Phase Shift: $-\frac{\pi}{3}$
Vertical Shift: 0



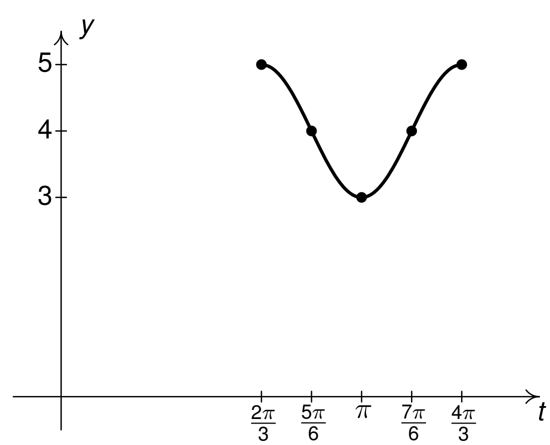
6. $h(t) = \sin(2t - \pi)$
Period: π
Amplitude: 1
Phase Shift: $\frac{\pi}{2}$
Vertical Shift: 0



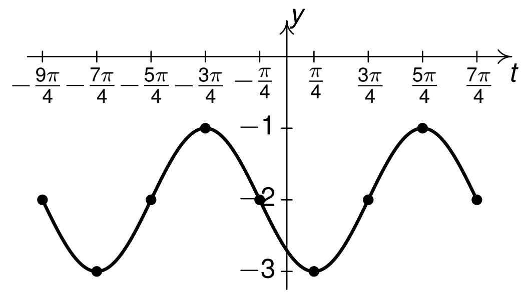
7. $f(t) = -\frac{1}{3}\cos\left(\frac{1}{2}t + \frac{\pi}{3}\right)$
Period: 4π
Amplitude: $\frac{1}{3}$
Phase Shift: $-\frac{2\pi}{3}$
Vertical Shift: 0



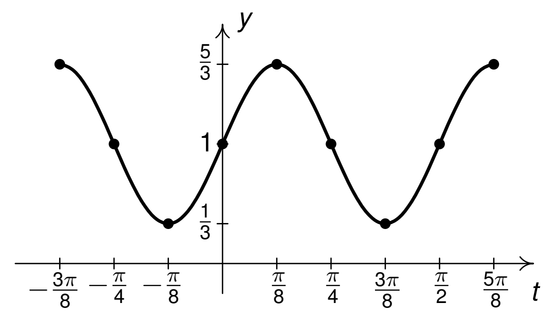
8. $g(t) = \cos(3t - 2\pi) + 4$
Period: $\frac{2\pi}{3}$
Amplitude: 1
Phase Shift: $\frac{2\pi}{3}$
Vertical Shift: 4



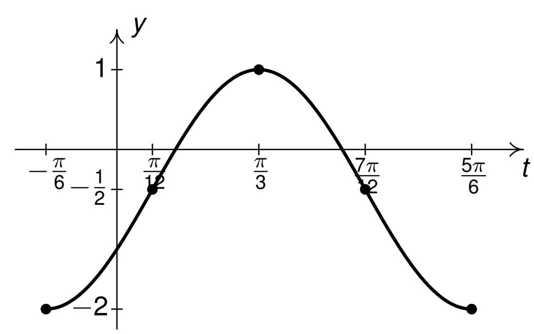
9. $h(t) = \sin\left(-t - \frac{\pi}{4}\right) - 2$
   Period: 2π
   Amplitude: 1
   Phase Shift: $-\frac{\pi}{4}$ (You need to use $y = -\sin\left(t + \frac{\pi}{4}\right) - 2$ to find this.)[28]
   Vertical Shift: -2



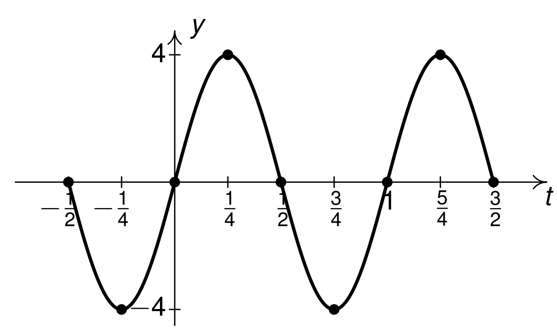
10. $f(t) = \frac{2}{3}\cos\left(\frac{\pi}{2} - 4t\right) + 1$
   Period: $\frac{\pi}{2}$
   Amplitude: $\frac{2}{3}$
   Phase Shift: $\frac{\pi}{8}$ (You need to use $y = \frac{2}{3}\cos\left(4t - \frac{\pi}{2}\right) + 1$ to find this.)[29]
   Vertical Shift: 1



11. $g(t) = -\frac{3}{2}\cos\left(2t + \frac{\pi}{3}\right) - \frac{1}{2}$
   Period: π
   Amplitude: $\frac{3}{2}$
   Phase Shift: $-\frac{\pi}{6}$
   Vertical Shift: $-\frac{1}{2}$



12. $h(t) = 4\sin(-2\pi t + \pi)$
   Period: 1
   Amplitude: 4
   Phase Shift: $\frac{1}{2}$ (You need to use $h(t) = -4\sin(2\pi t - \pi)$ to find this.)[30]
   Vertical Shift: 0



[28] Two cycles of the graph are shown to illustrate the discrepancy discussed on page 951.
[29] Again, we graph two cycles to illustrate the discrepancy discussed on page 951.
[30] This will be the last time we graph two cycles to illustrate the discrepancy discussed on page 951.

13. $S(t) = 4\sin\left(t + \frac{\pi}{4}\right)$, $C(t) = 4\cos\left(t - \frac{\pi}{4}\right)$

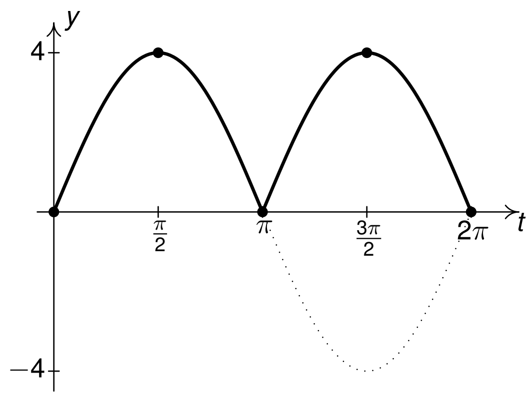
14. $S(t) = -3\sin(t) + 3$, $C(t) = -3\cos\left(t - \frac{\pi}{2}\right) + 3$

15. $S(t) = 3\sin\left(2t - \frac{\pi}{3}\right)$, $C(t) = 3\cos\left(2t - \frac{5\pi}{6}\right)$

16. $S(t) = \frac{7}{2}\sin(\pi t) + \frac{1}{2}$, $C(t) = \frac{7}{2}\cos\left(\pi t \frac{\pi}{2}\right) + \frac{1}{2}$

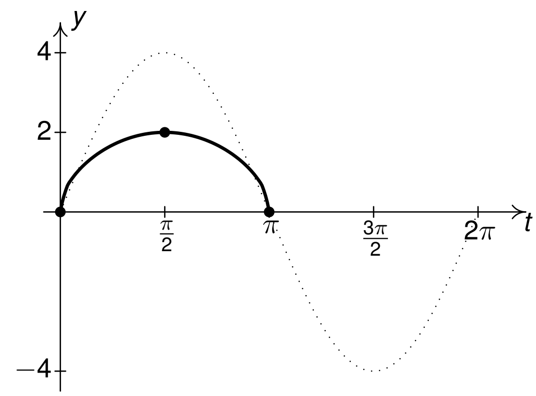
17. (a) $y = |4\sin(t)|$. Period: π.

Two cycles are graphed below.



(b) $y = \sqrt{4\sin(t)}$. Period: 2π.

One cycle is graphed below.



18. $f(t) = \cos(3t) + \sin(t)$ has period 2π.

19. The graph of $f(t) = t\sin(t)$ is bounded by the lines $y = \pm t$; f has a variable amplitude of t.

20. The graph of $f(t) = \frac{\sin(t)}{t}$ is bounded by the graphs of $y = \pm\frac{1}{t}$; f has a variable amplitude of $\frac{1}{t}$.

   (a) $\displaystyle\lim_{t\to\infty} f(t) = 0$. We have a horizontal asymptote $y = 0$.

   (b) Since $-\frac{1}{t} \leq \frac{\sin(t)}{t} \leq \frac{1}{t}$, $\displaystyle\lim_{t\to\infty}\left(-\frac{1}{t}\right) = \lim_{t\to\infty}\frac{1}{t} = 0$, $\displaystyle\lim_{t\to\infty}\frac{\sin(t)}{t} = 0$ by the Squeeze Theorem.

21. Graph $f(t) = \cos\left(\frac{1}{t}\right)$.

   (a) $\displaystyle\lim_{t\to 0} f(t)$ does not exist. We have infinitely many oscillations as $t \to 0$.

   (b) $\displaystyle\lim_{t\to\infty} f(t) = 1$ since as $t \to \infty$, $\frac{1}{t} \to 0$. Since cosine is continuous, $\cos\left(\frac{1}{t}\right) \to \cos(0) = 1$. We have a horizontal asymptote $y = 1$.

22. The graph of $f(t) = e^{-0.1t}(\cos(2t) + \sin(2t))$ lies between[31] the graphs of $y = \pm 2\,e^{-0.1t}$.

   (a) $\displaystyle\lim_{t\to\infty} f(t) = 0$. We have a horizontal asymptote $y = 0$.

   (b) Since $-2e^{-0.1t} \leq e^{-0.1t}(\cos(2t) + \sin(2t)) \leq 2e^{-0.1t}$, $\displaystyle\lim_{t\to\infty}\left(-2e^{-0.1t}\right) = \lim_{t\to\infty} 2e^{-0.1t} = 0$, the Squeeze Theorem gives $\displaystyle\lim_{t\to\infty} e^{-0.1t}(\cos(2t) + \sin(2t)) = 0$.

[31] We'll be able to show in Section 12.2.1 that the graph of f more perfectly lies between the graphs of $y = \pm\sqrt{2}\,e^{-0.1t}$ ...

23. $S(t) = \sin(880\pi t)$

24. $V(t) = 220\sqrt{2}\sin(120\pi t)$

25. $h(t) = 67.5\sin\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) + 67.5$

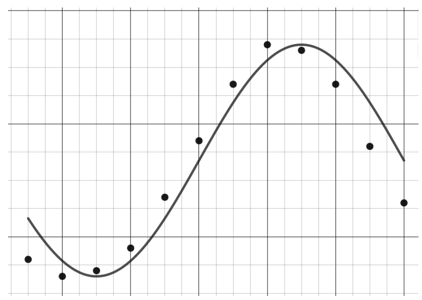
26. $x(t) = 67.5\cos\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) = 67.5\sin\left(\frac{\pi}{15}t\right)$

27. $h(t) = 28\sin\left(\frac{2\pi}{3}t - \frac{\pi}{2}\right) + 30$

28. (a) $\theta(t) = \theta_0\sin\left(\sqrt{\frac{g}{l}}\,t + \frac{\pi}{2}\right)$ (b) $\theta(t) = \frac{\pi}{12}\sin\left(4\pi t + \frac{\pi}{2}\right)$

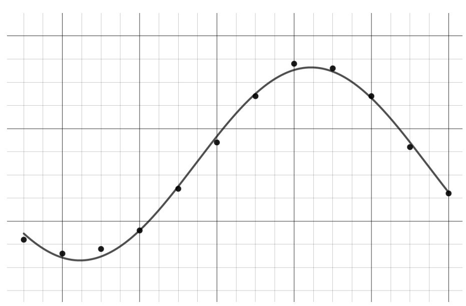
29. (a) $T(t) = 20.5\sin\left(\frac{\pi}{6}t - \pi\right) + 53.5$

(b) The model and data are graphed below. The sinusoid is shifted to the right of our data.



(c) The average temperature on April 15th is approximately $T(4.5) \approx 39.00^\circ$F and the average temperature on September 15th is approximately $T(9.5) \approx 73.38^\circ$F.

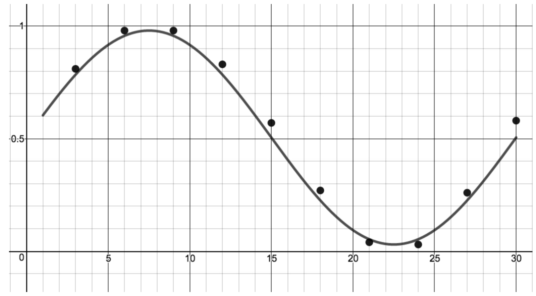
(d) Desmos gives: $T(t) = 20.8374\sin(-0.5251t - 0.2812) + 52.3659$.



This model predicts the average temperature for April 15th to be approximately 42.42°F and the average temperature on September 15th to be approximately 70.05°F. This model appears to be more accurate.

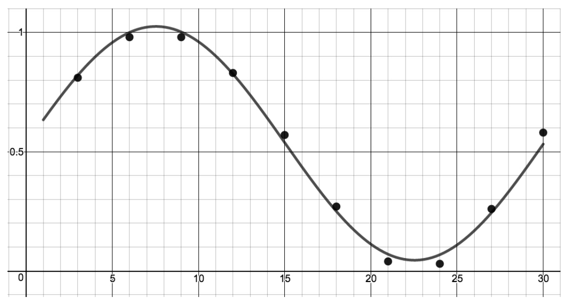
30. (a) Based on the shape of the data, we either choose $A < 0$ or we find the *second* value of t which closely approximates the 'baseline' value, $F = 0.505$. We choose the latter to obtain $F(t) = 0.475\sin\left(\frac{\pi}{15}t - 2\pi\right) + 0.505 = 0.475\sin\left(\frac{\pi}{15}t\right) + 0.505$

(b) Our function and the data set are graphed below. It's a pretty good fit.



(c) The fraction of the moon illuminated on June 1st, 2009 is approximately $F(1) \approx 0.60$

(d) Using desmos,[32] we get $F(t) = 0.49 \sin\left(\frac{\pi}{15} t - 6.29\right) + 0.535$.



This model predicts that the fraction of the moon illuminated on June 1st, 2009 is approximately 0.63. This appears to be a better fit to the data than our first model.

[32]...specifying $\omega = \frac{\pi}{15}$ ...

11.4 The Circular Functions: Tangent, Secant, Cosecant, and Cotangent

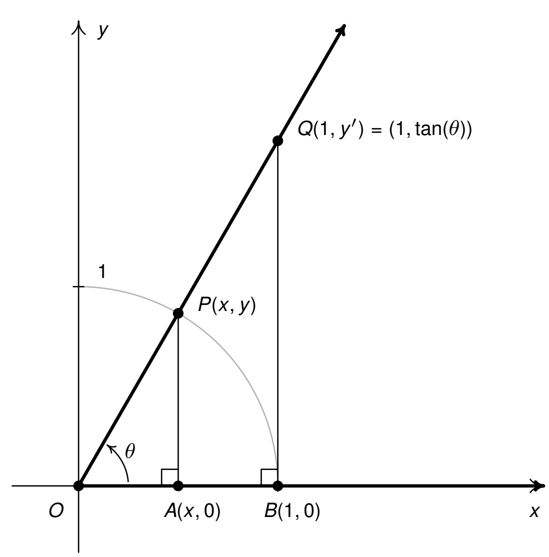
In section 11.2, we extended the notion of $\sin(\theta)$ and $\cos(\theta)$ from acute angles to any angles using the coordinate values of points on the Unit Circle. In total, there are six circular functions, as listed below.

Definition 11.4. The Circular Functions: Suppose an angle θ is graphed in standard position. Let $P(x, y)$ be the point of intersection of the terminal side of P and the Unit Circle.

- The **sine** of θ, denoted $\sin(\theta)$, is defined by $\sin(\theta) = y$.
- The **cosine** of θ, denoted $\cos(\theta)$, is defined by $\cos(\theta) = x$.
- The **tangent** of θ, denoted $\tan(\theta)$, is defined by $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- The **secant** of θ, denoted $\sec(\theta)$, is defined by $\sec(\theta) = \dfrac{1}{x}$, provided $x \neq 0$.
- The **cosecant** of θ, denoted $\csc(\theta)$, is defined by $\csc(\theta) = \dfrac{1}{y}$, provided $y \neq 0$.
- The **cotangent** of θ, denoted $\cot(\theta)$, is defined by $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

While we left the history of the name 'sine' as an interesting research project in Section 11.2,we take a slight detour here to explain the origin of the names 'tangent' and 'secant.'

Consider the acute angle θ in standard position sketched in the diagram below.



As usual, $P(x, y)$ denotes the point on the terminal side of θ which lies on the Unit Circle, but we also consider the point $Q(1, y')$, the point on the terminal side of θ which lies on the vertical line $x = 1$.

The word 'tangent' comes from the Latin meaning 'to touch,' and for this reason, the line $x = 1$ is called a *tangent* line to the Unit Circle since it intersects, or 'touches', the circle at only one point, namely $(1, 0)$.

Dropping perpendiculars from P and Q creates a pair of similar triangles ΔOPA and ΔOQB. Hence the corresponding sides are proportional. We get $\frac{y'}{y} = \frac{1}{x}$ which gives $y' = \frac{y}{x} = \tan(\theta)$.

We have just shown that for acute angles θ, $\tan(\theta)$ is the y-coordinate of the point on the terminal side of θ which lies on the line $x = 1$ which is *tangent* to the Unit Circle.

The word 'secant' means 'to cut', so a secant line is any line that 'cuts through' a circle at two points.[1] The line containing the terminal side of θ (not just the terminal side itself) is one such secant line since it intersects the Unit Circle in Quadrants I and III.

With the point P lying on the Unit Circle, the length of the hypotenuse of ΔOPA is 1. If we let h denote the length of the hypotenuse of ΔOQB, we have from similar triangles that $\frac{h}{1} = \frac{1}{x}$, or $h = \frac{1}{x} = \sec(\theta)$.

Hence for an acute angle θ, $\sec(\theta)$ is the length of the line segment which lies on the secant line determined by the terminal side of θ and 'cuts off' the tangent line $x = 1$.

As we mentioned in Definition 11.2, the 'co' in 'cosecant' and 'cotangent' tie back to the concept of 'co'mplementary angles and is explained in detail in Section 12.2.

Not only do these observations help explain the names of these functions, they serve as the basis for a fundamental inequality needed for Calculus which we'll explore in the Exercises.

Of the six circular functions, only sine and cosine are defined for all angles θ. Since $x = \cos(\theta)$ and $y = \sin(\theta)$ in Definition 11.4, it is customary to rephrase the remaining four circular functions Definition 11.4 in terms of sine and cosine.

Theorem 11.7. Reciprocal and Quotient Identities:

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\sec(\theta)$ is undefined.
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\csc(\theta)$ is undefined.
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\tan(\theta)$ is undefined.
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\cot(\theta)$ is undefined.

We call the equations listed in Theorem 11.7 **identities** since they are relationships which are true regardless of the values of θ. This is in contrast to **conditional equations** such as $\sin(\theta) = 1$ which are true for only **some** values of θ. We will study identities more extensively in Sections 12.1 and 12.2.

While the Reciprocal and Quotient Identities presented in Theorem 11.7 allow us to always reduce problems involving secant, cosecant, tangent and cotangent to problems involving sine and cosine, it is not

[1] Compare this with the definition given in Section 1.2.4.

always convenient to do so.[2] It is worth taking the time to memorize the tangent and cotangent values of the common angles summarized below.

Tangent and Cotangent Values of Common Angles

θ(degrees)	θ(radians)	$\tan(\theta)$	$\cot(\theta)$
0°	0	0	undefined
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	1	1
60°	$\frac{\pi}{3}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	undefined	0

Coupling Theorem 11.7 with the Reference Angle Theorem, Theorem 11.1, we get the following.

Theorem 11.8. Generalized Reference Angle Theorem. The values of the circular functions of an angle, if they exist, are the same, up to a sign, of the corresponding circular functions of its reference angle.

More specifically, if α is the reference angle for θ, then:

$$\sin(\theta) = \pm\sin(\alpha),\ \cos(\theta) = \pm\cos(\alpha),\ \tan(\theta) = \pm\tan(\alpha)$$

and

$$\sec(\theta) = \pm\sec(\alpha),\ \csc(\theta) = \pm\csc(\alpha),\ \cot(\theta) = \pm\cot(\alpha)$$

where the choice of the $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

It is high time for an example.

Example 11.4.1.

1. Find the exact value of the following, if it exists:

   (a) $\sec(60^\circ)$ (b) $\csc\left(\frac{7\pi}{4}\right)$ (c) $\tan(225^\circ)$ (d) $\cot\left(-\frac{7\pi}{6}\right)$

2. Find all angles which satisfy the given equation.

   (a) $\sec(\theta) = 2$ (b) $\csc(\theta) = -\sqrt{2}$ (c) $\tan(\theta) = \sqrt{3}$ (d) $\cot(\theta) = -1$.

Solution.

1. (a) According to Theorem 11.7, $\sec(60^\circ) = \frac{1}{\cos(60^\circ)}$. Hence, $\sec(60^\circ) = \frac{1}{(1/2)} = 2$.

   (b) Since $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\csc\left(\frac{7\pi}{4}\right) = \frac{1}{\sin\left(\frac{7\pi}{4}\right)} = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$.

[2] As we shall see shortly, when solving equations involving secant and cosecant, we usually convert back to cosines and sines. However, when solving for tangent or cotangent, we usually stick with what we're dealt.

(c) We have two ways to proceed to determine $\tan(225^\circ)$. First, we can use Theorem 11.7 and note that $\tan(225^\circ) = \frac{\sin(225^\circ)}{\cos(225^\circ)}$. Since $\sin(225^\circ) = \cos(225^\circ) = -\frac{\sqrt{2}}{2}$, $\tan(225^\circ) = 1$.

Another way to proceed is to note that 225° has a reference angle of 45°. Per Theorem 11.8, $\tan(225^\circ) = \pm\tan(45^\circ) = \pm 1$. Since 225° is a Quadrant III angle, where both the x and y coordinates of points are both negative, and tangent is defined as the *ratio* of coordinates $\frac{y}{x}$, we know $\tan(225^\circ) > 0$. Hence, $\tan(225^\circ) = 1$.

(d) As with the previous example, we have two ways to proceed. Using Theorem 11.7, we have $\cot\left(-\frac{7\pi}{6}\right) = \frac{\cos\left(-\frac{7\pi}{6}\right)}{\sin\left(-\frac{7\pi}{6}\right)}$. Since $\cos\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{7\pi}{6}\right) = \frac{1}{2}$, we get $\cot\left(-\frac{7\pi}{6}\right) = -\sqrt{3}$.

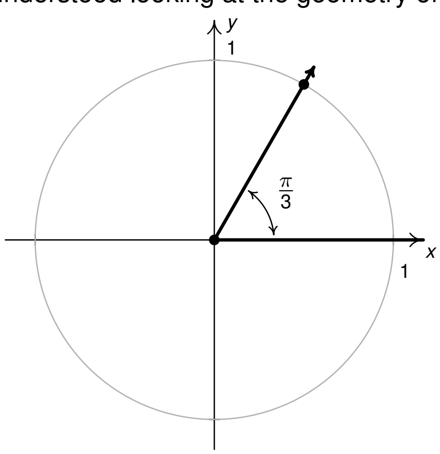
Alternatively, we note $-\frac{7\pi}{6}$ is a Quadrant II angle with reference angle $\frac{\pi}{6}$. Hence, Theorem 11.8 tells us $\cot\left(-\frac{7\pi}{6}\right) = \pm\cot\left(\frac{\pi}{6}\right) = \pm\sqrt{3}$. Since $-\frac{7\pi}{6}$ is a Quadrant II angle, where the x and y coordinates have different signs, and cotangent is defined as the ratio of coordinates $\frac{x}{y}$, we know $\cot\left(-\frac{7\pi}{6}\right) < 0$. Hence, $\cot\left(-\frac{7\pi}{6}\right) = -\sqrt{3}$.

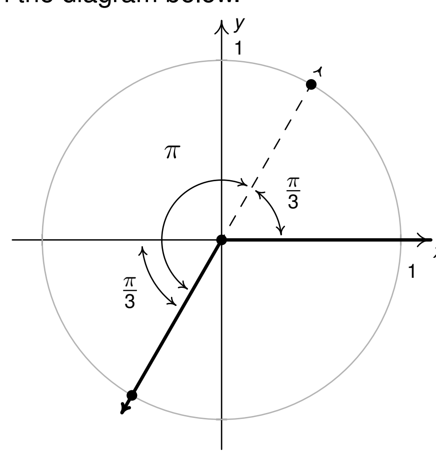
2. (a) To solve $\sec(\theta) = 2$, we convert to cosines and get $\frac{1}{\cos(\theta)} = 2$ or $\cos(\theta) = \frac{1}{2}$. This is the exact same equation we solved in Example 11.2.4, number 1, so we know the answer is: $\theta = \frac{\pi}{3} + 2\pi k$ or $\theta = \frac{5\pi}{3} + 2\pi k$ for integers k.

(b) From the table of common values, we see $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$. According to Theorem 11.8, we know the solutions to $\tan(\theta) = \sqrt{3}$ must, therefore, have a reference angle of $\frac{\pi}{3}$.

To find the quadrants in which our solutions lie, we note that tangent is defined as the ratio $\frac{y}{x}$ of points (x, y) on the Unit Circle. Hence, tangent is positive when x and y have the same sign (i.e., when they are both positive or both negative.) This happens in Quadrants I and III.

In Quadrant I, we get the solutions: $\theta = \frac{\pi}{3} + 2\pi k$ for integers k, and for Quadrant III, we get $\theta = \frac{4\pi}{3} + 2\pi k$ for integers k. While these descriptions of the solutions are correct, they can be combined into one list as $\theta = \frac{\pi}{3} + \pi k$ for integers k. The latter form of the solution is best understood looking at the geometry of the situation in the diagram below.[3]



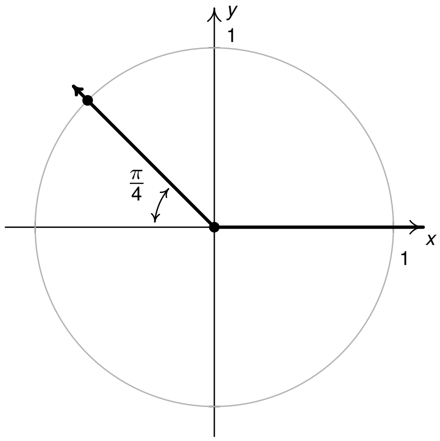


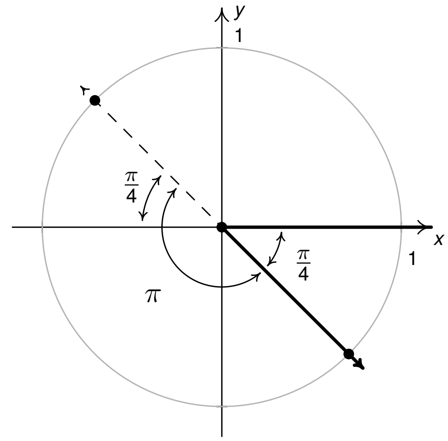
[3]See Example 11.2.4 number 3 in Section 11.2 for another example of this kind of simplification of the solution.

(c) From the table of common values, we see that $\frac{\pi}{4}$ has a cotangent of 1, which means the solutions to $\cot(\theta) = -1$ have a reference angle of $\frac{\pi}{4}$.

To find the quadrants in which our solutions lie, we note that $\cot(\theta) = \frac{x}{y}$ for a point (x, y) on the Unit Circle where $y \neq 0$. If $\cot(\theta)$ is negative, then x and y must have different signs (i.e., one positive and one negative.) Hence, our solutions lie in Quadrants II and IV.

Our Quadrant II solution is $\theta = \frac{3\pi}{4} + 2\pi k$, and for Quadrant IV, we get $\theta = \frac{7\pi}{4} + 2\pi k$ for integers k. As in the previous problem, we can combine these solutions as: $\theta = \frac{3\pi}{4} + \pi k$ for integers k.



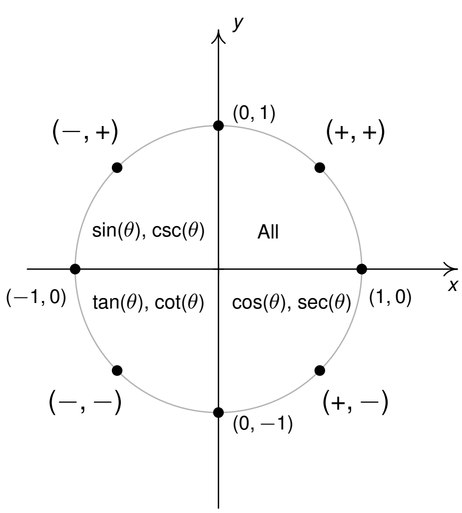


□

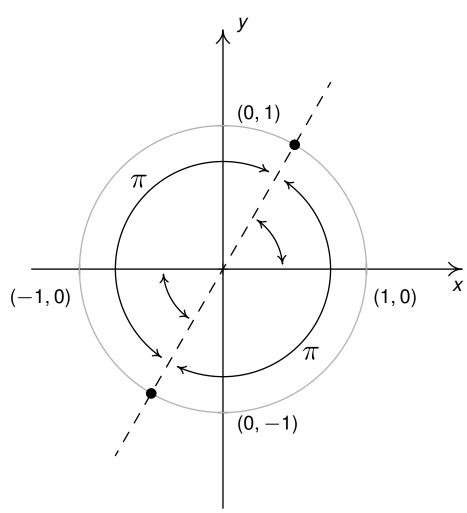
A few remarks about Example 11.4.1 are in order. First note that the signs ($\pm$) of secant and cosecant are the same as the signs of cosine and sine, respectively.

On the other hand, since tangent and cotangent are defined in terms of the *ratios* of coordinates x and y, tangent and cotangent are positive in Quadrants I and III (where both x and y have the same sign) and negative in Quadrants II and IV (where x and y have opposite signs.)

The diagram below on the left summarizes which circular functions are positive in which quadrants.



Postive Circular Functions



The period of $\tan(\theta)$ and $\cot(\theta)$ is π

Also note it is no coincidence that both of our solutions to the equations involving tangent and cotangent in Example 11.4.1 could be simplified to just one list of angles differing by multiples of π.

Indeed, any two angles that are π units apart will not only have the same reference angle, but points on their terminal sides on the Unit Circle will be reflections through the origin, as illustrated above on the right.

It follows that the tangent and cotangent of such angles (if defined) will be the same, which means the period of these function is (at most) π.

Using an argument similar to the one we used to establish the period of sine and cosine in Section 11.3, we note that if $\tan(x+p) = \tan(x)$ for all real numbers x, then, in particular, $\tan(p) = \tan(0+p) = \tan(0) = 0$. Hence, p is a multiple of π, and the smallest multiple of π is π itself.

Hence, the period of tangent (and cotangent) is π, and we will see the consequences of this both when solving equations in this section and when graphing these functions in Section 11.5.

As with sine and cosine, the circular functions defined in Definition 11.4 agree with those put forth in Definitions B.1 and B.2 in Section B.2 for acute angles situated in right triangles. The argument is identical to the one given in Section 11.2 and is left to the reader.

Moreover, Definition 11.4 can be extended to circles of arbitrary radius $r > 0$ using the same similarity arguments in Section 11.2.1 to generalize Definition 11.2 to Theorem 11.3 as summarized below.

Theorem 11.9. Suppose $Q(x, y)$ is the point on the terminal side of an angle θ (plotted in standard position) which lies on the circle of radius r, $x^2 + y^2 = r^2$. Then:

- $\sin(\theta) = \dfrac{y}{r} = \dfrac{y}{\sqrt{x^2+y^2}}$
- $\cos(\theta) = \dfrac{x}{r} = \dfrac{x}{\sqrt{x^2+y^2}}$
- $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- $\sec(\theta) = \dfrac{r}{x} = \dfrac{\sqrt{x^2+y^2}}{x}$, provided $x \neq 0$.
- $\csc(\theta) = \dfrac{r}{y} = \dfrac{\sqrt{x^2+y^2}}{y}$, provided $y \neq 0$.
- $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

We make good use of Theorem 11.9 in the following example.

Example 11.4.2. Use Theorem 11.9 to solve the following.

1. Suppose the terminal side of θ, when plotted in standard position, contains the point $Q(3, 4)$. Find the values of the six circular functions of θ.

2. Suppose θ is a Quadrant IV angle with $\cot(\theta) = -4$. Find the values of the five remaining circular functions of θ.

3. Find $\sin(\theta)$, where $\sec(\theta) = -\sqrt{5}$ and θ is a Quadrant II angle.

4. Find $\cos(\theta)$, where $\tan(\theta) = 3$ and $\pi < \theta < \frac{3\pi}{2}$.

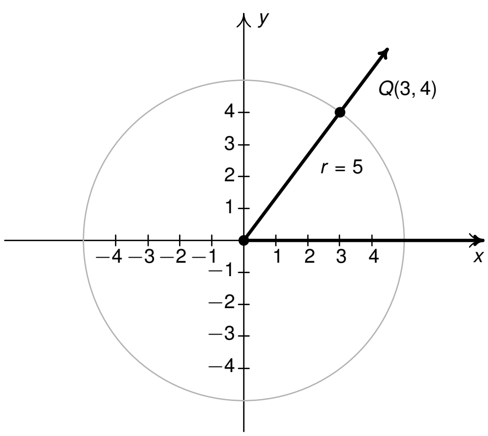
Solution.

1. Since $x = 3$ and $y = 4$, from $x^2 + y^2 = r^2$, $(3)^2 + (4)^2 = r^2$ so $r^2 = 25$, or $r = 5$. Theorem 11.9 tells us $\sin(\theta) = \frac{4}{5}$, $\cos(\theta) = \frac{3}{5}$, $\tan(\theta) = \frac{4}{3}$, $\sec(\theta) = \frac{5}{3}$, $\csc(\theta) = \frac{5}{4}$, and $\cot(\theta) = \frac{3}{4}$.

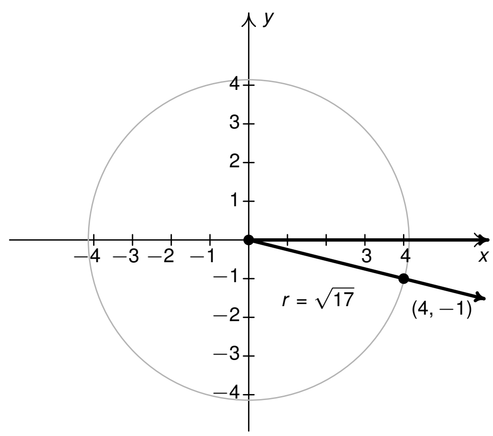
2. In order to use Theorem 11.9, we need to find a point $Q(x, y)$ which lies on the terminal side of θ, when θ is plotted in standard position.

   We have that $\cot(\theta) = -4 = \frac{x}{y}$. Since θ is a Quadrant IV angle, we also know $x > 0$ and $y < 0$. Rewriting $-4 = \frac{4}{-1}$, we choose[4] $x = 4$ and $y = -1$ so that $r = \sqrt{x^2 + y^2} = \sqrt{(4)^2 + (-1)^2} = \sqrt{17}$.

   Applying Theorem 11.9, we find $\sin(\theta) = -\frac{1}{\sqrt{17}} = -\frac{\sqrt{17}}{17}$, $\cos(\theta) = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17}$, $\tan(\theta) = -\frac{1}{4}$, $\sec(\theta) = \frac{\sqrt{17}}{4}$, and $\csc(\theta) = -\sqrt{17}$.



$Q(3, 4)$ lies on a circle of radius 5 units,



θ is Quadrant IV with $\cot(\theta) = -4$.

3. To find $\sin(\theta)$ using Theorem 11.9, we need to determine the y-coordinate of a point $Q(x, y)$ on the terminal side of θ, when θ is plotted in standard position, and the corresponding radius r.

   Since $\sec(\theta) = \frac{r}{x}$ and $r > 0$, we rewrite $\sec(\theta) = \frac{r}{x} = -\sqrt{5} = \frac{\sqrt{5}}{-1}$ and take $r = \sqrt{5}$ and $x = -1$.

   To find y, we substitute $x = -1$ and $r = \sqrt{5}$ into $x^2 + y^2 = r^2$ to get $(-1)^2 + y^2 = (\sqrt{5})^2$. We find $y^2 = 4$ or $y = \pm 2$. Since θ is a Quadrant II angle, we select $y = 2$.

   Hence, $\sin(\theta) = \frac{y}{r} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$.

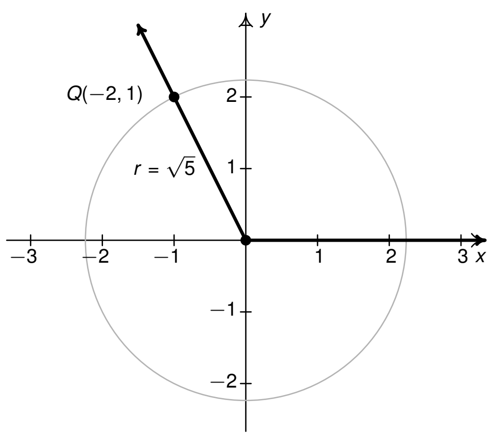
4. We are told $\tan(\theta) = 3$ and $\pi < \theta < \frac{3\pi}{2}$, so we know θ is a Quadrant III angle.

[4] We could have just as easily chosen $x = 8$ and $y = -2$ - just so long as $x > 0$, $y < 0$ and $\frac{x}{y} = -4$.

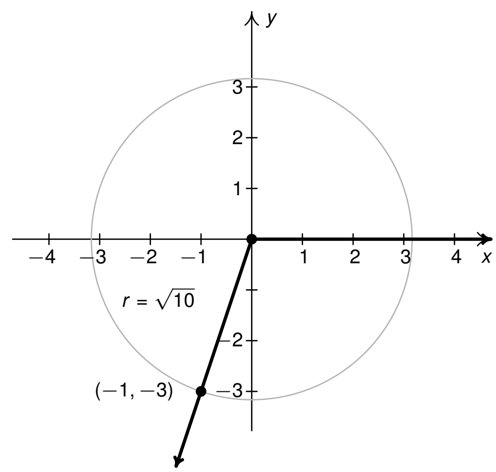
To find $\cos(\theta)$ using Theorem 11.9, we need to find the x-coordinate of a point $Q(x, y)$ on the terminal side of θ, when θ is plotted in standard position, and the corresponding radius, r.

Since $\tan(\theta) = \frac{y}{x}$ and θ is a Quadrant III angle, we rewrite $\tan(\theta) = 3 = \frac{-3}{-1} = \frac{y}{x}$ and choose $x = -1$ and $y = -3$. From $x^2 + y^2 = r^2$, we get $r = \sqrt{10}$.

Hence, $\cos(\theta) = \frac{x}{r} = \frac{-1}{\sqrt{10}} = -\frac{\sqrt{10}}{10}$.



θ is Quadrant II with $\sec(\theta) = -\sqrt{5}$



θ is Quadrant III with $\tan(\theta) = 3$.

□

As we did in Section 11.2.1, we may consider $\tan(t)$, $\sec(t)$, $\csc(t)$, and $\cot(t)$ as functions *real numbers* by associating each real number t with an angle θ measuring t radians as discussed on page 933 and using Definition 11.4, or, more generally, Theorem 11.9.

Alternatively, we could define each of these four functions in terms of $f(t) = \sin(t)$ and $g(t) = \cos(t)$ as demonstrated in Theorem 11.7. For example, we could simply *define* $\sec(t) = \frac{1}{\cos(t)}$ so long as $\cos(t) \neq 0$.

Either way, we have the means to explore these functions in greater detail. Before doing so, we'll need practice with these additional four circular functions courtesy of the Exercises.

11.4.1 Exercises

In Exercises 1 - 20, find the exact value or state that it is undefined.

1. $\tan\left(\frac{\pi}{4}\right)$
2. $\sec\left(\frac{\pi}{6}\right)$
3. $\csc\left(\frac{5\pi}{6}\right)$
4. $\cot\left(\frac{4\pi}{3}\right)$
5. $\tan\left(-\frac{11\pi}{6}\right)$
6. $\sec\left(-\frac{3\pi}{2}\right)$
7. $\csc\left(-\frac{\pi}{3}\right)$
8. $\cot\left(\frac{13\pi}{2}\right)$
9. $\tan(117\pi)$
10. $\sec\left(-\frac{5\pi}{3}\right)$
11. $\csc(3\pi)$
12. $\cot(-5\pi)$
13. $\tan\left(\frac{31\pi}{2}\right)$
14. $\sec\left(\frac{\pi}{4}\right)$
15. $\csc\left(-\frac{7\pi}{4}\right)$
16. $\cot\left(\frac{7\pi}{6}\right)$
17. $\tan\left(\frac{2\pi}{3}\right)$
18. $\sec(-7\pi)$
19. $\csc\left(\frac{\pi}{2}\right)$
20. $\cot\left(\frac{3\pi}{4}\right)$

In Exercises 21 - 24, use the given the information to determine the quadrant in which the terminal side of the angle lies when plotted in standard position.

21. $\sin(\theta) > 0$ but $\tan(\theta) < 0$.
22. $\cot(\alpha) > 0$ but $\cos(\alpha) < 0$.
23. $\sin(\beta) > 0$ and $\tan(\beta) > 0$.
24. $\cos(\gamma) > 0$ but $\cot(\gamma) < 0$.

In Exercises 25 - 38, use the given the information to find the exact values of the circular functions of θ.

25. $\sin(\theta) = \frac{3}{5}$ with θ in Quadrant II
26. $\tan(\theta) = \frac{12}{5}$ with θ in Quadrant III
27. $\csc(\theta) = \frac{25}{24}$ with θ in Quadrant I
28. $\sec(\theta) = 7$ with θ in Quadrant IV
29. $\csc(\theta) = -\frac{10\sqrt{91}}{91}$ with θ in Quadrant III
30. $\cot(\theta) = -23$ with θ in Quadrant II
31. $\tan(\theta) = -2$ with θ in Quadrant IV.
32. $\sec(\theta) = -4$ with θ in Quadrant II.
33. $\cot(\theta) = \sqrt{5}$ with θ in Quadrant III.
34. $\cos(\theta) = \frac{1}{3}$ with θ in Quadrant I.
35. $\cot(\theta) = 2$ with $0 < \theta < \frac{\pi}{2}$.
36. $\csc(\theta) = 5$ with $\frac{\pi}{2} < \theta < \pi$.
37. $\tan(\theta) = \sqrt{10}$ with $\pi < \theta < \frac{3\pi}{2}$.
38. $\sec(\theta) = 2\sqrt{5}$ with $\frac{3\pi}{2} < \theta < 2\pi$.

In Exercises 39 - 46, use your calculator to approximate the given value to three decimal places. Make sure your calculator is in the proper angle measurement mode!

39. $\csc(78.95^\circ)$
40. $\tan(-2.01)$
41. $\cot(392.994)$
42. $\sec(207^\circ)$
43. $\csc(5.902)$
44. $\tan(39.672^\circ)$
45. $\cot(3^\circ)$
46. $\sec(0.45)$

In Exercises 47 - 61, find all of the angles which satisfy the equation.

47. $\tan(\theta) = \sqrt{3}$
48. $\sec(\theta) = 2$
49. $\csc(\theta) = -1$
50. $\cot(\theta) = \dfrac{\sqrt{3}}{3}$
51. $\tan(\theta) = 0$
52. $\sec(\theta) = 1$
53. $\csc(\theta) = 2$
54. $\cot(\theta) = 0$
55. $\tan(\theta) = -1$
56. $\sec(\theta) = 0$
57. $\csc(\theta) = -\dfrac{1}{2}$
58. $\sec(\theta) = -1$
59. $\tan(\theta) = -\sqrt{3}$
60. $\csc(\theta) = -2$
61. $\cot(\theta) = -1$

In Exercises 62 - 69, solve the equation for t. Give exact values.

62. $\cot(t) = 1$
63. $\tan(t) = \dfrac{\sqrt{3}}{3}$
64. $\sec(t) = -\dfrac{2\sqrt{3}}{3}$
65. $\csc(t) = 0$
66. $\cot(t) = -\sqrt{3}$
67. $\tan(t) = -\dfrac{\sqrt{3}}{3}$
68. $\sec(t) = \dfrac{2\sqrt{3}}{3}$
69. $\csc(t) = \dfrac{2\sqrt{3}}{3}$

In Exercises 70 - 77, write the given function as a nontrivial decomposition of functions as directed.

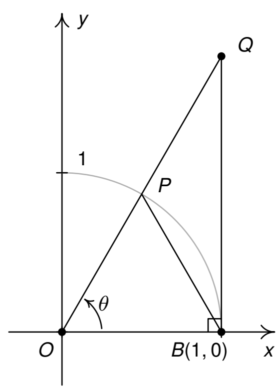
70. For $f(t) = 3t^2 + 2\tan(3t)$, find functions g and h so that $f = g + h$.
71. For $f(\theta) = \sec(\theta) - \tan(\theta)$, find functions g and h so that $f = g - h$.
72. For $f(t) = -\csc(t)\cot(t)$, find functions g and h so that $f = gh$.
73. For $r(t) = \dfrac{\tan(3t)}{t}$, find functions f and g so $r = \dfrac{f}{g}$.
74. For $T(\theta) = \tan(4\theta)$, find functions f and g so $T = g \circ f$.
75. For $s(\theta) = \sec^2(\theta)$, find functions f and g so $s = g \circ f$.
76. For $L(x) = \ln(\sin(x))$, find functions f and g so $L = g \circ f$.
77. For $\ell(\theta) = \ln|\sec(\theta) - \tan(\theta)|$, find find functions f, g, and h so $\ell = h \circ (f - g)$.
78. Let $S(t) = \sin(t)$ and $C(t) = \cos(t)$, $F(t) = \tan(t)$, and $G(t) = \cot(t)$. Explain why $F = \dfrac{S}{C}$ but $F \neq \dfrac{1}{G}$.

    HINT: Think about domains ...

79. For each function $T(t)$ listed below, compute the average rate of change over the indicated interval.[5] What trends do you notice? Compare your answer with what you discovered in Section 11.2 number 64. Be sure your calculator is in radian mode!

$T(t)$	$[-0.1, 0.1]$	$[-0.01, 0.01]$	$[-0.001, 0.001]$
$\tan(t)$			
$\tan(2t)$			
$\tan(3t)$			
$\tan(4t)$			

80. We wish to establish the inequality $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$ for $0 < \theta < \frac{\pi}{2}$. Use the diagram from the beginning of the section, partially reproduced below, to answer the following.



(a) Show that triangle OPB has area $\frac{1}{2}\sin(\theta)$ and triangle OQB has area $\frac{1}{2}\tan(\theta)$.

(b) Show that the circular sector OPB with central angle θ has area $\frac{1}{2}\theta$.

(c) Comparing areas, show that $\sin(\theta) < \theta < \tan(\theta)$ for $0 < \theta < \frac{\pi}{2}$.

(d) Use the inequality $\sin(\theta) < \theta$ to show that $\frac{\sin(\theta)}{\theta} < 1$ for $0 < \theta < \frac{\pi}{2}$.

(e) Use the inequality $\theta < \tan(\theta)$ to show that $\cos(\theta) < \frac{\sin(\theta)}{\theta}$ for $0 < \theta < \frac{\pi}{2}$. Combine this with the previous part to complete the proof.

81. Show that $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$ also holds for $-\frac{\pi}{2} < \theta < 0$.

82. Use the results from Exercises 80 and 81 along with the Squeeze Theorem,[6] to prove $\displaystyle\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$.

[5] See Definition 1.8 in Section 1.2.4 for a review of this concept, as needed.

[6] Theorem 10.2

11.4.2 Answers

1. $\tan\left(\frac{\pi}{4}\right) = 1$
2. $\sec\left(\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$
3. $\csc\left(\frac{5\pi}{6}\right) = 2$
4. $\cot\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{3}$
5. $\tan\left(-\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{3}$
6. $\sec\left(-\frac{3\pi}{2}\right)$ is undefined
7. $\csc\left(-\frac{\pi}{3}\right) = -\frac{2\sqrt{3}}{3}$
8. $\cot\left(\frac{13\pi}{2}\right) = 0$
9. $\tan(117\pi) = 0$
10. $\sec\left(-\frac{5\pi}{3}\right) = 2$
11. $\csc(3\pi)$ is undefined
12. $\cot(-5\pi)$ is undefined
13. $\tan\left(\frac{31\pi}{2}\right)$ is undefined
14. $\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$
15. $\csc\left(-\frac{7\pi}{4}\right) = \sqrt{2}$
16. $\cot\left(\frac{7\pi}{6}\right) = \sqrt{3}$
17. $\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$
18. $\sec(-7\pi) = -1$
19. $\csc\left(\frac{\pi}{2}\right) = 1$
20. $\cot\left(\frac{3\pi}{4}\right) = -1$
21. Quadrant II.
22. Quadrant III.
23. Quadrant I.
24. Quadrant IV.
25. $\sin(\theta) = \frac{3}{5}, \cos(\theta) = -\frac{4}{5}, \tan(\theta) = -\frac{3}{4}, \csc(\theta) = \frac{5}{3}, \sec(\theta) = -\frac{5}{4}, \cot(\theta) = -\frac{4}{3}$
26. $\sin(\theta) = -\frac{12}{13}, \cos(\theta) = -\frac{5}{13}, \tan(\theta) = \frac{12}{5}, \csc(\theta) = -\frac{13}{12}, \sec(\theta) = -\frac{13}{5}, \cot(\theta) = \frac{5}{12}$
27. $\sin(\theta) = \frac{24}{25}, \cos(\theta) = \frac{7}{25}, \tan(\theta) = \frac{24}{7}, \csc(\theta) = \frac{25}{24}, \sec(\theta) = \frac{25}{7}, \cot(\theta) = \frac{7}{24}$
28. $\sin(\theta) = \frac{-4\sqrt{3}}{7}, \cos(\theta) = \frac{1}{7}, \tan(\theta) = -4\sqrt{3}, \csc(\theta) = -\frac{7\sqrt{3}}{12}, \sec(\theta) = 7, \cot(\theta) = -\frac{\sqrt{3}}{12}$
29. $\sin(\theta) = -\frac{\sqrt{91}}{10}, \cos(\theta) = -\frac{3}{10}, \tan(\theta) = \frac{\sqrt{91}}{3}, \csc(\theta) = -\frac{10\sqrt{91}}{91}, \sec(\theta) = -\frac{10}{3}, \cot(\theta) = \frac{3\sqrt{91}}{91}$
30. $\sin(\theta) = \frac{\sqrt{530}}{530}, \cos(\theta) = -\frac{23\sqrt{530}}{530}, \tan(\theta) = -\frac{1}{23}, \csc(\theta) = \sqrt{530}, \sec(\theta) = -\frac{\sqrt{530}}{23}, \cot(\theta) = -23$
31. $\sin(\theta) = -\frac{2\sqrt{5}}{5}, \cos(\theta) = \frac{\sqrt{5}}{5}, \tan(\theta) = -2, \csc(\theta) = -\frac{\sqrt{5}}{2}, \sec(\theta) = \sqrt{5}, \cot(\theta) = -\frac{1}{2}$
32. $\sin(\theta) = \frac{\sqrt{15}}{4}, \cos(\theta) = -\frac{1}{4}, \tan(\theta) = -\sqrt{15}, \csc(\theta) = \frac{4\sqrt{15}}{15}, \sec(\theta) = -4, \cot(\theta) = -\frac{\sqrt{15}}{15}$
33. $\sin(\theta) = -\frac{\sqrt{6}}{6}, \cos(\theta) = -\frac{\sqrt{30}}{6}, \tan(\theta) = \frac{\sqrt{5}}{5}, \csc(\theta) = -\sqrt{6}, \sec(\theta) = -\frac{\sqrt{30}}{5}, \cot(\theta) = \sqrt{5}$
34. $\sin(\theta) = \frac{2\sqrt{2}}{3}, \cos(\theta) = \frac{1}{3}, \tan(\theta) = 2\sqrt{2}, \csc(\theta) = \frac{3\sqrt{2}}{4}, \sec(\theta) = 3, \cot(\theta) = \frac{\sqrt{2}}{4}$
35. $\sin(\theta) = \frac{\sqrt{5}}{5}, \cos(\theta) = \frac{2\sqrt{5}}{5}, \tan(\theta) = \frac{1}{2}, \csc(\theta) = \sqrt{5}, \sec(\theta) = \frac{\sqrt{5}}{2}, \cot(\theta) = 2$
36. $\sin(\theta) = \frac{1}{5}, \cos(\theta) = -\frac{2\sqrt{6}}{5}, \tan(\theta) = -\frac{\sqrt{6}}{12}, \csc(\theta) = 5, \sec(\theta) = -\frac{5\sqrt{6}}{12}, \cot(\theta) = -2\sqrt{6}$

37. $\sin(\theta) = -\frac{\sqrt{110}}{11}, \cos(\theta) = -\frac{\sqrt{11}}{11}, \tan(\theta) = \sqrt{10}, \csc(\theta) = -\frac{\sqrt{110}}{10}, \sec(\theta) = -\sqrt{11}, \cot(\theta) = \frac{\sqrt{10}}{10}$

38. $\sin(\theta) = -\frac{\sqrt{95}}{10}, \cos(\theta) = \frac{\sqrt{5}}{10}, \tan(\theta) = -\sqrt{19}, \csc(\theta) = -\frac{2\sqrt{95}}{19}, \sec(\theta) = 2\sqrt{5}, \cot(\theta) = -\frac{\sqrt{19}}{19}$

39. $\csc(78.95^\circ) \approx 1.019$

40. $\tan(-2.01) \approx 2.129$

41. $\cot(392.994) \approx 3.292$

42. $\sec(207^\circ) \approx -1.122$

43. $\csc(5.902) \approx -2.688$

44. $\tan(39.672^\circ) \approx 0.829$

45. $\cot(3^\circ) \approx 19.081$

46. $\sec(0.45) \approx 1.111$

47. $\tan(\theta) = \sqrt{3}$ when $\theta = \dfrac{\pi}{3} + \pi k$ for any integer k

48. $\sec(\theta) = 2$ when $\theta = \dfrac{\pi}{3} + 2\pi k$ or $\theta = \dfrac{5\pi}{3} + 2\pi k$ for any integer k

49. $\csc(\theta) = -1$ when $\theta = \dfrac{3\pi}{2} + 2\pi k$ for any integer k.

50. $\cot(\theta) = \dfrac{\sqrt{3}}{3}$ when $\theta = \dfrac{\pi}{3} + \pi k$ for any integer k

51. $\tan(\theta) = 0$ when $\theta = \pi k$ for any integer k

52. $\sec(\theta) = 1$ when $\theta = 2\pi k$ for any integer k

53. $\csc(\theta) = 2$ when $\theta = \dfrac{\pi}{6} + 2\pi k$ or $\theta = \dfrac{5\pi}{6} + 2\pi k$ for any integer k.

54. $\cot(\theta) = 0$ when $\theta = \dfrac{\pi}{2} + \pi k$ for any integer k

55. $\tan(\theta) = -1$ when $\theta = \dfrac{3\pi}{4} + \pi k$ for any integer k

56. $\sec(\theta) = 0$ never happens

57. $\csc(\theta) = -\dfrac{1}{2}$ never happens

58. $\sec(\theta) = -1$ when $\theta = \pi + 2\pi k = (2k+1)\pi$ for any integer k

59. $\tan(\theta) = -\sqrt{3}$ when $\theta = \dfrac{2\pi}{3} + \pi k$ for any integer k

60. $\csc(\theta) = -2$ when $\theta = \dfrac{7\pi}{6} + 2\pi k$ or $\theta = \dfrac{11\pi}{6} + 2\pi k$ for any integer k

61. $\cot(\theta) = -1$ when $\theta = \dfrac{3\pi}{4} + \pi k$ for any integer k

62. $\cot(t) = 1$ when $t = \dfrac{\pi}{4} + \pi k$ for any integer k

63. $\tan(t) = \dfrac{\sqrt{3}}{3}$ when $t = \dfrac{\pi}{6} + \pi k$ for any integer k

64. $\sec(t) = -\dfrac{2\sqrt{3}}{3}$ when $t = \dfrac{5\pi}{6} + 2\pi k$ or $t = \dfrac{7\pi}{6} + 2\pi k$ for any integer k

65. $\csc(t) = 0$ never happens

66. $\cot(t) = -\sqrt{3}$ when $t = \dfrac{5\pi}{6} + \pi k$ for any integer k

67. $\tan(t) = -\dfrac{\sqrt{3}}{3}$ when $t = \dfrac{5\pi}{6} + \pi k$ for any integer k

68. $\sec(t) = \dfrac{2\sqrt{3}}{3}$ when $t = \dfrac{\pi}{6} + 2\pi k$ or $t = \dfrac{11\pi}{6} + 2\pi k$ for any integer k

69. $\csc(t) = \dfrac{2\sqrt{3}}{3}$ when $t = \dfrac{\pi}{3} + 2\pi k$ or $t = \dfrac{2\pi}{3} + 2\pi k$ for any integer k

70. One solution is $g(t) = 3t^2$ and $h(t) = 2\tan(3t)$.

71. One solution is $g(\theta) = \sec(\theta)$ and $h(\theta) = \tan(\theta)$.

72. One solution is $g(t) = -\csc(t)$ and $h(t) = \cot(t)$.

73. One solution is $f(t) = \tan(3t)$ and $g(t) = t$.

74. One solution is $f(\theta) = 4\theta$ and $g(\theta) = \tan(\theta)$.

75. Since $\sec^2(\theta) = (\sec(\theta))^2$, one solution is $f(\theta) = \sec(\theta)$ and $g(\theta) = \theta^2$.

76. One solution is $f(x) = \sin(x)$ and $g(x) = \ln(x)$.

77. One solution is $f(\theta) = \sec(\theta)$, $g(\theta) = \tan(\theta)$, and $h(\theta) = \ln|\theta|$.

79. As we zoom in towards 0, the average rate of change of $\tan(kt)$ approaches k. This is the same trend we observed for $\sin(kt)$ in Section 11.2 number 64.

$T(t)$	$[-0.1, 0.1]$	$[-0.01, 0.01]$	$[-0.001, 0.001]$
$\tan(t)$	≈ 1.0033	≈ 1	≈ 1
$\tan(2t)$	≈ 2.0271	≈ 2.0003	≈ 2
$\tan(3t)$	≈ 3.0933	≈ 3.0009	≈ 3
$\tan(4t)$	≈ 4.2279	≈ 4.0021	≈ 4

11.5 Graphs of Secant, Cosecant, Tangent, and Cotangent Functions

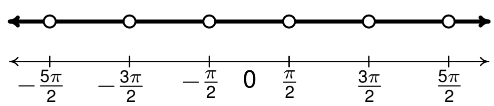
11.5.1 Graphs of the Secant and Cosecant Functions

As mentioned at the end of Section 11.4, one way to proceed with our analysis of the circular functions is to use what we know about the functions $\sin(t)$ and $\cos(t)$ to rewrite the four additional circular functions in terms of sine and cosine with help from Theorem 11.7. We use this approach to analyze $F(t) = \sec(t)$.

Rewriting $F(t) = \sec(t) = \frac{1}{\cos(t)}$, we first note that $F(t)$ is undefined whenever $\cos(t) = 0$. Thanks to Example 11.2.4 number 3, we know $\cos(t) = 0$ whenever $t = \frac{\pi}{2} + \pi k$ for integers k.

This gives us one way to describe the domain of F: $\{t \mid t \neq \frac{\pi}{2} + \pi k, \text{for integers } k\}$. To get a better feel for the set of real numbers we're dealing with, we write out and graph the domain on the number line.

Running through a few values of k, we find some of the values excluded from the domain: $t \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$. Using these we can graph the domain on the number line below.



Expressing this set using interval notation is a bit of a challenge, owing to the infinitely many intervals present. As a first attempt, we have: $\ldots \cup \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \ldots$, where, as usual, the periods of ellipsis indicate the pattern continues indefinitely.[1] Hence, for now, it suffices to know that the domain of $F(t) = \sec(t)$ excludes the odd multiples of $\frac{\pi}{2}$.

To find the range of F, we find it helpful once again to view $F(t) = \sec(t) = \frac{1}{\cos(t)}$. We know the range of $\cos(t)$ is $[-1, 1]$, and since $F(t) = \sec(t) = \frac{1}{\cos(t)}$ is undefined when $\cos(t) = 0$, we split our discussion into two cases: when $0 < \cos(t) \leq 1$ and when $-1 \leq \cos(t) < 0$.

If $0 < \cos(t) \leq 1$, then we can divide the inequality $\cos(t) \leq 1$ by $\cos(t)$ to obtain $\sec(t) = \frac{1}{\cos(t)} \geq 1$. Moreover, we see as $\cos(t) \to 0^+$, $\sec(t) \to \infty$. If, on the other hand, if $-1 \leq \cos(t) < 0$, then dividing by $\cos(t)$ causes a reversal of the inequality so that $\sec(t) = \frac{1}{\cos(t)} \leq -1$. In this case, as $\cos(t) \to 0^-$, $\sec(t) \to -\infty$. Since $\cos(t)$ admits all of the values in $[-1, 1]$, the function $F(t) = \sec(t)$ admits all of the values in $(-\infty, -1] \cup [1, \infty)$.

Since $\cos(t)$ is periodic with period 2π, it shoudn't be too surprising to find that $\sec(t)$ is also. Indeed, provided $\sec(\alpha)$ and $\sec(\beta)$ are defined, $\sec(\alpha) = \sec(\beta)$ if and only if $\cos(\alpha) = \cos(\beta)$. Said differently, $\sec(t)$ 'inherits' its period from $\cos(t)$.

We now turn our attention to graphing $F(t) = \sec(t)$. Using the table of values we tabulated when graphing $y = \cos(t)$ in Section 11.3, we can generate points on the graph of $y = \sec(t)$ by taking reciprocals.

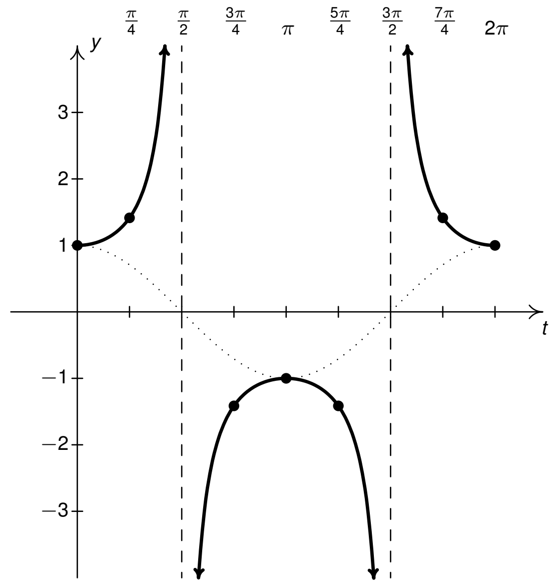
Using the techniques developed in Section 3.1, we can more closely analyze the behavior of F near the values excluded from its domain. We find as $t \to \frac{\pi}{2}^-$, $\cos(t) \to 0^+$, so $\displaystyle\lim_{t \to \frac{\pi}{2}^-} \sec(t) = \infty$. Similarly, we get $\displaystyle\lim_{t \to \frac{\pi}{2}^+} \sec(t) = -\infty$, $\displaystyle\lim_{t \to \frac{3\pi}{2}^-} \sec(t) = -\infty$, and $\displaystyle\lim_{t \to \frac{3\pi}{2}^+} \sec(t) = \infty$. This means the lines $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ are vertical asymptotes to the graph of $y = \sec(t)$.

---

[1] We introduce an extended form of interval notation in Section 11.5.3 which gives us a more compact way to express this set.

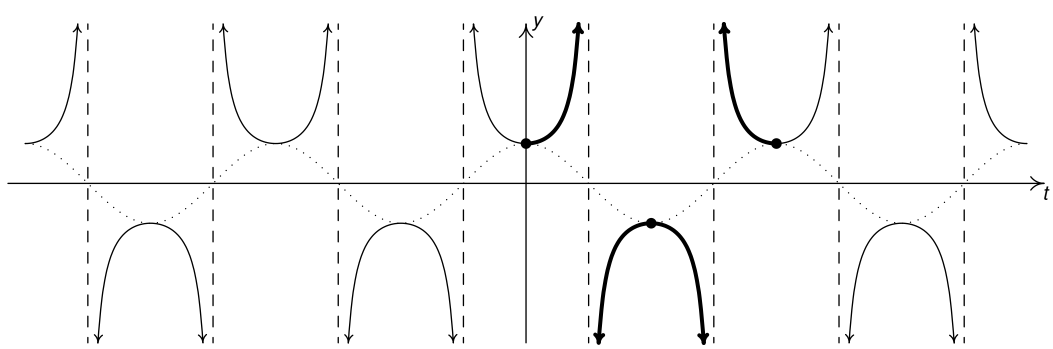
Below on the right we graph a fundamental cycle of $y = \sec(t)$ with the graph of the fundamental cycle of $y = \cos(t)$ dotted for reference.

t	$\cos(t)$	$\sec(t)$	$(t, \sec(t))$
0	1	1	$(0, 1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{\pi}{4}, \sqrt{2}\right)$
$\frac{\pi}{2}$	0	undefined	
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{3\pi}{4}, -\sqrt{2}\right)$
π	-1	-1	$(\pi, -1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$
$\frac{3\pi}{2}$	0	undefined	
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{7\pi}{4}, \sqrt{2}\right)$
2π	1	1	$(2\pi, 1)$



The ‘fundamental cycle’ of $y = \sec(t)$.

To get a graph of the entire secant function, we paste copies of the fundamental cycle end to end to produce the graph below. The graph suggests that $F(t) = \sec(t)$ is even. Indeed, since $\cos(t)$ is even, that is, $\cos(-t) = \cos(t)$, we have $\sec(-t) = \frac{1}{\cos(-t)} = \frac{1}{\cos(t)} = \sec(t)$. Hence, along with its period, the secant function inherits its symmetry from the cosine function.

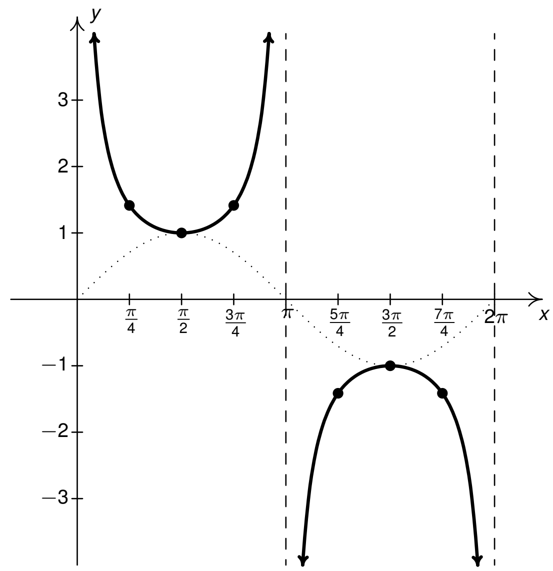


The graph of $y = \sec(t)$.

As one would expect, to graph $G(t) = \csc(t)$ we begin with $y = \sin(t)$ and take reciprocals of the corresponding y-values. Here, we encounter issues at $t = 0$, $t = \pi$, $t = 2\pi$, and, in general, at all whole number multiples of π, so the domain of G is $\{t \,|\, t \neq \pi k, \text{for integers } k\}$. Not surprisingly, these values produce vertical asymptotes.

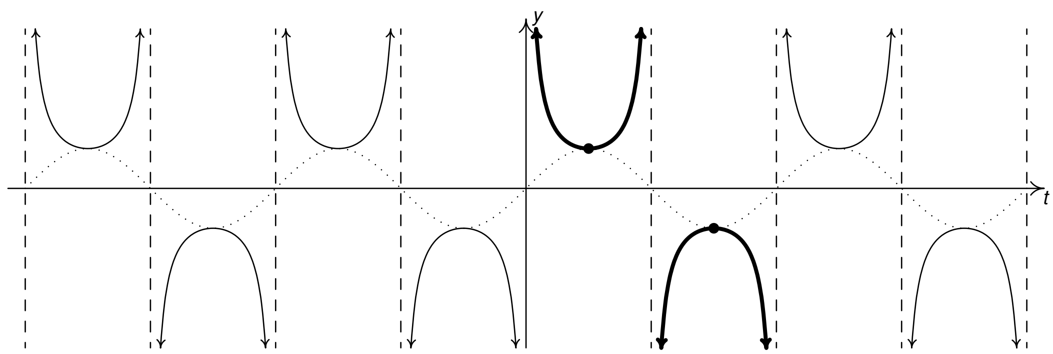
Proceeding as above, we graph produce the graph of the fundamental cycle of $y = \csc(t)$ below along with the dotted graph of $y = \sin(t)$ for reference.

x	$\sin(x)$	$\csc(x)$	$(x, \csc(x))$
0	0	undefined	
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{\pi}{4}, \sqrt{2}\right)$
$\frac{\pi}{2}$	1	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{3\pi}{4}, \sqrt{2}\right)$
π	0	undefined	
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$
$\frac{3\pi}{2}$	-1	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{7\pi}{4}, -\sqrt{2}\right)$
2π	0	undefined	



The 'fundamental cycle' of $y = \csc(t)$.

Pasting copies of the fundamental period of $y = \csc(t)$ end to end produces the graph below. Since the graphs of $y = \sin(t)$ and $y = \cos(t)$ are merely phase shifts of each other, it is not too surprising to find the graphs of $y = \csc(t)$ and $y = \sec(t)$ are as well.



The graph of $y = \csc(t)$.

As with the graph of secant, the graph below suggests symmetry. Indeed, since the sine function is odd, that is $\sin(-t) = -\sin(t)$, so too is the cosecant function: $\csc(-t) = \frac{1}{\sin(-t)} = -\frac{1}{\sin(t)} = -\csc(t)$. Hence, the graph of $G(t) = \csc(t)$ is symmetric about the origin.

Note that, on the intervals between the vertical asymptotes, both $F(t) = \sec(t)$ and $G(t) = \csc(t)$ are continuous and smooth. In other words, they are continuous and smooth *on their domains*.[2]

The following theorem summarizes the properties of the secant and cosecant functions. Note that all of these properties are direct results of them being reciprocals of the cosine and sine functions, respectively.

[2]Just like the rational functions in Chapter 3 are continuous and smooth on their domains because polynomials are continuous and smooth everywhere, the secant and cosecant functions are continuous and smooth on their domains since the cosine and sine functions are continuous and smooth everywhere.

Theorem 11.10. Properties of the Secant and Cosecant Functions

- The function $F(t) = \sec(t)$
  - has domain $\left\{t \,|\, t \neq \frac{\pi}{2} + \pi k, \;\; k \text{ is an integer}\right\}$
  - has range $(-\infty, -1] \cup [1, \infty)$
  - is continuous and smooth on its domain
  - is even
  - has period 2π
- The function $G(t) = \csc(t)$
  - has domain $\{t \,|\, t \neq \pi k, \;\; k \text{ is an integer}\}$
  - has range $(-\infty, -1] \cup [1, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period 2π

In the next example, we discuss graphing more general secant and cosecant curves. We make heavy use of the fact they are reciprocals of sine and cosine functions and apply what we learned in Section 11.3.

Example 11.5.1. Graph one cycle of the following functions. State the period of each.

1. $f(t) = 1 - 2\sec(2t)$
2. $g(t) = \dfrac{\csc(-\pi t - \pi) - 5}{3}$

Solution.

1. To graph $f(t) = 1 - 2\sec(2t)$, we follow the same procedure as in Example 11.3.2. That is, we use the concept of frequency and phase shift to identify quarter marks, then substitute these values into the function to obtain the corresponding points.

   If we think about a related *cosine* curve, $y = 1 - 2\cos(2t) = -2\cos(2t) + 1$, we know from Section 11.3, that the frequency is $\omega = 2$, so the period is $T = \frac{2\pi}{2} = \pi$. Since the phase $\phi = 0$, there is no phase shift. Hence, the new quarter marks for this curve are $t = 0$, $t = \frac{\pi}{4}$, $t = \frac{\pi}{2}$, $t = \frac{3\pi}{4}$, and $t = \pi$.

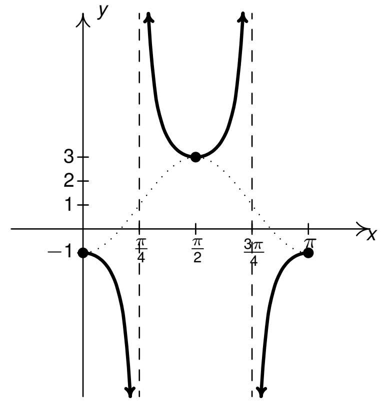
   Since we obtained the fundamental cycle of the secant curve from the fundamental cycle of the cosine curve, these same t-values are the new quarter marks for $f(t) = 1 - 2\sec(2t)$.

   Substituting these t values $f(t)$, we get the table below on the left. Note that if $f(t)$ exists, we have a point on the graph; otherwise, we have found a vertical asymptote.[3]

[3] As with the examples in Section 11.3, note that we can partially check our answer since the argument of the secant function should simplify to the 'original' quarter marks - the quadrantal angles.

We graph one cycle of $f(t) = 1 - 2\sec(2t)$ below on the right along with the associated cosine curve, $y = 1 - 2\cos(2t)$ which is dotted, and confirm the period is $\pi - 0 = \pi$.

t	$f(t)$	$(t, f(t))$
0	-1	$(0, -1)$
$\frac{\pi}{4}$	undefined	
$\frac{\pi}{2}$	3	$\left(\frac{\pi}{2}, 3\right)$
$\frac{3\pi}{4}$	undefined	
π	-1	$(\pi, -1)$



One cycle of $y = 1 - 2\sec(2t)$.

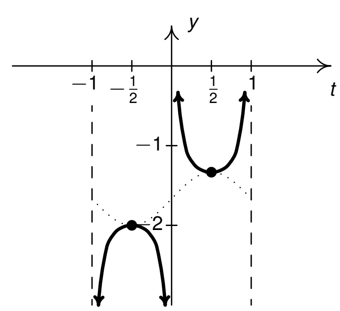
2. As with the previous example, we start graphing $g(t) = \frac{\csc(-\pi t - \pi) - 5}{3}$ by first finding the quarter marks of the associated sine curve: $y = \frac{\sin(-\pi t - \pi) - 5}{3} = \frac{1}{3}\sin(-\pi t - \pi) - \frac{5}{3}$.

   Since the coefficient of t is negative, we make use of the odd property of sine to rewrite the function as: $y = \frac{1}{3}\sin(-\pi t - \pi) - \frac{5}{3} = \frac{1}{3}\sin(-(\pi t + \pi)) - \frac{5}{3} = -\frac{1}{3}\sin(\pi t + \pi) - \frac{5}{3}$.

   We find the frequency is $\omega = \pi$, so the period is $T = \frac{2\pi}{\pi} = 2$. The phase is $\phi = \pi$, so the phase shift is $-\frac{\pi}{\pi} = -1$. Hence the fundamental cycle $[0, 2\pi]$ is shifted to the interval $[-1, 1]$ with quarter marks $t = -1$, $t = -\frac{1}{2}$, $t = 0$, $t = \frac{1}{2}$ and $t = 1$.

   Substituting these t-values into $g(t)$, we generate the graph below on the right confirm the period is $1 - (-1) = 2$. The associated sine curve, $y = \frac{\sin(-\pi t - \pi) - 5}{3}$, is dotted in as a reference.

t	$g(t)$	$(t, g(t))$
-1	undefined	
$-\frac{1}{2}$	-2	$\left(-\frac{1}{2}, -2\right)$
0	undefined	
$\frac{1}{2}$	$-\frac{4}{3}$	$\left(\frac{1}{2}, -\frac{4}{3}\right)$
1	undefined	



One cycle of $y = \frac{\csc(-\pi t - \pi) - 5}{3}$.

$\square$

As suggested in Example 11.5.1, the concepts of frequency, period, phase shift, and baseline are alive and well with graphs of the secant and cosecant functions. Since the secant and cosecant curves are unbounded, we do not have the concept of 'amplitude' for these curves. That being said, the amplitudes of the corresponding cosine and sine curves do play a role here - they measure how wide the gap is between the baseline and the curve.

We gather these observations in the following result whose proof is a consequence of Theorem 11.6 and is relegated to Exercise 29.

Theorem 11.11. For $\omega > 0$, the graphs of

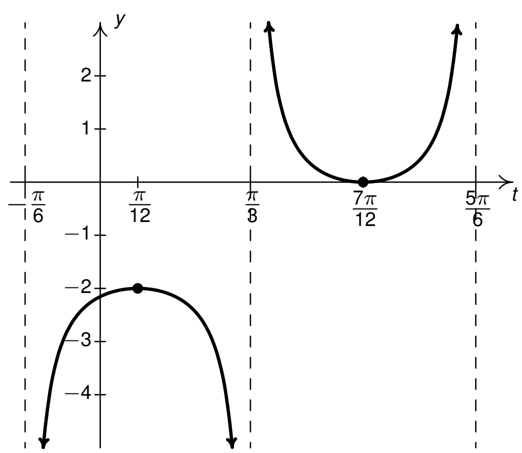
$$F(t) = A\sec(\omega t + \phi) + B \quad \text{and} \quad G(t) = A\csc(\omega t + \phi) + B$$

- have period $T = \dfrac{2\pi}{\omega}$
- have phase shift $-\dfrac{\phi}{\omega}$
- have 'baseline' B and have a vertical gap $|A|$ units between the the baseline and the graph.[a]

[a] In other words, the range of these functions is $(-\infty, B - |A|] \cup [B + |A|, \infty)$.

We put Theorem 11.11 to good use in the next example.

Example 11.5.2. Below is the graph of one cycle of a secant (cosecant) function, $y = f(t)$.



1. Write $f(t)$ in the form $F(t) = A\sec(\omega t + \phi) + B$ for $\omega > 0$.
2. Write $f(t)$ in the form $G(t) = A\csc(\omega t + \phi) + B$ for $\omega > 0$.

Solution.

1. We first note the period: $T = \frac{5\pi}{6} - \left(-\frac{\pi}{6}\right) = \pi$. Since $T = \frac{2\pi}{\omega} = \pi$, we get $\omega = 2$.

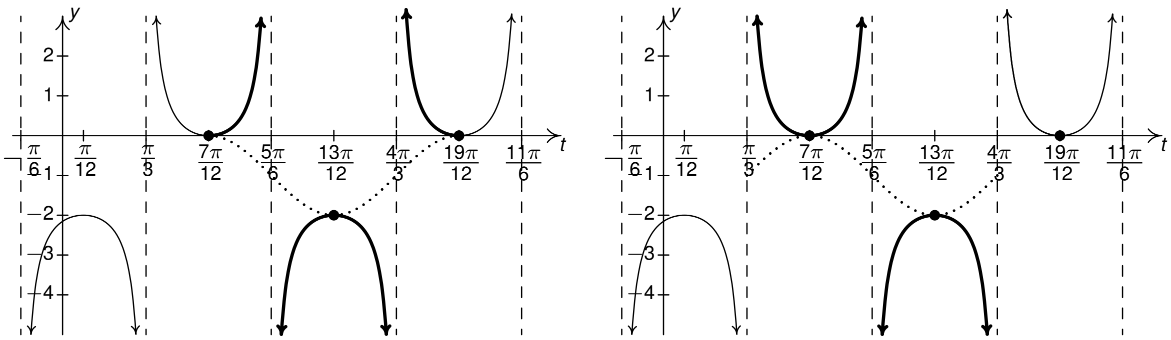
   To find the phase ϕ, we need to first determine the phase shift. Recall that what is graphed here is only one cycle of the function, so by copying and pasting one more cycle, we identify what looks like a fundamental cycle of the secant function to us[4] as highlighted below on the left.

   We get the phase shift is $\frac{7\pi}{12}$ so solving $-\frac{\phi}{2} = \frac{7\pi}{12}$, we get $\phi = -\frac{7\pi}{6}$.

[4] Assuming $A > 0$, that is.

To find the baseline, B, we take a cue from our work in Example 11.3.3 in Section 11.3. We find the average of the local minimums and maximums to be $\frac{-2+0}{2} = -1$, so $B = -1$. Since there is a 1 unit gap between the baseline and the graph of the function, we have $A = 1$. Alternatively, we can sketch the corresponding cosine curve (dotted in the figure below) and determine B and A that way.

We find our final answer to be $f(t) = \sec\left(2t - \frac{7\pi}{6}\right) - 1$. As usual, we check our answer by graphing.



Extending the graph one more cycle.

2. Since the secant and cosecant curves are phase shifts of each other, we could find a formula for $f(t)$ in terms of cosecants by shifting our formula $F(t) = \sec\left(2t - \frac{7\pi}{6}\right) - 1$. We leave this to the reader.[5]

   Working 'from scratch,' we would find $T = \pi$, $\omega = 2$, $B = -1$, and $A = 1$ the same as above.[6] To determine the phase shift, we refer to the figure above on the right.

   Since the phase shift is $\frac{\pi}{3}$, we solve $-\frac{\phi}{2} = \frac{\pi}{3}$ to get $\phi = -\frac{2\pi}{3}$. Putting all our work together, we get our final answer: $f(t) = \csc\left(2t - \frac{2\pi}{3}\right) - 1$. Again, our best check here is to graph. □

We cannot stress enough that our answers to Example 11.5.2 are one of many. For example, in Exercise 28, we ask you to rework this example choosing $A < 0$. It is well worth the time to think about what relationships exist between the different answers, however. For now, we move on to graphing the last pair of circular functions: tangent and cotangent curves.

11.5.2 Graphs of the Tangent and Cotangent Functions

Next, we turn our attention to the tangent and cotangent functions. Viewing $J(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$, we find the domain of J excludes all values where $\cos(t) = 0$. Hence, the domain of J is $\{t \,|\, t \neq \frac{\pi}{2} + \pi k, \text{for integers } k\}$. Using this information along with the common values we derived in Section 11.4, we create the table of values below on the left.

---

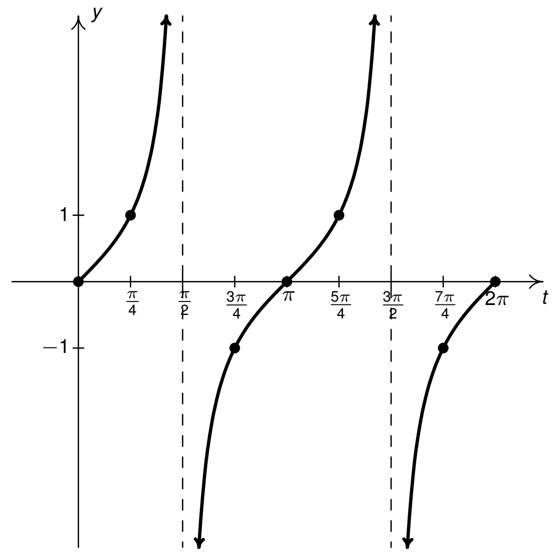
[5]See Exercise 27.

[6]Again, assuming we want $A > 0$.

Investigating the behavior near the values excluded from the domain, we find as $t \to \frac{\pi}{2}^-$, $\sin(t) \to 1^-$ and $\cos(t) \to 0^+$. Hence, $\displaystyle\lim_{t\to\frac{\pi}{2}^-} \tan(t) = \infty$ producing a vertical asymptote to the graph at $t = \frac{\pi}{2}$. Similarly, we get that as $\displaystyle\lim_{t\to\frac{\pi}{2}^+} \tan(t) = -\infty$, $\displaystyle\lim_{t\to\frac{3\pi}{2}^-} \tan(t) = \infty$, and $\displaystyle\lim_{t\to\frac{3\pi}{2}^+} \tan(t) = -\infty$.

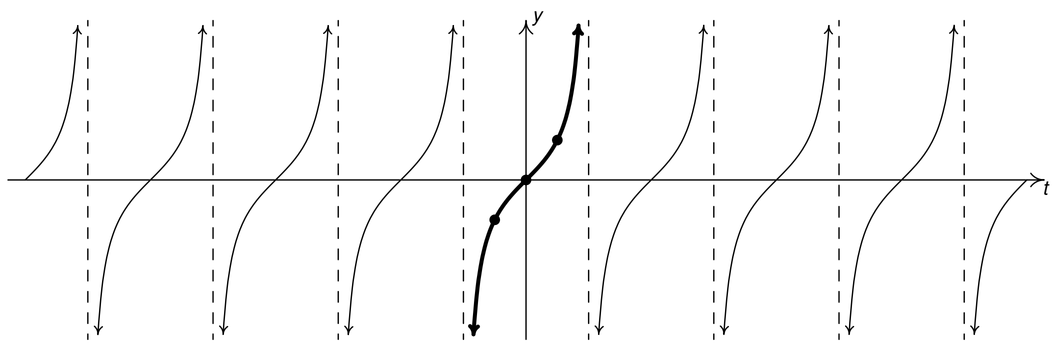
Putting all of this information together, we graph $y = \tan(t)$ over the interval $[0, 2\pi]$ below on the right.

t	$\tan(t)$	$(t, \tan(t))$
0	0	$(0,0)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	undefined	
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
π	0	$(\pi, 0)$
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	undefined	
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4}, -1\right)$
2π	0	$(2\pi, 0)$



The graph of $y = \tan(t)$ over $[0, 2\pi]$.

After the usual 'copy and paste' procedure, we create the graph of $y = \tan(t)$ below:



The graph of $y = \tan(t)$.

The graph of $y = \tan(t)$ suggests symmetry through the origin. Indeed, tangent is odd since sine is odd and cosine is even: $\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin(t)}{\cos(t)} = -\tan(t)$.

We also see the graph suggests the range of $J(t) = \tan(t)$ is all real numbers, $(-\infty, \infty)$. We present one proof of this fact in Exercise 31.

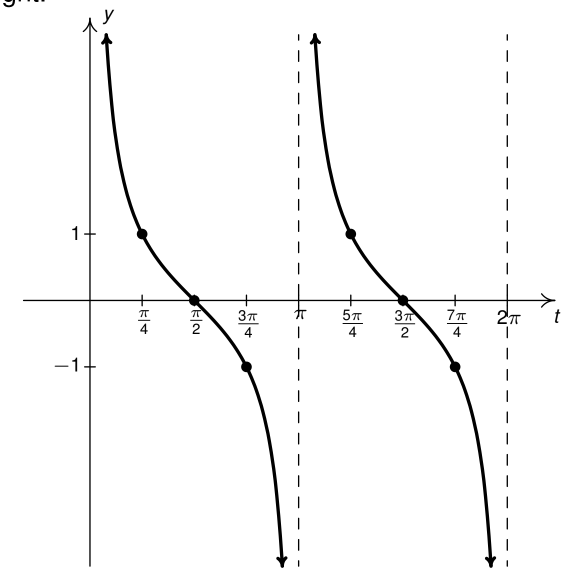
Moreover, as noted in Section 11.4, the period of the tangent function is π, and we see that reflected in the graph. This means we can choose *any* interval of length π to serve as our 'fundamental cycle.'

We choose the cycle traced out over the (open) interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as highlighted above. In addition to the asymptotes at the endpoints $t = \pm\frac{\pi}{2}$, we use the 'quarter marks' $t = \pm\frac{\pi}{4}$ and $t = 0$.

It should be no surprise that $K(t) = \cot(t)$ behaves similarly to $J(t) = \tan(t)$. Since $\cot(t) = \frac{\cos(t)}{\sin(t)}$, the domain of K excludes the values where $\sin(t) = 0$: $\{t \mid t \neq \pi k, \text{for integers } k\}$.

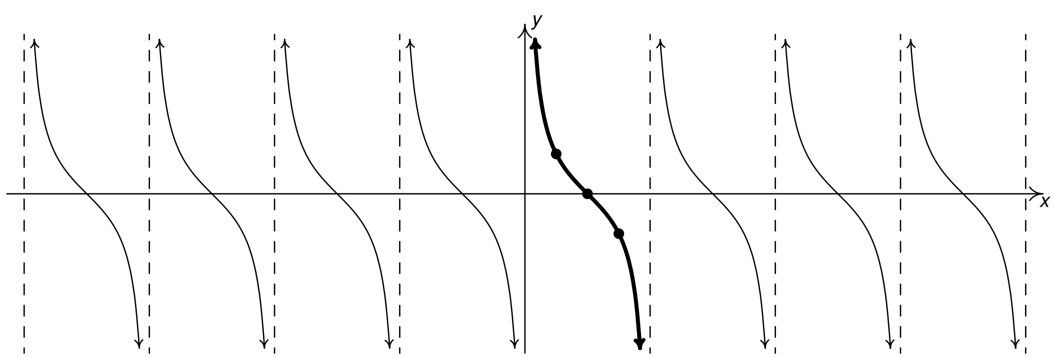
After analyzing the behavior of K near the values excluded from its domain along with plotting points, we graph $y = \cot(t)$ over the interval $[0, 2\pi]$ below on the right.

t	$\cot(t)$	$(t, \cot(t))$
0	undefined	
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
π	undefined	
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4}, -1\right)$
2π	undefined	



The graph of $y = \cot(t)$ over $[0, 2\pi]$.

As usual, pasting copies end to end produces the graph of $K(t) = \cot(t)$ below.



The graph of $y = \cot(x)$.

As with $J(t) = \tan(t)$, the graph of $K(t) = \cot(t)$ suggests K is odd, a fact we leave to the reader to prove in Exercise 32. Also, we see that the period of cotangent (like tangent) is π and the range is $(-\infty, \infty)$.

We take as one fundamental cycle the graph as traced out over the interval $(0, \pi)$, highlighted above, with quarter marks: $t = 0$, $t = \frac{\pi}{4}$, $t = \frac{\pi}{2}$, $t = \frac{3\pi}{4}$ and $t = \pi$.

The properties of the tangent and cotangent functions are summarized below. As with Theorem 11.10, each of the results below can be traced back to properties of the cosine and sine functions and the definition of the tangent and cotangent functions as quotients thereof.

Theorem 11.12. Properties of the Tangent and Cotangent Functions

- The function $J(t) = \tan(t)$
  - has domain $\left\{t \,|\, t \neq \frac{\pi}{2} + \pi k, \;\; k \text{ is an integer}\right\}$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π
- The function $K(t) = \cot(t)$
  - has domain $\{t \,|\, t \neq \pi k, \;\; k \text{ is an integer}\}$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π

Unlike the secant and cosecant functions, the tangent and cotangent functions have different periods than sine and cosine. Moreover, in the case of the tangent function, the fundamental cycle we've chosen starts at $-\frac{\pi}{2}$ instead of 0. Nevertheless, we can use the same notions of period and phase shift to graph transformed versions of tangent and cotangent functions, since these results ultimately trace back to applying Theorem 5.11. We state a version of Theorem 11.6 for tangent and cotangent functions below.

Theorem 11.13. For $\omega > 0$, the functions

$$J(t) = A\tan(\omega t + \phi) + B \quad \text{and} \quad K(t) = A\cot(\omega t + \phi) + B$$

- have period $T = \dfrac{\pi}{\omega}$
- have vertical shift or 'baseline' B
- The phase shift for $y = J(t)$ is $-\dfrac{\phi}{\omega} - \dfrac{\pi}{2\omega}$.
- The phase shift for $y = K(t)$ is $-\dfrac{\phi}{\omega}$.

The proof of the proof of Theorem 11.13 is left to the reader in Exercise 30.

We put Theorem 11.13 to good use in the following example.

Example 11.5.3. Graph one cycle of the following functions. Find the period.

1. $f(t) = 1 - \tan\left(\frac{t}{2} - \pi\right)$.
2. $g(t) = 2\cot(2\pi - \pi t) - 1$.

Solution.

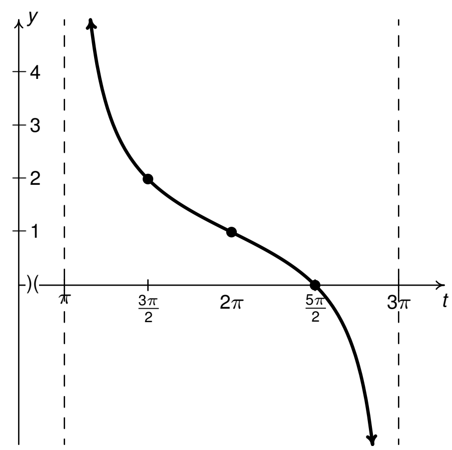
1. Rewriting $f(t)$ so it fits the form in Theorem 11.13, we get $f(t) = -\tan\left(\frac{1}{2}t + (-\pi)\right) + 1$.

   With $\omega = \frac{1}{2}$, we find the period $T = \frac{\pi}{1/2} = 2\pi$. Since $\phi = -\pi$, the phase shift is $-\frac{(-\pi)}{1/2} - \frac{\pi}{2(1/2)} = \pi$.

   Hence, one cycle of $f(t)$ starts at $t = \pi$ and finishes at $t = \pi + 2\pi = 3\pi$. Our quarter marks are $\frac{2\pi}{4} = \frac{\pi}{2}$ units apart and are $t = \pi$, $t = \frac{3\pi}{2}$, $t = 2\pi$, $t = \frac{5\pi}{2}$, and, finally, $t = 3\pi$.

   Substituting these t-values into $f(t)$, we find points on the graph and the vertical asymptotes.[7]

t	$f(t)$	$(t, f(t))$
π	undefined	
$\frac{3\pi}{2}$	2	$\left(\frac{3\pi}{2}, 2\right)$
2π	1	$(2\pi, 1)$
$\frac{5\pi}{2}$	0	$\left(\frac{5\pi}{2}, 0\right)$
3π	undefined	



One cycle of $y = 1 - \tan\left(\frac{t}{2} - \pi\right)$.

   We confirm that the period is $3\pi - \pi = 2\pi$.

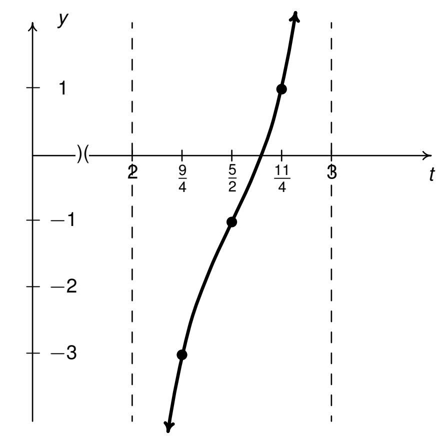
2. To put $g(t)$ into the form prescribed by Theorem 11.13, we make use of the odd property of cotangent: $g(t) = 2\cot(2\pi - \pi t) - 1 = 2\cot(-[\pi t - 2\pi]) - 1 = -2\cot(\pi t - 2\pi) - 1 = -2\cot(\pi t + (-2\pi)) - 1$.

   We identify $\omega = \pi$ so the period is $T = \frac{\pi}{\pi} = 1$. Since $\phi = -2\pi$, the phase shift is $-\frac{-2\pi}{\pi} = 2$. Hence, one cycle of $g(t)$ starts at $t = 2$ and ends at $t = 2 + 1 = 3$.

   Our quarter marks are $\frac{1}{4}$ units apart and are $t = 2$, $t = \frac{9}{4}$, $t = \frac{5}{2}$, $t = \frac{11}{4}$, and $t = 3$. We generate the graph below.

[7] Here, as with all tangent functions, we can partially check our new quarter marks by noting the argument of the tangent function simplifies, in each case, to one of the original quarter marks of the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

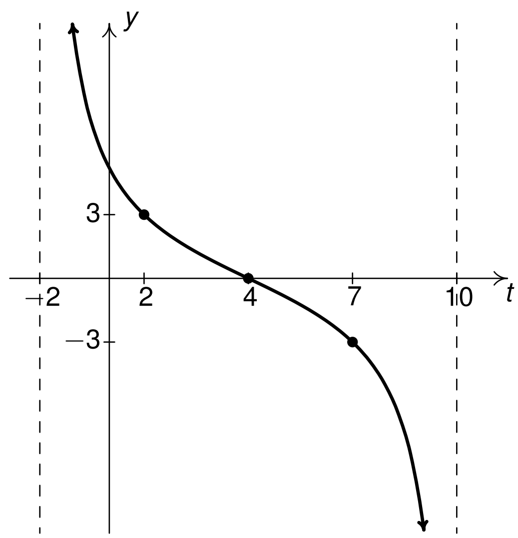
t	$g(t)$	$(t, g(t))$
2	undefined	
$\frac{9}{4}$	-3	$\left(\frac{9}{4}, -3\right)$
$\frac{5}{2}$	-1	$\left(\frac{5}{2}, -1\right)$
$\frac{11}{4}$	1	$\left(\frac{11}{4}, 1\right)$
3	undefined	



One cycle of $y = 2\cot(2\pi - \pi t) - 1$.

We confirm the period is $3 - 2 = 1$. □

Example 11.5.4. Below is the graph of one cycle of a tangent (cotangent) function, $y = f(t)$.



1. Write $f(t)$ in the form $J(t) = A\tan(\omega t + \phi) + B$ for $\omega > 0$.
2. Write $f(t)$ in the form $K(t) = A\cot(\omega t + \phi) + B$ for $\omega > 0$.

Solution.

1. We first find the period $T = 10 - (-2) = 12$. Per Theorem 11.13, we know $\frac{\pi}{\omega} = 12$, or $\omega = \frac{\pi}{12}$.

   Next, we look for the phase shift. We notice the cycle graphed for us is decreasing instead of the usual increasing we expect for a standard tangent cycle. When this sort of thing happened in Examples 11.3.3 and 11.5.2, we pasted another cycle of the function and used that to help identify the phase shift in order to keep the value of $A > 0$. Here, no amount of 'copying and pasting' will produce an increasing cycle (do you see why?), so we know $A < 0$ and use -2 as the phase shift.

The formula given in Theorem 11.13 tells us $-\frac{\phi}{\omega} - \frac{\pi}{2\omega} = -2$ so substituting $\omega = \frac{\pi}{12}$ gives $\phi = -\frac{\pi}{3}$.

Next, we see the baseline here is still the t-axis, so $B = 0$. This means all that's left to find is A. We have already established that $A < 0$ to account for the reflection across the t-axis. Moreover, the y-values of the points off of the baseline are 3 units from the baseline, indicating a vertical stretch by a factor of 3. Hence, $A = -3$ and $f(t) = -3\tan\left(\frac{\pi}{12}t - \frac{\pi}{3}\right)$. As usual, the ultimate check is to graph.

2. We find $T = 12$, $\omega = \frac{\pi}{12}$, and $B = 0$ as above. Since the fundamental cycle of cotangent is decreasing, we know $A > 0$ and identify the phase shift as -2.

   Using Theorem 11.13, we know $-\frac{\phi}{\omega} = -2$ so substituting $\omega = \frac{\pi}{12}$, we get $\phi = \frac{\pi}{6}$.

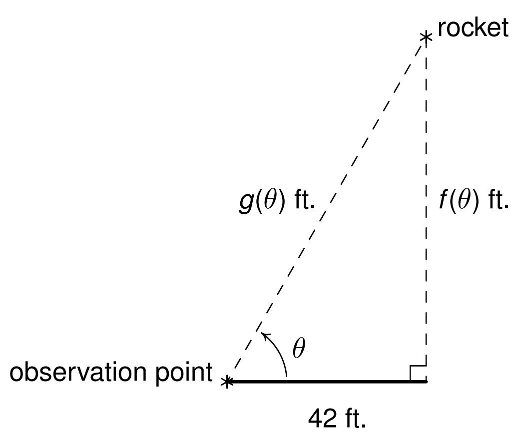
   As above, the vertical stretch is by a factor of 3, so we take $A = 3$ for our final answer: $f(t) = 3\cot\left(\frac{\pi}{12}t + \frac{\pi}{6}\right)$. As always, we check our answer by graphing. □

Once again, our answers to Example 11.5.4 are one of many, and we invite the reader to think about what all of the solutions would have in common. We close this section with an application.

Example 11.5.5. Let θ be the angle of inclination from an observation point on the ground 42 feet away from the launch site of a model rocket. Assuming the rocket is launched directly upwards:

1. Find a formula for $f(\theta)$, the distance from the rocket to the ground (in feet) as a function of θ. Find and interpret $f\left(\frac{\pi}{3}\right)$.
2. Find a formula for $g(\theta)$, the distance from the rocket to the observation point on the ground (in feet) as a function of θ. Find and interpret $g\left(\frac{\pi}{3}\right)$.
3. Find and interpret $\displaystyle\lim_{\theta \to \frac{\pi}{2}^-} f(\theta)$ and $\displaystyle\lim_{\theta \to \frac{\pi}{2}^-} g(\theta)$.

Solution.
We begin by sketching the scenario below. Since the rocket us launched 'directly upwards,' we may assume the rocket is launched at a 90° angle which provides us with a right triangle.



1. From the remarks preceding Theorem 11.9, we know the definitions of the circular functions agree with those specified for acute angles in right triangles as described in Definition B.2 in Section B.2. Hence, $\tan(\theta) = \frac{f(\theta)}{42}$, so $f(\theta) = 42\tan(\theta)$.

   We find $f\left(\frac{\pi}{3}\right) = 42\tan\left(\frac{\pi}{3}\right) = 30\sqrt{3}$. This means when the angle of inclination is $\frac{\pi}{3}$ or 60°, the rocket is or $30\sqrt{3} \approx 73$ feet off of the ground.

2. Again, working with the triangle, we find $\sec(\theta) = \frac{g(\theta)}{42}$ so that $g(\theta) = 42\sec(\theta)$. We find $g\left(\frac{\pi}{3}\right) = 42\sec\left(\frac{\pi}{3}\right) = 84$, so when the angle of inclination is 60°, the rocket is 84 feet from the observation point on the ground.

3. We find both $\displaystyle\lim_{\theta\to\frac{\pi}{2}^-} f(\theta) = \infty$ and $\displaystyle\lim_{\theta\to\frac{\pi}{2}^-} g(\theta) = \infty$, which we can verify graphically. This means as the angle of inclination approaches $\frac{\pi}{2}$ or 90°, the distances from the rocket to the ground and from to the rocket to the observation point increase without bound. Barring the effects of drift or the curvature of space, this matches our intuition. □

11.5.3 Extended Interval Notation

Using interval notation to describe the domains of the secant, cosecant, tangent, and cotangent functions is complicated by the fact there are infinitely many intervals to represent. In this section, we introduce **extended interval notation** to handle these situations.

Let us return to the domain of $F(t) = \sec(t)$, $\left\{t \,|\, t \neq \frac{\pi}{2} + \pi k, \text{for integers } k\right\}$. Using interval notation, we describe this set as: $\ldots \cup \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \ldots$

In order to write this set in a more compact way, we let t_k denote the the kth real number excluded from the domain. That is, $t_k = \frac{\pi}{2} + \pi k$. (This is sequence notation from Chapter 10.)

Getting a common denominator and factoring out the π in the numerator, we get $t_k = \frac{(2k+1)\pi}{2}$. The set we're after consists of the union of intervals determined by the successive points t_k: $(t_k, t_{k+1}) = \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$ where k ranges through the integers. We denote this union as:

$$\bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right).$$

The reader should compare this notation with summation notation introduced in Section 10.2, in particular the notation used to describe geometric series in Theorem 10.5. In the same way the index k in the series

$$\sum_{k=1}^{\infty} ar^{k-1}$$

never equals ∞, but rather, ranges through all of the natural numbers, the index k in the union

$$\bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$$

never equals ∞ or $-\infty$, but rather, this conveys the idea that k ranges through all of the integers.

Using extended interval notation, we summarize the domains and ranges of all six circular functions below.

Theorem 11.14. Domains and Ranges of the Circular Functions

- The function $f(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $g(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $F(t) = \sec(t)$
  - has domain $\displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2} \right)$
  - has range $(-\infty, -1] \cup [1, \infty)$
- The function $G(t) = \csc(t)$
  - has domain $\displaystyle\bigcup_{k=-\infty}^{\infty} (k\pi, (k+1)\pi)$
  - has range $(-\infty, -1] \cup [1, \infty)$
- The function $J(t) = \tan(t)$
  - has domain $\displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2} \right)$
  - has range $(-\infty, \infty)$
- The function $K(t) = \cot(t)$
  - has domain $\displaystyle\bigcup_{k=-\infty}^{\infty} (k\pi, (k+1)\pi)$
  - has range $(-\infty, \infty)$

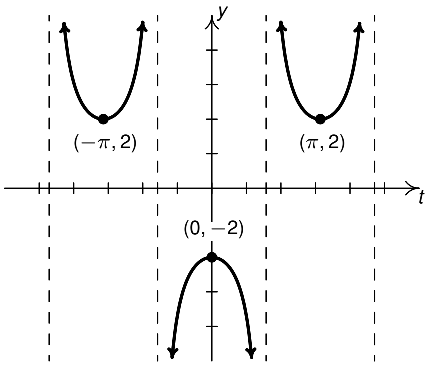
11.5.4 Exercises

In Exercises 1 - 12, graph one cycle of the given function. State the period of the function.

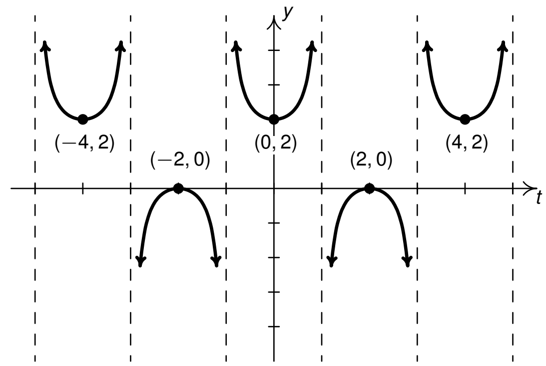
1. $y = \tan\left(t - \dfrac{\pi}{3}\right)$
2. $y = 2\tan\left(\dfrac{1}{4}t\right) - 3$
3. $y = \dfrac{1}{3}\tan(-2t - \pi) + 1$
4. $y = \sec\left(t - \dfrac{\pi}{2}\right)$
5. $y = -\csc\left(t + \dfrac{\pi}{3}\right)$
6. $y = -\dfrac{1}{3}\sec\left(\dfrac{1}{2}t + \dfrac{\pi}{3}\right)$
7. $y = \csc(2t - \pi)$
8. $y = \sec(3t - 2\pi) + 4$
9. $y = \csc\left(-t - \dfrac{\pi}{4}\right) - 2$
10. $y = \cot\left(t + \dfrac{\pi}{6}\right)$
11. $y = -11\cot\left(\dfrac{1}{5}t\right)$
12. $y = \dfrac{1}{3}\cot\left(2t + \dfrac{3\pi}{2}\right) + 1$

In Exercises 13 - 14, the graph of a (co)secant function is given. Find a formula for the function in the form $F(t) = A\sec(\omega t + \phi) + B$ and $G(t) = A\csc(\omega t + \phi) + B$. Select ω so $\omega > 0$. Check your answer by graphing.

13. Asymptotes: $t = \pm\frac{\pi}{2}$, $t = \pm\frac{3\pi}{2}$, $\ldots$

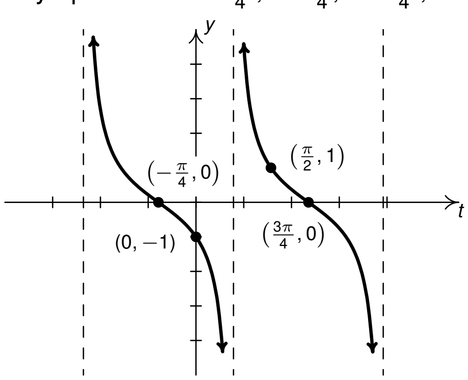


14. Asymptotes: $t = \pm 1$, $t = \pm 3$, $t = \pm 5$, $\ldots$

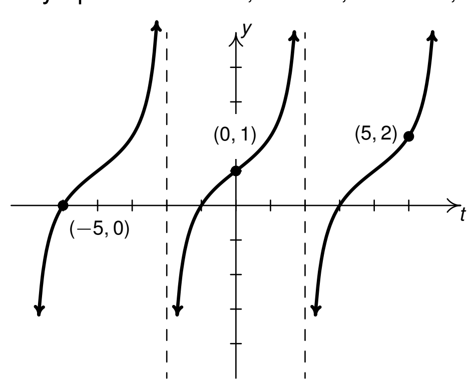


In Exercises 15 - 16, the graph of a (co)tangent function given. Find a formula the function in the form $J(t) = A\tan(\omega t + \phi) + B$ and $K(t) = A\cot(\omega t + \phi) + B$. Select ω so $\omega > 0$. Check your answer by graphing.

15. Asymptotes: $t = -\frac{3\pi}{4}$, $t = \frac{\pi}{4}$, $t = \frac{5\pi}{4}$, $\ldots$



16. Asymptotes: $t = \pm 2$, $t = \pm 6$, $t = \pm 10$, $\ldots$



In Section 11.3.2, we observed[8] that the cosine and sine functions are continuous. As such, the other four circular functions are continuous on their domains.[9]

In Exercises 17 - 25, determine the given limit. Use the symbols '$-\infty$' and '∞'as appropriate. Check your answers graphically.

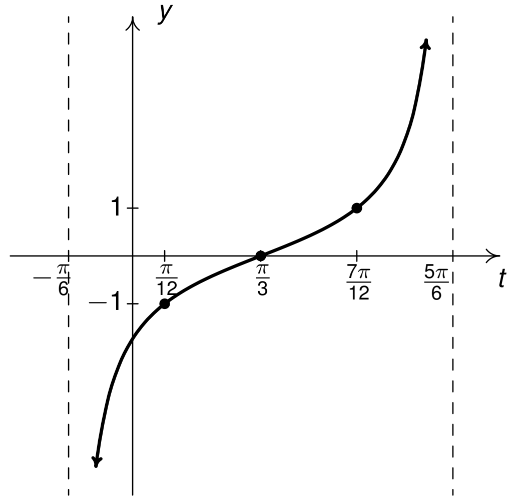
17. $\lim\limits_{t\to 0} \tan(t)$.
18. $\lim\limits_{t\to \pi} \sec(t)$
19. $\lim\limits_{t\to 3\pi} \cot\left(\frac{t}{2}\right)$
20. $\lim\limits_{\theta\to 0} \csc\left(2\theta + \frac{\pi}{4}\right)$.
21. $\lim\limits_{\theta\to \pi} (\cos(\theta) - \sec(\theta))$
22. $\lim\limits_{\theta\to \frac{\pi}{4}} \sec(\theta)\tan(\theta)$
23. $\lim\limits_{x\to 0^+} \csc(3x)$
24. $\lim\limits_{x\to \pi^-} (\sin(2x) - \tan(x))$
25. $\lim\limits_{x\to \pi^+} (\cos(x) + \cot(x))$
26. In Exercise 82 in Section 11.4, we proved $\lim\limits_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.
    (a) Use Theorem 6.2 from Section 6.1 to show $\lim\limits_{\theta\to 0} \frac{\theta}{\sin(\theta)} = 1$.
    (b) Which indeterminate form is present in the limit $\lim\limits_{\theta\to 0^+} 2\theta\csc(\theta)$?
    (c) Graph $f(\theta) = 2\theta\csc(\theta)$ near $\theta = 0$. What appears to be the limit?
    (d) Rewrite $2\theta\csc(\theta)$ in terms of θ and $\sin(\theta)$ and use part 26a to analytically find $\lim\limits_{\theta\to 0^+} 2\theta\csc(\theta)$.
    (e) Use the same methodology as above to help you analytically determine $\lim\limits_{\theta\to 0^+} 2\theta\cot(\theta)$
27. (a) Use the conversion formulas listed in Theorem 11.5 to create conversion formulas between secant and cosecant functions.
    (b) Use a conversion formula to rewrite our first answer to Example 11.5.2, $f(t) = \sec\left(2t - \frac{7\pi}{6}\right) - 1$, in terms of cosecants.
28. Rework Example 11.5.2 and find answers with $A < 0$.
29. Prove Theorem 11.11 using Theorem 11.6.
30. Prove Theorem 11.13 using Theorem 5.11.
31. In this Exercise, we argue the range of the tangent function is $(-\infty, \infty)$. Let M be a fixed, but arbitrary positive real number.
    (a) Show there is an acute angle θ with $\tan(\theta) = M$. (Hint: think right triangles.)
    (b) Using the symmetry of the Unit Circle, explain why there are angles θ with $\tan(\theta) = -M$.
    (c) Find angles with $\tan(\theta) = 0$.
    (d) Combine the three parts above to conclude the range of the tangent function is $(-\infty, \infty)$.
32. Prove $\cot(t)$ is odd. (Hint: mimic the proof given in the text that $\tan(t)$ is odd.)

[8] insisted?

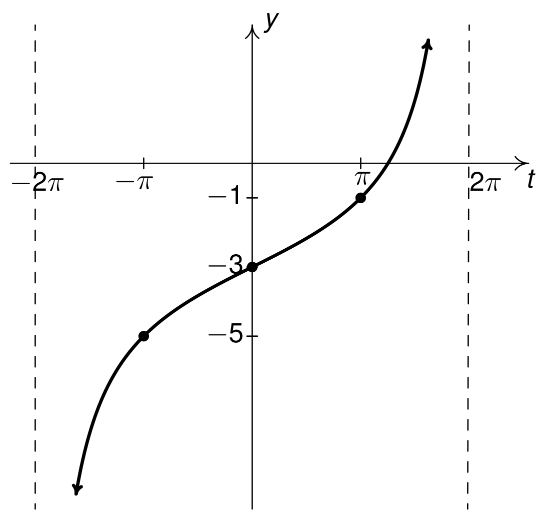
[9] See the remarks following Definition 6.4 in Section 6.1.2.

11.5.5 Answers

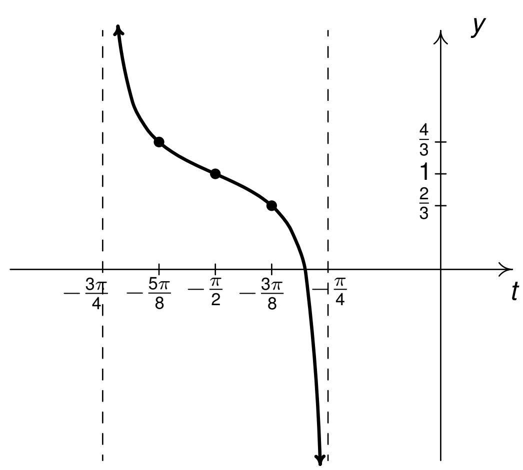
1. $y = \tan\left(t - \dfrac{\pi}{3}\right)$
   Period: π



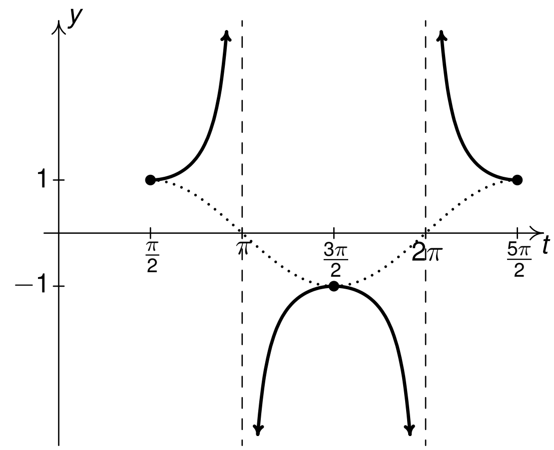
2. $y = 2\tan\left(\dfrac{1}{4}t\right) - 3$
   Period: 4π



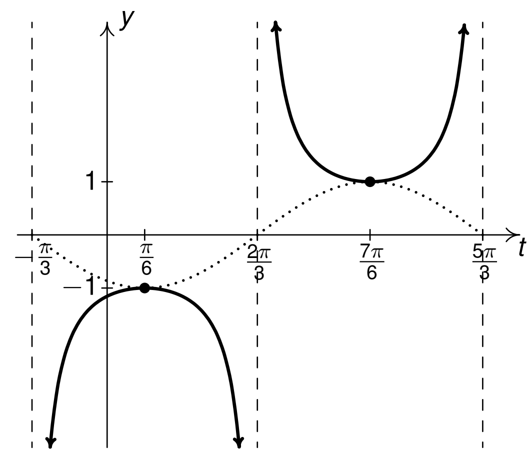
3. $y = \dfrac{1}{3}\tan(-2t - \pi) + 1$
   is equivalent to
   $y = -\dfrac{1}{3}\tan(2t + \pi) + 1$
   via the Even / Odd identity for tangent.
   Period: $\dfrac{\pi}{2}$



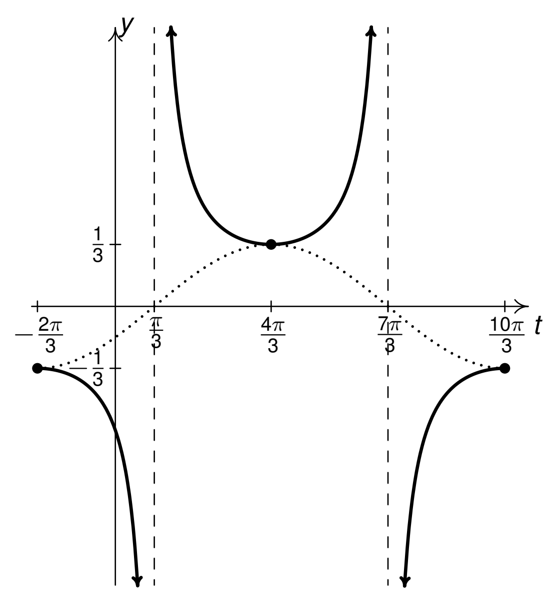
4. $y = \sec\left(t - \frac{\pi}{2}\right)$
   Start with $y = \cos\left(t - \frac{\pi}{2}\right)$
   Period: 2π



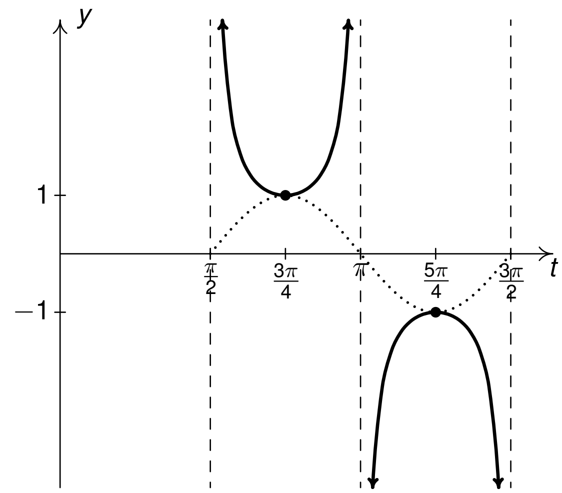
5. $y = -\csc\left(t + \dfrac{\pi}{3}\right)$
   Start with $y = -\sin\left(t + \dfrac{\pi}{3}\right)$
   Period: 2π



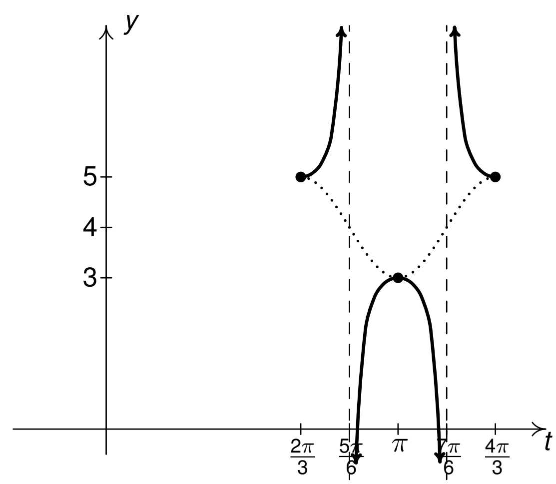
6. $y = -\dfrac{1}{3}\sec\left(\dfrac{1}{2}t + \dfrac{\pi}{3}\right)$
   Start with $y = -\dfrac{1}{3}\cos\left(\dfrac{1}{2}t + \dfrac{\pi}{3}\right)$
   Period: 4π



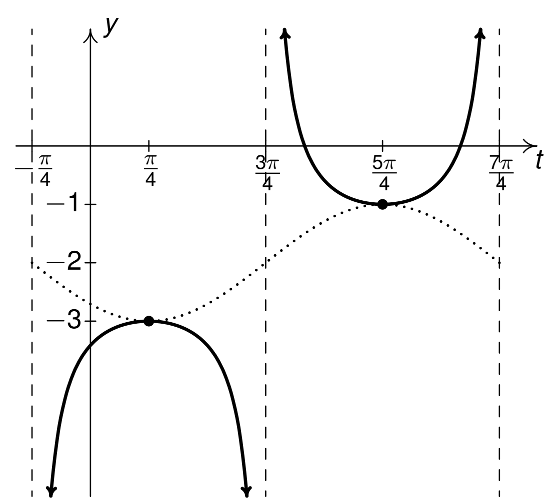
7. $y = \csc(2t - \pi)$
   Start with $y = \sin(2t - \pi)$
   Period: π



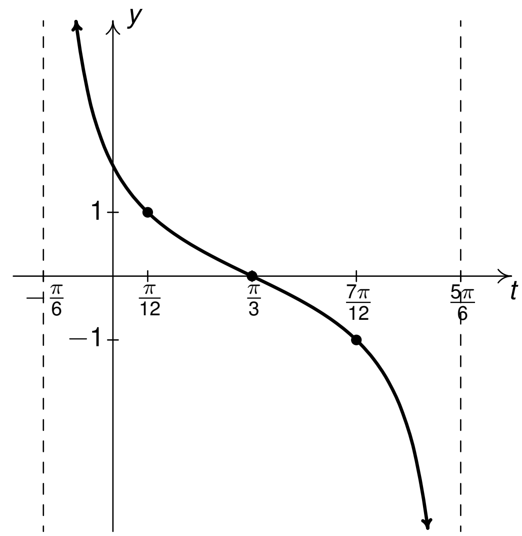
8. $y = \sec(3t - 2\pi) + 4$
   Start with $y = \cos(3t - 2\pi) + 4$
   Period: $\dfrac{2\pi}{3}$



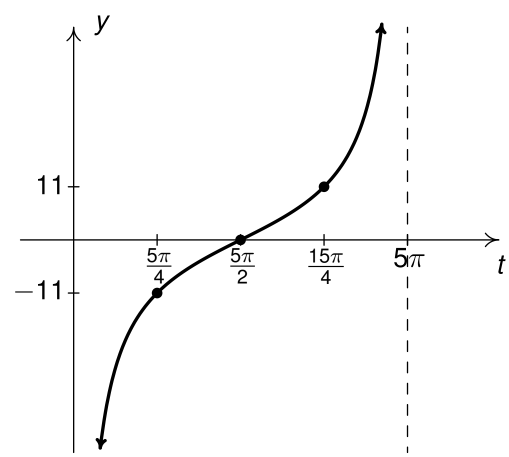
9. $y = \csc\left(-t - \dfrac{\pi}{4}\right) - 2$
   Start with $y = \sin\left(-t - \dfrac{\pi}{4}\right) - 2$
   Period: 2π



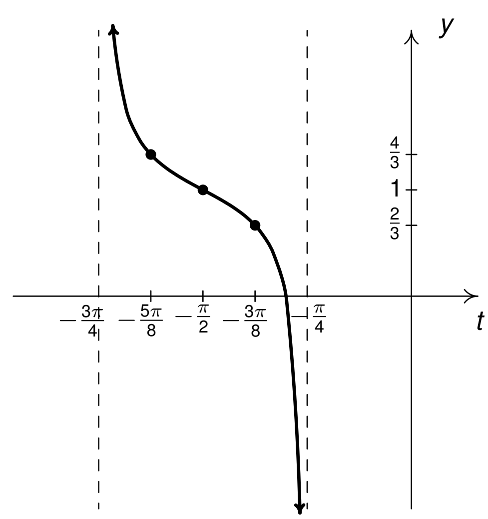
10. $y = \cot\left(t + \frac{\pi}{6}\right)$
Period: π



11. $y = -11\cot\left(\frac{1}{5}t\right)$
Period: 5π



12. $y = \frac{1}{3}\cot\left(2t + \frac{3\pi}{2}\right) + 1$
Period: $\frac{\pi}{2}$



13. $F(t) = 2\sec(t-\pi)$, $G(t) = 2\csc\left(t-\frac{\pi}{2}\right)$

14. $F(t) = \sec\left(\frac{\pi}{2}t\right)+1$, $G(t) = \csc\left(\frac{\pi}{2}t+\frac{\pi}{2}\right)+1$

15. $J(t) = -\tan\left(t+\frac{\pi}{4}\right)$, $K(t) = \cot\left(t-\frac{\pi}{4}\right)$

16. $J(t) = \tan\left(\frac{\pi}{4}t\right)+1$, $K(t) = -\cot\left(\frac{\pi}{4}t+\frac{\pi}{2}\right)+1$

17. $\lim\limits_{t\to 0}\tan(t) = \tan(0) = 0.$

18. $\lim\limits_{t\to \pi}\sec(t) = \sec(\pi) = -1$

19. $\lim\limits_{t\to 3\pi}\cot\left(\frac{t}{2}\right) = \cot\left(\frac{3\pi}{2}\right) = 0$

20. $\lim\limits_{\theta\to 0}\csc\left(2\theta+\frac{\pi}{4}\right) = \csc\left(\frac{\pi}{4}\right) = \sqrt{2}.$

21. $\lim\limits_{\theta\to \pi}(\cos(\theta)-\sec(\theta)) = \cos(\pi)-\sec(\pi) = -1-(-1) = 0$

22. $\lim\limits_{\theta\to \frac{\pi}{4}}(\sec(\theta)\,\tan(\theta)) = \sec\left(\dfrac{\pi}{4}\right)\tan\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

23. $\lim\limits_{x\to 0^+}\csc(3x) = \infty$

24. $\lim\limits_{x\to \pi^-}(\sin(2x)-\tan(x)) = \sin(2\pi)-\tan(\pi) = 0$

25. $\lim\limits_{x\to \pi^+}(\cos(2x)+\cot(x)) = \infty$

26. (a) $\lim\limits_{\theta\to 0}\frac{\theta}{\sin(\theta)} = \lim\limits_{\theta\to 0}\left[\frac{\sin(\theta)}{\theta}\right]^{-1} = 1^{-1} = 1$

  (b) As $\theta\to 0^+$, $2\theta\,\csc(\theta)\to 0\cdot\infty$.

  (c) The graph of $f(\theta) = 2\theta\,\csc(\theta)$ approaches $(0,2)$ so $\lim\limits_{\theta\to 0^+}2\theta\,\csc(\theta)$ appears to be 2.

  (d) $\lim\limits_{\theta\to 0^+}2\theta\,\csc(\theta) = \lim\limits_{\theta\to 0^+}2\,\theta\frac{1}{\sin(\theta)} = \lim\limits_{\theta\to 0^+}2\,\frac{\theta}{\sin(\theta)} = 2(1) = 2.$

  (e) $\lim\limits_{\theta\to 0^+}2\theta\,\cot(\theta) = \lim\limits_{\theta\to 0^+}2\,\theta\,\frac{\cos(\theta)}{\sin(\theta)} = \lim\limits_{\theta\to 0^+}2\,\cos(\theta)\,\frac{\theta}{\sin(\theta)} = 2\cos(0)(1) = 2(1)(1) = 2$

27. (a) $\csc\left(t+\frac{\pi}{2}\right) = \sec(t)$ and $\sec\left(t-\frac{\pi}{2}\right) = \csc(t)$.

  (b) $f(t) = \sec\left(2t-\frac{7\pi}{6}\right)-1 = \csc\left(\left[2t-\frac{7\pi}{6}\right]+\frac{\pi}{2}\right)-1 = \csc\left(2t-\frac{2\pi}{3}\right)-1$, in terms of cosecants.

28. $f(t) = -\sec\left(2t-\frac{\pi}{6}\right)-1$ and $f(t) = -\csc\left(2t+\frac{\pi}{3}\right)-1$ are two answers

Chapter 12

Analytical Trigonometry

12.1 The Pythagorean Identities

In section Section 11.4, we first encountered the concept of an **identity** when discussing Theorem 11.7. Recall that an identity is an equation which is true regardless of the choice of variable. Identities are important in mathematics because they facilitate changing forms.[1]

We take a moment to generalize Theorem 11.7 below.

Theorem 12.1. Reciprocal and Quotient Identities: The following relationships hold for all angles θ provided each side of each equation is defined.

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$
- $\cos(\theta) = \dfrac{1}{\sec(\theta)}$
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$
- $\sin(\theta) = \dfrac{1}{\csc(\theta)}$
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$
- $\cot(\theta) = \dfrac{1}{\tan(\theta)}$
- $\tan(\theta) = \dfrac{1}{\cot(\theta)}$

It is important to remember that the equivalences stated in Theorem 12.1 are valid only when *all* quantities described therein are defined. As an example, $\tan(0) = 0$, but $\tan(0) \neq \frac{1}{\cot(0)}$ since $\cot(0)$ is undefined.

When it comes down to it, the Reciprocal and Quotient Identities amount to giving different ratios on the Unit Circle different names. The main focus of this section is on a more algebraic relationship between certain pairs of the circular functions: the **Pythagorean Identities**.

Recall in Definition 11.2, the cosine and sine of an angle is defined as the x and y-coordinate, respectively, of a point on the Unit Circle. Since the coordinates of all points (x, y) on the Unit Circle satisfy the equation $x^2 + y^2 = 1$, we get for all angles θ, $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. An unfortunate[2] convention, which the authors are compelled to perpetuate, is to write $(\cos(\theta))^2$ as $\cos^2(\theta)$ and $(\sin(\theta))^2$ as $\sin^2(\theta)$. Rewriting the identity using this convention results in the following theorem, which is without a doubt one of the most important results in Trigonometry.

[1]We've seen the utility of changing form throughout the text, most recently when we completed the square in Chapter 8 to put general quadratic equations into standard form in order to graph them.

[2]This is unfortunate from a 'function notation' perspective. See Section 12.3.

Theorem 12.2. The Pythagorean Identity: For any angle θ, $\cos^2(\theta) + \sin^2(\theta) = 1$.

The moniker 'Pythagorean' brings to mind the Pythagorean Theorem, from which both the Distance Formula and the equation for a circle are ultimately derived.[3] The word 'Identity' reminds us that, regardless of the angle θ, the equation in Theorem 12.2 is always true.

If one of $\cos(\theta)$ or $\sin(\theta)$ is known, Theorem 12.2 can be used to determine the other, up to a $(\pm)$ sign. If, in addition, we know where the terminal side of θ lies when in standard position, then we can remove the ambiguity of the $(\pm)$ and completely determine the missing value.[4] We illustrate this approach in the following example.

Example 12.1.1. Use Theorem 12.2 and the given information to find the indicated value.

1. If θ is a Quadrant II angle with $\sin(\theta) = \frac{3}{5}$, find $\cos(\theta)$.
2. If $\pi < t < \frac{3\pi}{2}$ with $\cos(t) = -\frac{\sqrt{5}}{5}$, find $\sin(t)$.
3. If $\sin(\theta) = 1$, find $\cos(\theta)$.

Solution.

1. When we substitute $\sin(\theta) = \frac{3}{5}$ into The Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$, we obtain $\cos^2(\theta) + \frac{9}{25} = 1$. Solving, we find $\cos(\theta) = \pm\frac{4}{5}$. Since θ is a Quadrant II angle, we know $\cos(\theta) < 0$. Hence, we select $\cos(\theta) = -\frac{4}{5}$.
2. Here we're using the variable t instead θ which usually corresponds to a real number variable instead of an angle. As usual, we associate real numbers t with angles θ measuring t radians,[5] so the the Pythagorean Identity works equally well for all real numbers t as it does for all angles θ.

   Substituting $\cos(t) = -\frac{\sqrt{5}}{5}$ into $\cos^2(t) + \sin^2(t) = 1$ gives $\sin(t) = \pm\frac{2}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5}$. Since $\pi < t < \frac{3\pi}{2}$, we know t corresponds to a Quadrant III angle, so $\sin(t) < 0$. Hence, $\sin(t) = -\frac{2\sqrt{5}}{5}$.
3. When we substitute $\sin(\theta) = 1$ into $\cos^2(\theta) + \sin^2(\theta) = 1$, we find $\cos(\theta) = 0$. $\square$

The reader is encouraged to compare and contrast the solution strategies demonstrated in Example 12.1.1 with those showcases in Examples 11.2.3 and 11.2.5 in Section 11.2.

As with many tools in mathematics, identities give us a different way to approach and solve problems.[6] As always, the key is to determine which approach makes the most sense (is more efficient, for instance) in the given scenario.

Our next task is to use use the Reciprocal and Quotient Identities found in Theorem 12.1 coupled with the Pythagorean Identity found in Theorem 12.2 to derive new Pythagorean-like identities for the remaining four circular functions.

[3] See Sections A.3 and 8.3 for details.

[4] See the illustration following Example 11.4.1 to refresh yourself which circular functions are positive in which quadrants.

[5] See page 909 if you need a review of how we associate real numbers with angles in radian measure.

[6] For example, factoring, completing the square, and the quadratic formula are three different (yet equivalent) ways to solve a quadratic equation. See Section A.10 for a refresher.

Assuming $\cos(\theta) \neq 0$, we may start with $\cos^2(\theta) + \sin^2(\theta) = 1$ and divide both sides by $\cos^2(\theta)$ to obtain $1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$. Using properties of exponents along with the Reciprocal and Quotient Identities, this reduces to $1 + \tan^2(\theta) = \sec^2(\theta)$.

If $\sin(\theta) \neq 0$, we can divide both sides of the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ by $\sin^2(\theta)$, apply Theorem 12.1 once again, and obtain $\cot^2(\theta) + 1 = \csc^2(\theta)$.

These three Pythagorean Identities are worth memorizing and they, along with some of their other common forms, are summarized in the following theorem.

Theorem 12.3. The Pythagorean Identities:

1. $\cos^2(\theta) + \sin^2(\theta) = 1$.

   Common Alternate Forms:

   - $1 - \sin^2(\theta) = \cos^2(\theta)$
   - $1 - \cos^2(\theta) = \sin^2(\theta)$

2. $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.

   Common Alternate Forms:

   - $\sec^2(\theta) - \tan^2(\theta) = 1$
   - $\sec^2(\theta) - 1 = \tan^2(\theta)$

3. $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.

   Common Alternate Forms:

   - $\csc^2(\theta) - \cot^2(\theta) = 1$
   - $\csc^2(\theta) - 1 = \cot^2(\theta)$

As usual, the formulas states in Theorem 12.3 work equally well for (the applicable) angles as well as real numbers.

Example 12.1.2. Use Theorems 12.1 and 12.3 to find the indicated values.

1. If θ is a Quadrant IV angle with $\sec(\theta) = 3$, find $\tan(\theta)$.
2. Find $\csc(t)$ if $\pi < t < \frac{3\pi}{2}$ and $\cot(t) = 2$.
3. If θ is a Quadrant II angle with $\cos(\theta) = -\frac{3}{5}$, find the exact values of the remaining circular functions.

Solution.

1. Per Theorem 12.3, $\tan^2(\theta) = \sec^2(\theta) - 1$. Since $\sec(\theta) = 3$, we have $\tan^2(\theta) = (3)^2 - 1 = 8$, or $\tan(\theta) = \pm\sqrt{8} = \pm 2\sqrt{2}$. Since θ is a Quadrant IV angle, we know $\tan(\theta) < 0$ so $\tan(\theta) = -2\sqrt{2}$.

2. Again, using Theorem 12.3, we have $\csc^2(t) = 1 + \cot^2(t)$, so we have $\csc^2(t) = 1 + (2)^2 = 5$. This gives $\csc(t) = \pm\sqrt{5}$. Since $\pi < t < \frac{3\pi}{2}$, t corresponds to a Quadrant III angle, so $\csc(t) = -\sqrt{5}$.

3. With five function values to find, we have our work cut out for us. From Theorem 12.1, we know $\sec(\theta) = \frac{1}{\cos(\theta)}$, so we (quickly) get $\sec(\theta) = \frac{1}{-3/5} = -\frac{5}{3}$.

   Next, we go after $\sin(\theta)$ since between $\sin(\theta)$ and $\cos(\theta)$, we can get all of the remaining values courtesy of Theorem 12.1.

   From Theorem 12.3, we have $\sin^2(\theta) = 1 - \cos^2(\theta)$, so $\sin^2(\theta) = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$. Hence, $\sin(\theta) = \pm\frac{4}{5}$ but since θ is a Quadrant II angle, we select $\sin(\theta) = \frac{4}{5}$.

   Back to Theorem 12.1, we get $\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{4/5} = \frac{5}{4}$, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{4/5}{-3/5} = -\frac{4}{3}$, and $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{-3/5}{4/5} = -\frac{3}{4}$. □

Again, the reader is encouraged to study the solution methodology illustrated in Example 12.1.2 as compared with that employed in Example 11.4.2 in Section 11.4.

Trigonometric identities play an important role in not just Trigonometry, but in Calculus as well. We'll use them in this book to find the values of the circular functions of an angle and solve equations and inequalities. In Calculus, they are needed to simplify otherwise complicated expressions. In the next example, we make good use of the Theorems 12.1 and 12.3.

Example 12.1.3. Verify the following identities. Assume that all quantities are defined.

1. $\tan(\theta) = \sin(\theta)\sec(\theta)$
2. $(\tan(t) - \sec(t))(\tan(t) + \sec(t)) = -1$
3. $\sin^2(x)\cos^3(x) = \sin^2(x)\left(1 - \sin^2(x)\right)\cos(x)$
4. $\dfrac{\sec(t)}{1 - \tan(t)} = \dfrac{1}{\cos(t) - \sin(t)}$
5. $6\sec(x)\tan(x) = \dfrac{3}{1 - \sin(x)} - \dfrac{3}{1 + \sin(x)}$
6. $\dfrac{\sin(\theta)}{1 - \cos(\theta)} = \dfrac{1 + \cos(\theta)}{\sin(\theta)}$

Solution. In verifying identities, we typically start with the more complicated side of the equation and use known identities to *transform* it into the other side of the equation.

1. Starting with the right hand side of $\tan(\theta) = \sin(\theta)\sec(\theta)$, we use $\sec(\theta) = \frac{1}{\cos(\theta)}$ and find:

$$\sin(\theta)\sec(\theta) = \sin(\theta)\frac{1}{\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta),$$

   where the last equality is courtesy of Theorem 12.1.

2. Expanding the left hand side, we get: $(\tan(t) - \sec(t))(\tan(t) + \sec(t)) = \tan^2(t) - \sec^2(t)$. From Theorem 12.3, we know $\sec^2(t) - \tan^2(t) = 1$, which isn't *quite* what we have. We are off by a negative sign $(-)$, so we factor it out:

$$(\tan(t) - \sec(t))(\tan(t) + \sec(t)) = \tan^2(t) - \sec^2(t) = (-1)(\sec^2(t) - \tan^2(t)) = (-1)(1) = -1.$$

3. Starting with the right hand side,[7] we notice we have a quantity we can immediately simplify per Theorem 12.3: $1 - \sin^2(x) = \cos^2(x)$. This increases the number of factors of cosine, (which is part of our goal in looking at the left hand side), so we proceed:

$$\sin^2(x)\left(1 - \sin^2(x)\right)\cos(x) = \sin^2(x)\cos^2(x)\cos(x) = \sin^2(x)\cos^3(x).$$

4. While both sides of our next identity contain fractions, the left side affords us more opportunities to use our identities.[8] Substituting $\sec(t) = \frac{1}{\cos(t)}$ and $\tan(t) = \frac{\sin(t)}{\cos(t)}$, we get:

$$\begin{aligned}
\frac{\sec(t)}{1-\tan(t)} &= \frac{\dfrac{1}{\cos(t)}}{1-\dfrac{\sin(t)}{\cos(t)}} = \frac{\dfrac{1}{\cos(t)}}{1-\dfrac{\sin(t)}{\cos(t)}} \cdot \frac{\cos(t)}{\cos(t)} \\
&= \frac{\left(\dfrac{1}{\cos(t)}\right)(\cos(t))}{\left(1-\dfrac{\sin(t)}{\cos(t)}\right)(\cos(t))} = \frac{1}{(1)(\cos(t)) - \left(\dfrac{\sin(t)}{\cos(t)}\right)(\cos(t))} \\
&= \frac{1}{\cos(t)-\sin(t)},
\end{aligned}$$

which is exactly what we had set out to show.

5. Starting with the right hand side, we can get started by obtaining common denominators to add:

$$\begin{aligned}
\frac{3}{1-\sin(x)} - \frac{3}{1+\sin(x)} &= \frac{3(1+\sin(x))}{(1-\sin(x))(1+\sin(x))} - \frac{3(1-\sin(x))}{(1+\sin(x))(1-\sin(x))} \\
&= \frac{3+3\sin(x)}{1-\sin^2(x)} - \frac{3-3\sin(x)}{1-\sin^2(x)} \\
&= \frac{(3+3\sin(x))-(3-3\sin(x))}{1-\sin^2(x)} \\
&= \frac{6\sin(x)}{1-\sin^2(x)}
\end{aligned}$$

At this point, we have at least reduced the number of fractions from two to one, it may not be clear how to proceed. When this happens, it isn't a bad idea to start working with the other side of the identity to get some clues how to proceed.

Using a reciprocal and quotient identity, we find $6\sec(x)\tan(x) = 6\left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{6\sin(x)}{\cos^2(x)}$.

[7] We hope by this point a shift of variable to 'x' instead of 'θ' or 't' is a non-issue.

[8] Or, to put to another way, earn more partial credit if this were an exam question!

Theorem 12.3 tells us $1 - \sin^2(x) = \cos^2(x)$, which means to our surprise and delight, we are much closer to our goal that we may have originally thought:

$$\begin{aligned}
\frac{3}{1-\sin(x)} - \frac{3}{1+\sin(x)} &= \frac{6\sin(x)}{1-\sin^2(x)} = \frac{6\sin(x)}{\cos^2(x)} \\
&= 6\left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{\cos(x)}\right) = 6\sec(x)\tan(x).
\end{aligned}$$

6. It is debatable which side of the identity is more complicated. One thing which stands out is that the denominator on the left hand side is $1 - \cos(\theta)$, while the numerator of the right hand side is $1+\cos(\theta)$. This suggests the strategy of starting with the left hand side and multiplying the numerator and denominator by the quantity $1 + \cos(\theta)$. Theorem 12.3 comes to our aid once more when we simplify $1 - \cos^2(\theta) = \sin^2(\theta)$:

$$\begin{aligned}
\frac{\sin(\theta)}{1-\cos(\theta)} &= \frac{\sin(\theta)}{(1-\cos(\theta))} \cdot \frac{(1+\cos(\theta))}{(1+\cos(\theta))} = \frac{\sin(\theta)(1+\cos(\theta))}{(1-\cos(\theta))(1+\cos(\theta))} \\
&= \frac{\sin(\theta)(1+\cos(\theta))}{1-\cos^2(\theta)} = \frac{\sin(\theta)(1+\cos(\theta))}{\sin^2(\theta)} \\
&= \frac{\cancel{\sin(\theta)}(1+\cos(\theta))}{\cancel{\sin(\theta)}\sin(\theta)} = \frac{1+\cos(\theta)}{\sin(\theta)}
\end{aligned}$$

□

In Example 12.1.3 number 6 above, we see that multiplying $1 - \cos(\theta)$ by $1 + \cos(\theta)$ produces a difference of squares that can be simplified to one term using Theorem 12.3.

This is exactly the same kind of phenomenon that occurs when we multiply expressions such as $1 - \sqrt{2}$ by $1 + \sqrt{2}$ or $3 - 4i$ by $3 + 4i$. In algebra, these sorts of expressions were called 'conjugates.'[9]

For this reason, the quantities $(1 - \cos(\theta))$ and $(1 + \cos(\theta))$ are called 'Pythagorean Conjugates.' Below is a list of other common Pythagorean Conjugates.

Pythagorean Conjugates

- $1 - \cos(\theta)$ and $1 + \cos(\theta)$: $(1 - \cos(\theta))(1 + \cos(\theta)) = 1 - \cos^2(\theta) = \sin^2(\theta)$
- $1 - \sin(\theta)$ and $1 + \sin(\theta)$: $(1 - \sin(\theta))(1 + \sin(\theta)) = 1 - \sin^2(\theta) = \cos^2(\theta)$
- $\sec(\theta) - 1$ and $\sec(\theta) + 1$: $(\sec(\theta) - 1)(\sec(\theta) + 1) = \sec^2(\theta) - 1 = \tan^2(\theta)$
- $\sec(\theta) - \tan(\theta)$ and $\sec(\theta) + \tan(\theta)$: $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta) = 1$
- $\csc(\theta) - 1$ and $\csc(\theta) + 1$: $(\csc(\theta) - 1)(\csc(\theta) + 1) = \csc^2(\theta) - 1 = \cot^2(\theta)$
- $\csc(\theta) - \cot(\theta)$ and $\csc(\theta) + \cot(\theta)$: $(\csc(\theta) - \cot(\theta))(\csc(\theta) + \cot(\theta)) = \csc^2(\theta) - \cot^2(\theta) = 1$

[9]See Sections A.11 and A.13.

Verifying trigonometric identities requires a healthy mix of tenacity and inspiration. You will need to spend many hours struggling with them just to become proficient in the basics.

Like many things in life, there is no short-cut here – there is no complete algorithm for verifying identities. Nevertheless, a summary of some strategies which may be helpful (depending on the situation) is provided below and ample practice is provided for you in the Exercises.

Strategies for Verifying Identities

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities in Theorem 12.1 to write functions on one side of the identity in terms of the functions on the other side of the identity.

  Simplify the resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean Identities in Theorem 12.3 to 'exchange' sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator **and** denominator by Pythagorean Conjugates in order to take advantage of the Pythagorean Identities in Theorem 12.3.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.
- Try *something*. The more you work with identities, the better you'll get with identities.

12.1.1 Exercises

In Exercises 1 - 11, use the Reciprocal and Quotient Identities (Theorem 12.1) along with the Pythagorean Identities (Theorem 12.3), to find the value of the circular function requested below. (Find the exact value unless otherwise indicated.)

1. If $\sin(\theta) = \frac{\sqrt{5}}{5}$, find $\csc(\theta)$.
2. If $\sec(\theta) = -4$, find $\cos(\theta)$.
3. If $\tan(t) = 3$, find $\cot(t)$.
4. If θ is a Quadrant IV angle with $\cos(\theta) = \frac{5}{13}$, find $\sin(\theta)$.
5. If θ is a Quadrant III angle with $\tan(\theta) = 2$, find $\sec(\theta)$.
6. If $\frac{\pi}{2} < t < \pi$ with $\cot(t) = -2$, find $\csc(t)$.
7. If $\sec(\theta) = 3$ and $\sin(\theta) < 0$, find $\tan(\theta)$.
8. If $\sin(\theta) = -\frac{2}{3}$ but $\tan(\theta) > 0$, find $\cos(\theta)$.
9. If $0 < t < \frac{\pi}{2}$ and $\sin(t) = 0.42$, find $\cos(t)$, rounded to four decimal places.
10. If θ is Quadrant IV angle with $\sec(\theta) = 1.17$, find $\tan(\theta)$, rounded to four decimal places.
11. If $\pi < t < \frac{3\pi}{2}$ with $\cot(t) = 4.2$, find $\csc(t)$, rounded to four decimal places.

In Exercises 12 - 25, use the Reciprocal and Quotient Identities (Theorem 12.1) along with the Pythagorean Identities (Theorem 12.3), to find the exact values of the remaining circular functions. (Compare your methods with how you solved Exercises 25 - 38 in Section 11.4.)

12. $\sin(\theta) = \dfrac{3}{5}$ with θ in Quadrant II
13. $\tan(\theta) = \dfrac{12}{5}$ with θ in Quadrant III
14. $\csc(\theta) = \dfrac{25}{24}$ with θ in Quadrant I
15. $\sec(\theta) = 7$ with θ in Quadrant IV
16. $\csc(\theta) = -\dfrac{10\sqrt{91}}{91}$ with θ in Quadrant III
17. $\cot(\theta) = -23$ with θ in Quadrant II
18. $\tan(\theta) = -2$ with θ in Quadrant IV.
19. $\sec(\theta) = -4$ with θ in Quadrant II.
20. $\cot(\theta) = \sqrt{5}$ with θ in Quadrant III.
21. $\cos(\theta) = \dfrac{1}{3}$ with θ in Quadrant I.
22. $\cot(t) = 2$ with $0 < t < \dfrac{\pi}{2}$.
23. $\csc(t) = 5$ with $\dfrac{\pi}{2} < t < \pi$.
24. $\tan(t) = \sqrt{10}$ with $\pi < t < \dfrac{3\pi}{2}$.
25. $\sec(t) = 2\sqrt{5}$ with $\dfrac{3\pi}{2} < t < 2\pi$.
26. Skippy claims $\cos(\theta) + \sin(\theta) = 1$ is an identity because when $\theta = 0$, the equation is true. Is Skippy correct? Explain.

In Exercises 27 - 73, verify the identity. Assume that all quantities are defined.

27. $\cos(\theta)\sec(\theta) = 1$

28. $\tan(t)\cos(t) = \sin(t)$

29. $\sin(\theta)\csc(\theta) = 1$

30. $\tan(t)\cot(t) = 1$

31. $\csc(x)\cos(x) = \cot(x)$

32. $\dfrac{\sin(t)}{\cos^2(t)} = \sec(t)\tan(t)$

33. $\dfrac{\cos(\theta)}{\sin^2(\theta)} = \csc(\theta)\cot(\theta)$

34. $\dfrac{1+\sin(x)}{\cos(x)} = \sec(x) + \tan(x)$

35. $\dfrac{1-\cos(\theta)}{\sin(\theta)} = \csc(\theta) - \cot(\theta)$

36. $\dfrac{\cos(t)}{1-\sin^2(t)} = \sec(t)$

37. $\dfrac{\sin(x)}{1-\cos^2(x)} = \csc(x)$

38. $\dfrac{\sec(t)}{1+\tan^2(t)} = \cos(t)$

39. $\dfrac{\csc(\theta)}{1+\cot^2(\theta)} = \sin(\theta)$

40. $\dfrac{\tan(x)}{\sec^2(x)-1} = \cot(x)$

41. $\dfrac{\cot(t)}{\csc^2(t)-1} = \tan(t)$

42. $4\cos^2(\theta) + 4\sin^2(\theta) = 4$

43. $9 - \cos^2(t) - \sin^2(t) = 8$

44. $\tan^3(t) = \tan(t)\sec^2(t) - \tan(t)$

45. $\sin^5(x) = \left(1-\cos^2(x)\right)^2\sin(x)$

46. $\sec^{10}(t) = \left(1+\tan^2(t)\right)^4\sec^2(t)$

47. $\cos^2(x)\tan^3(x) = \tan(x) - \sin(x)\cos(x)$

48. $\sec^4(t) - \sec^2(t) = \tan^2(t) + \tan^4(t)$

49. $\dfrac{\cos(\theta)+1}{\cos(\theta)-1} = \dfrac{1+\sec(\theta)}{1-\sec(\theta)}$

50. $\dfrac{\sin(t)+1}{\sin(t)-1} = \dfrac{1+\csc(t)}{1-\csc(t)}$

51. $\dfrac{1-\cot(x)}{1+\cot(x)} = \dfrac{\tan(x)-1}{\tan(x)+1}$

52. $\dfrac{1-\tan(t)}{1+\tan(t)} = \dfrac{\cos(t)-\sin(t)}{\cos(t)+\sin(t)}$

53. $\tan(\theta) + \cot(\theta) = \sec(\theta)\csc(\theta)$

54. $\csc(t) - \sin(t) = \cot(t)\cos(t)$

55. $\cos(x) - \sec(x) = -\tan(x)\sin(x)$

56. $\cos(x)(\tan(x) + \cot(x)) = \csc(x)$

57. $\sin(t)(\tan(t) + \cot(t)) = \sec(t)$

58. $\dfrac{1}{1-\cos(\theta)} + \dfrac{1}{1+\cos(\theta)} = 2\csc^2(\theta)$

59. $\dfrac{1}{\sec(t)+1} + \dfrac{1}{\sec(t)-1} = 2\csc(t)\cot(t)$

60. $\dfrac{1}{\csc(x)+1} + \dfrac{1}{\csc(x)-1} = 2\sec(x)\tan(x)$

61. $\dfrac{1}{\csc(t)-\cot(t)} - \dfrac{1}{\csc(t)+\cot(t)} = 2\cot(t)$

62. $\dfrac{\cos(\theta)}{1-\tan(\theta)} + \dfrac{\sin(\theta)}{1-\cot(\theta)} = \sin(\theta)+\cos(\theta)$

63. $\dfrac{1}{\sec(t)+\tan(t)} = \sec(t)-\tan(t)$

64. $\dfrac{1}{\sec(x)-\tan(x)} = \sec(x)+\tan(x)$

65. $\dfrac{1}{\csc(t)-\cot(t)} = \csc(t)+\cot(t)$

66. $\dfrac{1}{\csc(\theta)+\cot(\theta)} = \csc(\theta)-\cot(\theta)$

67. $\dfrac{1}{1-\sin(x)} = \sec^2(x)+\sec(x)\tan(x)$

68. $\dfrac{1}{1+\sin(t)} = \sec^2(t)-\sec(t)\tan(t)$

69. $\dfrac{1}{1-\cos(\theta)} = \csc^2(\theta)+\csc(\theta)\cot(\theta)$

70. $\dfrac{1}{1+\cos(x)} = \csc^2(x)-\csc(x)\cot(x)$

71. $\dfrac{\cos(t)}{1+\sin(t)} = \dfrac{1-\sin(t)}{\cos(t)}$

72. $\csc(\theta)-\cot(\theta) = \dfrac{\sin(\theta)}{1+\cos(\theta)}$

73. $\dfrac{1-\sin(x)}{1+\sin(x)} = (\sec(x)-\tan(x))^2$

In Exercises 74 - 77, verify the identity. You may need to consult Sections 1.3 and 7.3 for a review of the properties of absolute value and logarithms before proceeding.

74. $\ln|\sec(x)| = -\ln|\cos(x)|$

75. $-\ln|\csc(x)| = \ln|\sin(x)|$

76. $-\ln|\sec(x)-\tan(x)| = \ln|\sec(x)+\tan(x)|$

77. $-\ln|\csc(x)+\cot(x)| = \ln|\csc(x)-\cot(x)|$

78. (a) What indeterminate form is present in the limit $\displaystyle\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta}$?

(b) Graph $f(\theta)=\frac{1-\cos(\theta)}{\theta}$ near $\theta=0$. What appears to be $\displaystyle\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta}$?

(c) Verify the identity: $\frac{1-\cos(\theta)}{\theta} = \frac{\sin(\theta)}{1+\cos(\theta)}\,\frac{\sin(\theta)}{\theta}$.

(d) Use the fact[10] that $\displaystyle\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1$ along with part 78c to help you find $\displaystyle\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta}$.

[10] See Exercise 82 in Section 11.4.

12.1.2 Answers

1. $\csc(\theta) = \sqrt{5}$.
2. $\cos(\theta) = -\frac{1}{4}$.
3. $\cot(t) = \frac{1}{3}$.
4. $\sin(\theta) = -\frac{12}{13}$.
5. $\sec(\theta) = -\sqrt{5}$.
6. $\csc(t) = \sqrt{5}$.
7. $\tan(\theta) = -2\sqrt{2}$.
8. $\cos(\theta) = -\frac{\sqrt{5}}{3}$.
9. $\cos(t) \approx 0.9075$.
10. $\tan(\theta) \approx -0.6074$.
11. $\csc(t) \approx -4.079$.
12. $\sin(\theta) = \frac{3}{5}, \cos(\theta) = -\frac{4}{5}, \tan(\theta) = -\frac{3}{4}, \csc(\theta) = \frac{5}{3}, \sec(\theta) = -\frac{5}{4}, \cot(\theta) = -\frac{4}{3}$
13. $\sin(\theta) = -\frac{12}{13}, \cos(\theta) = -\frac{5}{13}, \tan(\theta) = \frac{12}{5}, \csc(\theta) = -\frac{13}{12}, \sec(\theta) = -\frac{13}{5}, \cot(\theta) = \frac{5}{12}$
14. $\sin(\theta) = \frac{24}{25}, \cos(\theta) = \frac{7}{25}, \tan(\theta) = \frac{24}{7}, \csc(\theta) = \frac{25}{24}, \sec(\theta) = \frac{25}{7}, \cot(\theta) = \frac{7}{24}$
15. $\sin(\theta) = \frac{-4\sqrt{3}}{7}, \cos(\theta) = \frac{1}{7}, \tan(\theta) = -4\sqrt{3}, \csc(\theta) = -\frac{7\sqrt{3}}{12}, \sec(\theta) = 7, \cot(\theta) = -\frac{\sqrt{3}}{12}$
16. $\sin(\theta) = -\frac{\sqrt{91}}{10}, \cos(\theta) = -\frac{3}{10}, \tan(\theta) = \frac{\sqrt{91}}{3}, \csc(\theta) = -\frac{10\sqrt{91}}{91}, \sec(\theta) = -\frac{10}{3}, \cot(\theta) = \frac{3\sqrt{91}}{91}$
17. $\sin(\theta) = \frac{\sqrt{530}}{530}, \cos(\theta) = -\frac{23\sqrt{530}}{530}, \tan(\theta) = -\frac{1}{23}, \csc(\theta) = \sqrt{530}, \sec(\theta) = -\frac{\sqrt{530}}{23}, \cot(\theta) = -23$
18. $\sin(\theta) = -\frac{2\sqrt{5}}{5}, \cos(\theta) = \frac{\sqrt{5}}{5}, \tan(\theta) = -2, \csc(\theta) = -\frac{\sqrt{5}}{2}, \sec(\theta) = \sqrt{5}, \cot(\theta) = -\frac{1}{2}$
19. $\sin(\theta) = \frac{\sqrt{15}}{4}, \cos(\theta) = -\frac{1}{4}, \tan(\theta) = -\sqrt{15}, \csc(\theta) = \frac{4\sqrt{15}}{15}, \sec(\theta) = -4, \cot(\theta) = -\frac{\sqrt{15}}{15}$
20. $\sin(\theta) = -\frac{\sqrt{6}}{6}, \cos(\theta) = -\frac{\sqrt{30}}{6}, \tan(\theta) = \frac{\sqrt{5}}{5}, \csc(\theta) = -\sqrt{6}, \sec(\theta) = -\frac{\sqrt{30}}{5}, \cot(\theta) = \sqrt{5}$
21. $\sin(\theta) = \frac{2\sqrt{2}}{3}, \cos(\theta) = \frac{1}{3}, \tan(\theta) = 2\sqrt{2}, \csc(\theta) = \frac{3\sqrt{2}}{4}, \sec(\theta) = 3, \cot(\theta) = \frac{\sqrt{2}}{4}$
22. $\sin(t) = \frac{\sqrt{5}}{5}, \cos(t) = \frac{2\sqrt{5}}{5}, \tan(t) = \frac{1}{2}, \csc(t) = \sqrt{5}, \sec(t) = \frac{\sqrt{5}}{2}, \cot(t) = 2$
23. $\sin(t) = \frac{1}{5}, \cos(t) = -\frac{2\sqrt{6}}{5}, \tan(t) = -\frac{\sqrt{6}}{12}, \csc(t) = 5, \sec(t) = -\frac{5\sqrt{6}}{12}, \cot(t) = -2\sqrt{6}$
24. $\sin(t) = -\frac{\sqrt{110}}{11}, \cos(t) = -\frac{\sqrt{11}}{11}, \tan(t) = \sqrt{10}, \csc(t) = -\frac{\sqrt{110}}{10}, \sec(t) = -\sqrt{11}, \cot(t) = \frac{\sqrt{10}}{10}$
25. $\sin(t) = -\frac{\sqrt{95}}{10}, \cos(t) = \frac{\sqrt{5}}{10}, \tan(t) = -\sqrt{19}, \csc(t) = -\frac{2\sqrt{95}}{19}, \sec(t) = 2\sqrt{5}, \cot(t) = -\frac{\sqrt{19}}{19}$
26. No, Skippy is not correct. In order to be an identity, an equation must hold for *all* applicable angles. For example, $\cos(\theta) + \sin(\theta) = 1$ does not hold when $\theta = \pi$.

78. (a) As $\theta \to 0$, $\frac{1-\cos(\theta)}{\theta} \to \frac{0}{0}$.
    (b) The graph of $f(\theta) = \frac{1-\cos(\theta)}{\theta}$ approaches $(0, 0)$, so $\lim\limits_{\theta\to 0} \frac{1-\cos(\theta)}{\theta}$ appears to be 0.
    (d) $\lim\limits_{\theta\to 0} \frac{1-\cos(\theta)}{\theta} = \lim\limits_{\theta\to 0} \frac{\sin(\theta)}{1+\cos(\theta)} \frac{\sin(\theta)}{\theta} = \left(\frac{0}{1+\cos(0)}\right)(1) = \left(\frac{0}{2}\right)(1) = 0.$

12.2 More Identities

In Section 12.1, we saw the utility of identities in finding the values of the circular functions of a given angle as well as simplifying expressions involving the circular functions. In this section, we introduce several collections of identities which have uses in this course and beyond.

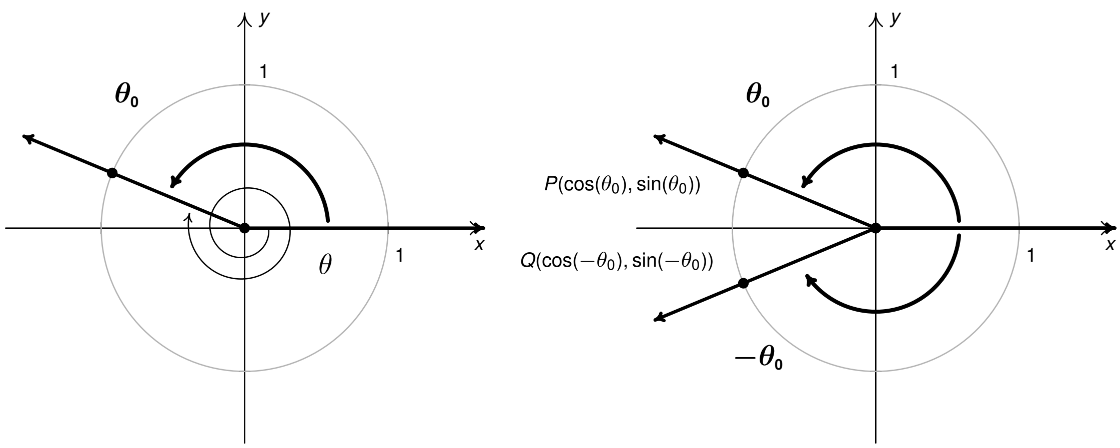
Our first set of identities is the 'Even / Odd' identities. We *observed* the even and odd properties of the circular functions graphically in Sections 11.3 and 11.5. Here, we take the time to *prove* these properties from first principles. We state the theorem below for reference.

Theorem 12.4. Even / Odd Identities: For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

We start by proving $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.

Consider an angle θ plotted in standard position. Let θ_0 be the angle coterminal with θ with $0 \leq \theta_0 < 2\pi$. (We can construct the angle θ_0 by rotating counter-clockwise from the positive x-axis to the terminal side of θ as pictured below.) Since θ and θ_0 are coterminal, $\cos(\theta) = \cos(\theta_0)$ and $\sin(\theta) = \sin(\theta_0)$.



We now consider the angles $-\theta$ and $-\theta_0$. Since θ is coterminal with θ_0, there is some integer k so that $\theta = \theta_0 + 2\pi \cdot k$. Hence, $-\theta = -\theta_0 - 2\pi \cdot k = -\theta_0 + 2\pi \cdot (-k)$. Since k is an integer, so is $(-k)$, which means $-\theta$ is coterminal with $-\theta_0$. Therefore, $\cos(-\theta) = \cos(-\theta_0)$ and $\sin(-\theta) = \sin(-\theta_0)$.

Let P and Q denote the points on the terminal sides of θ_0 and $-\theta_0$, respectively, which lie on the Unit Circle. By definition, the coordinates of P are $(\cos(\theta_0), \sin(\theta_0))$ and the coordinates of Q are $(\cos(-\theta_0), \sin(-\theta_0))$.

Since θ_0 and $-\theta_0$ sweep out congruent central sectors of the Unit Circle, it follows that the points P and Q are symmetric about the x-axis. Thus, $\cos(-\theta_0) = \cos(\theta_0)$ and $\sin(-\theta_0) = -\sin(\theta_0)$.

Since the cosines and sines of θ_0 and $-\theta_0$ are the same as those for θ and $-\theta$, respectively, we get $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, as required.

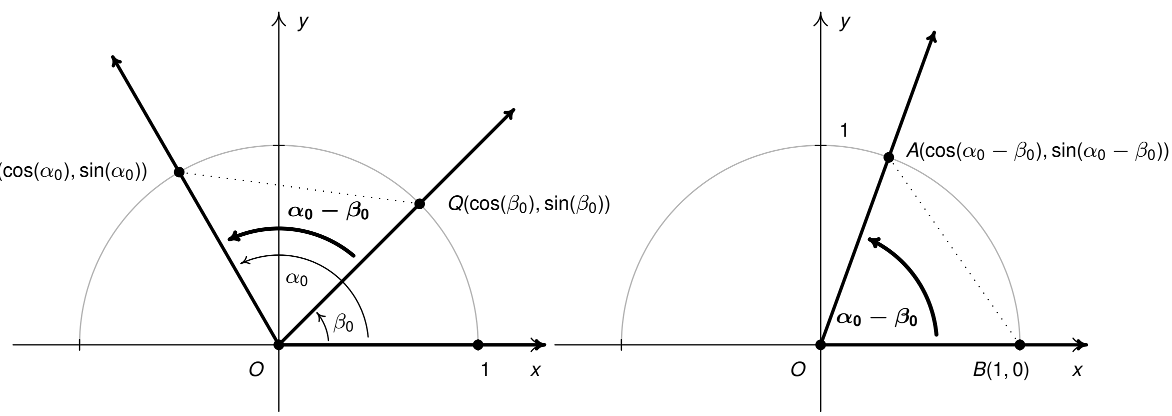
As we saw in Section 11.5, the remaining four circular functions 'inherit' their even/odd nature from sine and cosine courtesy of the Reciprocal and Quotient Identities, Theorem 12.1.

Our next set of identities establish how the cosine function handles sums and differences of angles.

Theorem 12.5. Sum and Difference Identities for Cosine: For all angles α and β,

- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

We first prove the result for differences. As in the proof of the Even / Odd Identities, we can reduce the proof for general angles α and β to angles α_0 and β_0, coterminal with α and β, respectively, each of which measure between 0 and 2π radians. Since α and α_0 are coterminal, as are β and β_0, it follows that $(\alpha-\beta)$ is coterminal with $(\alpha_0-\beta_0)$. Consider the case below where $\alpha_0 \geq \beta_0$.



Since the angles POQ and AOB are congruent, the distance between P and Q is equal to the distance between A and B.[1] The distance formula, Equation A.1, yields

$$\sqrt{(\cos(\alpha_0)-\cos(\beta_0))^2+(\sin(\alpha_0)-\sin(\beta_0))^2} = \sqrt{(\cos(\alpha_0-\beta_0)-1)^2+(\sin(\alpha_0-\beta_0)-0)^2}$$

Squaring both sides, we expand the left hand side of this equation as

$$\begin{aligned}(\cos(\alpha_0)-\cos(\beta_0))^2+(\sin(\alpha_0)-\sin(\beta_0))^2 &= \cos^2(\alpha_0)-2\cos(\alpha_0)\cos(\beta_0)+\cos^2(\beta_0)\\ &\quad+\sin^2(\alpha_0)-2\sin(\alpha_0)\sin(\beta_0)+\sin^2(\beta_0)\\ &= \cos^2(\alpha_0)+\sin^2(\alpha_0)+\cos^2(\beta_0)+\sin^2(\beta_0)\\ &\quad-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)\end{aligned}$$

From the Pythagorean Identities, $\cos^2(\alpha_0)+\sin^2(\alpha_0)=1$ and $\cos^2(\beta_0)+\sin^2(\beta_0)=1$, so

$$(\cos(\alpha_0)-\cos(\beta_0))^2+(\sin(\alpha_0)-\sin(\beta_0))^2 = 2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)$$

[1] In the picture we've drawn, the <u>tri</u>angles POQ and AOB are congruent, which is even better. However, $\alpha_0-\beta_0$ could be 0 or it could be π, neither of which makes a triangle. It could also be larger than π, which makes a triangle, just not the one we've drawn. You should think about those three cases.

Turning our attention to the right hand side of our equation, we find

$$\begin{aligned}(\cos(\alpha_0-\beta_0)-1)^2+(\sin(\alpha_0-\beta_0)-0)^2 &= \cos^2(\alpha_0-\beta_0)-2\cos(\alpha_0-\beta_0)+1+\sin^2(\alpha_0-\beta_0)\\ &= 1+\cos^2(\alpha_0-\beta_0)+\sin^2(\alpha_0-\beta_0)-2\cos(\alpha_0-\beta_0)\end{aligned}$$

Once again, we simplify $\cos^2(\alpha_0-\beta_0)+\sin^2(\alpha_0-\beta_0)=1$, so that

$$(\cos(\alpha_0-\beta_0)-1)^2+(\sin(\alpha_0-\beta_0)-0)^2 = 2-2\cos(\alpha_0-\beta_0)$$

Putting it all together, we get $2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)=2-2\cos(\alpha_0-\beta_0)$, which simplifies to: $\cos(\alpha_0-\beta_0)=\cos(\alpha_0)\cos(\beta_0)+\sin(\alpha_0)\sin(\beta_0)$.

Since α and α_0, β and β_0, and $(\alpha-\beta)$ and $(\alpha_0-\beta_0)$ are all coterminal pairs of angles, we have established the identity: $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$.

For the case where $\alpha_0\le\beta_0$, we can apply the above argument to the angle $\beta_0-\alpha_0$ to obtain the identity $\cos(\beta_0-\alpha_0)=\cos(\beta_0)\cos(\alpha_0)+\sin(\beta_0)\sin(\alpha_0)$. Using this formula in conjunction with the Even Identity of cosine gives us the result in this case, too:

$$\begin{aligned}\cos(\alpha_0-\beta_0)=\cos(-(\alpha_0-\beta_0))=\cos(\beta_0-\alpha_0) &= \cos(\beta_0)\cos(\alpha_0)+\sin(\beta_0)\sin(\alpha_0)\\ &= \cos(\alpha_0)\cos(\beta_0)+\sin(\alpha_0)\sin(\beta_0).\end{aligned}$$

To get the sum identity for cosine, we use the difference formula along with the Even/Odd Identities

$$\cos(\alpha+\beta)=\cos(\alpha-(-\beta))=\cos(\alpha)\cos(-\beta)+\sin(\alpha)\sin(-\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).$$

We put these newfound identities to good use in the following example.

Example 12.2.1.

1. Find the exact value of $\cos(15^\circ)$.
2. Verify the identity: $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$.
3. Suppose α is a Quadrant I angle with $\sin(\alpha)=\frac{3}{5}$ and β is a Quadrant IV angle with $\sec(\beta)=4$. Find the exact value of $\cos(\alpha+\beta)$.

Solution.

1. In order to use Theorem 12.5 to find $\cos(15^\circ)$, we need to write 15° as a sum or difference of angles whose cosines and sines we know. One way to do so is to write $15^\circ=45^\circ-30^\circ$. We find:

$$\begin{aligned}\cos(15^\circ) &= \cos(45^\circ-30^\circ)\\ &= \cos(45^\circ)\cos(30^\circ)+\sin(45^\circ)\sin(30^\circ)\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

2. Using Theorem 12.5 gives:

$$\begin{aligned}
\cos\left(\frac{\pi}{2}-\theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta)+\sin\left(\frac{\pi}{2}\right)\sin(\theta) \\
&= (0)(\cos(\theta))+(1)(\sin(\theta)) \\
&= \sin(\theta).
\end{aligned}$$

3. Per Theorem 12.5, we know $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$. Hence, we need to find the sines and cosines of α and β to complete the problem.

   We are given $\sin(\alpha)=\frac{3}{5}$, so our first task is to find $\cos(\alpha)$. We can quickly get $\cos(\alpha)$ using the Pythagorean Identity $\cos^2(\alpha)=1-\sin^2(\alpha)=1-\left(\frac{3}{5}\right)^2=\frac{16}{25}$. We get $\cos(\alpha)=\frac{4}{5}$, choosing the positive root since α is a Quadrant I angle.

   Next, we need the $\sin(\beta)$ and $\cos(\beta)$. Since $\sec(\beta)=4$, we immediately get $\cos(\beta)=\frac{1}{4}$ courtesy of the Reciprocal and Quotient Identities.

   To get $\sin(\beta)$, we employ the Pythagorean Identity: $\sin^2(\beta)=1-\cos^2(\beta)=1-\left(\frac{1}{4}\right)^2=\frac{15}{16}$. Here, since β is a Quadrant IV angle, we get $\sin(\beta)=-\frac{\sqrt{15}}{4}$.

   Finally, we get: $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)=\left(\frac{4}{5}\right)\left(\frac{1}{4}\right)-\left(\frac{3}{5}\right)\left(-\frac{\sqrt{15}}{4}\right)=\frac{4+3\sqrt{15}}{20}$. □

The identity verified in Example 12.2.1, namely, $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$, is the first of the celebrated 'cofunction' identities. These identities were first hinted at in Exercise 44 in Section B.2.
From $\sin(\theta)=\cos\left(\frac{\pi}{2}-\theta\right)$, we get: $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\left(\frac{\pi}{2}-\left[\frac{\pi}{2}-\theta\right]\right)=\cos(\theta)$, which says, in words, that the 'co'sine of an angle is the sine of its 'co'mplement. Now that these identities have been established for cosine and sine, the remaining circular functions follow suit. The remaining proofs are left as exercises.

Theorem 12.6. Cofunction Identities: For all applicable angles θ,

- $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$
- $\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$
- $\sec\left(\frac{\pi}{2}-\theta\right)=\csc(\theta)$
- $\csc\left(\frac{\pi}{2}-\theta\right)=\sec(\theta)$
- $\tan\left(\frac{\pi}{2}-\theta\right)=\cot(\theta)$
- $\cot\left(\frac{\pi}{2}-\theta\right)=\tan(\theta)$

The Cofunction Identities enable us to derive the sum and difference formulas for sine. We first convert to sine to cosine and expand:

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2}-(\alpha+\beta)\right) \\
&= \cos\left(\left[\frac{\pi}{2}-\alpha\right]-\beta\right) \\
&= \cos\left(\frac{\pi}{2}-\alpha\right)\cos(\beta)+\sin\left(\frac{\pi}{2}-\alpha\right)\sin(\beta) \\
&= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)
\end{aligned}$$

We can derive the difference formula for sine by rewriting $\sin(\alpha - \beta)$ as $\sin(\alpha + (-\beta))$ and using the sum formula and the Even / Odd Identities. Again, we leave the details to the reader.

Theorem 12.7. Sum and Difference Identities for Sine: For all angles α and β,

- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

We try out these new identities in the next example.

Example 12.2.2.

1. Find the exact value of $\sin\left(\frac{19\pi}{12}\right)$
2. Suppose α is a Quadrant II angle with $\sin(\alpha) = \frac{5}{13}$, and β is a Quadrant III angle with $\tan(\beta) = 2$. Find the exact value of $\sin(\alpha - \beta)$.
3. Derive a formula for $\tan(\alpha + \beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

Solution.

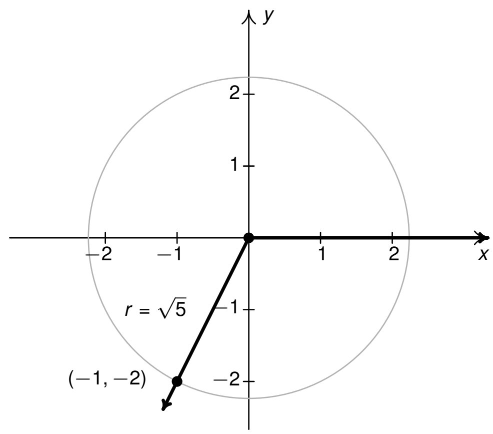
1. As in Example 12.2.1, we need to write the angle $\frac{19\pi}{12}$ as a sum or difference of common angles. The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination[2] is $\frac{19\pi}{12} = \frac{4\pi}{3} + \frac{\pi}{4}$. Applying Theorem 12.7, we get
$$\begin{aligned} \sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3} + \frac{\pi}{4}\right) \\ &= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{-\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$
2. In order to find $\sin(\alpha - \beta)$ using Theorem 12.7, we need to find $\cos(\alpha)$ and both $\cos(\beta)$ and $\sin(\beta)$.

   To find $\cos(\alpha)$, we use the Pythagorean Identity $\cos^2(\alpha) = 1 - \sin^2(\alpha) = 1 - \left(\frac{5}{13}\right)^2 = \frac{144}{169}$. We get $\cos(\alpha) = -\frac{12}{13}$, the negative, here, owing to the fact that α is a Quadrant II angle.

   We now set about finding $\sin(\beta)$ and $\cos(\beta)$. We have several ways to proceed at this point, but since there isn't a direct way to get from $\tan(\beta) = 2$ to either $\sin(\beta)$ or $\cos(\beta)$, we opt for a more geometric approach as presented in Section 11.4.

[2]It takes some trial and error to find this combination. One alternative is to convert to degrees . . .

Since β is a Quadrant III angle with $\tan(\beta) = 2 = \frac{-2}{-1}$, we know the point $Q(x, y) = (-1, -2)$ is on the terminal side of β as illustrated below.[3]



the terminal side of β contains $Q(-1, -2)$

We find $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$, so per Theorem 11.9, $\sin(\beta) = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$ and $\cos(\beta) = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$.

At last, we have $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) = \left(\frac{5}{13}\right)\left(-\frac{\sqrt{5}}{5}\right) - \left(-\frac{12}{13}\right)\left(-\frac{2\sqrt{5}}{5}\right) = -\frac{29\sqrt{5}}{65}$.

3. We can start expanding $\tan(\alpha + \beta)$ using a quotient identity and our sum formulas

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\ &= \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)} \end{aligned}$$

Since $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$ and $\tan(\beta) = \frac{\sin(\beta)}{\cos(\beta)}$, it looks as though if we divide both numerator and denominator by $\cos(\alpha)\cos(\beta)$ we will have what we want

[3]Note that even though $\tan(\beta) = \frac{\sin(\beta)}{\cos(\beta)}$, we *cannot* take $\sin(\beta) = -2$ and $\cos(\beta) = -1$. Recall that $\sin(\beta)$ and $\cos(\beta)$ are the y and x coordinates on a *specific* circle, the Unit Circle. As we'll see shortly, $(-1, -2)$ lies on a circle of $\sqrt{5}$, so *not* the Unit Circle.

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \cdot \frac{\dfrac{1}{\cos(\alpha)\cos(\beta)}}{\dfrac{1}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\dfrac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} + \dfrac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\dfrac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} - \dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\dfrac{\sin(\alpha)\cancel{\cos(\beta)}}{\cos(\alpha)\cancel{\cos(\beta)}} + \dfrac{\cancel{\cos(\alpha)}\sin(\beta)}{\cancel{\cos(\alpha)}\cos(\beta)}}{\dfrac{\cancel{\cos(\alpha)\cos(\beta)}}{\cancel{\cos(\alpha)\cos(\beta)}} - \dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}
\end{aligned}$$

Naturally, this formula is limited to those cases where all of the tangents are defined. □

The formula developed in Exercise 12.2.2 for $\tan(\alpha+\beta)$ can be used to find a formula for $\tan(\alpha-\beta)$ by rewriting the difference as a sum, $\tan(\alpha+(-\beta))$ and using the odd property of tangent. (The reader is encouraged to fill in the details.) Below we summarize all of the sum and difference formulas.

Theorem 12.8. Sum and Difference Identities: For all applicable angles α and β,

- $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

In the statement of Theorem 12.8, we have combined the cases for the sum '+' and difference '−' of angles into one formula. The convention here is that if you want the formula for the sum '+' of two angles, you use the top sign in the formula; for the difference, '−', use the bottom sign. For example,

$$\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$$

If we set $\alpha = \beta$ in the sum formulas in Theorem 12.8, we obtain the following 'Double Angle' Identities:

Theorem 12.9. Double Angle Identities: For all applicable angles θ,

- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 2\cos^2(\theta) - 1 \\ 1 - 2\sin^2(\theta) \end{cases}$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1 - \tan^2(\theta)}$

The three different forms for $\cos(2\theta)$ can be explained by our ability to 'exchange' squares of cosine and sine via the Pythagorean Identity. For instance, if we substitute $\sin^2(\theta) = 1 - \cos^2(\theta)$ into the first formula for $\cos(2\theta)$, we get $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \cos^2(\theta) - (1 - \cos^2(\theta)) = 2\cos^2(\theta) - 1$.

It is interesting to note that to determine the value of $\cos(2\theta)$, only *one* piece of information is required: either $\cos(\theta)$ or $\sin(\theta)$. To determine $\sin(2\theta)$, however, it appears that we must know both $\sin(\theta)$ and $\cos(\theta)$. In the next example, we show how we can find $\sin(2\theta)$ knowing just one piece of information, namely $\tan(\theta)$.

Example 12.2.3.

1. Suppose $P(-3, 4)$ lies on the terminal side of θ when θ is plotted in standard position.

   Find $\cos(2\theta)$ and $\sin(2\theta)$ and determine the quadrant in which the terminal side of the angle 2θ lies when it is plotted in standard position.

2. If $\sin(\theta) = x$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

3. Verify the identity: $\sin(2\theta) = \dfrac{2\tan(\theta)}{1 + \tan^2(\theta)}$.

4. Express $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$.

Solution.

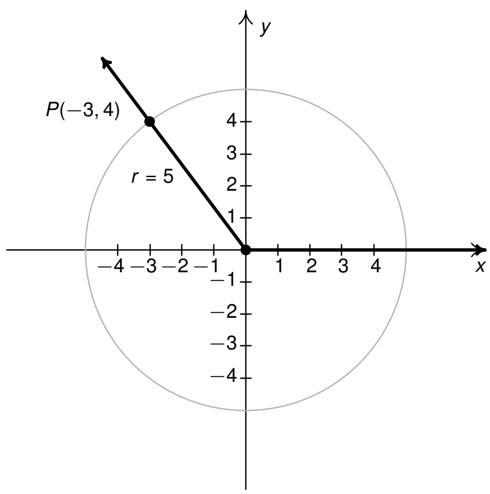
1. We sketch the terminal side of θ below on the left. Using Theorem 11.3 from Section 11.2 with $x = -3$ and $y = 4$, we find $r = \sqrt{x^2 + y^2} = 5$. Hence, $\cos(\theta) = -\frac{3}{5}$ and $\sin(\theta) = \frac{4}{5}$.

   Theorem 12.9 gives us three different formulas to choose from to find $\cos(2\theta)$. Using the first formula,[4] we get: $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = -\frac{7}{25}$. For $\sin(2\theta)$, we get $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$.

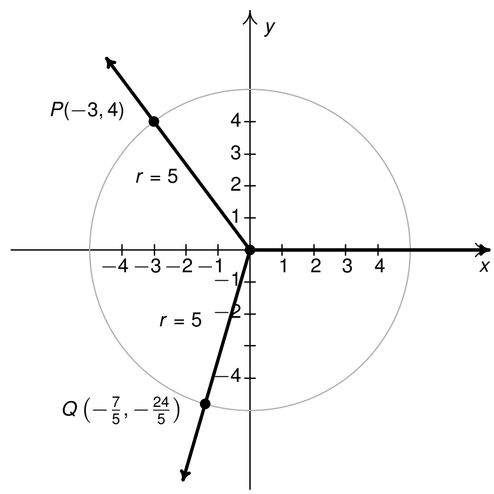
   Since both cosine and sine of 2θ are negative, the terminal side of 2θ, when plotted in standard position, lies in Quadrant III. To see this more clearly, we plot the terminal side of 2θ, along with the terminal side of θ below on the right.

[4] We invite the reader to check this answer using the other two formulas.

Note that in order to find the point $Q(x, y)$ on the terminal side of 2θ of a circle of radius 5, we use Theorem 11.3 again and find $x = r\cos(2\theta) = 5\left(-\frac{7}{25}\right) = -\frac{7}{5}$ and $y = r\sin(2\theta) = 5\left(-\frac{24}{25}\right) = -\frac{24}{5}$.



$P(-3, 4)$ is on the terminal side of θ.



$Q\left(-\frac{7}{5}, -\frac{24}{5}\right)$ is on the terminal side of 2θ.

2. If your first reaction to '$\sin(\theta) = x$' is 'No it's not, $\cos(\theta) = x$!' then you have indeed learned something, and we take comfort in that.

   While we have mostly used 'x' to represent the x-coordinate of the point the terminal side of an angle θ, here, 'x' represents the quantity $\sin(\theta)$ and our task is to express $\sin(2\theta)$ in terms of x.

   Since $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2x\cos(\theta)$, what remains is to express $\cos(\theta)$ in terms of x.

   Substituting $\sin(\theta) = x$ into the Pythagorean Identity, we get $\cos^2(\theta) = 1 - \sin^2(\theta) = 1 - x^2$, or $\cos(\theta) = \pm\sqrt{1 - x^2}$. Since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos(\theta) \geq 0$, and thus $\cos(\theta) = \sqrt{1 - x^2}$.

   Our final answer is $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2x\sqrt{1 - x^2}$.

3. We start with the right hand side of the identity and note that $1 + \tan^2(\theta) = \sec^2(\theta)$. Next, we use the Reciprocal and Quotient Identities to rewrite $\tan(\theta)$ and $\sec(\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$:

$$\begin{aligned}
\frac{2\tan(\theta)}{1 + \tan^2(\theta)} &= \frac{2\tan(\theta)}{\sec^2(\theta)} = \frac{2\left(\dfrac{\sin(\theta)}{\cos(\theta)}\right)}{\dfrac{1}{\cos^2(\theta)}} = 2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos^2(\theta) \\
&= 2\left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cancel{\cos(\theta)}\cos(\theta) = 2\sin(\theta)\cos(\theta) = \sin(2\theta).
\end{aligned}$$

4. In Theorem 12.9, one of the formulas for $\cos(2\theta)$, namely $\cos(2\theta) = 2\cos^2(\theta) - 1$, expresses $\cos(2\theta)$ as a polynomial in terms of $\cos(\theta)$. We are now asked to find such an identity for $\cos(3\theta)$.

   Using the sum formula for cosine, we begin with

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta+\theta)\\ &= \cos(2\theta)\cos(\theta)-\sin(2\theta)\sin(\theta).\end{aligned}$$

Our ultimate goal is to express the right hand side in terms of $\cos(\theta)$ only. To that end, we substitute $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ which yields:

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta)\cos(\theta)-\sin(2\theta)\sin(\theta)\\ &= \left(2\cos^2(\theta)-1\right)\cos(\theta)-(2\sin(\theta)\cos(\theta))\sin(\theta)\\ &= 2\cos^3(\theta)-\cos(\theta)-2\sin^2(\theta)\cos(\theta)\end{aligned}$$

Finally, we exchange $\sin^2(\theta) = 1 - \cos^2(\theta)$ courtesy of the Pythagorean Identity, and get

$$\begin{aligned}\cos(3\theta) &= 2\cos^3(\theta)-\cos(\theta)-2\sin^2(\theta)\cos(\theta)\\ &= 2\cos^3(\theta)-\cos(\theta)-2\left(1-\cos^2(\theta)\right)\cos(\theta)\\ &= 2\cos^3(\theta)-\cos(\theta)-2\cos(\theta)+2\cos^3(\theta)\\ &= 4\cos^3(\theta)-3\cos(\theta).\end{aligned}$$

Hence, $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$. □

In the last problem in Example 12.2.3, we saw how we could rewrite $\cos(3\theta)$ as sums of powers of $\cos(\theta)$. In Calculus, we have occasion to do the reverse; that is, reduce the power of cosine and sine. Solving the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ for $\cos^2(\theta)$ and the identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ for $\sin^2(\theta)$ results in the aptly-named 'Power Reduction' formulas below.

Theorem 12.10. Power Reduction Formulas: For all angles θ,

- $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$
- $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$

Our next example is a typical application of Theorem 12.10 that you'll likely see in Calculus.

Example 12.2.4. Rewrite $\sin^2(\theta)\cos^2(\theta)$ as a sum and difference of cosines to the first power.

Solution. We begin with a straightforward application of Theorem 12.10

$$\begin{aligned}\sin^2(\theta)\cos^2(\theta) &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right)\\ &= \frac{1}{4}\left(1-\cos^2(2\theta)\right)\\ &= \frac{1}{4}-\frac{1}{4}\cos^2(2\theta)\end{aligned}$$

Next, we apply the power reduction formula to $\cos^2(2\theta)$ to finish the reduction

$$
\begin{aligned}
\sin^2(\theta)\cos^2(\theta) &= \frac{1}{4} - \frac{1}{4}\cos^2(2\theta) \\
&= \frac{1}{4} - \frac{1}{4}\left(\frac{1+\cos(2(2\theta))}{2}\right) \\
&= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4\theta) \\
&= \frac{1}{8} - \frac{1}{8}\cos(4\theta)
\end{aligned}
$$

□

Another application of the Power Reduction Formulas is the Half Angle Formulas. To start, we apply the Power Reduction Formula to $\cos^2\left(\frac{\theta}{2}\right)$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2} = \frac{1+\cos(\theta)}{2}.$$

We can obtain a formula for $\cos\left(\frac{\theta}{2}\right)$ by extracting square roots. In a similar fashion, we may obtain a half angle formula for sine, and by using a quotient formula, obtain a half angle formula for tangent.

We summarize these formulas below.

Theorem 12.11. Half Angle Formulas: For all applicable angles θ,

- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

where the choice of $\pm$ depends on the quadrant in which the terminal side of $\frac{\theta}{2}$ lies.

Example 12.2.5.

1. Use a half angle formula to find the exact value of $\cos(15^\circ)$.
2. Suppose $-\pi \leq t \leq 0$ with $\cos(t) = -\frac{3}{5}$. Find $\sin\left(\frac{t}{2}\right)$.
3. Use the identity given in number 3 of Example 12.2.3 to derive the identity

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$$

Solution.

1. To use the half angle formula, we note that $15^\circ = \frac{30^\circ}{2}$ and since 15° is a Quadrant I angle, its cosine is positive. Thus we have

$$\begin{aligned} \cos\left(15^\circ\right) &= +\sqrt{\frac{1+\cos\left(30^\circ\right)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\ &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}\cdot\frac{2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2} \end{aligned}$$

   Back in Example 12.2.1, we found $\cos\left(15^\circ\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$ by using the difference formula for cosine. The reader is encouraged to prove that these two expressions are equal algebraically.

2. If $-\pi \leq t \leq 0$, then $-\frac{\pi}{2} \leq \frac{t}{2} \leq 0$, which means $\frac{t}{2}$ corresponds to a Quadrant IV angle. Hence, $\sin\left(\frac{t}{2}\right) < 0$, so we choose the negative root formula from Theorem 12.11:

$$\begin{aligned} \sin\left(\frac{t}{2}\right) &= -\sqrt{\frac{1-\cos\left(t\right)}{2}} = -\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} \\ &= -\sqrt{\frac{1+\frac{3}{5}}{2}\cdot\frac{5}{5}} = -\sqrt{\frac{8}{10}} = -\frac{2\sqrt{5}}{5} \end{aligned}$$

3. Instead of our usual approach to verifying identities, namely starting with one side of the equation and trying to transform it into the other, we will start with the identity we proved in number 3 of Example 12.2.3 and manipulate it into the identity we are asked to prove.

   The identity we are asked to start with is $\sin(2\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}$. If we are to use this to derive an identity for $\tan\left(\frac{\theta}{2}\right)$, it seems reasonable to proceed by replacing each occurrence of θ with $\frac{\theta}{2}$

$$\begin{aligned} \sin\left(2\left(\frac{\theta}{2}\right)\right) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \end{aligned}$$

   We now have the $\sin(\theta)$ we need, but we somehow need to get a factor of $1+\cos(\theta)$ involved. We substitute $1+\tan^2\left(\frac{\theta}{2}\right) = \sec^2\left(\frac{\theta}{2}\right)$, and continue to manipulate our given identity by converting secants to cosines.

$$\begin{aligned} \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= 2\tan\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) \end{aligned}$$

Finally, we apply a power reduction formula, and then solve for $\tan\left(\frac{\theta}{2}\right)$

$$\begin{aligned}
\sin(\theta) &= 2\tan\left(\tfrac{\theta}{2}\right)\cos^2\left(\tfrac{\theta}{2}\right) \\
\sin(\theta) &= 2\tan\left(\tfrac{\theta}{2}\right)\left(\frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}\right) \\
\sin(\theta) &= \tan\left(\tfrac{\theta}{2}\right)(1+\cos(\theta)) \\
\tan\left(\frac{\theta}{2}\right) &= \frac{\sin(\theta)}{1+\cos(\theta)}
\end{aligned}$$

□

Our next batch of identities, the Product to Sum Formulas,[5] are easily verified by expanding each of the right hand sides in accordance with Theorem 12.8 and as you should expect by now we leave the details as exercises. They are of particular use in Calculus, and we list them here for reference.

Theorem 12.12. Product to Sum Formulas: For all angles α and β,

- $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

Related to the Product to Sum Formulas are the Sum to Product Formulas, which we will have need of in Section 12.4. These are essentially restatements of the Product to Sum Formulas (by re-labeling the arguments of the sine and cosine functions) and as such, their proofs are left as exercises.

Theorem 12.13. Sum to Product Formulas: For all angles α and β,

- $\cos(\alpha)+\cos(\beta) = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$
- $\cos(\alpha)-\cos(\beta) = -2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$
- $\sin(\alpha)\pm\sin(\beta) = 2\sin\left(\dfrac{\alpha\pm\beta}{2}\right)\cos\left(\dfrac{\alpha\mp\beta}{2}\right)$

Example 12.2.6.

1. Write $\cos(2\theta)\cos(6\theta)$ as a sum.
2. Write $\sin(\theta)-\sin(3\theta)$ as a product.

[5]These are also known as the Prosthaphaeresis Formulas and have a rich history. The authors recommend that you conduct some research on them as your schedule allows.

Solution.

1. Identifying $\alpha = 2\theta$ and $\beta = 6\theta$, we find

$$\begin{aligned} \cos(2\theta)\cos(6\theta) &= \tfrac{1}{2}\left[\cos(2\theta - 6\theta) + \cos(2\theta + 6\theta)\right] \\ &= \tfrac{1}{2}\cos(-4\theta) + \tfrac{1}{2}\cos(8\theta) \\ &= \tfrac{1}{2}\cos(4\theta) + \tfrac{1}{2}\cos(8\theta), \end{aligned}$$

   where the last equality is courtesy of the even identity for cosine, $\cos(-4\theta) = \cos(4\theta)$.

2. Identifying $\alpha = \theta$ and $\beta = 3\theta$ yields

$$\begin{aligned} \sin(\theta) - \sin(3\theta) &= 2\sin\left(\frac{\theta - 3\theta}{2}\right)\cos\left(\frac{\theta + 3\theta}{2}\right) \\ &= 2\sin(-\theta)\cos(2\theta) \\ &= -2\sin(\theta)\cos(2\theta), \end{aligned}$$

   where the last equality is courtesy of the odd identity for sine, $\sin(-\theta) = -\sin(\theta)$. □

The reader is reminded that all of the identities presented in this section which regard the circular functions as functions of angles (in radian measure) apply equally well to the circular (trigonometric) functions regarded as functions of real numbers.

12.2.1 Sinusoids, Revisited

We first studied sinusoids in Section 11.3. Using the sum formulas for sine and cosine, we can expand the forms given to us in Theorem 11.6:

$$S(t) = A\sin(\omega t + \phi) + B = A\sin(\omega t)\cos(\phi) + A\cos(\omega t)\sin(\phi) + B,$$

and

$$C(t) = A\cos(\omega t + \phi) + B = A\cos(\omega t)\cos(\phi) - A\sin(\omega t)\sin(\phi) + B.$$

As we'll see in the next example, recognizing these 'expanded' forms of sinusoids allows us to graph functions as sinusoids which, at first glance, don't appear to fit the forms of either $C(t)$ or $S(t)$.

Example 12.2.7. Consider the function $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$. Find a formula for $f(t)$:

1. in the form $C(t) = A\cos(\omega t + \phi) + B$ for $\omega > 0$
2. in the form $S(t) = A\sin(\omega t + \phi) + B$ for $\omega > 0$

Check your answers analytically using identities and using a graphing utility.

Solution.

1. The key to this problem is to use the expanded forms of the sinusoid formulas and match up corresponding coefficients. We start by equating $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$ with the expanded form of $C(t) = A\cos(\omega t + \phi) + B$: $\cos(2t) - \sqrt{3}\sin(2t) = A\cos(\omega t)\cos(\phi) - A\sin(\omega t)\sin(\phi) + B$.

   If we take $\omega = 2$ and $B = 0$, we get: $\cos(2t) - \sqrt{3}\sin(2t) = A\cos(2t)\cos(\phi) - A\sin(2t)\sin(\phi)$.

   To determine A and ϕ, a bit more work is involved. We get started by equating the coefficients of the trigonometric functions on either side of the equation.

   On the left hand side, the coefficient of $\cos(2t)$ is 1, while on the right hand side, it is $A\cos(\phi)$. Since this equation is to hold for all real numbers, we must have[6] that $A\cos(\phi) = 1$.

   Similarly, we find by equating the coefficients of $\sin(2t)$ that $A\sin(\phi) = \sqrt{3}$. In conjunction with $A\cos(\phi) = 1$, we have a system of two (nonlinear) equations and two unknowns.

   As usual, our first task is to reduce this system of two equations and two unknowns to one equation and one unknown. We can temporarily eliminate the dependence on ϕ by using a Pythagorean Identity. From $\cos^2(\phi) + \sin^2(\phi) = 1$, we multiply through by A^2 to get $A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$.

   In our case, $A\cos(\phi) = 1$ and $A\sin(\phi) = \sqrt{3}$, hence $A^2 = A^2\cos^2(\phi) + A^2\sin^2(\phi) = 1^2 + (\sqrt{3})^2 = 4$ so $A = \pm 2$. In much the same way we fit a sinusoid to a graph in Example 11.3.3, we *choose* $A = 2$, and then find the phase angle ϕ associated with this choice.

   Substituting $A = 2$ into our two equations, $A\cos(\phi) = 1$ and $A\sin(\phi) = \sqrt{3}$, we get $2\cos(\phi) = 1$ and $2\sin(\phi) = \sqrt{3}$. After some rearrangement, $\cos(\phi) = \frac{1}{2}$ and $\sin(\phi) = \frac{\sqrt{3}}{2}$. One such angle ϕ which satisfies this criteria is $\phi = \frac{\pi}{3}$.

   Hence, one way to write $f(t)$ as a sinusoid is $f(t) = 2\cos\left(2t + \frac{\pi}{3}\right)$. We can check our answer using the sum formula for cosine :

$$\begin{aligned}
f(t) &= 2\cos\left(2t + \tfrac{\pi}{3}\right) \\
&= 2\left[\cos(2t)\cos\left(\tfrac{\pi}{3}\right) - \sin(2t)\sin\left(\tfrac{\pi}{3}\right)\right] \\
&= 2\left[\cos(2t)\left(\tfrac{1}{2}\right) - \sin(2t)\left(\tfrac{\sqrt{3}}{2}\right)\right] \\
&= \cos(2t) - \sqrt{3}\sin(2t).
\end{aligned}$$

2. Proceeding as before, we equate $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$ with the expanded form of of the sinusoid $S(t) = A\sin(\omega t + \phi) + B$ to get: $\cos(2t) - \sqrt{3}\sin(2t) = A\sin(\omega t)\cos(\phi) + A\cos(\omega t)\sin(\phi) + B$.

   Taking $\omega = 2$ and $B = 0$, we get $\cos(2t) - \sqrt{3}\sin(2t) = A\sin(2t)\cos(\phi) + A\cos(2t)\sin(\phi)$. We equate[7] the coefficients of $\cos(2t)$ on either side and get $A\sin(\phi) = 1$ and $A\cos(\phi) = -\sqrt{3}$.

   Using $A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$ as before, we get $A = \pm 2$, and again we choose $A = 2$.

[6] This should remind you of equation coefficients of like powers of x in Section 9.6.

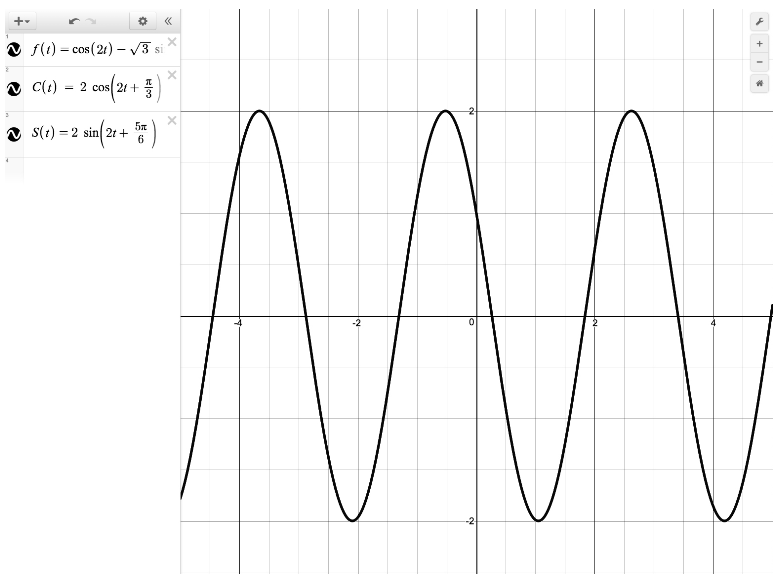
[7] Be careful here!

This means $2\sin(\phi) = 1$, or $\sin(\phi) = \frac{1}{2}$, and $2\cos(\phi) = -\sqrt{3}$, so $\cos(\phi) = -\frac{\sqrt{3}}{2}$. One such angle which meets these criteria is $\phi = \frac{5\pi}{6}$.

Hence, we have $f(t) = 2\sin\left(2t + \frac{5\pi}{6}\right)$. Checking our work analytically, we have

$$\begin{array}{rcl} f(t) & = & 2\sin\left(2t + \frac{5\pi}{6}\right) \\ & = & 2\left[\sin(2t)\cos\left(\frac{5\pi}{6}\right) + \cos(2t)\sin\left(\frac{5\pi}{6}\right)\right] \\ & = & 2\left[\sin(2t)\left(-\frac{\sqrt{3}}{2}\right) + \cos(2t)\left(\frac{1}{2}\right)\right] \\ & = & \cos(2t) - \sqrt{3}\sin(2t) \end{array}$$

Graphing the three formulas for $f(t)$ result in the identical curve, verifying the work done analytically.



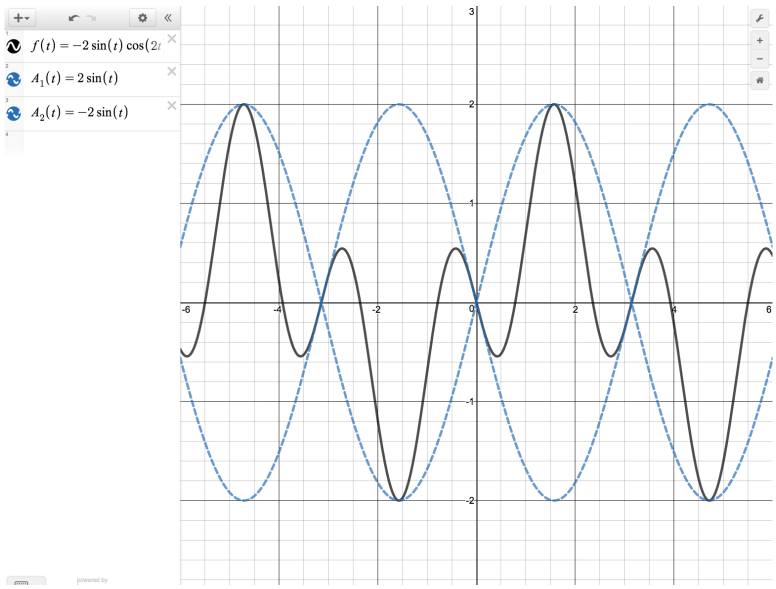
□

A couple of remarks about Example 12.2.7 are in order. First, had we chosen $A = -2$ instead of $A = 2$ as we worked through Example 12.2.7, our final answers would have *looked* different. The reader is encouraged to rework Example 12.2.7 using $A = -2$ to see what these differences are, and then for a challenging exercise, use identities to show that the formulas are all equivalent.[8]

It is important to note that in order for the technique presented in Example 12.2.7 to fit a function into one of the forms in Theorem 11.6, the *frequencies* of the sine and cosine terms much match. For example, in the Exercises, you'll be asked to write $f(t) = 3\sqrt{3}\sin(3t) - 3\cos(3t)$ in the form of $S(t)$ and $C(t)$ above, and since both the sine and cosine terms have frequency 3, this is possible.

[8]The general equations to fit a function of the form $f(x) = a\cos(\omega x) + b\sin(\omega x) + B$ into one of the forms in Theorem 11.6 are explored in Exercise 36.

However, a function such as $f(t) = \sin(t) - \sin(3t)$ cannot be written in the form of $S(t)$ or $C(t)$. The quickest way to see this is to examine its graph below which is decidedly not a sinusoid. That being said, we can still analyze this curve using identities.



Using our result from number 2 Example 12.2.6, we may rewrite $f(t) = \sin(t) - \sin(3t) = -2\sin(t)\cos(2t)$. Grouping factors, we can view $f(t) = [-2\sin(t)]\cos(2t) = A(t)\cos(2t)$ as the curve $y = \cos(2t)$ with a *variable* amplitude, $A(t) = -2\sin(t)$.

Overlaying the graphs of $f(t)$ with the (dashed) graphs of $A_1(t) = 2\sin(t)$ and $A_2(t) = -2\sin(t)$, we can see the role these two curves play in the graph of $y = f(t)$. They create a kind of 'wave envelope' for the graph of $y = f(t)$. This is an example of the beats phenomenon. Note that when written as a product of sinusoids, it is always the *lower* frequency factor which creates the 'wave-envelope' of the curve.

Note that in order to rewrite a sum or difference of sine and cosine functions with different frequencies into a product using the sum to product identities, Theorem 12.13, we need the *amplitudes* of each term to be the same. We explore more examples of these functions and this behavior in the Exercises.

12.2.2 Exercises

In Exercises 1 - 6, use the Even / Odd Identities to verify the identity. Assume all quantities are defined.

1. $\sin(3\pi - 2\theta) = -\sin(2\theta - 3\pi)$
2. $\cos\left(-\frac{\pi}{4} - 5t\right) = \cos\left(5t + \frac{\pi}{4}\right)$
3. $\tan(-x^2 + 1) = -\tan(x^2 - 1)$
4. $\csc(-\theta - 5) = -\csc(\theta + 5)$
5. $\sec(-6x) = \sec(6x)$
6. $\cot(9 - 7\theta) = -\cot(7\theta - 9)$

In Exercises 7 - 21, use the Sum and Difference Identities to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.

7. $\cos(75^\circ)$
8. $\sec(165^\circ)$
9. $\sin(105^\circ)$
10. $\csc(195^\circ)$
11. $\cot(255^\circ)$
12. $\tan(375^\circ)$
13. $\cos\left(\frac{13\pi}{12}\right)$
14. $\sin\left(\frac{11\pi}{12}\right)$
15. $\tan\left(\frac{13\pi}{12}\right)$
16. $\cos\left(\frac{7\pi}{12}\right)$
17. $\tan\left(\frac{17\pi}{12}\right)$
18. $\sin\left(\frac{\pi}{12}\right)$
19. $\cot\left(\frac{11\pi}{12}\right)$
20. $\csc\left(\frac{5\pi}{12}\right)$
21. $\sec\left(-\frac{\pi}{12}\right)$

22. If α is a Quadrant IV angle with $\cos(\alpha) = \frac{\sqrt{5}}{5}$, and $\sin(\beta) = \frac{\sqrt{10}}{10}$, where $\frac{\pi}{2} < \beta < \pi$, find
    (a) $\cos(\alpha + \beta)$
    (b) $\sin(\alpha + \beta)$
    (c) $\tan(\alpha + \beta)$
    (d) $\cos(\alpha - \beta)$
    (e) $\sin(\alpha - \beta)$
    (f) $\tan(\alpha - \beta)$

23. If $\csc(\alpha) = 3$, where $0 < \alpha < \frac{\pi}{2}$, and β is a Quadrant II angle with $\tan(\beta) = -7$, find
    (a) $\cos(\alpha + \beta)$
    (b) $\sin(\alpha + \beta)$
    (c) $\tan(\alpha + \beta)$
    (d) $\cos(\alpha - \beta)$
    (e) $\sin(\alpha - \beta)$
    (f) $\tan(\alpha - \beta)$

24. If $\sin(\alpha) = \frac{3}{5}$, where $0 < \alpha < \frac{\pi}{2}$, and $\cos(\beta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \beta < 2\pi$, find
    (a) $\sin(\alpha + \beta)$
    (b) $\cos(\alpha - \beta)$
    (c) $\tan(\alpha - \beta)$

25. If $\sec(\alpha) = -\frac{5}{3}$, where $\frac{\pi}{2} < \alpha < \pi$, and $\tan(\beta) = \frac{24}{7}$, where $\pi < \beta < \frac{3\pi}{2}$, find
    (a) $\csc(\alpha - \beta)$
    (b) $\sec(\alpha + \beta)$
    (c) $\cot(\alpha + \beta)$

In Exercises 26 - 35, use Example 12.2.7 as a guide to show that the function is a sinusoid by rewriting it in the forms $C(t) = A\cos(\omega t + \phi) + B$ and $S(t) = A\sin(\omega t + \phi) + B$ for $\omega > 0$ and $0 \leq \phi < 2\pi$.

26. $f(t) = \sqrt{2}\sin(t) + \sqrt{2}\cos(t) + 1$

27. $f(t) = 3\sqrt{3}\sin(3t) - 3\cos(3t)$

28. $f(t) = -\sin(t) + \cos(t) - 2$

29. $f(t) = -\frac{1}{2}\sin(2t) - \frac{\sqrt{3}}{2}\cos(2t)$

30. $f(t) = 2\sqrt{3}\cos(t) - 2\sin(t)$

31. $f(t) = \frac{3}{2}\cos(2t) - \frac{3\sqrt{3}}{2}\sin(2t) + 6$

32. $f(t) = -\frac{1}{2}\cos(5t) - \frac{\sqrt{3}}{2}\sin(5t)$

33. $f(t) = -6\sqrt{3}\cos(3t) - 6\sin(3t) - 3$

34. $f(t) = \frac{5\sqrt{2}}{2}\sin(t) - \frac{5\sqrt{2}}{2}\cos(t)$

35. $f(t) = 3\sin\left(\frac{t}{6}\right) - 3\sqrt{3}\cos\left(\frac{t}{6}\right)$

36. In Exercises 26 - 35, you should have noticed a relationship between the phases ϕ for the $S(t)$ and $C(t)$. Show that if $f(t) = A\sin(\omega t + \alpha) + B$, then $f(t) = A\cos(\omega t + \beta) + B$ where $\beta = \alpha - \frac{\pi}{2}$.

37. Let ϕ be an angle measured in radians and let $P(a, b)$ be a point on the terminal side of ϕ when it is drawn in standard position. Use Theorem 11.3 and the sum identity for sine in Theorem 12.7 to show that $f(t) = a\sin(\omega t) + b\cos(\omega t) + B$ (with $\omega > 0$) can be rewritten as $f(t) = \sqrt{a^2 + b^2}\sin(\omega t + \phi) + B$.

38. In Example 11.3.5 in Section 11.3, we developed two (seemingly) different formulas to model the hours of daylight, $H(t)$: $H_1(t) = 9.25\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 12.55$ and $H_2(t) = -8.13\sin\left(\frac{\pi}{6}t - 4.70\right) + 12.5$. Use the difference identities for sine to expand $H_1(t)$ and $H_2(t)$. How different are they?

In Exercises 39 - 50, verify the identity.[9]

39. $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(t)$

40. $\cos\left(\theta - \frac{\pi}{2}\right) = \sin(t)$

41. $\cos(\theta - \pi) = -\cos(\theta)$

42. $\sin(\pi - \theta) = \sin(\theta)$

43. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot(\theta)$

44. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$

45. $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos(\alpha)\sin(\beta)$

46. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos(\alpha)\cos(\beta)$

47. $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin(\alpha)\sin(\beta)$

48. $\dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \dfrac{1 + \cot(\alpha)\tan(\beta)}{1 - \cot(\alpha)\tan(\beta)}$

49. $\dfrac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \dfrac{1 - \tan(\alpha)\tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

50. $\dfrac{\tan(\alpha + \beta)}{\tan(\alpha - \beta)} = \dfrac{\sin(\alpha)\cos(\alpha) + \sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha) - \sin(\beta)\cos(\beta)}$

[9]Note: numbers 39 and 40 are the conversion formulas stated in Theorem 11.5 in Section 11.3.

In Exercise 51 - 52, use the results from Exercise 82 in Section 11.4: $\lim\limits_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$ and Exercise 78 in Section 12.1 : $\lim\limits_{\theta \to 0} \frac{1-\cos(\theta)}{\theta} = 0$ to help you find the derivatives of the sine and cosine functions.

51. (a) Verify for $f(t) = \sin(t)$, $\dfrac{f(t+h)-f(t)}{h} = \dfrac{\sin(t+h)-\sin(t)}{h} = \cos(t)\left(\dfrac{\sin(h)}{h}\right)+\sin(t)\left(\dfrac{\cos(h)-1}{h}\right)$

    (b) Show $f'(t) = \lim\limits_{h \to 0} \dfrac{f(t+h)-f(t)}{h} = \cos(t)$.

    HINT: $\lim\limits_{\theta \to 0} \frac{\cos(\theta)-1}{\theta} = \lim\limits_{\theta \to 0} \left(-\frac{1-\cos(\theta)}{\theta}\right) = -(0) = 0$

    (c) Find the equation of the tangent line to the graph of $y = \sin(t)$ at $(0,0)$, $\left(\frac{\pi}{2}, 1\right)$ and $(\pi, 0)$. Check your answers graphically.

52. (a) Verify for $g(t) = \cos(t)$, $\dfrac{g(t+h)-g(t)}{h} = \dfrac{\cos(t+h)-\cos(t)}{h} = \cos(t)\left(\dfrac{\cos(h)-1}{h}\right)-\sin(t)\left(\dfrac{\sin(h)}{h}\right)$

    (b) Show $g'(t) = \lim\limits_{h \to 0} \dfrac{g(t+h)-g(t)}{h} = -\sin(t)$.

    (c) Find the equation of the tangent line to the graph of $y = \cos(t)$ at $(0,1)$, $\left(\frac{\pi}{2}, 0\right)$ and $\left(-\frac{\pi}{2}, 0\right)$. Check your answers graphically.

53. (a) Use the fact[10] that $\lim\limits_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$ to prove $\lim\limits_{\theta \to 0} \frac{\tan(\theta)}{\theta} = 1$.

    HINT: $\frac{\tan(\theta)}{\theta} = \frac{\sin(\theta)}{\theta} \frac{1}{\cos(\theta)} \cdots$

    (b) For $f(t) = \tan(t)$, show $\dfrac{f(t+h)-f(t)}{h} = \dfrac{\tan(t+h)-\tan(t)}{h} = \left(\dfrac{\tan(h)}{h}\right)\left(\dfrac{\sec^2(t)}{1-\tan(t)\tan(h)}\right)$.

    (c) Use part 53a to show $f'(t) = \lim\limits_{h \to 0} \dfrac{f(t+h)-f(t)}{h} = \sec^2(t)$

    (d) Find the equation of the tangent line to the graph of $y = \tan(t)$ at $(0,0)$, $\left(\frac{\pi}{4}, 1\right)$ and $\left(-\frac{\pi}{4}, -1\right)$. Check your answers graphically.

In Exercises 54 - 63, use the Half Angle Formulas to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.

54. $\cos(75^\circ)$ (compare with Exercise 7)

55. $\sin(105^\circ)$ (compare with Exercise 9)

56. $\cos(67.5^\circ)$

57. $\sin(157.5^\circ)$

[10] See Exercise 82 in Section 11.4.

58. $\tan(112.5^\circ)$

59. $\cos\left(\frac{7\pi}{12}\right)$ (compare with Exercise 16)

60. $\sin\left(\frac{\pi}{12}\right)$ (compare with Exercise 18)

61. $\cos\left(\frac{\pi}{8}\right)$

62. $\sin\left(\frac{5\pi}{8}\right)$

63. $\tan\left(\frac{7\pi}{8}\right)$

In Exercises 64 - 73, use the given information about θ to find the exact values of

- $\sin(2\theta)$
- $\cos(2\theta)$
- $\tan(2\theta)$
- $\sin\left(\frac{\theta}{2}\right)$
- $\cos\left(\frac{\theta}{2}\right)$
- $\tan\left(\frac{\theta}{2}\right)$

64. $\sin(\theta) = -\frac{7}{25}$ where $\frac{3\pi}{2} < \theta < 2\pi$

65. $\cos(\theta) = \frac{28}{53}$ where $0 < \theta < \frac{\pi}{2}$

66. $\tan(\theta) = \frac{12}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

67. $\csc(\theta) = 4$ where $\frac{\pi}{2} < \theta < \pi$

68. $\cos(\theta) = \frac{3}{5}$ where $0 < \theta < \frac{\pi}{2}$

69. $\sin(\theta) = -\frac{4}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

70. $\cos(\theta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \theta < 2\pi$

71. $\sin(\theta) = \frac{5}{13}$ where $\frac{\pi}{2} < \theta < \pi$

72. $\sec(\theta) = \sqrt{5}$ where $\frac{3\pi}{2} < \theta < 2\pi$

73. $\tan(\theta) = -2$ where $\frac{\pi}{2} < \theta < \pi$

In Exercises 74 - 88, verify the identity. Assume all quantities are defined.

74. $(\cos(\theta) + \sin(\theta))^2 = 1 + \sin(2\theta)$

75. $(\cos(\theta) - \sin(\theta))^2 = 1 - \sin(2\theta)$

76. $\tan(2t) = \frac{1}{1-\tan(t)} - \frac{1}{1+\tan(t)}$

77. $\csc(2\theta) = \frac{\cot(\theta)+\tan(\theta)}{2}$

78. $8\sin^4(x) = \cos(4x) - 4\cos(2x) + 3$

79. $8\cos^4(x) = \cos(4x) + 4\cos(2x) + 3$

80. $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$

81. $\sin(4\theta) = 4\sin(\theta)\cos^3(\theta) - 4\sin^3(\theta)\cos(\theta)$

82. $32\sin^2(t)\cos^4(t) = 2 + \cos(2t) - 2\cos(4t) - \cos(6t)$

83. $32\sin^4(t)\cos^2(t) = 2 - \cos(2t) - 2\cos(4t) + \cos(6t)$

84. $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$

85. $\cos(8\theta) = 128\cos^8(\theta) - 256\cos^6(\theta) + 160\cos^4(\theta) - 32\cos^2(\theta) + 1$ (HINT: Use the result to 84.)

86. $\sec(2x) = \dfrac{\cos(x)}{\cos(x)+\sin(x)} + \dfrac{\sin(x)}{\cos(x)-\sin(x)}$

87. $\dfrac{1}{\cos(\theta)-\sin(\theta)} + \dfrac{1}{\cos(\theta)+\sin(\theta)} = \dfrac{2\cos(\theta)}{\cos(2\theta)}$

88. $\dfrac{1}{\cos(\theta)-\sin(\theta)} - \dfrac{1}{\cos(\theta)+\sin(\theta)} = \dfrac{2\sin(\theta)}{\cos(2\theta)}$

89. Suppose θ is a Quadrant I angle with $\sin(\theta) = x$. Verify the following formulas

(a) $\cos(\theta) = \sqrt{1 - x^2}$ (b) $\sin(2\theta) = 2x\sqrt{1 - x^2}$ (c) $\cos(2\theta) = 1 - 2x^2$

90. Discuss with your classmates how each of the formulas, if any, in Exercise 89 change if we change assume θ is a Quadrant II, III, or IV angle.

91. Suppose θ is a Quadrant I angle with $\tan(\theta) = x$. Verify the following formulas

(a) $\cos(\theta) = \dfrac{1}{\sqrt{x^2+1}}$ (b) $\sin(\theta) = \dfrac{x}{\sqrt{x^2+1}}$

(c) $\sin(2\theta) = \dfrac{2x}{x^2+1}$ (d) $\cos(2\theta) = \dfrac{1-x^2}{x^2+1}$

92. Discuss with your classmates how each of the formulas, if any, in Exercise 91 change if we change assume θ is a Quadrant II, III, or IV angle.

93. If $\sin(t) = x$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, find an expression for $\tan(t)$ in terms of x.

94. If $\tan(\theta) = x$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sec(\theta)$ in terms of x.

95. If $\sec(\theta) = x$ where θ is a Quadrant II angle, find an expression for $\tan(\theta)$ in terns of x.

96. If $\sin(t) = \frac{x}{2}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, find an expression for $\cos(2t)$ in terms of x.

97. If $\tan(\theta) = \frac{x}{7}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

98. If $\sec(t) = \frac{x}{4}$ for $0 < t < \frac{\pi}{2}$, find an expression for $\ln|\sec(t) + \tan(t)|$ in terms of x.

99. Show that $\cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$ for all θ.

100. Let θ be a Quadrant III angle with $\cos(\theta) = -\frac{1}{5}$. Show that this is not enough information to determine the sign of $\sin\left(\frac{\theta}{2}\right)$ by first assuming $3\pi < \theta < \frac{7\pi}{2}$ and then assuming $\pi < \theta < \frac{3\pi}{2}$ and computing $\sin\left(\frac{\theta}{2}\right)$ in both cases.

101. Without using your calculator, show that $\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$

102. In part 4 of Example 12.2.3, we wrote $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$. In Exercise 84, we had you verify an identity which expresses $\cos(4\theta)$ as a polynomial in terms of $\cos(\theta)$. Can you find a polynomial in terms of $\cos(\theta)$ for $\cos(5\theta)$? $\cos(6\theta)$? Can you find a pattern so that $\cos(n\theta)$ could be written as a polynomial in cosine for any natural number n?

103. In Exercise 80, we has you verify an identity which expresses $\sin(3\theta)$ as a polynomial in terms of $\sin(\theta)$. Can you do the same for $\sin(5\theta)$? What about for $\sin(4\theta)$? If not, what goes wrong?

In Exercises 104 - 109, verify the identity by graphing the right and left hand using a graphing utility.

104. $\sin^2(t) + \cos^2(t) = 1$

105. $\sec^2(x) - \tan^2(x) = 1$

106. $\cos(t) = \sin\left(\frac{\pi}{2} - t\right)$

107. $\tan(x + \pi) = \tan(x)$

108. $\sin(2t) = 2\sin(t)\cos(t)$

109. $\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1+\cos(x)}$

In Exercises 110 - 115, write the given product as a sum. Note: you may need to use an Even/Odd Identity to match the answer provided.

110. $\cos(3\theta)\cos(5\theta)$

111. $\sin(2t)\sin(7t)$

112. $\sin(9x)\cos(x)$

113. $\cos(2\theta)\cos(6\theta)$

114. $\sin(3t)\sin(2t)$

115. $\cos(x)\sin(3x)$

In Exercises 116 - 121, write the given sum as a product. Note: you may need to use an Even/Odd or Cofunction Identity to match the answer provided.

116. $\cos(3\theta) + \cos(5\theta)$

117. $\sin(2t) - \sin(7t)$

118. $\cos(5x) - \cos(6x)$

119. $\sin(9\theta) - \sin(-\theta)$

120. $\sin(t) + \cos(t)$

121. $\cos(x) - \sin(x)$

In Exercises 122 - 125, using the remarks following Example 12.2.7 on page 1034 as a guide, rewrite the given function $f(t)$ as a product of sinusoids. Identify the functions which create the 'wave envelope.' Check your answer by graphing the function along with the 'wave-envelope' using a graphing utility.

122. $f(t) = \cos(3t) + \cos(5t)$

123. $f(t) = 3\cos(5t) - 3\cos(6t)$

124. $f(t) = \frac{1}{2}\sin(9t) + \frac{1}{2}\sin(t)$

125. $f(t) = \frac{2}{3}\sin(2t) - \frac{2}{3}\sin(7t)$

126. Verify the Even / Odd Identities for tangent, secant, cosecant and cotangent.

127. Verify the Cofunction Identities for tangent, secant, cosecant and cotangent.

128. Verify the Difference Identities for sine and tangent.

129. Verify the Product to Sum Identities.

130. Verify the Sum to Product Identities.

12.2.3 Answers

7. $\cos(75^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}$

8. $\sec(165^\circ) = -\frac{4}{\sqrt{2}+\sqrt{6}} = \sqrt{2} - \sqrt{6}$

9. $\sin(105^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}$

10. $\csc(195^\circ) = \frac{4}{\sqrt{2}-\sqrt{6}} = -(\sqrt{2} + \sqrt{6})$

11. $\cot(255^\circ) = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$

12. $\tan(375^\circ) = \frac{3-\sqrt{3}}{3+\sqrt{3}} = 2 - \sqrt{3}$

13. $\cos\left(\frac{13\pi}{12}\right) = -\frac{\sqrt{6}+\sqrt{2}}{4}$

14. $\sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

15. $\tan\left(\frac{13\pi}{12}\right) = \frac{3-\sqrt{3}}{3+\sqrt{3}} = 2 - \sqrt{3}$

16. $\cos\left(\frac{7\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$

17. $\tan\left(\frac{17\pi}{12}\right) = 2 + \sqrt{3}$

18. $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

19. $\cot\left(\frac{11\pi}{12}\right) = -(2 + \sqrt{3})$

20. $\csc\left(\frac{5\pi}{12}\right) = \sqrt{6} - \sqrt{2}$

21. $\sec\left(-\frac{\pi}{12}\right) = \sqrt{6} - \sqrt{2}$

22. (a) $\cos(\alpha + \beta) = -\frac{\sqrt{2}}{10}$
    (b) $\sin(\alpha + \beta) = \frac{7\sqrt{2}}{10}$
    (c) $\tan(\alpha + \beta) = -7$
    (d) $\cos(\alpha - \beta) = -\frac{\sqrt{2}}{2}$
    (e) $\sin(\alpha - \beta) = \frac{\sqrt{2}}{2}$
    (f) $\tan(\alpha - \beta) = -1$

23. (a) $\cos(\alpha + \beta) = -\frac{4+7\sqrt{2}}{30}$
    (b) $\sin(\alpha + \beta) = \frac{28-\sqrt{2}}{30}$
    (c) $\tan(\alpha + \beta) = \frac{-28+\sqrt{2}}{4+7\sqrt{2}} = \frac{63-100\sqrt{2}}{41}$
    (d) $\cos(\alpha - \beta) = \frac{-4+7\sqrt{2}}{30}$
    (e) $\sin(\alpha - \beta) = -\frac{28+\sqrt{2}}{30}$
    (f) $\tan(\alpha - \beta) = \frac{28+\sqrt{2}}{4-7\sqrt{2}} = -\frac{63+100\sqrt{2}}{41}$

24. (a) $\sin(\alpha + \beta) = \frac{16}{65}$
    (b) $\cos(\alpha - \beta) = \frac{33}{65}$
    (c) $\tan(\alpha - \beta) = \frac{56}{33}$

25. (a) $\csc(\alpha - \beta) = -\frac{5}{4}$
    (b) $\sec(\alpha + \beta) = \frac{125}{117}$
    (c) $\cot(\alpha + \beta) = \frac{117}{44}$

26. $f(t) = \sqrt{2}\sin(t) + \sqrt{2}\cos(t) + 1 = 2\sin\left(t + \frac{\pi}{4}\right) + 1 = 2\cos\left(t + \frac{7\pi}{4}\right) + 1$

27. $f(t) = 3\sqrt{3}\sin(3t) - 3\cos(3t) = 6\sin\left(3t + \frac{11\pi}{6}\right) = 6\cos\left(3t + \frac{4\pi}{3}\right)$

28. $f(t) = -\sin(t) + \cos(t) - 2 = \sqrt{2}\sin\left(t + \frac{3\pi}{4}\right) - 2 = \sqrt{2}\cos\left(t + \frac{\pi}{4}\right) - 2$

29. $f(t) = -\frac{1}{2}\sin(2t) - \frac{\sqrt{3}}{2}\cos(2t) = \sin\left(2t + \frac{4\pi}{3}\right) = \cos\left(2t + \frac{5\pi}{6}\right)$

30. $f(t) = 2\sqrt{3}\cos(t) - 2\sin(t) = 4\sin\left(t + \frac{2\pi}{3}\right) = 4\cos\left(t + \frac{\pi}{6}\right)$

31. $f(t) = \frac{3}{2}\cos(2t) - \frac{3\sqrt{3}}{2}\sin(2t) + 6 = 3\sin\left(2t + \frac{5\pi}{6}\right) + 6 = 3\cos\left(2t + \frac{\pi}{3}\right) + 6$

32. $f(t) = -\frac{1}{2}\cos(5t) - \frac{\sqrt{3}}{2}\sin(5t) = \sin\left(5t + \frac{7\pi}{6}\right) = \cos\left(5t + \frac{2\pi}{3}\right)$

33. $f(t) = -6\sqrt{3}\cos(3t) - 6\sin(3t) - 3 = 12\sin\left(3t + \frac{4\pi}{3}\right) - 3 = 12\cos\left(3t + \frac{5\pi}{6}\right) - 3$

34. $f(t) = \frac{5\sqrt{2}}{2}\sin(t) - \frac{5\sqrt{2}}{2}\cos(t) = 5\sin\left(t + \frac{7\pi}{4}\right) = 5\cos\left(t + \frac{5\pi}{4}\right)$

35. $f(t) = 3\sin\left(\frac{t}{6}\right) - 3\sqrt{3}\cos\left(\frac{t}{6}\right) = 6\sin\left(\frac{t}{6} + \frac{5\pi}{3}\right) = 6\cos\left(\frac{t}{6} + \frac{7\pi}{6}\right)$

51. (d) at $(0, 0)$: $y = x$; at $\left(\frac{\pi}{2}, 1\right)$: $y = 1$ at $(\pi, 0)$: $y = -x + \pi$

52. (d) at $(0, 1)$: $y = 1$; at $\left(\frac{\pi}{2}, 0\right)$: $y = -x + \frac{\pi}{2}$ at $\left(-\frac{\pi}{2}, 0\right)$: $y = x + \frac{\pi}{2}$

53. (e) at $(0, 0)$: $y = x$; at $\left(\frac{\pi}{4}, 1\right)$: $y = 2x - \frac{\pi}{2} + 1$ at $\left(-\frac{\pi}{4}, -1\right)$: $y = 2x + \frac{\pi}{2} - 1$

54. $\cos(75^\circ) = \frac{\sqrt{2-\sqrt{3}}}{2}$

55. $\sin(105^\circ) = \frac{\sqrt{2+\sqrt{3}}}{2}$

56. $\cos(67.5^\circ) = \frac{\sqrt{2-\sqrt{2}}}{2}$

57. $\sin(157.5^\circ) = \frac{\sqrt{2-\sqrt{2}}}{2}$

58. $\tan(112.5^\circ) = -\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = -1 - \sqrt{2}$

59. $\cos\left(\frac{7\pi}{12}\right) = -\frac{\sqrt{2-\sqrt{3}}}{2}$

60. $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$

61. $\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$

62. $\sin\left(\frac{5\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$

63. $\tan\left(\frac{7\pi}{8}\right) = -\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} = 1 - \sqrt{2}$

64.
- $\sin(2\theta) = -\frac{336}{625}$
- $\cos(2\theta) = \frac{527}{625}$
- $\tan(2\theta) = -\frac{336}{527}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{2}}{10}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{7\sqrt{2}}{10}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{1}{7}$

65.
- $\sin(2\theta) = \frac{2520}{2809}$
- $\cos(2\theta) = -\frac{1241}{2809}$
- $\tan(2\theta) = -\frac{2520}{1241}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{5\sqrt{106}}{106}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{9\sqrt{106}}{106}$
- $\tan\left(\frac{\theta}{2}\right) = \frac{5}{9}$

66.
- $\sin(2\theta) = \frac{120}{169}$
- $\cos(2\theta) = -\frac{119}{169}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{3\sqrt{13}}{13}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{2\sqrt{13}}{13}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{3}{2}$

67.
- $\sin(2\theta) = -\frac{\sqrt{15}}{8}$
- $\cos(2\theta) = \frac{7}{8}$
- $\tan(2\theta) = -\frac{\sqrt{15}}{7}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{8+2\sqrt{15}}}{4}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{8-2\sqrt{15}}}{4}$
- $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}}}$
  $\tan\left(\frac{\theta}{2}\right) = 4 + \sqrt{15}$

68.
- $\sin(2\theta) = \frac{24}{25}$
- $\cos(2\theta) = -\frac{7}{25}$
- $\tan(2\theta) = -\frac{24}{7}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{5}}{5}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{2\sqrt{5}}{5}$
- $\tan\left(\frac{\theta}{2}\right) = \frac{1}{2}$

69.
- $\sin(2\theta) = \frac{24}{25}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{2\sqrt{5}}{5}$
- $\cos(2\theta) = -\frac{7}{25}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{\sqrt{5}}{5}$
- $\tan(2\theta) = -\frac{24}{7}$
- $\tan\left(\frac{\theta}{2}\right) = -2$

70.
- $\sin(2\theta) = -\frac{120}{169}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{26}}{26}$
- $\cos(2\theta) = \frac{119}{169}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{5\sqrt{26}}{26}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{1}{5}$

71.
- $\sin(2\theta) = -\frac{120}{169}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{5\sqrt{26}}{26}$
- $\cos(2\theta) = \frac{119}{169}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{26}}{26}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\tan\left(\frac{\theta}{2}\right) = 5$

72.
- $\sin(2\theta) = -\frac{4}{5}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{50-10\sqrt{5}}}{10}$
- $\cos(2\theta) = -\frac{3}{5}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{\sqrt{50+10\sqrt{5}}}{10}$
- $\tan(2\theta) = \frac{4}{3}$
- $\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{5-\sqrt{5}}{5+\sqrt{5}}}$
  $\tan\left(\frac{\theta}{2}\right) = \frac{5-5\sqrt{5}}{10}$

73.
- $\sin(2\theta) = -\frac{4}{5}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{50+10\sqrt{5}}}{10}$
- $\cos(2\theta) = -\frac{3}{5}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{50-10\sqrt{5}}}{10}$
- $\tan(2\theta) = \frac{4}{3}$
- $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}}$
  $\tan\left(\frac{\theta}{2}\right) = \frac{5+5\sqrt{5}}{10}$

93. $\tan(t) = \frac{x}{\sqrt{1-x^2}}$

94. $\sec(\theta) = \sqrt{1+x^2}$

95. $\tan(\theta) = -\sqrt{x^2-1}$

96. $\cos(2t) = 1 - \dfrac{x^2}{2}$

97. $\sin(2\theta) = \dfrac{14x}{x^2+49}$

98. $\ln|\sec(t)+\tan(t)| = \ln\left|x+\sqrt{x^2+16}\right| - \ln(4)$

110. $\dfrac{\cos(2\theta)+\cos(8\theta)}{2}$

111. $\dfrac{\cos(5t)-\cos(9t)}{2}$

112. $\dfrac{\sin(8x)+\sin(10x)}{2}$

113. $\dfrac{\cos(4\theta)+\cos(8\theta)}{2}$

114. $\dfrac{\cos(t)-\cos(5t)}{2}$

115. $\dfrac{\sin(2x)+\sin(4x)}{2}$

116. $2\cos(4\theta)\cos(\theta)$

117. $-2\cos\left(\frac{9}{2}t\right)\sin\left(\frac{5}{2}t\right)$

118. $2\sin\left(\frac{11}{2}x\right)\sin\left(\frac{1}{2}x\right)$

119. $2\cos(4\theta)\sin(5\theta)$

120. $\sqrt{2}\cos\left(t-\frac{\pi}{4}\right)$

121. $-\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$

122. $f(t) = [2\cos(t)]\cos(4t)$, $A(t) = 2\cos(t)$, wave-envelope: $y = \pm 2\cos(t)$.

123. $f(t) = \left[6\sin\left(\frac{1}{2}t\right)\right]\sin\left(\frac{11}{2}t\right)$, $A(t) = 6\sin\left(\frac{1}{2}t\right)$, wave-envelope: $y = \pm 6\sin\left(\frac{1}{2}t\right)$.

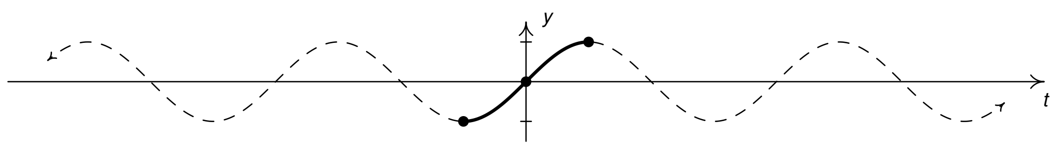
124. $f(t) = [\cos(4t)]\sin(5t)$, $A(t) = \cos(4t)$, wave-envelope: $y = \pm\cos(4t)$.

125. $f(t) = \left[-\frac{4}{3}\sin\left(\frac{5}{2}t\right)\right]\cos\left(\frac{9}{2}t\right)$, $A(t) = -\frac{4}{3}\sin\left(\frac{5}{2}t\right)$, wave-envelope: $y = \pm\frac{4}{3}\sin\left(\frac{5}{2}t\right)$.

12.3 The Inverse Circular Functions

In this section we concern ourselves with finding inverses of the circular (trigonometric) functions.[1] Our immediate problem is that, owing to their periodic nature, none of the six circular functions is one-to-one. To remedy this, we restrict the domains of the circular functions in the same way we restricted the domain of the quadratic function in Example 5.6.3 in Section 5.6 to obtain a one-to-one function.

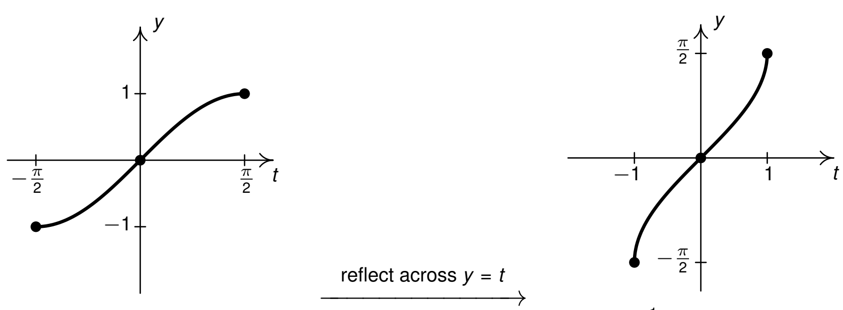
We start with $f(t) = \sin(t)$ and restrict our domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in order to keep the range as $[-1, 1]$ as well as the properties of being smooth and continuous.



Restricting the domain of $f(t) = \sin(t)$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

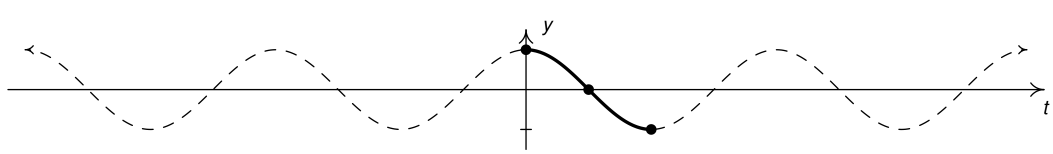
Recall from Section 5.6 that the inverse of a function f is typically denoted f^{-1}. For this reason, some textbooks use the notation $f^{-1}(t) = \sin^{-1}(t)$ for the inverse of $f(t) = \sin(t)$. The obvious pitfall here is our convention of writing $(\sin(t))^2$ as $\sin^2(t)$, $(\sin(t))^3$ as $\sin^3(t)$ and so on. It is far too easy to confuse $\sin^{-1}(t)$ with $\frac{1}{\sin(t)} = \csc(t)$ so we will not use this notation in our text.[2]

Instead, we use the notation $f^{-1}(t) = \arcsin(t)$, read 'arc-sine of t'. We'll explain the 'arc' in 'arcsine' shortly. For now, we graph $f(t) = \sin(t)$ and $f^{-1}(t) = \arcsin(t)$, where we obtain the latter from the former by reflecting it across the line $y = t$, in accordance with Theorem 5.13.



$f(t) = \sin(t)$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. $\qquad$ $f^{-1}(t) = \arcsin(t)$.

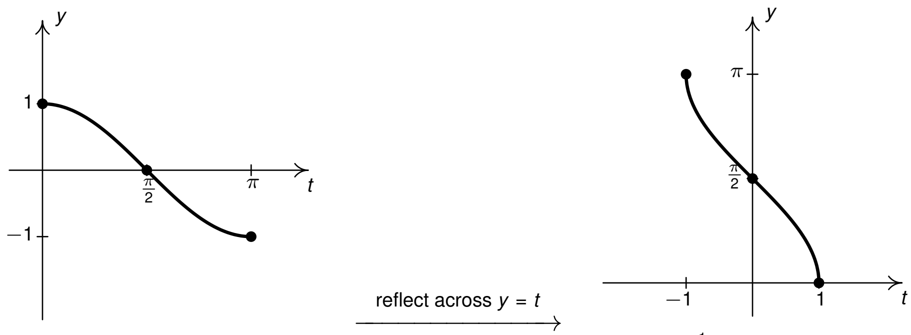
Next, we consider $g(t) = \cos(t)$. Here, we select the interval $[0, \pi]$ for our restriction.



Restricting the domain of $f(t) = \cos(t)$ to $[0, \pi]$.

[1]We have already discussed this concept in Section B.2 as the 'angle finder' in the context of acute angles in right triangles.
[2]But be aware that many books do! As always, be sure to check the context!

Reflecting the across the line $y = t$ produces the graph $y = g^{-1}(t) = \arccos(t)$.



$f(t) = \cos(t)$, $0 \leq t \leq \pi$ $\qquad$ $f^{-1}(t) = \arccos(t)$.

We list some important facts about the arcsine and arccosine functions in the following theorem.[3] Everything in Theorem 12.14 is a direct consequence of Theorem 5.13 as applied to the (restricted) sine and cosine functions, and as such, its proof is left to the reader.

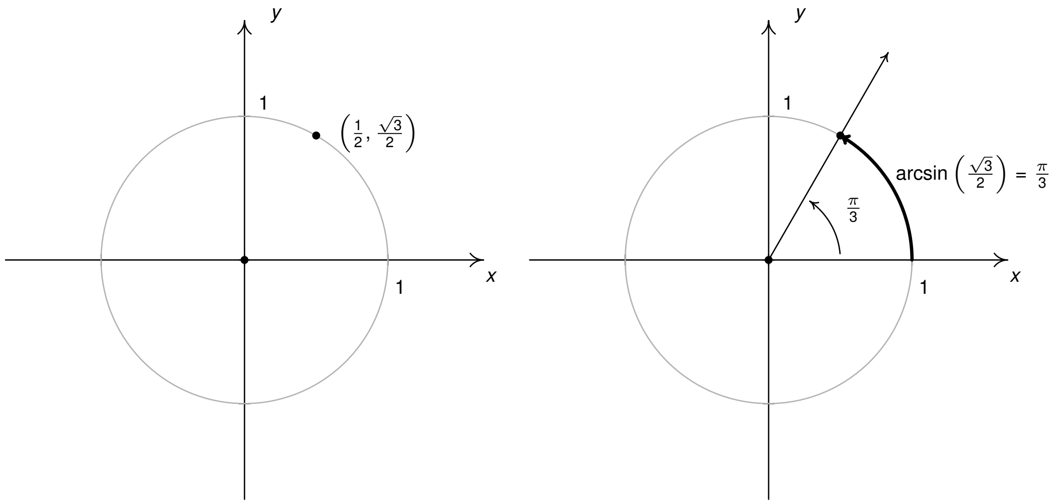
Theorem 12.14. Properties of the Arccosine and Arcsine Functions

- Properties of $F(x) = \arcsin(x)$
  - Domain: $[-1, 1]$
  - Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\arcsin(x) = t$ if and only if $\sin(t) = x$ and $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
  - $\sin(\arcsin(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arcsin(\sin(t)) = t$ provided $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
  - $F(x) = \arcsin(x)$ is odd
- Properties of $G(x) = \arccos(x)$
  - Domain: $[-1, 1]$
  - Range: $[0, \pi]$
  - $\arccos(x) = t$ if and only if $\cos(t) = x$ and $0 \leq t \leq \pi$
  - $\cos(\arccos(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arccos(\cos(t)) = t$ provided $0 \leq t \leq \pi$

Before moving to an example, we take a moment to understand the 'arc' in 'arcsine.' Consider the figure below which illustrates the specific case of $\arcsin\left(\frac{\sqrt{3}}{2}\right)$.

[3]We switch the input variable to the arcsine and arccosine functions to 'x' to avoid confusion with the outputs we label 't.'

By definition, the real number $t = \arcsin\left(\frac{\sqrt{3}}{2}\right)$ satisfies $\sin(t) = \frac{\sqrt{3}}{2}$ with $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. In other words, we are looking for angle measuring t radians between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with a sine of $\frac{\sqrt{3}}{2}$. Hence, $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. In terms of oriented arcs[4], if we start at $(1, 0)$ and travel along the Unit Circle in the positive (counterclockwise) direction for $\frac{\pi}{3}$ units, we will arrive at the point whose y-coordinate is $\frac{\sqrt{3}}{2}$. Hence, the real number $\frac{\pi}{3}$ also corresponds to 'arc' corresponding to the 'sine' that is $\frac{\sqrt{3}}{2}$.



y-value on Unit Circle is $\frac{\sqrt{3}}{2}$. corresponding oriented arc: $t = \frac{\pi}{3}$

In general, the function $f(t) = \sin(t)$ takes a real number input t, associates it with the angle $\theta = t$ radians, and returns the value $\sin(\theta)$. The value $\sin(\theta) = \sin(t)$ is the y-coordinate of the terminal point on the Unit Circle of an oriented arc of length $|t|$ whose initial point is $(1, 0)$.

Hence, we may view the inputs to $f(t) = \sin(t)$ as oriented arcs and the outputs as y-coordinates on the Unit Circle. Therefore, the function f^{-1} reverses this process and takes y-coordinates on the Unit Circle and return oriented arcs, hence the 'arc' in arcsine.

It is high time for an example.

Example 12.3.1.

1. Find the exact values of the following.

(a) $\arcsin\left(\frac{\sqrt{2}}{2}\right)$

(b) $\arccos\left(\frac{1}{2}\right)$

(c) $\arcsin\left(-\frac{1}{2}\right)$

(d) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$

[4]See page 909 if you need a review of how we associate real numbers with angles in radian measure.

(e) $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$

(f) $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right)$

(g) $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$

(h) $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$

2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

   (a) $f(x) = \tan(\arccos(x))$

   (b) $g(x) = \cos(2\arcsin(x))$

Solution.
The best way to approach these problems is to remember that $\arcsin(x)$ and $\arccos(x)$ are real numbers which correspond to the radian measure of angles that fall within a certain prescribed range.

1. (a) To find $\arcsin\left(\frac{\sqrt{2}}{2}\right)$, we need the angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin(t) = \frac{\sqrt{2}}{2}$. Hence, $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

   (b) To find $\arccos\left(\frac{1}{2}\right)$, we are looking for the angle measuring t radians which lies between 0 and π that has $\cos(t) = \frac{1}{2}$. Our answer is $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (c) For $\arcsin\left(-\frac{1}{2}\right)$, we are looking for an angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin(t) = -\frac{1}{2}$. Hence, $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

   Alternatively, we could use the fact that the arcsine function is odd, so $\arcsin\left(-\frac{1}{2}\right) = -\arcsin\left(\frac{1}{2}\right)$. We find $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$, so $\arcsin\left(-\frac{1}{2}\right) = -\arcsin\left(\frac{1}{2}\right) = -\frac{\pi}{6}$.

   (d) For $\arccos\left(-\frac{\sqrt{2}}{2}\right)$, we need the angle measuring t radians which lies between 0 and π with $\cos(t) = -\frac{\sqrt{2}}{2}$. Hence, $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

   (e) Since $0 \leq \frac{\pi}{6} \leq \pi$, we could simply invoke Theorem 12.14 to get $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$.

   However, in order to make sure we understand *why* this is the case, we choose to work the example through using the definition of arccosine.

   Working from the inside out, $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. To find $\arccos\left(\frac{\sqrt{3}}{2}\right)$, we need an angle measuring t radians which lies between 0 and π that has $\cos(t) = \frac{\sqrt{3}}{2}$. We get $t = \frac{\pi}{6}$, so that $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

   (f) Since $\frac{11\pi}{6}$ does not fall between 0 and π, Theorem 12.14 does not apply. We are forced to work through from the inside out starting with $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. From the previous problem, we know $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$. Hence, $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$.

   (g) One way to simplify $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$ is to use Theorem 12.14 directly. Since $-\frac{3}{5}$ is between -1 and 1, we have that $\cos\left(\arccos\left(-\frac{3}{5}\right)\right) = -\frac{3}{5}$ and we are done.

   However, as before, to really understand *why* this cancellation occurs, we let $t = \arccos\left(-\frac{3}{5}\right)$. By definition, $\cos(t) = -\frac{3}{5}$. Hence, $\cos\left(\arccos\left(-\frac{3}{5}\right)\right) = \cos(t) = -\frac{3}{5}$, and we are finished in (nearly) the same amount of time.

(h) As in the previous example, we let $t = \arccos\left(-\frac{3}{5}\right)$ so that $\cos(t) = -\frac{3}{5}$ for some angle measuring t radians between 0 and π.

Since $\cos(t) < 0$, we can narrow this down a bit and conclude that $\frac{\pi}{2} < t < \pi$, so that t corresponds to an angle in Quadrant II.

In terms of t, then, we need to find $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \sin(t)$, and since we know $\cos(t)$, the fastest route is through the Pythagorean Identity.

We get $\sin^2(t) = 1 - \cos^2(t) = 1 - \left(-\frac{3}{5}\right)^2 = \frac{16}{25}$. Since t corresponds to a Quadrant II angle, we choose the positive root, $\sin(t) = \frac{4}{5}$, so $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \frac{4}{5}$.

2. (a) We begin this problem in the same manner we began the previous two problems. We let $t = \arccos(x)$, so our goal is to find a way to express $\tan(\arccos(x)) = \tan(t)$ in terms of x.

Since $t = \arccos(x)$, we know $\cos(t) = x$ where $0 \leq t \leq \pi$. One approach[5] to finding $\tan(t)$ is to use the quotient identity $\tan(t) = \frac{\sin(t)}{\cos(t)}$. Since we know $\cos(t)$, we just need to find $\sin(t)$.

Using the Pythagorean Identity, we get $\sin^2(t) = 1 - \cos^2(t) = 1 - x^2$ so that $\sin(t) = \pm\sqrt{1-x^2}$. Since $0 \leq t \leq \pi$, $\sin(t) \geq 0$, so we choose $\sin(t) = \sqrt{1-x^2}$.

Thus, $\tan(t) = \frac{\sin(t)}{\cos(t)} = \frac{\sqrt{1-x^2}}{x}$, so $f(x) = \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$.

To determine the domain, we harken back to Section 5.3. The function $f(x) = \tan(\arccos(x))$ can be thought of as a two step process: first, take the arccosine of a number, and second, take the tangent of whatever comes out of the arccosine.

Since the domain of $\arccos(x)$ is $-1 \leq x \leq 1$, the domain of f will be some subset of $[-1, 1]$. The range of $\arccos(x)$ is $[0, \pi]$, and of these values, only $\frac{\pi}{2}$ will cause a problem for the tangent function. Since $\arccos(x) = \frac{\pi}{2}$ happens when $x = \cos\left(\frac{\pi}{2}\right) = 0$, we exclude $x = 0$ from our domain. Hence, the domain of $f(x) = \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$ is $[-1, 0) \cup (0, 1]$.

Note that *in this particular case*, we could have obtained the correct domain of f using its algebraic description: $f(x) = \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$. This is not always true, however, as we'll see in the next problem.

(b) We proceed as in the previous problem by writing $t = \arcsin(x)$ so that t lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(t) = x$. We aim to express $\cos(2\arcsin(x)) = \cos(2t)$ in terms of x.

Thanks to Theorem 12.9, we have three choices for rewriting $\cos(2t)$: $\cos(2t) = \cos^2(t) - \sin^2(t)$, $\cos(2t) = 2\cos^2(t) - 1$ and $\cos(2t) = 1 - 2\sin^2(t)$.

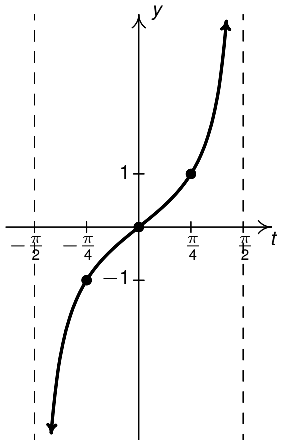
Since we know $x = \sin(t)$, we choose: $\cos(2\arcsin(x)) = \cos(2t) = 1 - 2\sin^2(t) = 1 - 2x^2$. Hence, $g(x) = \cos(2\arcsin(x)) = 1 - 2x^2$.

To find the domain of $g(x) = \cos(2\arcsin(x))$, we once again appeal to what we learned in Section 5.3. The domain of $\arcsin(x)$ is $[-1, 1]$, and since there are no domain restrictions on cosine, the domain of g is $[-1, 1]$.

[5] Alternatively, we could use the identity: $1 + \tan^2(t) = \sec^2(t)$. Since $x = \cos(t)$, $\sec(t) = \frac{1}{\cos(t)} = \frac{1}{x}$. The reader is invited to work through this approach to see what, if any, difficulties arise.

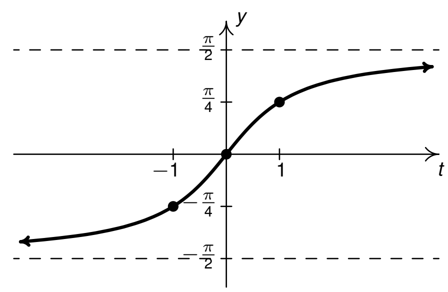
It is important to note that in this case, even though the algebraic expression $1 - 2x^2$ is defined for all real numbers, the domain of g is limited to that of $\arcsin(x)$, namely $[-1, 1]$. The adage 'find the domain *before* you simplify' rings as true here as it did in Chapter 5. □

The next pair of functions we wish to discuss are the inverses of tangent and cotangent. First, we restrict $f(t) = \tan(t)$ to its fundamental cycle on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to obtain the arctangent function, $f^{-1}(t) = \arctan(t)$. Among other things, note that the *vertical* asymptotes $t = -\frac{\pi}{2}$ and $t = \frac{\pi}{2}$ of the graph of $f(t) = \tan(t)$ become the *horizontal* asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ of the graph of $f^{-1}(t) = \arctan(t)$.



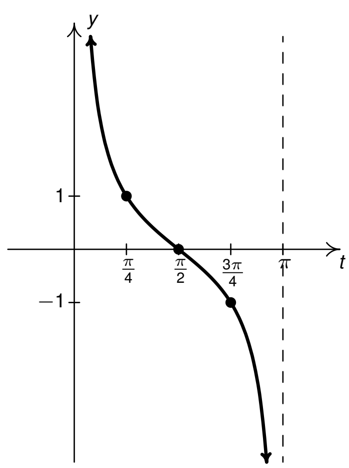
$f(t) = \tan(t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

reflect across $y = t$ $\longrightarrow$ switch t and y coordinates



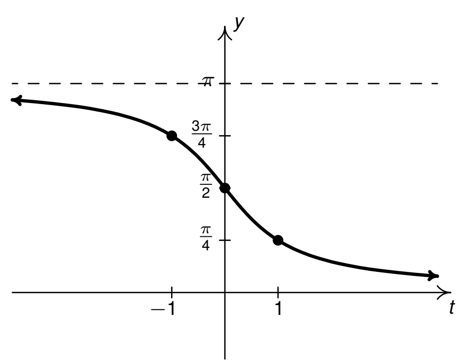
$f^{-1}(t) = \arctan(t)$.

Next, we restrict $g(t) = \cot(t)$ to its fundamental cycle on $(0, \pi)$ to obtain $g^{-1}(t) = \operatorname{arccot}(t)$, the arccotangent function. Once again, the vertical asymptotes $t = 0$ and $t = \pi$ of the graph of $g(t) = \cot(t)$ become the horizontal asymptotes $y = 0$ and $y = \pi$ of the graph of $g^{-1}(t) = \operatorname{arccot}(t)$.



$g(t) = \cot(t)$, $0 < t < \pi$.

reflect across $y = t$ $\longrightarrow$ switch t and y coordinates



$g^{-1}(t) = \operatorname{arccot}(t)$.

Below we summarize the important properties of the arctangent and arccotangent functions.

Theorem 12.15. Properties of the Arctangent and Arccotangent Functions

- Properties of $F(x) = \arctan(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - $\displaystyle\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}$; $\displaystyle\lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}$
  - $\arctan(x) = t$ if and only if $\tan(t) = x$ and $-\frac{\pi}{2} < t < \frac{\pi}{2}$
  - $\arctan(x) = \operatorname{arccot}\left(\frac{1}{x}\right)$ for $x > 0$
  - $\tan(\arctan(x)) = x$ for all real numbers x
  - $\arctan(\tan(t)) = t$ provided $-\frac{\pi}{2} < t < \frac{\pi}{2}$
  - $F(x) = \arctan(x)$ is odd
- Properties of $G(x) = \operatorname{arccot}(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \pi)$
  - $\displaystyle\lim_{x \to -\infty} \operatorname{arccot}(x) = \pi$; $\displaystyle\lim_{x \to \infty} \operatorname{arccot}(x) = 0$
  - $\operatorname{arccot}(x) = t$ if and only if $\cot(t) = x$ and $0 < t < \pi$
  - $\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$ for $x > 0$
  - $\cot(\operatorname{arccot}(x)) = x$ for all real numbers x
  - $\operatorname{arccot}(\cot(t)) = t$ provided $0 < t < \pi$

The properties listed in Theorem 12.15 are consequences of the definitions of the arctangent and arccotangent functions along with Theorem 5.13, and its proof is left to the reader.

Example 12.3.2.

1. Find the exact values of the following.

   (a) $\arctan(\sqrt{3})$

   (b) $\operatorname{arccot}(-\sqrt{3})$

   (c) $\cot(\operatorname{arccot}(-5))$

   (d) $\sin\left(\arctan\left(-\frac{4}{3}\right)\right)$

2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

   (a) $\tan(2\arctan(x))$ (b) $\cos(\operatorname{arccot}(2x))$

Solution.

1. (a) To find $\arctan(\sqrt{3})$, we need the angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan(t) = \sqrt{3}$. We find $\arctan(\sqrt{3}) = \frac{\pi}{3}$.

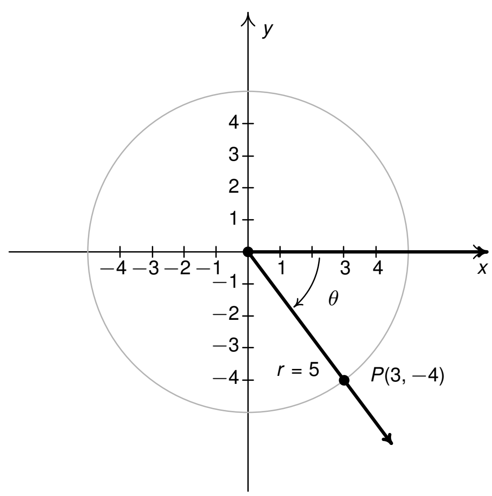
   (b) To find $\operatorname{arccot}(-\sqrt{3})$, we need the angle measuring t radians which lies between 0 and π with $\cot(t) = -\sqrt{3}$. Hence, $\operatorname{arccot}(-\sqrt{3}) = \frac{5\pi}{6}$.

   (c) We can apply Theorem 12.15 directly and obtain $\cot(\operatorname{arccot}(-5)) = -5$. However, working it through provides us with yet another opportunity to understand why this is the case.

   Letting $t = \operatorname{arccot}(-5)$, by definition, $\cot(t) = -5$. Hence, $\cot(\operatorname{arccot}(-5)) = \cot(t) = -5$.

   (d) We start simplifying $\sin\left(\arctan\left(-\frac{4}{3}\right)\right)$ by letting $t = \arctan\left(-\frac{4}{3}\right)$. By definition, $\tan(t) = -\frac{4}{3}$ for some angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since $\tan(t) < 0$, we know, in fact, t corresponds to a Quadrant IV angle.

   We are given $\tan(t)$ but wish to know $\sin(t)$. Since there is no direct identity to marry the two, we make a quick sketch of the situation below. Since $\tan(t) = -\frac{4}{3} = \frac{-4}{3}$, we take $P(3, -4)$ as a point on the terminal side of $\theta = t = \arctan\left(-\frac{4}{3}\right)$ radians.



$P(3, -4)$ is on the terminal side of θ.

   We find $r = \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = 5$, so $\sin(t) = -\frac{4}{5}$. Hence, $\sin\left(\arctan\left(-\frac{4}{3}\right)\right) = -\frac{4}{5}$.

2. (a) We proceed as above and let $t = \arctan(x)$. We have $\tan(t) = x$ where $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Our goal is to express $\tan(2\arctan(x)) = \tan(2t)$ in terms of x.

   From Theorem 12.9, we know $\tan(2t) = \frac{2\tan(t)}{1-\tan^2(t)} = \frac{2x}{1-x^2}$. Hence $f(x) = \tan(2\arctan(x)) = \frac{2x}{1-x^2}$.

To find the domain, we once again think of $f(x) = \tan(2\arctan(x))$ as a sequence of steps and work from the inside out.

The first step is to find the arctangent of a real number. Since the domain of $\arctan(x)$ is all real numbers, we have no restrictions here and we get out all values $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The next step is to multiply $\arctan(x)$ by 2. There are no restrictions here, either. Since the range of $\arctan(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the range of $2\arctan(x)$ is $(-\pi, \pi)$.

The last step is to take the tangent of $2\arctan(x)$. Since we are taking the tangent of values in the interval $(-\pi, \pi)$, we will run into trouble if $2\arctan(x) = \pm\frac{\pi}{2}$, that is, if $\arctan(x) = \pm\frac{\pi}{4}$. Since this happens exactly when $x = \tan\left(\pm\frac{\pi}{4}\right) = \pm 1$, we must exclude $x = \pm 1$ from the domain of f.

Hence, the domain of $f(x) = \tan(2\arctan(x))$ is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. In this example, we could have obtained the correct answer by looking at the algebraic equivalence, $f(x) = \frac{2x}{1-x^2}$. However, as we saw in Example 12.3.1, number 2b, this is not always the case.

(b) To get started, we let $t = \operatorname{arccot}(2x)$ so that $\cot(t) = 2x$ where $0 < t < \pi$. In terms of t, $\cos(\operatorname{arccot}(2x)) = \cos(t)$, and our goal is to express the latter in terms of x.

One way to proceed is to rewrite the identity $\cot(t) = \frac{\cos(t)}{\sin(t)}$ as $\cos(t) = \cot(t)\sin(t)$ and use the fact that $\cot(t) = 2x$ to find $\sin(t)$ in terms of x. This isn't as hopeless as it might seem, since the Pythagorean Identity $\csc^2(t) = 1 + \cot^2(t)$ relates cotangent to cosecant, and $\sin(t) = \frac{1}{\csc(t)}$.

Following this strategy, we get $\csc^2(t) = 1 + \cot^2(t) = 1 + (2x)^2 = 1 + 4x^2$ so $\csc(t) = \pm\sqrt{4x^2+1}$. Since t is between 0 and π, $\csc(t) > 0$. Hence, $\csc(t) = \sqrt{4x^2+1}$, so $\sin(t) = \frac{1}{\csc(t)} = \frac{1}{\sqrt{4x^2+1}}$.

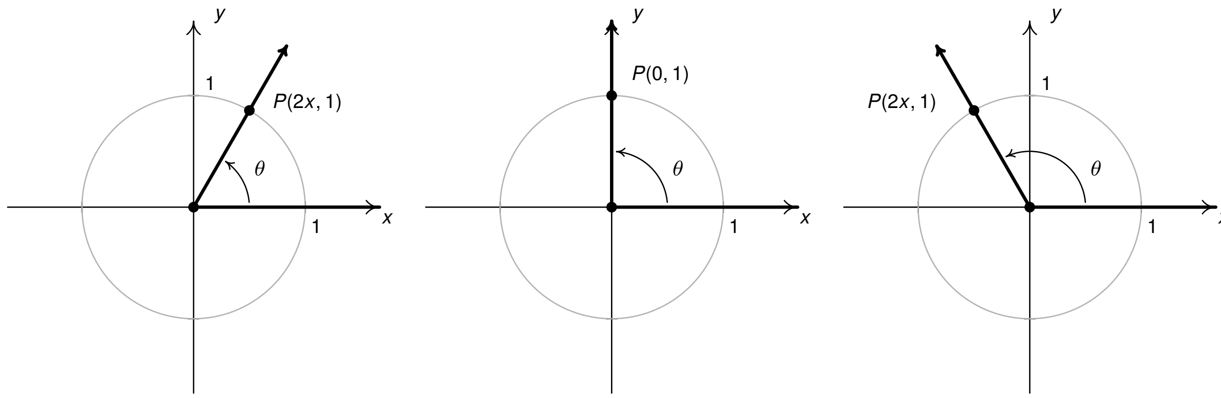
We find $\cos(t) = \cot(t)\sin(t) = \frac{2x}{\sqrt{4x^2+1}}$. Hence, $g(x) = \cos(\operatorname{arccot}(2x)) = \frac{2x}{\sqrt{4x^2+1}}$.

Viewing $g(x) = \cos(\operatorname{arccot}(2x))$ as a sequence of steps, we see we first double the input x, then take the arccotangent, and, finally, take the cosine. Since each of these processes are valid for all real numbers, the domain of g is $(-\infty, \infty)$. $\square$

The reader may well wonder if there isn't a more direct way to handle Example 12.3.2 number 2b. Indeed, we can take some inspiration from Section 11.4 and imagine an angle θ measuring t radians so that $\cot(\theta) = \cot(t) = 2x$ where $0 < \theta < \pi$.

Thinking of $\cot(\theta)$ as a ratio of coordinates on a circle, we may rewrite $\cot(\theta) = 2x = \frac{2x}{1}$ and we would like to identify a point $P(2x, 1)$ on the terminal side of θ.

We need to be careful here. Since $\cot(\theta) = 2x$, $x = \frac{1}{2}\cot(\theta)$, so as θ ranges between 0 and π, x can take on positive, negative or 0 values. We need to argue that the point $P(2x, 1)$ lies in the quadrant we expect (as depicted below) in all cases before we delve too far into our analysis.



$0 < \theta < \frac{\pi}{2}$, $x > 0$. $\theta = \frac{\pi}{2}$, $x = 0$. $\frac{\pi}{2} < \theta < \pi$, $x < 0$.

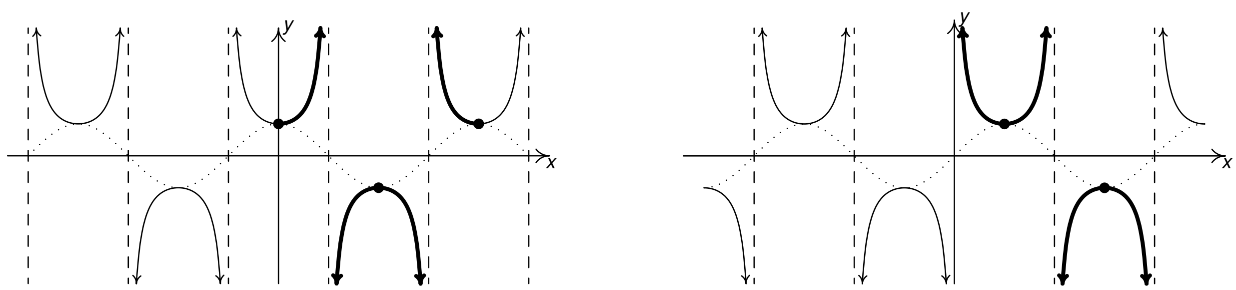
If $0 < \theta < \frac{\pi}{2}$, then $\cot(\theta) > 0$. Hence, $x > 0$ so the point $P(2x, 1)$ is in Quadrant I, as required. If $\theta = \frac{\pi}{2}$, then $x = 0$, and our point $P(2x, 1) = (0, 1)$, as required. If $\frac{\pi}{2} < \theta < \pi$, then $\cot(\theta) < 0$. Hence, $x < 0$, so $P(2x, 1)$ is in Quadrant II, as required.

Hence, in all three cases, our formula for the point $P(2x, 1)$ determines a point in the same quadrant as the terminal side of θ, as illustrated above.

This allows us to use Theorem 11.9 from Section 11.4. We find $r = \sqrt{(2x)^2 + 1^2} = \sqrt{4x^2 + 1}$, and hence, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$, which agrees with our answer from Example 12.3.2.

It shouldn't surprise the reader that there are some cases where the approach outlined above doesn't go as smoothly (as we'll see in the discussion following Example 12.3.3.)

The last two functions to invert are secant and cosecant. A portion of each of their graphs, which were first discussed in Subsection 11.5.1, are given below with the fundamental cycles highlighted.



The graph of $y = \sec(x)$. The graph of $y = \csc(x)$.

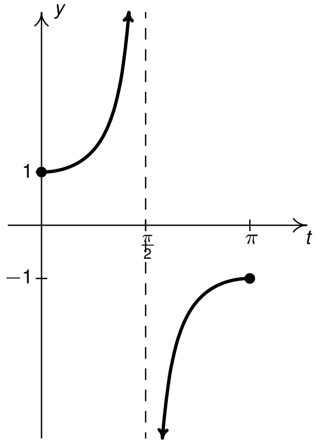
It is clear from the graph of secant that we cannot find one single continuous piece of its graph which covers its entire range of $(-\infty, -1] \cup [1, \infty)$ and restricts the domain of the function so that it is one-to-one. The same is true for cosecant.

Thus in order to define the arcsecant and arccosecant functions, we must settle for a piecewise approach wherein we choose one piece to cover the top of the range, namely $[1, \infty)$, and another piece to cover the bottom, namely $(-\infty, -1]$.

There are two generally accepted ways make these choices which restrict the domains of these functions so that they are one-to-one. One approach simplifies the Trigonometry associated with the inverse functions, but complicates the Calculus; the other makes the Calculus easier, but the Trigonometry less so. For completeness, we present both points of view, each in its own subsection.

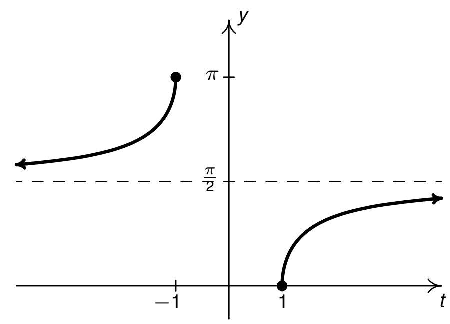
12.3.1 Inverses of Secant and Cosecant: Trigonometry Friendly Approach

In this subsection, we restrict the secant and cosecant functions to coincide with the restrictions on cosine and sine, respectively. For $f(t) = \sec(t)$, we restrict the domain to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



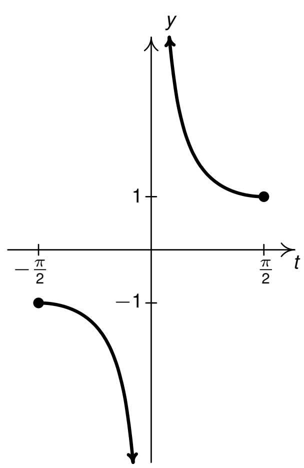
$f(t) = \sec(t)$ on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

reflect across $y = t$ $\xrightarrow{\hspace{2cm}}$ switch t and y coordinates



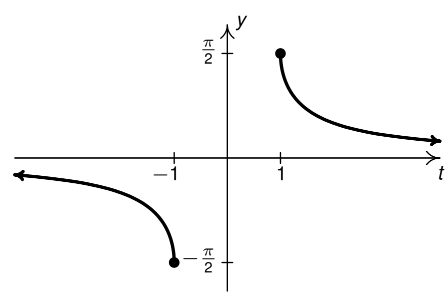
$f^{-1}(t) = \operatorname{arcsec}(t)$

and we restrict $g(t) = \csc(t)$ to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.



$g(t) = \csc(t)$ on $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

reflect across $y = t$ $\xrightarrow{\hspace{2cm}}$ switch t and y coordinates



$g^{-1}(t) = \operatorname{arccsc}(t)$

Note that for both arcsecant and arccosecant, the domain is $(-\infty, -1] \cup [1, \infty)$. Taking a page from Section A.7, we can rewrite this as $\{x \,|\, |x| \geq 1\}$. (This is often done in Calculus textbooks, so we include it here for completeness.)

Using these definitions along with Theorem 5.13, we get the following properties of the arcsecant and arccosecant functions.

Theorem 12.16. Properties of the Arcsecant and Arccosecant Functions[a]

- Properties of $F(x) = \operatorname{arcsec}(x)$
  - Domain: $\{x \mid |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
  - $\lim\limits_{x \to -\infty} \operatorname{arcsec}(x) = \frac{\pi}{2}$; $\lim\limits_{x \to \infty} \operatorname{arcsec}(x) = \frac{\pi}{2}$
  - $\operatorname{arcsec}(x) = t$ if and only if $\sec(t) = x$ and $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$
  - $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\sec(\operatorname{arcsec}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arcsec}(\sec(t)) = t$ provided $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$
- Properties of $G(x) = \operatorname{arccsc}(x)$
  - Domain: $\{x \mid |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
  - $\lim\limits_{x \to -\infty} \operatorname{arccsc}(x) = 0$; $\lim\limits_{x \to \infty} \operatorname{arccsc}(x) = 0$
  - $\operatorname{arccsc}(x) = t$ if and only if $\csc(t) = x$ and $-\frac{\pi}{2} \leq t < 0$ or $0 < t \leq \frac{\pi}{2}$
  - $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\csc(\operatorname{arccsc}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arccsc}(\csc(t)) = t$ provided $-\frac{\pi}{2} \leq t < 0$ or $0 < t \leq \frac{\pi}{2}$
  - $G(x) = \operatorname{arccsc}(x)$ is odd

[a]. . . assuming the "Trigonometry Friendly" ranges are used.

The reason the ranges here are called 'Trigonometry Friendly' is specifically because of two properties listed in Theorem 12.16: $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ and $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$.

These formulas essentially allow us to always convert arcsecants and arccosecants back to arccosines and arcsines, respectively. We see this play out in our next example.

Example 12.3.3.

1. Find the exact values of the following.

   (a) $\text{arcsec}(2)$ (b) $\text{arccsc}(-2)$ (c) $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$ (d) $\cot\left(\text{arccsc}\left(-3\right)\right)$

2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

   (a) $f(x) = \tan(\text{arcsec}(x))$ (b) $g(x) = \cos(\text{arccsc}(4x))$

Solution.

1. (a) Using Theorem 12.16, we have $\text{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.
   (b) Once again, Theorem 12.16 comes to our aid giving $\text{arccsc}(-2) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.
   (c) Since $\frac{5\pi}{4}$ doesn't fall between 0 and $\frac{\pi}{2}$ or $\frac{\pi}{2}$ and π, we cannot use the inverse property stated in Theorem 12.16. Hence, we work from the 'inside out.'

   We get: $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \text{arcsec}(-\sqrt{2}) = \arccos\left(-\frac{1}{\sqrt{2}}\right) = \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.
   (d) We begin simplifying $\cot\left(\text{arccsc}\left(-3\right)\right)$ by letting $t = \text{arccsc}(-3)$. Then, $\csc(t) = -3$. Since $\csc(t) < 0$, t lies in the interval $\left[-\frac{\pi}{2}, 0\right)$, so t corresponds to a Quadrant IV angle.

   To find $\cot\left(\text{arccsc}\left(-3\right)\right) = \cot(t)$, we use the Pythagorean Identity: $\cot^2(t) = \csc^2(t) - 1$. We get $\csc^2(t) = (-3)^2 - 1 = 8$, or $\cot(t) = \pm\sqrt{8} = \pm 2\sqrt{2}$.

   Since t corresponds to a Quadrant IV angle, $\cot(t) < 0$. Hence, $\cot\left(\text{arccsc}\left(-3\right)\right) = -2\sqrt{2}$.

2. (a) Proceeding as above, we let $t = \text{arcsec}(x)$. Then, $\sec(t) = x$ for t in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. We seek a formula for $\tan(\text{arcsec}(x)) = \tan(t)$ in terms of x.

   To relate $\sec(t)$ to $\tan(t)$, we use the Pythagorean Identity: $\tan^2(t) = \sec^2(t) - 1$. Substituting $\sec(t) = x$, we get $\tan^2(t) = \sec^2(t) - 1 = x^2 - 1$, so $\tan(t) = \pm\sqrt{x^2 - 1}$.

   If t belongs to $\left[0, \frac{\pi}{2}\right)$ then $\tan(t) \geq 0$. On the the other hand, if t belongs to $\left(\frac{\pi}{2}, \pi\right]$ then $\tan(t) \leq 0$. As a result, we get a *piecewise defined* function for $\tan(t)$:

$$\tan(t) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } 0 \leq t < \frac{\pi}{2} \\ -\sqrt{x^2 - 1}, & \text{if } \frac{\pi}{2} < t \leq \pi \end{cases}$$

   Now we need to determine what these conditions on t mean for x. Since $x = \sec(t)$, when $0 \leq t < \frac{\pi}{2}$, $x \geq 1$, and when $\frac{\pi}{2} < t \leq \pi$, $x \leq -1$. Hence,

$$f(x) = \tan(\text{arcsec}(x)) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } x \geq 1 \\ -\sqrt{x^2 - 1}, & \text{if } x \leq -1 \end{cases}$$

   To find the domain of f, we consider $f(x) = \tan(\text{arcsec}(x))$ as a two step process. First, we have the arcsecant function, whose domain is $(\infty, -1] \cup [1, \infty)$.

   Since the range of $\text{arcsec}(x)$ is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, taking the tangent of any output from $\text{arcsec}(x)$ is defined. Hence, the domain of f is $(-\infty, -1] \cup [1, \infty)$.

(b) Taking a cue from the previous problem, we start by letting $t = \operatorname{arccsc}(4x)$. Then $\csc(t) = 4x$ for t in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. Our goal is to rewrite $\cos(\operatorname{arccsc}(4x)) = \cos(t)$ in terms of x.

From $\csc(t) = 4x$, we get $\sin(t) = \frac{1}{4x}$, so to find $\cos(t)$, we can make use of the Pythagorean Identity: $\cos^2(t) = 1 - \sin^2(t)$. Substituting $\sin(t) = \frac{1}{4x}$ gives $\cos^2(t) = 1 - \left(\frac{1}{4x}\right)^2 = 1 - \frac{1}{16x^2}$. Getting a common denominator and extracting square roots, we obtain:

$$\cos(t) = \pm\sqrt{\frac{16x^2 - 1}{16x^2}} = \pm\frac{\sqrt{16x^2 - 1}}{4|x|}.$$

Since t belongs to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, we know $\cos(t) \geq 0$, so we choose $\cos(t) = \frac{\sqrt{16-x^2}}{4|x|}$. (The absolute values here are necessary, since x could be negative.) Hence,

$$g(x) = \cos(\operatorname{arccsc}(4x)) = \frac{\sqrt{16 - x^2}}{4|x|}.$$

To find the domain of $g(x) = \cos(\operatorname{arccsc}(4x))$, as usual, we think of g as a series of processes. First, we take the input, x, and multiply it by 4. Since this can be done to any real number, we have no restrictions here.

Next, we take the arccosecant of $4x$. Using interval notation, the domain of the arccosecant function is written as: $(-\infty, -1] \cup [1, \infty)$. Hence to take the arccosecant of $4x$, the quantity $4x$ must lie in one of these two intervals.[6] That is, $4x \leq -1$ or $4x \geq 1$, so $x \leq -\frac{1}{4}$ or $x \geq \frac{1}{4}$.

The third and final process coded in $g(x) = \cos(\operatorname{arccsc}(4x))$ is to take the cosine of $\operatorname{arccsc}(4x)$. Since the cosine accepts any real number, we have no additional restrictions. Hence, the domain of g is $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$. □

As promised in the discussion following Example 12.3.2, in which we used the methods from Section 11.4 to circumvent some onerous identity work, we take some time here to revisit number 2a to see what issues arise when we take a Section 11.4 approach here.

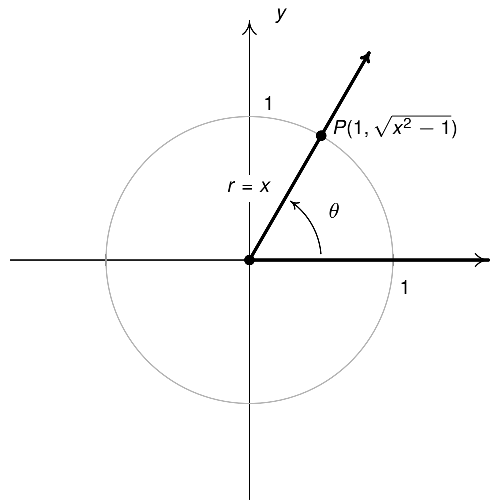
As above, we start rewriting $f(x) = \tan(\operatorname{arcsec}(x))$ by letting $t = \operatorname{arcsec}(x)$ so that $\sec(t) = x$ where $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$. We let $\theta = t$ radians and wish to view $\sec(\theta) = \sec(t) = x$ as described in Theorem 11.9: the ratio of the radius of a circle, r centered at the origin, divided by the abscissa[7] of a point on the terminal side of θ which intersects said circle.

If we make the usual identification $\sec(\theta) = x = \frac{x}{1}$, we see that if $0 \leq \theta < \frac{\pi}{2}$, then $x = \sec\theta \geq 1$, so it makes sense to identify the quantity x as the radius of the circle with 1 as the abscissa of the point where the terminal side of θ intersects said circle. To find the associated ordinate (y-coordinate), we have $1^2 + y^2 = x^2$ so $y = \sqrt{x^2 - 1}$, where we have chosen the positive root since we are in Quadrant I. We sketch out this scenario below on the left.

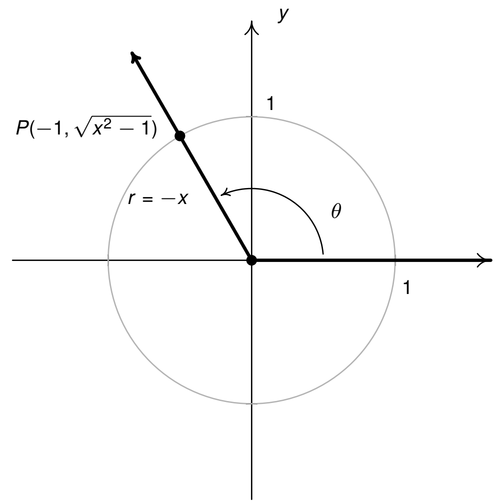
If, however, $\frac{\pi}{2} < t \leq \pi$, then $x = \sec(t) \leq -1$, so we need to rewrite $\sec(\theta) = x = \frac{x}{1} = \frac{-x}{-1}$ in order to keep the radius of the circle, $r = -x > 0$ and the abscissa, $-1 < 0$. From $(-1)^2 + y^2 = (-x)^2$, we still get $y = \sqrt{x^2 - 1}$, as shown below on the right.

[6] Alternatively, we can write the domain of $\operatorname{arccsc}(x)$ as $|x| \geq 1$, so the domain of $\operatorname{arccsc}(4x)$ is $|4x| \geq 1$.

[7] We'll avoid the label 'x-coordinate' here since as we'll see, the quantity x in this problem is tied to the radius as opposed to the coordinates of points on the terminal side of θ.



$0 < \theta < \frac{\pi}{2}$, $x \geq 1$, $r = x$.

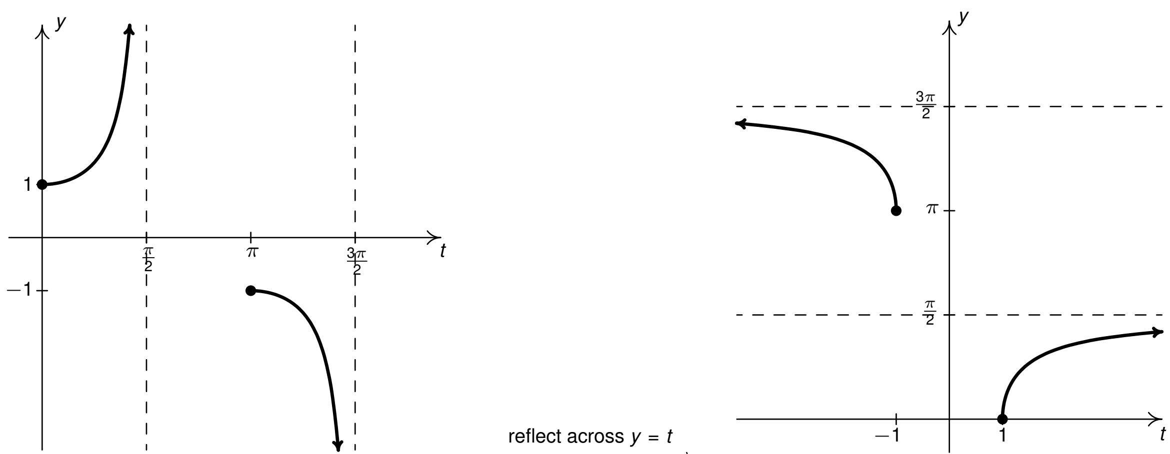


$\frac{\pi}{2} < \theta < \pi$, $x \leq -1$, $r = -x$.

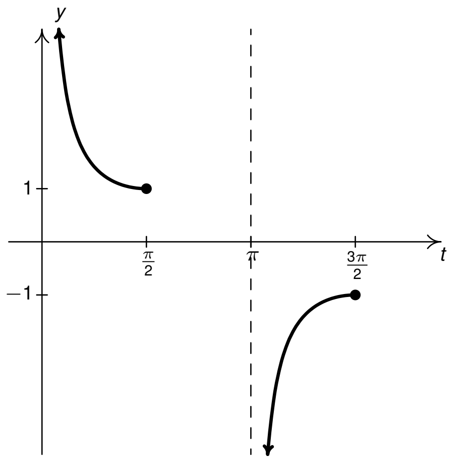
In the Quadrant I case, when $x \geq 1$, we get $\tan(\theta) = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$. In Quadrant II, when $x \leq -1$, we obtain $\tan(\theta) = \frac{\sqrt{x^2-1}}{-1} = -\sqrt{x^2-1}$. Hence, we get the piecewise definition for $f(x)$ as we did in number 2a above: $f(x) = \tan(\text{arcsec}(x)) = \sqrt{x^2-1}$ if $x \geq 1$ and $f(x) = \tan(\text{arcsec}(x)) = -\sqrt{x^2-1}$ if $x \leq -1$. The moral of the story here is that you are free to choose whichever route you like to simplify expressions like those found in Example 12.3.3 number 2a. Whether you choose identities or a more geometric route, just be careful to keep in mind which quadrants are in play, which variables represent which quantities, and what signs ($\pm$) each should have.

12.3.2 Inverses of Secant and Cosecant: Calculus Friendly Approach

In this subsection, we restrict $f(t) = \sec(t)$ to $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and $g(t) = \csc(t)$ to $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$. Using these restrictions we get the graphs and properties below.

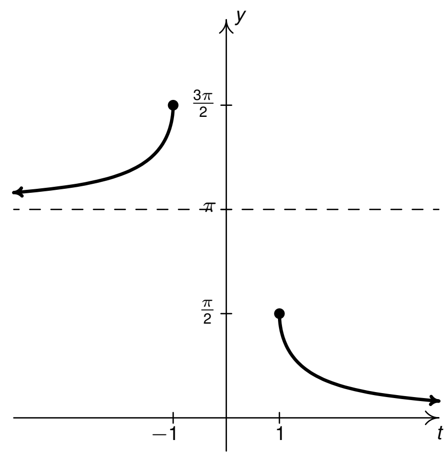


$f(t) = \sec(t)$ on $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ switch t and y coordinates $f^{-1}(t) = \text{arcsec}(t)$



$g(t) = \csc(t)$ on $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$

reflect across $y = t$ $\xrightarrow{\hspace{2cm}}$ switch x and y coordinates



$g^{-1}(t) = \operatorname{arccsc}(t)$

Theorem 12.17. Properties of the Arcsecant and Arccosecant Functions[a]

- Properties of $F(x) = \operatorname{arcsec}(x)$
  - Domain: $\{x \mid |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$
  - $\displaystyle\lim_{x\to-\infty} \operatorname{arcsec}(x) = \frac{3\pi}{2}$; $\displaystyle\lim_{x\to\infty} \operatorname{arcsec}(x) = \frac{\pi}{2}$
  - $\operatorname{arcsec}(x) = t$ if and only if $\sec(t) = x$ and $0 \leq t < \frac{\pi}{2}$ or $\pi \leq t < \frac{3\pi}{2}$.
  - $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ for $x \geq 1$ only
  - $\sec(\operatorname{arcsec}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arcsec}(\sec(t)) = t$ provided $0 \leq t < \frac{\pi}{2}$ or $\pi \leq t < \frac{3\pi}{2}$
- Properties of $G(x) = \operatorname{arccsc}(x)$
  - Domain: $\{x \mid |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$
  - $\displaystyle\lim_{x\to-\infty} \operatorname{arccsc}(x) = \pi$; $\displaystyle\lim_{x\to\infty} \operatorname{arccsc}(x) = 0$
  - $\operatorname{arccsc}(x) = t$ if and only if $\csc(t) = x$ and $0 < t \leq \frac{\pi}{2}$ or $\pi < t \leq \frac{3\pi}{2}$
  - $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ for $x \geq 1$ only
  - $\csc(\operatorname{arccsc}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arccsc}(\csc(t)) = t$ provided $0 < t \leq \frac{\pi}{2}$ or $\pi < t \leq \frac{3\pi}{2}$

[a] . . . assuming the “Calculus Friendly” ranges are used.

While it is difficult to explain why the choices here for the ranges for the arcsecant an arccosecant are, indeed, 'Calculus Friendly,' we can demonstrate how they are slightly less 'Trigonometry Friendly.' Note the equivalences $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ and $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ hold for $x \geq 1$ only, and not for all x in the domain. We will need to remember this as we work through the problems in the next example.

Speaking of which, our next example is a duplicate of Example 12.3.3. The interested reader is invited to see what differences are to be had as a consequence of the change in ranges.

Example 12.3.4.

1. Find the exact values of the following.

   (a) $\text{arcsec}(2)$ (b) $\text{arccsc}(-2)$ (c) $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$ (d) $\cot\left(\text{arccsc}\left(-3\right)\right)$

2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

   (a) $\tan(\text{arcsec}(x))$ (b) $\cos(\text{arccsc}(4x))$

Solution.

1. (a) Since $2 \geq 1$, we may invoke Theorem 12.17 to get $\text{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (b) Unfortunately, -2 is not greater to or equal to 1, so we cannot apply Theorem 12.17 to $\text{arccsc}(-2)$ and convert this into an arcsine problem. Instead, we appeal to the definition.

   To find $t = \text{arccsc}(-2)$, we need the angle measuring t radians with $\csc(t) = -2$ and is either between 0 and $\frac{\pi}{2}$ or between π and $\frac{3\pi}{2}$.

   Since $\csc(t) < 0$, we know t corresponds to an angle between π and $\frac{3\pi}{2}$, and since $\csc(t) = -2$, we know $\sin(t) = -\frac{1}{2}$. Hence, $t = \text{arccsc}(-2) = \frac{7\pi}{6}$.

   (c) Since $\frac{5\pi}{4}$ lies between π and $\frac{3\pi}{2}$, Theorem 12.17 applies: $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \frac{5\pi}{4}$.

   We encourage the reader to work this through using the definition as we have done in the previous examples to see how it goes.

   (d) To simplify $\cot\left(\text{arccsc}\left(-3\right)\right)$ we let $t = \text{arccsc}\left(-3\right)$ so that $\cot\left(\text{arccsc}\left(-3\right)\right) = \cot(t)$.

   We know $\csc(t) = -3$, and since this is negative, t lies in $\left(\pi, \frac{3\pi}{2}\right]$. To get from $\csc(t)$ to $\cot(t)$, we use the Pythagorean Identity: $\cot^2(t) = \csc^2(t) - 1$. We find $\cot^2(t) = (-3)^2 - 1 = 8$ so that $\cot(t) = \pm\sqrt{8} = \pm 2\sqrt{2}$.

   Since t is in the interval $\left(\pi, \frac{3\pi}{2}\right]$, t corresponds to a Quadrant III angle, so we know $\cot(t) > 0$. Hence, our answer is $\cot\left(\text{arccsc}\left(-3\right)\right) = 2\sqrt{2}$.

2. (a) We begin rewriting $f(x) = \tan(\text{arcsec}(x))$ by letting $t = \text{arcsec}(x)$. Hence, $\sec(t) = x$ where either $0 \leq t < \frac{\pi}{2}$ or $\pi \leq t < \frac{3\pi}{2}$. Our goal is to find an expression for $\tan(t)$ in terms of x.

   To relate $\sec(t)$ to $\tan(t)$, we use the Pythagorean Identity: $\tan^2(t) = \sec^2(t) - 1 = x^2 - 1$ so that $\tan(t) = \pm\sqrt{x^2 - 1}$. Since t lies in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, $\tan(t) \geq 0$, so we choose $\tan(t) = \sqrt{x^2 - 1}$. Hence, $f(x) = \tan(\text{arcsec}(x)) = \sqrt{x^2 - 1}$.

For the domain of f, we note that the domain of $\operatorname{arcsec}(x)$ is $(-\infty,-1]\cup[1,\infty)$. Since all values in the range of arcsecant, $\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$, are in the domain of the tangent function, we find the domain of f is $(-\infty,-1]\cup[1,\infty)$.

(b) To rewrite $g(x)=\cos(\operatorname{arccsc}(4x))$, we start by letting $t=\operatorname{arccsc}(4x)$. Then $\csc(t)=4x$ where either $0<t\leq\frac{\pi}{2}$ or $\pi<t\leq\frac{3\pi}{2}$. Our goal is to find an expression for $\cos(t)$ in terms of x.

From $\csc(t)=4x$, we get $\sin(t)=\frac{1}{4x}$, so to find $\cos(t)$, we can make use of the Pythagorean Identity: $\cos^2(t)=1-\sin^2(t)$. Substituting $\sin(t)=\frac{1}{4x}$ gives $\cos^2(t)=1-\left(\frac{1}{4x}\right)^2=1-\frac{1}{16x^2}$. Getting a common denominator and extracting square roots, we obtain:

$$\cos(t)=\pm\sqrt{\frac{16x^2-1}{16x^2}}=\pm\frac{\sqrt{16x^2-1}}{4|x|}.$$

If t lies in $\left(0,\frac{\pi}{2}\right]$, then $\cos(t)\geq 0$, and we choose $\cos(t)=\frac{\sqrt{16x^2-1}}{4|x|}$. Here, $x=\frac{1}{4}\csc(t)>0$ as well, so we can disregard the absolute values here and write $\cos(t)=\frac{\sqrt{16x^2-1}}{4x}$.

If t belongs to $\left(\pi,\frac{3\pi}{2}\right]$, then $\cos(t)\leq 0$, so, we choose $\cos(t)=-\frac{\sqrt{16x^2-1}}{4|x|}$. In this case, $x=\frac{1}{4}\csc(t)<0$, so $|x|=-x$ (see Section 1.3 for a refresher, if needs be!) and so,

$$\cos(t)=-\frac{\sqrt{16x^2-1}}{4|x|}=-\frac{\sqrt{16x^2-1}}{4(-x)}=\frac{\sqrt{16x^2-1}}{4x}.$$

Hence, in both cases, we get

$$g(x)=\cos(\operatorname{arccsc}(4x))=\frac{\sqrt{16x^2-1}}{4x}.$$

To find the domain of $g(x)=\cos(\operatorname{arccsc}(4x))$, as usual, we think of g as a series of processes. First, we take the input, x, and multiply it by 4. Since this can be done to any real number, we have no restrictions here.

Next, we take the arccosecant of $4x$. Using interval notation, the domain of the arccosecant function is written as: $(-\infty,-1]\cup[1,\infty)$. Hence to take the arccosecant of $4x$, the quantity $4x$ must lie in one of these two intervals.[8] That is, $4x\leq -1$ or $4x\geq 1$, so $x\leq-\frac{1}{4}$ or $x\geq\frac{1}{4}$.

The third and final process coded in $g(x)=\cos(\operatorname{arccsc}(4x))$ is to take the cosine of $\operatorname{arccsc}(4x)$. Since the cosine accepts any real number, we have no additional restrictions. Hence, the domain of g is $\left(-\infty,-\frac{1}{4}\right]\cup\left[\frac{1}{4},\infty\right)$. □

For completeness, we embark here on a discussion of how the techniques from Section 11.4, in particular Theorem 11.9 can be used to circumvent some of the identity work in number 2a above.[9]

As above, we start rewriting $f(x)=\tan(\operatorname{arcsec}(x))$ by letting $t=\operatorname{arcsec}(x)$ so that $\sec(t)=x$ where $0\leq t<\frac{\pi}{2}$ or $\pi\leq t<\frac{3\pi}{2}$. We let $\theta=t$ radians and wish to view $\sec(\theta)=\sec(t)=x$ as described in

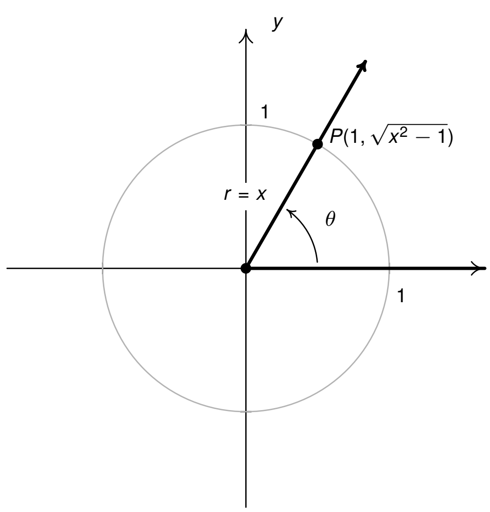
[8] Alternatively, we can write the domain of $\operatorname{arccsc}(x)$ as $|x|\geq 1$, so the domain of $\operatorname{arccsc}(4x)$ is $|4x|\geq 1$.

[9] See also the remarks following Examples 12.3.2 and 12.3.3.

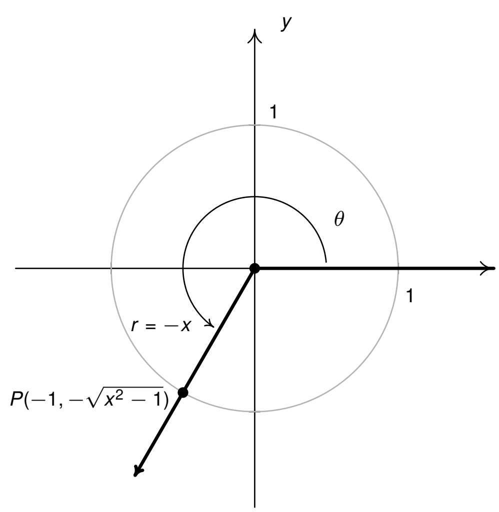
Theorem 11.9: the ratio of the radius of a circle, r centered at the origin, divided by the abscissa[10] of a point on the terminal side of θ which intersects said circle.

If we make the usual identification $\sec(\theta) = x = \frac{x}{1}$, we see that if $0 \leq \theta < \frac{\pi}{2}$, then $x = \sec\theta \geq 1$, so it makes sense to identify the quantity x as the radius of the circle with 1 as the abscissa of the point where the terminal side of θ intersects said circle. To find the associated ordinate (y-coordinate), we have $1^2 + y^2 = x^2$ so $y = \sqrt{x^2 - 1}$, where we have chosen the positive root since we are in Quadrant I. We sketch out this scenario below on the left.

If, however, $\pi \leq t < \frac{3\pi}{2}$, then $x = \sec(t) \leq -1$, so we need to rewrite $\sec(\theta) = x = \frac{x}{1} = \frac{-x}{-1}$ in order to keep the radius of the circle, $r = -x > 0$ and the abscissa, $-1 < 0$. From $(-1)^2 + y^2 = (-x)^2$, we get $y = -\sqrt{x^2 - 1}$, in this case choosing the negative root since we are in Quadrant III.



$0 < \theta < \frac{\pi}{2}$, $x \geq 1$, $r = x$.



$\pi < \theta < \frac{3\pi}{2}$, $x \leq -1$, $r = -x$.

In the Quadrant I case, when $x \geq 1$, we get $\tan(\theta) = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2 - 1}$. In Quadrant III, when $x \leq -1$, we obtain $\tan(\theta) = \frac{-\sqrt{x^2-1}}{-1} = \sqrt{x^2 - 1}$. Hence, in both cases, we obtain the same answer as we did in number 2a above: $f(x) = \tan(\text{arcsec}(x)) = \sqrt{x^2 - 1}$ for x in $(-\infty, -1] \cup [1, \infty)$.

12.3.3 Calculators and the Inverse Circular Functions.

In the sections to come, we will have need to approximate the values of the inverse circular functions. On most calculators, only the arcsine, arccosine and arctangent functions are available and they are usually labeled as $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, respectively. If we are asked to approximate these values, it is a simple matter to punch up the appropriate decimal on the calculator.

If we are asked for an arccotangent, arcsecant or arccosecant, however, we often need to employ some ingenuity, as our next example illustrates.

[10] We'll avoid the label 'x-coordinate' here since as we'll see, the quantity x in this problem is tied to the radius as opposed to the coordinates of points on the terminal side of θ.

Example 12.3.5.

1. Use a calculator to approximate the following values to four decimal places.

   (a) arccot(2) (b) arcsec(5) (c) arccot(-2) (d) arccsc$\left(-\dfrac{3}{2}\right)$

2. Find the domain and range of the following functions. Check your answers using a graphing utility.

   (a) $f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$ (b) $g(x) = 3\arctan(4x)$.

Solution.

1. (a) Since $2 > 0$, we can use the property listed in Theorem 12.15 to get: $\operatorname{arccot}(2) = \arctan\left(\frac{1}{2}\right)$. In 'radian' mode, we find $\operatorname{arccot}(2) = \arctan\left(\frac{1}{2}\right) \approx 0.4636$.

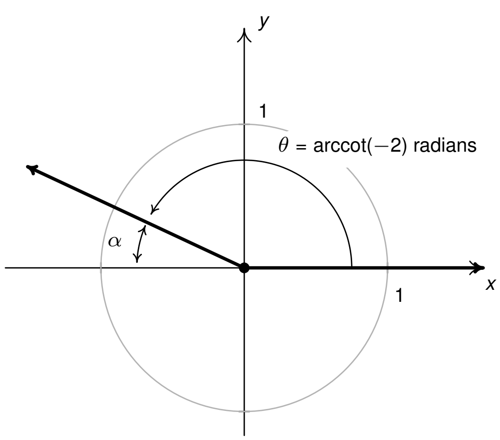
   (b) Since $5 \geq 1$, we can use the property from either Theorem 12.16 or Theorem 12.17 to write $\operatorname{arcsec}(5) = \arccos\left(\frac{1}{5}\right) \approx 1.3694$.

   (c) Since the argument -2 is negative, we cannot directly apply Theorem 12.15 to help us find $\operatorname{arccot}(-2)$, so we appeal to the definition.

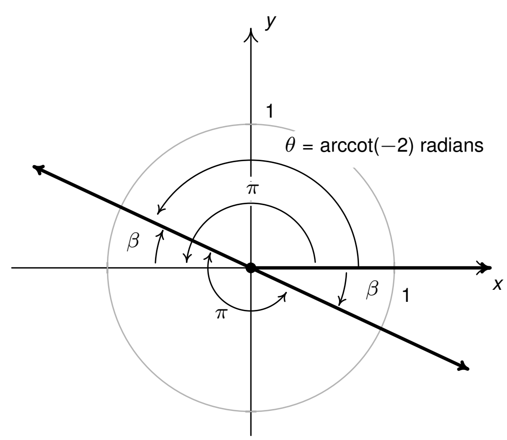
   The number $t = \operatorname{arccot}(-2)$ corresponds to an angle $\theta = t$ radians with $\cot(\theta) = -2$ which lies between 0 and π. Moreover, since $\cot(\theta) < 0$, we know θ is a Quadrant II angle.

   Let α be the reference angle for θ, as pictured below on the left. By definition, α is an acute angle which means $0 < \alpha < \frac{\pi}{2}$. By The Reference Angle Theorem, Theorem 11.1, we also know $\cot(\alpha) = 2$. Hence, by definition, $\alpha = \operatorname{arccot}(2)$ radians.

   Since the argument of arccotangent is now a *positive* 2, we can use Theorem 12.15 to get $\alpha = \operatorname{arccot}(2) = \arctan\left(\frac{1}{2}\right)$ radians. Since $\theta = \pi - \alpha = \pi - \arctan\left(\frac{1}{2}\right) \approx 2.6779$ radians, we get $\operatorname{arccot}(-2) \approx 2.6779$.



α is a reference angle of θ



$\beta = \arctan\left(-\frac{1}{2}\right)$ radians

Another way to attack the problem is to use $\arctan\left(-\frac{1}{2}\right)$. By definition, we have the real number $t = \arctan\left(-\frac{1}{2}\right)$ satisfies $\tan(t) = -\frac{1}{2}$ with $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Since $\tan(t) < 0$, we know more specifically that $-\frac{\pi}{2} < t < 0$, so t corresponds to an angle β in Quadrant IV. We sketch β along with $\theta = \text{arccot}(-2)$ radians above on the right.

To find the value of $\text{arccot}(-2)$, we once again visualize the angle $\theta = \text{arccot}(-2)$ radians and note that it is a Quadrant II angle with $\tan(\theta) = -\frac{1}{2}$. This means it is exactly π units away from β, and we get $\theta = \pi + \beta = \pi + \arctan\left(-\frac{1}{2}\right) \approx 2.6779$ radians. Hence, as before, $\text{arccot}(-2) \approx 2.6779$.

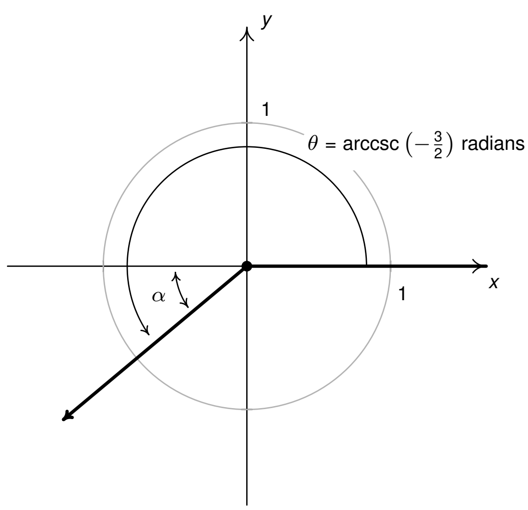
(d) If the range of arccosecant is taken to be $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, we can use Theorem 12.16 to get $\text{arccsc}\left(-\frac{3}{2}\right) = \arcsin\left(-\frac{2}{3}\right) \approx -0.7297$.

If, on the other hand, the range of arccosecant is taken to be $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$, then we proceed as in the previous problem by letting $t = \text{arccsc}\left(-\frac{3}{2}\right)$.

Then t is a real number with $\csc(t) = -\frac{3}{2}$ and since $\csc(t) < 0$, we know $\pi < \theta \leq \frac{3\pi}{2}$, Hence, t corresponds to a Quadrant III angle, θ, as depicted below.

As above, we let α be the reference angle for θ. Then $0 < \alpha < \frac{\pi}{2}$ and $\csc(\alpha) = \frac{3}{2}$, which means $\alpha = \text{arccsc}\left(\frac{3}{2}\right)$ radians.

Since the argument of arccosecant is now positive, Theorem 12.17 applies so we can rewrite $\alpha = \text{arccsc}\left(\frac{3}{2}\right) = \arcsin\left(\frac{2}{3}\right)$ radians. Since $\theta = \pi + \alpha = \pi + \arcsin\left(\frac{2}{3}\right) \approx 3.8713$ radians, we have that in this case, $\text{arccsc}\left(-\frac{3}{2}\right) \approx 3.8713$.



2. (a) To find the domain of f, we can think of the function as a sequence of steps and track our inputs through each step and track the restrictions that arise. To that end, we rewrite $f(x)$ as we did in Section 5.4: $f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right) = -\arccos\left(\frac{x}{5}\right) + \frac{\pi}{2}$.

Starting with a real number x, we divide by 5 to obtain $\frac{x}{5}$. So far, we have no restrictions.

Next, we take the arccosine of $\frac{x}{5}$. Since the arccosine function only admits inputs between -1 and 1 inclusive, we require that $-1 \leq \frac{x}{5} \leq 1$. Solving, we get $-5 \leq x \leq 5$.

Moving outside the arccosine, we multiply the outputs from the arccosine by -1 and then add $\frac{\pi}{2}$. Since these are defined for all real numbers, we have our domain restricted only by the arccosine itself. Hence, the domain of f is $[-5, 5]$.

To determine the range of f, we work through the steps above, this time paying attention to the outputs from each step. We know our domain is restricted to $[-5, 5]$ due to the arccosine, so we start with the range of arccosine: $[0, \pi]$.

Let y represent a typical output from $\arccos(x)$. Then $0 \leq y \leq \pi$. From our work above, we know the arccosine values are first multiplied by -1 and then added to $\frac{\pi}{2}$, so we apply these same operations to the inequality $0 \leq y \leq \pi$.

Multiplying this inequality through by -1 gives $-\pi \leq -y \leq 0$. Adding through by $\frac{\pi}{2}$ gives $-\frac{\pi}{2} \leq -y + \frac{\pi}{2} \leq \frac{\pi}{2}$. Hence the range of $f(x) = -\arccos\left(\frac{x}{5}\right) + \frac{\pi}{2}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Our graph below on the left confirms our results.[11]

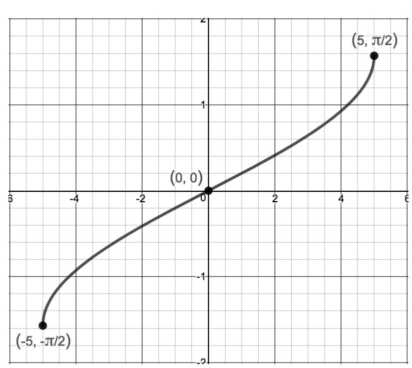
(b) As with the previous example, we think of $g(x) = 3\arctan(4x)$ as a series of steps in order to find the domain and track the range.

Starting with an input x, we multiply it by 4, take the arctangent, and then multiply that result by 3. Since all of these operations are defined for all real numbers, we conclude the domain of g is also all real numbers, or $(-\infty, \infty)$.

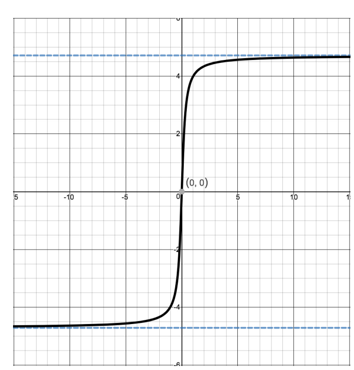
Looking at outputs, we find that the range first becomes limited when taking the arctangent. If y represent a typical output from $\arctan(x)$, then $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

In our formula for $g(x)$, the outputs from the arctangent are multiplied by 3. Multiplying the inequality $-\frac{\pi}{2} < y < \frac{\pi}{2}$ through by 3 gives $-\frac{3\pi}{2} < 3y < \frac{3\pi}{2}$.

Hence the range of g is $\left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$. Our answers are confirmed by examining the graph of g below on the right.



$y = f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$



$y = g(x) = 3\arctan(4x)$

□

[11] If this sort of analysis sounds familiar, it should. We are really just tracking the effect of transformations as in Section 5.4.

12.3.4 Solving Equations Using the Inverse Trigonometric Functions.

In Sections 11.2 and 11.4, we learned how to solve equations like $\sin(\theta) = \frac{1}{2}$ and $\tan(t) = -1$. In each case, we ultimately appealed to the Unit Circle and relied on the fact that the answers corresponded to a set of 'common angles' listed on page 926.

If, on the other hand, we had been asked to find all angles with $\sin(\theta) = \frac{1}{3}$ or solve $\tan(t) = -2$ for real numbers t, we would have been hard-pressed to do so. With the introduction of the inverse trigonometric functions, however, we are now in a position to solve these equations.

A good parallel to keep in mind is how the square root function can be used to solve certain quadratic equations. The equation $x^2 = 4$ is a lot like $\sin(\theta) = \frac{1}{2}$ in that it has friendly, 'common value' answers $x = \pm 2$. The equation $x^2 = 7$, on the other hand, is a lot like $\sin(\theta) = \frac{1}{3}$. We know there are answers, but we can't express them using 'friendly' numbers.

To solve $x^2 = 7$, we make use of the square root function (which is an inverse to $f(x) = x^2$ on a restricted domain) and write our answer as $x = \pm\sqrt{7}$. We need the $\pm$ to adjust for the fact that $\sqrt{7}$ is defined to be positive only, but we know we have two solutions, one positive and one negative. Using a calculator, we can certainly *approximate* the values $\pm\sqrt{7}$, but as far as exact answers go, we leave them as $x = \pm\sqrt{7}$.

In the same way, we will use the arcsine function (the inverse to the sine function on a restricted domain) to solve $\sin(\theta) = \frac{1}{3}$. However, we will need to adjust for the fact that there is more than one answer to this equation (infinitely many, in fact!) As it turns out, we will be able to express every solution in terms of $\arcsin\left(\frac{1}{3}\right)$, as our next example illustrates.

Example 12.3.6. Solve the following equations.

1. Find all angles θ for which $\sin(\theta) = \frac{1}{3}$.
2. Find all real numbers t for which $\tan(t) = -2$
3. Solve $\sec(x) = -\frac{5}{3}$ for x.

Solution.

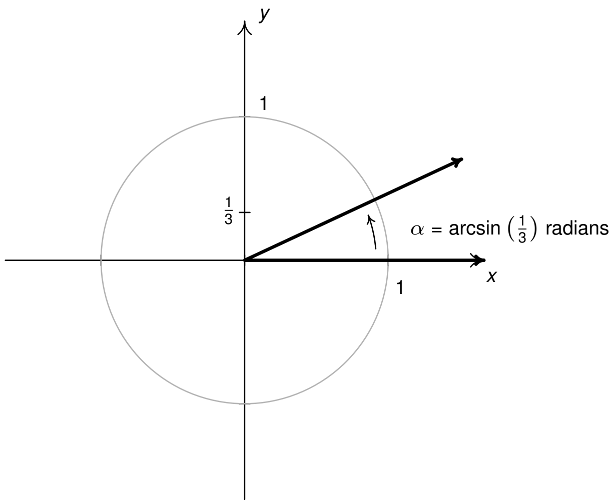
1. If $\sin(\theta) = \frac{1}{3}$, then the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $y = \frac{1}{3}$. Geometrically, we see that this happens at two places: in Quadrant I and Quadrant II.

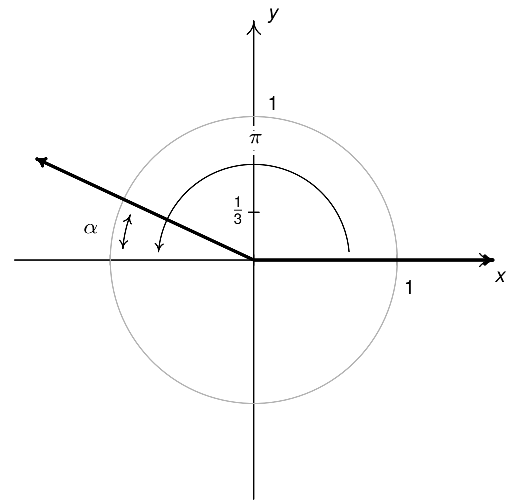
   If we let α denote the acute solution to the equation, then all the solutions to this equation in Quadrant I are coterminal with α, and α serves as the reference angle for all of the solutions to this equation in Quadrant II as seen below.

   Since $\frac{1}{3}$ isn't the sine of any of the 'common angles' we've encountered, we use the arcsine functions to express our answers. By definition, real number $t = \arcsin\left(\frac{1}{3}\right)$ $\sin(t) = \frac{1}{3}$ with $0 < t < \frac{\pi}{2}$.

   Hence, $\alpha = \arcsin\left(\frac{1}{3}\right)$ radians is an acute angle with $\sin(\alpha) = \frac{1}{3}$. Since all of the Quadrant I solutions θ are all coterminal with α, we get $\theta = \alpha + 2\pi k = \arcsin\left(\frac{1}{3}\right) + 2\pi k$ for integers k.

   Turning our attention to Quadrant II, we get one solution to be $\pi - \alpha$. Hence, the Quadrant II solutions are $\theta = \pi - \alpha + 2\pi k = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k$, for integers k.

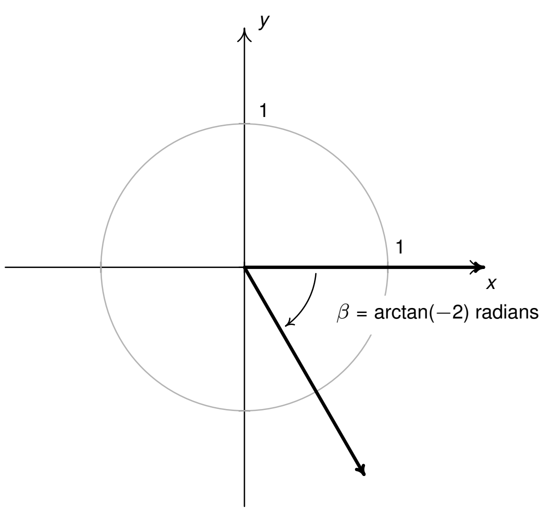


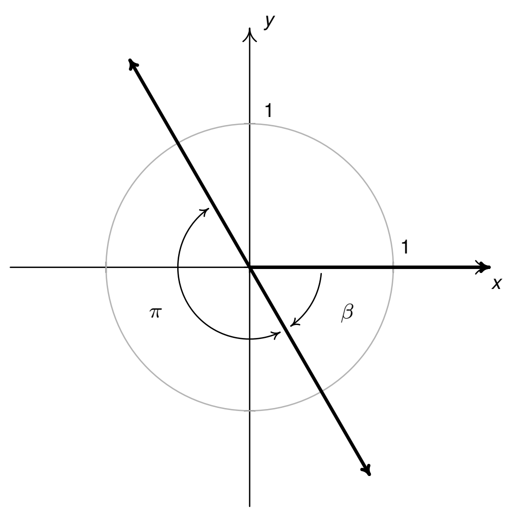


2. The real number solutions to $\tan(t) = -2$ correspond to angles θ with $\tan(\theta) = -2$. Since tangent is negative only in Quadrants II and IV, we focus our efforts there.

   The real number $t = \arctan(-2)$ satisfies $\tan(t) = -2$ and $-\frac{\pi}{2} < t < 0$. If we let $\beta = \arctan(-2)$ radians, then all of the Quadrant IV solutions to $\tan(\theta) = -2$ are coterminal with β.

   Moreover, as seen below on the right, the solutions from Quadrant II differ by exactly π units from the solutions in Quadrant IV (recall, the period of the tangent function is π.) Hence, all of the solutions to $\tan(\theta) = -2$ are of the form $\theta = \beta + \pi k = \arctan(-2) + \pi k$ for some integer k. Switching back to the variable t, we record our final answer to $\tan(t) = -2$ as $t = \arctan(-2) + \pi k$ for integers k.

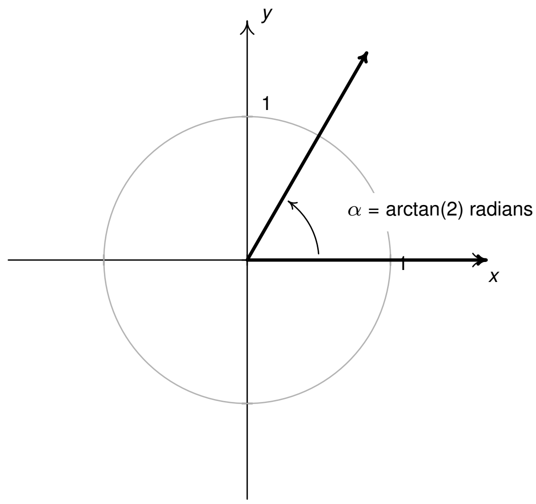


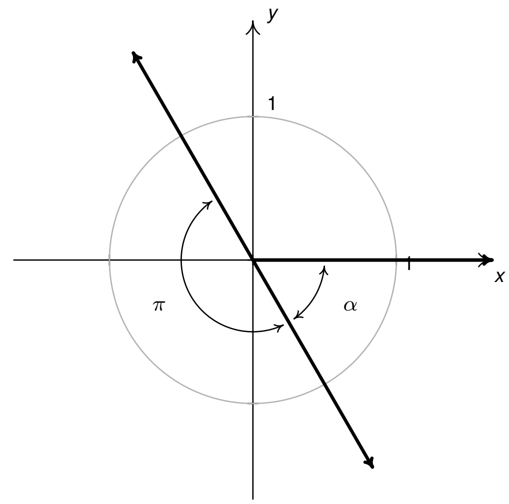


   Another tact we could have taken to solve this problem is to use reference angles. Consider the (angle) equation: $\tan(\theta) = -2$. If we let α be the reference angle for the solutions θ, we know α is an acute angle with $\tan(\alpha) = 2$.

By definition, the real number $t = \arctan(2)$ satisfies $0 < t < \frac{\pi}{2}$ with $\tan(t) = 2$. Hence, the angle $\alpha = \arctan(2)$ radians is the reference angle for our solutions to $\tan(\theta) = -2$.

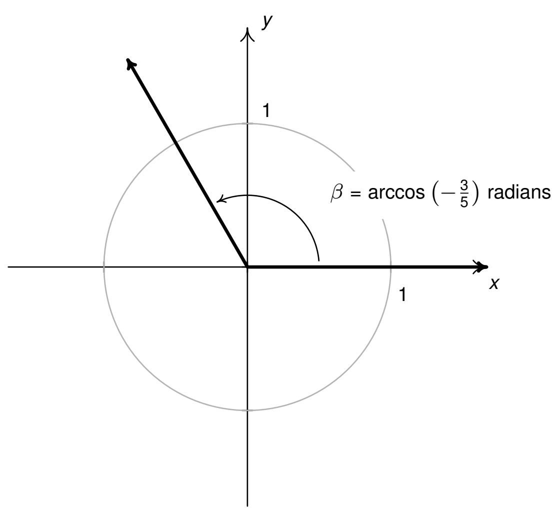
Adjusting for quadrants, we get our answers to $\tan(\theta) = -2$ are $\theta = -\alpha + \pi k = -\arctan(2) + \pi k$ for integers k. Again, we cosmetically change the variable from θ back to t so our answer to $\tan(t) = -2$ is $t = -\arctan(2) + \pi k$. Thanks to the odd property of arctangent, $\arctan(-2) = -\arctan(2)$ and we see this family of solutions is identical to what we obtained earlier.

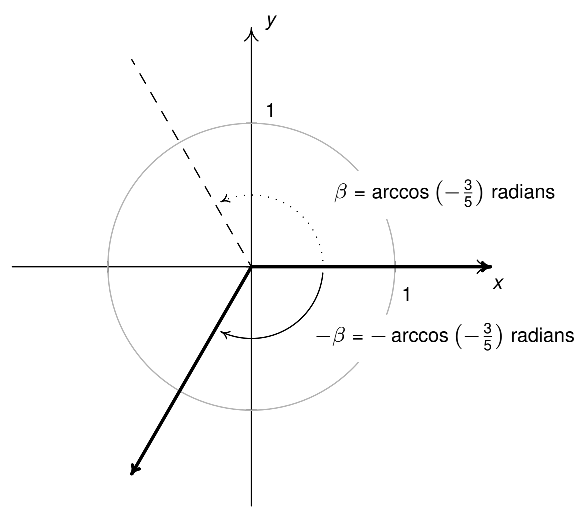




3. In the last equation, $\sec(x) = -\frac{5}{3}$, we are not told whether or not x represents an angle or a real number. This isn't really much of an issue, since we attack both problems the same way.

   Taking a cue from our work in Section 11.4 and use a Reciprocal Identity to convert the equation $\sec(x) = -\frac{5}{3}$ to $\cos(x) = -\frac{3}{5}$. Thinking geometrically, we are looking for angles θ with $\cos(\theta) = -\frac{3}{5}$. Since $\cos(\theta) < 0$, we are looking for solutions in Quadrants II and III. Since $-\frac{3}{5}$ isn't the cosine of any of the 'common angles', we'll need to express our solutions in terms of the arccosine function.





$\square$

The real number $t = \arccos\left(-\frac{3}{5}\right)$ is defined so that $\frac{\pi}{2} < t < \pi$ with $\cos(t) = -\frac{3}{5}$. Hence, the angle $\beta = \arccos\left(-\frac{3}{5}\right)$ radians is a Quadrant II angle which satisfies $\cos(\beta) = -\frac{3}{5}$. To obtain a Quadrant III angle solution, we may simply use $-\beta = -\arccos\left(-\frac{3}{5}\right)$ as seen above on the right.

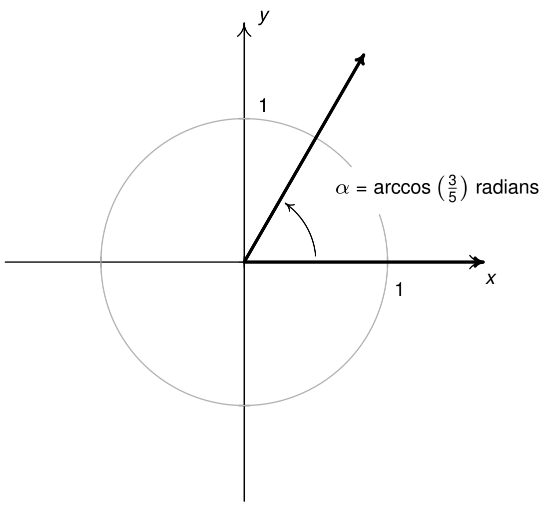
Since all angle solutions are coterminal with β or $-\beta$, we get our solutions to $\cos(\theta) = -\frac{3}{5}$ to be $\theta = \beta + 2\pi k = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $\theta = -\beta + 2\pi k = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k.

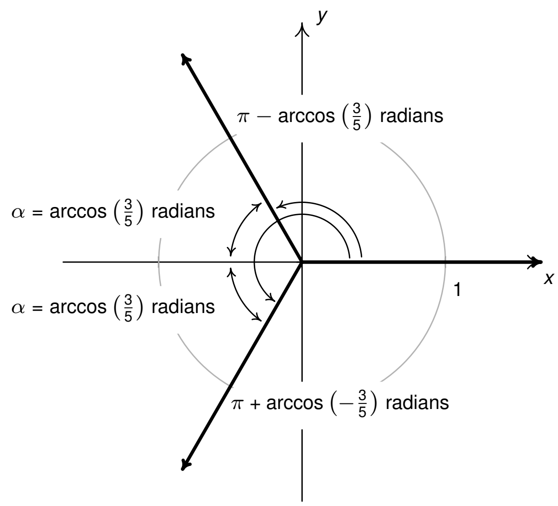
Switching back to the variable x, we record our final answer to $\sec(x) = -\frac{5}{3}$ as $x = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $x = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k.

As with the previous problem, we can approach solving $\cos(\theta) = -\frac{3}{5}$ using reference angles. Letting α represent the reference angle for the solutions θ, we know α is an acute angle with $\cos(\alpha) = \frac{3}{5}$.

We know the real number $t = \arccos\left(\frac{3}{5}\right)$ satisfies $\cos(t) = \frac{3}{5}$ and $0 < t < \frac{\pi}{2}$, hence $\alpha = \arccos\left(\frac{3}{5}\right)$ radians is the reference angle for the solutions to $\cos(\theta) = -\frac{3}{5}$.

Hence, the Quadrant II solutions to $\cos(\theta) = -\frac{3}{5}$ are $\theta = \pi - \alpha + 2\pi k = \pi - \arccos\left(\frac{3}{5}\right) + 2\pi k$ while the Quadrant IV solutions to $\cos(\theta) = -\frac{3}{5}$ are $\theta = \pi + \alpha + 2\pi k = \pi + \arccos\left(\frac{3}{5}\right) + 2\pi k$ for integers k.





□

Shifting back to the variable x, we get our solution to $\sec(x) = -\frac{5}{3}$ are $x = \pi - \arccos\left(\frac{3}{5}\right) + 2\pi k$ or $x = \pi + \arccos\left(\frac{3}{5}\right) + 2\pi k$ for integers k. While these certainly look quite different than what we obtained before, $x = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $x = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k, they are, in fact, equivalent. To show this, we start with $\arccos\left(-\frac{3}{5}\right) = \pi - \arccos\left(\frac{3}{5}\right)$ and begin writing out specific solutions from each family by choosing specific values of k. We leave these details to the reader. □

We close this section with one last sinusoid example.

Example 12.3.7. Consider the function $f(t) = 3\cos(6t) - 4\sin(6t)$. Find a formula for $f(t)$:

1. in the form $C(t) = A\cos(\omega t + \phi) + B$ for $\omega > 0$
2. in the form $S(t) = A\sin(\omega t + \phi) + B$ for $\omega > 0$

Solution.

1. As in Example 12.2.7, we compare the expanded form of $C(t) = A\cos(\omega t)\cos(\phi) - A\sin(\omega t)\sin(\phi) + B$ with $f(t) = 3\cos(6t) - 4\sin(6t)$. We identify $\omega = 6$ and $B = 0$ and by equating coefficients of $\cos(6t)$ and $\sin(6t)$ get the two equations: $A\cos(\phi) = 3$ and $A\sin(\phi) = 4$.

   Using the Pythagorean Identity to eliminate ϕ, we get $A^2 = (A\cos(\phi))^2 + (A\sin(\phi))^2 = 3^2 + 4^2 = 25$. We choose $A = 5$ and work to find the phase angle ϕ.

   Substituting $A = 5$ into our two equations relating A and ϕ, we get $5\cos(\phi) = 3$, or $\cos(\phi) = \frac{3}{5}$ and $5\sin(\phi) = 4$, so $\sin(\phi) = \frac{4}{5}$. Since both $\sin(\phi)$ and $\cos(\phi)$ are positive, we know ϕ is a Quadrant I angle. However, since neither the sine nor cosine value of ϕ corresponds to a common angle, we need to express ϕ in terms of either an arcsine or arccosine.

   Since the real number $t = \arccos\left(\frac{3}{5}\right)$ satisfies $\cos(t) = \frac{3}{5}$ and $0 < t < \frac{\pi}{2}$, we know the angle $\phi = \arccos\left(\frac{3}{5}\right)$ radians is an acute (Quadrant I) angle which satisfies $\cos(\phi) = \frac{3}{5}$. Hence, we can take $\phi = \arccos\left(\frac{3}{5}\right)$ and write $f(t) = 5\cos\left(6t + \arccos\left(\frac{3}{5}\right)\right)$.

   In addition, the real number $t = \arcsin\left(\frac{4}{5}\right)$ satisfies $\sin(t) = \frac{4}{5}$ and $0 < t < \frac{\pi}{2}$. Hence $\phi = \arcsin\left(\frac{4}{5}\right)$ radians is Quadrant I angle with $\sin(\phi) = \frac{4}{5}$. This means we could also take $\phi = \arcsin\left(\frac{4}{5}\right)$ and write $f(t) = 5\cos\left(6t + \arcsin\left(\frac{4}{5}\right)\right)$. (We could also express ϕ in terms of arctangents, if we wanted!)

   We leave it to the reader to verify (both) solutions analytically and graphically.

2. Once again, we equate the expanded form of $S(t) = A\sin(\omega t)\cos(\phi) + A\cos(\omega t)\sin(\phi) + B$ with $f(t) = 3\cos(6t) - 4\sin(6t)$. Once again, we get $\omega = 6$ and $B = 0$. Here, our two equations for A and ϕ are $A\cos(\phi) = -4$ and $A\sin(\phi) = 3$.

   As before, we get $A^2 = (A\cos(\phi))^2 + (A\sin(\phi))^2 = (-4)^2 + 3^2 = 25$, and we choose $A = 5$. Our equations for ϕ become: $\cos(\phi) = -\frac{4}{5}$ and $\sin(\phi) = \frac{3}{5}$. Since $\cos(\phi) < 0$ but $\sin(\phi) > 0$, we know ϕ is a Quadrant II angle. As before, since neither the sine nor cosine value of ϕ corresponds to a common angle, we need to express ϕ in terms of either an arcsine or arccosine.

   Here, we opt to use the arccosine function, since the range of arccosine, $[0, \pi]$ covers Quadrant II. From $\cos(\phi) = -\frac{4}{5}$, we get $\phi = \arccos\left(-\frac{4}{5}\right)$, so $f(t) = 5\sin\left(6t + \arccos\left(-\frac{4}{5}\right)\right)$.

   Had we chosen to work with arcsines, we would need a Quadrant II solution to $\sin(\phi) = \frac{3}{5}$. Going through the usual machinations, we arrive at $\phi = \pi - \arcsin\left(\frac{3}{5}\right)$. Hence, an alternative form of our answer is $f(t) = 5\sin\left(6t + \pi - \arcsin\left(\frac{3}{5}\right)\right)$. We leave the checks to the reader. $\square$

12.3.5 Exercises

In Exercises 1 - 40, find the exact value.

1. $\arcsin(-1)$
2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$
3. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$
4. $\arcsin\left(-\frac{1}{2}\right)$
5. $\arcsin(0)$
6. $\arcsin\left(\frac{1}{2}\right)$
7. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$
8. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
9. $\arcsin(1)$
10. $\arccos(-1)$
11. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$
12. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$
13. $\arccos\left(-\frac{1}{2}\right)$
14. $\arccos(0)$
15. $\arccos\left(\frac{1}{2}\right)$
16. $\arccos\left(\frac{\sqrt{2}}{2}\right)$
17. $\arccos\left(\frac{\sqrt{3}}{2}\right)$
18. $\arccos(1)$
19. $\arctan\left(-\sqrt{3}\right)$
20. $\arctan(-1)$
21. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$
22. $\arctan(0)$
23. $\arctan\left(\frac{\sqrt{3}}{3}\right)$
24. $\arctan(1)$
25. $\arctan\left(\sqrt{3}\right)$
26. $\operatorname{arccot}\left(-\sqrt{3}\right)$
27. $\operatorname{arccot}(-1)$
28. $\operatorname{arccot}\left(-\frac{\sqrt{3}}{3}\right)$
29. $\operatorname{arccot}(0)$
30. $\operatorname{arccot}\left(\frac{\sqrt{3}}{3}\right)$
31. $\operatorname{arccot}(1)$
32. $\operatorname{arccot}\left(\sqrt{3}\right)$
33. $\operatorname{arcsec}(2)$
34. $\operatorname{arccsc}(2)$
35. $\operatorname{arcsec}\left(\sqrt{2}\right)$
36. $\operatorname{arccsc}\left(\sqrt{2}\right)$
37. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right)$
38. $\operatorname{arccsc}\left(\frac{2\sqrt{3}}{3}\right)$
39. $\operatorname{arcsec}(1)$
40. $\operatorname{arccsc}(1)$

In Exercises 41 - 48, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and that the range of arccosecant is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ when finding the exact value. (See Section 12.3.1.)

41. $\operatorname{arcsec}(-2)$
42. $\operatorname{arcsec}\left(-\sqrt{2}\right)$
43. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$
44. $\operatorname{arcsec}(-1)$
45. $\operatorname{arccsc}(-2)$
46. $\operatorname{arccsc}\left(-\sqrt{2}\right)$
47. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$
48. $\operatorname{arccsc}(-1)$

In Exercises 49 - 56, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and that the range of arccosecant is $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ when finding the exact value. (See Section 12.3.2.)

49. $\text{arcsec}\,(-2)$
50. $\text{arcsec}\left(-\sqrt{2}\right)$
51. $\text{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$
52. $\text{arcsec}\,(-1)$
53. $\text{arccsc}\,(-2)$
54. $\text{arccsc}\left(-\sqrt{2}\right)$
55. $\text{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$
56. $\text{arccsc}\,(-1)$

In Exercises 57 - 86, find the exact value or state that it is undefined.

57. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$
58. $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right)$
59. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right)$
60. $\sin\left(\arcsin\left(-0.42\right)\right)$
61. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$
62. $\cos\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right)$
63. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$
64. $\cos\left(\arccos\left(\frac{5}{13}\right)\right)$
65. $\cos\left(\arccos\left(-0.998\right)\right)$
66. $\cos\left(\arccos\left(\pi\right)\right)$
67. $\tan\left(\arctan\left(-1\right)\right)$
68. $\tan\left(\arctan\left(\sqrt{3}\right)\right)$
69. $\tan\left(\arctan\left(\frac{5}{12}\right)\right)$
70. $\tan\left(\arctan\left(0.965\right)\right)$
71. $\tan\left(\arctan\left(3\pi\right)\right)$
72. $\cot\left(\text{arccot}\left(1\right)\right)$
73. $\cot\left(\text{arccot}\left(-\sqrt{3}\right)\right)$
74. $\cot\left(\text{arccot}\left(-\frac{7}{24}\right)\right)$
75. $\cot\left(\text{arccot}\left(-0.001\right)\right)$
76. $\cot\left(\text{arccot}\left(\frac{17\pi}{4}\right)\right)$
77. $\sec\left(\text{arcsec}\left(2\right)\right)$
78. $\sec\left(\text{arcsec}\left(-1\right)\right)$
79. $\sec\left(\text{arcsec}\left(\frac{1}{2}\right)\right)$
80. $\sec\left(\text{arcsec}\left(0.75\right)\right)$
81. $\sec\left(\text{arcsec}\left(117\pi\right)\right)$
82. $\csc\left(\text{arccsc}\left(\sqrt{2}\right)\right)$
83. $\csc\left(\text{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right)$
84. $\csc\left(\text{arccsc}\left(\frac{\sqrt{2}}{2}\right)\right)$
85. $\csc\left(\text{arccsc}\left(1.0001\right)\right)$
86. $\csc\left(\text{arccsc}\left(\frac{\pi}{4}\right)\right)$

In Exercises 87 - 106, find the exact value or state that it is undefined.

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right)$
88. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right)$
89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$

90. $\arcsin\left(\sin\left(\frac{11\pi}{6}\right)\right)$

91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$

92. $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right)$

93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$

94. $\arccos\left(\cos\left(\frac{3\pi}{2}\right)\right)$

95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right)$

96. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right)$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$

98. $\arctan\left(\tan\left(-\frac{\pi}{4}\right)\right)$

99. $\arctan(\tan(\pi))$

100. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$

101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right)$

102. $\text{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right)$

103. $\text{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right)$

104. $\text{arccot}(\cot(\pi))$

105. $\text{arccot}\left(\cot\left(\frac{\pi}{2}\right)\right)$

106. $\text{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right)$

In Exercises 107 - 118, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and that the range of arccosecant is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ when finding the exact value. (See Section 12.3.1.)

107. $\text{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$

108. $\text{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

109. $\text{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$

110. $\text{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$

111. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$

112. $\text{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$

113. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$

114. $\text{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$

115. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$

116. $\text{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$

117. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

118. $\text{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

In Exercises 119 - 130, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and that the range of arccosecant is $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ when finding the exact value. (See Section 12.3.2.)

119. $\text{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$

120. $\text{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

121. $\text{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$

122. $\text{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$

123. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$

124. $\text{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$

125. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$

126. $\text{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$

127. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$

128. $\text{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$

129. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

130. $\text{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

In Exercises 131 - 154, find the exact value or state that it is undefined.

131. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$
132. $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$
133. $\sin\left(\arctan\left(-2\right)\right)$
134. $\sin\left(\operatorname{arccot}\left(\sqrt{5}\right)\right)$
135. $\sin\left(\operatorname{arccsc}\left(-3\right)\right)$
136. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right)$
137. $\cos\left(\arctan\left(\sqrt{7}\right)\right)$
138. $\cos\left(\operatorname{arccot}\left(3\right)\right)$
139. $\cos\left(\operatorname{arcsec}\left(5\right)\right)$
140. $\tan\left(\arcsin\left(-\frac{2\sqrt{5}}{5}\right)\right)$
141. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right)$
142. $\tan\left(\operatorname{arcsec}\left(\frac{5}{3}\right)\right)$
143. $\tan\left(\operatorname{arccot}\left(12\right)\right)$
144. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right)$
145. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$
146. $\cot\left(\operatorname{arccsc}\left(\sqrt{5}\right)\right)$
147. $\cot\left(\arctan\left(0.25\right)\right)$
148. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$
149. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right)$
150. $\sec\left(\arctan\left(10\right)\right)$
151. $\sec\left(\operatorname{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right)$
152. $\csc\left(\operatorname{arccot}\left(9\right)\right)$
153. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right)$
154. $\csc\left(\arctan\left(-\frac{2}{3}\right)\right)$

In Exercises 155 - 164, find the exact value or state that it is undefined.

155. $\sin\left(\arcsin\left(\frac{5}{13}\right)+\frac{\pi}{4}\right)$
156. $\cos\left(\operatorname{arcsec}(3)+\arctan(2)\right)$
157. $\tan\left(\arctan(3)+\arccos\left(-\frac{3}{5}\right)\right)$
158. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right)$
159. $\sin\left(2\operatorname{arccsc}\left(\frac{13}{5}\right)\right)$
160. $\sin\left(2\arctan\left(2\right)\right)$
161. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right)$
162. $\cos\left(2\operatorname{arcsec}\left(\frac{25}{7}\right)\right)$
163. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right)$
164. $\sin\left(\frac{\arctan(2)}{2}\right)$

In Exercises 165 - 184, rewrite each of the following composite functions as algebraic functions of x and state the domain.

165. $f(x) = \sin(\arccos(x))$

166. $f(x) = \cos(\arctan(x))$

167. $f(x) = \tan(\arcsin(x))$

168. $f(x) = \sec(\arctan(x))$

169. $f(x) = \csc(\arccos(x))$

170. $f(x) = \sin(2\arctan(x))$

171. $f(x) = \sin(2\arccos(x))$

172. $f(x) = \cos(2\arctan(x))$

173. $f(x) = \sin(\arccos(2x))$

174. $f(x) = \sin\left(\arccos\left(\dfrac{x}{5}\right)\right)$

175. $f(x) = \cos\left(\arcsin\left(\dfrac{x}{2}\right)\right)$

176. $f(x) = \cos(\arctan(3x))$

177. $f(x) = \sin(2\arcsin(7x))$

178. $f(x) = \sin\left(2\arcsin\left(\dfrac{x\sqrt{3}}{3}\right)\right)$

179. $f(x) = \cos(2\arcsin(4x))$

180. $f(x) = \sec(\arctan(2x))\tan(\arctan(2x))$

181. $f(x) = \sin(\arcsin(x) + \arccos(x))$

182. $f(x) = \cos(\arcsin(x) + \arctan(x))$

183. $f(x) = \tan(2\arcsin(x))$

184. $f(x) = \sin\left(\dfrac{1}{2}\arctan(x)\right)$

185. If $\theta = \arcsin\left(\dfrac{x}{2}\right)$, find an expression for $\theta + \sin(2\theta)$ in terms of x.

186. If $\theta = \arctan\left(\dfrac{x}{7}\right)$, find an expression for $\dfrac{1}{2}\theta - \dfrac{1}{2}\sin(2\theta)$ in terms of x.

187. If $\theta = \operatorname{arcsec}\left(\dfrac{x}{4}\right)$, find an expression for $4\tan(\theta) - 4\theta$ in terms of x assuming $x \geq 4$.

In Exercises 188 - 207, solve the equation using the techniques discussed in Example 12.3.6 then approximate the solutions which lie in the interval $[0, 2\pi)$ to four decimal places.

188. $\sin(\theta) = \dfrac{7}{11}$

189. $\cos(\theta) = -\dfrac{2}{9}$

190. $\sin(\theta) = -0.569$

191. $\cos(\theta) = 0.117$

192. $\sin(\theta) = 0.008$

193. $\cos(\theta) = \dfrac{359}{360}$

194. $\tan(t) = 117$

195. $\cot(t) = -12$

196. $\sec(t) = \dfrac{3}{2}$

197. $\csc(t) = -\dfrac{90}{17}$

198. $\tan(t) = -\sqrt{10}$

199. $\sin(t) = \dfrac{3}{8}$

200. $\cos(x) = -\dfrac{7}{16}$

201. $\tan(x) = 0.03$

202. $\sin(x) = 0.3502$

203. $\sin(x) = -0.721$

204. $\cos(x) = 0.9824$

205. $\cos(x) = -0.5637$

206. $\cot(x) = \dfrac{1}{117}$

207. $\tan(x) = -0.6109$

In Exercises 208 - 213, rewrite the given function as a sinusoid of the form $C(t) = A\cos(\omega t + \phi)$ and $S(t) = A\sin(\omega t + \phi)$ (See Example 12.3.7.) Approximate the value of ϕ (which is in radians, of course) to four decimal places.

208. $f(t) = 5\sin(3t) + 12\cos(3t)$

209. $f(t) = 3\cos(2t) + 4\sin(2t)$

210. $f(t) = \cos(t) - 3\sin(t)$

211. $f(t) = 7\sin(10t) - 24\cos(10t)$

212. $f(t) = -\cos(t) - 2\sqrt{2}\sin(t)$

213. $f(t) = 2\sin(t) - \cos(t)$

In Exercises 214 - 225, find the domain of the given function. Write your answers in interval notation.

214. $f(x) = \arcsin(5x)$

215. $f(x) = \arccos\left(\dfrac{3x-1}{2}\right)$

216. $f(x) = \arcsin\left(2x^2\right)$

217. $f(x) = \arccos\left(\dfrac{1}{x^2-4}\right)$

218. $f(x) = \arctan(4x)$

219. $f(x) = \operatorname{arccot}\left(\dfrac{2x}{x^2-9}\right)$

220. $f(x) = \arctan(\ln(2x-1))$

221. $f(x) = \operatorname{arccot}(\sqrt{2x-1})$

222. $f(x) = \operatorname{arcsec}(12x)$

223. $f(x) = \operatorname{arccsc}(x+5)$

224. $f(x) = \operatorname{arcsec}\left(\dfrac{x^3}{8}\right)$

225. $f(x) = \operatorname{arccsc}\left(e^{2x}\right)$

226. Find the following limits.

(a) $\displaystyle\lim_{x\to 1^-} \arcsin(x)$

(b) $\displaystyle\lim_{x\to -\infty} \arctan(3x)$

(c) $\displaystyle\lim_{x\to 1} \operatorname{arcsec}(2x)$

227. Find a nonzero number x where $\operatorname{arccot}(x) \neq \arctan\left(\dfrac{1}{x}\right)$.

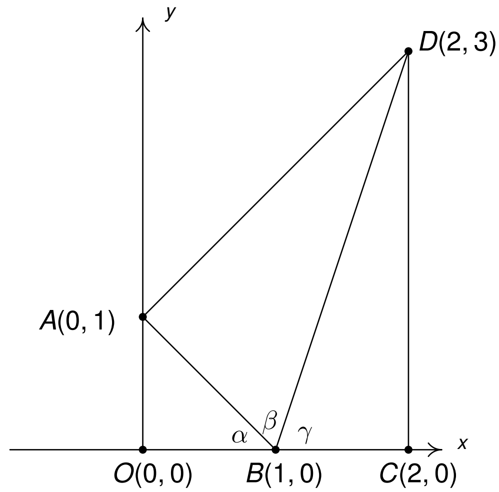
228. Find an example where $\operatorname{arcsec}(x) \neq \arccos\left(\dfrac{1}{x}\right)$ if we use $\left[0, \dfrac{\pi}{2}\right) \cup \left[\pi, \dfrac{3\pi}{2}\right)$ as the range of $f(x) = \operatorname{arcsec}(x)$.

229. Show that $\arcsin(x) + \arccos(x) = \dfrac{\pi}{2}$ for $-1 \leq x \leq 1$.

230. Discuss with your classmates why $\arcsin\left(\dfrac{1}{2}\right) \neq 30$.

231. Use the diagram below along with the accompanying questions to show:

$$\arctan(1) + \arctan(2) + \arctan(3) = \pi$$



(a) Clearly $\triangle AOB$ and $\triangle BCD$ are right triangles because the line through O and A and the line through C and D are perpendicular to the x-axis. Use the distance formula to show that $\triangle BAD$ is also a right triangle (with $\angle BAD$ being the right angle) by showing that the sides of the triangle satisfy the Pythagorean Theorem.

(b) Use $\triangle AOB$ to show that $\alpha = \arctan(1)$

(c) Use $\triangle BAD$ to show that $\beta = \arctan(2)$

(d) Use $\triangle BCD$ to show that $\gamma = \arctan(3)$

(e) Use the fact that O, B and C all lie on the x-axis to conclude that $\alpha + \beta + \gamma = \pi$. Thus $\arctan(1) + \arctan(2) + \arctan(3) = \pi$.

12.3.6 Answers

1. $\arcsin(-1) = -\frac{\pi}{2}$
2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$
3. $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$
4. $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$
5. $\arcsin(0) = 0$
6. $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$
7. $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$
8. $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$
9. $\arcsin(1) = \frac{\pi}{2}$
10. $\arccos(-1) = \pi$
11. $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$
12. $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$
13. $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$
14. $\arccos(0) = \frac{\pi}{2}$
15. $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$
16. $\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$
17. $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$
18. $\arccos(1) = 0$
19. $\arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$
20. $\arctan(-1) = -\frac{\pi}{4}$
21. $\arctan\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$
22. $\arctan(0) = 0$
23. $\arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$
24. $\arctan(1) = \frac{\pi}{4}$
25. $\arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$
26. $\operatorname{arccot}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$
27. $\operatorname{arccot}(-1) = \frac{3\pi}{4}$
28. $\operatorname{arccot}\left(-\frac{\sqrt{3}}{3}\right) = \frac{2\pi}{3}$
29. $\operatorname{arccot}(0) = \frac{\pi}{2}$
30. $\operatorname{arccot}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{3}$
31. $\operatorname{arccot}(1) = \frac{\pi}{4}$
32. $\operatorname{arccot}\left(\sqrt{3}\right) = \frac{\pi}{6}$
33. $\operatorname{arcsec}(2) = \frac{\pi}{3}$
34. $\operatorname{arccsc}(2) = \frac{\pi}{6}$
35. $\operatorname{arcsec}\left(\sqrt{2}\right) = \frac{\pi}{4}$
36. $\operatorname{arccsc}\left(\sqrt{2}\right) = \frac{\pi}{4}$
37. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right) = \frac{\pi}{6}$
38. $\operatorname{arccsc}\left(\frac{2\sqrt{3}}{3}\right) = \frac{\pi}{3}$
39. $\operatorname{arcsec}(1) = 0$
40. $\operatorname{arccsc}(1) = \frac{\pi}{2}$
41. $\operatorname{arcsec}(-2) = \frac{2\pi}{3}$
42. $\operatorname{arcsec}\left(-\sqrt{2}\right) = \frac{3\pi}{4}$
43. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{5\pi}{6}$
44. $\operatorname{arcsec}(-1) = \pi$
45. $\operatorname{arccsc}(-2) = -\frac{\pi}{6}$

46. $\text{arccsc}\left(-\sqrt{2}\right) = -\frac{\pi}{4}$

47. $\text{arccsc}\left(-\frac{2\sqrt{3}}{3}\right) = -\frac{\pi}{3}$

48. $\text{arccsc}\,(-1) = -\frac{\pi}{2}$

49. $\text{arcsec}\,(-2) = \frac{4\pi}{3}$

50. $\text{arcsec}\left(-\sqrt{2}\right) = \frac{5\pi}{4}$

51. $\text{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{7\pi}{6}$

52. $\text{arcsec}\,(-1) = \pi$

53. $\text{arccsc}\,(-2) = \frac{7\pi}{6}$

54. $\text{arccsc}\left(-\sqrt{2}\right) = \frac{5\pi}{4}$

55. $\text{arccsc}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{4\pi}{3}$

56. $\text{arccsc}\,(-1) = \frac{3\pi}{2}$

57. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right) = \frac{1}{2}$

58. $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right) = -\frac{\sqrt{2}}{2}$

59. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{3}{5}$

60. $\sin\left(\arcsin\left(-0.42\right)\right) = -0.42$

61. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$ is undefined.

62. $\cos\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right) = \frac{\sqrt{2}}{2}$

63. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$

64. $\cos\left(\arccos\left(\frac{5}{13}\right)\right) = \frac{5}{13}$

65. $\cos\left(\arccos\left(-0.998\right)\right) = -0.998$

66. $\cos\left(\arccos\left(\pi\right)\right)$ is undefined.

67. $\tan\left(\arctan\left(-1\right)\right) = -1$

68. $\tan\left(\arctan\left(\sqrt{3}\right)\right) = \sqrt{3}$

69. $\tan\left(\arctan\left(\frac{5}{12}\right)\right) = \frac{5}{12}$

70. $\tan\left(\arctan\left(0.965\right)\right) = 0.965$

71. $\tan\left(\arctan\left(3\pi\right)\right) = 3\pi$

72. $\cot\left(\text{arccot}\left(1\right)\right) = 1$

73. $\cot\left(\text{arccot}\left(-\sqrt{3}\right)\right) = -\sqrt{3}$

74. $\cot\left(\text{arccot}\left(-\frac{7}{24}\right)\right) = -\frac{7}{24}$

75. $\cot\left(\text{arccot}\left(-0.001\right)\right) = -0.001$

76. $\cot\left(\text{arccot}\left(\frac{17\pi}{4}\right)\right) = \frac{17\pi}{4}$

77. $\sec\left(\text{arcsec}\left(2\right)\right) = 2$

78. $\sec\left(\text{arcsec}\left(-1\right)\right) = -1$

79. $\sec\left(\text{arcsec}\left(\frac{1}{2}\right)\right)$ is undefined.

80. $\sec\left(\text{arcsec}\left(0.75\right)\right)$ is undefined.

81. $\sec\left(\text{arcsec}\left(117\pi\right)\right) = 117\pi$

82. $\csc\left(\text{arccsc}\left(\sqrt{2}\right)\right) = \sqrt{2}$

83. $\csc\left(\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right) = -\frac{2\sqrt{3}}{3}$

84. $\csc\left(\operatorname{arccsc}\left(\frac{\sqrt{2}}{2}\right)\right)$ is undefined.

85. $\csc(\operatorname{arccsc}(1.0001)) = 1.0001$

86. $\csc\left(\operatorname{arccsc}\left(\frac{\pi}{4}\right)\right)$ is undefined.

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

88. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$

89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$

90. $\arcsin\left(\sin\left(\frac{11\pi}{6}\right)\right) = -\frac{\pi}{6}$

91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{\pi}{3}$

92. $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$

94. $\arccos\left(\cos\left(\frac{3\pi}{2}\right)\right) = \frac{\pi}{2}$

95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

96. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right) = \frac{3\pi}{4}$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$

98. $\arctan\left(\tan\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$

99. $\arctan(\tan(\pi)) = 0$

100. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$ is undefined

101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right) = -\frac{\pi}{3}$

102. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$

103. $\operatorname{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right) = \frac{3\pi}{4}$

104. $\operatorname{arccot}(\cot(\pi))$ is undefined

105. $\operatorname{arccot}\left(\cot\left(\frac{3\pi}{2}\right)\right) = \frac{\pi}{2}$

106. $\operatorname{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$

107. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

108. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right) = \frac{2\pi}{3}$

109. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right) = \frac{5\pi}{6}$

110. $\operatorname{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$ is undefined.

111. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right) = \frac{\pi}{3}$

112. $\operatorname{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

113. $\operatorname{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right) = -\frac{\pi}{4}$

114. $\operatorname{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$

115. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right) = -\frac{\pi}{2}$

116. $\text{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right) = -\frac{\pi}{6}$

117. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right) = \frac{11\pi}{12}$

118. $\text{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right) = -\frac{\pi}{8}$

119. $\text{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

120. $\text{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right) = \frac{4\pi}{3}$

121. $\text{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right) = \frac{7\pi}{6}$

122. $\text{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$ is undefined.

123. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right) = \frac{\pi}{3}$

124. $\text{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

125. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right) = \frac{5\pi}{4}$

126. $\text{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$

127. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right) = \frac{3\pi}{2}$

128. $\text{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right) = \frac{7\pi}{6}$

129. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right) = \frac{13\pi}{12}$

130. $\text{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right) = \frac{9\pi}{8}$

131. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right) = \frac{\sqrt{3}}{2}$

132. $\sin\left(\arccos\left(\frac{3}{5}\right)\right) = \frac{4}{5}$

133. $\sin(\arctan(-2)) = -\frac{2\sqrt{5}}{5}$

134. $\sin\left(\text{arccot}\left(\sqrt{5}\right)\right) = \frac{\sqrt{6}}{6}$

135. $\sin(\text{arccsc}(-3)) = -\frac{1}{3}$

136. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right) = \frac{12}{13}$

137. $\cos\left(\arctan\left(\sqrt{7}\right)\right) = \frac{\sqrt{2}}{4}$

138. $\cos(\text{arccot}(3)) = \frac{3\sqrt{10}}{10}$

139. $\cos(\text{arcsec}(5)) = \frac{1}{5}$

140. $\tan\left(\arcsin\left(-\frac{2\sqrt{5}}{5}\right)\right) = -2$

141. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right) = -\sqrt{3}$

142. $\tan\left(\text{arcsec}\left(\frac{5}{3}\right)\right) = \frac{4}{3}$

143. $\tan(\text{arccot}(12)) = \frac{1}{12}$

144. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right) = \frac{5}{12}$

145. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right) = \sqrt{3}$

146. $\cot\left(\text{arccsc}\left(\sqrt{5}\right)\right) = 2$

147. $\cot(\arctan(0.25)) = 4$

148. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right) = \frac{2\sqrt{3}}{3}$

149. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right) = \frac{13}{5}$

150. $\sec(\arctan(10)) = \sqrt{101}$

151. $\sec\left(\operatorname{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right) = -\sqrt{11}$

152. $\csc(\operatorname{arccot}(9)) = \sqrt{82}$

153. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{5}{3}$

154. $\csc\left(\arctan\left(-\frac{2}{3}\right)\right) = -\frac{\sqrt{13}}{2}$

155. $\sin\left(\arcsin\left(\frac{5}{13}\right) + \frac{\pi}{4}\right) = \frac{17\sqrt{2}}{26}$

156. $\cos(\operatorname{arcsec}(3) + \arctan(2)) = \frac{\sqrt{5} - 4\sqrt{10}}{15}$

157. $\tan\left(\arctan(3) + \arccos\left(-\frac{3}{5}\right)\right) = \frac{1}{3}$

158. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right) = -\frac{24}{25}$

159. $\sin\left(2\operatorname{arccsc}\left(\frac{13}{5}\right)\right) = \frac{120}{169}$

160. $\sin(2\arctan(2)) = \frac{4}{5}$

161. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right) = \frac{7}{25}$

162. $\cos\left(2\operatorname{arcsec}\left(\frac{25}{7}\right)\right) = -\frac{527}{625}$

163. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right) = \frac{2}{3}$

164. $\sin\left(\frac{\arctan(2)}{2}\right) = \sqrt{\frac{5-\sqrt{5}}{10}}$

165. $f(x) = \sin(\arccos(x)) = \sqrt{1-x^2}$ for $-1 \leq x \leq 1$

166. $f(x) = \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$ for all x

167. $f(x) = \tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$ for $-1 < x < 1$

168. $f(x) = \sec(\arctan(x)) = \sqrt{1+x^2}$ for all x

169. $f(x) = \csc(\arccos(x)) = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$

170. $f(x) = \sin(2\arctan(x)) = \frac{2x}{x^2+1}$ for all x

171. $f(x) = \sin(2\arccos(x)) = 2x\sqrt{1-x^2}$ for $-1 \leq x \leq 1$

172. $f(x) = \cos(2\arctan(x)) = \frac{1-x^2}{1+x^2}$ for all x

173. $f(x) = \sin(\arccos(2x)) = \sqrt{1-4x^2}$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$

174. $f(x) = \sin\left(\arccos\left(\frac{x}{5}\right)\right) = \frac{\sqrt{25 - x^2}}{5}$ for $-5 \leq x \leq 5$

175. $f(x) = \cos\left(\arcsin\left(\frac{x}{2}\right)\right) = \frac{\sqrt{4 - x^2}}{2}$ for $-2 \leq x \leq 2$

176. $f(x) = \cos\left(\arctan\left(3x\right)\right) = \frac{1}{\sqrt{1 + 9x^2}}$ for all x

177. $f(x) = \sin(2\arcsin(7x)) = 14x\sqrt{1 - 49x^2}$ for $-\frac{1}{7} \leq x \leq \frac{1}{7}$

178. $f(x) = \sin\left(2\arcsin\left(\frac{x\sqrt{3}}{3}\right)\right) = \frac{2x\sqrt{3 - x^2}}{3}$ for $-\sqrt{3} \leq x \leq \sqrt{3}$

179. $f(x) = \cos(2\arcsin(4x)) = 1 - 32x^2$ for $-\frac{1}{4} \leq x \leq \frac{1}{4}$

180. $f(x) = \sec(\arctan(2x))\tan(\arctan(2x)) = 2x\sqrt{1 + 4x^2}$ for all x

181. $f(x) = \sin\left(\arcsin(x) + \arccos(x)\right) = 1$ for $-1 \leq x \leq 1$

182. $f(x) = \cos\left(\arcsin(x) + \arctan(x)\right) = \frac{\sqrt{1 - x^2} - x^2}{\sqrt{1 + x^2}}$ for $-1 \leq x \leq 1$

183.[12] $f(x) = \tan\left(2\arcsin(x)\right) = \frac{2x\sqrt{1 - x^2}}{1 - 2x^2}$ for x in $\left(-1, -\frac{\sqrt{2}}{2}\right) \cup \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, 1\right)$

184. $f(x) = \sin\left(\frac{1}{2}\arctan(x)\right) = \begin{cases} \sqrt{\dfrac{\sqrt{x^2 + 1} - 1}{2\sqrt{x^2 + 1}}} & \text{for } x \geq 0 \\ -\sqrt{\dfrac{\sqrt{x^2 + 1} - 1}{2\sqrt{x^2 + 1}}} & \text{for } x < 0 \end{cases}$

185. $\theta + \sin(2\theta) = \arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4 - x^2}}{2}$

186. $\frac{1}{2}\theta - \frac{1}{2}\sin(2\theta) = \frac{1}{2}\arctan\left(\frac{x}{7}\right) - \frac{7x}{x^2 + 49}$

187. $4\tan(\theta) - 4\theta = \sqrt{x^2 - 16} - 4\text{arcsec}\left(\frac{x}{4}\right)$

188. $\theta = \arcsin\left(\frac{7}{11}\right) + 2\pi k$ or $\theta = \pi - \arcsin\left(\frac{7}{11}\right) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 0.6898,\ 2.4518$

[12] The equivalence for $x = \pm 1$ can be verified independently of the derivation of the formula, but Calculus is required to fully understand what is happening at those x values. You'll see what we mean when you work through the details of the identity for $\tan(2t)$. For now, we exclude $x = \pm 1$ from our answer.

189. $\theta = \arccos\left(-\dfrac{2}{9}\right) + 2\pi k$ or $\theta = -\arccos\left(-\dfrac{2}{9}\right) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 1.7949,\ 4.4883$

190. $\theta = \pi + \arcsin(0.569) + 2\pi k$ or $\theta = 2\pi - \arcsin(0.569) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 3.7469,\ 5.6779$

191. $\theta = \arccos(0.117) + 2\pi k$ or $\theta = 2\pi - \arccos(0.117) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 1.4535,\ 4.8297$

192. $\theta = \arcsin(0.008) + 2\pi k$ or $\theta = \pi - \arcsin(0.008) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 0.0080,\ 3.1336$

193. $\theta = \arccos\left(\dfrac{359}{360}\right) + 2\pi k$ or $\theta = 2\pi - \arccos\left(\dfrac{359}{360}\right) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 0.0746,\ 6.2086$

194. $t = \arctan(117) + \pi k$, in $[0, 2\pi)$, $t \approx 1.56225,\ 4.70384$

195. $t = \arctan\left(-\dfrac{1}{12}\right) + \pi k$, in $[0, 2\pi)$, $t \approx 3.0585,\ 6.2000$

196. $t = \arccos\left(\dfrac{2}{3}\right) + 2\pi k$ or $t = 2\pi - \arccos\left(\dfrac{2}{3}\right) + 2\pi k$, in $[0, 2\pi)$, $t \approx 0.8411,\ 5.4422$

197. $t = \pi + \arcsin\left(\dfrac{17}{90}\right) + 2\pi k$ or $t = 2\pi - \arcsin\left(\dfrac{17}{90}\right) + 2\pi k$, in $[0, 2\pi)$, $t \approx 3.3316,\ 6.0932$

198. $t = \arctan\left(-\sqrt{10}\right) + \pi k$, in $[0, 2\pi)$, $t \approx 1.8771,\ 5.0187$

199. $t = \arcsin\left(\dfrac{3}{8}\right) + 2\pi k$ or $t = \pi - \arcsin\left(\dfrac{3}{8}\right) + 2\pi k$, in $[0, 2\pi)$, $t \approx 0.3844,\ 2.7572$

200. $x = \arccos\left(-\dfrac{7}{16}\right) + 2\pi k$ or $x = -\arccos\left(-\dfrac{7}{16}\right) + 2\pi k$, in $[0, 2\pi)$, $x \approx 2.0236,\ 4.2596$

201. $x = \arctan(0.03) + \pi k$, in $[0, 2\pi)$, $x \approx 0.0300,\ 3.1716$

202. $x = \arcsin(0.3502) + 2\pi k$ or $x = \pi - \arcsin(0.3502) + 2\pi k$, in $[0, 2\pi)$, $x \approx 0.3578,\ 2.784$

203. $x = \pi + \arcsin(0.721) + 2\pi k$ or $x = 2\pi - \arcsin(0.721) + 2\pi k$, in $[0, 2\pi)$, $x \approx 3.9468,\ 5.4780$

204. $x = \arccos(0.9824) + 2\pi k$ or $x = 2\pi - \arccos(0.9824) + 2\pi k$, in $[0, 2\pi)$, $x \approx 0.1879,\ 6.0953$

205. $x = \arccos(-0.5637) + 2\pi k$ or $x = -\arccos(-0.5637) + 2\pi k$, in $[0, 2\pi)$, $x \approx 2.1697,\ 4.1135$

206. $x = \arctan(117) + \pi k$, in $[0, 2\pi)$, $x \approx 1.5622,\ 4.7038$

207. $x = \arctan(-0.6109) + \pi k$, in $[0, 2\pi)$, $x \approx 2.5932,\ 5.7348$

208. $f(t) = 5\sin(3t) + 12\cos(3t) = 13\sin\left(3t + \arcsin\left(\dfrac{12}{13}\right)\right) \approx 13\sin(3t + 1.1760)$

$f(t) = 5\sin(3t) + 12\cos(3t) = 13\cos\left(3t + \arcsin\left(-\dfrac{5}{13}\right)\right) \approx 13\cos(3t - 0.3948)$

209. $f(t) = 3\cos(2t) + 4\sin(2t) = 5\sin\left(2t + \arcsin\left(\frac{3}{5}\right)\right) \approx 5\sin(2t + 0.6435)$

$f(t) = 3\cos(2t) + 4\sin(2t) = 5\cos\left(2t + \arcsin\left(-\frac{4}{5}\right)\right) \approx 5\cos(2t - 0.9273)$

210. $f(t) = \cos(t) - 3\sin(t) = \sqrt{10}\sin\left(t + \arccos\left(-\frac{3\sqrt{10}}{10}\right)\right) \approx \sqrt{10}\sin(t + 2.8198)$

$f(t) = \cos(t) - 3\sin(t) = \sqrt{10}\cos\left(t + \arcsin\left(\frac{3\sqrt{10}}{10}\right)\right) \approx \sqrt{10}\cos(t + 1.2490)$

211. $f(t) = 7\sin(10t) - 24\cos(10t) = 25\sin\left(10t + \arcsin\left(-\frac{24}{25}\right)\right) \approx 25\sin(10t - 1.2870)$

$f(t) = 7\sin(10t) - 24\cos(10t) = 25\cos\left(10t + \pi + \arcsin\left(\frac{7}{25}\right)\right) \approx 25\cos(10t + 3.4254)$

212. $f(t) = -\cos(t) - 2\sqrt{2}\sin(t) = 3\sin\left(t + \pi + \arcsin\left(\frac{1}{3}\right)\right) \approx 3\sin(t + 3.4814)$

$f(t) = -\cos(t) - 2\sqrt{2}\sin(t) = 3\cos\left(t + \arccos\left(-\frac{1}{3}\right)\right) \approx 3\sin(t + 1.9106)$

213. $f(t) = 2\sin(t) - \cos(t) = \sqrt{5}\sin\left(t + \arcsin\left(-\frac{\sqrt{5}}{5}\right)\right) \approx \sqrt{5}\sin(t - 0.4636)$

$f(t) = 2\sin(t) - \cos(t) = \sqrt{5}\cos\left(t + \pi + \arcsin\left(\frac{2\sqrt{5}}{5}\right)\right) \approx \sqrt{5}\cos(t + 4.2487)$

214. $\left[-\frac{1}{5}, \frac{1}{5}\right]$

215. $\left[-\frac{1}{3}, 1\right]$

216. $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$

217. $(-\infty, -\sqrt{5}] \cup [-\sqrt{3}, \sqrt{3}] \cup [\sqrt{5}, \infty)$

218. $(-\infty, \infty)$

219. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

220. $\left(\frac{1}{2}, \infty\right)$

221. $\left[\frac{1}{2}, \infty\right)$

222. $\left(-\infty, -\frac{1}{12}\right] \cup \left[\frac{1}{12}, \infty\right)$

223. $(-\infty, -6] \cup [-4, \infty)$

224. $(-\infty, -2] \cup [2, \infty)$

225. $[0, \infty)$

226. (a) $\lim\limits_{x \to 1^-} \arcsin(x) = \frac{\pi}{2}$ (b) $\lim\limits_{x \to -\infty} \arctan(3x) = -\infty$ (c) $\lim\limits_{x \to 1} \operatorname{arcsec}(2x) = \frac{\pi}{3}$

12.4 Equations and Inequalities Involving the Circular Functions

In Sections 11.2, 11.4 and most recently 12.3, we solved some basic equations involving the trigonometric functions. Below we summarize the techniques we've employed thus far. Note that we use the neutral letter 'u' as the argument of each circular function for generality.

Strategies for Solving Basic Equations Involving the Circular Functions

- To solve $\cos(u) = c$ or $\sin(u) = c$ for $-1 \leq c \leq 1$, first solve for u in the interval $[0, 2\pi)$ and add integer multiples of the period 2π. If $c < -1$ or of $c > 1$, there are no real solutions.
- To solve $\sec(u) = c$ or $\csc(u) = c$ for $c \leq -1$ or $c \geq 1$, convert to cosine or sine, respectively, and solve as above. If $-1 < c < 1$, there are no real solutions.
- To solve $\tan(u) = c$ for any real number c, first solve for u in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and add integer multiples of the period π.
- To solve $\cot(u) = c$ for $c \neq 0$, convert to tangent and solve as above. If $c = 0$, the solution to $\cot(u) = 0$ is $u = \frac{\pi}{2} + \pi k$ for integers k.

Using the above guidelines, we can comfortably solve $\sin(x) = \frac{1}{2}$ and find the solution $x = \frac{\pi}{6} + 2\pi k$ or $x = \frac{5\pi}{6} + 2\pi k$ for integers k. But how do we solve the related equation $\sin(3x) = \frac{1}{2}$?

Since this equation has the *form* $\sin(u) = \frac{1}{2}$, we know the solutions take the form $u = \frac{\pi}{6} + 2\pi k$ or $u = \frac{5\pi}{6} + 2\pi k$ for integers k. Since the argument of sine here is $3x$, we have $3x = \frac{\pi}{6} + 2\pi k$ or $3x = \frac{5\pi}{6} + 2\pi k$.

To solve for x, we divide both sides[1] of these equations by 3, and obtain $x = \frac{\pi}{18} + \frac{2\pi}{3}k$ or $x = \frac{5\pi}{18} + \frac{2\pi}{3}k$ for integers k. This is the technique employed in the example below.

Example 12.4.1. Solve the following equations and check your answers analytically. List the solutions which lie in the interval $[0, 2\pi)$ and verify them using a graphing utility.

1. $\cos(2\theta) = -\frac{\sqrt{3}}{2}$
2. $\csc\left(\frac{1}{3}\theta - \pi\right) = \sqrt{2}$
3. $\cot(3t) = 0$
4. $\sec^2(t) = 4$
5. $\tan\left(\frac{x}{2}\right) = -3$
6. $\sin(2x) = 0.87$

Solution.

1. The solutions to $\cos(u) = -\frac{\sqrt{3}}{2}$ are $u = \frac{5\pi}{6} + 2\pi k$ or $u = \frac{7\pi}{6} + 2\pi k$ for integers k.

   Since the argument of cosine here is 2θ, this means $2\theta = \frac{5\pi}{6} + 2\pi k$ or $2\theta = \frac{7\pi}{6} + 2\pi k$ for integers k. Solving for θ gives $\theta = \frac{5\pi}{12} + \pi k$ or $\theta = \frac{7\pi}{12} + \pi k$ for integers k.

   To check these answers analytically, we substitute them into the original equation. For any integer k:

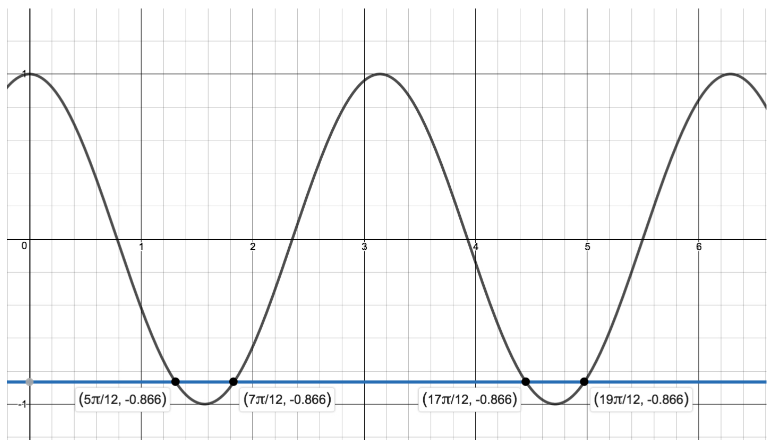
[1] Don't forget to divide the $2\pi k$ by 3 as well!

$$\begin{aligned}
\cos\left(2\left[\tfrac{5\pi}{12}+\pi k\right]\right) &= \cos\left(\tfrac{5\pi}{6}+2\pi k\right) \\
&= \cos\left(\tfrac{5\pi}{6}\right) \qquad \text{(the period of cosine is } 2\pi\text{)} \\
&= -\tfrac{\sqrt{3}}{2}
\end{aligned}$$

Similarly, we find $\cos\left(2\left[\frac{7\pi}{12}+\pi k\right]\right) = \cos\left(\frac{7\pi}{6}+2\pi k\right) = \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

To determine which of our solutions lie in $[0, 2\pi)$, we substitute integer values for k. The solutions we keep come from the values of $k = 0$ and $k = 1$ and are $\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}$ and $\frac{19\pi}{12}$.

Using a calculator, we graph $y = \cos(2\theta)$ and $y = -\frac{\sqrt{3}}{2}$ over $[0, 2\pi)$ and examine where these two graphs intersect to verify our answers.



$\boldsymbol{y = \cos(2\theta)}$ and $y = -\frac{\sqrt{3}}{2}$

2. Since this equation has the form $\csc(u) = \sqrt{2}$, we rewrite this as $\sin(u) = \frac{\sqrt{2}}{2}$ and find $u = \frac{\pi}{4} + 2\pi k$ or $u = \frac{3\pi}{4} + 2\pi k$ for integers k.

   Since the argument of cosecant here is $\left(\frac{1}{3}\theta - \pi\right)$, $\frac{1}{3}\theta - \pi = \frac{\pi}{4} + 2\pi k$ or $\frac{1}{3}\theta - \pi = \frac{3\pi}{4} + 2\pi k$.

   To solve $\frac{1}{3}\theta - \pi = \frac{\pi}{4} + 2\pi k$, we first add π to both sides to get $\frac{1}{3}\theta = \frac{\pi}{4} + 2\pi k + \pi$. A common error is to treat the '$2\pi k$' and 'π' terms as 'like' terms and try to combine them when they are not.

   We can, however, combine the 'π' and '$\frac{\pi}{4}$' terms to get $\frac{1}{3}\theta = \frac{5\pi}{4} + 2\pi k$.

   We now finish by multiplying both sides by 3 to get $\theta = 3\left(\frac{5\pi}{4} + 2\pi k\right) = \frac{15\pi}{4} + 6\pi k$, where k, as always, runs through the integers.

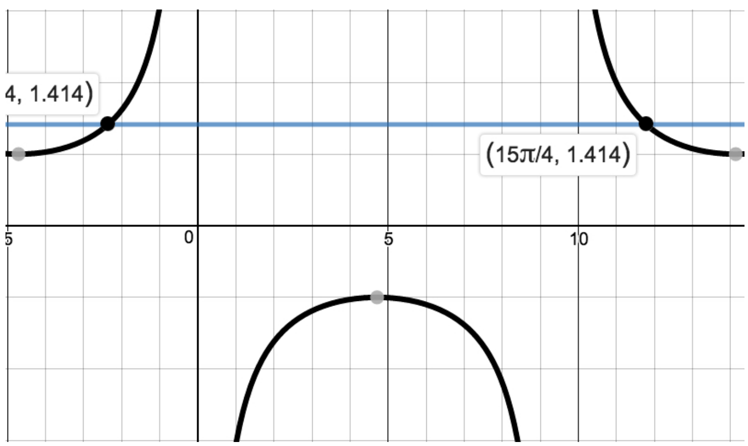
   Solving the other equation, $\frac{1}{3}\theta - \pi = \frac{3\pi}{4} + 2\pi k$ produces $\theta = \frac{21\pi}{4} + 6\pi k$ for integers k. To check the first family of answers, we substitute, combine line terms, and simplify.

$$\begin{aligned}
\csc\left(\tfrac{1}{3}\left[\tfrac{15\pi}{4}+6\pi k\right]-\pi\right) &= \csc\left(\tfrac{5\pi}{4}+2\pi k-\pi\right) \\
&= \csc\left(\tfrac{\pi}{4}+2\pi k\right) \\
&= \csc\left(\tfrac{\pi}{4}\right) \qquad \text{(the period of cosecant is } 2\pi) \\
&= \sqrt{2}
\end{aligned}$$

The family $\theta = \frac{21\pi}{4} + 6\pi k$ checks similarly.

Despite having infinitely many solutions, we find that *none* of them lie in $[0, 2\pi)$.

To verify this graphically, we check that $y = \csc\left(\frac{1}{3}\theta - \pi\right)$ and $y = \sqrt{2}$ do not intersect at all over the interval $[0, 2\pi)$.



$y = \csc\left(\frac{1}{3}\theta - \pi\right)$ and $y = \sqrt{2}$

3. Since $\cot(3t) = 0$ has the form $\cot(u) = 0$, we know $u = \frac{\pi}{2} + \pi k$, so, in this case, $3t = \frac{\pi}{2} + \pi k$ for integers k.

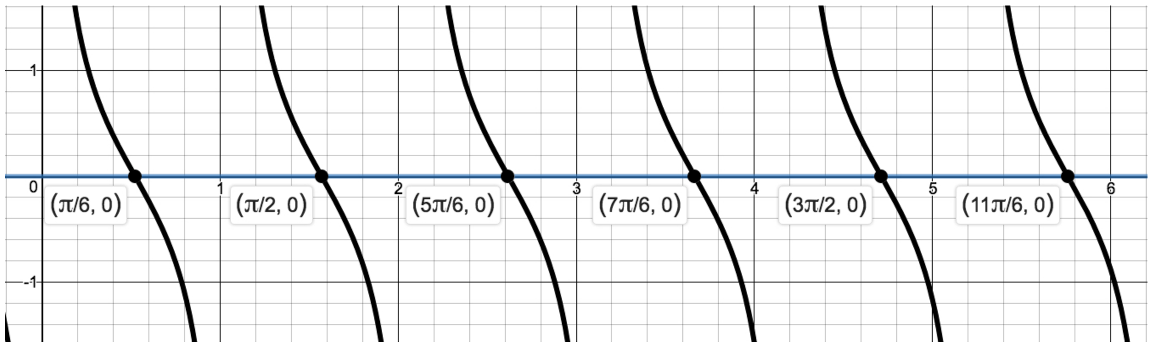
   Solving for t yields $t = \frac{\pi}{6} + \frac{\pi}{3}k$. Checking our answers, we get

$$\begin{aligned}
\cot\left(3\left[\tfrac{\pi}{6}+\tfrac{\pi}{3}k\right]\right) &= \cot\left(\tfrac{\pi}{2}+\pi k\right) \\
&= \cot\left(\tfrac{\pi}{2}\right) \qquad \text{(the period of cotangent is } \pi) \\
&= 0
\end{aligned}$$

   As k runs through the integers, we obtain six answers, corresponding to $k = 0$ through $k = 5$, which lie in $[0, 2\pi)$: $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$ and $\frac{11\pi}{6}$.

   Graphing $y = \cot(3t)$ and $y = 0$ (the t-axis), we confirm our result.[2]

[2]On many calculators, there is no function button for cotangent. In that case, we would use the quotient identity and graph $y = \frac{\cos(3t)}{\sin(3t)}$ instead. The reader is invited to see what happens if we would graph $y = \frac{1}{\tan(3t)}$ instead.



$y = \cot(3t)$ and $y = 0$

4. The complication in solving an equation like $\sec^2(t) = 4$ comes not from the argument of secant, which is just t, but rather, the fact the secant is being squared: $\sec^2(t) = (\sec(t))^2 = 4$.

   To get this equation to look like one of the forms listed on page 1086, we extract square roots to get $\sec(t) = \pm 2$. Converting to cosines, we have $\cos(t) = \pm \frac{1}{2}$.

   For $\cos(t) = \frac{1}{2}$, we get $t = \frac{\pi}{3} + 2\pi k$ or $t = \frac{5\pi}{3} + 2\pi k$ for integers k. For $\cos(t) = -\frac{1}{2}$, we get $t = \frac{2\pi}{3} + 2\pi k$ or $t = \frac{4\pi}{3} + 2\pi k$ for integers k.

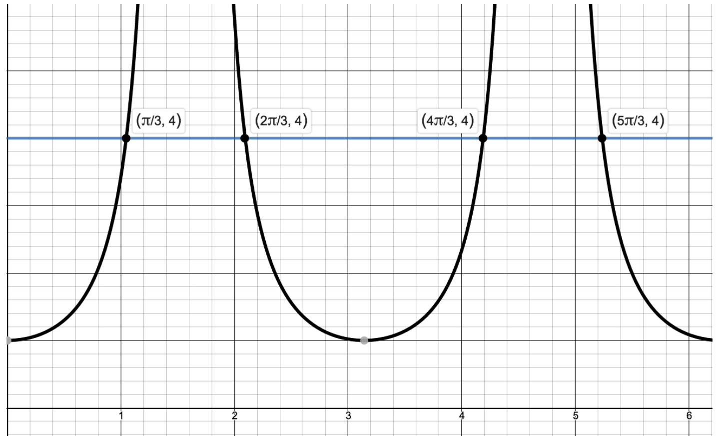
   If we take a step back and think of these families of solutions geometrically, we see we are finding the measures of all angles with a reference angle of $\frac{\pi}{3}$. As a result, these solutions can be combined and we may write our solutions as $t = \frac{\pi}{3} + \pi k$ and $t = \frac{2\pi}{3} + \pi k$ for integers k.

   To check the first family of solutions, we note that, depending on the integer k, $\sec\left(\frac{\pi}{3} + \pi k\right)$ doesn't always equal $\sec\left(\frac{\pi}{3}\right)$. It is true, though, that for all integers k, $\sec\left(\frac{\pi}{3} + \pi k\right) = \pm \sec\left(\frac{\pi}{3}\right) = \pm 2$. (Can you show this?) Hence, checking our first family of solutions gives:

$$\begin{aligned}
\sec^2\left(\frac{\pi}{3} + \pi k\right) &= \left(\pm \sec\left(\frac{\pi}{3}\right)\right)^2 \\
&= (\pm 2)^2 \\
&= 4
\end{aligned}$$

   The check for the family of solutions $t = \frac{2\pi}{3} + \pi k$ is similar.

   The solutions which lie in $[0, 2\pi)$ come from the values $k = 0$ and $k = 1$, namely $t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$. Graphing $y = (\sec(t))^2$ and $y = 4$ confirms our results.



***y* = (sec(*t*))^2** and $y = 4$

5. The equation $\tan\left(\frac{x}{2}\right) = -3$ has the form $\tan(u) = -3$, whose solution is $u = \arctan(-3) + \pi k$.

   Hence, $\frac{x}{2} = \arctan(-3) + \pi k$, so $x = 2\arctan(-3) + 2\pi k$ for integers k. To check, we note

$$\begin{array}{rcll}\tan\left(\frac{2\arctan(-3)+2\pi k}{2}\right) & = & \tan\left(\arctan(-3) + \pi k\right) & \\ & = & \tan\left(\arctan(-3)\right) & \text{(the period of tangent is } \pi) \\ & = & -3 & \text{(See Theorem 12.15)}\end{array}$$

   To determine which of our answers lie in the interval $[0, 2\pi)$, we first need to get an idea of the value of $2\arctan(-3)$. While we could easily find an approximation using a calculator,[3] we proceed analytically, as is our custom.

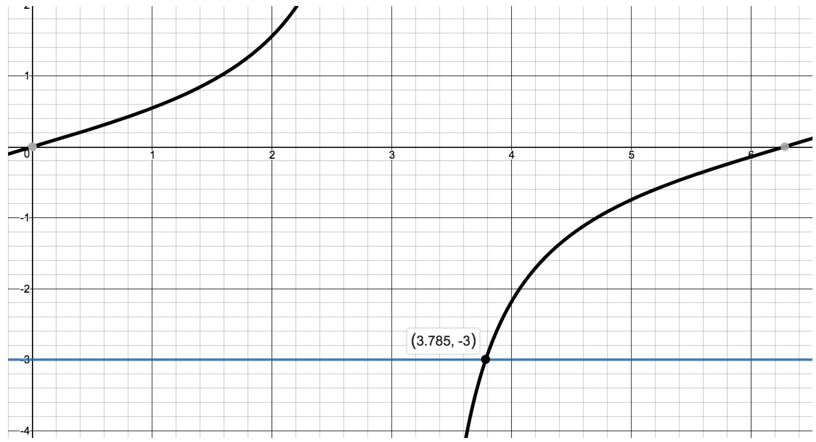
   To get started, we note that since $-3 < 0$, it $-\frac{\pi}{2} < \arctan(-3) < 0$. Hence, $-\pi < 2\arctan(-3) < 0$. With regard to our solutions, $x = 2\arctan(-3) + 2\pi k$, we see for $k = 0$, we get $x = 2\arctan(-3) < 0$, so we discard this answer and all answers $x = 2\arctan(-3) + 2\pi k$ where $k < 0$.

   Next, we turn our attention to $k = 1$ and get $x = 2\arctan(-3) + 2\pi$. Starting with the inequality $-\pi < 2\arctan(-3) < 0$, we add through 2π and get $\pi < 2\arctan(-3) + 2\pi < 2\pi$. This means $x = 2\arctan(-3) + 2\pi$ lies in $[0, 2\pi)$.

   Advancing k to 2 produces $x = 2\arctan(-3) + 4\pi$. Once again, we get from $-\pi < 2\arctan(-3) < 0$ that $3\pi < 2\arctan(-3) + 4\pi < 4\pi$. Since this is outside the interval of interest, $[0, 2\pi)$, we discard $x = 2\arctan(-3) + 4\pi$ and all solutions of the form $x = 2\arctan(-3) + 2\pi k$ for $k > 2$.

   Graphically, $y = \tan\left(\frac{x}{2}\right)$ and $y = -3$ intersect only once on $[0, 2\pi)$ at $x = 2\arctan(-3) + 2\pi \approx 3.785$.

[3] Your instructor will let you know if you should abandon the analytic route at this point and use your calculator.



$y = \tan\left(\frac{x}{2}\right)$ and $y = -3$

6. To solve $\sin(2x) = 0.87$, we first note that it has the form $\sin(u) = 0.87$, which has the family of solutions $u = \arcsin(0.87) + 2\pi k$ or $u = \pi - \arcsin(0.87) + 2\pi k$ for integers k.

   Since the argument of sine here is $2x$, we get $2x = \arcsin(0.87) + 2\pi k$ or $2x = \pi - \arcsin(0.87) + 2\pi k$ which gives $x = \frac{1}{2}\arcsin(0.87) + \pi k$ or $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k$ for integers k. To check,

$$\begin{array}{rcll} \sin\left(2\left[\frac{1}{2}\arcsin(0.87) + \pi k\right]\right) & = & \sin\left(\arcsin(0.87) + 2\pi k\right) & \\ & = & \sin\left(\arcsin(0.87)\right) & \text{(the period of sine is } 2\pi\text{)} \\ & = & 0.87 & \text{(See Theorem 12.14)} \end{array}$$

   For the family $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k$, we get

$$\begin{array}{rcll} \sin\left(2\left[\frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k\right]\right) & = & \sin\left(\pi - \arcsin(0.87) + 2\pi k\right) & \\ & = & \sin\left(\pi - \arcsin(0.87)\right) & \text{(the period of sine is } 2\pi\text{)} \\ & = & \sin\left(\arcsin(0.87)\right) & (\sin(\pi - t) = \sin(t)) \\ & = & 0.87 & \text{(See Theorem 12.14)} \end{array}$$

   To determine which of these solutions lie in $[0, 2\pi)$, we first need to get an idea of the value of $x = \frac{1}{2}\arcsin(0.87)$. Once again, we could use the calculator, but we adopt an analytic route here.

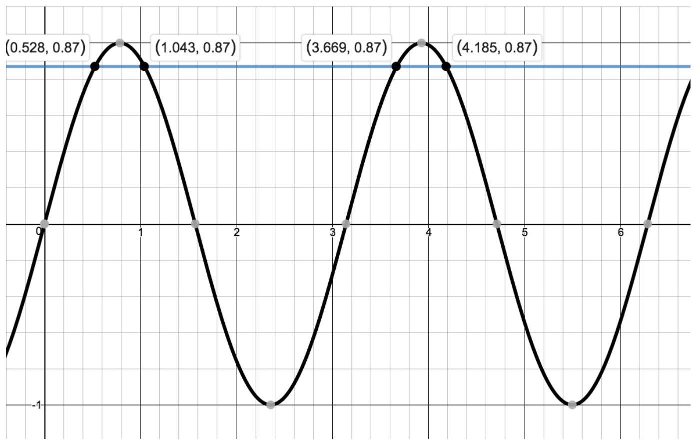
   By definition, $0 < \arcsin(0.87) < \frac{\pi}{2}$ so that multiplying through by $\frac{1}{2}$ gives us $0 < \frac{1}{2}\arcsin(0.87) < \frac{\pi}{4}$.

   Starting with the family of solutions $x = \frac{1}{2}\arcsin(0.87) + \pi k$, we use the same kind of arguments as in our solution to number 5 above and find only the solutions corresponding to $k = 0$ and $k = 1$ lie in $[0, 2\pi)$: $x = \frac{1}{2}\arcsin(0.87)$ and $x = \frac{1}{2}\arcsin(0.87) + \pi$.

   Next, we move to the family $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k$ for integers k. Here, we need to get a better estimate of $\frac{\pi}{2} - \frac{1}{2}\arcsin(0.87)$. From the inequality $0 < \frac{1}{2}\arcsin(0.87) < \frac{\pi}{4}$, we first multiply through by -1 and then add $\frac{\pi}{2}$ to get $\frac{\pi}{2} > \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) > \frac{\pi}{4}$, or $\frac{\pi}{4} < \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) < \frac{\pi}{2}$.

Proceeding with the usual arguments, we find the only solutions which lie in $[0, 2\pi)$ correspond to $k = 0$ and $k = 1$, namely $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87)$ and $x = \frac{3\pi}{2} - \frac{1}{2}\arcsin(0.87)$.

All told, we have found four solutions to $\sin(2x) = 0.87$ in $[0, 2\pi)$: $x = \frac{1}{2}\arcsin(0.87) \approx 0.528$, $x = \frac{1}{2}\arcsin(0.87) + \pi \approx 3.669$, $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) \approx 1.043$ and $x = \frac{3\pi}{2} - \frac{1}{2}\arcsin(0.87) \approx 4.185$. By graphing $y = \sin(2x)$ and $y = 0.87$, we confirm our results.



$y = \sin(2x)$ and $y = 0.87$

□

If one looks closely at the equations and solutions in Example 12.4.1, an interesting relationship evolves between the frequency of the circular function involved in the equation and how many solutions one can expect in the interval $[0, 2\pi)$. This relationship is explored in Exercise 108.

Each of the problems in Example 12.4.1 featured one circular function. If an equation involves two different circular functions or if the equation contains the same circular function but with different arguments, we will need to employ identities and Algebra to reduce the equation to the same form as those given on page 1086. We demonstrate these techniques in the following example.

Example 12.4.2. Solve the following equations and list the solutions which lie in the interval $[0, 2\pi)$. Verify your solutions on $[0, 2\pi)$ graphically.

1. $3\sin^3(\theta) = \sin^2(\theta)$
2. $\sec^2(\theta) = \tan(\theta) + 3$
3. $\cos(2t) = 3\cos(t) - 2$
4. $\cos(3t) = 2 - \cos(t)$
5. $\cos(3x) = \cos(5x)$
6. $\sin(2x) = \sqrt{3}\cos(x)$
7. $\sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) = 1$
8. $\cos(x) - \sqrt{3}\sin(x) = 2$

Solution.

1. One approach to solving $3\sin^3(\theta) = \sin^2(\theta)$ begins with dividing both sides by $\sin^2(\theta)$. Doing so, however, assumes that $\sin^2(\theta) \neq 0$ which means we risk losing solutions.

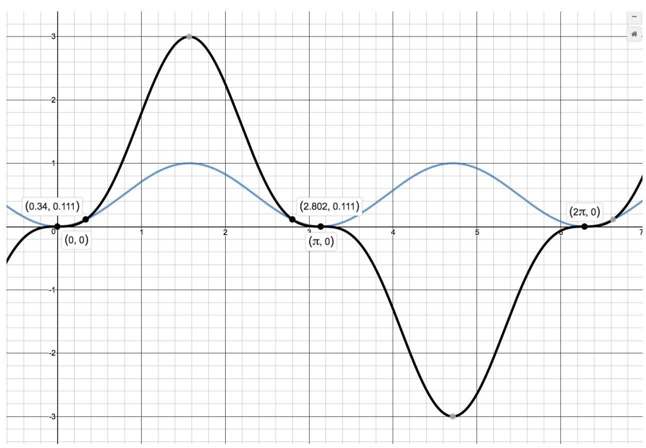
   Instead, we take a cue from Chapter 2 (since what we have here is a polynomial equation in terms of $\sin(\theta)$) and gather all the nonzero terms on one side and factor:

$$\begin{array}{rcll} 3\sin^3(\theta) & = & \sin^2(\theta) & \\ 3\sin^3(\theta) - \sin^2(\theta) & = & 0 & \\ \sin^2(\theta)(3\sin(\theta) - 1) & = & 0 & \quad \text{Factor out } \sin^2(\theta) \text{ from both terms.} \end{array}$$

   We get $\sin^2(\theta) = 0$ or $3\sin(\theta) - 1 = 0$, so $\sin(\theta) = 0$ or $\sin(\theta) = \frac{1}{3}$. The solution to $\sin(\theta) = 0$ is $\theta = \pi k$, with $\theta = 0$ and $\theta = \pi$ being the two solutions which lie in $[0, 2\pi)$.

   To solve $\sin(\theta) = \frac{1}{3}$, we use the arcsine function to get $\theta = \arcsin\left(\frac{1}{3}\right) + 2\pi k$ or $\theta = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k$ for integers k. We find the two solutions here which lie in $[0, 2\pi)$ to be $\theta = \arcsin\left(\frac{1}{3}\right) \approx 0.34$ and $\theta = \pi - \arcsin\left(\frac{1}{3}\right) \approx 2.80$.

   To check graphically, we plot $y = 3(\sin(\theta))^3$ and $y = (\sin(\theta))^2$ and find the θ-coordinates of the intersection points of these two curves.[4] (Some extra zooming may be required near $\theta = 0$ and $\theta = \pi$ to verify that these two curves do in fact intersect four times.)



$y = 3(\sin(\theta))^3$ and $y = (\sin(\theta))^2$

2. We see immediately in the equation $\sec^2(\theta) = \tan(\theta) + 3$ that there are two different circular functions present, so we look for an identity to express both sides in terms of the same function.

   We use the Pythagorean Identity $\sec^2(\theta) = 1 + \tan^2(\theta)$ to exchange $\sec^2(\theta)$ for tangents. What results is a 'quadratic in disguise:'[5]

[4] Note that we do *not* list $\theta = 2\pi$ as part of the solution over the interval $[0, 2\pi)$ since 2π is not in $[0, 2\pi)$.

[5] See Section A.10 for a review of this concept.

$$\begin{array}{rcll}
\sec^2(\theta) & = & \tan(\theta) + 3 & \\
1 + \tan^2(\theta) & = & \tan(\theta) + 3 & (\text{Since } \sec^2(\theta) = 1 + \tan^2(\theta).) \\
\tan^2(\theta) - \tan(\theta) - 2 & = & 0 & \\
u^2 - u - 2 & = & 0 & \text{Let } u = \tan(\theta). \\
(u+1)(u-2) & = & 0 &
\end{array}$$

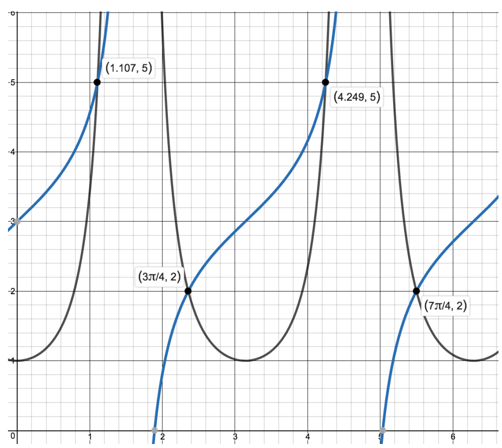
This gives $u = -1$ or $u = 2$. Since $u = \tan(\theta)$, we have $\tan(\theta) = -1$ or $\tan(\theta) = 2$.

From $\tan(\theta) = -1$, we get $\theta = -\frac{\pi}{4} + \pi k$ for integers k. To solve $\tan(\theta) = 2$, we employ the arctangent function and get $\theta = \arctan(2) + \pi k$ for integers k.

From the first set of solutions, we get $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$ as our answers which lie in $[0, 2\pi)$.

Using the same sort of argument we saw in Example 12.4.1, we get $\theta = \arctan(2) \approx 1.107$ and $\theta = \pi + \arctan(2) \approx 4.249$ as answers from our second set of solutions which lie in $[0, 2\pi)$.

We verify our solutions below graphically.



$y = (\sec(\theta))^2$ and $y = \tan(\theta) + 3$

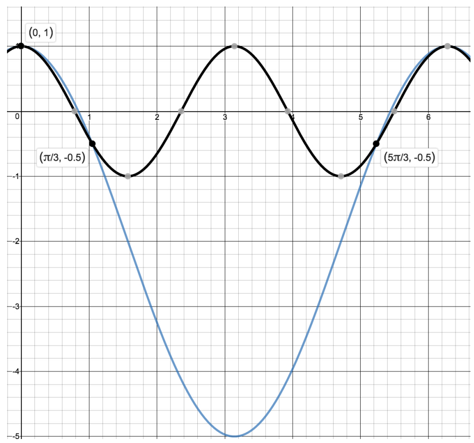
3. The good news is that in the equation $\cos(2t) = 3\cos(t) - 2$, we have the same circular function, cosine, throughout. The bad news is that we have different arguments, $2t$ and t.

   Using the double angle identity $\cos(2t) = 2\cos^2(t) - 1$ results in another quadratic in disguise:'

$$\begin{array}{rcll}
\cos(2t) & = & 3\cos(t) - 2 & \\
2\cos^2(t) - 1 & = & 3\cos(t) - 2 & (\text{Since } \cos(2t) = 2\cos^2(t) - 1.) \\
2\cos^2(t) - 3\cos(t) + 1 & = & 0 & \\
2u^2 - 3u + 1 & = & 0 & \text{Let } u = \cos(t). \\
(2u-1)(u-1) & = & 0 &
\end{array}$$

   We get $u = \frac{1}{2}$ or $u = 1$, so $\cos(t) = \frac{1}{2}$ or $\cos(t) = 1$. Solving $\cos(t) = \frac{1}{2}$, we get $t = \frac{\pi}{3} + 2\pi k$ or $t = \frac{5\pi}{3} + 2\pi k$ for integers k. From $\cos(t) = 1$, we get $t = 2\pi k$ for integers k.

The answers which lie in $[0, 2\pi)$ are $t = 0$, $\frac{\pi}{3}$, and $\frac{5\pi}{3}$. Graphing $y = \cos(2t)$ and $y = 3\cos(t) - 2$, we find that the curves intersect in three places on $[0, 2\pi)$ and confirm our results.



$\boldsymbol{y = \cos(2t)}$ and $y = 3\cos(t) - 2$

4. To solve $\cos(3t) = 2 - \cos(t)$, we take a cue from the previous problem and look for an identity to rewrite $\cos(3t)$ in terms of $\cos(t)$.

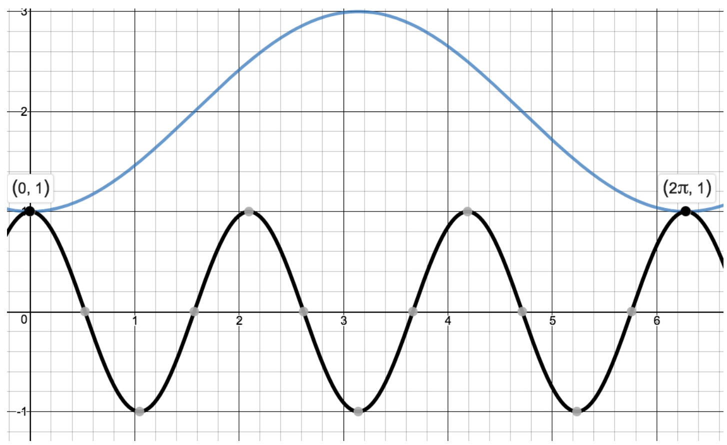
   From Example 12.2.3, number 4, we know that $\cos(3t) = 4\cos^3(t) - 3\cos(t)$. This transforms the equation into a polynomial in terms of $\cos(t)$.

$$\begin{array}{rcll} \cos(3t) & = & 2 - \cos(t) & \\ 4\cos^3(t) - 3\cos(t) & = & 2 - \cos(t) & \\ 2\cos^3(t) - 2\cos(t) - 2 & = & 0 & \\ 4u^3 - 2u - 2 & = & 0 & \text{Let } u = \cos(t). \end{array}$$

   Using what we know from Chapter 2, we factor $4u^3 - 2u - 2$ as $(u - 1)\left(4u^2 + 4u + 2\right)$ and set each factor equal to 0.

   We get either $u - 1 = 0$ or $4u^2 + 2u + 2 = 0$, and since the discriminant of the latter is negative, the only real solution to $4u^3 - 2u - 2 = 0$ is $u = 1$.

   Since $u = \cos(t)$, we get $\cos(t) = 1$, so $t = 2\pi k$ for integers k. The only solution which lies in $[0, 2\pi)$ is $t = 0$. Our graph below confirms this.



***y* = cos(3*t*)** and $y = 2 - \cos(t)$

5. While we could approach solving the equation $\cos(3x) = \cos(5x)$ in the same manner as we did the previous two problems, we choose instead to showcase the utility of the Sum to Product Identities.[6]

   From $\cos(3x) = \cos(5x)$, we get $\cos(5x) - \cos(3x) = 0$, and it is the presence of 0 on the right hand side that indicates a switch to a product would be a good move.[7]

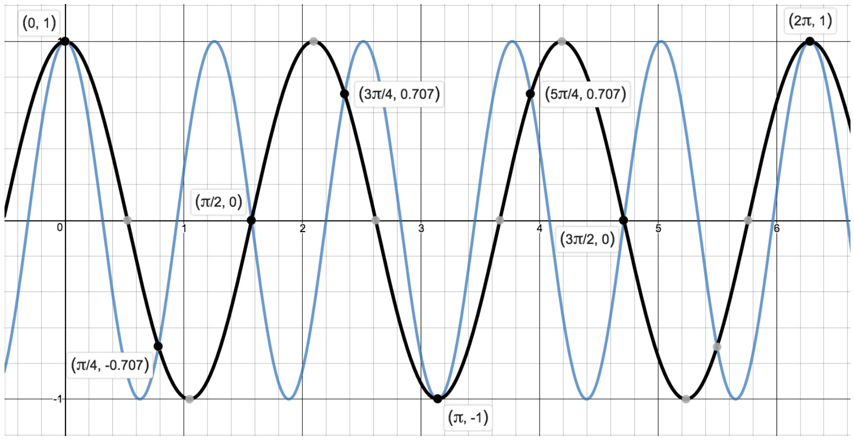
   Using Theorem 12.13, we rewrite $\cos(5x) - \cos(3x)$ as $-2\sin\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right) = -2\sin(4x)\sin(x)$. Hence, our original equation $\cos(3x) = \cos(5x)$ is equivalent to $-2\sin(4x)\sin(x) = 0$.

   From $-2\sin(4x)\sin(x) = 0$, we get either $\sin(4x) = 0$ or $\sin(x) = 0$. Solving $\sin(4x) = 0$ gives $x = \frac{\pi}{4}k$ for integers k, and the solution to $\sin(x) = 0$ is $x = \pi k$ for integers k.

   The second set of solutions is contained in the first set of solutions,[8] so our final solution to $\cos(5x) = \cos(3x)$ is $x = \frac{\pi}{4}k$ for integers k.

   There are eight of these answers which lie in $[0, 2\pi)$: $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}$ and $\frac{7\pi}{4}$. Our plot of the graphs of $y = \cos(3x)$ and $y = \cos(5x)$ below (after some careful zooming) bears this out.

[6] We invite the reader to try the 'polynomial approach' used in the previous problem to see what difficulties are encountered.
[7] Since a *product* equalling zero means, necessarily, one or both *factors* is 0. See page A.2.
[8] As always, when in doubt, write it out!



$y = \cos(3x)$ and $y = cos(5x)$

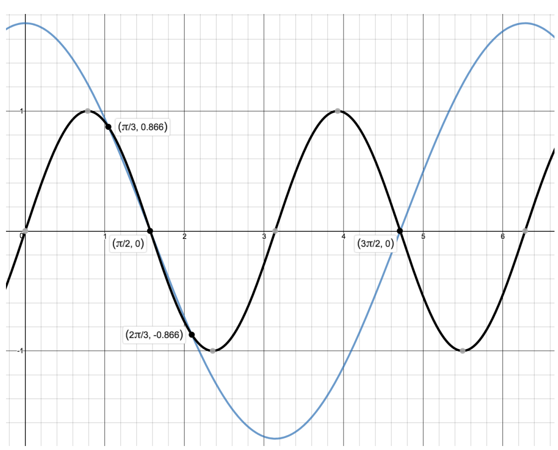
6. In the equation $\sin(2x) = \sqrt{3}\cos(x)$, we not only have different circular functions involved, but we also have different arguments to contend with.

   Using the double angle identity $\sin(2x) = 2\sin(x)\cos(x)$ makes all of the arguments the same and we proceed to gather all of the nonzero terms on one side of the equation and factor.

$$\begin{array}{rcll} \sin(2x) & = & \sqrt{3}\cos(x) & \\ 2\sin(x)\cos(x) & = & \sqrt{3}\cos(x) & \text{(Since } \sin(2x) = 2\sin(x)\cos(x).\text{)} \\ 2\sin(x)\cos(x) - \sqrt{3}\cos(x) & = & 0 & \\ \cos(x)(2\sin(x) - \sqrt{3}) & = & 0 & \end{array}$$

   We get $\cos(x) = 0$ or $\sin(x) = \frac{\sqrt{3}}{2}$. From $\cos(x) = 0$, we obtain $x = \frac{\pi}{2} + \pi k$ for integers k. From $\sin(x) = \frac{\sqrt{3}}{2}$, we get $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{2\pi}{3} + 2\pi k$ for integers k.

   The answers which lie in $[0, 2\pi)$ are $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}$ and $\frac{2\pi}{3}$, as verified graphically below.

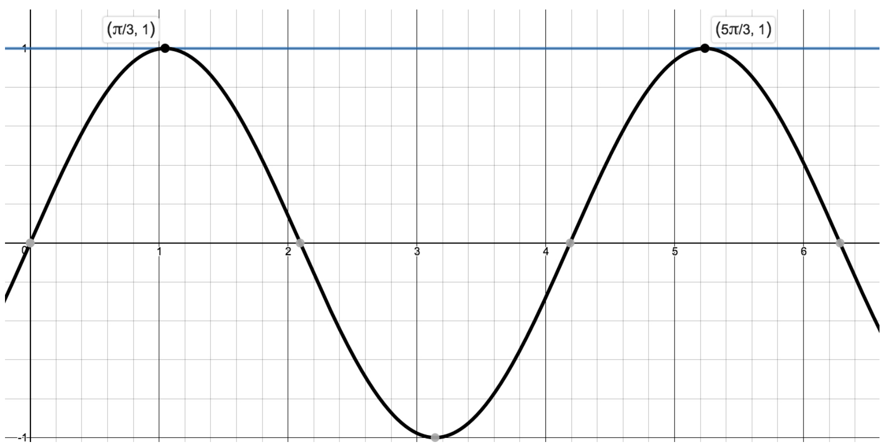


$y = \sin(2x)$ and $y = \sqrt{3}\cos(x)$

7. Unlike the previous problem, there seems to be no quick way to get the circular functions or their arguments to match in the equation $\sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) = 1$.

   If we stare at it long enough, however, we realize that the left hand side is the expanded form of the sum formula for $\sin\left(x + \frac{x}{2}\right)$. Hence, our original equation is equivalent to $\sin\left(\frac{3}{2}x\right) = 1$.

   Solving, we find $x = \frac{\pi}{3} + \frac{4\pi}{3}k$ for integers k. Two of these solutions lie in $[0, 2\pi)$: $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$. Graphing $y = \sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right)$ and $y = 1$ validates our solutions.



$\boldsymbol{y = \sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right)}$ and $y = 1$

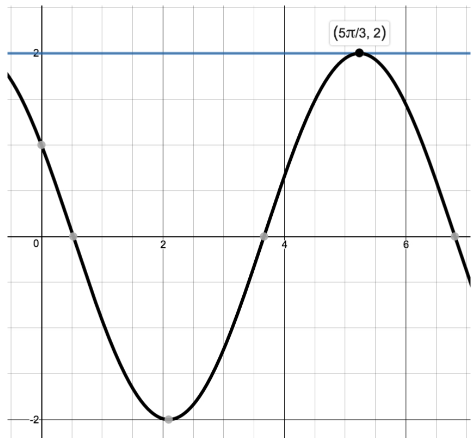
8. With the absence of double angles or squares, there doesn't seem to be much we can do with the equation $\cos(x) - \sqrt{3}\sin(x) = 2$.

   However, since the frequencies of the sine and cosine terms are the same, we can rewrite the left hand side of this equation as a sinusoid.

   To fit $f(x) = \cos(x) - \sqrt{3}\sin(x)$ to the form $A\sin(\omega t + \phi) + B$, we use what we learned in Example 12.2.7 and find $A = 2$, $B = 0$, $\omega = 1$ and $\phi = \frac{5\pi}{6}$.

   Hence, we can rewrite the equation $\cos(x) - \sqrt{3}\sin(x) = 2$ as $2\sin\left(x + \frac{5\pi}{6}\right) = 2$, or $\sin\left(x + \frac{5\pi}{6}\right) = 1$. Solving, we get $x = -\frac{\pi}{3} + 2\pi k$ for integers k.

   Only one of our solutions, $x = \frac{5\pi}{3}$, which corresponds to $k = 1$, lies in $[0, 2\pi)$. Geometrically, we see that $y = \cos(x) - \sqrt{3}\sin(x)$ and $y = 2$ intersect just once, supporting our answer.



$y = \cos(x) - \sqrt{3}\sin(x)$ and $y = 2$

An alternative way to solve this problem is to *introduce* squares in order to exchange sines and cosines using a Pythagorean Identity.

From $\cos(x) - \sqrt{3}\sin(x) = 2$ we get $\sqrt{3}\sin(x) = \cos(x) - 2$ so that $\left(\sqrt{3}\sin(x)\right)^2 = (\cos(x) - 2)^2$. Simplifying, we get: $3\sin^2(x) = \cos^2(x) - 4\cos(x) + 4$.

Substituting $\sin^2(x) = 1 - \cos^2(x)$, we get $3(1 - \cos^2(x)) = \cos^2(x) - 4\cos(x) + 4$ which results in the quadratic equation: $4\cos^2(x) - 4\cos(x) + 1 = 0$.

Letting $u = \cos(x)$, we get $4u^2 - 4u + 1 = 0$ or $(2u - 1)^2 = 0$. We get $u = \cos(x) = \frac{1}{2}$. Solving $\cos(x) = \frac{1}{2}$ gives $x = \frac{\pi}{3} + 2\pi k$ as well as $x = \frac{5\pi}{3} + 2\pi k$ for integers, k.

Of these two families, only solutions of the form $x = \frac{5\pi}{3} + 2\pi k$ checks in our original equation.[9] We leave it the reader to verify this representation of solutions to $\cos(x) - \sqrt{3}\sin(x) = 2$ is equivalent to the one we found previously. □

We repeat here the advice given when solving systems of nonlinear equations in section 9.7 – when it comes to solving equations involving the circular functions, it helps to just try something.

Next, we focus on solving inequalities involving the circular functions. Since these functions are continuous on their domains, we may use the sign diagram technique we've used in the past to solve the inequalities.[10]

Example 12.4.3. Solve the following inequalities on $[0, 2\pi)$. Express your answers using interval notation and verify your answers graphically.

1. $2\sin(t) \leq 1$
2. $\sin(2x) > \cos(x)$
3. $\tan(x) \geq 3$

[9]We've seen how squaring both sides can lead to extraneous solutions in Section A.13 and Chapter 4. Here, squaring both sides admits an entire *family* of extraneous solutions.

[10]See pages 113, 253, 333, as well as Examples 7.4.2 and 7.5.2 for a review of this technique, as needed.

Solution.

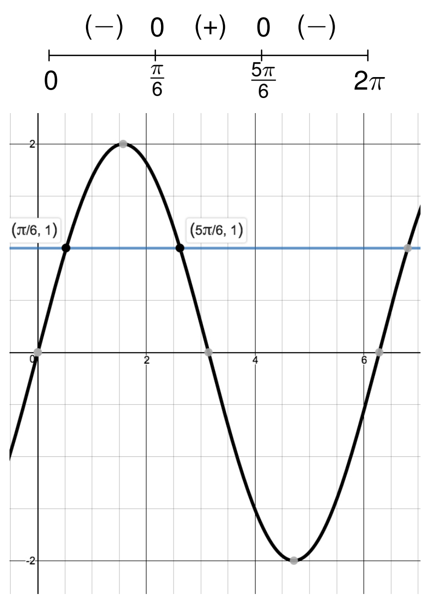
1. We begin solving $2\sin(t) \leq 1$ by collecting all of the terms on one side of the equation and zero on the other to get $2\sin(t) - 1 \leq 0$.

   Next, we let $f(t) = 2\sin(t) - 1$ and note that our original inequality is equivalent to solving $f(t) \leq 0$. We now look to see where, if ever, f is undefined and where $f(t) = 0$.

   Since the domain of f is all real numbers, we can immediately set about finding the zeros of f. Solving $f(t) = 0$, we have $2\sin(t) - 1 = 0$ or $\sin(t) = \frac{1}{2}$. The solutions here are $t = \frac{\pi}{6} + 2\pi k$ and $t = \frac{5\pi}{6} + 2\pi k$ for integers k. Since we are restricting our attention to $[0, 2\pi)$, only $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$ are of concern.

   Next, we choose test values in $[0, 2\pi)$ other than the zeros and determine if f is positive or negative there. For $t = 0$ we have $f(0) = -1$, for $t = \frac{\pi}{2}$ we get $f\left(\frac{\pi}{2}\right) = 1$ and for $t = \pi$ we get $f(\pi) = -1$.

   Since our original inequality is equivalent to $f(t) \leq 0$, we are looking for where the function is negative $(-)$ or 0, and we get the intervals $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right)$. We can confirm our answer graphically by seeing where the graph of $y = 2\sin(t)$ crosses or is below the graph of $y = 1$.



$\boldsymbol{y = 2\sin(t)}$ and $y = 1$

2. We first rewrite $\sin(2x) > \cos(x)$ as $\sin(2x) - \cos(x) > 0$ and let $f(x) = \sin(2x) - \cos(x)$.

   Our original inequality is thus equivalent to $f(x) > 0$. The domain of f is all real numbers, so we can advance to finding the zeros of f.

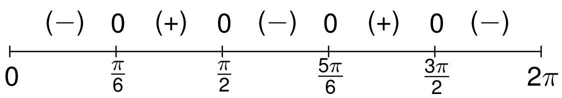
   Setting $f(x) = 0$ yields $\sin(2x) - \cos(x) = 0$, which, by way of the double angle identity for sine, becomes $2\sin(x)\cos(x) - \cos(x) = 0$ or $\cos(x)(2\sin(x) - 1) = 0$.

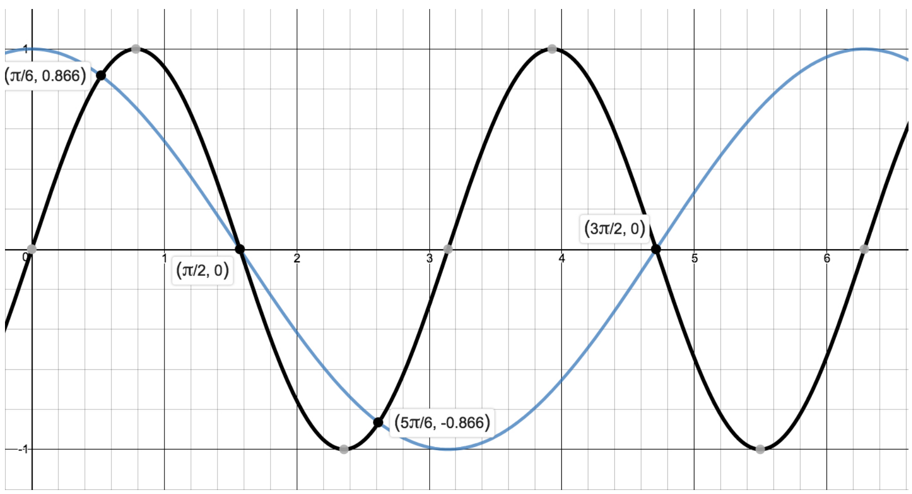
From $\cos(x) = 0$, we get $x = \frac{\pi}{2} + \pi k$ for integers k of which only $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ lie in $[0, 2\pi)$.

For $2\sin(x) - 1 = 0$, we get $\sin(x) = \frac{1}{2}$ which gives $x = \frac{\pi}{6} + 2\pi k$ or $x = \frac{5\pi}{6} + 2\pi k$ for integers k. Of those, only $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ lie in $[0, 2\pi)$.

Choosing test values, we get: for $x = 0$ we find $f(0) = -1$; when $x = \frac{\pi}{4}$ we get $f\left(\frac{\pi}{4}\right) = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$; for $x = \frac{3\pi}{4}$ we get $f\left(\frac{3\pi}{4}\right) = -1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-2}{2}$; when $x = \pi$ we have $f(\pi) = 1$, and lastly, for $x = \frac{7\pi}{4}$ we get $f\left(\frac{7\pi}{4}\right) = -1 - \frac{\sqrt{2}}{2} = \frac{-2-\sqrt{2}}{2}$.

We see $f(x) > 0$ on $\left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, so this is our answer. Geometrically, we see the graph of $y = \sin(2x)$ is indeed above the graph of $y = \cos(x)$ on those intervals.





$\boldsymbol{y = \sin(2x)}$ and $y = \cos(x)$

3. Proceeding as above, we rewrite $\tan(x) \geq 3$ as $\tan(x) - 3 \geq 0$ and let $f(x) = \tan(x) - 3$.

   We note that on $[0, 2\pi)$, f is undefined at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, so those values will need the usual disclaimer on the sign diagram.[11]

   Moving along to zeros, solving $f(x) = \tan(x) - 3 = 0$ requires the arctangent function. We find $x = \arctan(3) + \pi k$ for integers k and of these, only $x = \arctan(3)$ and $x = \arctan(3) + \pi$ lie in $[0, 2\pi)$. Since $3 > 0$, we know $0 < \arctan(3) < \frac{\pi}{2}$ which allows us to position these zeros correctly on the sign diagram.

   To choose test values, we begin with $x = 0$ and find $f(0) = -3$. Finding a convenient test value in the interval $\left(\arctan(3), \frac{\pi}{2}\right)$ is a bit more challenging. Since the arctangent function is increasing and

[11] See page 253 for a discussion of the non-standard character known as the interrobang.

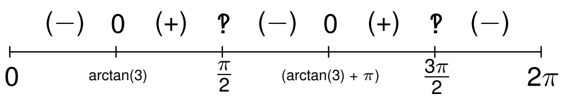
is bounded above by $\frac{\pi}{2}$, the number $x = \arctan(117)$ is guaranteed[12] to lie between $\arctan(3)$ and $\frac{\pi}{2}$. We see that $f(\arctan(117)) = \tan(\arctan(117)) - 3 = 114$.

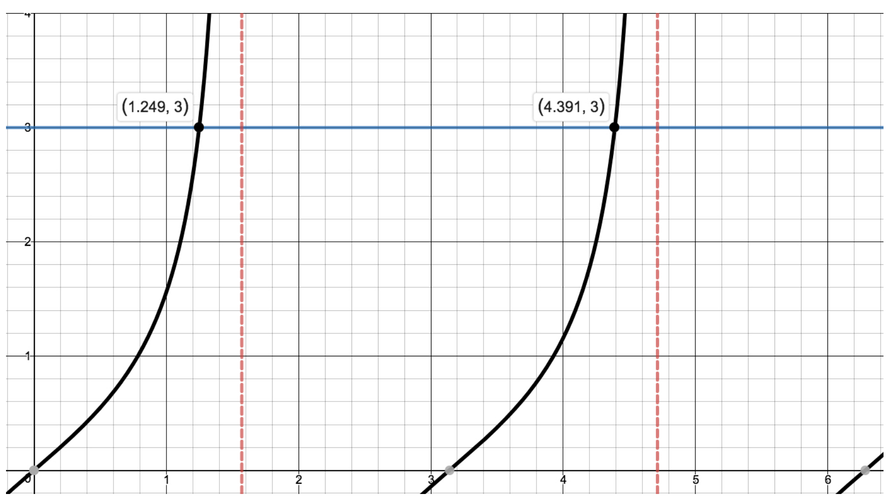
For our next test value, we take $x = \pi$ and find $f(\pi) = -3$, which brings us to finding a test value in the interval $\left(\arctan(3) + \pi, \frac{3\pi}{2}\right)$.

From $\arctan(3) < \arctan(117) < \frac{\pi}{2}$ we get $\arctan(3) + \pi < \arctan(117) + \pi < \frac{3\pi}{2}$ by adding π through the inequality. We find $f(\arctan(117) + \pi) = \tan(\arctan(117) + \pi) - 3 = \tan(\arctan(117)) - 3 = 114$.

For our last test value, we choose $x = \frac{7\pi}{4}$ and find $f\left(\frac{7\pi}{4}\right) = -4$.

Since we want $f(x) \geq 0$, we see that our answer is $\left[\arctan(3), \frac{\pi}{2}\right) \cup \left[\arctan(3) + \pi, \frac{3\pi}{2}\right)$. Using the graphs of $y = \tan(x)$ and $y = 3$, we see when the graph of the former is above (or meets) the graph of the latter. (Note, $\arctan(3) \approx 1.249$ and $\arctan(3) + \pi \approx 4.391$.)





$\boldsymbol{y = \tan(x)}$ and $y = 3$

$\square$

Our next example puts solving equations and inequalities to good use – finding domains of functions.

Example 12.4.4. Express the domain of the following functions using extended interval notation.[13]

1. $f(x) = \csc\left(2x + \frac{\pi}{3}\right)$
2. $f(t) = \dfrac{\sin(t)}{2\cos(t) - 1}$
3. $f(x) = \sqrt{1 - \cot(x)}$

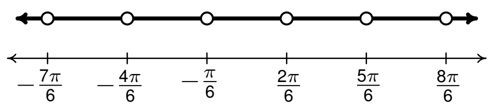
[12]We could have chosen any value $\arctan(t)$ where $t > 3$.

[13]See Section 11.5.3 for details about this notation.

Solution.

1. To find the domain of $f(x) = \csc\left(2x + \frac{\pi}{3}\right)$, we rewrite f in terms of sine as $f(x) = \frac{1}{\sin\left(2x+\frac{\pi}{3}\right)}$. Since the sine function is defined everywhere, our only concern comes from zeros in the denominator.

   Solving $\sin\left(2x + \frac{\pi}{3}\right) = 0$, we get $x = -\frac{\pi}{6} + \frac{\pi}{2}k$ for integers k. In set-builder notation, our domain is $\left\{x \mid x \neq -\frac{\pi}{6} + \frac{\pi}{2}k \text{ for integers } k\right\}$. To help visualize the domain, we follow the old mantra 'When in doubt, write it out!' We get $\left\{x \mid x \neq -\frac{\pi}{6}, \frac{2\pi}{6}, -\frac{4\pi}{6}, \frac{5\pi}{6}, -\frac{7\pi}{6}, \frac{8\pi}{6}, \ldots\right\}$, where we have kept the denominators 6 throughout to help see the pattern. Graphing the situation on a number line, we have

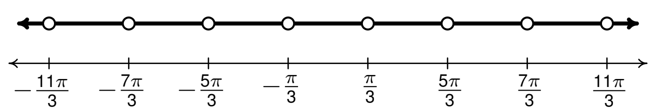
   Proceeding as in Section 11.5.3, we let x_k denote the kth number excluded from the domain and we have $x_k = -\frac{\pi}{6} + \frac{\pi}{2}k = \frac{(3k-1)\pi}{6}$ for integers k. The intervals which comprise the domain are of the form $(x_k, x_{k+1}) = \left(\frac{(3k-1)\pi}{6}, \frac{(3k+2)\pi}{6}\right)$ as k runs through the integers. Using extended interval notation, we have that the domain is

   $$\bigcup_{k=-\infty}^{\infty} \left(\frac{(3k-1)\pi}{6}, \frac{(3k+2)\pi}{6}\right)$$

   We can check our answer by substituting in values of k to see that it matches our diagram.

2. Since the domains of $\sin(t)$ and $\cos(t)$ are all real numbers, the only concern when finding the domain of $f(t) = \frac{\sin(t)}{2\cos(t)-1}$ is division by zero so we set the denominator equal to zero and solve.

   From $2\cos(t) - 1 = 0$ we get $\cos(t) = \frac{1}{2}$ so $t = \frac{\pi}{3} + 2\pi k$ or $t = \frac{5\pi}{3} + 2\pi k$ for integers k. Using set-builder notation, the domain is $\left\{t \mid t \neq \frac{\pi}{3} + 2\pi k \text{ and } t \neq \frac{5\pi}{3} + 2\pi k \text{ for integers } k\right\}$. Writing this out, we find the domain is $\left\{t \mid t \neq \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \pm\frac{7\pi}{3}, \pm\frac{11\pi}{3}, \ldots\right\}$, so we have

   Unlike the previous example, we have *two* different families of points to consider, and we present two ways of dealing with this kind of situation. One way is to generalize what we did in the previous example and use the formulas we found in our domain work to describe the intervals.

   To that end, we let $a_k = \frac{\pi}{3} + 2\pi k = \frac{(6k+1)\pi}{3}$ and $b_k = \frac{5\pi}{3} + 2\pi k = \frac{(6k+5)\pi}{3}$ for integers k. The goal now is to write the domain in terms of the a's an b's. We find $a_0 = \frac{\pi}{3}$, $a_1 = \frac{7\pi}{3}$, $a_{-1} = -\frac{5\pi}{3}$, $a_2 = \frac{13\pi}{3}$, $a_{-2} = -\frac{11\pi}{3}$, $b_0 = \frac{5\pi}{3}$, $b_1 = \frac{11\pi}{3}$, $b_{-1} = -\frac{\pi}{3}$, $b_2 = \frac{17\pi}{3}$ and $b_{-2} = -\frac{7\pi}{3}$.

   Hence, in terms of the a's and b's, our domain is

$$\ldots (a_{-2}, b_{-2}) \cup (b_{-2}, a_{-1}) \cup (a_{-1}, b_{-1}) \cup (b_{-1}, a_0) \cup (a_0, b_0) \cup (b_0, a_1) \cup (a_1, b_1) \cup \ldots$$

If we group these intervals in pairs, $(a_{-2}, b_{-2}) \cup (b_{-2}, a_{-1})$, $(a_{-1}, b_{-1}) \cup (b_{-1}, a_0)$, $(a_0, b_0) \cup (b_0, a_1)$ and so forth, we see a pattern emerge of the form $(a_k, b_k) \cup (b_k, a_{k+1})$ for integers k so that our domain can be written as

$$\bigcup_{k=-\infty}^{\infty} (a_k, b_k) \cup (b_k, a_{k+1}) = \bigcup_{k=-\infty}^{\infty} \left(\frac{(6k+1)\pi}{3}, \frac{(6k+5)\pi}{3}\right) \cup \left(\frac{(6k+5)\pi}{3}, \frac{(6k+7)\pi}{3}\right)$$

A second approach to the problem exploits the periodic nature of f. Since $\cos(t)$ and $\sin(t)$ have period 2π, it's not too difficult to show the function f repeats itself every 2π units.[14] This means if we can find a formula for the domain on an interval of length 2π, we can express the entire domain by translating our answer left and right on the t-axis by adding integer multiples of 2π.

One such interval that arises naturally from our domain work is $\left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$. The portion of the domain here is $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right) \cup \left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)$. Adding integer multiples of 2π, we obtain the family of intervals: $\left(\frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k\right) \cup \left(\frac{5\pi}{3} + 2\pi k, \frac{7\pi}{3} + 2\pi k\right)$ for integers k. We leave it to the reader to show that getting common denominators leads to our previous answer.

3. To find the domain of $f(x) = \sqrt{1 - \cot(x)}$, we first note that, due to the presence of the $\cot(x)$ term, $x \neq \pi k$ for integers k.

   Next, we recall that for the square root to be defined, we need $1 - \cot(x) \geq 0$. Unlike the inequalities we solved in Example 12.4.3, we are not restricted here to a given interval. Our strategy is to solve this inequality over $(0, \pi)$ (the same interval which generates a fundamental cycle of cotangent) and then add integer multiples of the period, in this case, π.

   We let $g(x) = 1 - \cot(x)$ and set about making a sign diagram for g over the interval $(0, \pi)$ to find where $g(x) \geq 0$. We note that g is undefined for $x = \pi k$ for integers k, in particular, at the endpoints of our interval $x = 0$ and $x = \pi$.

   Next, we look for the zeros of g. Solving $g(x) = 0$, we get $\cot(x) = 1$ or $x = \frac{\pi}{4} + \pi k$ for integers k and only one of these, $x = \frac{\pi}{4}$, lies in $(0, \pi)$. Choosing the test values $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$, we get $g\left(\frac{\pi}{6}\right) = 1 - \sqrt{3}$, and $g\left(\frac{\pi}{2}\right) = 1$. We construct the sign diagram for g over the interval $(0, \pi)$ below:

?	$(-)$	0	$(+)$	?
0		$\frac{\pi}{4}$		π

   We find $g(x) \geq 0$ on $\left[\frac{\pi}{4}, \pi\right)$. Adding multiples of the period we get our solution to consist of the intervals $\left[\frac{\pi}{4} + \pi k, \pi + \pi k\right) = \left[\frac{(4k+1)\pi}{4}, (k+1)\pi\right)$.

[14] This doesn't necessarily mean the period of f is 2π. The tangent function is comprised of sine and cosine, but its period is half theirs. The reader is invited to investigate the period of f.

Using extended interval notation, we have our final answer:

$$\bigcup_{k=-\infty}^{\infty} \left[\frac{(4k+1)\pi}{4}, (k+1)\pi \right)$$

□

In our next example, we solve equations and inequalities involving the *inverse* circular functions.

Example 12.4.5. Solve the following equations and inequalities analytically. Check your answers using a graphing utility.

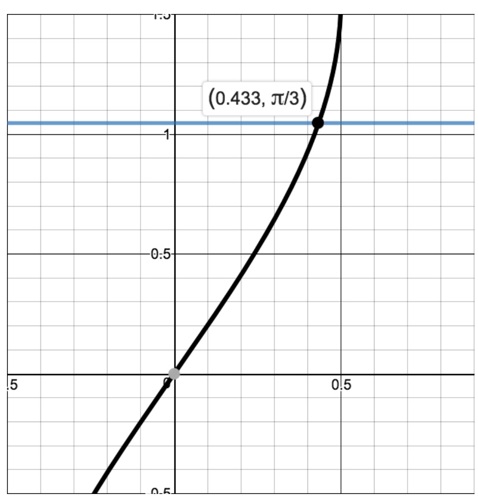
1. $\arcsin(2x) = \frac{\pi}{3}$
2. $4\arccos(t) - 3\pi = 0$
3. $3\,\text{arcsec}(2x-1) + \pi = 2\pi$
4. $4\arctan^2(t) - 3\pi\arctan(t) - \pi^2 = 0$
5. $\pi^2 - 4\arccos^2(x) < 0$
6. $4\,\text{arccot}(3t) > \pi$

Solution.

1. To solve $\arcsin(2x) = \frac{\pi}{3}$, we first note that $\frac{\pi}{3}$ is in the range of the arcsine function (so a solution exists!) Next, we exploit the inverse property of sine and arcsine from Theorem 12.14

$$\begin{array}{rcll} \arcsin(2x) & = & \frac{\pi}{3} & \\ \sin(\arcsin(2x)) & = & \sin\left(\frac{\pi}{3}\right) & \\ 2x & = & \frac{\sqrt{3}}{2} & \text{Since } \sin(\arcsin(u)) = u \\ x & = & \frac{\sqrt{3}}{4} & \end{array}$$

Below we see the graphs of $y = \arcsin(2x)$ and $y = \frac{\pi}{3}$, intersect at $x = \frac{\sqrt{3}}{4} \approx 0.4430$.

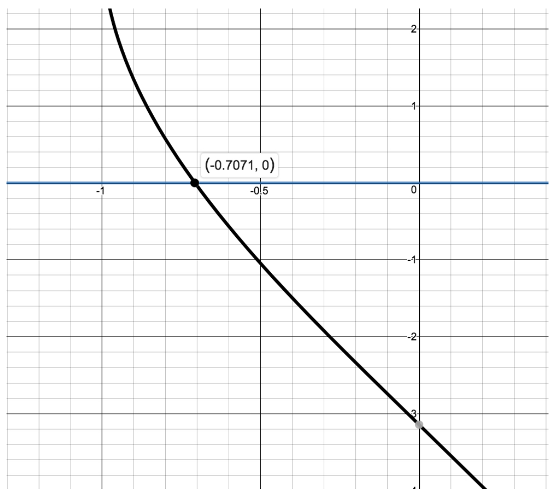


$\boldsymbol{y = \arcsin(2x)}$ and $y = \frac{\pi}{3}$

2. Our first step in solving $4\arccos(t) - 3\pi = 0$ is to isolate the arccosine. We get $\arccos(t) = \frac{3\pi}{4}$. Since $\frac{3\pi}{4}$ is in the range of arccosine, we may apply Theorem 12.14

$$\begin{array}{rcll} \arccos(t) & = & \frac{3\pi}{4} & \\ \cos(\arccos(t)) & = & \cos\left(\frac{3\pi}{4}\right) & \\ t & = & -\frac{\sqrt{2}}{2} & \text{Since } \cos(\arccos(u)) = u \end{array}$$

Below we see the graph of $y = 4\arccos(t) - 3\pi$ crosses $y = 0$ at $t = -\frac{\sqrt{2}}{2} \approx -0.7071$.

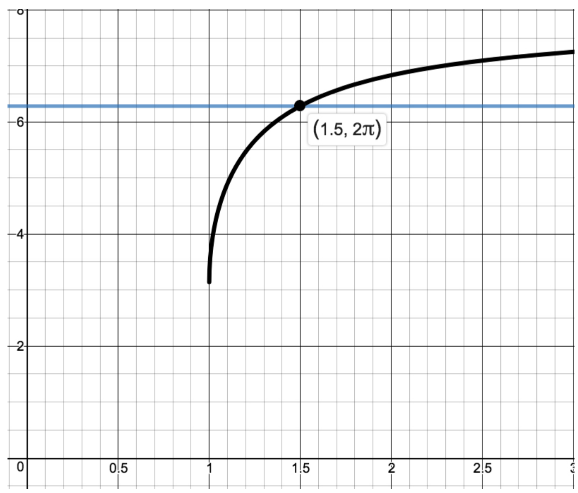


$\boldsymbol{y = 4\arccos(t) - 3\pi}$ and $y = 0$

3. From $3\,\mathrm{arcsec}(2x - 1) + \pi = 2\pi$, we get $\mathrm{arcsec}(2x - 1) = \frac{\pi}{3}$. Regardless of how the range of arcsecant is chosen, since $0 \leq \frac{\pi}{3} < \frac{\pi}{2}$, both Theorems 12.16 12.17, apply:

$$\begin{array}{rcll} \mathrm{arcsec}(2x-1) & = & \frac{\pi}{3} & \\ \sec(\mathrm{arcsec}(2x-1)) & = & \sec\left(\frac{\pi}{3}\right) & \\ 2x - 1 & = & 2 & \text{Since } \sec(\mathrm{arcsec}(u)) = u \\ x & = & \frac{3}{2} & \end{array}$$

Below we see the graphs of $y = 3\,\mathrm{arcsec}(2x - 1) + \pi$ and $y = 2\pi$ intersect at $x = \frac{3}{2} = 1.5$.



$y = 3\,\text{arcsec}(2x - 1) + \pi$ and $y = 2\pi$

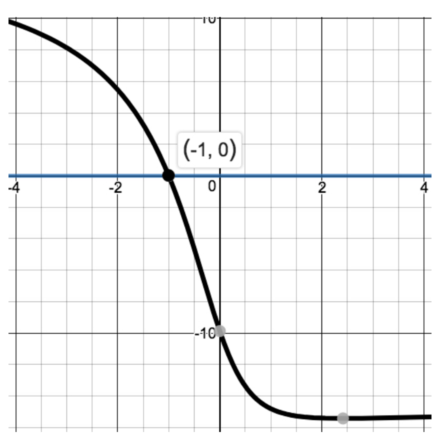
4. With the presence of both $\arctan^2(t)$ ($= (\arctan(t))^2$) and $\arctan(t)$, we substitute $u = \arctan(t)$ to reveal a quadratic in disguise: $4u^2 - 3\pi u - \pi^2 = 0$.

   Factoring, (don't let the π throw you!) we get $(4u + \pi)(u - \pi) = 0$, so $u = \arctan(t) = -\frac{\pi}{4}$ or $u = \arctan(t) = \pi$.

   Since $-\frac{\pi}{4}$ is in the range of arctangent, but π is not, we only get solutions from the first equation. Using Theorem 12.15, we get

$$\begin{array}{rcll} \arctan(t) & = & -\frac{\pi}{4} & \\ \tan(\arctan(t)) & = & \tan\left(-\frac{\pi}{4}\right) & \\ t & = & -1 & \text{Since } \tan(\arctan(u)) = u. \end{array}$$

   We verify this result graphically below.



$y = 4\arctan^2(t) - 3\pi\arctan(t) - \pi^2$ and $y = 0$.

5. Since the inverse circular functions are continuous on their domains, we can solve inequalities featuring these functions using sign diagrams.

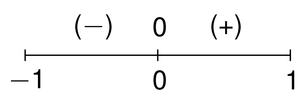
   Since all of the nonzero terms of $\pi^2 - 4\arccos^2(x) < 0$ are on one side of the inequality, we let $f(x) = \pi^2 - 4\arccos^2(x)$ and note the domain of f is limited by the $\arccos(x)$ to $[-1, 1]$.

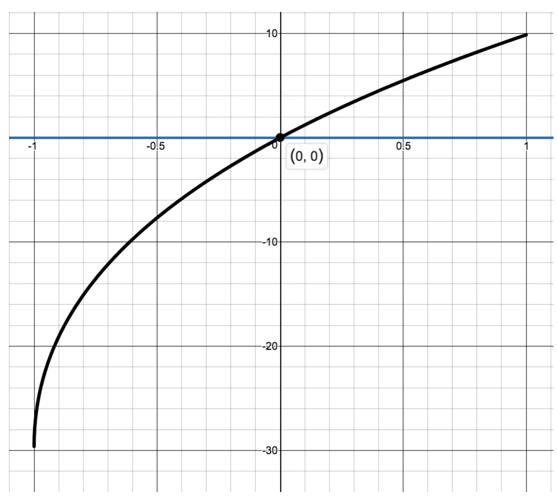
   Next, we find the zeros of f by setting $f(x) = \pi^2 - 4\arccos^2(x) = 0$. We get $\arccos(x) = \pm\frac{\pi}{2}$, and since the range of arccosine is $[0, \pi]$, we focus our attention on $\arccos(x) = \frac{\pi}{2}$.

   Using Theorem 12.14, we get $x = \cos\left(\frac{\pi}{2}\right) = 0$ as our only zero which breaks our domain $[-1, 1]$ into two test intervals: $[-1, 0)$ and $(0, 1]$.

   Choosing test values $x = \pm 1$, we get $f(-1) = -3\pi^2 < 0$ and $f(1) = \pi^2 > 0$. Since we are looking for where $f(x) = \pi^2 - 4\arccos^2(x) < 0$, our answer is $[-1, 0)$.

   Geometrically, we find the graph of $y = \pi^2 - 4\arccos^2(x)$ is below $y = 0$ (the x-axis) on $[-1, 0)$.





$\boldsymbol{y = \pi^2 - 4\arccos^2(x)}$ and $y = 0$

6. As in the previous problem, we will use a sign diagram to solve $4\operatorname{arccot}(3t) > \pi$. Our first step is to rewrite the inequality as $4\operatorname{arccot}(3t) - \pi > 0$.

   We let $f(t) = 4\operatorname{arccot}(3t) - \pi$, and find the domain of f is all real numbers, $(-\infty, \infty)$.

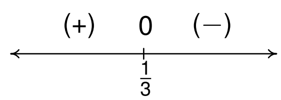
   To find the zeros of f, we set $f(t) = 4\operatorname{arccot}(3t) - \pi = 0$ and solve. We get $\operatorname{arccot}(3t) = \frac{\pi}{4}$, and since $\frac{\pi}{4}$ is in the range of arccotangent, we may apply Theorem 12.15 and solve

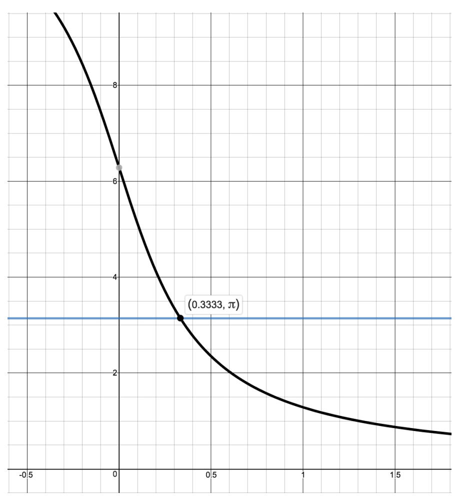
$$\begin{aligned} \operatorname{arccot}(3t) &= \frac{\pi}{4} \\ \cot(\operatorname{arccot}(3t)) &= \cot\left(\frac{\pi}{4}\right) \\ 3t &= 1 && \text{Since } \cot(\operatorname{arccot}(u)) = u. \\ t &= \frac{1}{3} \end{aligned}$$

Next, we make a sign diagram for f. Since the domain of f is all real numbers, and there is only one zero of f, $t = \frac{1}{3}$, we have two test intervals, $\left(-\infty, \frac{1}{3}\right)$ and $\left(\frac{1}{3}, \infty\right)$.

Ideally, we wish to find test values t in these intervals so that $\operatorname{arccot}(3t)$ corresponds to one of our oft-used 'common' angles. After a bit of computation,[15] we choose $t = 0$ for the interval $\left(-\infty, \frac{1}{3}\right)$ and $t = \frac{\sqrt{3}}{3}$ for the interval $\left(\frac{1}{3}, \infty\right)$.

We find $f(0) = \pi > 0$ and $f\left(\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{3} < 0$. Since we are looking for where $f(t) > 0$, we get our answer $\left(-\infty, \frac{1}{3}\right)$. Graphically, we see the graph of $y = 4\operatorname{arccot}(3t)$ is above the horizontal line $y = \pi$ on $\left(-\infty, \frac{1}{3}\right) = \left(-\infty, 0.\overline{3}\right)$.





$\boldsymbol{y = 4}\ \operatorname{arccot}\boldsymbol{(3t)}$ and $y = \pi$

□

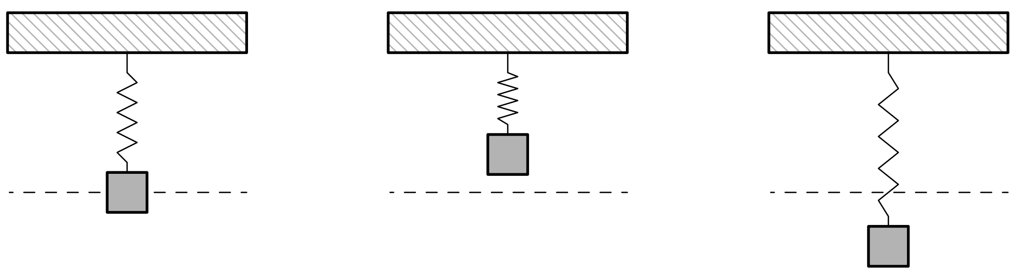
[15] Set $3t$ equal to the cotangents of the 'common angles' and choose accordingly.

12.4.1 Harmonic Motion

One of the major applications of the circular functions (sinusoids in particular!) in Science and Engineering is the study of **harmonic motion**, We close this chapter with a brief foray into this topic since it pulls together many important concepts from both Chapters 11 and 12. The equations for harmonic motion can be used to describe a wide range of phenomena, from the motion of an object on a spring, to the response of an electronic circuit. In this subsection, we restrict our attention to modeling a simple spring system. Before we jump into the Mathematics, there are some Physics terms and concepts we need to discuss.

In Physics, 'mass' is defined as a measure of an object's resistance to straight-line motion whereas 'weight' is the amount of force (pull) gravity exerts on an object. An object's mass cannot change,[16] while its weight could change. An object which weighs 6 pounds on the surface of the Earth would weigh 1 pound on the surface of the Moon, but its mass is the same in both places. In the English system of units, 'pounds' (lbs.) is a measure of force (weight), and the corresponding unit of mass is the 'slug'. In the SI system, the unit of force is 'Newtons' (N) and the associated unit of mass is the 'kilogram' (kg).

We convert between mass and weight using the formula[17] $w = mg$. Here, w is the weight of the object, m is the mass and g is the acceleration due to gravity. In the English system, $g = 32 \frac{\text{feet}}{\text{second}^2}$, and in the SI system, $g = 9.8 \frac{\text{meters}}{\text{second}^2}$. Hence, on Earth a *mass* of 1 slug *weighs* 32 lbs. and a *mass* of 1 kg *weighs* 9.8 N.[18] Suppose we attach an object with mass m to a spring as depicted below.



$x(t) = 0$ at the equilibrium position

$x(t) < 0$ above the equilibrium position

$x(t) > 0$ below the equilibrium position

The weight of the object will stretch the spring. The system is said to be in 'equilibrium' when the weight of the object is perfectly balanced with the restorative force of the spring. How far the spring stretches to reach equilibrium depends on the spring's 'spring constant'. Usually denoted by the letter k, the spring constant relates the force F applied to the spring to the amount d the spring stretches in accordance with Hooke's Law[19] $F = kd$.

If the object is released above or below the equilibrium position, or if the object is released with an upward or downward velocity, the object will bounce up and down on the end of the spring until some external force stops it. If we let $x(t)$ denote the object's displacement from the equilibrium position at time t, then $x(t) = 0$

[16] Well, assuming the object isn't subjected to relativistic speeds . . .

[17] This is a consequence of Newton's Second Law of Motion $F = ma$ where F is force, m is mass and a is acceleration. In our present setting, the force involved is weight which is caused by the acceleration due to gravity.

[18] Note that 1 pound = 1 $\frac{\text{slug foot}}{\text{second}^2}$ and 1 Newton = 1 $\frac{\text{kg meter}}{\text{second}^2}$.

[19] Look familiar? We saw Hooke's Law in Section A.14.

means the object is at the equilibrium position, $x(t) < 0$ means the object is *above* the equilibrium position, and $x(t) > 0$ means the object is *below* the equilibrium position. The function $x(t)$ is called the 'equation of motion' of the object.[20]

If we ignore all other influences on the system except gravity and the spring force, then Physics tells us that gravity and the spring force will battle each other forever and the object will oscillate indefinitely. In this case, we describe the motion as 'free' (meaning there is no external force causing the motion) and 'undamped' (meaning we ignore friction caused by surrounding medium, which in our case is air).

The following theorem, which comes from Differential Equations, gives $x(t)$ as a function of the mass m of the object, the spring constant k, the initial displacement x_0 of the object and initial velocity v_0 of the object. As with $x(t)$, $x_0 = 0$ means the object is released from the equilibrium position, $x_0 < 0$ means the object is released *above* the equilibrium position and $x_0 > 0$ means the object is released *below* the equilibrium position. As far as the initial velocity v_0 is concerned, $v_0 = 0$ means the object is released 'from rest,' $v_0 < 0$ means the object is heading *upwards* and $v_0 > 0$ means the object is heading *downwards*.[21]

Theorem 12.18. Equation for Free Undamped Harmonic Motion: Suppose an object of mass m is suspended from a spring with spring constant k. If the initial displacement from the equilibrium position is x_0 and the initial velocity of the object is v_0, then the displacement x from the equilibrium position at time t is given by $x(t) = A\sin(\omega t + \phi)$ where

- $\omega = \sqrt{\dfrac{k}{m}}$ and $A = \sqrt{x_0^2 + \left(\dfrac{v_0}{\omega}\right)^2}$
- $A\sin(\phi) = x_0$ and $A\omega\cos(\phi) = v_0$.

It is a great exercise in 'dimensional analysis' to verify that the formulas given in Theorem 12.18 work out so that ω has units $\frac{1}{s}$ and A has units ft. or m, depending on which system we choose.

Example 12.4.6. Suppose an object weighing 64 pounds stretches a spring 8 feet.

1. If the object is attached to the spring and released 3 feet below the equilibrium position from rest, find the equation of motion of the object, $x(t)$. When does the object first pass through the equilibrium position? Is the object heading upwards or downwards at this instant?

2. If the object is attached to the spring and released 3 feet below the equilibrium position with an upward velocity of 8 feet per second, find the equation of motion of the object, $x(t)$. What is the longest distance the object travels *above* the equilibrium position? When does this first happen? Confirm your result using a graphing utility.

[20]To keep units compatible, if we are using the English system, we use feet (ft.) to measure displacement. If we are in the SI system, we measure displacement in meters (m). Time is always measured in seconds (s).

[21]The sign conventions here are carried over from Physics. If not for the spring, the object would fall towards the ground, which is the 'natural' or 'positive' direction. Since the spring force acts in direct opposition to gravity, any movement upwards is considered 'negative'.

Solution. In order to use the formulas in Theorem 12.18, we first need to determine the spring constant k and the mass of the object m.

To find k, we use Hooke's Law $F = kd$. We know the object weighs 64 lbs. and stretches the spring 8 ft.. Using $F = 64$ and $d = 8$, we get $64 = k \cdot 8$, or $k = 8 \frac{\text{lbs.}}{\text{ft.}}$.

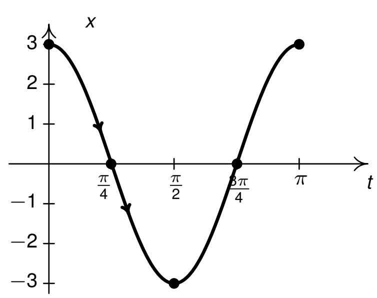
To find m, we use $w = mg$ with $w = 64$ lbs. and $g = 32 \frac{\text{ft.}}{s^2}$. We get $m = 2$ slugs. We can now proceed to apply Theorem 12.18.

1. With $k = 8$ and $m = 2$, we get $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2$. Since the object is released 3 feet *below* the equilibrium position 'from rest,' $x_0 = 3$ and $v_0 = 0$. Therefore, $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{3^2 + 0^2} = 3$.

   To determine the phase ϕ, we have $A\sin(\phi) = x_0$, which in this case gives $3\sin(\phi) = 3$ so $\sin(\phi) = 1$. Only $\phi = \frac{\pi}{2}$ and angles coterminal to it satisfy this condition, so we pick[22] the phase to be $\phi = \frac{\pi}{2}$. Hence, the equation of motion is $x(t) = 3\sin\left(2t + \frac{\pi}{2}\right)$.

   To find when the object passes through the equilibrium position we solve $x(t) = 3\sin\left(2t + \frac{\pi}{2}\right) = 0$. Going through the usual analysis we find $t = -\frac{\pi}{4} + \frac{\pi}{2}k$ for integers k. Since we are interested in the *first* time the object passes through the equilibrium position, we look for the smallest positive t value which in this case is $t = \frac{\pi}{4} \approx 0.78$ seconds after the start of the motion.

   Common sense suggests that if we release the object *below* the equilibrium position, the object should be traveling *upwards* when it first passes through it. To check this answer, we graph one cycle of $x(t)$. Since our applied domain in this situation is $t \geq 0$, and the period of $x(t)$ is $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$, we graph $x(t)$ over the interval $[0, \pi]$. Remembering that $x(t) > 0$ means the object is below the equilibrium position and $x(t) < 0$ means the object is above the equilibrium position, the fact our graph is crossing through the t-axis from positive x to negative x at $t = \frac{\pi}{4}$ confirms our answer.



$x(t) = 3\sin\left(2t + \frac{\pi}{2}\right)$

2. The only difference between this problem and the previous problem is that we now release the object with an upward velocity of $8 \frac{\text{ft}}{s}$. We still have $\omega = 2$ and $x_0 = 3$, but now we have $v_0 = -8$, the negative indicating the velocity is directed upwards.

[22]For confirmation, we note that $A\omega\cos(\phi) = v_0$, which in this case reduces to $6\cos(\phi) = 0$.

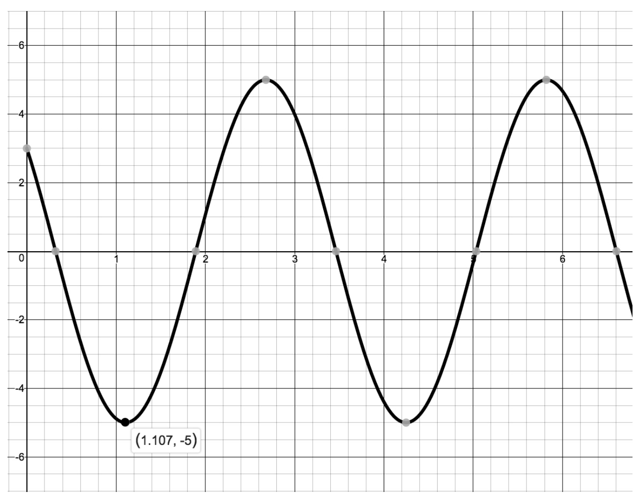
Here, we get $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{3^2 + (-4)^2} = 5$. From $A\sin(\phi) = x_0$, we get $5\sin(\phi) = 3$ which gives $\sin(\phi) = \frac{3}{5}$. From $A\omega\cos(\phi) = v_0$, we get $10\cos(\phi) = -8$, or $\cos(\phi) = -\frac{4}{5}$.

Hence, ϕ is a Quadrant II angle which we can describe in terms of either arcsine or arccosine. Since the range of arccosine covers Quadrant II, we choose to express ϕ in terms of the arccosine: $\phi = \arccos\left(-\frac{4}{5}\right)$. Hence, $x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right)$.

Since the amplitude of $x(t)$ is 5, the object will travel at most 5 feet above the equilibrium position. To find when this happens, we solve the equation $x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right) = -5$, the negative once again signifying that the object is *above* the equilibrium position.

Going through the usual machinations, we get $t = -\frac{1}{2}\arccos\left(-\frac{4}{5}\right) - \frac{\pi}{4} + \pi k$ for integers k. The smallest (positive) of these values occurs when $k = 1$, that is, $t = -\frac{1}{2}\arccos\left(-\frac{4}{5}\right) + \frac{3\pi}{4} \approx 1.107$ seconds after the start of the motion.

Graphing $x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right)$, we find the coordinates of the first relative minimum of to be approximately $(1.107, -5)$.



$x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right)$

□

Though beyond the scope of this course, it is possible to model the effects of friction and other external forces acting on the system.[23]

While we may not have the Physics and Calculus background to *derive* equations of motion for these scenarios, we can certainly analyze them. We examine three cases in the following example.

[23] Take a good Differential Equations class to see this!

Example 12.4.7.

1. Write $x(t) = 5e^{-t/5}\cos(t) + 5e^{-t/5}\sqrt{3}\sin(t)$ in the form $x(t) = A(t)\sin(\omega t + \phi)$. Graph $x(t)$ using a graphing utility.

2. Write $x(t) = (t + 3)\sqrt{2}\cos(2t) + (t + 3)\sqrt{2}\sin(2t)$ in the form $x(t) = A(t)\sin(\omega t + \phi)$. Graph $x(t)$ using a graphing utility.

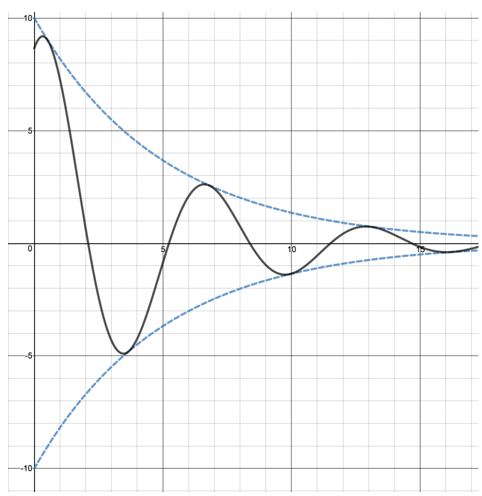
3. Find the period of $x(t) = 5\sin(6t) - 5\sin(8t)$. Graph $x(t)$ using a graphing utility.

Solution.

1. We start rewriting $x(t) = 5e^{-t/5}\cos(t) + 5e^{-t/5}\sqrt{3}\sin(t)$ by factoring out $5e^{-t/5}$ from both terms to get $x(t) = 5e^{-t/5}\left(\cos(t) + \sqrt{3}\sin(t)\right)$. We convert what's left in parentheses to the required form using the technique introduced in Example 12.2.1 from Section 12.2. We find $\left(\cos(t) + \sqrt{3}\sin(t)\right) = 2\sin\left(t + \frac{\pi}{3}\right)$ so that $x(t) = 10e^{-t/5}\sin\left(t + \frac{\pi}{3}\right)$.

   Graphing $x(t)$ reveals some interesting behavior. The sinusoidal nature continues indefinitely, but it is being attenuated. In the sinusoid $A\sin(\omega t + \phi)$, the coefficient A of the sine function is the amplitude. In the case of $x(t) = 10e^{-t/5}\sin\left(t + \frac{\pi}{3}\right)$, we can think of the *function* $A(t) = 10e^{-t/5}$ as the amplitude.[24] Since $\displaystyle\lim_{t\to\infty} 10e^{-t/5} = 0$, we can use the Squeeze Theorem, Theorem 10.2 that $\displaystyle\lim_{t\to\infty} x(t) = 0$. (See Exercise 113).

   Indeed, if we graph $x = \pm 10e^{-t/5}$ along with $x(t) = 10e^{-t/5}\sin\left(t + \frac{\pi}{3}\right)$, we see this attenuation taking place with the exponentials acting as a 'wave envelope.'



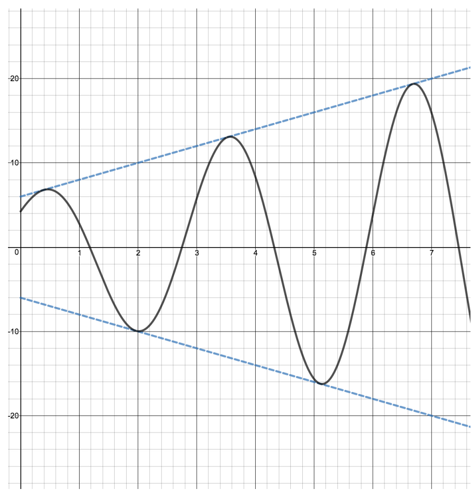
$x(t) = 10e^{-t/5}\sin\left(t + \frac{\pi}{3}\right)$ and $x = \pm 10e^{-t/5}$

[24] This is the same sort of phenomenon we saw on page 1034 in Section 12.2.1.

In this case, the function $x(t)$ corresponds to the motion of an object on a spring where there is a slight force which acts to 'damp', or slow the motion. An example of this kind of force would be the friction of the object against the air. According to this model, the object oscillates forever, but with increasingly smaller and smaller amplitude. This motion is often described as **underdamped** motion.

2. Proceeding as in the first example, we factor out $(t+3)\sqrt{2}$ from each term in the function $x(t)$ to get $x(t) = (t+3)\sqrt{2}(\cos(2t)+\sin(2t))$. We find $(\cos(2t)+\sin(2t)) = \sqrt{2}\sin\left(2t+\frac{\pi}{4}\right)$, so an equivalent form of $x(t)$ is $x(t) = 2(t+3)\sin\left(2t+\frac{\pi}{4}\right)$.

   Graphing $x(t)$, we find the sinusoid's amplitude growing. This isn't too surprising since our amplitude function here is $A(t) = 2(t+3) = 2t+6$, grows without bound as $t \to \infty$.

   $\boldsymbol{x(t) = 2(t+3)\sin\left(2t+\frac{\pi}{4}\right)}$ and $x = \pm 2(t+3)$

   The phenomenon illustrated here is 'forced' motion. That is, we imagine that the entire apparatus on which the spring is attached is oscillating as well.

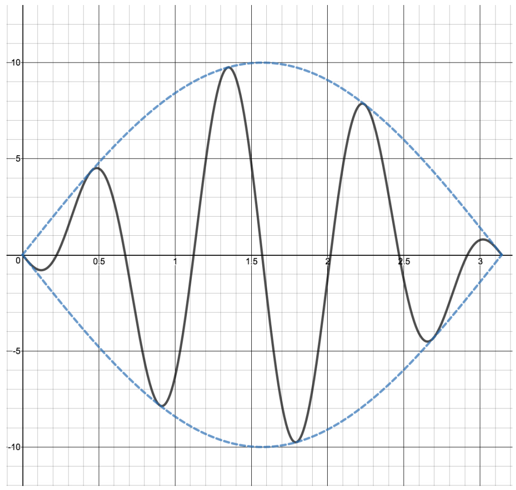
   In this particular case, we are witnessing a 'resonance' effect – the frequency of the external oscillation matches the frequency of the motion of the object on the spring. In a mechanical system, this will result in some sort of structural failure.[25]

3. Last, but not least, we come to $x(t) = 5\sin(6t) - 5\sin(8t)$. To find the period of this function, we need to determine the length of the smallest interval on which both $f(t) = 5\sin(6t)$ and $g(t) = 5\sin(8t)$ complete a whole number of cycles.

   To do this, we take the ratio of their frequencies and reduce to lowest terms: $\frac{6}{8} = \frac{3}{4}$. This tells us that for every 3 cycles f makes, g makes 4. Hence, the period of $x(t)$ is three times the period of $f(t)$ (which is four times the period of $g(t)$), or π. We check our work by graphing $x(t)$ over $[0, \pi]$

[25] The reader is invited to investigate the destructive implications of resonance.

The reader may recognize $x(t)$ an example of the 'beats' phenomenon we first saw on 1034 in Section 12.2.1. Indeed, using a sum to product identity, we may rewrite $x(t)$ as $x(t) = -10\sin(t)\cos(7t)$. As we saw on 1034 (and Exercises 122 - 125 in Section 12.2), the lower frequency factor, $-10\sin(t)$ determines the 'wave-envelope,' $x = \pm 10\sin(t)$.



$\boldsymbol{x(t) = 5\sin(6t) - 5\sin(8t)}$ and $x = \pm 10\sin(t)$ over $[0, \pi]$

This equation of motion also results from 'forced' motion, but here the frequency of the external oscillation is different than that of the object on the spring. Since the sinusoids here have different frequencies, they are 'out of sync' and do not amplify each other as in the previous example. Instead, through a combination of constructive and destructive interference, the mass continues to oscillate no more than 10 units from its equilibrium position indefinitely. □

Our last examples use the tools of this section along with those developed in Section 6.3.

Example 12.4.8. Let $f(x) = 2\sin(x) - \sin(2x)$ restricted to the interval $[0, 2\pi]$.

1. Given $f'(x) = 2\cos(x) - 2\cos(2x)$, find the open intervals over which f is increasing and decreasing.
2. Locate all relative extrema and check your answers graphically.

Solution.

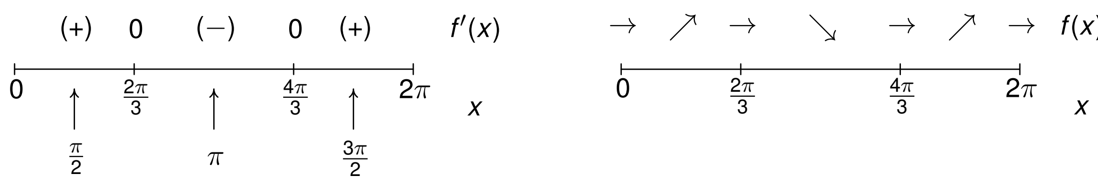
1. We begin making a sign diagram for $f'(x) = 2\cos(x) - 2\cos(2x)$ by solving $f'(x) = 0$:

$$\begin{array}{rcll} f'(x) & = & 0 & \\ 2\cos(x) - 2\cos(2x) & = & 0 & \\ 2\cos(x) - 2\left(2\cos^2(x) - 1\right) & = & 0 & \text{Double Angle Identity: } \cos(2x) = 2\cos^2(x) - 1 \end{array}$$

$$\begin{aligned} 2\cos(x) - 4\cos^2(x) + 2 &= 0 \\ -2\left(2\cos^2(x) - \cos(x) - 1\right) &= 0 \\ -2(2\cos(x)+1)(\cos(x)-1) &= 0 \end{aligned}$$

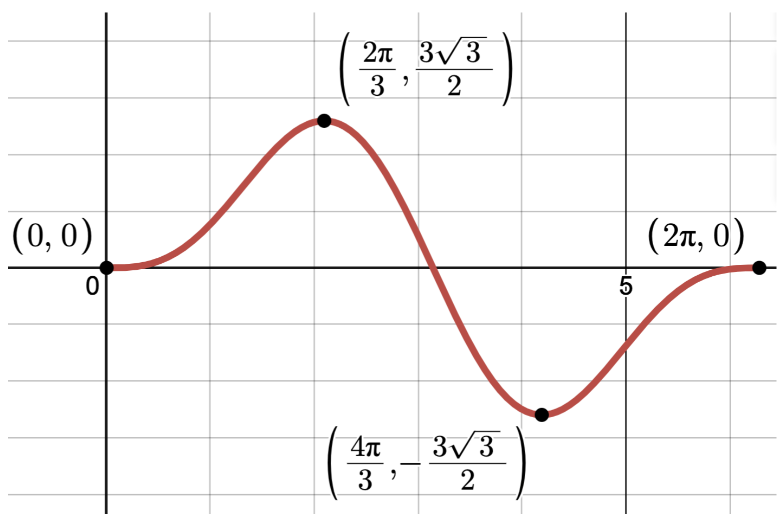
We get $2\cos(x) + 1 = 0$ or $\cos(x) = -\frac{1}{2}$. In the interval $(0, 2\pi)$, the only solutions are $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$. We also get $\cos(x) - 1 = 0$ or $\cos(x) = 1$ which occurs at the endpoints $x = 0$ and $x = 2\pi$.

We create our sign diagram for $f'(x)$ and interpret what it means for $f(x)$ below.



We find f is increasing on $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$ and f is decreasing on $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$.

2. Using the sign diagram along with the First Derivative Test, Theorem 6.5, we find that f has a local maximum: $\left(\frac{2\pi}{3}, f\left(\frac{2\pi}{3}\right)\right) = \left(\frac{2\pi}{3}, \frac{3\sqrt{3}}{2}\right)$ and a local minimum:$\left(\frac{4\pi}{3}, f\left(\frac{4\pi}{3}\right)\right) = \left(\frac{4\pi}{3}, -\frac{3\sqrt{3}}{2}\right)$. Both of these extrema are absolute (global) as well as local.



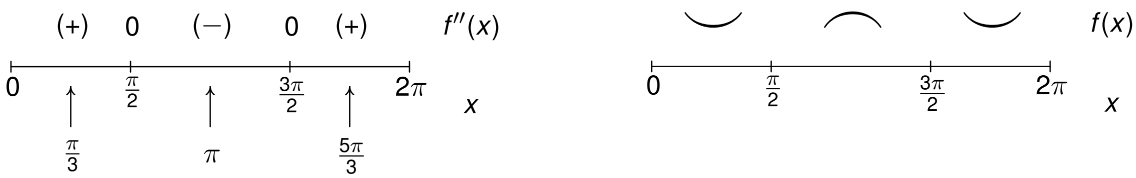
□

We'll continue our work with $f(x) = 2\sin(x) - \sin(2x)$ from Example 12.4.8 in Exercise 110.

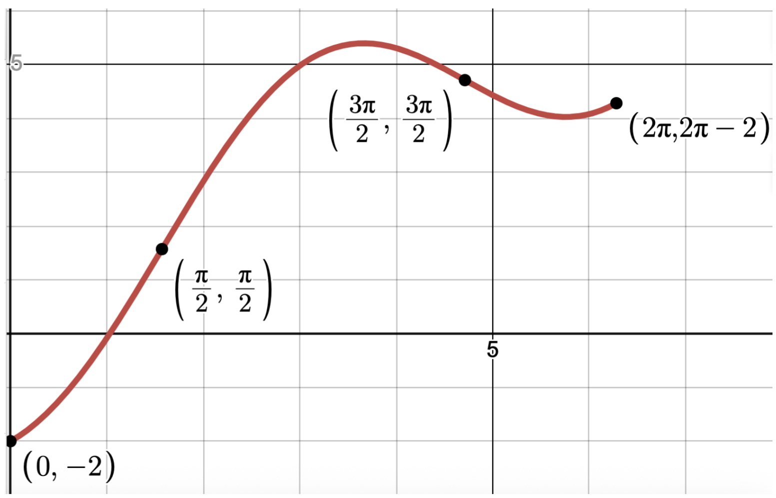
Example 12.4.9. Let $f(x) = x - 2\cos(x)$ restricted to the interval $0 \leq x \leq 2\pi$.

Given $f''(x) = 2\cos(x)$, find the inflection points of the graph. Check your answer graphically.

Solution. We make a sign digram for $f''(x)$, first solving $f''(x) = 2\cos(x) = 0$. We get $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.



We see the concavity changes at both $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ so we have inflection points: $\left(\frac{\pi}{2}, f\left(\frac{\pi}{2}\right)\right) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, f\left(\frac{3\pi}{2}\right)\right) = \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$.



□

We'll revisit $f(x) = x - 2\cos(x)$ from Example 12.4.9 in Exercise 111. Speaking of Exercises . . .

12.4.2 Exercises

In Exercises 1 - 18, find all of the exact solutions of the equation and then list those solutions which are in the interval $[0, 2\pi)$.

1. $\sin(5\theta) = 0$
2. $\cos(3t) = \frac{1}{2}$
3. $\sin(-2x) = \frac{\sqrt{3}}{2}$
4. $\tan(6\theta) = 1$
5. $\csc(4t) = -1$
6. $\sec(3x) = \sqrt{2}$
7. $\cot(2\theta) = -\frac{\sqrt{3}}{3}$
8. $\cos(9t) = 9$
9. $\sin\left(\frac{x}{3}\right) = \frac{\sqrt{2}}{2}$
10. $\cos\left(\theta + \frac{5\pi}{6}\right) = 0$
11. $\sin\left(2t - \frac{\pi}{3}\right) = -\frac{1}{2}$
12. $2\cos\left(x + \frac{7\pi}{4}\right) = \sqrt{3}$
13. $\csc(\theta) = 0$
14. $\tan(2t - \pi) = 1$
15. $\tan^2(x) = 3$
16. $\sec^2(\theta) = \frac{4}{3}$
17. $\cos^2(t) = \frac{1}{2}$
18. $\sin^2(x) = \frac{3}{4}$

In Exercises 19 - 42, solve the equation, giving the exact solutions which lie in $[0, 2\pi)$

19. $\sin(\theta) = \cos(\theta)$
20. $\sin(2t) = \sin(t)$
21. $\sin(2x) = \cos(x)$
22. $\cos(2\theta) = \sin(\theta)$
23. $\cos(2t) = \cos(t)$
24. $\cos(2x) = 2 - 5\cos(x)$
25. $3\cos(2\theta) + \cos(\theta) + 2 = 0$
26. $\cos(2t) = 5\sin(t) - 2$
27. $3\cos(2x) = \sin(x) + 2$
28. $2\sec^2(\theta) = 3 - \tan(\theta)$
29. $\tan^2(t) = 1 - \sec(t)$
30. $\cot^2(x) = 3\csc(x) - 3$
31. $\sec(\theta) = 2\csc(\theta)$
32. $\cos(t)\csc(t)\cot(t) = 6 - \cot^2(t)$
33. $\sin(2x) = \tan(x)$
34. $\cot^4(\theta) = 4\csc^2(\theta) - 7$
35. $\cos(2t) + \csc^2(t) = 0$
36. $\tan^3(x) = 3\tan(x)$
37. $\tan^2(\theta) = \frac{3}{2}\sec(\theta)$
38. $\cos^3(t) = -\cos(t)$
39. $\tan(2x) - 2\cos(x) = 0$
40. $\csc^3(\theta) + \csc^2(\theta) = 4\csc(\theta) + 4$
41. $2\tan(t) = 1 - \tan^2(t)$
42. $\tan(x) = \sec(x)$

In Exercises 43 - 58, solve the equation, giving the exact solutions which lie in $[0, 2\pi)$

43. $\sin(6\theta)\cos(\theta) = -\cos(6\theta)\sin(\theta)$

44. $\sin(3t)\cos(t) = \cos(3t)\sin(t)$

45. $\cos(2x)\cos(x) + \sin(2x)\sin(x) = 1$

46. $\cos(5\theta)\cos(3\theta) - \sin(5\theta)\sin(3\theta) = \dfrac{\sqrt{3}}{2}$

47. $\sin(t) + \cos(t) = 1$

48. $\sin(x) + \sqrt{3}\cos(x) = 1$

49. $\sqrt{2}\cos(\theta) - \sqrt{2}\sin(\theta) = 1$

50. $\sqrt{3}\sin(2t) + \cos(2t) = 1$

51. $\cos(2x) - \sqrt{3}\sin(2x) = \sqrt{2}$

52. $3\sqrt{3}\sin(3\theta) - 3\cos(3\theta) = 3\sqrt{3}$

53. $\cos(3t) = \cos(5t)$

54. $\cos(4x) = \cos(2x)$

55. $\sin(5\theta) = \sin(3\theta)$

56. $\cos(5t) = -\cos(2t)$

57. $\sin(6x) + \sin(x) = 0$

58. $\tan(x) = \cos(x)$

In Exercises 59 - 68, solve the equation.

59. $\arccos(2x) = \pi$

60. $\pi - 2\arcsin(t) = 2\pi$

61. $4\arctan(3x - 1) - \pi = 0$

62. $6\,\text{arccot}(2t) - 5\pi = 0$

63. $4\,\text{arcsec}\left(\frac{x}{2}\right) = \pi$

64. $12\,\text{arccsc}\left(\frac{t}{3}\right) = 2\pi$

65. $9\arcsin^2(x) - \pi^2 = 0$

66. $9\arccos^2(t) - \pi^2 = 0$

67. $8\,\text{arccot}^2(x) + 3\pi^2 = 10\pi\,\text{arccot}(x)$

68. $6\arctan(t)^2 = \pi\arctan(x) + \pi^2$

In Exercises 69 - 80, solve the inequality. Express the exact answer in <u>interval</u> notation, restricting your attention to $0 \leq x \leq 2\pi$.

69. $\sin(x) \leq 0$

70. $\tan(t) \geq \sqrt{3}$

71. $\sec^2(x) \leq 4$

72. $\cos^2(t) > \dfrac{1}{2}$

73. $\cos(2x) \leq 0$

74. $\sin\left(t + \dfrac{\pi}{3}\right) > \dfrac{1}{2}$

75. $\cot^2(x) \geq \dfrac{1}{3}$

76. $2\cos(t) \geq 1$

77. $\sin(5x) \geq 5$

78. $\cos(3t) \leq 1$

79. $\sec(x) \leq \sqrt{2}$

80. $\cot(t) \leq 4$

In Exercises 81 - 86, solve the inequality. Express the exact answer in interval notation, restricting your attention to $-\pi \leq x \leq \pi$.

81. $\cos(x) > \dfrac{\sqrt{3}}{2}$

82. $\sin(t) > \dfrac{1}{3}$

83. $\sec(x) \leq 2$

84. $\sin^2(t) < \dfrac{3}{4}$

85. $\cot(x) \geq -1$

86. $\cos(t) \geq \sin(t)$

In Exercises 87 - 92, solve the inequality. Express the exact answer in interval notation, restricting your attention to $-2\pi \leq x \leq 2\pi$.

87. $\csc(x) > 1$

88. $\cos(t) \leq \dfrac{5}{3}$

89. $\cot(x) \geq 5$

90. $\tan^2(t) \geq 1$

91. $\sin(2x) \geq \sin(x)$

92. $\cos(2t) \leq \sin(x)$

In Exercises 93 - 98, solve the given inequality.

93. $\arcsin(2x) > 0$

94. $3\arccos(t) \leq \pi$

95. $6\,\text{arccot}(7x) \geq \pi$

96. $\pi > 2\arctan(t)$

97. $2\arcsin(x)^2 > \pi\arcsin(x)$

98. $12\arccos(t)^2 + 2\pi^2 > 11\pi\arccos(t)$

In Exercises 99 - 107, express the domain of the function using the extended interval notation. (See Example 12.4.4 and Section 11.5.3 for details.)

99. $f(x) = \dfrac{1}{\cos(x) - 1}$

100. $f(t) = \dfrac{\cos(t)}{\sin(t) + 1}$

101. $f(x) = \sqrt{\tan^2(x) - 1}$

102. $f(t) = \sqrt{2 - \sec(t)}$

103. $f(x) = \csc(2x)$

104. $f(t) = \dfrac{\sin(t)}{2 + \cos(t)}$

105. $f(x) = 3\csc(x) + 4\sec(x)$

106. $f(t) = \ln\left(|\cos(t)|\right)$

107. $f(x) = \arcsin(\tan(x))$

108. (a) With the help of your classmates, determine the number of solutions to $\sin(x) = \frac{1}{2}$ in $[0, 2\pi)$. Then find the number of solutions to $\sin(2x) = \frac{1}{2}$, $\sin(3x) = \frac{1}{2}$ and $\sin(4x) = \frac{1}{2}$ in $[0, 2\pi)$. What pattern emerges? Explain how this pattern would help you solve equations like $\sin(11x) = \frac{1}{2}$.

(b) Repeat the above exercise focusing on $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$, $\sin\left(\frac{3x}{2}\right) = \frac{1}{2}$ and $\sin\left(\frac{5x}{2}\right) = \frac{1}{2}$. What pattern emerges here?

(c) Replace sine with tangent and $\frac{1}{2}$ with 1 and repeat the whole exploration.

109. Suppose an object weighing 10 pounds is suspended from the ceiling by a spring which stretches 2 feet to its equilibrium position when the object is attached.

   (a) Find the spring constant k in $\frac{\text{lbs.}}{\text{ft.}}$ and the mass of the object in slugs.
   (b) Find the equation of motion of the object if it is released from 1 foot *below* the equilibrium position from rest. When is the first time the object passes through the equilibrium position? In which direction is it heading?
   (c) Find the equation of motion of the object if it is released from 6 inches *above* the equilibrium position with a *downward* velocity of 2 feet per second. Find when the object passes through the equilibrium position heading downwards for the third time.

110. In Example 12.4.8, $f(x) = 2\sin(x) - \sin(2x)$ restricted to $0 \leq x \leq 2\pi$. If $f''(x) = 4\sin(2x) - 2\sin(x)$, find the inflection points of the graph of $y = f(x)$.

111. In Example 12.4.9, $f(x) = x - 2\cos(x)$ restricted to $0 \leq x \leq 2\pi$. If $f'(x) = 2\sin(x) + 1$, list the open intervals over which f is increasing and decreasing. Find the local extrema.

112. Let $f(x) = e^{-x}\sin(x)$ for $x > 0$.

   (a) Use the Squeeze Theorem, Theorem 10.2, to find $\displaystyle\lim_{x\to\infty} f(x)$. Interpret your answer graphically.

   HINT: Since $-1 \leq \sin(x) \leq 1$, $-e^{-x} \leq e^{-x}\sin(x) \leq e^{-x} \ldots$
   (b) Use the fact that $f'(x) = e^{-x}\cos(x) - e^{-x}\sin(x)$ to help you find the intervals over which f is increasing and decreasing.
   (c) Use the fact that $f''(x) = -2e^{-x}\cos(x)$ to help you find the intervals over which the graph of f is concave up and concave down.

113. Recall $x(t) = 10e^{-t/5}\sin\left(t + \frac{\pi}{3}\right)$ from Example 12.4.7 number 1 models underdamped motion. Use the Squeeze Theorem, Theorem 10.2, to prove $\displaystyle\lim_{t\to\infty} x(t) = 0$.

   HINT: Since $-1 \leq \sin\left(t + \frac{\pi}{3}\right) \leq 1$, $-10e^{-t/5} \leq 10e^{-t/5}\sin\left(t + \frac{\pi}{3}\right) \leq 10e^{-t/5} \ldots$

12.4.3 Answers

1. $\theta = \frac{\pi k}{5}$; $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5}$
2. $t = \frac{\pi}{9} + \frac{2\pi k}{3}$ or $t = \frac{5\pi}{9} + \frac{2\pi k}{3}$; $t = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$
3. $x = \frac{2\pi}{3} + \pi k$ or $x = \frac{5\pi}{6} + \pi k$; $x = \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$
4. $\theta = \frac{\pi}{24} + \frac{\pi k}{6}$; $\theta = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{3\pi}{8}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{7\pi}{8}, \frac{25\pi}{24}, \frac{29\pi}{24}, \frac{11\pi}{8}, \frac{37\pi}{24}, \frac{41\pi}{24}, \frac{15\pi}{8}$
5. $t = \frac{3\pi}{8} + \frac{\pi k}{2}$; $t = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$
6. $x = \frac{\pi}{12} + \frac{2\pi k}{3}$ or $x = \frac{7\pi}{12} + \frac{2\pi k}{3}$; $x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$
7. $\theta = \frac{\pi}{3} + \frac{\pi k}{2}$; $\theta = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$
8. No solution
9. $x = \frac{3\pi}{4} + 6\pi k$ or $x = \frac{9\pi}{4} + 6\pi k$; $x = \frac{3\pi}{4}$
10. $\theta = -\frac{\pi}{3} + \pi k$; $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$
11. $t = \frac{3\pi}{4} + \pi k$ or $t = \frac{13\pi}{12} + \pi k$; $t = \frac{\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{7\pi}{4}$
12. $x = -\frac{19\pi}{12} + 2\pi k$ or $x = \frac{\pi}{12} + 2\pi k$; $x = \frac{\pi}{12}, \frac{5\pi}{12}$
13. No solution
14. $t = \frac{5\pi}{8} + \frac{\pi k}{2}$; $t = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$
15. $x = \frac{\pi}{3} + \pi k$ or $x = \frac{2\pi}{3} + \pi k$; $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
16. $\theta = \frac{\pi}{6} + \pi k$ or $\theta = \frac{5\pi}{6} + \pi k$; $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
17. $t = \frac{\pi}{4} + \frac{\pi k}{2}$; $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
18. $x = \frac{\pi}{3} + \pi k$ or $x = \frac{2\pi}{3} + \pi k$; $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

19. $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

20. $t = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

21. $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

22. $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

23. $t = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

24. $x = \frac{\pi}{3}, \frac{5\pi}{3}$

25. $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \arccos\left(\frac{1}{3}\right), 2\pi - \arccos\left(\frac{1}{3}\right)$

26. $t = \frac{\pi}{6}, \frac{5\pi}{6}$

27. $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \arcsin\left(\frac{1}{3}\right), \pi - \arcsin\left(\frac{1}{3}\right)$

28. $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \arctan\left(\frac{1}{2}\right), \pi + \arctan\left(\frac{1}{2}\right)$

29. $t = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

30. $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$

31. $\theta = \arctan(2), \pi + \arctan(2)$

32. $t = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$

33. $x = 0, \pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

34. $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{6}$

35. $t = \frac{\pi}{2}, \frac{3\pi}{2}$

36. $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

37. $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

38. $t = \frac{\pi}{2}, \frac{3\pi}{2}$

39. $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

40. $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

41. $t = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

42. No solution

43. $\theta = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{8\pi}{7}, \frac{9\pi}{7}, \frac{10\pi}{7}, \frac{11\pi}{7}, \frac{12\pi}{7}, \frac{13\pi}{7}$

44. $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

45. $x = 0$

46. $\theta = \frac{\pi}{48}, \frac{11\pi}{48}, \frac{13\pi}{48}, \frac{23\pi}{48}, \frac{25\pi}{48}, \frac{35\pi}{48}, \frac{37\pi}{48}, \frac{47\pi}{48}, \frac{49\pi}{48}, \frac{59\pi}{48}, \frac{61\pi}{48}, \frac{71\pi}{48}, \frac{73\pi}{48}, \frac{83\pi}{48}, \frac{85\pi}{48}, \frac{95\pi}{48}$

47. $t = 0, \frac{\pi}{2}$

48. $x = \frac{\pi}{2}, \frac{11\pi}{6}$

49. $\theta = \frac{\pi}{12}, \frac{17\pi}{12}$

50. $t = 0, \pi, \frac{\pi}{3}, \frac{4\pi}{3}$

51. $x = \frac{17\pi}{24}, \frac{41\pi}{24}, \frac{23\pi}{24}, \frac{47\pi}{24}$

52. $\theta = \frac{\pi}{6}, \frac{5\pi}{18}, \frac{5\pi}{6}, \frac{17\pi}{18}, \frac{3\pi}{2}, \frac{29\pi}{18}$

53. $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

54. $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

55. $\theta = 0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \pi, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

56. $t = \frac{\pi}{7}, \frac{\pi}{3}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{11\pi}{7}, \frac{5\pi}{3}, \frac{13\pi}{7}$

57. $x = 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

58. $x = \arcsin\left(\frac{-1+\sqrt{5}}{2}\right) \approx 0.6662, \pi - \arcsin\left(\frac{-1+\sqrt{5}}{2}\right) \approx 2.4754$

59. $x = -\frac{1}{2}$

60. $t = -1$

61. $x = \frac{2}{3}$

62. $t = -\frac{\sqrt{3}}{2}$

63. $x = 2\sqrt{2}$

64. $t = 6$

65. $x = \pm\frac{\sqrt{3}}{2}$

66. $t = \frac{1}{2}$

67. $x = -1, 0$

68. $t = -\sqrt{3}$

69. $[\pi, 2\pi]$

70. $\left[\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left[\frac{4\pi}{3}, \frac{3\pi}{2}\right)$

71. $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$

72. $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right]$

73. $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right]$

74. $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{11\pi}{6}, 2\pi\right]$

75. $\left(0, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \pi\right) \cup \left(\pi, \frac{4\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right)$

76. $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$

77. No solution

78. $[0, 2\pi]$

79. $\left[0, \frac{\pi}{4}\right] \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left[\frac{7\pi}{4}, 2\pi\right]$

80. $[\operatorname{arccot}(4), \pi) \cup [\pi + \operatorname{arccot}(4), 2\pi)$

81. $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$

82. $\left(\arcsin\left(\frac{1}{3}\right), \pi - \arcsin\left(\frac{1}{3}\right)\right)$

83. $\left[-\pi, -\frac{\pi}{2}\right) \cup \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \cup \left(\frac{\pi}{2}, \pi\right]$

84. $\left(-\frac{2\pi}{3}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$

85. $\left(-\pi, -\frac{\pi}{4}\right] \cup \left(0, \frac{3\pi}{4}\right]$

86. $\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$

87. $\left(-2\pi, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\pi\right) \cup \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

88. $[-2\pi, 2\pi]$

89. $(-2\pi, \operatorname{arccot}(5) - 2\pi] \cup (-\pi, \operatorname{arccot}(5) - \pi] \cup (0, \operatorname{arccot}(5)] \cup (\pi, \pi + \operatorname{arccot}(5)]$

90. $\left[-\frac{7\pi}{4}, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\frac{5\pi}{4}\right] \cup \left[-\frac{3\pi}{4}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right] \cup \left[\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right]$

91. $\left[-2\pi, -\frac{5\pi}{3}\right] \cup \left[-\pi, -\frac{\pi}{3}\right] \cup \left[0, \frac{\pi}{3}\right] \cup \left[\pi, \frac{5\pi}{3}\right]$

92. $\left[-\frac{11\pi}{6}, -\frac{7\pi}{6}\right] \cup \left[\frac{\pi}{6}, \frac{5\pi}{6}\right] \cup, \left\{-\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

93. $\left(0, \frac{1}{2}\right]$

94. $\left[\frac{1}{2}, 1\right]$

95. $\left(-\infty, \frac{\sqrt{3}}{7}\right]$

96. $(-\infty, \infty)$

97. $[-1, 0)$

98. $\left[-1, -\frac{1}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, 1\right]$

99. $\displaystyle\bigcup_{k=-\infty}^{\infty} (2k\pi, (2k+2)\pi)$

100. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{(4k-1)\pi}{2}, \frac{(4k+3)\pi}{2}\right)$

101. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left\{\left[\frac{(4k+1)\pi}{4}, \frac{(2k+1)\pi}{2}\right) \cup \left(\frac{(2k+1)\pi}{2}, \frac{(4k+3)\pi}{4}\right]\right\}$

102. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left\{\left[\frac{(6k-1)\pi}{3}, \frac{(6k+1)\pi}{3}\right] \cup \left(\frac{(4k+1)\pi}{2}, \frac{(4k+3)\pi}{2}\right)\right\}$

103. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right)$

104. $(-\infty, \infty)$

105. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right)$

106. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}\right)$

107. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left[\frac{(4k-1)\pi}{4}, \frac{(4k+1)\pi}{4}\right]$

109. (a) $k = 5 \, \frac{\text{lbs.}}{\text{ft.}}$ and $m = \frac{5}{16}$ slugs

(b) $x(t) = \sin\left(4t + \frac{\pi}{2}\right)$. The object first passes through the equilibrium point when $t = \frac{\pi}{8} \approx 0.39$ seconds after the motion starts. At this time, the object is heading upwards.

(c) $x(t) = \frac{\sqrt{2}}{2} \sin\left(4t + \frac{7\pi}{4}\right)$. The object passes through the equilibrium point heading downwards for the third time when $t = \frac{17\pi}{16} \approx 3.34$ seconds.

110. The inflection points are: $\left(\arccos\left(\frac{1}{4}\right), \frac{3\sqrt{15}}{8}\right)$, $(\pi, 0)$, and $\left(2\pi - \arccos\left(\frac{1}{4}\right), -\frac{3\sqrt{15}}{8}\right)$

111. f is increasing on $\left(0, \frac{7\pi}{6}\right)$ and again on $\left(\frac{11\pi}{6}, 2\pi\right)$; f is decreasing on $\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$; local (absolute) max: $\left(\frac{7\pi}{6}, \frac{7\pi}{6} + \sqrt{3}\right)$; local min: $\left(\frac{11\pi}{6}, \frac{11\pi}{6} - \sqrt{3}\right)$

112. (a) $\lim\limits_{x\to\infty} f(x) = 0$. We have a horizontal asymptote $y = 0$.

(b)
- f is increasing on $\left(0, \frac{\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right), \left(\frac{13\pi}{4}, \frac{17\pi}{4}\right), \ldots$:
  In other words: on $\left(0, \frac{\pi}{4}\right)$ along with $\left(\frac{(8k+5)\pi}{4}, \frac{(8k+9)\pi}{4}\right)$ for $k = 0, 1, 2, 3, ...$
- f is decreasing on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{9\pi}{4}, \frac{13\pi}{4}\right), \left(\frac{17\pi}{4}, \frac{21\pi}{4}\right), \ldots$:
  In other words: on $\left(\frac{(8k+1)\pi}{4}, \frac{(8k+5)\pi}{4}\right)$ for $k = 0, 1, 2, 3 ...$

(c)
- the graph of f is concave up on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{5\pi}{2}, \frac{7\pi}{2}\right), \left(\frac{9\pi}{2}, \frac{11\pi}{2}\right), \ldots$:
  In other words: on $\left(\frac{(4k+1)\pi}{2}, \frac{(4k+3)\pi}{2}\right)$ for $k = 0, 1, 2, 3, ...$
- the graph of f is concave down on $\left(0, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \left(\frac{7\pi}{2}, \frac{9\pi}{2}\right), \ldots$:
  In other words: on $\left(0, \frac{\pi}{2}\right)$ along with $\left(\frac{(4k+3)\pi}{2}, \frac{(4k+5)\pi}{2}\right)$ for $k = 0, 1, 2, 3 ...$

Chapter 13

Geometric Applications of Trigonometry

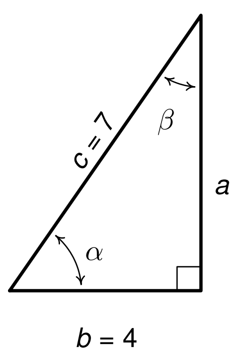
13.1 The Law of Sines

In this chapter, we showcase how the the tools we've developed in Chapters 11 and 12 can be applied to Geometry. Our first two sections focus specifically on solving oblique (non-right) Triangles.[1]

Our first example reviews the basics of right triangle trigonometry. The reader is referred to Section B.2 for more details and practice with these concepts.

Example 13.1.1. Given a right triangle with a hypotenuse of length 7 units and one leg of length 4 units, find the length of the remaining side and the measures of the remaining angles. Express the angles in decimal degrees, rounded to the nearest hundredth of a degree.

Solution. For definitiveness, we label the triangle below.



To find a, we use the Pythagorean Theorem, Theorem B.1: $a^2 + 4^2 = 7^2$, so $a = \sqrt{33}$ units.

Now that all three sides of the triangle are known, there are several ways we can find α using the inverse trigonometric functions.

To decrease the chances of propagating error, however, we stick to using the data given to us in the problem. In this case, the lengths 4 and 7 were given, so we want to relate these to α.

[1] Recall that the word 'Trigonometry' literally means 'measuring triangles' so we are returning to our roots here.

According to Definition B.1, $\cos(\alpha) = \frac{4}{7}$. Since α is an acute angle, $\alpha = \arccos\left(\frac{4}{7}\right)$ radians $\approx 55.15^{\circ}$.

Now that we have the measure of angle α, we could find the measure of angle β using the fact that α and β are complements so $\alpha + \beta = 90^{\circ}$.

Once again, in the interests of minimizing propagated error, we opt to use the data given to us in the problem. According to Definition B.1, $\sin(\beta) = \frac{4}{7}$ so $\beta = \arcsin\left(\frac{4}{7}\right)$ radians $\approx 34.85^{\circ}$. □

A few remarks about Example 13.1.1 are in order. First, we adhere to the convention that a lower case Greek letter denotes an angle (as well as the measure of said angle) and the corresponding lowercase English letter represents the side (as well as the length of said side) opposite that angle.

More specifically, a is the side opposite α, b is the side opposite β and c is the side opposite γ. Taken together, the pairs (α, a), (β, b) and (γ, c) are called *angle-side opposite pairs*.

Second, as mentioned earlier, we will strive to solve for quantities using the original data given in the problem whenever possible. While this is not always the easiest or fastest way to proceed, it minimizes the chances of propagated error.[2]

Third, since many of the applications which require solving triangles 'in the wild' rely on degree measure, we shall adopt this convention for the time being.

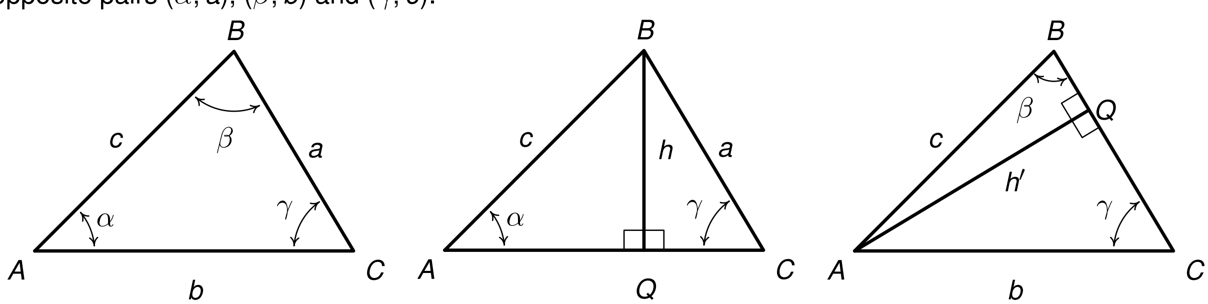
The Pythagorean Theorem along with Definition B.1 allow us to easily handle any given *right* triangle problem, but what if the triangle isn't a right triangle? In certain cases, we can use the **Law of Sines**.

Theorem 13.1. The Law of Sines: Given a triangle with angle-side opposite pairs (α, a), (β, b) and (γ, c), the following ratios hold:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} \quad \text{or, equivalently,} \quad \frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

The proof of the Law of Sines can be broken into three cases, and, as we'll see, ultimately relies on what we know about right triangles.

For our first case, consider the triangle $\triangle ABC$ below, all of whose angles are acute, with angle-side opposite pairs (α, a), (β, b) and (γ, c).



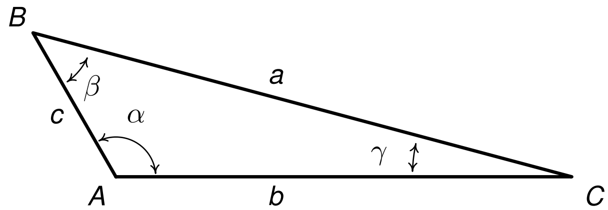
If we drop an altitude from vertex B, we divide the triangle into two right triangles: $\triangle ABQ$ and $\triangle BCQ$.

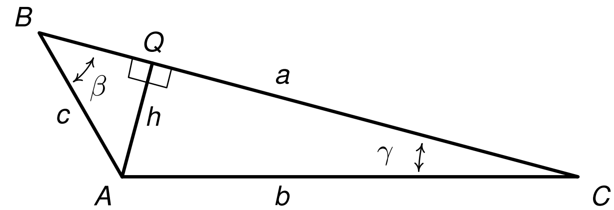
If we call the length of the altitude h (for height), we get from Definition B.1 that $\sin(\alpha) = \frac{h}{c}$ and $\sin(\gamma) = \frac{h}{a}$ so that $h = c\sin(\alpha) = a\sin(\gamma)$. Rearranging this last equation, we get $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$.

[2]Your Science teachers should thank us for this.

Dropping an altitude from vertex A, we can proceed as above using the triangles $\triangle ABQ$ and $\triangle ACQ$. We find that $\frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$, so we have shown $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$ as required.

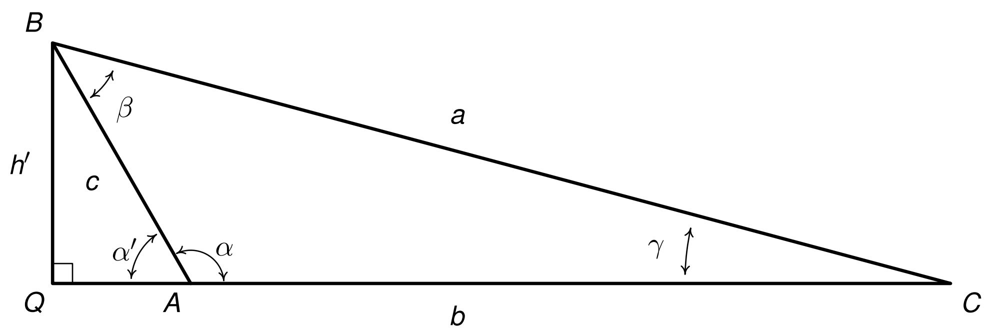
For our next case consider the triangle $\triangle ABC$ below with obtuse angle α.





Extending an altitude from vertex A gives two right triangles, as in the previous case: $\triangle ABQ$ and $\triangle ACQ$. Proceeding as before, we get $h = b\sin(\gamma)$ and $h = c\sin(\beta)$ so that $\frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$.

Dropping an altitude from vertex B also generates two right triangles, $\triangle ABQ$ and $\triangle BCQ$.



We see $\sin(\alpha') = \frac{h'}{c}$ so that $h' = c\sin(\alpha')$. Since $\alpha' = 180^\circ - \alpha$, $\sin(\alpha') = \sin(\alpha)$, so $h' = c\sin(\alpha)$.

Proceeding to $\triangle BCQ$, we get $\sin(\gamma) = \frac{h'}{a}$ so $h' = a\sin(\gamma)$.

As before, we get $\frac{\sin(\gamma)}{c} = \frac{\sin(\alpha)}{a}$, so $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$ in this case, too.

The remaining case is when $\triangle ABC$ is a right triangle. In this case, the Law of Sines reduces to the formulas given in Definition B.1 and is left to the reader.

In order to use the Law of Sines to solve a triangle, we need at least one angle-side opposite pair. The next example showcases some of the power, and the pitfalls, of the Law of Sines.

Example 13.1.2. Solve the following triangles. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

1. $\alpha = 120^\circ$, $a = 7$ units, $\beta = 45^\circ$
2. $\alpha = 85^\circ$, $\beta = 30^\circ$, $c = 5.25$ units
3. $\alpha = 30^\circ$, $a = 1$ units, $c = 4$ units
4. $\alpha = 30^\circ$, $a = 2$ units, $c = 4$ units

5. $\alpha = 30^\circ$, $a = 3$ units, $c = 4$ units

6. $\alpha = 30^\circ$, $a = 4$ units, $c = 4$ units

Solution.

1. Knowing an angle-side opposite pair, namely α and a, we may proceed in using the Law of Sines.

   Since $\beta = 45^\circ$, we use $\frac{b}{\sin(45^\circ)} = \frac{7}{\sin(120^\circ)}$ so $b = \frac{7\sin(45^\circ)}{\sin(120^\circ)} = \frac{7\sqrt{6}}{3} \approx 5.72$ units.

   To find γ, we use the fact that the sum of the measures of the angles in a triangle is 180°. Hence, $\gamma = 180^\circ - 120^\circ - 45^\circ = 15^\circ$.

   To find c, we have no choice but to used the derived value $\gamma = 15^\circ$, yet we can minimize the propagation of error here by using the given angle-side opposite pair (α, a).

   The Law of Sines gives us $\frac{c}{\sin(15^\circ)} = \frac{7}{\sin(120^\circ)}$ so that $c = \frac{7\sin(15^\circ)}{\sin(120^\circ)} \approx 2.09$ units.[3]

   We sketch this triangle below on the left.

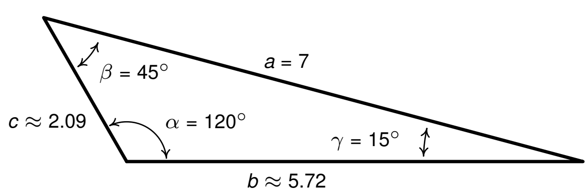
2. In this example, we are not immediately given an angle-side opposite pair, but as we have the measures of α and β, we can solve for γ since $\gamma = 180^\circ - 85^\circ - 30^\circ = 65^\circ$.

   As in the previous example, we are forced to use a derived value in our computations since the only angle-side pair available is (γ, c).

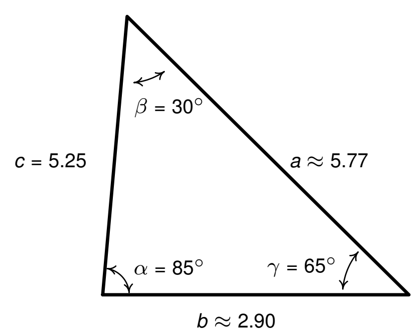
   The Law of Sines gives $\frac{a}{\sin(85^\circ)} = \frac{5.25}{\sin(65^\circ)}$. Solving, we get $a = \frac{5.25\sin(85^\circ)}{\sin(65^\circ)} \approx 5.77$ units.

   To find b we use the angle-side pair (γ, c): $\frac{b}{\sin(30^\circ)} = \frac{5.25}{\sin(65^\circ)}$. Hence $b = \frac{5.25\sin(30^\circ)}{\sin(65^\circ)} \approx 2.90$ units.

   We sketch this triangle below on the right.



Triangle for number 1



Triangle for number 2

3. Since we are given (α, a) and c, we use the Law of Sines to find the measure of γ.

   From $\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{1}$, we get $\sin(\gamma) = 4\sin(30^\circ) = 2$, which is impossible. (Why?) As seen below on the left, side a is just too short to make a triangle.

[3]The exact value of $\sin(15^\circ)$ could be found using the difference identity for sine or a half-angle formula, but that becomes unnecessarily messy for the discussion at hand. Thus "exact" here means $\frac{7\sin(15^\circ)}{\sin(120^\circ)}$.

The next three examples keep the same values for the measure of α and the length of c while varying the length of a. We will discuss this case in more detail after we see what happens in those examples.

4. In this case, we have the measure of $\alpha = 30^\circ$, $a = 2$ and $c = 4$. Using the Law of Sines, we get $\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{2}$ so $\sin(\gamma) = 2\sin(30^\circ) = 1$.

   Since γ is an angle in a triangle which also contains $\alpha = 30^\circ$, γ must measure between 0° and 150° in order to fit inside the triangle with α. The only angle that satisfies this requirement and has $\sin(\gamma) = 1$ is $\gamma = 90^\circ$, so we are working in a right triangle.

   We find the measure of β to be $\beta = 180^\circ - 30^\circ - 90^\circ = 60^\circ$. Using the Law of Sines, we get $b = \frac{2\sin(60^\circ)}{\sin(30^\circ)} = 2\sqrt{3} \approx 3.46$ units.

   As seen below on the right, the side a is just long enough to form a right triangle in this case.

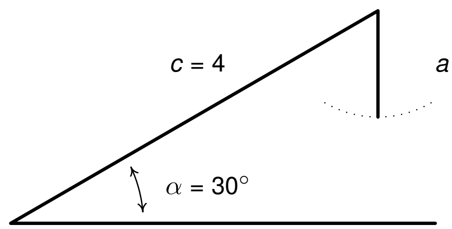
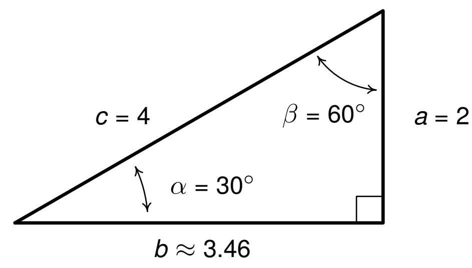


Diagram for number 3



Triangle for number 4

5. Proceeding as we have in the previous two examples, we use the Law of Sines to find γ.

   In this case, we have $\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{3}$ or $\sin(\gamma) = \frac{4\sin(30^\circ)}{3} = \frac{2}{3}$. Since γ lies in a triangle with $\alpha = 30^\circ$, we must have that $0^\circ < \gamma < 150^\circ$.

   In this case, there are *two* angles that fall in this range: $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians $\approx 41.81^\circ$ and $\gamma = \pi - \arcsin\left(\frac{2}{3}\right)$ radians $\approx 138.19^\circ$.

   At this point, we pause to see if it makes sense that we have two cases to consider.

   Since $c > a$, it must also be true that γ, which is opposite c, has greater measure than α which is opposite a. In both cases, $\gamma > \alpha$, so both candidates for γ make sense with the given value of c.

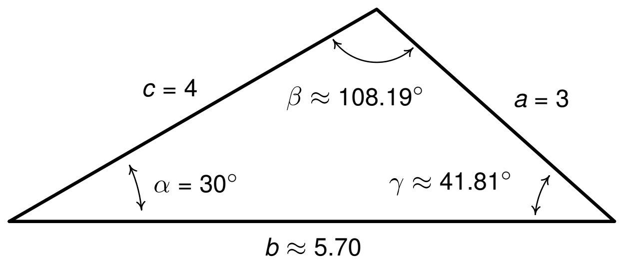
   Thus have two triangles on our hands. In the case $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians $\approx 41.81^\circ$, we find[4] $\beta \approx 180^\circ - 30^\circ - 41.81^\circ = 108.19^\circ$.

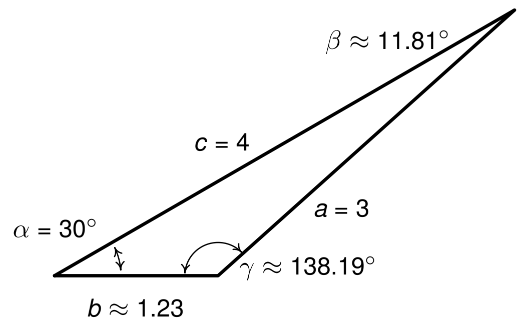
   The Law of Sines with the angle-side opposite pair (α, a) and β gives $b \approx \frac{3\sin(108.19^\circ)}{\sin(30^\circ)} \approx 5.70$ units. We sketch this triangle below on the left.

   In the case $\gamma = \pi - \arcsin\left(\frac{2}{3}\right)$ radians $\approx 138.19^\circ$, we repeat the same steps and find $\beta \approx 11.81^\circ$ and $b \approx 1.23$ units.[5] We sketch this triangle below on the right.

[4] To find an exact expression for β, we convert everything back to radians: $\alpha = 30^\circ = \frac{\pi}{6}$ radians, $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians and $180^\circ = \pi$ radians. Hence, $\beta = \pi - \frac{\pi}{6} - \arcsin\left(\frac{2}{3}\right) = \frac{5\pi}{6} - \arcsin\left(\frac{2}{3}\right)$ radians $\approx 108.19^\circ$.

[5] An exact answer for β in this case is $\beta = \arcsin\left(\frac{2}{3}\right) - \frac{\pi}{6}$ radians $\approx 11.81^\circ$.

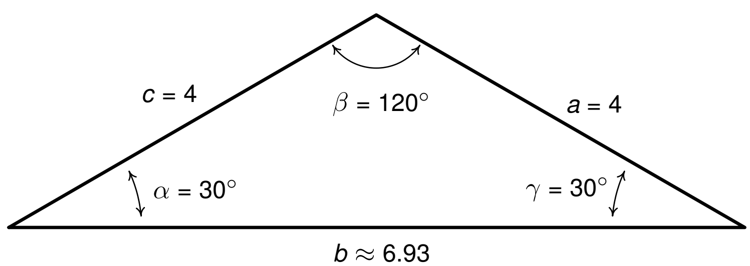




6. For this last problem, we repeat the usual Law of Sines routine to find that $\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{4}$ so that $\sin(\gamma) = \frac{1}{2}$. Since γ must inhabit a triangle with $\alpha = 30^\circ$, we must have $0^\circ < \gamma < 150^\circ$.

   Since the measure of γ must be *strictly less* than 150°, there is just *one* angle which satisfies both required conditions, namely $\gamma = 30^\circ$.

   Hence, $\beta = 180^\circ - 30^\circ - 30^\circ = 120^\circ$. The Law of Sines gives $b = \frac{4\sin(120^\circ)}{\sin(30^\circ)} = 4\sqrt{3} \approx 6.93$ units. We sketch this triangle below.



□

Some remarks about Example 13.1.2 are in order. First note that if we are given the measures of two of the angles in a triangle, say α and β, the measure of the third angle γ is uniquely determined using the equation $\gamma = 180^\circ - \alpha - \beta$. Knowing the measures of all three angles of a triangle completely determines the triangle's *shape*.

If in addition we are given the length of one of the sides of the triangle, we can then use the Law of Sines to find the lengths of the remaining two sides to determine the *size* of the triangle. Such is the case in numbers 1 and 2 above.

In number 1, the given side is adjacent to just one of the angles – this is called the 'Angle-Angle-Side' (AAS) case.[6] In number 2, the given side is adjacent to both angles which means we are in the so-called 'Angle-Side-Angle' (ASA) case.

[6]If this sounds familiar, it should. From Geometry, we know there are four congruence conditions for triangles: Angle-Angle-Side (AAS), Angle-Side-Angle (ASA), Side-Angle-Side (SAS) and Side-Side-Side (SSS). If we are given information about a triangle that meets one of these four criteria, then we are guaranteed that exactly one triangle exists which satisfies said criteria.

If, on the other hand, we are given the measure of just one of the angles in the triangle along with the length of two sides, only one of which is adjacent to the given angle, we are in the 'Angle-Side-Side' (ASS) case.[7] Such was the case in numbers 3, 4, 5, and 6 above.

In number 3, the length of the one given side a was too short to even form a triangle; in number 4, the length of a was just long enough to form a right triangle; in 5, a was long enough, but not too long, so that two triangles were possible; and in number 6, side a was long enough to form a triangle but too long to swing back and form two. These four cases exemplify all of the possibilities in the Angle-Side-Side case which are summarized in the following theorem.

Theorem 13.2. Suppose (α, a) and (γ, c) are intended to be angle-side pairs in a triangle where α, a and c are given. Let $h = c\sin(\alpha)$

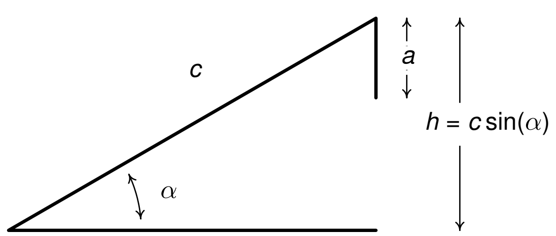
- If $a < h$, then no triangle exists which satisfies the given criteria.
- If $a = h$, then $\gamma = 90^\circ$ so exactly one (right) triangle exists which satisfies the criteria.
- If $h < a < c$, then two distinct triangles exist which satisfy the given criteria.
- If $a \geq c$, then γ is acute and exactly one triangle exists which satisfies the given criteria

Theorem 13.2 is proved on a case-by-case basis. If $a < h$, then $a < c\sin(\alpha)$. If a triangle were to exist, the Law of Sines would have $\frac{\sin(\gamma)}{c} = \frac{\sin(\alpha)}{a}$ so that $\sin(\gamma) = \frac{c\sin(\alpha)}{a} > \frac{a}{a} = 1$, which is impossible.

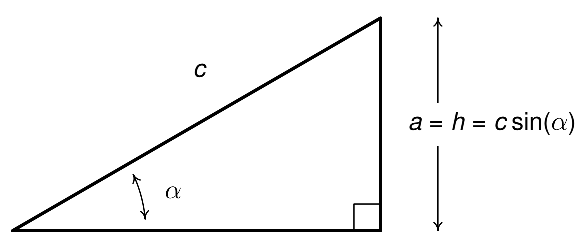
In the figure below on the left, we see geometrically why this is the case. Simply put, if $a < h$ the side a is too short to connect to form a triangle.

This means if $a \geq h$, we are always guaranteed to have at least one triangle, and the remaining parts of the theorem tell us what kind and how many triangles to expect in each case.

If $a = h$, then $a = c\sin(\alpha)$ and the Law of Sines gives $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$ so that $\sin(\gamma) = \frac{c\sin(\alpha)}{a} = \frac{a}{a} = 1$. Here, $\gamma = 90^\circ$ as required. This situation is sketched below on the right.



$a < h$, no triangle



$a = h$, $\gamma = 90^\circ$

Moving along, now suppose $h < a < c$. As before, the Law of Sines[8] gives $\sin(\gamma) = \frac{c\sin(\alpha)}{a}$.

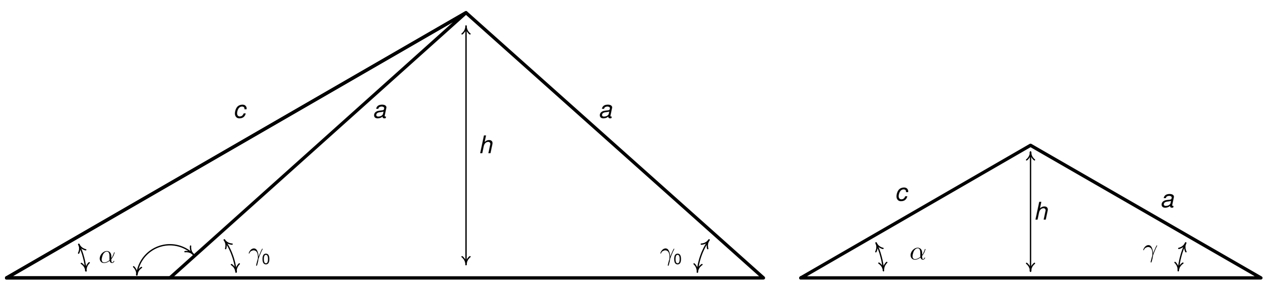
[7] In more reputable books, this is called the 'Side-Side-Angle' or SSA case.
[8] Remember, we have already argued that a triangle exists in this case!

Since $h < a$, $c \sin(\alpha) < a$ or $\frac{c \sin(\alpha)}{a} < 1$ which means there are two solutions to $\sin(\gamma) = \frac{c \sin(\alpha)}{a}$: an acute angle which we'll call γ_0, and its supplement, $180^\circ - \gamma_0$.

Our job now is to argue that each of these angles 'fit' into a triangle with α. Since (α, a) and (γ_0, c) are angle-side opposite pairs, the assumption $c > a$ in this case gives us $\gamma_0 > \alpha$. Since γ_0 is acute, we must have that α is acute as well. This means one triangle can contain both α and γ_0, giving us one of the triangles promised in the theorem.

If we manipulate the inequality $\gamma_0 > \alpha$ a bit, we have $180^\circ - \gamma_0 < 180^\circ - \alpha$. Adding α to both sides gives $(180^\circ - \gamma_0) + \alpha < 180^\circ$. This proves a triangle can contain both of the angles α and $(180^\circ - \gamma_0)$, giving us the second triangle predicted in the theorem. We sketch the two triangle case below on the left.

To prove the last case in the theorem, we assume $a \geq c$. Then $\alpha \geq \gamma$, which forces γ to be an acute angle. Hence, we get only one triangle in this case, completing the proof.



$h < a < c$, two triangles $\qquad$ $a \geq c$, one triangle

One last comment regarding the Angle-Side-Side case: if you are given an obtuse angle to begin with then it is impossible to have the two triangle case. Think about this before reading further.

In many of the derivations and arguments in this section, we used the height of a given triangle, h, as an intermediate variable to prove equivalences. Since the height of a triangle can be used to determine the area enclosed by said triangle, we can use the methods in this section to reformulate area in terms of side lengths and sines of angles. We state the following theorem and leave its proof as an exercise.

Theorem 13.3. Suppose (α, a), (β, b) and (γ, c) are the angle-side opposite pairs of a triangle. Then the area A enclosed by the triangle is given by

$$A = \frac{1}{2} bc \sin(\alpha) = \frac{1}{2} ac \sin(\beta) = \frac{1}{2} ab \sin(\gamma)$$

That is, the area enclosed by the triangle $A = \frac{1}{2}$ (the product of two sides) sin(of the included angle).

Example 13.1.3. Find the area of the triangle in Example 13.1.2 number 1.

Solution. From our work in Example 13.1.2 number 1, we have all three angles and all three sides to work with. However, to minimize propagated error, we choose $A = \frac{1}{2} ac \sin(\beta)$ from Theorem 13.3 because it uses the most pieces of given information.

We are given $a = 7$ and $\beta = 45^\circ$, and we calculated $c = \frac{7\sin(15^\circ)}{\sin(120^\circ)}$. Using these values, we find the area $A = \frac{1}{2}(7)\left(\frac{7\sin(15^\circ)}{\sin(120^\circ)}\right)\sin(45^\circ) =\approx 5.18$ square units. The reader is encouraged to check this answer against the results obtained using the other formulas in Theorem 13.3. □

13.1.1 Bearings

Our last example of the section uses the navigation tool known as **bearings**. Simply put, a bearing is the direction you are heading according to a compass.

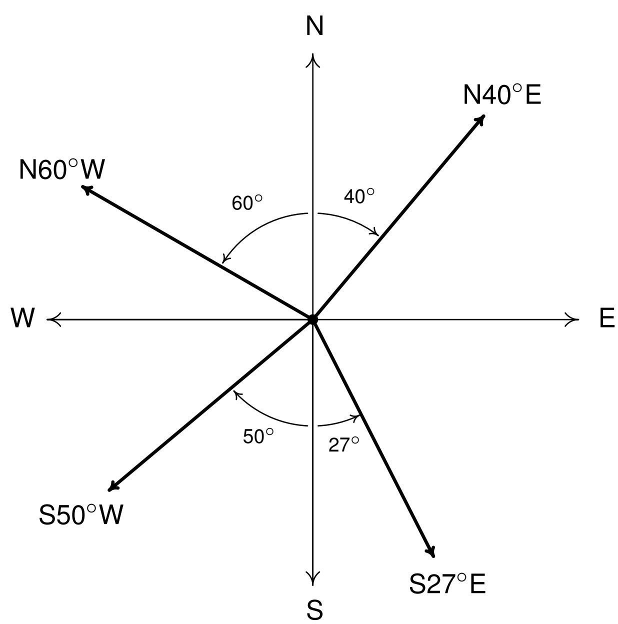
The classic nomenclature for bearings, however, is not given as an angle in standard position, so we must first understand the notation. A bearing is given as an acute angle of rotation (to the east or to the west) away from the north-south (up and down) line of a compass rose.

For example, N40°E (read "40° east of north") is a bearing which is rotated clockwise 40° from due north. If we imagine standing at the origin in the Cartesian Plane, this bearing would have us heading into Quadrant I along the terminal side of $\theta = 50^\circ$.

Similarly, S50°W would point into Quadrant III along the terminal side of $\theta = 220^\circ$ because we started out pointing due south (along $\theta = 270^\circ$) and rotated clockwise 50° back to 220°.

Counter-clockwise rotations would be found in the bearings N60°W (which is on the terminal side of $\theta = 150^\circ$) and S27°E (which lies along the terminal side of $\theta = 297^\circ$).

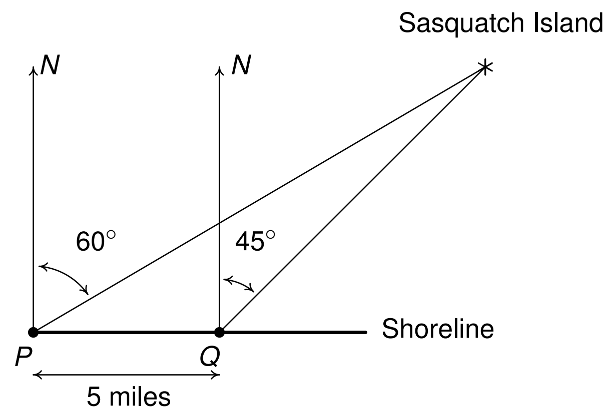
These four bearings are sketched in the plane below.



The cardinal directions north, south, east and west are usually not given as bearings in the fashion described above, but rather, one just refers to them as 'due north', 'due south', 'due east' and 'due west', respectively, and it is assumed that you know which quadrantal angle goes with each cardinal direction.

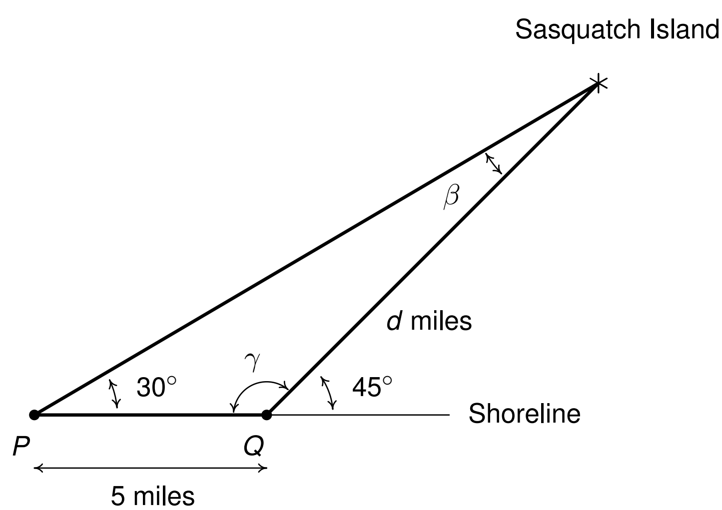
We make good use of bearings and the Law of Sines in our next example.

Example 13.1.4. Sasquatch Island lies off the coast of Ippizuti Lake. As illustrated below, from a point P on the shore, the bearing to Sasquatch Island is observed to be N60°E. From a point Q that is 5 miles due East of P, the bearing to the island is observed to be N45°E.



Assuming the coastline continues to run due East, find the distance from the point Q to the island. What point on the shore is closest to the island? How far is the island from this point?

Solution. As illustrated above, the points P, Q, and the location of the island (represented as a point) form a triangle. Using the bearings information given, we get that the angle between the shore and the island at point P is $90^\circ - 60^\circ = 30^\circ$ while the angle between the shore and the island at point Q is $90^\circ - 45^\circ = 45^\circ$. We pause for a moment to summarize our known (and label our unknown) information below.

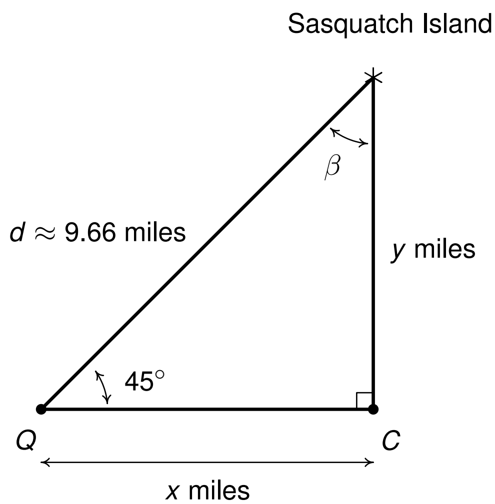


In order to use the Law of Sines to find the distance d from Q to the island, we first need to find the measure of β which is the angle opposite the side of length 5 miles.

Since the angles γ and 45° are supplemental, we get $\gamma = 180^\circ - 45^\circ = 135^\circ$. Knowing γ, we now find $\beta = 180^\circ - 30^\circ - \gamma = 180^\circ - 30^\circ - 135^\circ = 15^\circ$.

By the Law of Sines, we have $\frac{d}{\sin(30^\circ)} = \frac{5}{\sin(15^\circ)}$ which gives $d = \frac{5\sin(30^\circ)}{\sin(15^\circ)} \approx 9.66$ miles.

To find the point on the coast closest to the island, which we've labeled as C in the diagram below, we need to find the perpendicular distance from the island to the coast.[9] Let x denote the distance from the second observation point Q to the point C and let y denote the distance from C to the island.



Using Definition B.1, we get $\sin(45^\circ) = \frac{y}{d}$, so $y = d\sin(45^\circ) \approx 9.66\left(\frac{\sqrt{2}}{2}\right) \approx 6.83$ miles. Hence, the island is approximately 6.83 miles from the coast.

To find the distance from Q to C, we note that $\beta = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ so by symmetry,[10] we get $x = y \approx 6.83$ miles. Hence, the point on the shore closest to the island is approximately 6.83 miles down the coast from the second observation point Q. $\square$

[9] Do you see why C must lie to the right (East) of Q?

[10] Or by Definition B.1 again . . .

13.1.2 Exercises

In Exercises 1 - 20, solve for the remaining side(s) and angle(s) if possible. As in the text, (α, a), (β, b) and (γ, c) are angle-side opposite pairs.

1. $\alpha = 13^\circ$, $\beta = 17^\circ$, $a = 5$
2. $\alpha = 73.2^\circ$, $\beta = 54.1^\circ$, $a = 117$
3. $\alpha = 95^\circ$, $\beta = 85^\circ$, $a = 33.33$
4. $\alpha = 95^\circ$, $\beta = 62^\circ$, $a = 33.33$
5. $\alpha = 117^\circ$, $a = 35$, $b = 42$
6. $\alpha = 117^\circ$, $a = 45$, $b = 42$
7. $\alpha = 68.7^\circ$, $a = 88$, $b = 92$
8. $\alpha = 42^\circ$, $a = 17$, $b = 23.5$
9. $\alpha = 68.7^\circ$, $a = 70$, $b = 90$
10. $\alpha = 30^\circ$, $a = 7$, $b = 14$
11. $\alpha = 42^\circ$, $a = 39$, $b = 23.5$
12. $\gamma = 53^\circ$, $\alpha = 53^\circ$, $c = 28.01$
13. $\alpha = 6^\circ$, $a = 57$, $b = 100$
14. $\gamma = 74.6^\circ$, $c = 3$, $a = 3.05$
15. $\beta = 102^\circ$, $b = 16.75$, $c = 13$
16. $\beta = 102^\circ$, $b = 16.75$, $c = 18$
17. $\beta = 102^\circ$, $\gamma = 35^\circ$, $b = 16.75$
18. $\beta = 29.13^\circ$, $\gamma = 83.95^\circ$, $b = 314.15$
19. $\gamma = 120^\circ$, $\beta = 61^\circ$, $c = 4$
20. $\alpha = 50^\circ$, $a = 25$, $b = 12.5$
21. Find the area of the triangles given in Exercises 1, 12 and 20 above.

The Grade of a Road: The grade of a road is much like the pitch of a roof (See Example B.2.3) in that it expresses the ratio of rise/run. In the case of a road, this ratio is always positive because it is measured going uphill and it is usually given as a percentage. For example, a road which rises 7 feet for every 100 feet of (horizontal) forward progress is said to have a 7% grade. However, if we want to apply any Trigonometry to a story problem involving roads going uphill or downhill, we need to view the grade as an angle with respect to the horizontal. In Exercises 22 - 24, we first have you change road grades into angles and then use the Law of Sines in an application.

22. Using a right triangle with a horizontal leg of length 100 and vertical leg with length 7, show that a 7% grade means that the road (hypotenuse) makes about a 4° angle with the horizontal. (It will not be exactly 4°, but it's pretty close.)
23. What grade is given by a 9.65° angle made by the road and the horizontal?[11]
24. Along a long, straight stretch of mountain road with a 7% grade, you see a tall tree standing perfectly plumb alongside the road.[12] From a point 500 feet downhill from the tree, the angle of inclination from the road to the top of the tree is 6°. Use the Law of Sines to find the height of the tree. (Hint: First show that the tree makes a 94° angle with the road.)

[11] I have friends who live in Pacifica, CA and their road is actually this steep. It's not a nice road to drive.

[12] The word 'plumb' here means that the tree is perpendicular to the horizontal.

Exercises 25 - 31 use the concept of bearings as introduced in Section 13.1.1.

25. Find the angle θ in standard position with $0^\circ \leq \theta < 360^\circ$ which corresponds to each of the bearings given below.

(a) due west
(b) S83°E
(c) N5.5°E
(d) due south
(e) N31.25°W
(f) S72°41′12″W[13]
(g) N45°E
(h) S45°W

26. The Colonel spots a campfire at a of bearing N42°E from his current position. Sarge, who is positioned 3000 feet due east of the Colonel, reckons the bearing to the fire to be N20°W from his current position. Determine the distance from the campfire to each man, rounded to the nearest foot.

27. A hiker starts walking due west from Sasquatch Point and gets to the Chupacabra Trailhead before she realizes that she hasn't reset her pedometer. From the Chupacabra Trailhead she hikes for 5 miles along a bearing of N53°W which brings her to the Muffin Ridge Observatory. From there, she knows a bearing of S65°E will take her straight back to Sasquatch Point. How far will she have to walk to get from the Muffin Ridge Observatory to Sasquach Point? What is the distance between Sasquatch Point and the Chupacabra Trailhead?

28. The captain of the SS Bigfoot sees a signal flare at a bearing of N15°E from her current location. From his position, the captain of the HMS Sasquatch finds the signal flare to be at a bearing of N75°W. If the SS Bigfoot is 5 miles from the HMS Sasquatch and the bearing from the SS Bigfoot to the HMS Sasquatch is N50°E, find the distances from the flare to each vessel, rounded to the nearest tenth of a mile.

29. Carl spies a potential Sasquatch nest at a bearing of N10°E and radios Jeff, who is at a bearing of N50°E from Carl's position. From Jeff's position, the nest is at a bearing of S70°W. If Jeff and Carl are 500 feet apart, how far is Jeff from the Sasquatch nest? Round your answer to the nearest foot.

30. A hiker determines the bearing to a lodge from her current position is S40°W. She proceeds to hike 2 miles at a bearing of S20°E at which point she determines the bearing to the lodge is S75°W. How far is she from the lodge at this point? Round your answer to the nearest hundredth of a mile.

31. A watchtower spots a ship off shore at a bearing of N70°E. A second tower, which is 50 miles from the first at a bearing of S80°E from the first tower, determines the bearing to the ship to be N25°W. How far is the boat from the second tower? Round your answer to the nearest tenth of a mile.

[13]See Example B.1.1 in Section B.1 for a review of the DMS system.

Exercises 32 - 34 use the concepts of 'angle of inclination' and 'angle of depression' introduced in Section B.2 on page B.2 and Exercise 31, respectively.

32. Skippy and Sally decide to hunt UFOs. One night, they position themselves 2 miles apart on an abandoned stretch of desert runway. An hour into their investigation, Skippy spies a UFO hovering over a spot on the runway directly between him and Sally. He records the angle of inclination from the ground to the craft to be 75° and radios Sally immediately to find the angle of inclination from her position to the craft is 50°. How high off the ground is the UFO at this point? Round your answer to the nearest foot. (Recall: 1 mile is 5280 feet.)

33. The angle of depression from an observer in an apartment complex to a gargoyle on the building next door is 55°. From a point five stories below the original observer, the angle of inclination to the gargoyle is 20°. Find the distance from each observer to the gargoyle and the distance from the gargoyle to the apartment complex. Round your answers to the nearest foot. (Use the rule of thumb that one story of a building is 9 feet.)

34. A villainous trio from a copyrighted anime ascends vertically in their hot air balloon from a point on level ground at a constant rate of 6 feet per second. Let θ be the angle of inclination to the base of the balloon basket from a point on the ground 40 feet away from the launch point.

    (a) Let h denote the height of the balloon off of the ground. Show $h = 40\sin(\theta)$.

    (b) Use the related rate law:[14] $\frac{\Delta h}{\Delta t} = \frac{\Delta h}{\Delta \theta}\frac{\Delta \theta}{\Delta t}$ to help you find the rate of change of θ with respect to time as θ increases from 60° to 60.1°. Remember to give units.

35. It takes 2 minutes for the 160 foot Ashtabula Bascule Lift Bridge to rotate 45° from its horizontal position to its raised position, as seen below.[15]

    Assume the bridge casts a shadow directly below itself the entire time it is being raised,[16]

    (a) Let s denote the length of the shadow of the bridge on the water. Show $s = 160\cos(\theta)$.

    (b) Assuming the angle of elevation of the bridge changes at a constant rate, use the related rate law:[17] $\frac{\Delta s}{\Delta t} = \frac{\Delta s}{\Delta \theta}\frac{\Delta \theta}{\Delta t}$ to help you find the rate of change of the shadow length with respect to time as θ increases from 30° to 30.1°. Remember to give units.

[14] Theorem 5.5 in Section 5.3.1

[15] You can see a video of the bridge being raised here.

[16] That is, the sun is directly overhead of the bridge and is shining for an entire two minutes... which never actually happens.

[17] Theorem 5.5 in Section 5.3.1

36. Prove that the Law of Sines holds when $\triangle ABC$ is a right triangle.

37. Discuss with your classmates why knowing only the three angles of a triangle is not enough to determine any of the sides.

38. Discuss with your classmates why the Law of Sines cannot be used to find the angles in the triangle when only the three sides are given. Also discuss what happens if only two sides and the angle between them are given. (Said another way, explain why the Law of Sines cannot be used in the SSS and SAS cases.)

39. Given $\alpha = 30^\circ$ and $b = 10$, choose four different values for a so that

    (a) the information yields no triangle
    (b) the information yields exactly one right triangle
    (c) the information yields two distinct triangles
    (d) the information yields exactly one obtuse triangle

    Explain why you cannot choose a in such a way as to have $\alpha = 30^\circ$, $b = 10$ and your choice of a yield only one triangle where that unique triangle has three acute angles.

40. Use the cases and diagrams in the proof of the Law of Sines (Theorem 13.1) to prove the area formulas given in Theorem 13.3. Why do those formulas yield square units when four quantities are being multiplied together?

13.1.3 Answers

1. $\alpha = 13^\circ$ $\beta = 17^\circ$ $\gamma = 150^\circ$
   $a = 5$ $b \approx 6.50$ $c \approx 11.11$

2. $\alpha = 73.2^\circ$ $\beta = 54.1^\circ$ $\gamma = 52.7^\circ$
   $a = 117$ $b \approx 99.00$ $c \approx 97.22$

3. Information does not produce a triangle

4. $\alpha = 95^\circ$ $\beta = 62^\circ$ $\gamma = 23^\circ$
   $a = 33.33$ $b \approx 29.54$ $c \approx 13.07$

5. Information does not produce a triangle

6. $\alpha = 117^\circ$ $\beta \approx 56.3^\circ$ $\gamma \approx 6.7^\circ$
   $a = 45$ $b = 42$ $c \approx 5.89$

7. $\alpha = 68.7^\circ$ $\beta \approx 76.9^\circ$ $\gamma \approx 34.4^\circ$
   $a = 88$ $b = 92$ $c \approx 53.36$

   $\alpha = 68.7^\circ$ $\beta \approx 103.1^\circ$ $\gamma \approx 8.2^\circ$
   $a = 88$ $b = 92$ $c \approx 13.47$

8. $\alpha = 42^\circ$ $\beta \approx 67.66^\circ$ $\gamma \approx 70.34^\circ$
   $a = 17$ $b = 23.5$ $c \approx 23.93$

   $\alpha = 42^\circ$ $\beta \approx 112.34^\circ$ $\gamma \approx 25.66^\circ$
   $a = 17$ $b = 23.5$ $c \approx 11.00$

9. Information does not produce a triangle

10. $\alpha = 30^\circ$ $\beta = 90^\circ$ $\gamma = 60^\circ$
    $a = 7$ $b = 14$ $c = 7\sqrt{3}$

11. $\alpha = 42^\circ$ $\beta \approx 23.78^\circ$ $\gamma \approx 114.22^\circ$
    $a = 39$ $b = 23.5$ $c \approx 53.15$

12. $\alpha = 53^\circ$ $\beta = 74^\circ$ $\gamma = 53^\circ$
    $a = 28.01$ $b \approx 33.71$ $c = 28.01$

13. $\alpha = 6^\circ$ $\beta \approx 169.43^\circ$ $\gamma \approx 4.57^\circ$
    $a = 57$ $b = 100$ $c \approx 43.45$

    $\alpha = 6^\circ$ $\beta \approx 10.57^\circ$ $\gamma \approx 163.43^\circ$
    $a = 57$ $b = 100$ $c \approx 155.51$

14. $\alpha \approx 78.59^\circ$ $\beta \approx 26.81^\circ$ $\gamma = 74.6^\circ$
    $a = 3.05$ $b \approx 1.40$ $c = 3$

    $\alpha \approx 101.41^\circ$ $\beta \approx 3.99^\circ$ $\gamma = 74.6^\circ$
    $a = 3.05$ $b \approx 0.217$ $c = 3$

15. $\alpha \approx 28.61^\circ$ $\beta = 102^\circ$ $\gamma \approx 49.39^\circ$
    $a \approx 8.20$ $b = 16.75$ $c = 13$

16. Information does not produce a triangle

17. $\alpha = 43^\circ$ $\beta = 102^\circ$ $\gamma = 35^\circ$
    $a \approx 11.68$ $b = 16.75$ $c \approx 9.82$

18. $\alpha = 66.92^\circ$ $\beta = 29.13^\circ$ $\gamma = 83.95^\circ$
    $a \approx 593.69$ $b = 314.15$ $c \approx 641.75$

19. Information does not produce a triangle

20. $\alpha = 50^\circ$ $\beta \approx 22.52^\circ$ $\gamma \approx 107.48^\circ$
    $a = 25$ $b = 12.5$ $c \approx 31.13$

21. The area of the triangle from Exercise 1 is about 8.1 square units.
    The area of the triangle from Exercise 12 is about 377.1 square units.
    The area of the triangle from Exercise 20 is about 149 square units.

22. $\arctan\left(\frac{7}{100}\right) \approx 0.699$ radians, which is equivalent to 4.004°

23. About 17%

24. About 53 feet

25. (a) $\theta = 180^\circ$ (b) $\theta = 353^\circ$ (c) $\theta = 84.5^\circ$ (d) $\theta = 270^\circ$

(e) $\theta = 121.25^\circ$ (f) $\theta = 197^\circ 18' 48''$ (g) $\theta = 45^\circ$ (h) $\theta = 225^\circ$

26. The Colonel is about 3193 feet from the campfire.
Sarge is about 2525 feet to the campfire.

27. The distance from the Muffin Ridge Observatory to Sasquach Point is about 7.12 miles.
The distance from Sasquatch Point to the Chupacabra Trailhead is about 2.46 miles.

28. The SS Bigfoot is about 4.1 miles from the flare.
The HMS Sasquatch is about 2.9 miles from the flare.

29. Jeff is about 371 feet from the nest.

30. She is about 3.02 miles from the lodge

31. The boat is about 25.1 miles from the second tower.

32. The UFO is hovering about 9539 feet above the ground.

33. The gargoyle is about 44 feet from the observer on the upper floor.
The gargoyle is about 27 feet from the observer on the lower floor.
The gargoyle is about 25 feet from the other building.

34. (b) $\frac{\Delta h}{\Delta t}$ is given as a constant $6\,\frac{\text{ft}}{\text{s}}$. $\frac{\Delta h}{\Delta \theta} = \frac{h(60.1)-h(60)}{0.1} \approx 0.348\,\frac{\text{ft}}{\text{degree, }\circ}$.

Hence, $\frac{\Delta \theta}{\Delta t} = \frac{6\,\frac{\text{ft}}{\text{s}}}{0.348\,\frac{\text{ft}}{\text{degree, }\circ}} \approx 17.215\,\frac{\text{degree, }\circ}{s}$.

The angle of elevation is increasing at an average rate of 17.215 degrees per second.

WARNING: For (good) reasons you'll explore more deeply in Calculus, you'll usually stick with radians when the Calculus version of this problem rolls around . . .

35. (b) $\frac{\Delta s}{\Delta \theta} = \frac{s(30.1)-s(30)}{0.1} \approx -1.398\,\frac{\text{ft}}{\text{degree}\,\circ}$.

We are told to assume $\frac{\Delta \theta}{\Delta t}$ is a constant, so $\frac{\Delta s}{\Delta t} = \frac{45^\circ}{2\text{ minutes}} = 22.5\,\frac{\text{degree, }\circ}{\text{min}}$.

We get: $\frac{\Delta s}{\Delta t} = \frac{\Delta s}{\Delta \theta}\frac{\Delta \theta}{\Delta t} \approx \left(-1.398\,\frac{\text{ft}}{\text{degree, }\circ}\right)\left(22.5\,\frac{\text{degree, }\circ}{\text{min}}\right) \approx -31.436\,\frac{\text{ft}}{\text{min}}$.

This means the shadow is receding at a rate of approximately 31.436 feet per minute.

WARNING: For (good) reasons you'll explore more deeply in Calculus, you'll usually stick with radians when the Calculus version of this problem rolls around . . .

13.2 The Law of Cosines

In Section 13.1, we developed the Law of Sines (Theorem 13.1) to enable us to solve triangles in the 'Angle-Angle-Side' (AAS), the 'Angle-Side-Angle' (ASA) and the ambiguous 'Angle-Side-Side' (ASS) cases.

In this section, we develop the Law of Cosines which handles solving triangles in the 'Side-Angle-Side' (SAS) and 'Side-Side-Side' (SSS) cases.[1] We state and prove the theorem below.

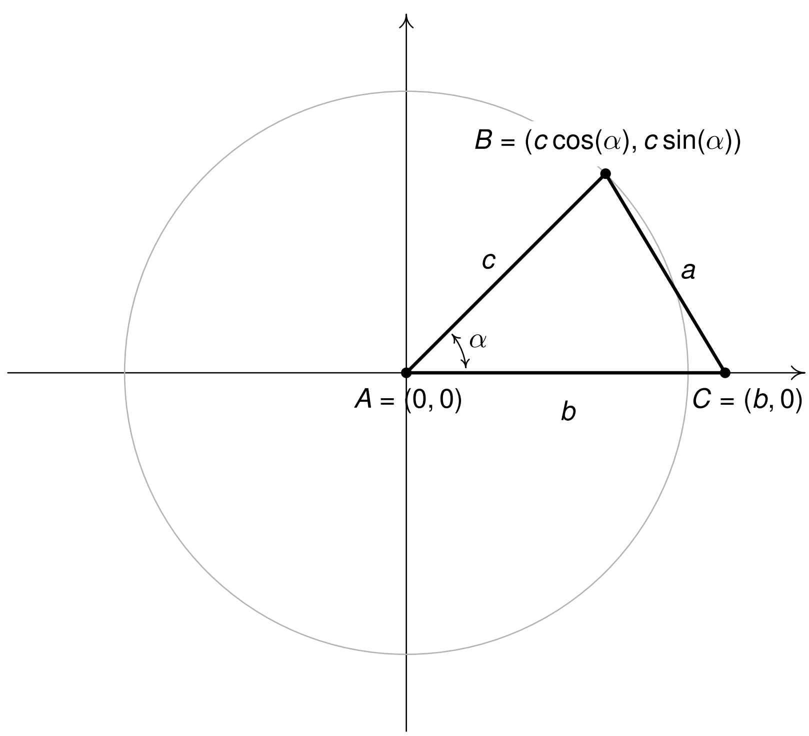
Theorem 13.4. Law of Cosines: Given a triangle with angle-side opposite pairs (α, a), (β, b) and (γ, c), the following equations hold

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha) \qquad b^2 = a^2 + c^2 - 2ac\cos(\beta) \qquad c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

or, solving for the cosine in each equation, we have

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos(\beta) = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}$$

To prove the theorem, we consider a generic triangle with the vertex of angle α at the origin with side b positioned along the positive x-axis as sketched in the diagram below.



[1] Here, 'Side-Angle-Side' means that we are given two sides and the 'included' angle - that is, the given angle is adjacent to both of the given sides.

From this set-up, we immediately find that the coordinates of A and C are $A(0,0)$ and $C(b,0)$. From Theorem 11.3, we know that since the point $B(x,y)$ lies on a circle of radius c, the coordinates of B are $B(x,y) = B(c\cos(\alpha), c\sin(\alpha))$. (This would be true even if α were an obtuse or right angle so although we have drawn the case when α is acute, the following computations hold for any angle α drawn in standard position where $0 < \alpha < 180^\circ$.)

We note that the distance between the points B and C is none other than the length of side a. Using the distance formula, Equation A.1, we get

$$\begin{array}{rcll}
a & = & \sqrt{(c\cos(\alpha) - b)^2 + (c\sin(\alpha) - 0)^2} & \\
a^2 & = & \left(\sqrt{(c\cos(\alpha) - b)^2 + c^2\sin^2(\alpha)}\right)^2 & \\
a^2 & = & (c\cos(\alpha) - b)^2 + c^2\sin^2(\alpha) & \\
a^2 & = & c^2\cos^2(\alpha) - 2bc\cos(\alpha) + b^2 + c^2\sin^2(\alpha) & \\
a^2 & = & c^2\left(\cos^2(\alpha) + \sin^2(\alpha)\right) + b^2 - 2bc\cos(\alpha) & \\
a^2 & = & c^2(1) + b^2 - 2bc\cos(\alpha) & \text{Since } \cos^2(\alpha) + \sin^2(\alpha) = 1 \\
a^2 & = & c^2 + b^2 - 2bc\cos(\alpha) &
\end{array}$$

The remaining formulas given in Theorem 13.4 can be shown by simply reorienting the triangle to place a different vertex at the origin. We leave these details to the reader.

What's important about a and α in the above proof is that (α, a) is an angle-side opposite pair and b and c are the sides adjacent to α – the same can be said of any other angle-side opposite pair in the triangle.

Notice that the proof of the Law of Cosines relies on the distance formula which has its roots in the Pythagorean Theorem. That being said, the Law of Cosines can be thought of as a *generalization* of the Pythagorean Theorem.

Indeed, in a triangle in which $\gamma = 90^\circ$, (i.e., a right triangle) then $\cos(\gamma) = \cos(90^\circ) = 0$ and we get the familiar relationship $c^2 = a^2 + b^2$. What this means is that in the larger mathematical sense, the Law of Cosines and the Pythagorean Theorem amount to pretty much the same thing.[2]

Example 13.2.1. Solve the following triangles. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

1. $\beta = 50^\circ$, $a = 7$ units, $c = 2$ units
2. $a = 4$ units, $b = 7$ units, $c = 5$ units

Solution.

1. We are given the lengths of two sides, $a = 7$ and $c = 2$, and the measure of the included angle, $\beta = 50^\circ$. With no angle-side opposite pair to use for the Law of Sines, we apply the Law of Cosines. We get $b^2 = 7^2 + 2^2 - 2(7)(2)\cos(50^\circ)$ which yields $b = \sqrt{53 - 28\cos(50^\circ)} \approx 5.92$ units.

   In order to determine the measures of the remaining angles α and γ, we are forced to used the derived value for b. There are two ways to proceed at this point. We could use the Law of Cosines again, or, since we have the angle-side opposite pair (β, b) we could use the Law of Sines.

[2]This shouldn't come as too much of a shock. All of the theorems in Trigonometry can ultimately be traced back to the definition of the circular functions along with the distance formula and hence, the Pythagorean Theorem.

The advantage to using the Law of Cosines over the Law of Sines in cases like this is that unlike the sine function, the cosine function distinguishes between acute and obtuse angles. The cosine of an acute is positive, whereas the cosine of an obtuse angle is negative. Since the sine of both acute and obtuse angles are positive, the sine of an angle alone is not enough to determine if the angle in question is acute or obtuse.

Since both authors of the textbook prefer the Law of Cosines, we proceed with this method first. When using the Law of Cosines, it's always best to find the measure of the largest unknown angle first, since this will give us the obtuse angle of the triangle if there is one.

Since the largest angle is opposite the longest side, we choose to find α first. To that end, we use the formula $\cos(\alpha) = \frac{b^2+c^2-a^2}{2bc}$ and substitute $a = 7$, $b = \sqrt{53 - 28\cos(50^\circ)}$ and $c = 2$. We get[3]

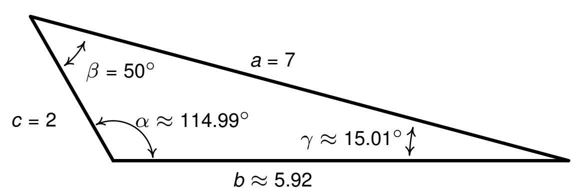
$$\cos(\alpha) = \frac{2 - 7\cos(50^\circ)}{\sqrt{53 - 28\cos(50^\circ)}}$$

Since α is an angle in a triangle, we know the radian measure of α must lie between 0 and π radians. This matches the range of the arccosine function, so we have

$$\alpha = \arccos\left(\frac{2 - 7\cos(50^\circ)}{\sqrt{53 - 28\cos(50^\circ)}}\right) \text{ radians } \approx 114.99^\circ$$

At this point, we could find γ using $\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 114.99^\circ - 50^\circ = 15.01^\circ$, that is if we trust our approximation for α.

To minimize propagation of error (and obtain an *exact* answer for γ), however, we could use the Law of Cosines again.[4] From $\cos(\gamma) = \frac{a^2+b^2-c^2}{2ab}$ with $a = 7$, $b = \sqrt{53 - 28\cos(50^\circ)}$ and $c = 2$, we get $\gamma = \arccos\left(\frac{7-2\cos(50^\circ)}{\sqrt{53-28\cos(50^\circ)}}\right)$ radians $\approx 15.01^\circ$. We sketch the triangle below.



As we mentioned earlier, once we've determined b it is possible to use the Law of Sines to find the remaining angles. Here, however, we must proceed with caution as we are in the ambiguous (ASS) case. Here it is advisable to first find the *smallest* of the unknown angles, since we are guaranteed it will be acute.[5]

In this case, we would find γ since the side opposite γ is smaller than the side opposite the other unknown angle, α. Using the angle-side opposite pair (β, b), we get $\frac{\sin(\gamma)}{2} = \frac{\sin(50^\circ)}{\sqrt{53-28\cos(50^\circ)}}$. The usual calculations produces $\gamma \approx 15.01^\circ$ and $\alpha = 180^\circ - \beta - \gamma \approx 180^\circ - 50^\circ - 15.01^\circ = 114.99^\circ$.

[3]after simplifying ...

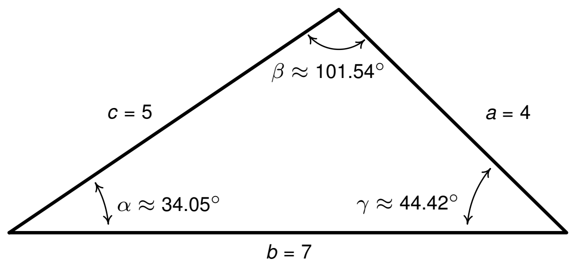
[4]Your instructor will let you know which procedure to use. It all boils down to how much you trust your calculator.

[5]There can only be one *obtuse* angle in the triangle, and if there is one, it must be the largest.

2. Since all three sides and no angles are given, we are forced to use the Law of Cosines. Following our discussion in the previous problem, we find β first, since it is opposite the longest side, b. We get $\cos(\beta) = \frac{a^2+c^2-b^2}{2ac} = -\frac{1}{5}$, so $\beta = \arccos\left(-\frac{1}{5}\right)$ radians $\approx 101.54^\circ$.

   Now that we have obtained an angle-side opposite pair (β, b), we could proceed using the Law of Sines. The Law of Cosines, however, offers us a rare opportunity to find the remaining angles using *only* the data given to us in the statement of the problem.

   Using the Law of Cosines, we get $\gamma = \arccos\left(\frac{5}{7}\right)$ radians $\approx 44.42^\circ$ and $\alpha = \arccos\left(\frac{29}{35}\right)$ radians $\approx 34.05^\circ$. We sketch this triangle below.

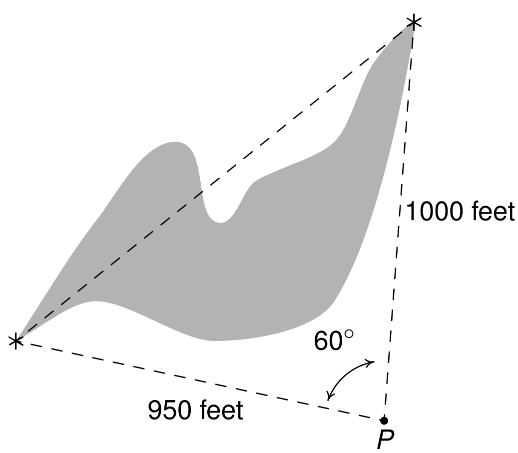


□

We note that, depending on how many decimal places are carried through successive calculations, and depending on which approach is used to solve the problem, the approximate answers you obtain may differ slightly from those the authors obtain in the Examples and the Exercises.

A great example of this is number 2 in Example 13.2.1, where the *approximate* values we record for the measures of the angles sum to 180.01°, which is geometrically impossible.

Example 13.2.2. A researcher wishes to determine the width of a vernal pond as drawn below. From a point P, he finds the distance to the eastern-most point of the pond to be 950 feet, while the distance to the western-most point of the pond from P is 1000 feet. If the angle between the two lines of sight is 60°, find the width of the pond.



Solution. We are given the lengths of two sides and the measure of an included angle, so we may apply the Law of Cosines to find the length of the missing side opposite the given angle.

Calling this length w (for *width*), we get $w^2 = 950^2 + 1000^2 - 2(950)(1000)\cos(60^\circ) = 952500$ from which we get $w = \sqrt{952500} \approx 976$ feet. $\Box$

In Section 13.1, we used the proof of the Law of Sines to develop Theorem 13.3 as an alternate formula for the area enclosed by a triangle. In this section, we use the Law of Cosines to derive another such formula, the so-called Heron's Formula.[6]

Theorem 13.5. Heron's Formula: Suppose a, b and c denote the lengths of the three sides of a triangle. Let s be the semiperimeter of the triangle, that is, let $s = \frac{1}{2}(a+b+c)$. Then the area A enclosed by the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

We prove Theorem 13.5 using Theorem 13.3. Using the convention that the angle γ is opposite the side c, we have $A = \frac{1}{2}ab\sin(\gamma)$ from Theorem 13.3.

In order to simplify computations, we start by manipulating the expression for A^2.

$$\begin{aligned}
A^2 &= \left(\frac{1}{2}ab\sin(\gamma)\right)^2 \\
&= \frac{1}{4}a^2b^2\sin^2(\gamma) \\
&= \frac{a^2b^2}{4}\left(1-\cos^2(\gamma)\right) \quad \text{since } \sin^2(\gamma) = 1-\cos^2(\gamma).
\end{aligned}$$

The Law of Cosines tells us $\cos(\gamma) = \frac{a^2+b^2-c^2}{2ab}$, so substituting this into our equation for A^2 gives

$$\begin{aligned}
A^2 &= \frac{a^2b^2}{4}\left(1-\cos^2(\gamma)\right) \\
&= \frac{a^2b^2}{4}\left[1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right] \\
&= \frac{a^2b^2}{4}\left[1-\frac{\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right] \\
&= \frac{a^2b^2}{4}\left[\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right] \\
&= \frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{16}
\end{aligned}$$

Recognizing $4a^2b^2$ as a perfect square, $4a^2b^2 = (2ab)^2$, we can factor the resulting difference of squares:

[6] Or 'Hero's Formula.'

$$\begin{aligned} A^2 &= \frac{(2ab)^2 - \left(a^2+b^2-c^2\right)^2}{16} \\ &= \frac{\left(2ab - \left[a^2+b^2-c^2\right]\right)\left(2ab + \left[a^2+b^2-c^2\right]\right)}{16} && \text{difference of squares.} \\ &= \frac{\left(c^2-a^2+2ab-b^2\right)\left(a^2+2ab+b^2-c^2\right)}{16} \end{aligned}$$

Next, we regroup $c^2-a^2+2ab-b^2 = c^2-\left[a^2-2ab+b^2\right]$ and $a^2+2ab+b^2-c^2 = \left[a^2+2ab+b^2\right]-c^2$. Recognizing $a^2-2ab+b^2=(a-b)^2$ and $a^2+2ab+b^2=(a+b)^2$, we continue factoring:

$$\begin{aligned} A^2 &= \frac{\left(c^2-\left[a^2-2ab+b^2\right]\right)\left(\left[a^2+2ab+b^2\right]-c^2\right)}{16} \\ &= \frac{\left(c^2-(a-b)^2\right)\left((a+b)^2-c^2\right)}{16} && \text{perfect square trinomials.} \\ &= \frac{(c-(a-b))(c+(a-b))((a+b)-c)((a+b)+c)}{16} && \text{difference of squares.} \\ &= \frac{(b+c-a)(a+c-b)(a+b-c)(a+b+c)}{16} \\ &= \frac{(b+c-a)}{2}\cdot\frac{(a+c-b)}{2}\cdot\frac{(a+b-c)}{2}\cdot\frac{(a+b+c)}{2} \end{aligned}$$

At this stage, we recognize the last factor as the semiperimeter, $s=\frac{1}{2}(a+b+c)=\frac{a+b+c}{2}$. To complete the proof, we note that

$$(s-a)=\frac{a+b+c}{2}-a=\frac{a+b+c-2a}{2}=\frac{b+c-a}{2}$$

Similarly, we find $(s-b)=\frac{a+c-b}{2}$ and $(s-c)=\frac{a+b-c}{2}$. Hence, we get

$$\begin{aligned} A^2 &= \frac{(b+c-a)}{2}\cdot\frac{(a+c-b)}{2}\cdot\frac{(a+b-c)}{2}\cdot\frac{(a+b+c)}{2} \\ &= (s-a)(s-b)(s-c)s \end{aligned}$$

so that $A=\sqrt{s(s-a)(s-b)(s-c)}$ as required.

We close with an example of Heron's Formula.

Example 13.2.3. Find the area enclosed of the triangle in Example 13.2.1 number 2.

Solution. We are given $a=4$, $b=7$ and $c=5$. Using these values, we find $s=\frac{1}{2}(4+7+5)=8$, $(s-a)=8-4=4$, $(s-b)=8-7=1$ and $(s-c)=8-5=3$.

Per Heron's Formula, $A=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{(8)(4)(1)(3)}=\sqrt{96}=4\sqrt{6}\approx 9.80$ square units. $\square$

13.2.1 Exercises

In Exercises 1 - 10, use the Law of Cosines to find the remaining side(s) and angle(s) if possible.

1. $a = 7, \; b = 12, \; \gamma = 59.3^\circ$
2. $\alpha = 104^\circ, \; b = 25, \; c = 37$
3. $a = 153, \; \beta = 8.2^\circ, \; c = 153$
4. $a = 3, \; b = 4, \; \gamma = 90^\circ$
5. $\alpha = 120^\circ, \; b = 3, \; c = 4$
6. $a = 7, \; b = 10, \; c = 13$
7. $a = 1, \; b = 2, \; c = 5$
8. $a = 300, \; b = 302, \; c = 48$
9. $a = 5, \; b = 5, \; c = 5$
10. $a = 5, \; b = 12, ; c = 13$

In Exercises 11 - 16, use any method to solve for the remaining side(s) and angle(s), if possible.

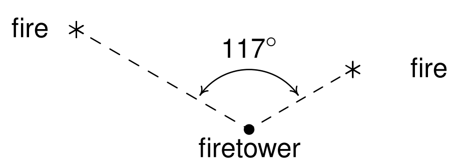
11. $a = 18, \; \alpha = 63^\circ, \; b = 20$
12. $a = 37, \; b = 45, \; c = 26$
13. $a = 16, \; \alpha = 63^\circ, \; b = 20$
14. $a = 22, \; \alpha = 63^\circ, \; b = 20$
15. $\alpha = 42^\circ, \; b = 117, \; c = 88$
16. $\beta = 7^\circ, \; \gamma = 170^\circ, \; c = 98.6$
17. Find the area of the triangles given in Exercises 6, 8 and 10 above.
18. The hour hand on my antique Seth Thomas schoolhouse clock in 4 inches long and the minute hand is 5.5 inches long. Find the distance between the ends of the hands when the clock reads four o'clock. Round your answer to the nearest hundredth of an inch.
19. A geologist wants to measure the diameter of an impact crater. From her camp, it is 4 miles to the northern-most point of the crater and 2 miles to the southern-most point. If the angle between the two lines of sight is 117°, what is the diameter of the crater? Round your answer to the nearest hundredth of a mile.
20. From the Pedimaxus International Airport a tour helicopter can fly to Cliffs of Insanity Point by following a bearing of N8.2°E for 192 miles and it can fly to Bigfoot Falls by following a bearing of S68.5°E for 207 miles.[7] Find the distance between Cliffs of Insanity Point and Bigfoot Falls. Round your answer to the nearest mile.
21. Cliffs of Insanity Point and Bigfoot Falls from Exericse 20 above both lie on a straight stretch of the Great Sasquatch Canyon. What bearing would the tour helicopter need to follow to go directly from Bigfoot Falls to Cliffs of Insanity Point? Round your angle to the nearest tenth of a degree.
22. A naturalist sets off on a hike from a lodge on a bearing of S80°W. After 1.5 miles, she changes her bearing to S17°W and continues hiking for 3 miles. Find her distance from the lodge at this point. Round your answer to the nearest hundredth of a mile. What bearing should she follow to return to the lodge? Round your angle to the nearest degree.

[7]Please refer to Section 13.1.1 for an introduction to bearings.

23. The HMS Sasquatch leaves port on a bearing of N23°E and travels for 5 miles. It then changes course and follows a heading of S41°E for 2 miles. How far is it from port? Round your answer to the nearest hundredth of a mile. What is its bearing to port? Round your angle to the nearest degree.

24. The SS Bigfoot leaves a harbor bound for Nessie Island which is 300 miles away at a bearing of N32°E. A storm moves in and after 100 miles, the captain of the Bigfoot finds he has drifted off course. If his bearing to the harbor is now S70°W, how far is the SS Bigfoot from Nessie Island? Round your answer to the nearest hundredth of a mile. What course should the captain set to head to the island? Round your angle to the nearest tenth of a degree.

25. From a point 300 feet above level ground in a firetower, a ranger spots two fires in the Yeti National Forest. The angle of depression[8] made by the line of sight from the ranger to the first fire is 2.5° and the angle of depression made by line of sight from the ranger to the second fire is 1.3°. The angle formed by the two lines of sight is 117°. Find the distance between the two fires. Round your answer to the nearest foot.



HINT: In order to use the 117° angle between the lines of sight, you will first need to use right angle Trigonometry to find the lengths of the lines of sight. This will give you a Side-Angle-Side case in which to apply the Law of Cosines.

26. If you apply the Law of Cosines to the ambiguous Angle-Side-Side (ASS) case, the result is a quadratic equation whose variable is that of the missing side. If the equation has no positive real zeros then the information given does not yield a triangle. If the equation has only one positive real zero then exactly one triangle is formed and if the equation has two distinct positive real zeros then two distinct triangles are formed. Apply the Law of Cosines to Exercises 11, 13 and 14 above in order to demonstrate this result.

27. Discuss with your classmates why Heron's Formula yields an area in square units even though four lengths are being multiplied together.

[8]See Exercise 31 in Section B.2 for the definition of this angle.

13.2.2 Answers

1. $\alpha \approx 35.54^\circ$ $\beta \approx 85.16^\circ$ $\gamma = 59.3^\circ$
   $a = 7$ $b = 12$ $c \approx 10.36$

2. $\alpha = 104^\circ$ $\beta \approx 29.40^\circ$ $\gamma \approx 46.60^\circ$
   $a \approx 49.41$ $b = 25$ $c = 37$

3. $\alpha \approx 85.90^\circ$ $\beta = 8.2^\circ$ $\gamma \approx 85.90^\circ$
   $a = 153$ $b \approx 21.88$ $c = 153$

4. $\alpha \approx 36.87^\circ$ $\beta \approx 53.13^\circ$ $\gamma = 90^\circ$
   $a = 3$ $b = 4$ $c = 5$

5. $\alpha = 120^\circ$ $\beta \approx 25.28^\circ$ $\gamma \approx 34.72^\circ$
   $a = \sqrt{37}$ $b = 3$ $c = 4$

6. $\alpha \approx 32.31^\circ$ $\beta \approx 49.58^\circ$ $\gamma \approx 98.21^\circ$
   $a = 7$ $b = 10$ $c = 13$

7. Information does not produce a triangle

8. $\alpha \approx 83.05^\circ$ $\beta \approx 87.81^\circ$ $\gamma \approx 9.14^\circ$
   $a = 300$ $b = 302$ $c = 48$

9. $\alpha = 60^\circ$ $\beta = 60^\circ$ $\gamma = 60^\circ$
   $a = 5$ $b = 5$ $c = 5$

10. $\alpha \approx 22.62^\circ$ $\beta \approx 67.38^\circ$ $\gamma = 90^\circ$
    $a = 5$ $b = 12$ $c = 13$

11. $\alpha = 63^\circ$ $\beta \approx 98.11^\circ$ $\gamma \approx 18.89^\circ$
    $a = 18$ $b = 20$ $c \approx 6.54$

    $\alpha = 63^\circ$ $\beta \approx 81.89^\circ$ $\gamma \approx 35.11^\circ$
    $a = 18$ $b = 20$ $c \approx 11.62$

12. $\alpha \approx 55.30^\circ$ $\beta \approx 89.40^\circ$ $\gamma \approx 35.30^\circ$
    $a = 37$ $b = 45$ $c = 26$

13. Information does not produce a triangle

14. $\alpha = 63^\circ$ $\beta \approx 54.1^\circ$ $\gamma \approx 62.9^\circ$
    $a = 22$ $b = 20$ $c \approx 21.98$

15. $\alpha = 42^\circ$ $\beta \approx 89.23^\circ$ $\gamma \approx 48.77^\circ$
    $a \approx 78.30$ $b = 117$ $c = 88$

16. $\alpha \approx 3^\circ$ $\beta = 7^\circ$ $\gamma = 170^\circ$
    $a \approx 29.72$ $b \approx 69.2$ $c = 98.6$

17. The area of the triangle given in Exercise 6 is $\sqrt{1200} = 20\sqrt{3} \approx 34.64$ square units.
    The area of the triangle given in Exercise 8 is $\sqrt{51764375} \approx 7194.75$ square units.
    The area of the triangle given in Exercise 10 is exactly 30 square units.

18. The distance between the ends of the hands at four o'clock is about 8.26 inches.

19. The diameter of the crater is about 5.22 miles.

20. About 313 miles

21. N31.8°W

22. She is about 3.92 miles from the lodge and her bearing to the lodge is N37°E.

23. It is about 4.50 miles from port and its heading to port is S47°W.

24. It is about 229.61 miles from the island and the captain should set a course of N16.4°E to reach the island.

25. The fires are about 17456 feet apart. (Try to avoid rounding errors.)

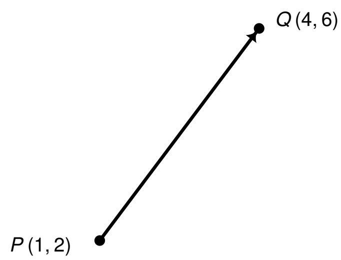
13.3 Vectors

As we have seen numerous times in this book, Mathematics can be used to model and solve real-world problems. For many applications, real numbers suffice; that is, real numbers with the appropriate units attached can be used to answer questions like "How close is the nearest Sasquatch nest?"

There are other times though, when these kinds of quantities do not suffice. Perhaps it is important to know, for instance, how close the nearest Sasquatch nest is as well as the direction in which it lies. To answer questions like these which involve both a quantitative answer, or *magnitude*, along with a *direction*, we use the mathematical objects called **vectors**.[1]

A vector is represented geometrically as a directed line segment where the magnitude of the vector is taken to be the length of the line segment and the direction is made clear with the use of an arrow at one endpoint of the segment. When referring to vectors in this text, we shall adopt[2] the 'arrow' notation, so the symbol $\vec{v}$ is read as 'the vector v'. Below is a typical vector $\vec{v}$ with endpoints $P\,(1,2)$ and $Q\,(4,6)$.

The point P is called the *initial point* or *tail* of $\vec{v}$ and the point Q is called the *terminal point* or *head* of $\vec{v}$. Since we can reconstruct $\vec{v}$ completely from P and Q, we write $\vec{v} = \overrightarrow{PQ}$, where the order of points P (initial point) and Q (terminal point) is important. (Think about this before moving on.)



$\vec{v} = \overrightarrow{PQ}$

While it is true that P and Q completely determine $\vec{v}$, it is important to note that since vectors are defined in terms of their two characteristics, magnitude and direction, any directed line segment with the same length and direction as $\vec{v}$ is considered to be the same vector as $\vec{v}$, regardless of its initial point.

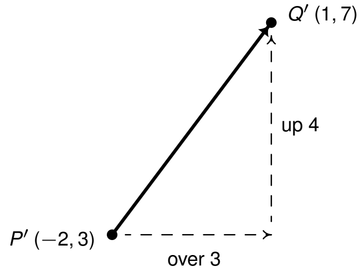
In the case of our vector $\vec{v}$ above, any vector which moves three units to the right and four up[3] from its initial point to arrive at its terminal point is considered the same vector as $\vec{v}$. The notation we use to capture this idea is the *component form* of the vector, $\vec{v} = \langle 3,4\rangle$, where the first number, 3, is called the *x-component* of $\vec{v}$ and the second number, 4, is called the *y-component* of $\vec{v}$.

For example, if we wanted to reconstruct $\vec{v} = \langle 3,4\rangle$ with initial point $P'(-2,3)$, then we would find the terminal point of $\vec{v}$ by adding 3 to the x-coordinate and adding 4 to the y-coordinate to obtain the terminal point $Q'(1,7)$, as seen below.

[1]The word 'vector' comes from the Latin *vehere* meaning 'to convey' or 'to carry.'

[2]Other textbook authors use bold vectors such as $\boldsymbol{v}$. We find that writing in bold font on the chalkboard is inconvenient at best, so we have chosen the 'arrow' notation.

[3]If this idea of 'over' and 'up' seems familiar, it should. The slope of the line segment containing $\vec{v}$ is $\frac{4}{3}$.



$\vec{v} = \langle 3, 4\rangle$ with initial point $P'\,(-2,3)$.

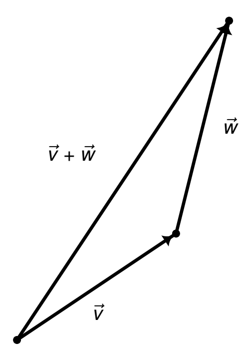
The component form of a vector is what ties these very geometric objects back to Algebra and ultimately Trigonometry. We generalize our example in our definition below.

Definition 13.1. Suppose $\vec{v}$ is represented by a directed line segment with initial point $P\,(x_0, y_0)$ and terminal point $Q\,(x_1, y_1)$. The **component form** of $\vec{v}$ is given by

$$\vec{v} = \overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0\rangle$$

Using the language of components, we have that two vectors are equal if and only if their corresponding components are equal. That is, $\langle v_1, v_2\rangle = \langle v_1', v_2'\rangle$ if and only if $v_1 = v_1'$ and $v_2 = v_2'$. (Again, think about this before reading on.)

We now set about defining operations on vectors. Suppose we are given two vectors $\vec{v}$ and $\vec{w}$. The sum, or *resultant* vector $\vec{v} + \vec{w}$ is obtained as follows. First, plot $\vec{v}$. Next, plot $\vec{w}$ so that its initial point is the terminal point of $\vec{v}$. To plot the vector $\vec{v} + \vec{w}$ we begin at the initial point of $\vec{v}$ and end at the terminal point of $\vec{w}$. It is helpful to think of the vector $\vec{v} + \vec{w}$ as the 'net result' of moving along $\vec{v}$ then moving along $\vec{w}$.



$\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$

Our next example makes good use of resultant vectors and reviews bearings and the Law of Cosines.[4]

Example 13.3.1. A plane leaves an airport with an airspeed[5] of 175 miles per hour at a bearing of N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

[4] If necessary, review Sections 13.1.1 and 13.2.

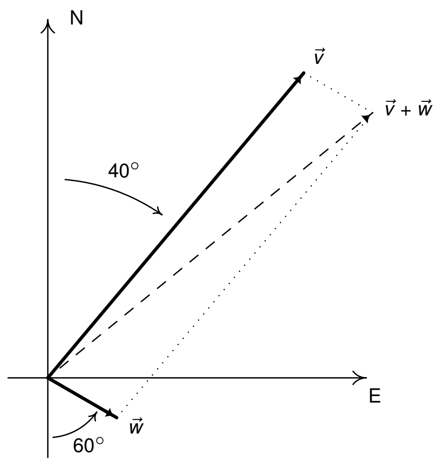
[5] That is, the speed of the plane relative to the air around it. If there were no wind, plane's airspeed would be the same as its speed as observed from the ground. How does wind affect this? Keep reading!

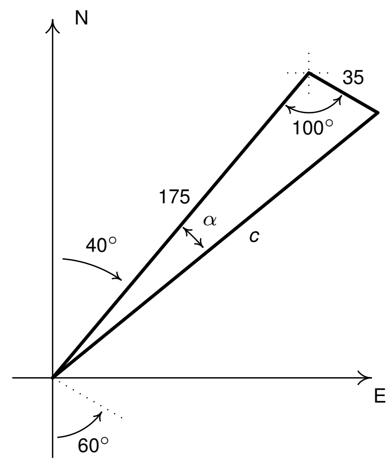
Solution: For both the plane and the wind, we are given their speeds and their directions. Coupling speed (as a magnitude) with direction is the concept of *velocity* which we've seen a few times before.[6]

We let $\vec{v}$ denote the plane's velocity and $\vec{w}$ denote the wind's velocity in the diagram below. The 'true' speed and bearing is found by analyzing the resultant vector, $\vec{v} + \vec{w}$.

From the vector diagram, we get a triangle, the lengths of whose sides are the magnitude of $\vec{v}$, which is 175, the magnitude of $\vec{w}$, which is 35, and the magnitude of $\vec{v} + \vec{w}$, which we'll call c.

From the given bearing information, we go through the usual geometry to determine that the angle between the sides of length 35 and 175 measures 100°.





From the Law of Cosines, we determine $c = \sqrt{31850 - 12250\cos(100^\circ)} \approx 184$, which means the true speed of the plane is (approximately) 184 miles per hour.

To determine the true bearing of the plane, we need to determine the angle α. Using the Law of Cosines once more,[7] we find $\cos(\alpha) = \frac{c^2+29400}{350c}$ so that $\alpha \approx 11^\circ$.

Given the geometry of the situation, we add α to the given 40° and find the true bearing of the plane to be (approximately) N51°E. □

Our next step is to define addition of vectors component-wise to match the geometric action.[8]

Definition 13.2. Suppose $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$. The vector $\vec{v} + \vec{w}$ is defined by

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$$

Example 13.3.2. Let $\vec{v} = \langle 3, 4 \rangle$ and suppose $\vec{w} = \overrightarrow{PQ}$ where $P(-3, 7)$ and $Q(-2, 5)$. Find $\vec{v} + \vec{w}$ and interpret this sum geometrically.

[6]See Section 11.1.1, for instance.

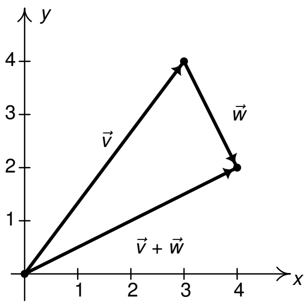
[7]Or, since our given angle, 100°, is obtuse, we could use the Law of Sines without any ambiguity here.

[8]Adding vectors 'component-wise' should seem hauntingly familiar. Compare this with how matrix addition was defined in section 9.3. In more advanced courses. chief among them Linear Algebra, vectors are actaually *defined* as $1 \times n$ or $n \times 1$ matrices, depending on the situation.

Solution. Before can add the vectors using Definition 13.2, we need to write $\vec{w}$ in component form. Using Definition 13.1, we get $\vec{w} = \langle -2-(-3), 5-7 \rangle = \langle 1, -2 \rangle$. Thus,

$$\vec{v} + \vec{w} = \langle 3, 4 \rangle + \langle 1, -2 \rangle = \langle 3+1, 4+(-2) \rangle = \langle 4, 2 \rangle.$$

To visualize this sum, we draw $\vec{v}$ with its initial point at $(0,0)$ (for convenience) so that its terminal point is $(3,4)$. Next, we graph $\vec{w}$ with its initial point at $(3,4)$. Moving one to the right and two down, we find the terminal point of $\vec{w}$ to be $(4,2)$.



We see the vector $\vec{v} + \vec{w}$ has initial point $(0,0)$ and terminal point $(4,2)$ so its component form is $\langle 4, 2 \rangle$. □

In order for vector addition to enjoy the same kinds of properties as real number addition, it is necessary to extend our definition of vectors to include a 'zero vector', $\vec{0} = \langle 0, 0 \rangle$.

Geometrically, $\vec{0}$ represents a point, which we can (very broadly) think of as a directed line segment with the same initial and terminal points. The reader may well object to the inclusion of $\vec{0}$, since after all, vectors are supposed to have both a magnitude (length) and a direction.

While it seems clear that the magnitude of $\vec{0}$ should be 0, it is not clear what its direction is. As we shall see, the direction of $\vec{0}$ is in fact undefined, but this minor hiccup in the natural flow of things is worth the benefits we reap by including $\vec{0}$ in our discussions. We have the following theorem.

Theorem 13.6. Properties of Vector Addition

- **Commutative Property:** For all vectors $\vec{v}$ and $\vec{w}$, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.
- **Associative Property:** For all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, $\left(\vec{u} + \vec{v}\right) + \vec{w} = \vec{u} + \left(\vec{v} + \vec{w}\right)$.
- **Identity Property:** For all vectors $\vec{v}$,

$$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}.$$

  The vector $\vec{0}$ acts as the additive identity for vector addition.
- **Inverse Property:** For every vector $\vec{v} = \langle v_1, v_2 \rangle$, the vector $\vec{w} = \langle -v_1, -v_2 \rangle$ satisfies

$$\vec{v} + \vec{w} = \vec{w} + \vec{v} = \vec{0}.$$

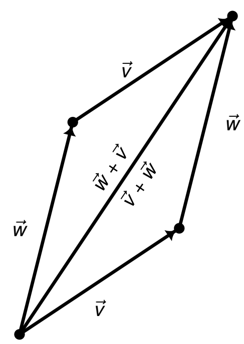
  That is, the additive inverse of a vector is the vector of the additive inverses of its components.

The properties in Theorem 13.6 are easily verified using the definition of vector addition,[9] and are a direct consequence of the definition of vector addition along with properties inherited from real number arithmetic.

For the commutative property, we note that if $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ then

$$\begin{aligned} \vec{v} + \vec{w} &= \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle \\ &= \langle v_1 + w_1, v_2 + w_2 \rangle \\ &= \langle w_1 + v_1, w_2 + v_2 \rangle \\ &= \vec{w} + \vec{v} \end{aligned}$$

Geometrically, we can 'see' the commutative property by realizing that the sums $\vec{v} + \vec{w}$ and $\vec{w} + \vec{v}$ are the same directed diagonal determined by the parallelogram below.



Demonstrating the commutative property of vector addition.

The proofs of the associative and identity properties proceed similarly, and the reader is encouraged to verify them and provide accompanying diagrams.

The additive identity property is likewise verified algebraically using a calculation. If $\vec{v} = \langle v_1, v_2 \rangle$, then

$$\vec{v} + \vec{0} = \langle v_1, v_2 \rangle + \langle 0, 0 \rangle = \langle v_1 + 0, v_2 + 0 \rangle = \langle v_1, v_2 \rangle = \vec{v}.$$

From the commutative property of vector addition, we get that $\vec{0} + \vec{v} = \vec{v}$ as well. Again, the reader is encouraged to visualize what this means geometrically.[10]

Regarding additive inverses, we can verify by direct computation that if $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle -v_1, -v_2 \rangle$,

$$\vec{v} + \vec{w} = \langle v_1, v_2 \rangle + \langle -v_1, -v_2 \rangle = \langle v_1 + (-v_1), v_2 + (-v_2) \rangle = < 0, 0 > = \vec{0}.$$

Once again, the commutative property of vector addition assures us that, likewise, $\vec{w} + \vec{v} = \vec{0}$.

Moreover, additive inverses of vectors are *unique*. That is, given a vector $\vec{v} = \langle v_1, v_2 \rangle$, there is precisely only *one* vector $\vec{w}$ so that $\vec{v} + \vec{w} = \vec{0}$.

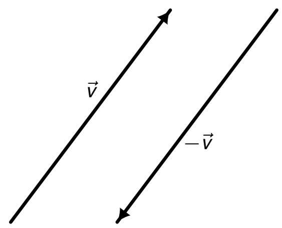
To see this, suppose a vector $\vec{w} = \langle w_1, w_2 \rangle$ satisfies $\vec{v} + \vec{w} = \vec{0}$. By the definition of vector addition, we have $\langle v_1 + w_1, v_2 + w_2 \rangle = \langle 0, 0 \rangle$. Hence, $v_1 + w_1 = 0$ and $v_2 + w_2 = 0$. We get $w_1 = -v_1$ and $w_2 = -v_2$ so that $\vec{w} = \langle -v_1, -v_2 \rangle$ as prescribed in Theorem 13.6.

[9] The interested reader is encouraged to compare Theorem 13.6 and the ensuing discussion with Theorem 9.3 in Section 9.3.

[10] Recall, $\vec{0}$ is represented geometrically as a point . . .

Hence, every vector $\vec{v}$ has one, and only one, additive inverse. In general, we denote the additive inverse of a vector $\vec{v}$ with the (highly suggestive) notation $-\vec{v}$.

Geometrically, the vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $-\vec{v} = \langle -v_1, -v_2 \rangle$ have the same length, but opposite directions. As a result, when adding the vectors geometrically, the sum $\vec{v} + (-\vec{v})$ results in starting at the initial point of $\vec{v}$ and ending back at the initial point of $\vec{v}$. That is, the net result of moving $\vec{v}$ then $-\vec{v}$ is not moving at all.



Using the additive inverse of a vector, we can define the difference of two vectors: $\vec{v} - \vec{w} = \vec{v} + (-\vec{w})$. Looking at this at the level of components, we see if $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ then

$$\begin{array}{rcl} \vec{v} - \vec{w} & = & \vec{v} + (-\vec{w}) \\ & = & \langle v_1, v_2 \rangle + \langle -w_1, -w_2 \rangle \\ & = & \langle v_1 + (-w_1), v_2 + (-w_2) \rangle \\ & = & \langle v_1 - w_1, v_2 - w_2 \rangle \end{array}$$

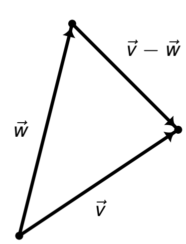
In other words, like vector addition, vector subtraction works component-wise.

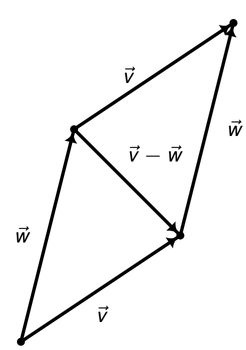
To interpret the vector $\vec{v} - \vec{w}$ geometrically, we note

$$\begin{array}{rcll} \vec{w} + \left(\vec{v} - \vec{w}\right) & = & \vec{w} + \left(\vec{v} + (-\vec{w})\right) & \text{Definition of Vector Subtraction} \\ & = & \vec{w} + \left((-\vec{w}) + \vec{v}\right) & \text{Commutativity of Vector Addition} \\ & = & (\vec{w} + (-\vec{w})) + \vec{v} & \text{Associativity of Vector Addition} \\ & = & \vec{0} + \vec{v} & \text{Definition of Additive Inverse} \\ & = & \vec{v} & \text{Definition of Additive Identity} \end{array}$$

This means that the 'net result' of moving along $\vec{w}$ then moving along $\vec{v} - \vec{w}$ is just $\vec{v}$ itself.

From the diagram below on the left, we see that $\vec{v} - \vec{w}$ may be interpreted as the vector whose initial point is the terminal point of $\vec{w}$ and whose terminal point is the terminal point of $\vec{v}$.





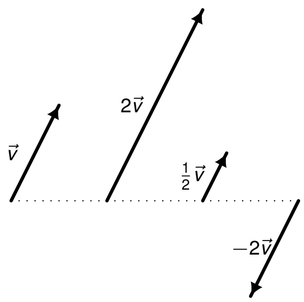
It is also worth mentioning that in the parallelogram determined by the vectors $\vec{v}$ and $\vec{w}$ above on the right, the vector $\vec{v} - \vec{w}$ is one of the diagonals – the other being $\vec{v} + \vec{w}$.

Next, we discuss *scalar* multiplication – that is, taking a real number times a vector. We define scalar multiplication for vectors in the same way we defined it for matrices in Section 9.3.

Definition 13.3. If k is a real number and $\vec{v} = \langle v_1, v_2 \rangle$, we define $k\vec{v}$ by

$$k\vec{v} = k\langle v_1, v_2 \rangle = \langle kv_1, kv_2 \rangle$$

Scalar multiplication by k in vectors can be understood geometrically as scaling the vector (if $k > 0$) or scaling the vector and reversing its direction (if $k < 0$) as demonstrated below.



Note by definition 13.3, $(-1)\vec{v} = (-1)\langle v_1, v_2 \rangle = \langle (-1)v_1, (-1)v_2 \rangle = \langle -v_1, -v_2 \rangle = -\vec{v}$, which is what we would expect. This and other properties of scalar multiplication are summarized in the theorem below.

Theorem 13.7. Properties of Scalar Multiplication

- **Associative Property:** For every vector $\vec{v}$ and scalars k and r, $(kr)\vec{v} = k(r\vec{v})$.
- **Identity Property:** For all vectors $\vec{v}$, $1\vec{v} = \vec{v}$.
- **Additive Inverse Property:** For all vectors $\vec{v}$, $-\vec{v} = (-1)\vec{v}$.
- **Distributive Property of Scalar Multiplication over Scalar Addition:**
  For every vector $\vec{v}$ and scalars k and r,
  $$(k + r)\vec{v} = k\vec{v} + r\vec{v}$$
- **Distributive Property of Scalar Multiplication over Vector Addition:**
  For all vectors $\vec{v}$ and $\vec{w}$ and scalars k,
  $$k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$$
- **Zero Product Property:** If $\vec{v}$ is vector and k is a scalar, then
  $$k\vec{v} = \vec{0} \quad \text{if and only if} \quad k = 0 \quad \text{or} \quad \vec{v} = \vec{0}$$

The proof of Theorem 13.7, like the proof of Theorem 13.6, ultimately boils down to the definition of scalar multiplication and properties of real numbers.

For example, to prove the associative property, we let $\vec{v} = \langle v_1, v_2 \rangle$. If k and r are scalars then

$$\begin{array}{rcll}
(kr)\vec{v} & = & (kr)\left\langle v_1, v_2 \right\rangle & \\
& = & \left\langle (kr)v_1, (kr)v_2 \right\rangle & \text{Definition of Scalar Multiplication} \\
& = & \left\langle k(rv_1), k(rv_2) \right\rangle & \text{Associative Property of Real Number Multiplication} \\
& = & k\left\langle rv_1, rv_2 \right\rangle & \text{Definition of Scalar Multiplication} \\
& = & k\left(r\left\langle v_1, v_2 \right\rangle\right) & \text{Definition of Scalar Multiplication} \\
& = & k(r\vec{v}) &
\end{array}$$

The reader is invited to think about what this property means geometrically. The remaining properties are proved similarly and are left as exercises.

Our next example demonstrates how Theorem 13.7 allows us to do the same kind of algebraic manipulations with vectors as we do with variables – multiplication and division of vectors notwithstanding. If the pedantry seems familiar, it should. This is the same treatment we gave Example 9.3.1 in Section 9.3. As in that example, we spell out the solution in excruciating detail to encourage the reader to think carefully about why each step is justified.

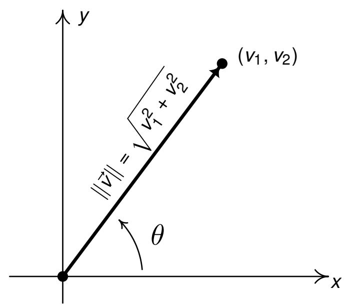
Example 13.3.3. Solve $5\vec{v} - 2\left(\vec{v} + \langle 1, -2 \rangle\right) = \vec{0}$ for $\vec{v}$.

Solution.

$$\begin{array}{rcl}
5\vec{v} - 2\left(\vec{v} + \langle 1, -2 \rangle\right) & = & \vec{0} \\
5\vec{v} + (-1)\left[2\left(\vec{v} + \langle 1, -2 \rangle\right)\right] & = & \vec{0} \\
5\vec{v} + [(-1)(2)]\left(\vec{v} + \langle 1, -2 \rangle\right) & = & \vec{0} \\
5\vec{v} + (-2)\left(\vec{v} + \langle 1, -2 \rangle\right) & = & \vec{0} \\
5\vec{v} + \left[(-2)\vec{v} + (-2)\langle 1, -2 \rangle\right] & = & \vec{0} \\
5\vec{v} + \left[(-2)\vec{v} + \langle (-2)(1), (-2)(-2) \rangle\right] & = & \vec{0} \\
\left[5\vec{v} + (-2)\vec{v}\right] + \langle -2, 4 \rangle & = & \vec{0} \\
(5 + (-2))\vec{v} + \langle -2, 4 \rangle & = & \vec{0} \\
3\vec{v} + \langle -2, 4 \rangle & = & \vec{0} \\
\left(3\vec{v} + \langle -2, 4 \rangle\right) + \left(-\langle -2, 4 \rangle\right) & = & \vec{0} + \left(-\langle -2, 4 \rangle\right) \\
3\vec{v} + \left[\langle -2, 4 \rangle + \left(-\langle -2, 4 \rangle\right)\right] & = & \vec{0} + (-1)\langle -2, 4 \rangle \\
3\vec{v} + \vec{0} & = & \vec{0} + \langle (-1)(-2), (-1)(4) \rangle \\
3\vec{v} & = & \langle 2, -4 \rangle \\
\frac{1}{3}\left(3\vec{v}\right) & = & \frac{1}{3}\left(\langle 2, -4 \rangle\right) \\
\left[\left(\frac{1}{3}\right)(3)\right]\vec{v} & = & \left\langle \left(\frac{1}{3}\right)(2), \left(\frac{1}{3}\right)(-4) \right\rangle \\
1\vec{v} & = & \left\langle \frac{2}{3}, -\frac{4}{3} \right\rangle \\
\vec{v} & = & \left\langle \frac{2}{3}, -\frac{4}{3} \right\rangle
\end{array}$$

The reader is invited to check our solution in the original equation. $\square$

A vector whose initial point is $(0,0)$ is said to be in **standard position**. If $\vec{v} = \langle v_1, v_2 \rangle$ is plotted in standard position, then its terminal point is necessarily (v_1, v_2). (Once more, think about this before reading on.)



$\vec{v} = \langle v_1, v_2 \rangle$ in standard position.

Plotting a vector in standard position enables us to more easily quantify the concepts of magnitude and direction of the vector.

Recall the magnitude of vector $\vec{v}$ is the length of the directed line segment representing $\vec{v}$. When plotted in standard position, the length of this line segment is none other than the distance from the origin $(0,0)$ to the point (v_1, v_2). Hence, the magnitude of $\vec{v}$, which we denote $\|\vec{v}\|$, is given by $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$.

Turning to the notion of direction, we note that the point (v_1, v_2) is on the terminal side of the angle θ depicted in the diagram above. From Theorem 11.3, we have $v_1 = \|\vec{v}\|\cos(\theta)$ and $v_2 = \|\vec{v}\|\sin(\theta)$. From the definition of scalar multiplication and vector equality, we get

$$\begin{aligned} \vec{v} &= \langle v_1, v_2 \rangle \\ &= \langle \|\vec{v}\|\cos(\theta), \|\vec{v}\|\sin(\theta) \rangle \\ &= \|\vec{v}\| \langle \cos(\theta), \sin(\theta) \rangle \end{aligned}$$

This motivates the following definition.

Definition 13.4. Suppose $\vec{v}$ is a vector with component form $\vec{v} = \langle v_1, v_2 \rangle$. Let θ be an angle in standard position whose terminal side contains the point (v_1, v_2).

- The **magnitude** of $\vec{v}$, denoted $\|\vec{v}\|$, is given by $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$
- If $\vec{v} \neq \vec{0}$, the **(vector) direction** of $\vec{v}$, denoted $\hat{v}$ is given by $\hat{v} = \langle \cos(\theta), \sin(\theta) \rangle$

Taken together, we get $\vec{v} = \langle \|\vec{v}\|\cos(\theta), \|\vec{v}\|\sin(\theta) \rangle$.

A few remarks are in order. First, we note that if $\vec{v} \neq 0$ then there are infinitely many angles θ which satisfy Definition 13.4. However, the fact that all of them must contain the same point (v_1, v_2) on their terminal sides means they are all coterminal.

Hence, if θ and θ' both satisfy the conditions of Definition 13.4, then $\cos(\theta) = \cos(\theta')$ and $\sin(\theta) = \sin(\theta')$, and as such, $\langle \cos(\theta), \sin(\theta) \rangle = \langle \cos(\theta'), \sin(\theta') \rangle$ making $\hat{v}$ is well-defined.

For $\vec{0} = \langle 0, 0 \rangle$, note that $\|\vec{0}\| = \sqrt{0^2 + 0^2} = 0$. Hence, $\|\vec{0}\| \langle \cos(\theta), \sin(\theta) \rangle = 0 \langle \cos(\theta), \sin(\theta) \rangle =< 0, 0 >$ for *every* angle θ. In other words, every angle θ satisfies the equation $\vec{v} = \langle \|\vec{v}\| \cos(\theta), \|\vec{v}\| \sin(\theta) \rangle$ in Definition 13.4, so for this reason, $\hat{0}$ is undefined.

The following theorem summarizes the important facts about the magnitude and direction of a vector.

Theorem 13.8. Properties of Magnitude and Direction: Suppose $\vec{v}$ is a vector.

- $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$
- For all scalars k, $\|k\,\vec{v}\| = |k|\|\vec{v}\|$.
- If $\vec{v} \neq \vec{0}$ then $\vec{v} = \|\vec{v}\|\hat{v}$, so that $\hat{v} = \left(\dfrac{1}{\|\vec{v}\|}\right) \vec{v}$.

The proof of the first property in Theorem 13.8 is a direct consequence of the definition of $\|\vec{v}\|$. Given $\vec{v} = \langle v_1, v_2 \rangle$, then $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$ which is by definition greater than or equal to 0. Moreover, $\sqrt{v_1^2 + v_2^2} = 0$ if and only of $v_1^2 + v_2^2 = 0$ if and only if $v_1 = v_2 = 0$. Hence, $\|\vec{v}\| = 0$ if and only if $\vec{v} = \langle 0, 0 \rangle = \vec{0}$, as required.

The second property is a result of the definition of magnitude and scalar multiplication along with a property of radicals. If $\vec{v} = \langle v_1, v_2 \rangle$ and k is a scalar then

$$
\begin{array}{rcll}
\|k\,\vec{v}\| & = & \|k \langle v_1, v_2 \rangle \| & \\
 & = & \| \langle kv_1, kv_2 \rangle \| & \text{Definition of scalar multiplication} \\
 & = & \sqrt{(kv_1)^2 + (kv_2)^2} & \text{Definition of magnitude} \\
 & = & \sqrt{k^2 v_1^2 + k^2 v_2^2} & \\
 & = & \sqrt{k^2 (v_1^2 + v_2^2)} & \\
 & = & \sqrt{k^2}\sqrt{v_1^2 + v_2^2} & \text{Product Rule for Radicals} \\
 & = & |k|\sqrt{v_1^2 + v_2^2} & \text{Since } \sqrt{k^2} = |k| \\
 & = & |k|\|\vec{v}\| &
\end{array}
$$

The equation $\vec{v} = \|\vec{v}\|\hat{v}$ in Theorem 13.8 is a consequence of the definitions of $\|\vec{v}\|$ and $\hat{v}$ and was worked out in the discussion just prior to Definition 13.4 on page 1163. In words, the equation $\vec{v} = \|\vec{v}\|\hat{v}$ says that any given vector is the product of its magnitude and its direction – an important concept to keep in mind when studying and using vectors.

The formula for $\hat{v}$ stated in Theorem 13.8 is a consequence of solving $\vec{v} = \|\vec{v}\|\hat{v}$ for $\hat{v}$ by multiplying[11] both sides of the equation by $\frac{1}{\|\vec{v}\|}$ and using the properties of Theorem 13.7. We leave these details to the reader. We are overdue for an example.

[11] Of course, to go from $\vec{v} = \|\vec{v}\|\hat{v}$ to $\hat{v} = \left(\frac{1}{\|\vec{v}\|}\right) \vec{v}$, we are essentially 'dividing both sides' of the equation by the scalar $\|\vec{v}\|$. The authors encourage the reader, however, to work out the details carefully to gain an appreciation of the properties in play.

Example 13.3.4.

1. Find the component form of the vector $\vec{v}$ with $\|\vec{v}\| = 5$ so that when $\vec{v}$ is plotted in standard position, it lies in Quadrant II and makes a 60° angle[12] with the negative x-axis.

2. For $\vec{v} = \left\langle 3, -3\sqrt{3} \right\rangle$, find $\|\vec{v}\|$ and θ, $0 \leq \theta < 2\pi$ so that $\vec{v} = \|\vec{v}\| \left\langle \cos(\theta), \sin(\theta) \right\rangle$.

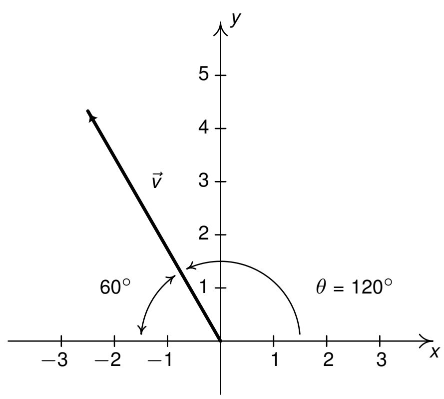
3. For the vectors $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 1, -2 \rangle$, find the following.

   (a) $\hat{v}$ (b) $\|\vec{v}\| - 2\|\vec{w}\|$ (c) $\|\vec{v} - 2\vec{w}\|$ (d) $\|\hat{w}\|$

Solution.

1. We are told that $\|\vec{v}\| = 5$ and are given information about its direction, so we can use the formula $\vec{v} = \|\vec{v}\|\hat{v}$ to get the component form of $\vec{v}$.

   To determine $\hat{v}$, we appeal to Definition 13.4. Since $\vec{v}$ lies in Quadrant II and makes a 60° angle with the negative x-axis, one angle θ satisfying the criteria of Definition 13.4 is $\theta = 120^\circ$.

   Hence, $\hat{v} = \left\langle \cos\left(120^\circ\right), \sin\left(120^\circ\right) \right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$, so $\vec{v} = \|\vec{v}\|\hat{v} = 5\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \left\langle -\frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle$.

2. For $\vec{v} = \left\langle 3, -3\sqrt{3} \right\rangle$, we get $\|\vec{v}\| = \sqrt{(3)^2 + (-3\sqrt{3})^2} = 6$. In light of Definition 13.4, we can find the θ we're after by finding a Quadrant IV angle whose terminal side contains the point $(3, -3\sqrt{3})$.

   Going through the usual calculations, we find $\cos(\theta) = \frac{1}{2}$ and $\sin(\theta) = -\frac{\sqrt{3}}{2}$. Hence, $\theta = \frac{5\pi}{3}$.

   We may check our answer by verifying $6\left\langle \cos\left(\frac{5\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right) \right\rangle = \left\langle 3, -3\sqrt{3} \right\rangle = \vec{v}$.

3. (a) Since we are given the component form of $\vec{v}$, we'll use the formula $\hat{v} = \left(\frac{1}{\|\vec{v}\|}\right)\vec{v}$. For $\vec{v} = \langle 3, 4 \rangle$, we have $\|\vec{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. Hence, $\hat{v} = \frac{1}{5}\langle 3, 4 \rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

[12] Due to the utility of vectors in 'real-world' applications, we will usually use degree measure for the angle when giving the vector's direction. That being said, since Carl doesn't want you to forget about radians, he's made sure there are examples and exercises which use them as well.

(b) We know from our work above that $\|\vec{v}\| = 5$, so to find $\|\vec{v}\| - 2\|\vec{w}\|$, we need only find $\|\vec{w}\|$. Since $\vec{w} = \langle 1, -2 \rangle$, we get $\|\vec{w}\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$. Hence, $\|\vec{v}\| - 2\|\vec{w}\| = 5 - 2\sqrt{5}$.

(c) In the expression $\|\vec{v} - 2\vec{w}\|$, notice that the arithmetic on the vectors comes first, then the magnitude. Hence, our first step is to find the component form of the vector $\vec{v} - 2\vec{w}$. We get $\vec{v} - 2\vec{w} = \langle 3, 4 \rangle - 2\langle 1, -2 \rangle = \langle 1, 8 \rangle$. Hence, $\|\vec{v} - 2\vec{w}\| = \|\langle 1, 8 \rangle\| = \sqrt{1^2 + 8^2} = \sqrt{65}$.

(d) One approach to find $\|\hat{w}\|$, is to first find $\hat{w}$ and then take the magnitude.

Using the formula $\hat{w} = \left(\frac{1}{\|\vec{w}\|}\right)\vec{w}$ along with $\|\vec{w}\| = \sqrt{5}$, which we found the in the previous problem, we get $\hat{w} = \frac{1}{\sqrt{5}}\langle 1, -2 \rangle = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle = \left\langle \frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5} \right\rangle$.

Hence, $\|\hat{w}\| = \sqrt{\left(\frac{\sqrt{5}}{5}\right)^2 + \left(-\frac{2\sqrt{5}}{5}\right)^2} = \sqrt{\frac{5}{25} + \frac{20}{25}} = \sqrt{1} = 1$.

Alternatively, we can use Theorem 13.8. Since $\hat{w} = \left(\frac{1}{\|\vec{w}\|}\right)\vec{w}$, where $\frac{1}{\|\vec{w}\|} > 0$ is a scalar,

$$\|\hat{w}\| = \left\|\left(\frac{1}{\|\vec{w}\|}\right)\vec{w}\right\| = \frac{1}{\|\vec{w}\|}\|\vec{w}\| = \frac{\|\vec{w}\|}{\|\vec{w}\|} = 1.$$

For a third way to show $\|\hat{w}\| = 1$, we can appeal to Definition 13.4. Since $\hat{w} = \langle \cos(\theta), \sin(\theta) \rangle$ for some angle θ, $\|\hat{w}\| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = \sqrt{1} = 1$, where we have used the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$. No matter how we approach the problem, $\|\hat{w}\| = 1$. □

Note that the second and third solutions to number 3d in Example 13.3.4 above work for *any* nonzero vector, $\vec{w}$. We will have more to say about this shortly.

The process exemplified by number 1 in Example 13.3.4 above by which we take information about the magnitude and direction of a vector and find the component form of a vector is called **resolving** a vector into its components. As an application of this process, we revisit Example 13.3.1 below.

Example 13.3.5. A plane leaves an airport with an airspeed of 175 miles per hour with bearing N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

Solution: We proceed as we did in Example 13.3.1 and let $\vec{v}$ denote the plane's velocity and $\vec{w}$ denote the wind's velocity, and set about determining $\vec{v} + \vec{w}$.

If we regard the airport as being at the origin, the positive y-axis acting as due north and the positive x-axis acting as due east, we see that the vectors $\vec{v}$ and $\vec{w}$ are in standard position and their directions correspond to the angles 50° and -30°, respectively.

Hence, the component form of $\vec{v} = 175\langle \cos(50^\circ), \sin(50^\circ) \rangle = \langle 175\cos(50^\circ), 175\sin(50^\circ) \rangle$ and the component form of $\vec{w} = \langle 35\cos(-30^\circ), 35\sin(-30^\circ) \rangle$.

Since we have no convenient way to express the exact values of cosine and sine of 50°, we leave both vectors in terms of cosines and sines.[13] Adding corresponding components, we find the resultant vector

[13]Keeping things 'calculator' friendly, for once!

$\vec{v} + \vec{w} = \langle 175\cos(50^\circ) + 35\cos(-30^\circ), 175\sin(50^\circ) + 35\sin(-30^\circ)\rangle$. To find the 'true' speed of the plane, we compute the magnitude of this resultant vector

$$\|\vec{v} + \vec{w}\| = \sqrt{(175\cos(50^\circ) + 35\cos(-30^\circ))^2 + (175\sin(50^\circ) + 35\sin(-30^\circ))^2} \approx 184$$

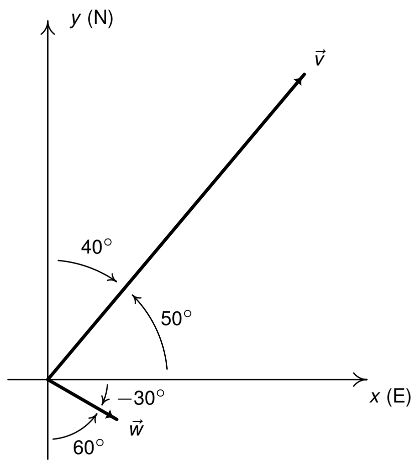
Hence, the 'true' speed of the plane is approximately 184 miles per hour.

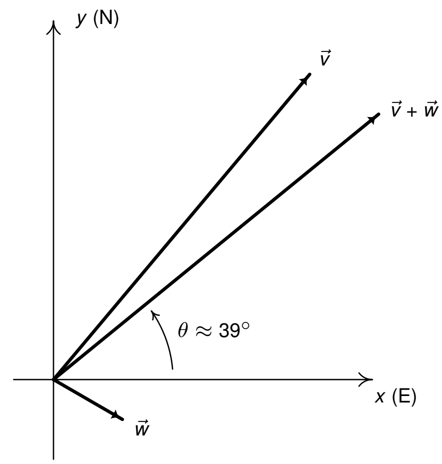
To find the true bearing, we need to find the angle θ whose terminal side when graphed in standard position contains $(x, y) = (175\cos(50^\circ) + 35\cos(-30^\circ), 175\sin(50^\circ) + 35\sin(-30^\circ))$.

Since both of these coordinates are positive,[14] we know θ is a Quadrant I angle, as depicted below. Furthermore,

$$\tan(\theta) = \frac{y}{x} = \frac{175\sin(50^\circ) + 35\sin(-30^\circ)}{175\cos(50^\circ) + 35\cos(-30^\circ)},$$

so using the arctangent function,[15] we get $\theta \approx 39^\circ$. Since, for the purposes of bearing, we need the angle between $\vec{v} + \vec{w}$ and the positive y-axis, we take the complement of θ and find the 'true' bearing of the plane to be approximately N51°E.





□

In part 3d of Example 13.3.4, we saw that the length of the direction vector, $\hat{w}$, $\|\hat{w}\| = 1$. Vectors of length 1 play such an important role that they are given a special name.

Definition 13.5. Unit Vectors: Let $\vec{v}$ be a vector. If $\|\vec{v}\| = 1$, we say that $\vec{v}$ is a **unit vector**.

Note that if $\vec{v}$ is a unit vector, then necessarily, $\vec{v} = \|\vec{v}\|\hat{v} = 1 \cdot \hat{v} = \hat{v}$. Conversely, in the solution of part 3d of Example 13.3.4, two different arguments show for any nonzero vector $\vec{v}$, $\|\hat{v}\| = 1$, so $\hat{v}$ is a unit vector.

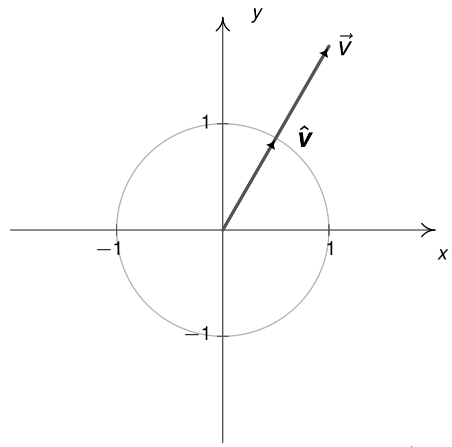
In other words, unit vectors are direction vectors and vice-versa. Indeed, the vector $\hat{v}$ which we have defined as 'the *direction* of $\vec{v}$' is often described as 'the *unit vector in the direction* of $\vec{v}$.'

[14] Yes, a calculator approximation is the quickest way to see this, but you can also use good old-fashioned inequalities and the fact that $45^\circ \leq 50^\circ \leq 60^\circ$.

[15] We could just have easily used arcsine or arccosine here . . .

In practice, if $\vec{v}$ is a unit vector we write it as $\hat{v}$ as opposed to $\vec{v}$ because we have reserved the '$\hat{\ }$' notation for unit vectors. The process of multiplying a nonzero vector by the factor $\frac{1}{\|\vec{v}\|}$ to produce a unit vector is called '**normalizing** the vector.'

The terminal points of unit vectors, when plotted in standard position, lie on the Unit Circle. (You should take the time to show this.) As a result, we visualize normalizing a nonzero vector $\vec{v}$ as shrinking[16] its terminal point, when plotted in standard position, back to the Unit Circle.



Visualizing vector normalization $\hat{v} = \left(\frac{1}{\|\vec{v}\|}\right)\vec{v}$

Of all of the unit vectors, two deserve special mention.

Definition 13.6. The Principal Unit Vectors:

- The vector $\hat{\imath}$ is defined by $\hat{\imath} = \langle 1, 0 \rangle$
- The vector $\hat{\jmath}$ is defined by $\hat{\jmath} = \langle 0, 1 \rangle$

Geometrically, in the xy-plane, the vector $\hat{\imath}$ as represents the positive x-direction, whereas the vector $\hat{\jmath}$ represents the positive y-direction. We have the following 'decomposition' theorem.[17]

Theorem 13.9. Principal Vector Decomposition Theorem:
Let $\vec{v}$ be a vector with component form $\vec{v} = \langle v_1, v_2 \rangle$. Then $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$.

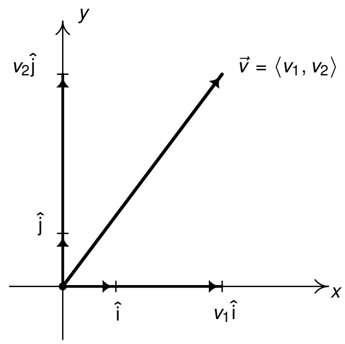
The proof of Theorem 13.9 is straightforward. Since $\hat{\imath} = \langle 1, 0 \rangle$ and $\hat{\jmath} = \langle 0, 1 \rangle$, we have from the definition of scalar multiplication and vector addition that

$$v_1\hat{\imath} + v_2\hat{\jmath} = v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle = \langle v_1, v_2 \rangle = \vec{v}$$

Geometrically, the situation looks like this:

[16]. . . if $\|\vec{v}\| > 1$ . . .

[17]We will see a generalization of Theorem 13.9 in Section 13.4. Stay tuned!



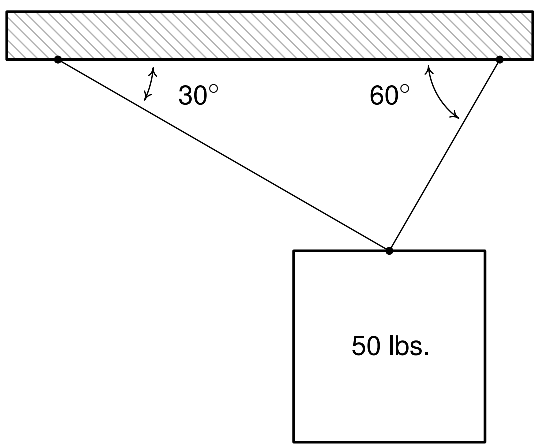
$$\vec{v} = \langle v_1, v_2 \rangle = v_1\hat{\imath} + v_2\hat{\jmath}.$$

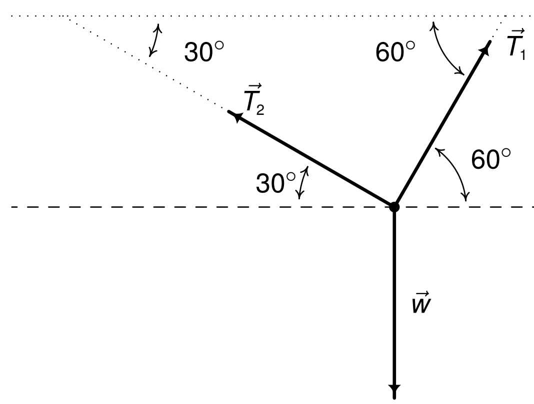
We conclude this section with a classic example which demonstrates how vectors are used in physics to study forces. A 'force' is defined as a 'push' or a 'pull.' The intensity of the push or pull is the magnitude of the force, and is measured in Netwons (N) in the SI system or pounds (lbs.) in the English system.[18]

The following example uses all of the concepts in this section, and should be studied in great detail.

Example 13.3.6. A 50 pound speaker is suspended from the ceiling by two support braces. If one of them makes a 60° angle with the ceiling and the other makes a 30° angle with the ceiling, what are the tensions on each of the supports?

Solution. We represent the problem schematically below along with the corresponding vector diagram.





We have three forces acting on the speaker: the weight of the speaker, which we'll call $\vec{w}$, pulling the speaker directly downward, and the forces on the support rods, which we'll call $\vec{T}_1$ and $\vec{T}_2$ (for 'tensions') acting upward at angles 60° and 30°, respectively.

We are looking for the tensions on the support, which are the magnitudes $\|\vec{T}_1\|$ and $\|\vec{T}_2\|$. In order for the speaker to remain stationary,[19] we require $\vec{w} + \vec{T}_1 + \vec{T}_2 = \vec{0}$.

Viewing the common initial point of these vectors as the origin and the dashed line as the x-axis, we use Theorem 13.8 to get component representations for the three vectors involved. We can model the weight

[18] See also Section 12.4.1.

[19] This is the criteria for 'static equilbrium'.

of the speaker as a vector pointing directly downwards with a magnitude of 50 pounds. That is, $\|\vec{w}\| = 50$ and $\hat{w} = -\hat{\jmath} = \langle 0, -1 \rangle$. Hence, $\vec{w} = 50 \langle 0, -1 \rangle = \langle 0, -50 \rangle$. For the force in the first support, we get

$$\begin{aligned} \vec{T}_1 &= \|\vec{T}_1\| \left\langle \cos(60^\circ), \sin(60^\circ) \right\rangle \\ &= \left\langle \frac{\|\vec{T}_1\|}{2}, \frac{\|\vec{T}_1\|\sqrt{3}}{2} \right\rangle \end{aligned}$$

For the second support, we note that the angle 30° is measured from the negative x-axis, so the angle needed to write $\vec{T}_2$ in component form is 150°. Hence

$$\begin{aligned} \vec{T}_2 &= \|\vec{T}_2\| \left\langle \cos(150^\circ), \sin(150^\circ) \right\rangle \\ &= \left\langle -\frac{\|\vec{T}_2\|\sqrt{3}}{2}, \frac{\|\vec{T}_2\|}{2} \right\rangle \end{aligned}$$

The requirement $\vec{w} + \vec{T}_1 + \vec{T}_2 = \vec{0}$ gives us the vector equation:

$$\begin{aligned} \vec{w} + \vec{T}_1 + \vec{T}_2 &= \vec{0} \\ \langle 0, -50 \rangle + \left\langle \frac{\|\vec{T}_1\|}{2}, \frac{\|\vec{T}_1\|\sqrt{3}}{2} \right\rangle + \left\langle -\frac{\|\vec{T}_2\|\sqrt{3}}{2}, \frac{\|\vec{T}_2\|}{2} \right\rangle &= \langle 0, 0 \rangle \\ \left\langle \frac{\|\vec{T}_1\|}{2} - \frac{\|\vec{T}_2\|\sqrt{3}}{2}, \frac{\|\vec{T}_1\|\sqrt{3}}{2} + \frac{\|\vec{T}_2\|}{2} - 50 \right\rangle &= \langle 0, 0 \rangle \end{aligned}$$

Equating the corresponding components of the vectors on each side, we get a system of linear equations in the variables $\|\vec{T}_1\|$ and $\|\vec{T}_2\|$.

$$\left\{ \begin{array}{lrcl} (E1) & \dfrac{\|\vec{T}_1\|}{2} - \dfrac{\|\vec{T}_2\|\sqrt{3}}{2} & = & 0 \\ (E2) & \dfrac{\|\vec{T}_1\|\sqrt{3}}{2} + \dfrac{\|\vec{T}_2\|}{2} - 50 & = & 0 \end{array} \right.$$

From $(E1)$, we get $\|\vec{T}_1\| = \|\vec{T}_2\|\sqrt{3}$. Substituting that into $(E2)$ gives $\frac{(\|\vec{T}_2\|\sqrt{3})\sqrt{3}}{2} + \frac{\|\vec{T}_2\|}{2} - 50 = 0$. Solving, we get $2\|\vec{T}_2\| - 50 = 0$, so $\|\vec{T}_2\| = 25$ pounds. Hence, $\|\vec{T}_1\| = \|\vec{T}_2\|\sqrt{3} = 25\sqrt{3}$ pounds. □

Note that the sum of the tensions on the wires in Example 13.3.6 exceed the 50 pounds of the speaker. Explaining why this happens is a good exercise and gets at the heart of the concept of vectors and resolution of forces. Speaking of exercises . . .

13.3.1 Exercises

In Exercises 1 - 10, use the given pair of vectors $\vec{v}$ and $\vec{w}$ to find the following quantities. State whether the result is a vector or a scalar.

- $\vec{v} + \vec{w}$
- $\vec{w} - 2\vec{v}$
- $\|\vec{v} + \vec{w}\|$
- $\|\vec{v}\| + \|\vec{w}\|$
- $\|\vec{v}\|\vec{w} - \|\vec{w}\|\vec{v}$
- $\|\vec{w}\|\hat{v}$

Finally, verify that the vectors satisfy the **Parallelogram Law**

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 = \frac{1}{2}\left[\|\vec{v} + \vec{w}\|^2 + \|\vec{v} - \vec{w}\|^2\right]$$

1. $\vec{v} = \langle 12, -5 \rangle$, $\vec{w} = \langle 3, 4 \rangle$
2. $\vec{v} = \langle -7, 24 \rangle$, $\vec{w} = \langle -5, -12 \rangle$
3. $\vec{v} = \langle 2, -1 \rangle$, $\vec{w} = \langle -2, 4 \rangle$
4. $\vec{v} = \langle 10, 4 \rangle$, $\vec{w} = \langle -2, 5 \rangle$
5. $\vec{v} = \left\langle -\sqrt{3}, 1 \right\rangle$, $\vec{w} = \left\langle 2\sqrt{3}, 2 \right\rangle$
6. $\vec{v} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, $\vec{w} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$
7. $\vec{v} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$, $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$
8. $\vec{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$, $\vec{w} = \left\langle -1, -\sqrt{3} \right\rangle$
9. $\vec{v} = 3\hat{\imath} + 4\hat{\jmath}$, $\vec{w} = -2\hat{\jmath}$
10. $\vec{v} = \frac{1}{2}\left(\hat{\imath} + \hat{\jmath}\right)$, $\vec{w} = \frac{1}{2}\left(\hat{\imath} - \hat{\jmath}\right)$

In Exercises 11 - 25, find the component form of the vector $\vec{v}$ using the information given about its magnitude and direction. Give exact values.

11. $\|\vec{v}\| = 6$; when drawn in standard position $\vec{v}$ lies in Quadrant I and makes a 60° angle with the positive x-axis
12. $\|\vec{v}\| = 3$; when drawn in standard position $\vec{v}$ lies in Quadrant I and makes a 45° angle with the positive x-axis
13. $\|\vec{v}\| = \frac{2}{3}$; when drawn in standard position $\vec{v}$ lies in Quadrant I and makes a 60° angle with the positive y-axis
14. $\|\vec{v}\| = 12$; when drawn in standard position $\vec{v}$ lies along the positive y-axis
15. $\|\vec{v}\| = 4$; when drawn in standard position $\vec{v}$ lies in Quadrant II and makes a 30° angle with the negative x-axis
16. $\|\vec{v}\| = 2\sqrt{3}$; when drawn in standard position $\vec{v}$ lies in Quadrant II and makes a 30° angle with the positive y-axis
17. $\|\vec{v}\| = \frac{7}{2}$; when drawn in standard position $\vec{v}$ lies along the negative x-axis
18. $\|\vec{v}\| = 5\sqrt{6}$; when drawn in standard position $\vec{v}$ lies in Quadrant III and makes a 45° angle with the negative x-axis
19. $\|\vec{v}\| = 6.25$; when drawn in standard position $\vec{v}$ lies along the negative y-axis

20. $\|\vec{v}\| = 4\sqrt{3}$; when drawn in standard position $\vec{v}$ lies in Quadrant IV and makes a 30° angle with the positive x-axis

21. $\|\vec{v}\| = 5\sqrt{2}$; when drawn in standard position $\vec{v}$ lies in Quadrant IV and makes a 45° angle with the negative y-axis

22. $\|\vec{v}\| = 2\sqrt{5}$; when drawn in standard position $\vec{v}$ lies in Quadrant I and makes an angle measuring $\arctan(2)$ with the positive x-axis

23. $\|\vec{v}\| = \sqrt{10}$; when drawn in standard position $\vec{v}$ lies in Quadrant II and makes an angle measuring $\arctan(3)$ with the negative x-axis

24. $\|\vec{v}\| = 5$; when drawn in standard position $\vec{v}$ lies in Quadrant III and makes an angle measuring $\arctan\left(\frac{4}{3}\right)$ with the negative x-axis

25. $\|\vec{v}\| = 26$; when drawn in standard position $\vec{v}$ lies in Quadrant IV and makes an angle measuring $\arctan\left(\frac{5}{12}\right)$ with the positive x-axis

In Exercises 26 - 31, approximate the component form of the vector $\vec{v}$ using the information given about its magnitude and direction. Round your approximations to two decimal places.

26. $\|\vec{v}\| = 392$; when drawn in standard position $\vec{v}$ makes a 117° angle with the positive x-axis

27. $\|\vec{v}\| = 63.92$; when drawn in standard position $\vec{v}$ makes a 78.3° angle with the positive x-axis

28. $\|\vec{v}\| = 5280$; when drawn in standard position $\vec{v}$ makes a 12° angle with the positive x-axis

29. $\|\vec{v}\| = 450$; when drawn in standard position $\vec{v}$ makes a 210.75° angle with the positive x-axis

30. $\|\vec{v}\| = 168.7$; when drawn in standard position $\vec{v}$ makes a 252° angle with the positive x-axis

31. $\|\vec{v}\| = 26$; when drawn in standard position $\vec{v}$ makes a 304.5° angle with the positive x-axis

In Exercises 32 - 52, for the given vector $\vec{v}$, find the magnitude $\|\vec{v}\|$ and an angle θ with $0 \leq \theta < 360^\circ$ so that $\vec{v} = \|\vec{v}\| \langle \cos(\theta), \sin(\theta) \rangle$ (See Definition 13.4.) Round approximations to two decimal places.

32. $\vec{v} = \left\langle 1, \sqrt{3} \right\rangle$

33. $\vec{v} = \langle 5, 5 \rangle$

34. $\vec{v} = \left\langle -2\sqrt{3}, 2 \right\rangle$

35. $\vec{v} = \left\langle -\sqrt{2}, \sqrt{2} \right\rangle$

36. $\vec{v} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$

37. $\vec{v} = \left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$

38. $\vec{v} = \langle 6, 0 \rangle$

39. $\vec{v} = \langle -2.5, 0 \rangle$

40. $\vec{v} = \left\langle 0, \sqrt{7} \right\rangle$

41. $\vec{v} = -10\hat{\jmath}$

42. $\vec{v} = \langle 3, 4 \rangle$

43. $\vec{v} = \langle 12, 5 \rangle$

44. $\vec{v} = \langle -4, 3 \rangle$

45. $\vec{v} = \langle -7, 24 \rangle$

46. $\vec{v} = \langle -2, -1 \rangle$

47. $\vec{v} = \langle -2, -6 \rangle$

48. $\vec{v} = \hat{\imath} + \hat{\jmath}$

49. $\vec{v} = \hat{\imath} - 4\hat{\jmath}$

50. $\vec{v} = \langle 123.4, -77.05 \rangle$

51. $\vec{v} = \langle 965.15, 831.6 \rangle$

52. $\vec{v} = \langle -114.1, 42.3 \rangle$

53. A small boat leaves the dock at Camp DuNuthin and heads across the Nessie River at 17 miles per hour (that is, with respect to the water) at a bearing of S68°W. The river is flowing due east at 8 miles per hour. What is the boat's true speed and heading? Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

54. The HMS Sasquatch leaves port with bearing S20°E maintaining a speed of 42 miles per hour (that is, with respect to the water). If the ocean current is 5 miles per hour with a bearing of N60°E, find the HMS Sasquatch's true speed and bearing. Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

55. If the captain of the HMS Sasquatch in Exercise 54 wishes to reach Chupacabra Cove, an island 100 miles away at a bearing of S20°E from port, in three hours, what speed and heading should she set to take into account the ocean current? Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

    HINT: If $\vec{v}$ denotes the velocity of the HMS Sasquatch and $\vec{w}$ denotes the velocity of the current, what does $\vec{v} + \vec{w}$ need to be to reach Chupacabra Cove in three hours?

56. In calm air, a plane flying from the Pedimaxus International Airport can reach Cliffs of Insanity Point in two hours by following a bearing of N8.2°E at 96 miles an hour. (The distance between the airport and the cliffs is 192 miles.) If the wind is blowing from the southeast at 25 miles per hour, what speed and bearing should the pilot take so that she makes the trip in two hours along the original heading? Round the speed to the nearest hundredth of a mile per hour and your angle to the nearest tenth of a degree.

57. The SS Bigfoot leaves Yeti Bay on a course of N37°W at a speed of 50 miles per hour. After traveling half an hour, the captain determines he is 30 miles from the bay and his bearing back to the bay is S40°E. What is the speed and bearing of the ocean current? Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

58. A 600 pound Sasquatch statue is suspended by two cables from a gymnasium ceiling. If each cable makes a 60° angle with the ceiling, find the tension on each cable. Round your answer to the nearest pound.

59. Two cables are to support an object hanging from a ceiling. If the cables are each to make a 42° angle with the ceiling, and each cable is rated to withstand a maximum tension of 100 pounds, what is the heaviest object that can be supported? Round your answer down to the nearest pound.

60. A 300 pound metal star is hanging on two cables which are attached to the ceiling. The left hand cable makes a 72° angle with the ceiling while the right hand cable makes a 18° angle with the ceiling. What is the tension on each of the cables? Round your answers to three decimal places.

61. Two drunken college students have filled an empty beer keg with rocks and tied ropes to it in order to drag it down the street in the middle of the night. The stronger of the two students pulls with a force of 100 pounds at a heading of N77°E and the other pulls at a heading of S68°E. What force should the weaker student apply to his rope so that the keg of rocks heads due east? What resultant force is applied to the keg? Round your answer to the nearest pound.

62. Emboldened by the success of their late night keg pull in Exercise 61 above, our intrepid young scholars have decided to pay homage to the chariot race scene from the movie 'Ben-Hur' by tying three ropes to a couch, loading the couch with all but one of their friends and pulling it due west down the street. The first rope points N80°W, the second points due west and the third points S80°W. The force applied to the first rope is 100 pounds, the force applied to the second rope is 40 pounds and the force applied (by the non-riding friend) to the third rope is 160 pounds. They need the resultant force to be at least 300 pounds otherwise the couch won't move. Does it move? If so, is it heading due west?

63. Let $\vec{v} = \langle v_1, v_2 \rangle$ be any non-zero vector. Show that $\dfrac{1}{\|\vec{v}\|}\vec{v}$ has length 1.

64. We say that two non-zero vectors $\vec{v}$ and $\vec{w}$ are **parallel** if they have same or opposite directions. That is, $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$ are parallel if either $\hat{v} = \hat{w}$ or $\hat{v} = -\hat{w}$. Show that this means $\vec{v} = k\vec{w}$ for some non-zero scalar k and that $k > 0$ if the vectors have the same direction and $k < 0$ if they point in opposite directions.

65. The goal of this exercise is to use vectors to describe non-vertical lines in the plane. To that end, consider the line $y = 2x - 4$. Let $\vec{v}_0 = \langle 0, -4 \rangle$ and let $\vec{s} = \langle 1, 2 \rangle$. Let t be any real number. Show that the vector defined by $\vec{v} = \vec{v}_0 + t\vec{s}$, when drawn in standard position, has its terminal point on the line $y = 2x - 4$. (Hint: Show that $\vec{v}_0 + t\vec{s} = \langle t, 2t - 4 \rangle$ for any real number t.) Now consider the non-vertical line $y = mx + b$. Repeat the previous analysis with $\vec{v}_0 = \langle 0, b \rangle$ and let $\vec{s} = \langle 1, m \rangle$. Thus any non-vertical line can be thought of as a collection of terminal points of the vector sum of $\langle 0, b \rangle$ (the position vector of the y-intercept) and a scalar multiple of the slope vector $\vec{s} = \langle 1, m \rangle$.

66. Prove the associative and identity properties of vector addition in Theorem 13.6.

67. Prove the properties of scalar multiplication in Theorem 13.7.

13.3.2 Answers

1.
- $\vec{v}+\vec{w}=\langle 15,-1\rangle$, vector
- $\|\vec{v}+\vec{w}\|=\sqrt{226}$, scalar
- $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\langle -21,77\rangle$, vector
- $\vec{w}-2\vec{v}=\langle -21,14\rangle$, vector
- $\|\vec{v}\|+\|\vec{w}\|=18$, scalar
- $\|w\|\hat{v}=\left\langle \frac{60}{13},-\frac{25}{13}\right\rangle$, vector

2.
- $\vec{v}+\vec{w}=\langle -12,12\rangle$, vector
- $\|\vec{v}+\vec{w}\|=12\sqrt{2}$, scalar
- $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\langle -34,-612\rangle$, vector
- $\vec{w}-2\vec{v}=\langle 9,-60\rangle$, vector
- $\|\vec{v}\|+\|\vec{w}\|=38$, scalar
- $\|w\|\hat{v}=\left\langle -\frac{91}{25},\frac{312}{25}\right\rangle$, vector

3.
- $\vec{v}+\vec{w}=\langle 0,3\rangle$, vector
- $\|\vec{v}+\vec{w}\|=3$, scalar
- $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\left\langle -6\sqrt{5},6\sqrt{5}\right\rangle$, vector
- $\vec{w}-2\vec{v}=\langle -6,6\rangle$, vector
- $\|\vec{v}\|+\|\vec{w}\|=3\sqrt{5}$, scalar
- $\|w\|\hat{v}=\langle 4,-2\rangle$, vector

4.
- $\vec{v}+\vec{w}=\langle 8,9\rangle$, vector
- $\|\vec{v}+\vec{w}\|=\sqrt{145}$, scalar
- $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\left\langle -14\sqrt{29},6\sqrt{29}\right\rangle$, vector
- $\vec{w}-2\vec{v}=\langle -22,-3\rangle$, vector
- $\|\vec{v}\|+\|\vec{w}\|=3\sqrt{29}$, scalar
- $\|w\|\hat{v}=\langle 5,2\rangle$, vector

5.
- $\vec{v}+\vec{w}=\left\langle \sqrt{3},3\right\rangle$, vector
- $\|\vec{v}+\vec{w}\|=2\sqrt{3}$, scalar
- $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\left\langle 8\sqrt{3},0\right\rangle$, vector
- $\vec{w}-2\vec{v}=\left\langle 4\sqrt{3},0\right\rangle$, vector
- $\|\vec{v}\|+\|\vec{w}\|=6$, scalar
- $\|w\|\hat{v}=\left\langle -2\sqrt{3},2\right\rangle$, vector

6.
- $\vec{v}+\vec{w}=\left\langle -\frac{1}{5},\frac{7}{5}\right\rangle$, vector
- $\|\vec{v}+\vec{w}\|=\sqrt{2}$, scalar
- $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\left\langle -\frac{7}{5},-\frac{1}{5}\right\rangle$, vector
- $\vec{w}-2\vec{v}=\langle -2,-1\rangle$, vector
- $\|\vec{v}\|+\|\vec{w}\|=2$, scalar
- $\|w\|\hat{v}=\left\langle \frac{3}{5},\frac{4}{5}\right\rangle$, vector

7.
- $\vec{v}+\vec{w}=\langle 0,0\rangle$, vector
- $\|\vec{v}+\vec{w}\|=0$, scalar
- $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\left\langle -\sqrt{2},\sqrt{2}\right\rangle$, vector
- $\vec{w}-2\vec{v}=\left\langle -\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right\rangle$, vector
- $\|\vec{v}\|+\|\vec{w}\|=2$, scalar
- $\|w\|\hat{v}=\left\langle \frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right\rangle$, vector

8.
- $\vec{v}+\vec{w}=\left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right\rangle$, vector
- $\|\vec{v}+\vec{w}\|=1$, scalar
- $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\left\langle -2, -2\sqrt{3}\right\rangle$, vector
- $\vec{w}-2\vec{v}=\left\langle -2, -2\sqrt{3}\right\rangle$, vector
- $\|\vec{v}\|+\|\vec{w}\|=3$, scalar
- $\|w\|\hat{v}=\left\langle 1, \sqrt{3}\right\rangle$, vector

9.
- $\vec{v}+\vec{w}=\langle 3, 2\rangle$, vector
- $\|\vec{v}+\vec{w}\|=\sqrt{13}$, scalar
- $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\langle -6, -18\rangle$, vector
- $\vec{w}-2\vec{v}=\langle -6, -10\rangle$, vector
- $\|\vec{v}\|+\|\vec{w}\|=7$, scalar
- $\|w\|\hat{v}=\left\langle \frac{6}{5}, \frac{8}{5}\right\rangle$, vector

10.
- $\vec{v}+\vec{w}=\langle 1, 0\rangle$, vector
- $\|\vec{v}+\vec{w}\|=1$, scalar
- $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\left\langle 0, -\frac{\sqrt{2}}{2}\right\rangle$, vector
- $\vec{w}-2\vec{v}=\left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle$, vector
- $\|\vec{v}\|+\|\vec{w}\|=\sqrt{2}$, scalar
- $\|w\|\hat{v}=\left\langle \frac{1}{2}, \frac{1}{2}\right\rangle$, vector

11. $\vec{v}=\left\langle 3, 3\sqrt{3}\right\rangle$
12. $\vec{v}=\left\langle \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right\rangle$
13. $\vec{v}=\left\langle \frac{\sqrt{3}}{3}, \frac{1}{3}\right\rangle$
14. $\vec{v}=\langle 0, 12\rangle$
15. $\vec{v}=\left\langle -2\sqrt{3}, 2\right\rangle$
16. $\vec{v}=\left\langle -\sqrt{3}, 3\right\rangle$
17. $\vec{v}=\left\langle -\frac{7}{2}, 0\right\rangle$
18. $\vec{v}=\left\langle -5\sqrt{3}, -5\sqrt{3}\right\rangle$
19. $\vec{v}=\langle 0, -6.25\rangle$
20. $\vec{v}=\left\langle 6, -2\sqrt{3}\right\rangle$
21. $\vec{v}=\langle 5, -5\rangle$
22. $\vec{v}=\langle 2, 4\rangle$
23. $\vec{v}=\langle -1, 3\rangle$
24. $\vec{v}=\langle -3, -4\rangle$
25. $\vec{v}=\langle 24, -10\rangle$
26. $\vec{v}\approx\langle -177.96, 349.27\rangle$
27. $\vec{v}\approx\langle 12.96, 62.59\rangle$
28. $\vec{v}\approx\langle 5164.62, 1097.77\rangle$
29. $\vec{v}\approx\langle -386.73, -230.08\rangle$
30. $\vec{v}\approx\langle -52.13, -160.44\rangle$
31. $\vec{v}\approx\langle 14.73, -21.43\rangle$
32. $\|\vec{v}\|=2$, $\theta=60^\circ$
33. $\|\vec{v}\|=5\sqrt{2}$, $\theta=45^\circ$
34. $\|\vec{v}\|=4$, $\theta=150^\circ$
35. $\|\vec{v}\|=2$, $\theta=135^\circ$
36. $\|\vec{v}\|=1$, $\theta=225^\circ$
37. $\|\vec{v}\|=1$, $\theta=240^\circ$
38. $\|\vec{v}\|=6$, $\theta=0^\circ$
39. $\|\vec{v}\|=2.5$, $\theta=180^\circ$
40. $\|\vec{v}\|=\sqrt{7}$, $\theta=90^\circ$
41. $\|\vec{v}\|=10$, $\theta=270^\circ$
42. $\|\vec{v}\|=5$, $\theta\approx 53.13^\circ$
43. $\|\vec{v}\|=13$, $\theta\approx 22.62^\circ$
44. $\|\vec{v}\|=5$, $\theta\approx 143.13^\circ$
45. $\|\vec{v}\|=25$, $\theta\approx 106.26^\circ$
46. $\|\vec{v}\|=\sqrt{5}$, $\theta\approx 206.57^\circ$

47. $\|\vec{v}\| = 2\sqrt{10}$, $\theta \approx 251.57^\circ$
48. $\|\vec{v}\| = \sqrt{2}$, $\theta \approx 45^\circ$
49. $\|\vec{v}\| = \sqrt{17}$, $\theta \approx 284.04^\circ$
50. $\|\vec{v}\| \approx 145.48$, $\theta \approx 328.02^\circ$
51. $\|\vec{v}\| \approx 1274.00$, $\theta \approx 40.75^\circ$
52. $\|\vec{v}\| \approx 121.69$, $\theta \approx 159.66^\circ$
53. The boat's true speed is about 10 miles per hour at a heading of S50.6°W.
54. The HMS Sasquatch's true speed is about 41 miles per hour at a heading of S26.8°E.
55. She should maintain a speed of about 35 miles per hour at a heading of S11.8°E.
56. She should fly at 83.46 miles per hour with a heading of N22.1°E
57. The current is moving at about 10 miles per hour bearing N54.6°W.
58. The tension on each of the cables is about 346 pounds.
59. The maximum weight that can be held by the cables in that configuration is about 133 pounds.
60. The tension on the left hand cable is 285.317 lbs. and on the right hand cable is 92.705 lbs.
61. The weaker student should pull about 60 pounds. The net force on the keg is about 153 pounds.
62. The resultant force is only about 296 pounds so the couch doesn't budge. Even if it did move, the stronger force on the third rope would have made the couch drift slightly to the south as it traveled down the street.

13.4 The Dot Product

In Section 13.3, we learned how add and subtract vectors and how to multiply vectors by scalars. In this section, we define a product of vectors. We begin with the following definition.

Definition 13.7. Given vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$, the **dot product** of $\vec{v}$ and $\vec{w}$ is given by

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle = v_1 w_1 + v_2 w_2$$

For example, if $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 1, -2 \rangle$,then $\vec{v} \cdot \vec{w} = \langle 3, 4 \rangle \cdot \langle 1, -2 \rangle = (3)(1) + (4)(-2) = -5$.
Note that the dot product takes two *vectors* and produces a *scalar*. For that reason, the quantity $\vec{v} \cdot \vec{w}$ is often called the **scalar product** of $\vec{v}$ and $\vec{w}$. The dot product enjoys the following properties.

Theorem 13.10. Properties of the Dot Product

- **Commutative Property:** For all vectors $\vec{v}$ and $\vec{w}$, $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
- **Distributive Property:** For all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, $\vec{u} \cdot \left(\vec{v} + \vec{w}\right) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
- **Scalar Property:** For all vectors $\vec{v}$ and $\vec{w}$ and scalars k, $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (k\vec{w})$.
- **Relation to Magnitude:** For all vectors $\vec{v}$, $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

Like most of the theorems involving vectors, the proof of Theorem 13.10 amounts to using the definition of the dot product and properties of real number arithmetic.
For example, to show the commutative property, let $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$. Then

$$\begin{array}{rcll} \vec{v} \cdot \vec{w} & = & \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle & \\ & = & v_1 w_1 + v_2 w_2 & \text{Definition of Dot Product} \\ & = & w_1 v_1 + w_2 v_2 & \text{Commutativity of Real Number Multiplication} \\ & = & \langle w_1, w_2 \rangle \cdot \langle v_1, v_2 \rangle & \text{Definition of Dot Product} \\ & = & \vec{w} \cdot \vec{v} & \end{array}$$

The distributive property is proved similarly and is left as an exercise.
For the scalar property, assume that $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ and k is a scalar. Then

$$\begin{array}{rcll} (k\vec{v}) \cdot \vec{w} & = & \left(k \langle v_1, v_2 \rangle\right) \cdot \langle w_1, w_2 \rangle & \\ & = & \langle kv_1, kv_2 \rangle \cdot \langle w_1, w_2 \rangle & \text{Definition of Scalar Multiplication} \\ & = & (kv_1)(w_1) + (kv_2)(w_2) & \text{Definition of Dot Product} \\ & = & k(v_1 w_1) + k(v_2 w_2) & \text{Associativity of Real Number Multiplication} \\ & = & k(v_1 w_1 + v_2 w_2) & \text{Distributive Law of Real Numbers} \\ & = & k \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle & \text{Definition of Dot Product} \\ & = & k(\vec{v} \cdot \vec{w}) & \end{array}$$

We leave the proof of $k(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (k\vec{w})$ as an exercise.

For the last property, we note that if $\vec{v} = \langle v_1, v_2 \rangle$, then $\vec{v} \cdot \vec{v} = \langle v_1, v_2 \rangle \cdot \langle v_1, v_2 \rangle = v_1^2 + v_2^2 = \|\vec{v}\|^2$, where the last equality comes courtesy of Definition 13.4.

The following example puts Theorem 13.10 to good use. As in Example 13.3.3, we work out the problem in great detail and encourage the reader to supply the justification for each step.

Example 13.4.1. Prove the identity: $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 - 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2$.

Solution. We begin by rewriting $\|\vec{v} - \vec{w}\|^2$ in terms of the dot product using Theorem 13.10.

$$
\begin{aligned}
\|\vec{v} - \vec{w}\|^2 &= (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) \\
&= (\vec{v} + [-\vec{w}]) \cdot (\vec{v} + [-\vec{w}]) \\
&= (\vec{v} + [-\vec{w}]) \cdot \vec{v} + (\vec{v} + [-\vec{w}]) \cdot [-\vec{w}] \\
&= \vec{v} \cdot (\vec{v} + [-\vec{w}]) + [-\vec{w}] \cdot (\vec{v} + [-\vec{w}]) \\
&= \vec{v} \cdot \vec{v} + \vec{v} \cdot [-\vec{w}] + [-\vec{w}] \cdot \vec{v} + [-\vec{w}] \cdot [-\vec{w}] \\
&= \vec{v} \cdot \vec{v} + \vec{v} \cdot [(-1)\vec{w}] + [(-1)\vec{w}] \cdot \vec{v} + [(-1)\vec{w}] \cdot [(-1)\vec{w}] \\
&= \vec{v} \cdot \vec{v} + (-1)(\vec{v} \cdot \vec{w}) + (-1)(\vec{w} \cdot \vec{v}) + [(-1)(-1)](\vec{w} \cdot \vec{w}) \\
&= \vec{v} \cdot \vec{v} + (-1)(\vec{v} \cdot \vec{w}) + (-1)(\vec{v} \cdot \vec{w}) + \vec{w} \cdot \vec{w} \\
&= \vec{v} \cdot \vec{v} - 2(\vec{v} \cdot \vec{w}) + \vec{w} \cdot \vec{w} \\
&= \|\vec{v}\|^2 - 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2
\end{aligned}
$$

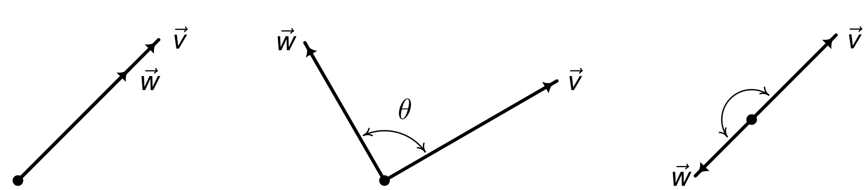
Hence, $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 - 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2$ as required. □

If we take a step back from the pedantry in Example 13.4.1, we see that the bulk of the work is needed to show that $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} - 2(\vec{v} \cdot \vec{w}) + \vec{w} \cdot \vec{w}$. If this looks familiar, it should.

Since the dot product enjoys many of the same properties enjoyed by real numbers, the machinations required to expand $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})$ for vectors $\vec{v}$ and $\vec{w}$ match those required to expand $(v - w)(v - w)$ for real numbers v and w, and hence we get similar looking results.

The identity verified in Example 13.4.1 plays a large role in the development of the geometric properties of the dot product, which we now explore.

Suppose $\vec{v}$ and $\vec{w}$ are two nonzero vectors. If we draw $\vec{v}$ and $\vec{w}$ with the same initial point, we define the **angle between** $\vec{v}$ and $\vec{w}$ to be the angle θ determined by the rays containing the vectors $\vec{v}$ and $\vec{w}$, as illustrated below. We require $0 \leq \theta \leq \pi$. (Think about why this is needed in the definition.)



$\theta = 0$ $\quad$ $0 < \theta < \pi$ $\quad$ $\theta = \pi$

The following theorem gives us some insight into the geometric role the dot product plays.

Theorem 13.11. Geometric Interpretation of Dot Product: If $\vec{v}$ and $\vec{w}$ are nonzero vectors then

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos(\theta),$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

We prove Theorem 13.11 in cases. If $\theta = 0$, then $\vec{v}$ and $\vec{w}$ have the same direction. It follows[1] that there is a real number $k > 0$ so that $\vec{w} = k\vec{v}$. Hence, $\vec{v} \cdot \vec{w} = \vec{v} \cdot (k\vec{v}) = k(\vec{v} \cdot \vec{v}) = k\|\vec{v}\|^2$.

Working from the other end of the equation, $\|\vec{v}\|\|\vec{w}\| \cos(\theta) = \|\vec{v}\|\|k\vec{v}\| \cos(0) = \|\vec{v}\|(|k|\|\vec{v}\|)(1) = k\|\vec{v}\|^2$, where $\|k\vec{v}\| = |k|\|\vec{v}\|$ courtesy of Theorem 13.8, and $|k| = k$ since $k > 0$.

Hence, in the case $\theta = 0$, we have shown $\vec{v} \cdot \vec{w} = k\|\vec{v}\|^2$ and $\|\vec{v}\|\|\vec{w}\| \cos(\theta) = k\|\vec{v}\|^2$. Putting these two equations together shows that $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos(\theta)$ holds in this case.

If $\theta = \pi$, $\vec{v}$ and $\vec{w}$ have the exact opposite directions, so there is a real number $k < 0$ with $\vec{w} = k\vec{v}$.

As before, we compute $\vec{v} \cdot \vec{w} = \vec{v} \cdot (k\vec{v}) = k(\vec{v} \cdot \vec{v}) = k\|\vec{v}\|^2$. Since $k < 0$ here, we have $|k| = -k$. Hence, we find $\|\vec{v}\|\|\vec{w}\| \cos(\theta) = \|\vec{v}\|\|k\vec{v}\| \cos(\pi) = \|\vec{v}\|(|k|\|\vec{v}\|)(-1) = \|\vec{v}\|(-k)\|\vec{v}\|(-1) = k\|\vec{v}\|^2$.

Once again, both $\vec{v} \cdot \vec{w} = k\|\vec{v}\|^2$ and $\|\vec{v}\|\|\vec{w}\| \cos(\theta) = k\|\vec{v}\|^2$, so $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos(\theta)$ in this case.

Next, if $0 < \theta < \pi$, the vectors $\vec{v}$, $\vec{w}$ and $\vec{v} - \vec{w}$ determine a triangle with side lengths $\|\vec{v}\|$, $\|\vec{w}\|$ and $\|\vec{v} - \vec{w}\|$, respectively, as seen in the diagram below.

The Law of Cosines yields $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos(\theta)$. From Example 13.4.1, we also have that $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 - 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2$.

Equating these two expressions for $\|\vec{v} - \vec{w}\|^2$ gives $\|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos(\theta) = \|\vec{v}\|^2 - 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2$ which reduces to $-2\|\vec{v}\|\|\vec{w}\| \cos(\theta) = -2(\vec{v} \cdot \vec{w})$. Hence, $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos(\theta)$, as required.

An immediate consequence of Theorem 13.11 is the following.

Theorem 13.12. Let $\vec{v}$ and $\vec{w}$ be nonzero vectors and let θ the angle between $\vec{v}$ and $\vec{w}$. Then

$$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos(\hat{v} \cdot \hat{w})$$

We obtain the formula in Theorem 13.12 by solving the equation given in Theorem 13.11 for θ.

Since $\vec{v}$ and $\vec{w}$ are nonzero, so are $\|\vec{v}\|$ and $\|\vec{w}\|$. Hence, we may divide both sides of $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos(\theta)$ by $\|\vec{v}\|\|\vec{w}\|$. Since $0 \leq \theta \leq \pi$ by definition, the values of θ exactly match the range of the arccosine

[1] Since $\vec{v} = \|\vec{v}\|\hat{v}$ and $\vec{w} = \|\vec{w}\|\hat{w}$, if $\hat{v} = \hat{w}$ then $\vec{w} = \|\vec{w}\|\hat{v} = \frac{\|\vec{w}\|}{\|\vec{v}\|}(\|\vec{v}\|\hat{v}) = \frac{\|\vec{w}\|}{\|\vec{v}\|}\vec{v}$. In this case, $k = \frac{\|\vec{w}\|}{\|\vec{v}\|} > 0$.

function. Hence,

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|} \Rightarrow \theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right).$$

Using Theorem 13.10, we can rewrite

$$\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \left(\frac{1}{\|\vec{v}\|}\vec{v}\right) \cdot \left(\frac{1}{\|\vec{w}\|}\vec{w}\right) = \hat{v} \cdot \hat{w},$$

giving us the alternative formula listed in Theorem 13.12: $\theta = \arccos(\hat{v} \cdot \hat{w})$. We are overdue for an example.

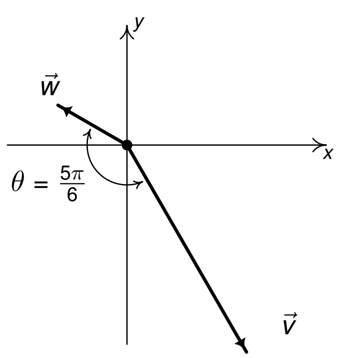
Example 13.4.2. Find the angle between the following pairs of vectors. Graph each pair of vectors in standard position to check the reasonableness of your answer.

1. $\vec{v} = \left\langle 3, -3\sqrt{3} \right\rangle$, and $\vec{w} = \left\langle -\sqrt{3}, 1 \right\rangle$
2. $\vec{v} = \langle 2, 2 \rangle$, and $\vec{w} = \langle 5, -5 \rangle$
3. $\vec{v} = \langle 3, -4 \rangle$, and $\vec{w} = \langle 2, 1 \rangle$

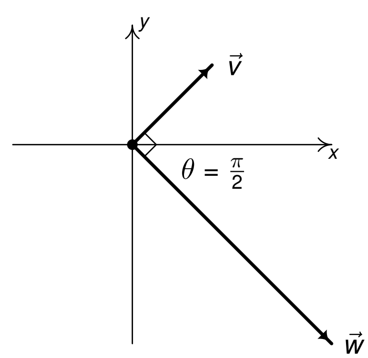
Solution. We use the formula $\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$ from Theorem 13.12 in each case below.

1. We have $\vec{v} \cdot \vec{w} = \left\langle 3, -3\sqrt{3} \right\rangle \cdot \left\langle -\sqrt{3}, 1 \right\rangle = -3\sqrt{3} - 3\sqrt{3} = -6\sqrt{3}$. Computing lengths of vectors, we find $\|\vec{v}\| = \sqrt{3^2 + (-3\sqrt{3})^2} = \sqrt{36} = 6$ and $\|\vec{w}\| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$. Hence, we find $\theta = \arccos\left(\frac{-6\sqrt{3}}{12}\right) = \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$. We check our answer geometrically by graphing this pair of vectors below on the left.

2. For $\vec{v} = \langle 2, 2 \rangle$ and $\vec{w} = \langle 5, -5 \rangle$, we find $\vec{v} \cdot \vec{w} = \langle 2, 2 \rangle \cdot \langle 5, -5 \rangle = 10 - 10 = 0$. Hence, it doesn't matter what $\|\vec{v}\|$ and $\|\vec{w}\|$ are, $\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos(0) = \frac{\pi}{2}$. We check our answer geometrically by graphing this pair of vectors below on the right.



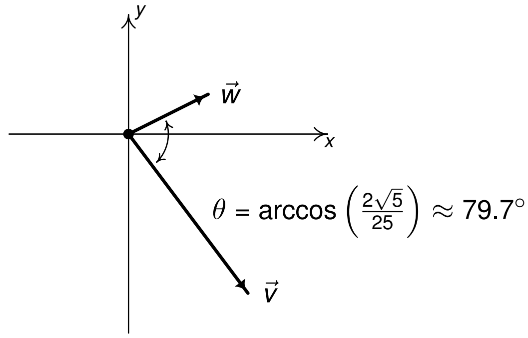
$\vec{v}$ and $\vec{w}$ from number 1



$\vec{v}$ and $\vec{w}$ from number 2

3. We find $\vec{v}\cdot\vec{w} = \langle 3, -4\rangle \cdot \langle 2, 1\rangle = 6-4 = 2$. Computing lengths, we find $\|\vec{v}\| = \sqrt{3^2+(-4)^2} = \sqrt{25} = 5$ and $\vec{w} = \sqrt{2^2+1^2} = \sqrt{5}$, so $\theta = \arccos\left(\frac{2}{5\sqrt{5}}\right) = \arccos\left(\frac{2\sqrt{5}}{25}\right)$.

   Since $\frac{2\sqrt{5}}{25}$ isn't the cosine of one of the 'common angles,' we leave our *exact* answer in terms of the arccosine function. For the purposes of checking our answer, however, we *approximate* $\theta \approx 79.7^\circ$.



$\vec{v}$ and $\vec{w}$ from number 3

□

A few remarks about Example 13.4.2 are in order. Note that for nonzero vectors $\vec{v}$ and $\vec{w}$, the lengths $\|\vec{v}\|$ and $\|\vec{w}\|$ are always positive. Since Theorem 13.11 tells us that $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta)$, we know the sign of $\vec{v}\cdot\vec{w}$ is the same as the sign of $\cos(\theta)$.

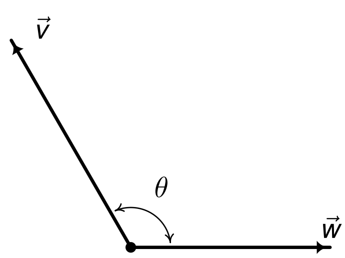
Geometrically, if $\vec{v}\cdot\vec{w} < 0$, then $\cos(\theta) < 0$ so θ is an obtuse angle, demonstrated number 1 above.

If $\vec{v}\cdot\vec{w} = 0$, then $\cos(\theta) = 0$ so $\theta = \frac{\pi}{2}$ as in number 2. In this case, the vectors $\vec{v}$ and $\vec{w}$ are called **orthogonal**. Geometrically, when orthogonal vectors are sketched with the same initial point, the lines containing the vectors are perpendicular. Hence, if $\vec{v}$ and $\vec{w}$ are orthogonal, we write $\vec{v} \perp \vec{w}$.

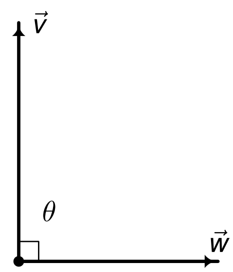
Note there is no 'zero product property' for the dot product. As with the vectors in number 2 above, it is quite possible to have $\vec{v}\cdot\vec{w} = 0$ but neither $\vec{v}$ nor $\vec{w}$ be $\vec{0}$.

Finally, if $\vec{v}\cdot\vec{w} > 0$, then $\cos(\theta) > 0$ so θ is an acute angle, as in the case of number 3 above.

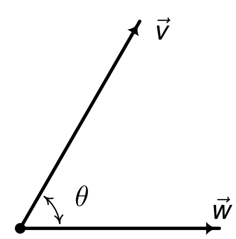
We summarize all of our observations in the schematic below.



$\vec{v}\cdot\vec{w} < 0$
θ is obtuse



$\vec{v}\cdot\vec{w} = 0$
$\theta = \frac{\pi}{2} = 90^\circ$



$\vec{v}\cdot\vec{w} > 0$
θ is acute

Of the three cases diagrammed above, the one which has the most mathematical significance moving forward is the orthogonal case. Hence, we state the corresponding theorem below.

Theorem 13.13. For nonzero vectors $\vec{v}$ and $\vec{w}$, $\vec{v} \perp \vec{w}$ if and only if $\vec{v} \cdot \vec{w} = 0$.

Basically, Theorem 13.13 tells us that 'the dot product detects orthogonality.' This is a helpful interpretation to keep in mind as you continue your study of vectors in later courses.

We have already argued one direction of Theorem 13.13, namely if $\vec{v} \cdot \vec{w} = 0$ then $\vec{v} \perp \vec{w}$ in the comments following Example 13.4.2.

To show the converse, we note if $\vec{v} \perp \vec{w}$, then the angle between $\vec{v}$ and $\vec{w}$, $\theta = \frac{\pi}{2}$. From Theorem 13.11, we have that $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\left(\frac{\pi}{2}\right) = \|\vec{v}\|\|\vec{w}\| \cdot (0) = 0$, as required.

We can use Theorem 13.13 in the following example to provide a different proof about the relationship between the slopes of perpendicular lines.[2]

Example 13.4.3. Let L_1 be the line $y = m_1 x + b_1$ and let L_2 be the line $y = m_2 x + b_2$. Prove that L_1 is perpendicular to L_2 if and only if $m_1 \cdot m_2 = -1$.

Solution. Our strategy is to find two vectors: $\vec{v}_1$, which has the same direction as L_1, and $\vec{v}_2$, which has the same direction as L_2 and show $\vec{v}_1 \perp \vec{v}_2$ if and only if $m_1 m_2 = -1$.

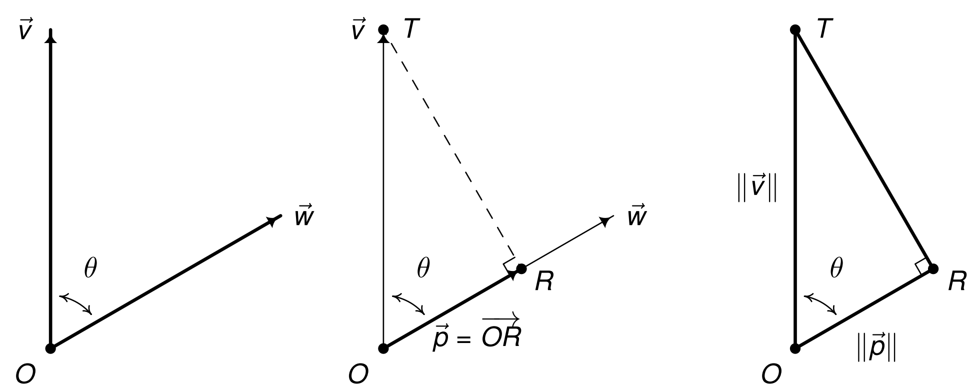
To that end, we substitute $x = 0$ and $x = 1$ into $y = m_1 x + b_1$ to find two points which lie on L_1, namely $P(0, b_1)$ and $Q(1, m_1 + b_1)$. We let $\vec{v}_1 = \overrightarrow{PQ} = \langle 1 - 0, (m_1 + b_1) - b_1 \rangle = \langle 1, m_1 \rangle$. Since $\vec{v}_1$ is determined by two points on L_1, it may be viewed as lying on L_1, so $\vec{v}_1$ has the same direction as L_1.

Similarly, we get the vector $\vec{v}_2 = \langle 1, m_2 \rangle$ which has the same direction as the line L_2. Hence, L_1 and L_2 are perpendicular if and only if $\vec{v}_1 \perp \vec{v}_2$. According to Theorem 13.13, $\vec{v}_1 \perp \vec{v}_2$ if and only if $\vec{v}_1 \cdot \vec{v}_2 = 0$.

Notice that $\vec{v}_1 \cdot \vec{v}_2 = \langle 1, m_1 \rangle \cdot \langle 1, m_2 \rangle = 1 + m_1 m_2$. Hence, $\vec{v}_1 \cdot \vec{v}_2 = 0$ if and only if $1 + m_1 m_2 = 0$, which is true if and only if $m_1 m_2 = -1$, as required. $\square$

13.4.1 Vector Projections

While Theorem 13.13 certainly gives us some insight into what the dot product means geometrically, there is more to the story of the dot product. Consider the two nonzero vectors $\vec{v}$ and $\vec{w}$ drawn with a common initial point O below. For the moment, assume that the angle between $\vec{v}$ and $\vec{w}$, θ, is acute.



[2] See Exercise 41 in Section A.5.

We wish to develop a formula for the vector $\vec{p}$, indicated below, which is called the **orthogonal projection of $\vec{v}$ onto $\vec{w}$**. The vector $\vec{p}$ is obtained geometrically as follows: drop a perpendicular from the terminal point T of $\vec{v}$ to the vector $\vec{w}$ and call the point of intersection R. The vector $\vec{p}$ is then defined as $\vec{p} = \overrightarrow{OR}$.

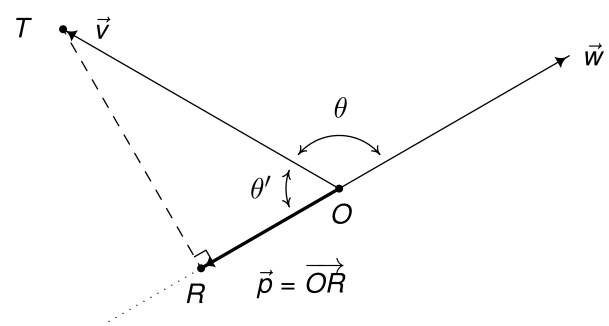
Like any vector, $\vec{p}$ is determined by its magnitude $\|\vec{p}\|$ and its direction $\hat{p}$ according to the formula $\vec{p} = \|\vec{p}\|\hat{p}$. Since we want $\hat{p}$ to have the same direction as $\vec{w}$, we have $\hat{p} = \hat{w}$.

To determine $\|\vec{p}\|$, we apply Definition B.1 to the right triangle $\triangle ORT$. We find $\cos(\theta) = \frac{\|\vec{p}\|}{\|\vec{v}\|}$, or, equivalently, $\|\vec{p}\| = \|\vec{v}\| \cos(\theta)$. Using Theorems 13.11 and 13.10, we get:

$$\|\vec{p}\| = \|\vec{v}\| \cos(\theta) = \frac{\|\vec{v}\|\|\vec{w}\| \cos(\theta)}{\|\vec{w}\|} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|} = \vec{v} \cdot \left(\frac{1}{\|\vec{w}\|} \vec{w} \right) = \vec{v} \cdot \hat{w}.$$

Hence, $\|\vec{p}\| = \vec{v} \cdot \hat{w}$, and since $\hat{p} = \hat{w}$, we have $\vec{p} = \|\vec{p}\|\hat{p} = (\vec{v} \cdot \hat{w})\hat{w}$.

Now suppose that the angle θ between $\vec{v}$ and $\vec{w}$ is obtuse, and consider the diagram below.



In this case, we see that $\hat{p} = -\hat{w}$ and using the triangle $\triangle ORT$, we find $\|\vec{p}\| = \|\vec{v}\| \cos(\theta')$. Since $\theta + \theta' = \pi$, it follows that $\cos(\theta') = -\cos(\theta)$, which means $\|\vec{p}\| = \|\vec{v}\| \cos(\theta') = -\|\vec{v}\| \cos(\theta)$.

Rewriting this last equation in terms of $\vec{v}$ and $\vec{w}$ as before, we get $\|\vec{p}\| = -(\vec{v} \cdot \hat{w})$. Putting this together with $\hat{p} = -\hat{w}$, we get $\vec{p} = \|\vec{p}\|\hat{p} = -(\vec{v} \cdot \hat{w})(-\hat{w}) = (\vec{v} \cdot \hat{w})\hat{w}$ in this case as well.

If the angle between $\vec{v}$ and $\vec{w}$ is $\frac{\pi}{2}$ then it is easy to show[3] that $\vec{p} = \vec{0}$. Since $\vec{v} \perp \vec{w}$ in this case, $\vec{v} \cdot \vec{w} = 0$. It follows that $\vec{v} \cdot \hat{w} = 0$ and $\vec{p} = \vec{0} = 0\hat{w} = (\vec{v} \cdot \hat{w})\hat{w}$ in this case, too. We have motivated the following.

Definition 13.8. Let $\vec{v}$ and $\vec{w}$ be nonzero vectors.

The **orthogonal projection of $\vec{v}$ onto $\vec{w}$**, denoted $\text{proj}_{\vec{w}}(\vec{v})$ is given by $\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \hat{w})\hat{w}$.

Definition 13.8 gives us a good idea what the dot product does. The scalar $\vec{v} \cdot \hat{w}$ is a measure of how much of the vector $\vec{v}$ is in the direction of the vector $\vec{w}$ and is thus called the **scalar projection** of $\vec{v}$ onto $\vec{w}$.

While the formula given in Definition 13.8 is theoretically appealing, because of the presence of the normalized unit vector $\hat{w}$, computing the projection using the formula $\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \hat{w})\hat{w}$ can be messy. We present two other formulas that are often used in practice.

[3] In this case, the point R coincides with the point O, so $\vec{p} = \overrightarrow{OR} = \overrightarrow{OO} = \vec{0}$.

Theorem 13.14. Alternate Formulas for Vector Projections: If $\vec{v}$ and $\vec{w}$ are nonzero vectors then

$$\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \hat{w})\hat{w} = \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\right)\vec{w} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}$$

The proof of Theorem 13.14, which we leave to the reader as an exercise, amounts to using the formula $\hat{w} = \left(\frac{1}{\|\vec{w}\|}\right)\vec{w}$ and properties of the dot product. It is time for an example.

Example 13.4.4. Let $\vec{v} = \langle 1, 8\rangle$ and $\vec{w} = \langle -1, 2\rangle$. Find $\vec{p} = \text{proj}_{\vec{w}}(\vec{v})$. Check your answer geometrically.

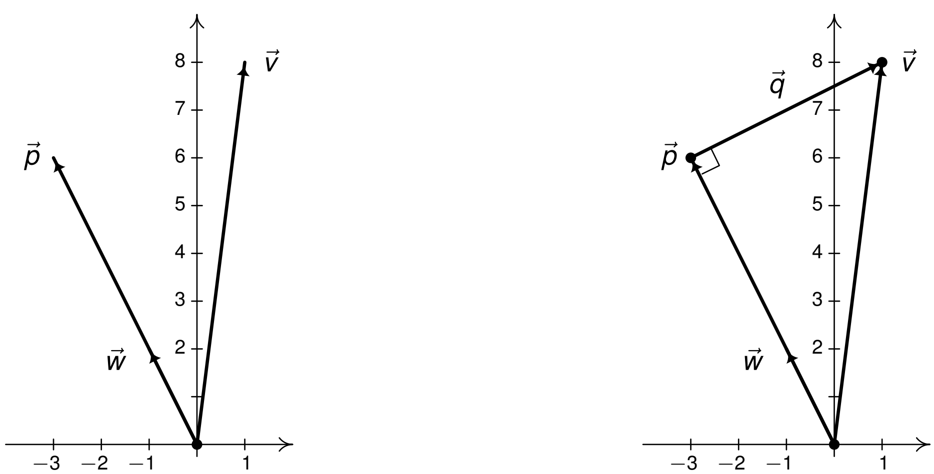
Solution. We find $\vec{v} \cdot \vec{w} = \langle 1, 8\rangle \cdot \langle -1, 2\rangle = (-1) + 16 = 15$ and $\vec{w} \cdot \vec{w} = \langle -1, 2\rangle \cdot \langle -1, 2\rangle = 1 + 4 = 5$. Hence,

$$\vec{p} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\vec{w} = \frac{15}{5}\langle -1, 2\rangle = \langle -3, 6\rangle.$$

We plot $\vec{v}$, $\vec{w}$ and $\vec{p}$ in standard position below on the left. We see $\vec{p}$ has the same direction as $\vec{w}$, but we need to do more to show $\vec{p}$ in is indeed the *orthogonal* projection of $\vec{v}$ onto $\vec{w}$.

Consider the vector $\vec{q}$ whose initial point is the terminal point of $\vec{p}$ and whose terminal point is the terminal point of $\vec{v}$. From the definition of vector arithmetic, $\vec{p} + \vec{q} = \vec{v}$, so that $\vec{q} = \vec{v} - \vec{p}$.

Since $\vec{v} = \langle 1, 8\rangle$ and $\vec{p} = \langle -3, 6\rangle$, $\vec{q} = \langle 1, 8\rangle - \langle -3, 6\rangle = \langle 4, 2\rangle$. To prove $\vec{q} \perp \vec{w}$, we compute the dot product: $\vec{q} \cdot \vec{w} = \langle 4, 2\rangle \cdot \langle -1, 2\rangle = (-4) + 4 = 0$. Hence, per Theorem 13.13, we know $\vec{q} \perp \vec{w}$ which completes our check.[4]



□

In Example 13.4.4 above, writing $\vec{v} = \vec{p} + \vec{q}$ is an example of what is called a **vector decomposition** of $\vec{v}$. We generalize this result in the following theorem.

Theorem 13.15. Generalized Decomposition Theorem: Let $\vec{v}$ and $\vec{w}$ be nonzero vectors. There are unique vectors $\vec{p}$ and $\vec{q}$ such that $\vec{v} = \vec{p} + \vec{q}$ where $\vec{p} = k\vec{w}$ for some scalar k, and $\vec{q} \cdot \vec{w} = 0$.

[4]Note that, necessarily, $\vec{q} \perp \vec{p}$ as well!

If the vectors $\vec{p}$ and $\vec{q}$ in Theorem 13.15 are nonzero, then we can say $\vec{p}$ is 'parallel'[5] to $\vec{w}$ and $\vec{q}$ is 'orthogonal' to $\vec{w}$. In this case, the vector $\vec{p}$ is sometimes called the 'vector component of $\vec{v}$ *parallel* to $\vec{w}$' and $\vec{q}$ is called the 'vector component of $\vec{v}$ *orthogonal* to $\vec{w}$.'

To prove Theorem 13.15, we take $\vec{p} = \text{proj}_{\vec{w}}(\vec{v})$ and $\vec{q} = \vec{v} - \vec{p}$. Then $\vec{p}$ is, by definition, a scalar multiple of $\vec{w}$. Next, we compute $\vec{q} \cdot \vec{w}$.

$$\begin{array}{rcll}
\vec{q} \cdot \vec{w} & = & (\vec{v} - \vec{p}) \cdot \vec{w} & \text{Definition of } \vec{q}. \\
& = & \vec{v} \cdot \vec{w} - \vec{p} \cdot \vec{w} & \text{Properties of Dot Product} \\
& = & \vec{v} \cdot \vec{w} - \left(\dfrac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} \right) \cdot \vec{w} & \text{Since } \vec{p} = \text{proj}_{\vec{w}}(\vec{v}). \\
& = & \vec{v} \cdot \vec{w} - \left(\dfrac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) (\vec{w} \cdot \vec{w}) & \text{Properties of Dot Product.} \\
& = & \vec{v} \cdot \vec{w} - \vec{v} \cdot \vec{w} & \\
& = & 0. &
\end{array}$$

Hence, $\vec{q} \cdot \vec{w} = 0$, as required. At this point, we have shown that the vectors $\vec{p}$ and $\vec{q}$ guaranteed by Theorem 13.15 *exist*. Now we need to show that they are *unique* - that is, there is only *one* such way to decompose $\vec{v}$ in the manner described in Theorem 13.15.

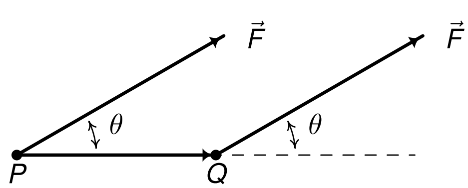
Suppose $\vec{v} = \vec{p} + \vec{q} = \vec{p}\,' + \vec{q}\,'$ where the vectors $\vec{p}\,'$ and $\vec{q}\,'$ satisfy the same properties described in Theorem 13.15 as $\vec{p}$ and $\vec{q}$. Then $\vec{p} - \vec{p}\,' = \vec{q}\,' - \vec{q}$, so $\vec{w} \cdot (\vec{p} - \vec{p}\,') = \vec{w} \cdot (\vec{q}\,' - \vec{q}) = \vec{w} \cdot \vec{q}\,' - \vec{w} \cdot \vec{q} = 0 - 0 = 0$. The long and short of this computation is that $\vec{w} \cdot (\vec{p} - \vec{p}\,') = 0$.

Now there are scalars k and k' so that $\vec{p} = k\vec{w}$ and $\vec{p}\,' = k'\vec{w}$. This means $\vec{w} \cdot (\vec{p} - \vec{p}\,') = \vec{w} \cdot (k\vec{w} - k'\vec{w}) = \vec{w} \cdot ([k - k']\vec{w}) = (k - k')(\vec{w} \cdot \vec{w}) = (k - k')\|\vec{w}\|^2$.

Since $\vec{w} \neq \vec{0}$, $\|\vec{w}\|^2 \neq 0$, which means the only way $\vec{w} \cdot (\vec{p} - \vec{p}\,') = (k - k')\|\vec{w}\|^2 = 0$ is for $k - k' = 0$, or $k = k'$. This means $\vec{p} = k\vec{w} = k'\vec{w} = \vec{p}\,'$. Since $\vec{q}\,' - \vec{q} = \vec{p} - \vec{p}\,' = \vec{p} - \vec{p} = \vec{0}$, it must be that $\vec{q}\,' = \vec{q}$ as well. Hence, we have shown there is only one way to write $\vec{v}$ as a sum of vectors as described in Theorem 13.15, so the decomposition listed there is unique.

We close this section with an application of the dot product. In Physics, if a constant force F is exerted over a distance d, the **work** W done by the force is given by $W = Fd$. Here, the assumption is that the force is being applied in the direction of the motion. If the force applied is not in the direction of the motion, we can use the dot product to find the work done.

Consider the scenario sketched below in which the constant force $\vec{F}$ is applied to move an object from the point P to the point Q. Here the force is being applied at an angle θ as opposed to being applied directly in the direction of the motion.



[5] See Exercise 64 in Section 13.3.

To find the work W done in this scenario, we need to find how much of the force $\vec{F}$ is in the direction of the motion $\overrightarrow{PQ}$. This is precisely what the dot product $\vec{F} \cdot \widehat{PQ}$ represents.

Since the distance the object travels is $\|\overrightarrow{PQ}\|$, we get $W = (\vec{F} \cdot \widehat{PQ})\|\overrightarrow{PQ}\|$. Since $\overrightarrow{PQ} = \|\overrightarrow{PQ}\|\widehat{PQ}$, we can simplify this formula as follows: $W = (\vec{F} \cdot \widehat{PQ})\|\overrightarrow{PQ}\| = \vec{F} \cdot (\|\overrightarrow{PQ}\|\widehat{PQ}) = \vec{F} \cdot \overrightarrow{PQ}$.

Using Theorem 13.11, we can rewrite $W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\|\|\overrightarrow{PQ}\|\cos(\theta)$, where θ is the angle between the applied force $\vec{F}$ and the trajectory of the motion $\overrightarrow{PQ}$. We have proved the following.

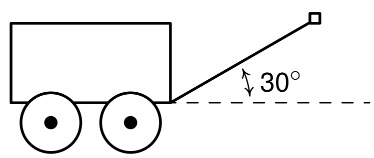
Theorem 13.16. Work as a Dot Product: Suppose a constant force $\vec{F}$ is applied along the vector $\overrightarrow{PQ}$. The work W done by $\vec{F}$ is given by

$$W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\|\|\overrightarrow{PQ}\|\cos(\theta),$$

where θ is the angle between $\vec{F}$ and $\overrightarrow{PQ}$.

We test out our formula for work in the following example.

Example 13.4.5. Taylor exerts a force of 10 pounds to pull her wagon a distance of 50 feet over level ground. If the handle of the wagon makes a 30° angle with the horizontal, how much work did Taylor do pulling the wagon? Assume the force of 10 pounds is exerted at a 30° angle for the duration of the 50 feet.



Solution. There are (at least) two ways to attack this problem. One way is to find the vectors $\vec{F}$ and $\overrightarrow{PQ}$ mentioned in Theorem 13.16 and compute $W = \vec{F} \cdot \overrightarrow{PQ}$.

To do this, we assume the origin is at the point where the handle of the wagon meets the wagon and the positive x-axis lies along the dashed line in the figure above.

To find the force vector $\vec{F}$, we note the force in this situation is a constant 10 pounds, so $\|\vec{F}\| = 10$. Moreover, the force is being applied at a constant angle of $\theta = 30^\circ$ with respect to the positive x-axis. Definition 13.4 gives us $\vec{F} = \|\vec{F}\| \langle\cos(\theta), \sin(\theta)\rangle = 10 \langle\cos(30^\circ, \sin(30^\circ)\rangle = \left\langle 5\sqrt{3}, 5\right\rangle$.

Since the wagon is being pulled along 50 feet in the positive x-direction, we find the displacement vector is $\overrightarrow{PQ} = 50\hat{\imath} = 50\langle 1, 0\rangle = \langle 50, 0\rangle$.

Per Theorem 13.16, $W = \vec{F} \cdot \overrightarrow{PQ} = \left\langle 5\sqrt{3}, 5\right\rangle \cdot \langle 50, 0\rangle = 250\sqrt{3}$. Since force is measured in pounds and distance is measured in feet, we get $W = 250\sqrt{3}$ foot-pounds.

Alternatively, we can use the formula $W = \|\vec{F}\|\|\overrightarrow{PQ}\|\cos(\theta)$. With $\|\vec{F}\| = 10$ pounds, $\|\overrightarrow{PQ}\| = 50$ feet and $\theta = 30^\circ$, we get $W = (10\,\text{pounds})(50\,\text{feet})\cos(30^\circ) = 250\sqrt{3}$ foot-pounds of work. $\square$

13.4.2 Exercises

In Exercises 1 - 20, use the pair of vectors $\vec{v}$ and $\vec{w}$ to find the following quantities.

- $\vec{v} \cdot \vec{w}$
- $\text{proj}_{\vec{w}}(\vec{v})$
- The angle θ (in degrees) between $\vec{v}$ and $\vec{w}$
- $\vec{q} = \vec{v} - \text{proj}_{\vec{w}}(\vec{v})$ (Show that $\vec{q} \cdot \vec{w} = 0$.)

1. $\vec{v} = \langle -2, -7 \rangle$ and $\vec{w} = \langle 5, -9 \rangle$
2. $\vec{v} = \langle -6, -5 \rangle$ and $\vec{w} = \langle 10, -12 \rangle$
3. $\vec{v} = \left\langle 1, \sqrt{3} \right\rangle$ and $\vec{w} = \left\langle 1, -\sqrt{3} \right\rangle$
4. $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle -6, -8 \rangle$
5. $\vec{v} = \langle -2, 1 \rangle$ and $\vec{w} = \langle 3, 6 \rangle$
6. $\vec{v} = \left\langle -3\sqrt{3}, 3 \right\rangle$ and $\vec{w} = \left\langle -\sqrt{3}, -1 \right\rangle$
7. $\vec{v} = \langle 1, 17 \rangle$ and $\vec{w} = \langle -1, 0 \rangle$
8. $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 5, 12 \rangle$
9. $\vec{v} = \langle -4, -2 \rangle$ and $\vec{w} = \langle 1, -5 \rangle$
10. $\vec{v} = \langle -5, 6 \rangle$ and $\vec{w} = \langle 4, -7 \rangle$
11. $\vec{v} = \langle -8, 3 \rangle$ and $\vec{w} = \langle 2, 6 \rangle$
12. $\vec{v} = \langle 34, -91 \rangle$ and $\vec{w} = \langle 0, 1 \rangle$
13. $\vec{v} = 3\hat{\imath} - \hat{\jmath}$ and $\vec{w} = 4\hat{\jmath}$
14. $\vec{v} = -24\hat{\imath} + 7\hat{\jmath}$ and $\vec{w} = 2\hat{\imath}$
15. $\vec{v} = \frac{3}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath}$ and $\vec{w} = \hat{\imath} - \hat{\jmath}$
16. $\vec{v} = 5\hat{\imath} + 12\hat{\jmath}$ and $\vec{w} = -3\hat{\imath} + 4\hat{\jmath}$
17. $\vec{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$
18. $\vec{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ and $\vec{w} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$
19. $\vec{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
20. $\vec{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$ and $\vec{w} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
21. A force of 1500 pounds is required to tow a trailer. Find the work done towing the trailer along a flat stretch of road 300 feet. Assume the force is applied in the direction of the motion.
22. Find the work done lifting a 10 pound book 3 feet straight up into the air. Assume the force of gravity is acting straight downwards.
23. Suppose Taylor fills her wagon with rocks and must exert a force of 13 pounds to pull her wagon across the yard. If she maintains a 15° angle between the handle of the wagon and the horizontal, compute how much work Taylor does pulling her wagon 25 feet. Round your answer to two decimal places.
24. In Exercise 61 in Section 13.3, two drunken college students have filled an empty beer keg with rocks which they drag down the street by pulling on two attached ropes. The stronger of the two students pulls with a force of 100 pounds on a rope which makes a 13° angle with the direction of motion. (In this case, the keg was being pulled due east and the student's heading was N77°E.) Find the work done by this student if the keg is dragged 42 feet.

25. Find the work done pushing a 200 pound barrel 10 feet up a 12.5° incline. Ignore all forces acting on the barrel except gravity, which acts downwards. Round your answer to two decimal places.

    HINT: Since you are working to overcome gravity only, the force being applied acts directly upwards. This means that the angle between the applied force in this case and the motion of the object is *not* the 12.5° of the incline!

26. Prove the distributive property of the dot product in Theorem 13.10.

27. Finish the proof of the scalar property of the dot product in Theorem 13.10.

28. Show Theorem 13.15 reduces to Theorem 13.9 in the case $\vec{w} = \hat{\imath}$.

29. Use the identity in Example 13.4.1 to prove the **Parallelogram Law**

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 = \frac{1}{2}\left[\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2\right]$$

30. We know that $|x+y| \leq |x| + |y|$ for all real numbers x and y by the Triangle Inequality established in Exercise 55 in Section 1.3. We can now establish a Triangle Inequality for vectors. In this exercise, we prove that $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ for all pairs of vectors $\vec{u}$ and $\vec{v}$.

    (a) (Step 1) Show that $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$.

    (b) (Step 2) Show that $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$. This is the celebrated Cauchy-Schwarz Inequality.[6]

    HINT: Start with $|\vec{u}\cdot\vec{v}| = \big|\, \|\vec{u}\|\|\vec{v}\|\cos(\theta)\,\big|$ and use the fact that $|\cos(\theta)| \leq 1$ for all θ.

    (c) (Step 3) Show:

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \leq \|\vec{u}\|^2 + 2|\vec{u}\cdot\vec{v}| + \|\vec{v}\|^2 \leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 = (\|\vec{u}\| + \|\vec{v}\|)^2.$$

    (d) (Step 4) Use Step 3 to show that $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ for all pairs of vectors $\vec{u}$ and $\vec{v}$.

[6] It is also known by other names. Check out this site for details.

13.4.3 Answers

1. $\vec{v} = \langle -2, -7 \rangle$ and $\vec{w} = \langle 5, -9 \rangle$
   $\vec{v} \cdot \vec{w} = 53$
   $\theta = 45^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{5}{2}, -\frac{9}{2} \right\rangle$
   $\vec{q} = \left\langle -\frac{9}{2}, -\frac{5}{2} \right\rangle$

2. $\vec{v} = \langle -6, -5 \rangle$ and $\vec{w} = \langle 10, -12 \rangle$
   $\vec{v} \cdot \vec{w} = 0$
   $\theta = 90^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, 0 \rangle$
   $\vec{q} = \langle -6, -5 \rangle$

3. $\vec{v} = \left\langle 1, \sqrt{3} \right\rangle$ and $\vec{w} = \left\langle 1, -\sqrt{3} \right\rangle$
   $\vec{v} \cdot \vec{w} = -2$
   $\theta = 120^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$
   $\vec{q} = \left\langle \frac{3}{2}, \frac{\sqrt{3}}{2} \right\rangle$

4. $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle -6, -8 \rangle$
   $\vec{v} \cdot \vec{w} = -50$
   $\theta = 180^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \langle 3, 4 \rangle$
   $\vec{q} = \langle 0, 0 \rangle$

5. $\vec{v} = \langle -2, 1 \rangle$ and $\vec{w} = \langle 3, 6 \rangle$
   $\vec{v} \cdot \vec{w} = 0$
   $\theta = 90^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, 0 \rangle$
   $\vec{q} = \langle -2, 1 \rangle$

6. $\vec{v} = \left\langle -3\sqrt{3}, 3 \right\rangle$ and $\vec{w} = \left\langle -\sqrt{3}, -1 \right\rangle$
   $\vec{v} \cdot \vec{w} = 6$
   $\theta = 60^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{3\sqrt{3}}{2}, -\frac{3}{2} \right\rangle$
   $\vec{q} = \left\langle -\frac{3\sqrt{3}}{2}, \frac{9}{2} \right\rangle$

7. $\vec{v} = \langle 1, 17 \rangle$ and $\vec{w} = \langle -1, 0 \rangle$
   $\vec{v} \cdot \vec{w} = -1$
   $\theta \approx 93.37^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \langle 1, 0 \rangle$
   $\vec{q} = \langle 0, 17 \rangle$

8. $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 5, 12 \rangle$
   $\vec{v} \cdot \vec{w} = 63$
   $\theta \approx 14.25^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{315}{169}, \frac{756}{169} \right\rangle$
   $\vec{q} = \left\langle \frac{192}{169}, -\frac{80}{169} \right\rangle$

9. $\vec{v} = \langle -4, -2 \rangle$ and $\vec{w} = \langle 1, -5 \rangle$
   $\vec{v} \cdot \vec{w} = 6$
   $\theta \approx 74.74^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{3}{13}, -\frac{15}{13} \right\rangle$
   $\vec{q} = \left\langle -\frac{55}{13}, -\frac{11}{13} \right\rangle$

10. $\vec{v} = \langle -5, 6 \rangle$ and $\vec{w} = \langle 4, -7 \rangle$
    $\vec{v} \cdot \vec{w} = -62$
    $\theta \approx 169.94^\circ$
    $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{248}{65}, \frac{434}{65} \right\rangle$
    $\vec{q} = \left\langle -\frac{77}{65}, -\frac{44}{65} \right\rangle$

11. $\vec{v} = \langle -8, 3 \rangle$ and $\vec{w} = \langle 2, 6 \rangle$
$\vec{v} \cdot \vec{w} = 2$
$\theta \approx 87.88^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{1}{10}, \frac{3}{10} \right\rangle$
$\vec{q} = \left\langle -\frac{81}{10}, \frac{27}{10} \right\rangle$

12. $\vec{v} = \langle 34, -91 \rangle$ and $\vec{w} = \langle 0, 1 \rangle$
$\vec{v} \cdot \vec{w} = -91$
$\theta \approx 159.51^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, -91 \rangle$
$\vec{q} = \langle 34, 0 \rangle$

13. $\vec{v} = 3\hat{\imath} - \hat{\jmath}$ and $\vec{w} = 4\hat{\jmath}$
$\vec{v} \cdot \vec{w} = -4$
$\theta \approx 108.43^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, -1 \rangle$
$\vec{q} = \langle 3, 0 \rangle$

14. $\vec{v} = -24\hat{\imath} + 7\hat{\jmath}$ and $\vec{w} = 2\hat{\imath}$
$\vec{v} \cdot \vec{w} = -48$
$\theta \approx 163.74^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \langle -24, 0 \rangle$
$\vec{q} = \langle 0, 7 \rangle$

15. $\vec{v} = \frac{3}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath}$ and $\vec{w} = \hat{\imath} - \hat{\jmath}$
$\vec{v} \cdot \vec{w} = 0$
$\theta = 90^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, 0 \rangle$
$\vec{q} = \left\langle \frac{3}{2}, \frac{3}{2} \right\rangle$

16. $\vec{v} = 5\hat{\imath} + 12\hat{\jmath}$ and $\vec{w} = -3\hat{\imath} + 4\hat{\jmath}$
$\vec{v} \cdot \vec{w} = 33$
$\theta \approx 59.49^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{99}{25}, \frac{132}{25} \right\rangle$
$\vec{q} = \left\langle \frac{224}{25}, \frac{168}{25} \right\rangle$

17. $\vec{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$
$\vec{v} \cdot \vec{w} = \frac{\sqrt{6}-\sqrt{2}}{4}$
$\theta = 75^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{1-\sqrt{3}}{4}, \frac{\sqrt{3}-1}{4} \right\rangle$
$\vec{q} = \left\langle \frac{1+\sqrt{3}}{4}, \frac{1+\sqrt{3}}{4} \right\rangle$

18. $\vec{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ and $\vec{w} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$
$\vec{v} \cdot \vec{w} = \frac{\sqrt{2}-\sqrt{6}}{4}$
$\theta = 105^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{\sqrt{2}-\sqrt{6}}{8}, \frac{3\sqrt{2}-\sqrt{6}}{8} \right\rangle$
$\vec{q} = \left\langle \frac{3\sqrt{2}+\sqrt{6}}{8}, \frac{\sqrt{2}+\sqrt{6}}{8} \right\rangle$

19. $\vec{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
$\vec{v} \cdot \vec{w} = -\frac{\sqrt{6}+\sqrt{2}}{4}$
$\theta = 165^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{\sqrt{3}+1}{4}, \frac{\sqrt{3}+1}{4} \right\rangle$
$\vec{q} = \left\langle \frac{\sqrt{3}-1}{4}, \frac{1-\sqrt{3}}{4} \right\rangle$

20. $\vec{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$ and $\vec{w} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
$\vec{v} \cdot \vec{w} = \frac{\sqrt{6}+\sqrt{2}}{4}$
$\theta = 15^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{\sqrt{3}+1}{4}, -\frac{\sqrt{3}+1}{4} \right\rangle$
$\vec{q} = \left\langle \frac{1-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4} \right\rangle$

21. $(1500 \text{ pounds})(300 \text{ feet}) \cos(0^\circ) = 450,000$ foot-pounds

22. $(10 \text{ pounds})(3 \text{ feet}) \cos(0^\circ) = 30$ foot-pounds

23. (13 pounds)(25 feet) $\cos(15^\circ) \approx 313.92$ foot-pounds

24. (100 pounds)(42 feet) $\cos(13^\circ) \approx 4092.35$ foot-pounds

25. (200 pounds)(10 feet) $\cos(77.5^\circ) \approx 432.88$ foot-pounds

Chapter 14

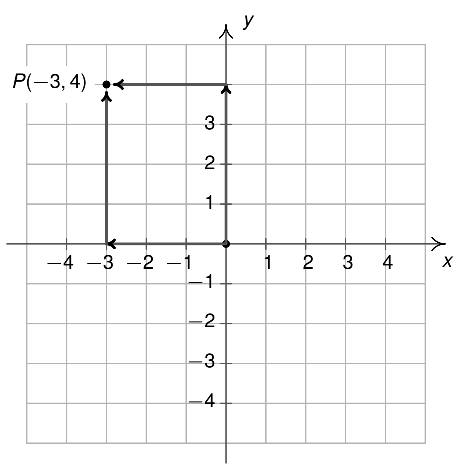
Polar Coordinates and Parametric Equations

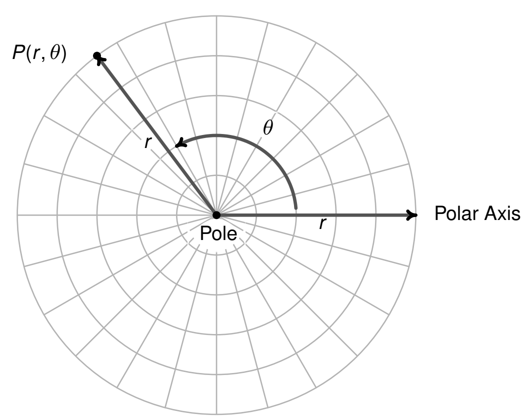
14.1 Polar Coordinates

In Section A.3, we introduced the notion of assigning ordered pairs of real numbers called 'coordinates' to points in the plane. Recall the Cartesian coordinate plane is defined using two number lines – one horizontal and one vertical – which intersect at right angles at a point called the 'origin'.

As seen below on the left, to plot a point with Cartesian coordinates, say $P(-3, 4)$, we start at the origin, travel horizontally to the left 3 units, then up 4 units. Alternatively, we could start at the origin, travel up 4 units, then to the left 3 units and arrive at the same location.

For the most part, the 'motions' of the Cartesian system (over and up) describe a rectangle, and most points can be thought of as the corner diagonally across the rectangle from the origin.[1] For this reason, the Cartesian coordinates of a point are often called 'rectangular' coordinates.





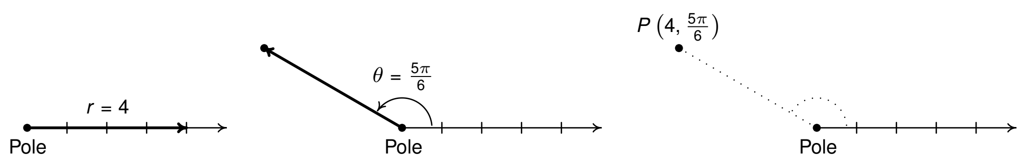
[1]Excluding, of course, the points in which one or both coordinates are 0.

In this section, we introduce a new system for assigning coordinates to points in the plane – **polar coordinates** as diagrammed above on the right. We start with an origin point, called the **pole**, and a ray called the **polar axis**.

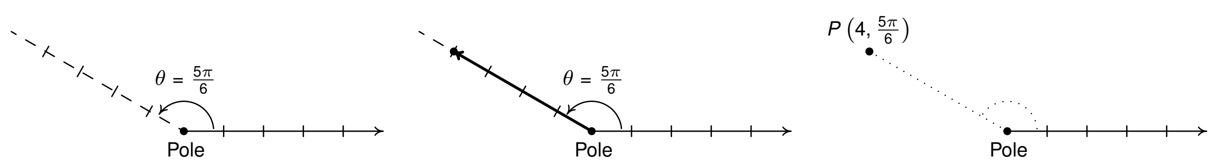
We locate a point P using two coordinates, (r, θ), where r represents a *directed* distance from the pole[2] and θ is a measure of counter-clockwise rotation from the polar axis.

Roughly speaking, the polar coordinates (r, θ) of a point measure 'how far out' the point is from the pole (that's r), and 'how far to rotate' from the polar axis, (that's θ).

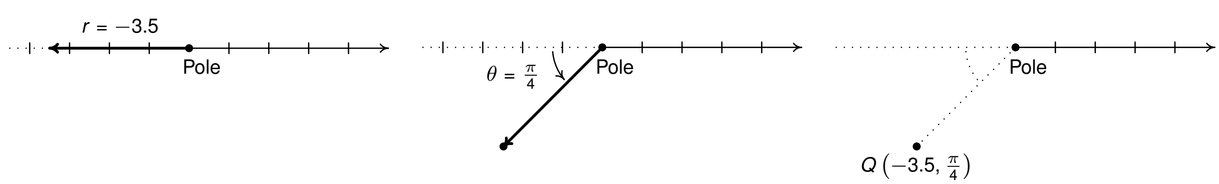
For example, if we wished to plot the point P with polar coordinates $\left(4, \frac{5\pi}{6}\right)$, we'd start at the pole, move out along the polar axis 4 units, then rotate $\frac{5\pi}{6}$ radians counter-clockwise.



We may also visualize this process by thinking of the rotation first.[3] To plot $P\left(4, \frac{5\pi}{6}\right)$ this way, we rotate $\frac{5\pi}{6}$ counter-clockwise from the polar axis, then move outwards from the pole 4 units. Essentially we are locating a point on the terminal side of $\frac{5\pi}{6}$ which is 4 units away from the pole.



If $r < 0$, we begin by moving in the opposite direction on the polar axis from the pole. For example, to plot the point with polar coordinates $Q\left(-3.5, \frac{\pi}{4}\right)$ we have



If we interpret the angle first, we rotate $\frac{\pi}{4}$ radians, then move back through the pole 3.5 units. Here we are locating a point 3.5 units away from the pole on the terminal side of $\frac{5\pi}{4}$, not $\frac{\pi}{4}$.

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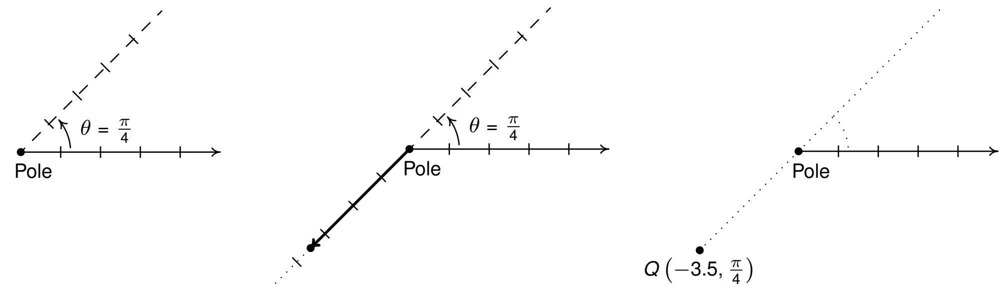
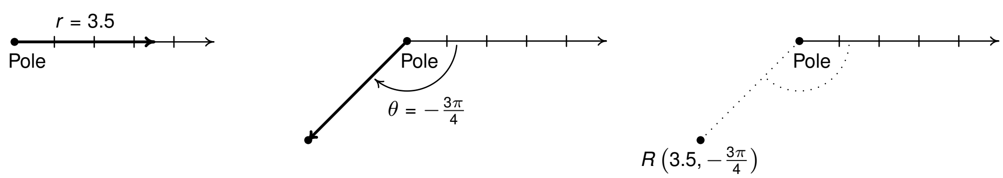
[2] We will explain more about this momentarily.

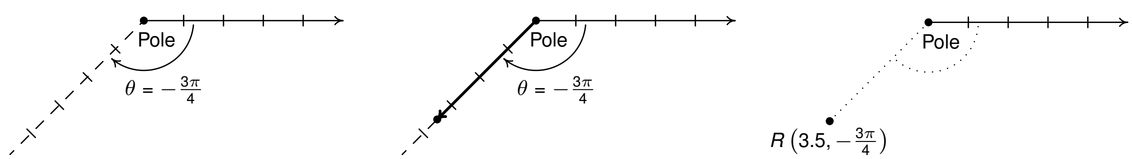
[3] As with anything in Mathematics, the more ways you have to look at something, the better. The authors encourage the reader to take time to think about both approaches to plotting points given in polar coordinates.



As you may have guessed, $\theta < 0$ means the rotation away from the polar axis is clockwise instead of counter-clockwise. Hence, to plot $R\left(3.5, -\frac{3\pi}{4}\right)$ we have the following.



From an 'angles first' approach, we rotate $-\frac{3\pi}{4}$ then move out 3.5 units from the pole. We see that R is the point on the terminal side of $\theta = -\frac{3\pi}{4}$ which is 3.5 units from the pole.



The points Q and R above are, in fact, the same point despite the fact that their polar coordinate representations are different. Unlike Cartesian coordinates where (a, b) and (c, d) represent the same point if and only if $a = c$ and $b = d$, a point can be represented by infinitely many polar coordinate pairs.

We explore this notion more in the following example.

Example 14.1.1. For each point in polar coordinates given below plot the point and then give two additional expressions for the point, one of which has $r > 0$ and the other with $r < 0$.

1. $P(2, 240^\circ)$
2. $P\left(-4, \frac{7\pi}{6}\right)$
3. $P\left(117, -\frac{5\pi}{2}\right)$
4. $P\left(-3, -\frac{\pi}{4}\right)$

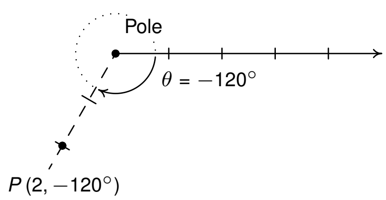
Solution.

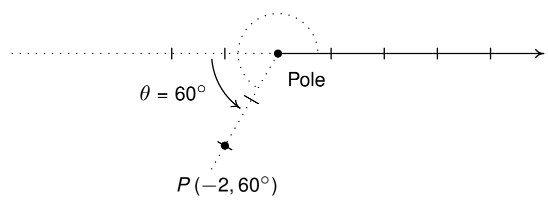
1. Whether we move 2 units along the polar axis and then rotate 240° or rotate 240° then move out 2 units from the pole, we plot $P(2, 240^\circ)$ below.

We now set about finding alternate descriptions (r, θ) for the point P. Since P is 2 units from the pole, $r = \pm 2$. Next, we choose angles θ for each of the r values.

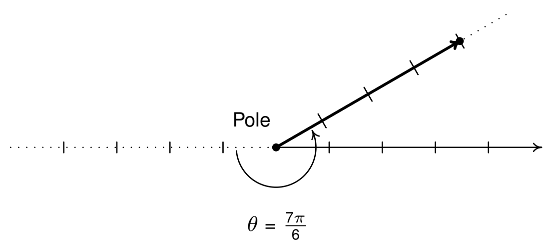
The given representation for P is $(2, 240^\circ)$ so the angle θ we choose for the $r = 2$ case must be coterminal with 240°. (Can you see why?) We choose $\theta = -120^\circ$, so one answer is $(2, -120^\circ)$.

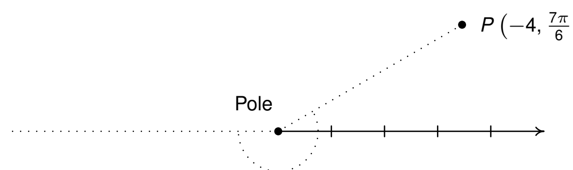
For the case $r = -2$, we visualize our rotation starting 2 units to the left of the pole. From this position, we need only to rotate $\theta = 60^\circ$ to arrive at location coterminal with 240°. Hence, our answer here is $(-2, 60^\circ)$. We check our answers by plotting them.





2. We plot $\left(-4, \frac{7\pi}{6}\right)$ by first moving 4 units to the left of the pole and then rotating $\frac{7\pi}{6}$ radians. Since $r = -4 < 0$, we find our point lies 4 units from the pole on the terminal side of $\frac{\pi}{6}$.

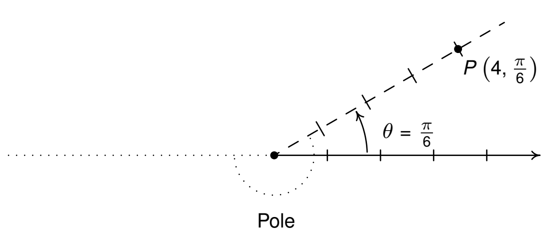


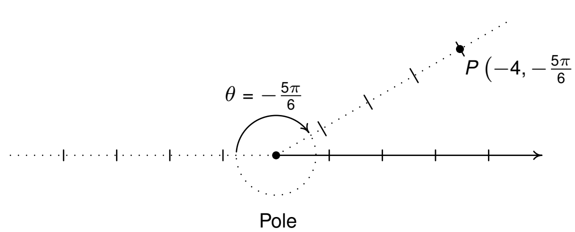


To find alternate descriptions for P, we note that the distance from P to the pole is 4 units, so any representation (r, θ) for P must have $r = \pm 4$.

As noted above, P lies on the terminal side of $\frac{\pi}{6}$, so this, coupled with $r = 4$, gives us $\left(4, \frac{\pi}{6}\right)$ as one of our answers.

To find a different representation for P with $r = -4$, we may choose any angle coterminal with the angle $\theta = \frac{7\pi}{6}$. We pick $-\frac{5\pi}{6}$ and get $\left(-4, -\frac{5\pi}{6}\right)$ as our second answer.



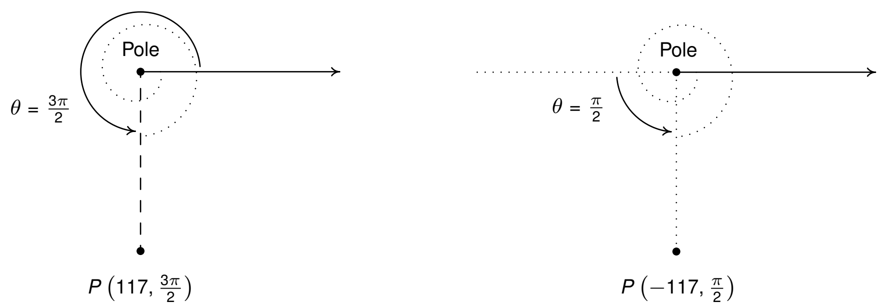


3. To plot $P\left(117, -\frac{5\pi}{2}\right)$, we move along the polar axis 117 units from the pole and rotate *clockwise* $\frac{5\pi}{2}$ radians as illustrated below.

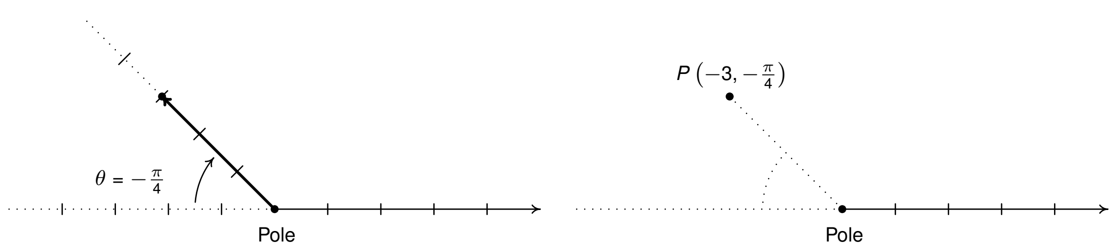
Since P is 117 units from the pole, any representation (r, θ) for P satisfies $r = \pm 117$.

For the $r = 117$ case, we can take θ to be any angle coterminal with $-\frac{5\pi}{2}$. In this case, we choose $\theta = \frac{3\pi}{2}$, and get $\left(117, \frac{3\pi}{2}\right)$ as one answer.

For the $r = -117$ case, we visualize moving left 117 units from the pole and then rotating through an angle θ to reach P. We find that $\theta = \frac{\pi}{2}$ works here, so our second answer is $\left(-117, \frac{\pi}{2}\right)$.

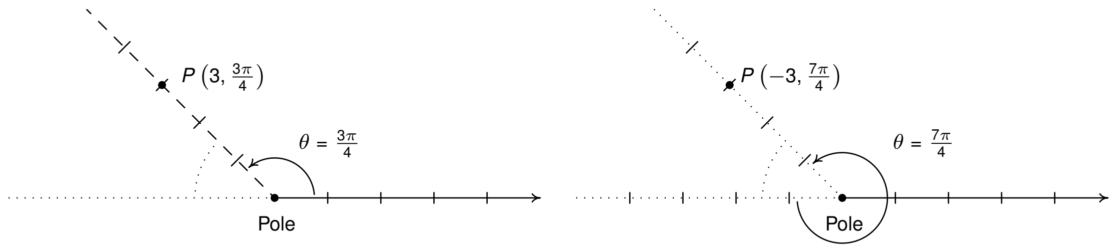


4. We move three units to the left of the pole and follow up with a clockwise rotation of $\frac{\pi}{4}$ radians to plot $P\left(-3, -\frac{\pi}{4}\right)$. We see that P lies on the terminal side of $\frac{3\pi}{4}$.



Since P lies on the terminal side of $\frac{3\pi}{4}$, one alternative representation for P is $\left(3, \frac{3\pi}{4}\right)$.

To find a different representation for P with $r = -3$, we may choose any angle coterminal with $-\frac{\pi}{4}$. We choose $\theta = \frac{7\pi}{4}$ for our final answer $\left(-3, \frac{7\pi}{4}\right)$.



□

In light of our work in Example 14.1.1, it should come as no surprise that any given point expressed in polar coordinates has infinitely many other representations in polar coordinates.

The following result characterizes when two sets of polar coordinates determine the same point in the plane. It could be considered as a definition or a theorem, depending on your point of view. We choose to state it as a property of the polar coordinate system.

Equivalent Representations of Points in Polar Coordinates

Suppose (r, θ) and $\left(r', \theta'\right)$ are polar coordinates where $r \neq 0$, $r' \neq 0$ and the angles are measured in radians. Then (r, θ) and $\left(r', \theta'\right)$ determine the same point P if and only if one of the following is true:

- $r' = r$ and $\theta' = \theta + 2\pi k$ for some integer k
- $r' = -r$ and $\theta' = \theta + (2k + 1)\pi$ for some integer k

All polar coordinates of the form $(0, \theta)$ represent the pole regardless of the value of θ.

The key to understanding this result, and indeed the whole polar coordinate system, is to keep in mind that (r, θ) means (directed distance from pole, angle of rotation).

If $r = 0$, then no matter how much rotation is performed, the point never leaves the pole. Thus $(0, \theta)$ is the pole for all values of θ.

Now let's assume that neither r nor r' is zero. If (r, θ) and $\left(r', \theta'\right)$ determine the same point P then the (non-zero) distance from P to the pole in each case must be the same. Since this distance is controlled by the first coordinate, we have that either $r' = r$ or $r' = -r$.

If $r' = r$, then when plotting (r, θ) and $\left(r', \theta'\right)$, the angles θ and θ' have the same initial side. Hence, if (r, θ) and $\left(r', \theta'\right)$ determine the same point, we must have that θ' is coterminal with θ. We know that this means $\theta' = \theta + 2\pi k$ for some integer k, as required.

If, on the other hand, $r' = -r$, then when plotting (r, θ) and $\left(r', \theta'\right)$, the initial side of θ' is rotated π radians away from the initial side of θ. In this case, θ' must be coterminal with $\pi + \theta$. Hence, $\theta' = \pi + \theta + 2\pi k$ which we rewrite as $\theta' = \theta + (2k + 1)\pi$ for some integer k.

Conversely, if $r' = r$ and $\theta' = \theta + 2\pi k$ for some integer k, then the points $P(r, \theta)$ and $P'\left(r', \theta'\right)$ lie the same (directed) distance from the pole on the terminal sides of coterminal angles, and hence are the same point.

Now suppose $r' = -r$ and $\theta' = \theta + (2k+1)\pi$ for some integer k. To plot P, we first move a directed distance r from the pole; to plot P', our first step is to move the same distance from the pole as P, but in the opposite direction. At this intermediate stage, we have two points equidistant from the pole rotated exactly π radians apart. Since $\theta' = \theta + (2k+1)\pi = (\theta + \pi) + 2\pi k$ for some integer k, we see that θ' is coterminal to $(\theta + \pi)$ and it is this extra π radians of rotation which aligns the points P and P'.

Next, we marry the polar coordinate system with the Cartesian (rectangular) coordinate system. To do so, we identify the pole and polar axis in the polar system to the origin and positive x-axis, respectively, in the rectangular system. We get the following result.

Theorem 14.1. Conversion Between Rectangular and Polar Coordinates:
Suppose P is represented in rectangular coordinates as (x, y) and in polar coordinates as (r, θ). Then

- $x = r\cos(\theta)$ and $y = r\sin(\theta)$
- $x^2 + y^2 = r^2$ and $\tan(\theta) = \dfrac{y}{x}$ (provided $x \neq 0$)

In the case $r > 0$, Theorem 14.1 is an immediate consequence of Theorems 11.3 and 11.9.

If $r < 0$, then we know an alternate representation for (r, θ) is $(-r, \theta + \pi)$. Since in this case, $-r > 0$, we know the theorem as stated is true for the representation $(-r, \theta + \pi)$ so we apply it here.

Applying Theorem 14.1 to $(-r, \theta + \pi)$ gives[4] $x = (-r)\cos(\theta + \pi) = (-r)(-\cos(\theta)) = r\cos(\theta)$ as well as $y = (-r)\sin(\theta + \pi) = (-r)(-\sin(\theta)) = r\sin(\theta)$.

Moreover, $x^2 + y^2 = (-r)^2 = r^2$, and $\frac{y}{x} = \tan(\theta + \pi) = \tan(\theta)$, so the theorem is true in this case, too.

The remaining case is $r = 0$, in which case $(r, \theta) = (0, \theta)$ is the pole. Since the pole is identified with the origin $(0, 0)$ in rectangular coordinates, the theorem in this case amounts to checking '0 = 0.'

Since we have argued that Theorem 14.1 is true in all cases, we put it to good use in the following example.

Example 14.1.2. Convert each point in rectangular coordinates given below into polar coordinates with $r \geq 0$ and $0 \leq \theta < 2\pi$. Use exact values if possible and round any approximate values to two decimal places. Check your answer by converting them back to rectangular coordinates.

1. $P\left(2, -2\sqrt{3}\right)$
2. $Q(-3, -3)$
3. $R(0, -3)$
4. $S(-3, 4)$

Solution.
Even though we are not explicitly told to do so, we can avoid many common mistakes by taking the time to plot the points before we do any calculations.

1. Plotting $P\left(2, -2\sqrt{3}\right)$, we find P lies in Quadrant IV. With $x = 2$ and $y = -2\sqrt{3}$, we calculate $r^2 = x^2 + y^2 = (2)^2 + \left(-2\sqrt{3}\right)^2 = 4 + 12 = 16$ so $r = \pm 4$. To satisfy $r \geq 0$, we choose $r = 4$.

   To find θ, know $\tan(\theta) = \frac{y}{x} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$. This tells us θ has a reference angle of $\frac{\pi}{3}$, and since P lies in Quadrant IV, we know θ is a Quadrant IV angle. To satisfy the stipulation that $0 \leq \theta < 2\pi$, we choose $\theta = \frac{5\pi}{3}$. Hence, our answer is $\left(4, \frac{5\pi}{3}\right)$.

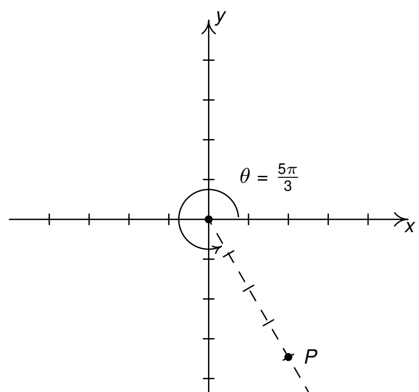
[4] Well, Theorem 14.1 along with the identities $\cos(\theta + \pi) = -\cos(\theta)$ and $\sin(\theta + \pi) = -\sin(\theta)$ ...

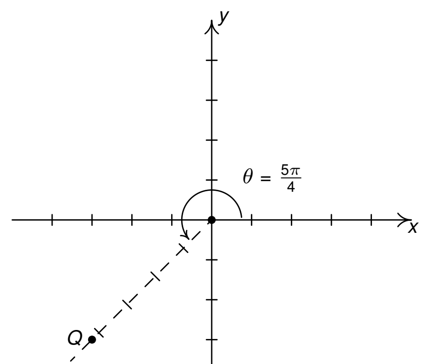
To check, we convert the polar representation $(r, \theta) = \left(4, \frac{5\pi}{3}\right)$ back to rectangular coordinates. We find $x = r\cos(\theta) = 4\cos\left(\frac{5\pi}{3}\right) = 4\left(\frac{1}{2}\right) = 2$ and $y = r\sin(\theta) = 4\sin\left(\frac{5\pi}{3}\right) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$.

2. The point $Q(-3, -3)$ lies in Quadrant III. Using $x = y = -3$, we get $r^2 = (-3)^2 + (-3)^2 = 18$ so $r = \pm\sqrt{18} = \pm 3\sqrt{2}$. To satisfy $r \geq 0$, we choose $r = 3\sqrt{2}$.

   We find $\tan(\theta) = \frac{-3}{-3} = 1$, which means θ has a reference angle of $\frac{\pi}{4}$. Since Q lies in Quadrant III, we choose $\theta = \frac{5\pi}{4}$, to satisfy the requirement that $0 \leq \theta < 2\pi$. Our final answer is $\left(3\sqrt{2}, \frac{5\pi}{4}\right)$.

   Checking our answer, we find that $x = r\cos(\theta) = (3\sqrt{2})\cos\left(\frac{5\pi}{4}\right) = (3\sqrt{2})\left(-\frac{\sqrt{2}}{2}\right) = -3$ and compute $y = r\sin(\theta) = (3\sqrt{2})\sin\left(\frac{5\pi}{4}\right) = (3\sqrt{2})\left(-\frac{\sqrt{2}}{2}\right) = -3$, so we are done.





P has rectangular coordinates $(2, -2\sqrt{3})$ Q has rectangular coordinates $(-3, -3)$
P has polar coordinates $\left(4, \frac{5\pi}{3}\right)$ Q has polar coordinates $\left(3\sqrt{2}, \frac{5\pi}{4}\right)$

3. The point $R(0, -3)$ lies along the negative y-axis. While we could go through the usual computations[5] to find the polar form of R, in this case it is much more efficient to find the polar coordinates of R using the definition.

   Since the pole is identified with the origin, we see the point R is 3 units from the pole. Hence in the polar representation (r, θ) of R we know $r = \pm 3$. To satisfy $r \geq 0$, we choose $r = 3$.

   Concerning θ, we once again find more or less 'by inspection' that $\theta = \frac{3\pi}{2}$ satisfies $0 \leq \theta < 2\pi$ with its terminal side along the negative y-axis. Hence, our answer is $\left(3, \frac{3\pi}{2}\right)$.

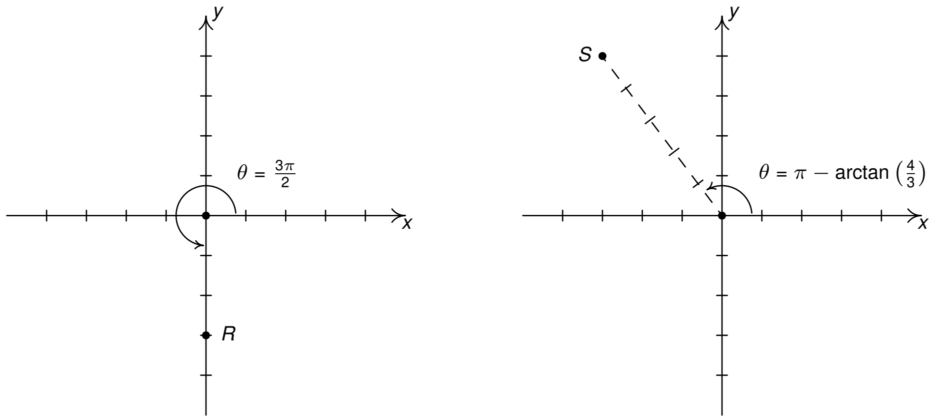
   To check, we note $x = r\cos(\theta) = 3\cos\left(\frac{3\pi}{2}\right) = (3)(0) = 0$ and $y = r\sin(\theta) = 3\sin\left(\frac{3\pi}{2}\right) = 3(-1) = -3$.

4. The point $S(-3, 4)$ lies in Quadrant II. With $x = -3$ and $y = 4$, we get $r^2 = (-3)^2 + (4)^2 = 25$ so $r = \pm 5$. As usual, we choose $r = 5 \geq 0$ and proceed to determine θ.

[5]Since $x = 0$, we would have to determine θ geometrically.

We have $\tan(\theta) = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$, and since this isn't the tangent of one the common angles, we resort to using the arctangent function. Since θ lies in Quadrant II and must satisfy $0 \leq \theta < 2\pi$, we choose $\theta = \pi - \arctan\left(\frac{4}{3}\right)$ radians. Hence, our answer is $(r, \theta) = \left(5, \pi - \arctan\left(\frac{4}{3}\right)\right) \approx (5, 2.21)$.

To check our answers requires a bit of tenacity since we need to simplify expressions of the form: $\cos\left(\pi - \arctan\left(\frac{4}{3}\right)\right)$ and $\sin\left(\pi - \arctan\left(\frac{4}{3}\right)\right)$. These are good review exercises (see Section 12.3) and are hence left to the reader. We find $\cos\left(\pi - \arctan\left(\frac{4}{3}\right)\right) = -\frac{3}{5}$ and $\sin\left(\pi - \arctan\left(\frac{4}{3}\right)\right) = \frac{4}{5}$, so that $x = r\cos(\theta) = (5)\left(-\frac{3}{5}\right) = -3$ and $y = r\sin(\theta) = (5)\left(\frac{4}{5}\right) = 4$ which confirms our answer.



R has rectangular coordinates $(0, -3)$
R has polar coordinates $\left(3, \frac{3\pi}{2}\right)$

S has rectangular coordinates $(-3, 4)$
S has polar coordinates $\left(5, \pi - \arctan\left(\frac{4}{3}\right)\right)$

□

Now that we've had practice converting representations of *points* between the rectangular and polar coordinate systems, we now set about converting *equations* from one system to another.

Just as we've used equations in x and y to represent relations in rectangular coordinates (see Section 5.5), equations in the variables r and θ represent relations in polar coordinates. We convert equations between the two systems using Theorem 14.1 as the next example illustrates.

Example 14.1.3.

1. Convert each equation in rectangular coordinates into an equation in polar coordinates.

   (a) $(x-3)^2 + y^2 = 9$ (b) $y = -x$ (c) $y = x^2$

2. Convert each equation in polar coordinates into an equation in rectangular coordinates.

   (a) $r = -3$ (b) $\theta = \frac{4\pi}{3}$ (c) $r = 1 - \cos(\theta)$

Solution.

1. One strategy to convert an equation from rectangular to polar coordinates is to replace every occurrence of x with $r\cos(\theta)$ and every occurrence of y with $r\sin(\theta)$ and use identities to simplify. This is the technique we employ below.

(a) We start by substituting $x = r\cos(\theta)$ and $y = \sin(\theta)$ into $(x-3)^2 + y^2 = 9$ and simplifying. With no real direction in which to proceed, we follow our mathematical instincts.[6]

$$\begin{array}{rcll} (r\cos(\theta)-3)^2 + (r\sin(\theta))^2 & = & 9 & \\ r^2\cos^2(\theta) - 6r\cos(\theta) + 9 + r^2\sin^2(\theta) & = & 9 & \\ r^2\left(\cos^2(\theta)+\sin^2(\theta)\right) - 6r\cos(\theta) & = & 0 & \text{Subtract 9 from both sides.} \\ r^2 - 6r\cos(\theta) & = & 0 & \text{Since } \cos^2(\theta)+\sin^2(\theta) = 1 \\ r(r-6\cos(\theta)) & = & 0 & \text{Factor.} \end{array}$$

We get $r = 0$ or $r = 6\cos(\theta)$. From Section 8.3 we know the equation $(x-3)^2+y^2 = 9$ describes a circle, and since $r = 0$ describes just a point (namely the pole/origin), we choose $r = 6\cos(\theta)$ for our final answer.[7]

(b) Substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into $y = -x$ gives $r\sin(\theta) = -r\cos(\theta)$. Rearranging, we get $r\cos(\theta) + r\sin(\theta) = 0$ or $r(\cos(\theta)+\sin(\theta)) = 0$. This gives $r = 0$ or $\cos(\theta)+\sin(\theta) = 0$. Solving the latter equation for θ, we get $\theta = -\frac{\pi}{4} + \pi k$ for integers k.

As we did in the previous example, we take a step back and think geometrically. We know $y = -x$ describes a line through the origin. As before, $r = 0$ describes the origin, but nothing else. Consider the equation $\theta = -\frac{\pi}{4}$. In this equation, the variable r is free,[8] meaning it can assume any and all values including $r = 0$.

If we imagine plotting points $(r, -\frac{\pi}{4})$ for all conceivable values of r (positive, negative and zero), we are essentially drawing the line containing the terminal side of $\theta = -\frac{\pi}{4}$ which is none other than $y = -x$. Hence, we can take as our final answer $\theta = -\frac{\pi}{4}$ here.[9]

(c) Substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into $y = x^2$ gives $r\sin(\theta) = (r\cos(\theta))^2$, or $r^2\cos^2(\theta) - r\sin(\theta) = 0$. Factoring, we get $r(r\cos^2(\theta) - \sin(\theta)) = 0$ so either $r = 0$ or $r\cos^2(\theta) = \sin(\theta)$.

We can solve $r\cos^2(\theta) = \sin(\theta)$ for r by dividing both sides of the equation by $\cos^2(\theta)$, but as a general rule, we never divide through by a quantity that may be 0.

In this particular case, we are safe since if $\cos^2(\theta) = 0$, then $\cos(\theta) = 0$, and for the equation $r\cos^2(\theta) = \sin(\theta)$ to hold, then $\sin(\theta)$ would also have to be 0. Since there are no angles with both $\cos(\theta) = 0$ and $\sin(\theta) = 0$, we are not losing any information by dividing both sides of $r\cos^2(\theta) = \sin(\theta)$ by $\cos^2(\theta)$.

Solving $r\cos^2(\theta) = \sin(\theta)$ gives $r = \frac{\sin(\theta)}{\cos^2(\theta)}$, or $r = \sec(\theta)\tan(\theta)$. As before, the $r = 0$ case is recovered in the solution $r = \sec(\theta)\tan(\theta)$ (let $\theta = 0$), so $r = \sec(\theta)\tan(\theta)$ is our final answer.

2. As a general rule, converting equations from polar to rectangular coordinates isn't as straight forward as the reverse process. We could solve $r^2 = x^2+y^2$ for r to get $r = \pm\sqrt{x^2+y^2}$ and solving $\tan(\theta) = \frac{y}{x}$ requires the arctangent function to get $\theta = \arctan\left(\frac{y}{x}\right) + \pi k$ for integers k.

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[6] Experience is the mother of all instinct, and necessity is the mother of invention. Study this example and see what techniques are employed, then try your best to get your answers in the homework to match Jeff's.

[7] Note when we substitute $\theta = \frac{\pi}{2}$ into the equation $r = 6\cos(\theta)$, we recover the point $r = 0$.

[8] See Section 9.1.

[9] We could take it to be *any* of $\theta = -\frac{\pi}{4} + \pi k$ for integers k.

Since neither of these expressions for r and θ are especially user-friendly, so we opt for a second strategy – rearrange the given polar equation so that the expressions $r^2 = x^2 + y^2$, $r\cos(\theta) = x$, $r\sin(\theta) = y$ and/or $\tan(\theta) = \frac{y}{x}$ present themselves.

(a) Starting with $r = -3$, we can square both sides to get $r^2 = (-3)^2$ or $r^2 = 9$. We may now substitute $r^2 = x^2 + y^2$ to get the equation $x^2 + y^2 = 9$.

As we have seen,[10] squaring an equation does not, in general, produce an equivalent equation. The concern here is that the equation $r^2 = 9$ might be satisfied by more points than $r = -3$.

On the surface, this certainly appears to be the case since $r^2 = 9$ is equivalent to $r = \pm 3$, not just $r = -3$. That being said, any point with polar coordinates $(3, \theta)$ can be represented as $(-3, \theta + \pi)$, which means any point (r, θ) whose polar coordinates satisfy the relation $r = \pm 3$ has an equivalent[11] representation which satisfies $r = -3$.

(b) We take the tangent of both sides the equation $\theta = \frac{4\pi}{3}$ to get $\tan(\theta) = \tan\left(\frac{4\pi}{3}\right) = \sqrt{3}$. Since $\tan(\theta) = \frac{y}{x}$, we get $\frac{y}{x} = \sqrt{3}$ or $y = x\sqrt{3}$. Of course, we pause a moment to wonder if, geometrically, the equations $\theta = \frac{4\pi}{3}$ and $y = x\sqrt{3}$ generate the same set of points.[12] The same argument presented in number 1b applies equally well here so we are done.

(c) Once again, we need to manipulate $r = 1 - \cos(\theta)$ a bit before using the conversion formulas given in Theorem 14.1. We could square both sides of this equation like we did in part 2a above to obtain an r^2 on the left hand side, but that does nothing helpful for the right hand side.

Instead, we multiply both sides by r to obtain $r^2 = r - r\cos(\theta)$. We now have an r^2 and an $r\cos(\theta)$ in the equation, which we can easily handle, but we also have another r to deal with. Rewriting the equation as $r = r^2 + r\cos(\theta)$ and squaring both sides yields $r^2 = \left(r^2 + r\cos(\theta)\right)^2$. Substituting $r^2 = x^2 + y^2$ and $r\cos(\theta) = x$ gives $x^2 + y^2 = \left(x^2 + y^2 + x\right)^2$.

Once again, we have performed some algebraic maneuvers which may have altered the set of points described by the original equation. First, we multiplied both sides by r. This means that now $r = 0$ is a viable solution to the equation. In the original equation, $r = 1 - \cos(\theta)$, we see that $\theta = 0$ gives $r = 0$, so the multiplication by r doesn't introduce any new points.

The squaring of both sides of this equation is also a reason to pause. That is, are there points with coordinates (r, θ) which satisfy $r^2 = \left(r^2 + r\cos(\theta)\right)^2$ but do not satisfy $r = r^2 + r\cos(\theta)$?

Suppose $\left(r', \theta'\right)$ satisfies $r^2 = \left(r^2 + r\cos(\theta)\right)^2$. Then $r' = \pm\left((r')^2 + r'\cos(\theta')\right)$. If it turns out that $r' = (r')^2 + r'\cos(\theta')$, then we are done.

[10] See Exercises 14 - 25 in Section A.13, for instance ...

[11] As *ordered pairs*, $(3, 0)$ and $(-3, \pi)$ are different, but since they correspond to the same point in the plane, we consider them 'equivalent' in this context. Technically speaking, the equations $r^2 = 9$ and $r = -3$ represent different relations per Definition 5.3 in Section 5.5 since they generate different sets of ordered pairs. Since polar coordinates were defined geometrically to describe the location of points in the plane, however, we concern ourselves only with ensuring that the sets of *points* in the plane generated by two equations are the same.

[12] In addition to taking the tangent of both sides of an equation (There are infinitely many solutions to $\tan(\theta) = \sqrt{3}$, and $\theta = \frac{4\pi}{3}$ is only one of them!), we also went from $\frac{y}{x} = \sqrt{3}$, in which x cannot be 0, to $y = x\sqrt{3}$ in which we assume x can be 0.

If $r' = -\left((r')^2 + r'\cos(\theta')\right) = -(r')^2 - r'\cos(\theta')$, we claim that the coordinates $(-r', \theta' + \pi)$, which determine the same point as (r', θ'), satisfy $r = r^2 + r\cos(\theta)$. To show this, we substitute $r = -r'$ and $\theta = \theta' + \pi$ into the equation $r = r^2 + r\cos(\theta)$:

$$\begin{array}{rcll} -r' & \stackrel{?}{=} & \left(-r'\right)^2 + \left(-r'\cos(\theta' + \pi)\right) & \\ -\left(-(r')^2 - r'\cos(\theta')\right) & \stackrel{?}{=} & (r')^2 - r'\cos(\theta' + \pi) & \text{Since } r' = -(r')^2 - r'\cos(\theta') \\ (r')^2 + r'\cos(\theta') & \stackrel{?}{=} & (r')^2 - r'(-\cos(\theta')) & \text{Since } \cos(\theta' + \pi) = -\cos(\theta') \\ (r')^2 + r'\cos(\theta') & \stackrel{\checkmark}{=} & (r')^2 + r'\cos(\theta') & \end{array}$$

Since both sides worked out to be equal, $(-r', \theta' + \pi)$ satisfies $r = r^2 + r\cos(\theta)$.

Hence, any point (r, θ) which satisfies $r^2 = \left(r^2 + r\cos(\theta)\right)^2$ has a representation which satisfies $r = r^2 + r\cos(\theta)$, so a rectangular representation of $r = 1 - \cos(\theta)$ is $x^2 + y^2 = \left(x^2 + y^2 + x\right)^2$. □

In practice, much of the pedantic verification of the equivalence of equations in Example 14.1.3 is left unsaid. Indeed, in most textbooks, squaring equations like $r = -3$ to arrive at $r^2 = 9$ happens without a second thought. Your instructor will ultimately decide how much, if any, justification is warranted.

If you take anything away from Example 14.1.3, it should be that relatively simple equations in rectangular coordinates, such as $y = x^2$, can become quite complicated in polar coordinates, and vice-versa.

In the next section, we devote our attention to graphing equations like the ones given in Example 14.1.3 number 2 on the Cartesian coordinate plane without converting back to rectangular coordinates. If nothing else, number 2c above shows the price we pay if we insist on always converting to back to the more familiar rectangular coordinate system.

14.1.1 Exercises

In Exercises 1 - 16, plot the point given in polar coordinates and then give three different expressions for the point such that (a) $r < 0$ and $0 \leq \theta \leq 2\pi$, (b) $r > 0$ and $\theta \leq 0$ (c) $r > 0$ and $\theta \geq 2\pi$

1. $\left(2, \frac{\pi}{3}\right)$
2. $\left(5, \frac{7\pi}{4}\right)$
3. $\left(\frac{1}{3}, \frac{3\pi}{2}\right)$
4. $\left(\frac{5}{2}, \frac{5\pi}{6}\right)$
5. $\left(12, -\frac{7\pi}{6}\right)$
6. $\left(3, -\frac{5\pi}{4}\right)$
7. $\left(2\sqrt{2}, -\pi\right)$
8. $\left(\frac{7}{2}, -\frac{13\pi}{6}\right)$
9. $(-20, 3\pi)$
10. $\left(-4, \frac{5\pi}{4}\right)$
11. $\left(-1, \frac{2\pi}{3}\right)$
12. $\left(-3, \frac{\pi}{2}\right)$
13. $\left(-3, -\frac{11\pi}{6}\right)$
14. $\left(-2.5, -\frac{\pi}{4}\right)$
15. $\left(-\sqrt{5}, -\frac{4\pi}{3}\right)$
16. $(-\pi, -\pi)$

In Exercises 17 - 36, convert the point from polar coordinates into rectangular coordinates.

17. $\left(5, \frac{7\pi}{4}\right)$
18. $\left(2, \frac{\pi}{3}\right)$
19. $\left(11, -\frac{7\pi}{6}\right)$
20. $(-20, 3\pi)$
21. $\left(\frac{3}{5}, \frac{\pi}{2}\right)$
22. $\left(-4, \frac{5\pi}{6}\right)$
23. $\left(9, \frac{7\pi}{2}\right)$
24. $\left(-5, -\frac{9\pi}{4}\right)$
25. $\left(42, \frac{13\pi}{6}\right)$
26. $(-117, 117\pi)$
27. $(6, \arctan(2))$
28. $(10, \arctan(3))$
29. $\left(-3, \arctan\left(\frac{4}{3}\right)\right)$
30. $\left(5, \arctan\left(-\frac{4}{3}\right)\right)$
31. $\left(2, \pi - \arctan\left(\frac{1}{2}\right)\right)$
32. $\left(-\frac{1}{2}, \pi - \arctan(5)\right)$
33. $\left(-1, \pi + \arctan\left(\frac{3}{4}\right)\right)$
34. $\left(\frac{2}{3}, \pi + \arctan\left(2\sqrt{2}\right)\right)$
35. $(\pi, \arctan(\pi))$
36. $\left(13, \arctan\left(\frac{12}{5}\right)\right)$

In Exercises 37 - 56, plot each point given in rectangular coordinates and convert to polar coordinates. Choose $r \geq 0$ and $0 \leq \theta < 2\pi$.

37. $(0, 5)$
38. $(3, \sqrt{3})$
39. $(7, -7)$
40. $(-3, -\sqrt{3})$
41. $(-3, 0)$
42. $(-\sqrt{2}, \sqrt{2})$
43. $(-4, -4\sqrt{3})$
44. $\left(\frac{\sqrt{3}}{4}, -\frac{1}{4}\right)$

45. $\left(-\frac{3}{10}, -\frac{3\sqrt{3}}{10}\right)$
46. $\left(-\sqrt{5}, -\sqrt{5}\right)$
47. $(6, 8)$
48. $(\sqrt{5}, 2\sqrt{5})$
49. $(-8, 1)$
50. $(-2\sqrt{10}, 6\sqrt{10})$
51. $(-5, -12)$
52. $\left(-\frac{\sqrt{5}}{15}, -\frac{2\sqrt{5}}{15}\right)$
53. $(24, -7)$
54. $(12, -9)$
55. $\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{6}}{4}\right)$
56. $\left(-\frac{\sqrt{65}}{5}, \frac{2\sqrt{65}}{5}\right)$

In Exercises 57 - 76, convert the equation from rectangular coordinates into polar coordinates. Solve for r in all but 60 through 63. In Exercises 60 - 63, solve for θ

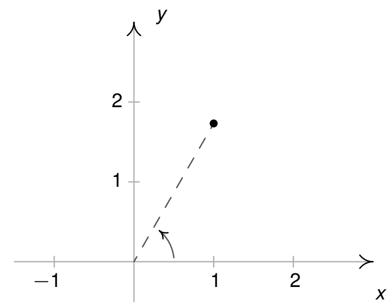
57. $x = 6$
58. $x = -3$
59. $y = 7$
60. $y = 0$
61. $y = -x$
62. $y = x\sqrt{3}$
63. $y = 2x$
64. $x^2 + y^2 = 25$
65. $x^2 + y^2 = 117$
66. $y = 4x - 19$
67. $x = 3y + 1$
68. $y = -3x^2$
69. $4x = y^2$
70. $x^2 + y^2 - 2y = 0$
71. $x^2 - 4x + y^2 = 0$
72. $x^2 + y^2 = x$
73. $y^2 = 7y - x^2$
74. $(x + 2)^2 + y^2 = 4$
75. $x^2 + (y - 3)^2 = 9$
76. $4x^2 + 4\left(y - \frac{1}{2}\right)^2 = 1$

In Exercises 77 - 96, convert the equation from polar coordinates into rectangular coordinates.

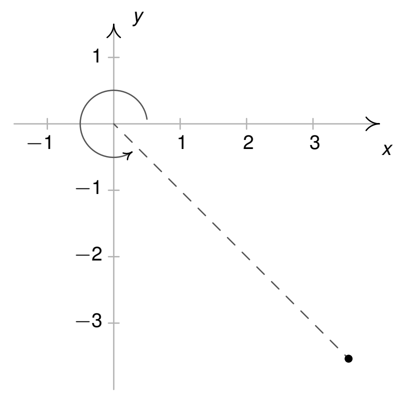
77. $r = 7$
78. $r = -3$
79. $r = \sqrt{2}$
80. $\theta = \frac{\pi}{4}$
81. $\theta = \frac{2\pi}{3}$
82. $\theta = \pi$
83. $\theta = \frac{3\pi}{2}$
84. $r = 4\cos(\theta)$
85. $5r = \cos(\theta)$
86. $r = 3\sin(\theta)$
87. $r = -2\sin(\theta)$
88. $r = 7\sec(\theta)$
89. $12r = \csc(\theta)$
90. $r = -2\sec(\theta)$
91. $r = -\sqrt{5}\csc(\theta)$
92. $r = 2\sec(\theta)\tan(\theta)$
93. $r = -\csc(\theta)\cot(\theta)$
94. $r^2 = \sin(2\theta)$
95. $r = 1 - 2\cos(\theta)$
96. $r = 1 + \sin(\theta)$
97. Convert the origin $(0, 0)$ into polar coordinates in four different ways.
98. With the help of your classmates, use the Law of Cosines to develop a formula for the distance between two points in polar coordinates.

14.1.2 Answers

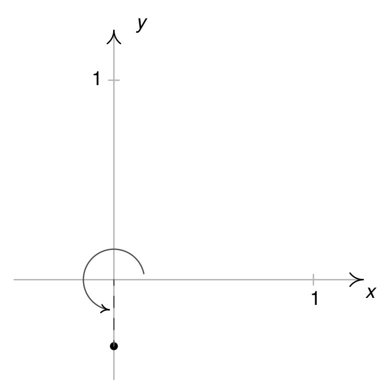
1. $\left(2, \dfrac{\pi}{3}\right)$, $\left(-2, \dfrac{4\pi}{3}\right)$
$\left(2, -\dfrac{5\pi}{3}\right)$, $\left(2, \dfrac{7\pi}{3}\right)$



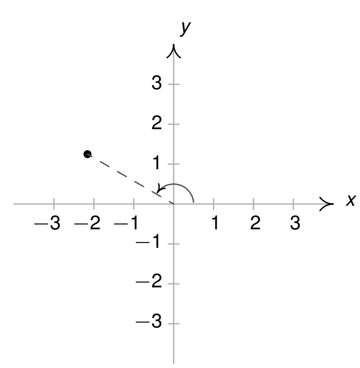
2. $\left(5, \dfrac{7\pi}{4}\right)$, $\left(-5, \dfrac{3\pi}{4}\right)$
$\left(5, -\dfrac{\pi}{4}\right)$, $\left(5, \dfrac{15\pi}{4}\right)$



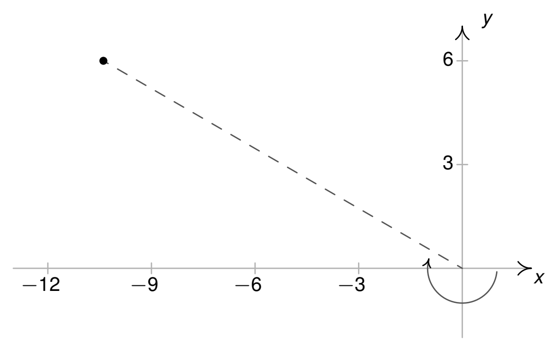
3. $\left(\dfrac{1}{3}, \dfrac{3\pi}{2}\right)$, $\left(-\dfrac{1}{3}, \dfrac{\pi}{2}\right)$
$\left(\dfrac{1}{3}, -\dfrac{\pi}{2}\right)$, $\left(\dfrac{1}{3}, \dfrac{7\pi}{2}\right)$



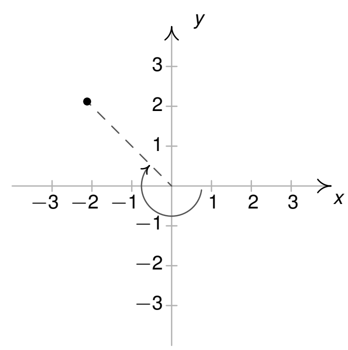
4. $\left(\dfrac{5}{2}, \dfrac{5\pi}{6}\right)$, $\left(-\dfrac{5}{2}, \dfrac{11\pi}{6}\right)$
$\left(\dfrac{5}{2}, -\dfrac{7\pi}{6}\right)$, $\left(\dfrac{5}{2}, \dfrac{17\pi}{6}\right)$



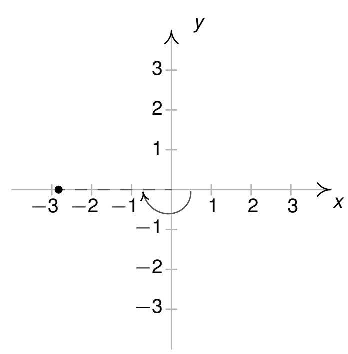
5. $\left(12, -\frac{7\pi}{6}\right)$, $\left(-12, \frac{11\pi}{6}\right)$
$\left(12, -\frac{19\pi}{6}\right)$, $\left(12, \frac{17\pi}{6}\right)$



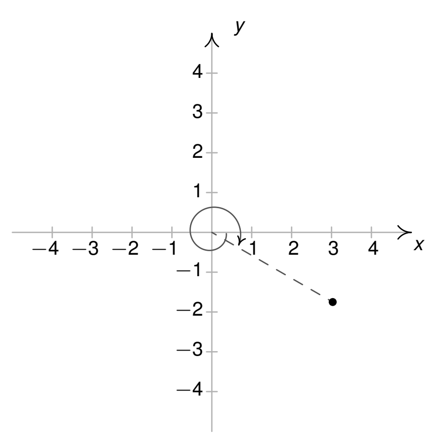
6. $\left(3, -\frac{5\pi}{4}\right)$, $\left(-3, \frac{7\pi}{4}\right)$
$\left(3, -\frac{13\pi}{4}\right)$, $\left(3, \frac{11\pi}{4}\right)$



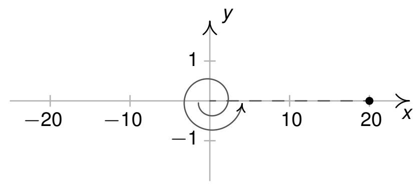
7. $\left(2\sqrt{2}, -\pi\right)$, $\left(-2\sqrt{2}, 0\right)$
$\left(2\sqrt{2}, -3\pi\right)$, $\left(2\sqrt{2}, 3\pi\right)$



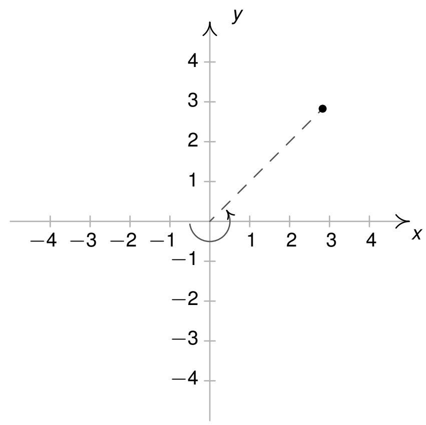
8. $\left(\frac{7}{2}, -\frac{13\pi}{6}\right)$, $\left(-\frac{7}{2}, \frac{5\pi}{6}\right)$
$\left(\frac{7}{2}, -\frac{\pi}{6}\right)$, $\left(\frac{7}{2}, \frac{23\pi}{6}\right)$



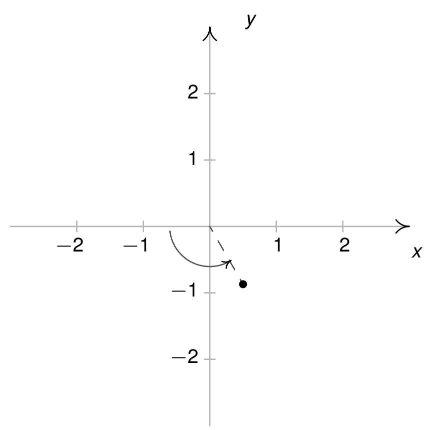
9. $(-20, 3\pi)$, $(-20, \pi)$
$(20, -2\pi)$, $(20, 4\pi)$



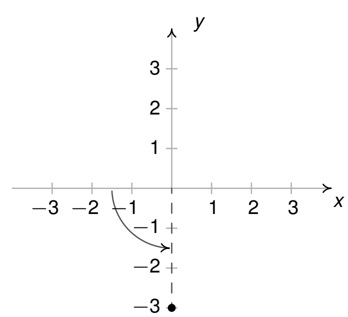
10. $\left(-4, \frac{5\pi}{4}\right)$, $\left(-4, \frac{5\pi}{4}\right)$
$\left(4, -\frac{7\pi}{4}\right)$, $\left(4, \frac{9\pi}{4}\right)$



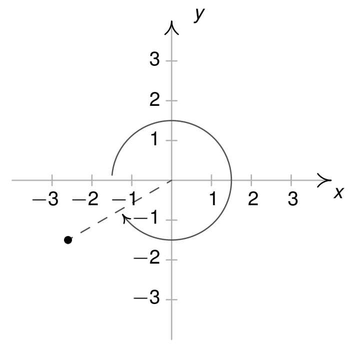
11. $\left(-1, \frac{2\pi}{3}\right)$, $\left(-1, \frac{2\pi}{3}\right)$
$\left(1, -\frac{\pi}{3}\right)$, $\left(1, \frac{11\pi}{3}\right)$



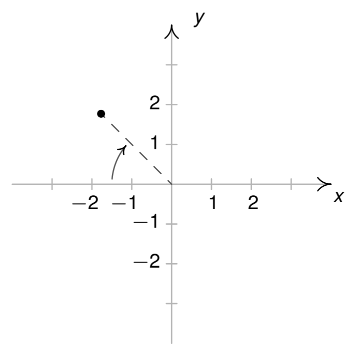
12. $\left(-3, \frac{\pi}{2}\right)$, $\left(-3, \frac{\pi}{2}\right)$
$\left(3, -\frac{\pi}{2}\right)$, $\left(3, \frac{7\pi}{2}\right)$



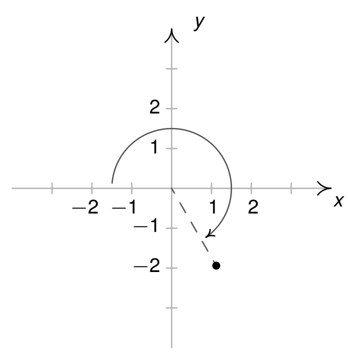
13. $\left(-3, -\dfrac{11\pi}{6}\right)$, $\left(-3, \dfrac{\pi}{6}\right)$
$\left(3, -\dfrac{5\pi}{6}\right)$, $\left(3, \dfrac{19\pi}{6}\right)$



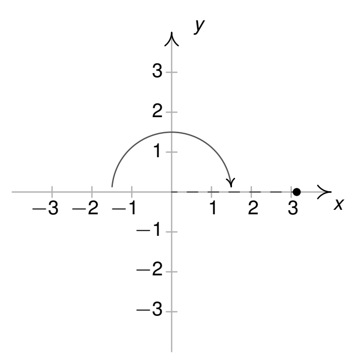
14. $\left(-2.5, -\dfrac{\pi}{4}\right)$, $\left(-2.5, \dfrac{7\pi}{4}\right)$
$\left(2.5, -\dfrac{5\pi}{4}\right)$, $\left(2.5, \dfrac{11\pi}{4}\right)$



15. $\left(-\sqrt{5}, -\dfrac{4\pi}{3}\right)$, $\left(-\sqrt{5}, \dfrac{2\pi}{3}\right)$
$\left(\sqrt{5}, -\dfrac{\pi}{3}\right)$, $\left(\sqrt{5}, \dfrac{11\pi}{3}\right)$



16. $(-\pi, -\pi)$, $(-\pi, \pi)$
$(\pi, -2\pi)$, $(\pi, 2\pi)$



17. $\left(\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$

18. $\left(1, \sqrt{3}\right)$

19. $\left(-\frac{11\sqrt{3}}{2}, \frac{11}{2}\right)$

20. $(20, 0)$

21. $\left(0, \frac{3}{5}\right)$

22. $\left(2\sqrt{3}, -2\right)$

23. $(0, -9)$

24. $\left(-\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$

25. $\left(21\sqrt{3}, 21\right)$

26. $(117, 0)$

27. $\left(\frac{6\sqrt{5}}{5}, \frac{12\sqrt{5}}{5}\right)$

28. $\left(\sqrt{10}, 3\sqrt{10}\right)$

29. $\left(-\frac{9}{5}, -\frac{12}{5}\right)$

30. $(3, -4)$

31. $\left(-\frac{4\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$

32. $\left(\frac{\sqrt{26}}{52}, -\frac{5\sqrt{26}}{52}\right)$

33. $\left(\frac{4}{5}, \frac{3}{5}\right)$

34. $\left(-\frac{2}{9}, -\frac{4\sqrt{2}}{9}\right)$

35. $\left(\frac{\pi}{\sqrt{1+\pi^2}}, \frac{\pi^2}{\sqrt{1+\pi^2}}\right)$

36. $(5, 12)$

37. $\left(5, \frac{\pi}{2}\right)$

38. $\left(2\sqrt{3}, \frac{\pi}{6}\right)$

39. $\left(7\sqrt{2}, \frac{7\pi}{4}\right)$

40. $\left(2\sqrt{3}, \frac{7\pi}{6}\right)$

41. $(3, \pi)$

42. $\left(2, \frac{3\pi}{4}\right)$

43. $\left(8, \frac{4\pi}{3}\right)$

44. $\left(\frac{1}{2}, \frac{11\pi}{6}\right)$

45. $\left(\frac{3}{5}, \frac{4\pi}{3}\right)$

46. $\left(\sqrt{10}, \frac{5\pi}{4}\right)$

47. $\left(10, \arctan\left(\frac{4}{3}\right)\right)$

48. $(5, \arctan(2))$

49. $\left(\sqrt{65}, \pi - \arctan\left(\frac{1}{8}\right)\right)$

50. $(20, \pi - \arctan(3))$

51. $\left(13, \pi + \arctan\left(\frac{12}{5}\right)\right)$

52. $\left(\frac{1}{3}, \pi + \arctan(2)\right)$

53. $\left(25, 2\pi - \arctan\left(\frac{7}{24}\right)\right)$

54. $\left(15, 2\pi - \arctan\left(\frac{3}{4}\right)\right)$

55. $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{3}\right)$

56. $\left(\sqrt{13}, \pi - \arctan(2)\right)$

57. $r = 6\sec(\theta)$

58. $r = -3\sec(\theta)$

59. $r = 7\csc(\theta)$

60. $\theta = 0$

61. $\theta = \frac{3\pi}{4}$

62. $\theta = \frac{\pi}{3}$

63. $\theta = \arctan(2)$

64. $r = 5$

65. $r = \sqrt{117}$

66. $r = \frac{19}{4\cos(\theta) - \sin(\theta)}$

67. $x = \frac{1}{\cos(\theta) - 3\sin(\theta)}$

68. $r = \frac{-\sec(\theta)\tan(\theta)}{3}$

69. $r = 4\csc(\theta)\cot(\theta)$

70. $r = 2\sin(\theta)$

71. $r = 4\cos(\theta)$

72. $r = \cos(\theta)$

73. $r = 7\sin(\theta)$

74. $r = -4\cos(\theta)$

75. $r = 6\sin(\theta)$

76. $r = \sin(\theta)$

77. $x^2 + y^2 = 49$

78. $x^2 + y^2 = 9$

79. $x^2 + y^2 = 2$

80. $y = x$

81. $y = -\sqrt{3}x$

82. $y = 0$

83. $x = 0$

84. $x^2 + y^2 = 4x$ or $(x-2)^2 + y^2 = 4$

85. $5x^2 + 5y^2 = x$ or $\left(x - \frac{1}{10}\right)^2 + y^2 = \frac{1}{100}$

86. $x^2 + y^2 = 3y$ or $x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$

87. $x^2 + y^2 = -2y$ or $x^2 + (y+1)^2 = 1$

88. $x = 7$

89. $y = \frac{1}{12}$

90. $x = -2$

91. $y = -\sqrt{5}$

92. $x^2 = 2y$

93. $y^2 = -x$

94. $\left(x^2 + y^2\right)^2 = 2xy$

95. $\left(x^2 + 2x + y^2\right)^2 = x^2 + y^2$

96. $\left(x^2 + y^2 - y\right)^2 = x^2 + y^2$

97. Any point of the form $(0, \theta)$ will work, e.g. $(0, \pi)$, $(0, -117)$, $\left(0, \frac{23\pi}{4}\right)$ and $(0, 0)$.

14.2 The Graphs of Polar Equations

In this section, we discuss how to graph equations relating the *polar coordinate* variables r and θ on the *rectangular coordinate* plane. Since every point in the plane has infinitely many different representations in polar coordinates, in order for a point P to be on the graph of a given equation, there must be *at least one* representation of $P(r, \theta)$ that satisfies that equation.

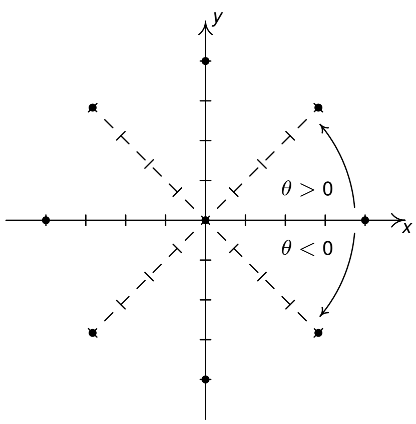
In our first example, only one of the variables r and θ is present making the other variable free.[1] This makes these graphs easier to visualize than others.

Example 14.2.1. Graph the following polar equations in the xy-plane.

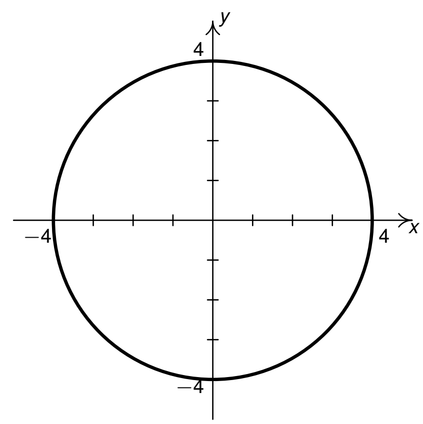
1. $r = 4$
2. $r = -3\sqrt{2}$
3. $\theta = \frac{5\pi}{4}$
4. $\theta = -\frac{3\pi}{2}$

Solution.

1. In the equation $r = 4$, θ is free. The graph of this equation is, therefore, all points which have a polar coordinate representation $(4, \theta)$, for any choice of θ. In other words, we trace out all of the points 4 units away from the origin. This is exactly the definition of circle, centered at the origin, with a radius of 4.



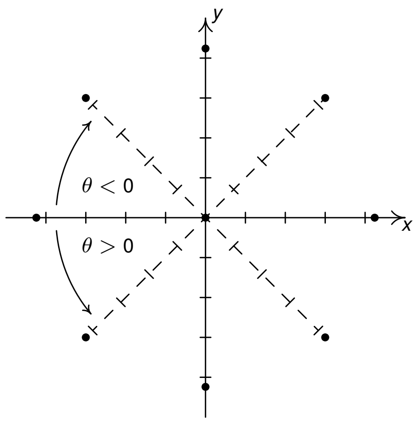
In $r = 4$, θ is free



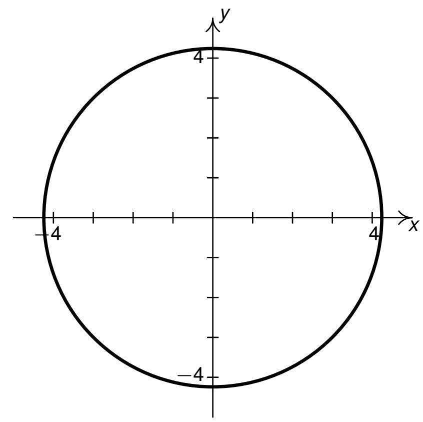
The graph of $r = 4$

2. Once again we have θ being free in the equation $r = -3\sqrt{2}$. Plotting all of the points of the form $(-3\sqrt{2}, \theta)$ gives us a circle of radius $3\sqrt{2}$ centered at the origin.

[1] See the discussion in Example 14.1.3 number 2a in Section 14.1.

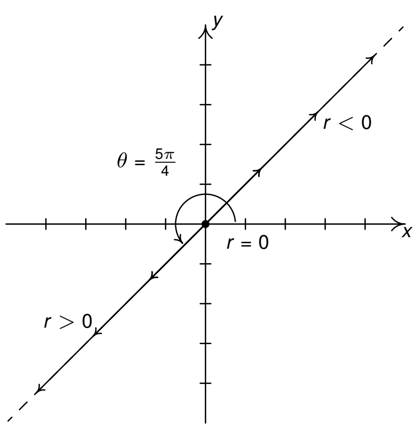


In $r = -3\sqrt{2}$, θ is free

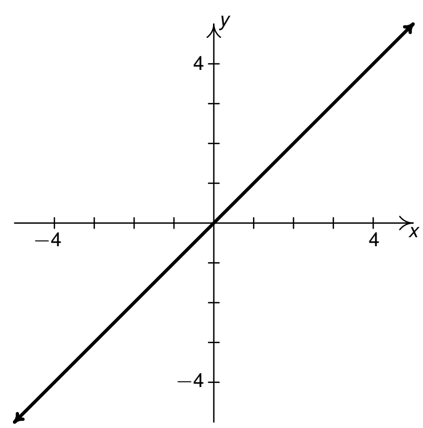


The graph of $r = -3\sqrt{2}$

3. In the equation $\theta = \frac{5\pi}{4}$, r is free, so we plot all of the points with polar representation $\left(r, \frac{5\pi}{4}\right)$. The result is the line containing the terminal side of $\theta = \frac{5\pi}{4}$, when plotted in standard position.

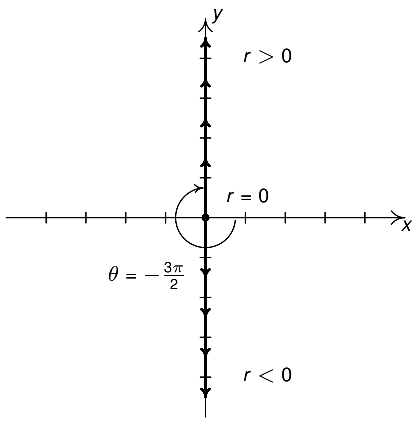


In $\theta = \frac{5\pi}{4}$, r is free

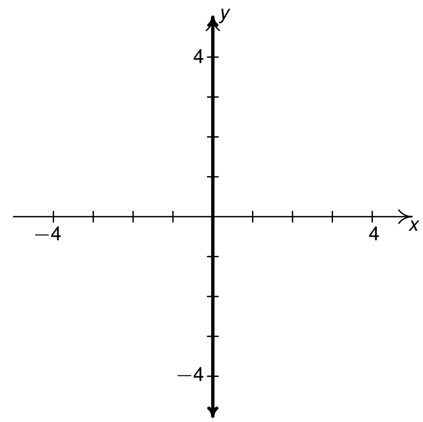


The graph of $\theta = \frac{5\pi}{4}$

4. As in the previous example, the variable r is free in the equation $\theta = -\frac{3\pi}{2}$. Plotting $\left(r, -\frac{3\pi}{2}\right)$ for various values of r shows us that we are tracing out the y-axis.



In $\theta = -\frac{3\pi}{2}$, r is free



The graph of $\theta = -\frac{3\pi}{2}$

$\square$

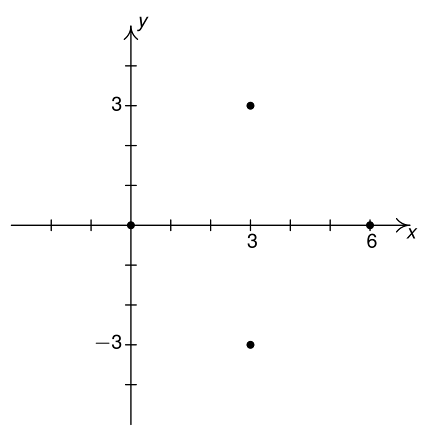
Hopefully, our experience in Example 14.2.1 makes the following result clear.

Theorem 14.2. Graphs of Constant r and θ: Suppose a and α are constants, $a \neq 0$.

- The graph of the polar equation $r = a$ on the Cartesian plane is a circle centered at the origin of radius $|a|$.
- The graph of the polar equation $\theta = \alpha$ on the Cartesian plane is the line containing the terminal side of α when plotted in standard position.

Suppose we wish to graph $r = 6\cos(\theta)$. A reasonable way to start is to treat θ as the independent variable, r as the dependent variable, evaluate $r = f(\theta)$ at some 'friendly' values of θ and plot the resulting points.[2]

θ	$r = 6\cos(\theta)$	(r, θ)
0	6	$(6, 0)$
$\frac{\pi}{4}$	$3\sqrt{2}$	$\left(3\sqrt{2}, \frac{\pi}{4}\right)$
$\frac{\pi}{2}$	0	$\left(0, \frac{\pi}{2}\right)$
$\frac{3\pi}{4}$	$-3\sqrt{2}$	$\left(-3\sqrt{2}, \frac{3\pi}{4}\right)$
π	-6	$(-6, \pi)$
$\frac{5\pi}{4}$	$-3\sqrt{2}$	$\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$
$\frac{3\pi}{2}$	0	$\left(0, \frac{3\pi}{2}\right)$
$\frac{7\pi}{4}$	$3\sqrt{2}$	$\left(3\sqrt{2}, \frac{7\pi}{4}\right)$
2π	6	$(6, 2\pi)$

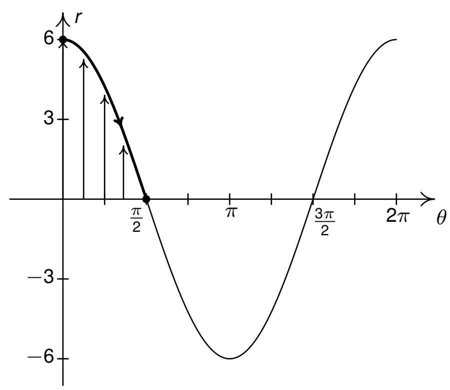


[2]For a review of these concepts and this process, see Section 1.1.

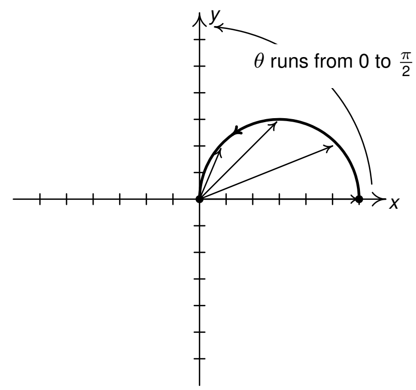
Despite having nine ordered pairs, we get only four distinct points on the graph. For this reason, we employ a slightly different strategy. We graph one cycle of $r = 6\cos(\theta)$ on the θr-plane[3] below on the left and use it to help graph the equation on the xy-plane below on the right.

We see that as θ ranges from 0 to $\frac{\pi}{2}$, r ranges from 6 to 0. In the xy-plane, this means that the curve starts 6 units from the origin on the positive x-axis ($\theta = 0$) and gradually returns to the origin by the time the curve reaches the y-axis ($\theta = \frac{\pi}{2}$).

The arrows drawn in the figure below are meant to help you visualize this process. In the θr-plane, the arrows are drawn from the θ-axis to the curve $r = 6\cos(\theta)$. In the xy-plane, each of these arrows starts at the origin and is rotated through the corresponding angle θ, in accordance with how we plot polar coordinates. This method is less precise than plotting actual function values, but much faster.

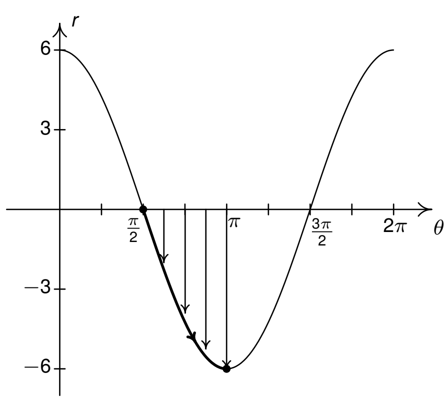


$r = 6\cos(\theta)$ in the θr-plane

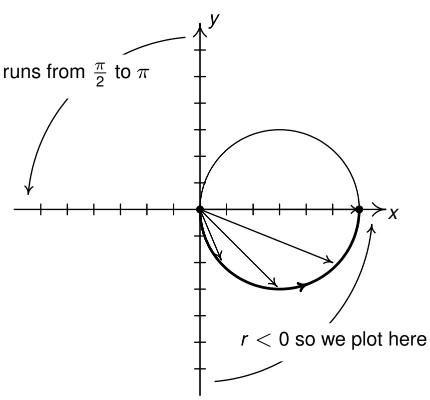


$r = 6\cos(\theta)$ in the xy-plane

Next, we repeat the process as θ ranges from $\frac{\pi}{2}$ to π. Here, the r values are all negative. This means that in the xy-plane, instead of graphing in Quadrant II, we graph in Quadrant IV, with all of the angle rotations starting from the negative x-axis.



$r = 6\cos(\theta)$ in the θr-plane

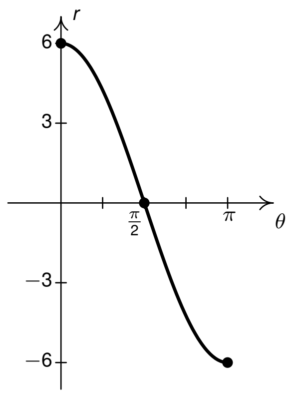


$r = 6\cos(\theta)$ in the xy-plane

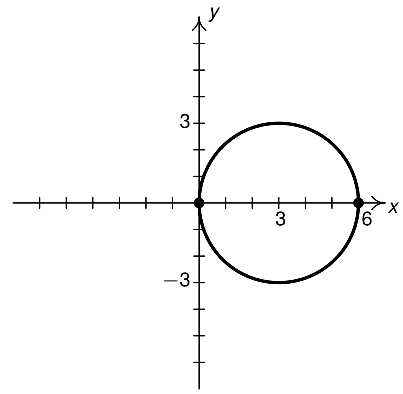
[3]The graph looks exactly like $y = 6\cos(x)$ in the xy-plane, and for good reason. At this stage, we are just graphing the relationship between r and θ before we interpret them as polar coordinates (r, θ) on the xy-plane.

As θ ranges from π to $\frac{3\pi}{2}$, the r values are still negative, which means the graph is traced out in Quadrant I instead of Quadrant III. Since the $|r|$ for these values of θ match the r values for θ in $\left[0, \frac{\pi}{2}\right]$, we have that the curve begins to retrace itself at this point.

Proceeding further, we find that when $\frac{3\pi}{2} \leq \theta \leq 2\pi$, we retrace the portion of the curve in Quadrant IV that we first traced out as $\frac{\pi}{2} \leq \theta \leq \pi$. The reader is invited to verify that plotting any range of θ outside the interval $[0, \pi]$ results in retracting some portion of the curve.[4] We present the final graph below.



$r = 6\cos(\theta)$ in the θr-plane



$r = 6\cos(\theta)$ in the xy-plane

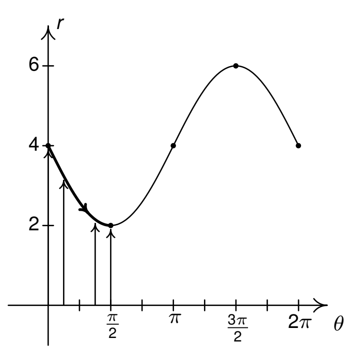
Example 14.2.2. Graph the following polar equations in the xy-plane.

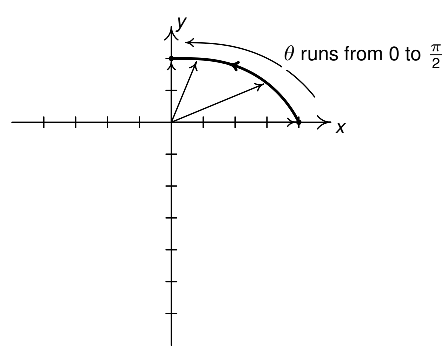
1. $r = 4 - 2\sin(\theta)$
2. $r = 2 + 4\cos(\theta)$
3. $r = 5\sin(2\theta)$
4. $r^2 = 16\cos(2\theta)$

Solution.

1. We first plot the fundamental cycle of $r = 4 - 2\sin(\theta)$ on the θr-axes. To help us visualize what is going on graphically, we divide up $[0, 2\pi]$ into the usual four subintervals $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$, $\left[\pi, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$, and proceed as we did above.

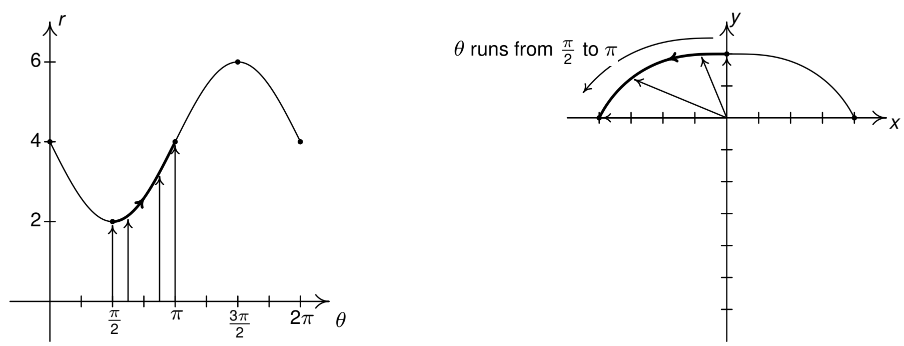
   As θ ranges from 0 to $\frac{\pi}{2}$, r decreases from 4 to 2. This means that the curve in the xy-plane starts 4 units from the origin on the positive x-axis and gradually pulls in towards the origin as it moves towards the positive y-axis.



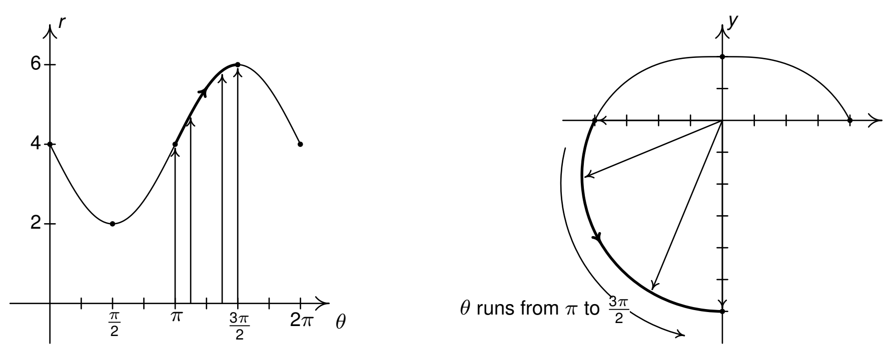


[4] The graph of $r = 6\cos(\theta)$ looks suspiciously like a circle, for good reason. See number 1a in Example 14.1.3.

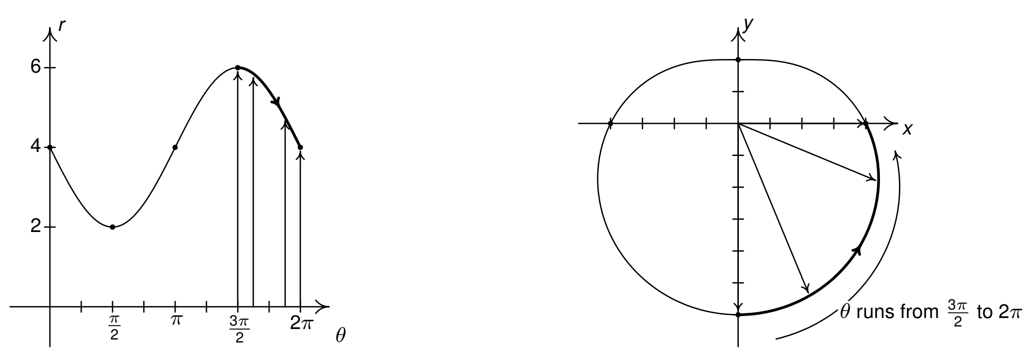
Next, as θ runs from $\frac{\pi}{2}$ to π, we see that r increases from 2 to 4. Picking up where we left off, we gradually pull the graph away from the origin until we reach the negative x-axis.



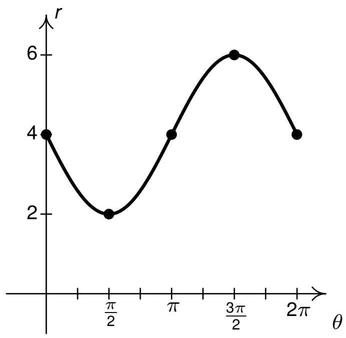
Over the interval $\left[\pi, \frac{3\pi}{2}\right]$, we see that r increases from 4 to 6. On the xy-plane, the curve sweeps out away from the origin as it travels from the negative x-axis to the negative y-axis.



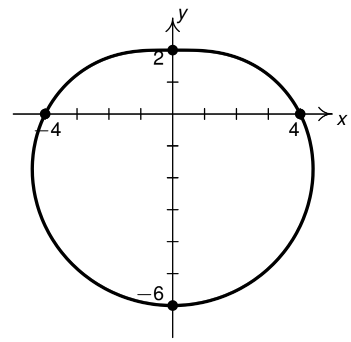
Finally, as θ takes on values from $\frac{3\pi}{2}$ to 2π, r decreases from 6 back to 4. The graph on the xy-plane pulls in from the negative y-axis to finish where we started.



We leave it to the reader to verify that plotting points corresponding to values of θ outside the interval $[0, 2\pi]$ results in retracing portions of the curve, so we are finished.



$r = 4 - 2\sin(\theta)$ in the θr-plane

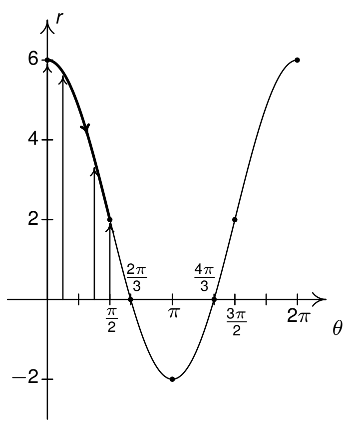


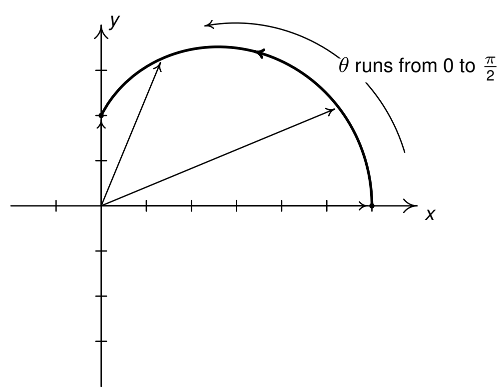
$r = 4 - 2\sin(\theta)$ in the xy-plane.

2. The first thing to note when graphing $r = 2 + 4\cos(\theta)$ on the θr-plane over the interval $[0, 2\pi]$ is that the graph crosses through the θ-axis. This corresponds to the graph of the curve passing through the origin in the xy-plane, so our first task is to determine when this happens.

   Setting $r = 0$ we get $2 + 4\cos(\theta) = 0$, or $\cos(\theta) = -\frac{1}{2}$. Solving for θ in $[0, 2\pi]$ gives $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. Since these values of θ are important geometrically, we break the interval $[0, 2\pi]$ into six subintervals: $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, $\left[\frac{2\pi}{3}, \pi\right]$, $\left[\pi, \frac{4\pi}{3}\right]$, $\left[\frac{4\pi}{3}, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$.

   As θ ranges from 0 to $\frac{\pi}{2}$, r decreases from 6 to 2. Plotting this on the xy-plane, we start 6 units out from the origin on the positive x-axis and slowly pull in towards the positive y-axis.

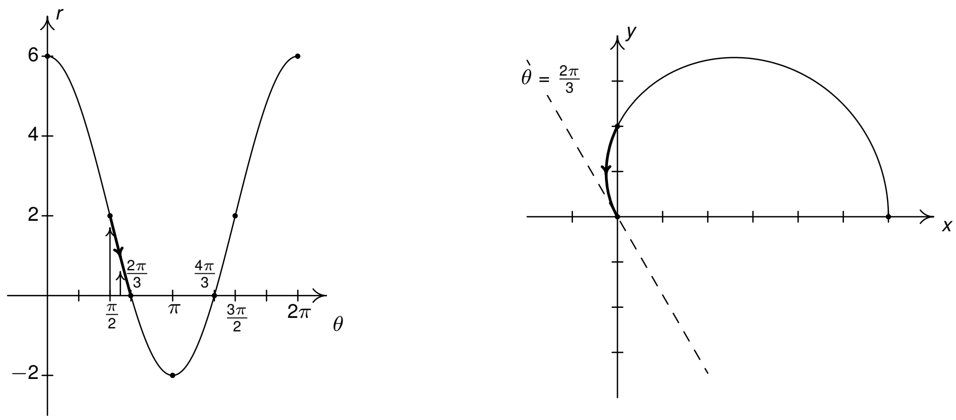




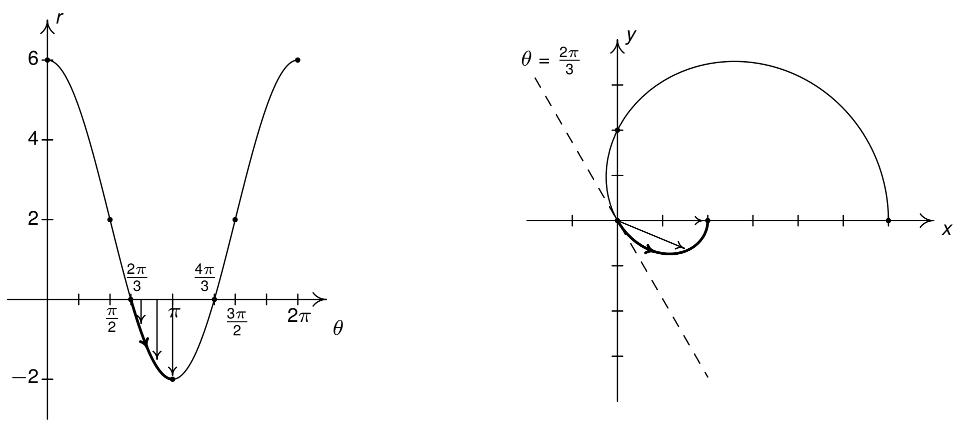
   On the interval $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, r decreases from 2 to 0, which means the graph is heading into (and will eventually cross through) the origin.

   Not only do we reach the origin when $\theta = \frac{2\pi}{3}$, a theorem from Calculus[5] states that the curve hugs the line $\theta = \frac{2\pi}{3}$ as it approaches the origin.

[5] The 'tangents at the pole' theorem from second semester Calculus.

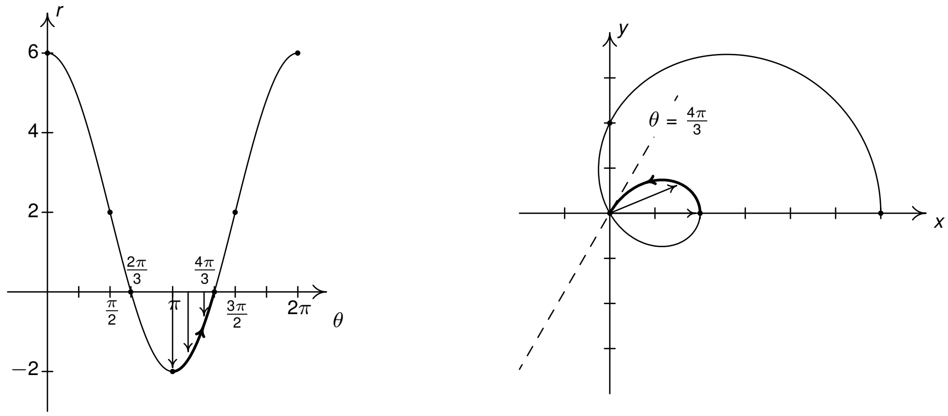


On the interval $\left[\frac{2\pi}{3}, \pi\right]$, r ranges from 0 to -2. Since $r \leq 0$, the curve passes through the origin in the xy-plane,[6] following the line $\theta = \frac{2\pi}{3}$. Since $|r|$ is increasing from 0 to 2, the curve pulls away from the origin and continues upwards through Quadrant IV to finish at a point on the positive x-axis.

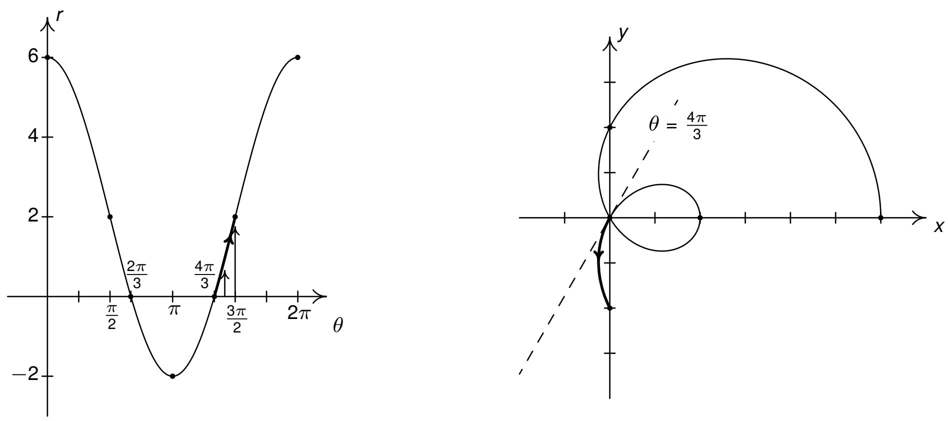


Next, as θ progresses from π to $\frac{4\pi}{3}$, r ranges from -2 to 0. Since $r \leq 0$, we continue our graph in the first quadrant, heading into the origin along the line $\theta = \frac{4\pi}{3}$.

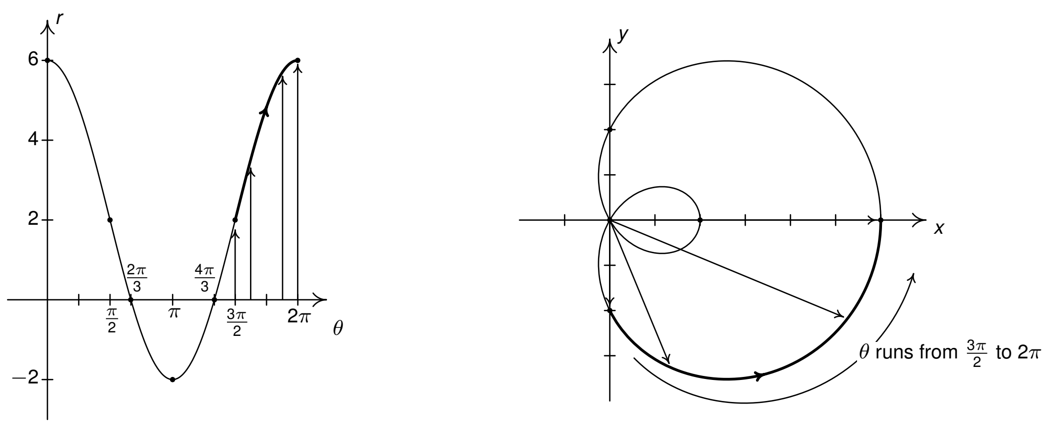
[6]Recall that one way to visualize plotting polar coordinates (r, θ) with $r < 0$ is to start the rotation from the left side of the pole, in this case, the negative x-axis. Rotating between $\frac{2\pi}{3}$ and π radians from the negative x-axis in this case determines the region between the line $\theta = \frac{2\pi}{3}$ and the x-axis in Quadrant IV.



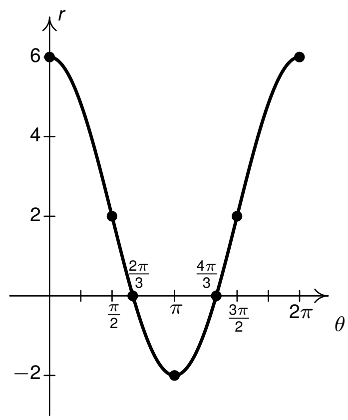
On the interval $\left[\frac{4\pi}{3}, \frac{3\pi}{2}\right]$, r returns to positive values and increases from 0 to 2. We hug the line $\theta = \frac{4\pi}{3}$ as we move through the origin and head towards the negative y-axis.



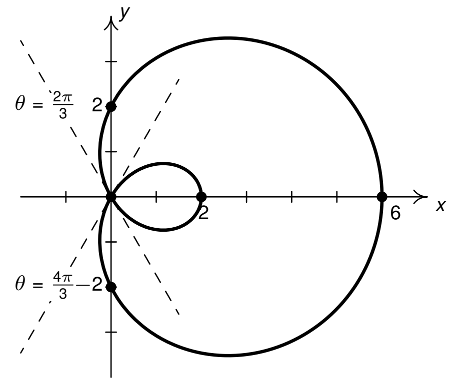
As we round out the interval, we find that as θ runs through $\frac{3\pi}{2}$ to 2π, r increases from 2 out to 6, and we end up back where we started, 6 units from the origin on the positive x-axis.



Again, we invite the reader to show that plotting the curve for values of θ outside $[0, 2\pi]$ results in retracing a portion of the curve already traced. Our final graph is below.



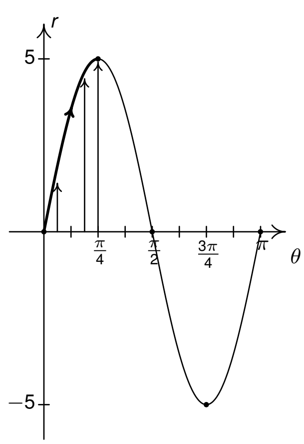
$r = 2 + 4\cos(\theta)$ in the θr-plane

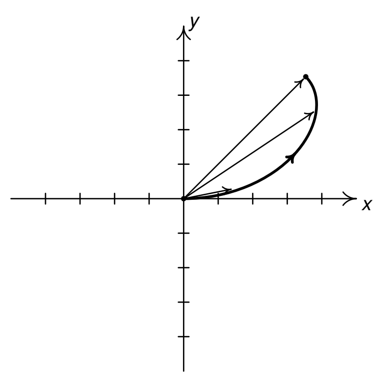


$r = 2 + 4\cos(\theta)$ in the xy-plane

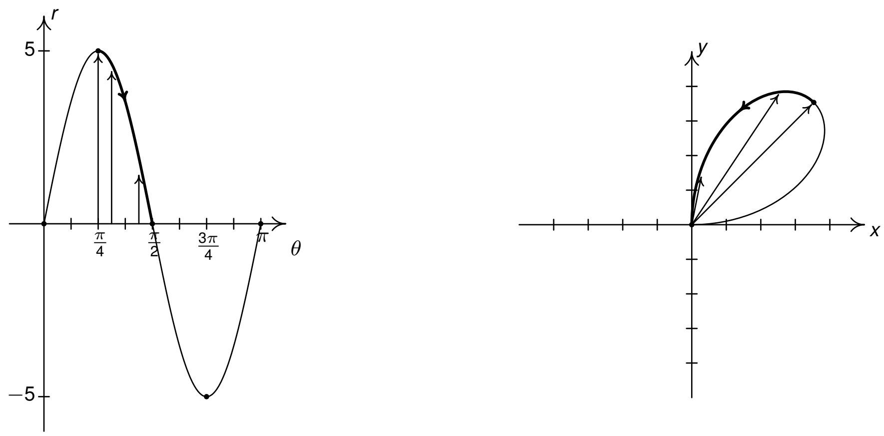
3. As usual, we start by graphing a fundamental cycle of $r = 5\sin(2\theta)$ in the θr-plane, which in this case, occurs as θ ranges from 0 to π. We partition our interval into subintervals to help us with the graphing, namely $\left[0, \frac{\pi}{4}\right]$, $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$ and $\left[\frac{3\pi}{4}, \pi\right]$.

   As θ ranges from 0 to $\frac{\pi}{4}$, r increases from 0 to 5. Hence the graph of $r = 5\sin(2\theta)$ in the xy-plane starts at the origin and gradually sweeps out so it is 5 units away from the origin on the line $\theta = \frac{\pi}{4}$.

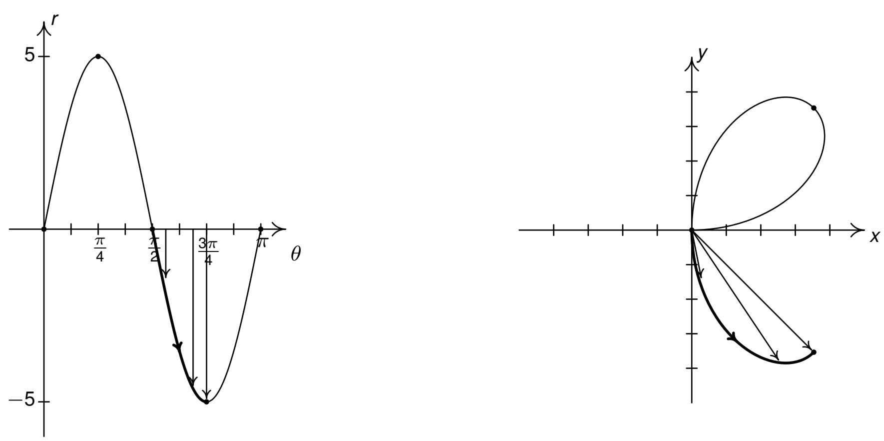




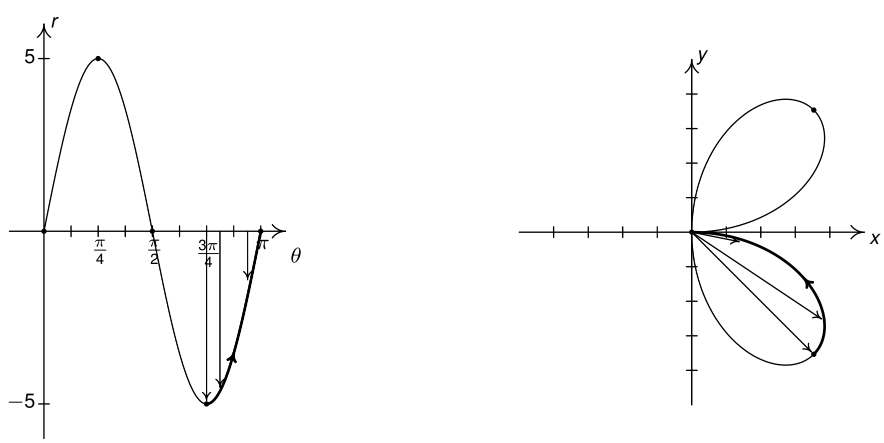
   Next, we see that r decreases from 5 to 0 as θ runs through $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. Moreover, r is becomes negative as θ crosses $\frac{\pi}{2}$. Hence, we draw the curve hugging the line $\theta = \frac{\pi}{2}$ (the y-axis) as the curve heads to the origin.



As θ runs from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, *r* becomes negative and ranges from 0 to -5. Since $r \leq 0$, the curve pulls away from the negative *y*-axis into Quadrant IV.

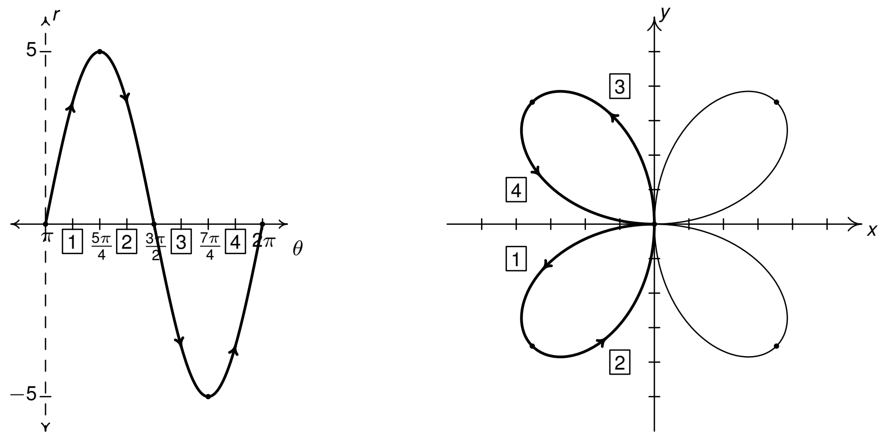


For $\frac{3\pi}{4} \leq \theta \leq \pi$, *r* increases from -5 to 0, so the curve pulls back to the origin.

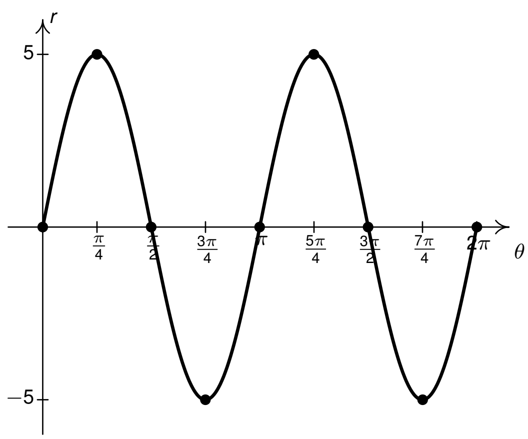


Even though we have finished with one complete cycle of $r = 5\sin(2\theta)$, if we continue plotting beyond $\theta = \pi$, we find to our surprise and delight that the curve continues into the third quadrant!

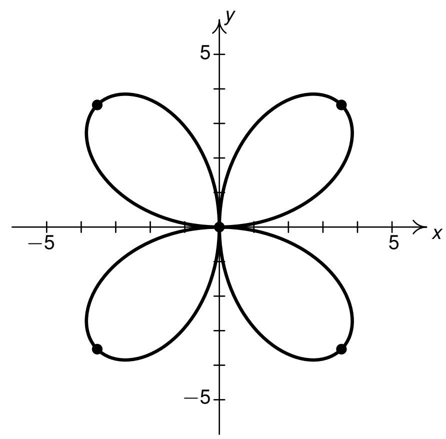
Below we present a graph of a second cycle of $r = 5\sin(2\theta)$ as continued from the first. The boxed labels on the θ-axis correspond to the portions with matching labels on the curve in the xy-plane.



We have the final graph below.



$r = 5\sin(2\theta)$ in the θr-plane



$r = 5\sin(2\theta)$ in the xy-plane

4. Graphing $r^2 = 16\cos(2\theta)$ is complicated by the r^2, so we solve for r by extracting square roots and get $r = \pm\sqrt{16\cos(2\theta)} = \pm 4\sqrt{\cos(2\theta)}$.

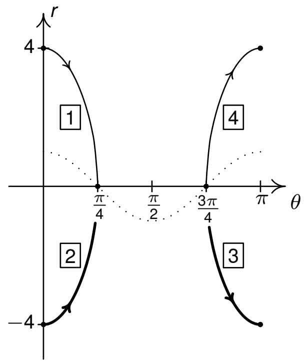
   How do we sketch such a curve? First off, we sketch a fundamental period of $r = \cos(2\theta)$ which we have dotted in the figure below. When $\cos(2\theta) < 0$, $\sqrt{\cos(2\theta)}$ is undefined, so we don't have any values on the interval $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$.

   On the intervals which remain, $\cos(2\theta)$ ranges from 0 to 1, inclusive. Hence, $\sqrt{\cos(2\theta)}$ ranges from 0 to 1 as well.[7] From this, we know $r = \pm 4\sqrt{\cos(2\theta)}$ ranges continuously from 0 to ± 4, respectively.

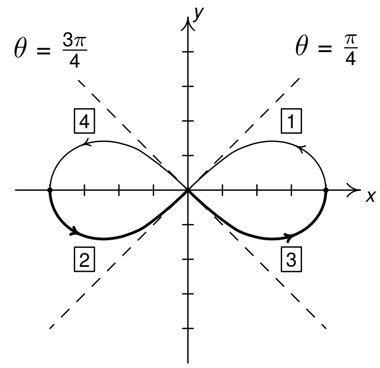
[7] Owing to the relationship between $y = x$ and $y = \sqrt{x}$ over $[0, 1]$, we know $\sqrt{\cos(2\theta)} \geq \cos(2\theta)$ wherever the former is defined.

Below we graph both $r = 4\sqrt{\cos(2\theta)}$ and $r = -4\sqrt{\cos(2\theta)}$ on the θr plane and use them to sketch the corresponding pieces of the curve $r^2 = 16\cos(2\theta)$ in the xy-plane.

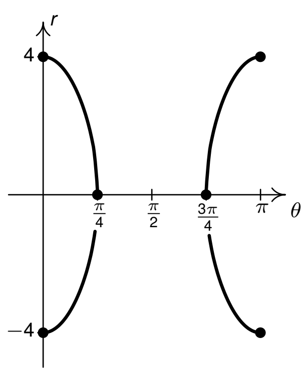
As we have seen in earlier examples, the lines $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$, which are the zeros of the functions $r = \pm 4\sqrt{\cos(2\theta)}$, serve as guides for us to draw the curve as is passes through the origin.



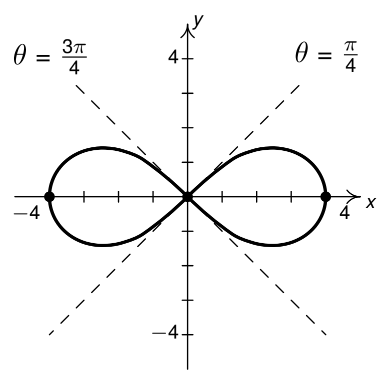
$r = 4\sqrt{\cos(2\theta)}$ and
$\boldsymbol{r = -4\sqrt{\cos(2\theta)}}$



As we plot points corresponding to values of θ outside of the interval $[0, \pi]$, we find ourselves retracing parts of the curve,[8] so our final answer is below.



$r = \pm 4\sqrt{\cos(2\theta)}$
in the θr-plane



$r^2 = 16\cos(2\theta)$
in the xy-plane

□

A few remarks are in order. First, there is no relation, in general, between the period of the function $f(\theta)$ and the length of the interval required to sketch the complete graph of $r = f(\theta)$ in the xy-plane.

As we saw on page 1217, despite the fact that the period of $f(\theta) = 6\cos(\theta)$ is 2π, we sketched the complete graph of $r = 6\cos(\theta)$ in the xy-plane just using the values of θ as θ ranged from 0 to π.

On the other hand, in Example 14.2.2, number 3, the period of $f(\theta) = 5\sin(2\theta)$ is π, but in order to obtain the complete graph of $r = 5\sin(2\theta)$, we needed to run θ from 0 to 2π.

Second, the symmetry seen in the examples is also a common occurrence when graphing polar equations.

[8]In this case, we could have generated the entire graph by using just the plot $r = 4\sqrt{\cos(2\theta)}$, but graphed over the interval $[0, 2\pi]$ in the θr-plane. We leave the details to the reader.

In addition symmetry about each axis and the origin, it is possible to talk about *rotational* symmetry with these curves. We leave the exploration of symmetry to Exercises 63 - 65.

Last we note that while many of the 'common' polar graphs can be grouped into families,[9] the authors truly feel that taking the time to work through each graph in the manner presented here is the best way to not only understand the polar coordinate system, but also prepare you for what is needed in Calculus.

Next we turn our attention to finding the intersection points of polar curves. What complicates matters in polar coordinates is that any given point has infinitely many representations. As a result, if a point P is on the graph of two different polar equations, it is entirely possible that the representation $P(r,\theta)$ which satisfies one of the equations does not satisfy the other equation.

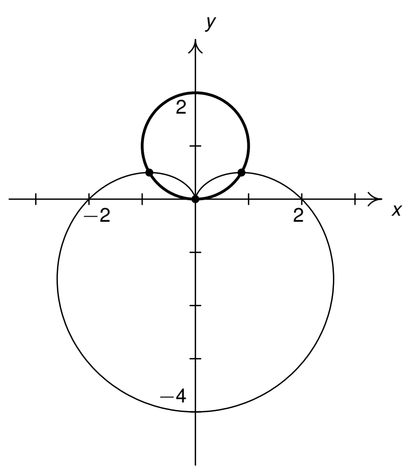
In our next example, we see the need to rely on Geometry as much as Algebra to solve each problem.

Example 14.2.3. Find the points of intersection of the graphs of the following polar equations.

1. $r = 2\sin(\theta)$ and $r = 2 - 2\sin(\theta)$
2. $r = 2$ and $r = 3\cos(\theta)$
3. $r = 3$ and $r = 6\cos(2\theta)$
4. $r = 3\sin\left(\frac{\theta}{2}\right)$ and $r = 3\cos\left(\frac{\theta}{2}\right)$

Solution.

1. Following the procedure in Example 14.2.2, we graph $r = 2\sin(\theta)$ and find it to be a circle centered at the point with rectangular coordinates $(0,1)$ with a radius of 1. The graph of $r = 2 - 2\sin(\theta)$ is a special kind of limaçon called a 'cardioid.'[10]



$r = 2 - 2\sin(\theta)$ and $\boldsymbol{r = 2\sin(\theta)}$

[9]Numbers 1 and 2 in Example 14.2.2 are examples of 'limaçons,' number 3 is an example of a 'polar rose,' and number 4 is the famous 'Lemniscate of Bernoulli.'

[10]Presumably, the name is derived from its resemblance to a stylized human heart.

It appears as if there are three intersection points: one in the first quadrant, one in the second quadrant, and the origin. Our next task is to find polar representations of these points.

In order for a point P to be on the graph of $r = 2\sin(\theta)$, it must have a representation $P(r, \theta)$ which satisfies $r = 2\sin(\theta)$. If P is also on the graph of $r = 2 - 2\sin(\theta)$, then P has a (possibly different) representation $P(r', \theta')$ which satisfies $r' = 2\sin(\theta')$. We first try to see if we can find any points which have a single representation $P(r, \theta)$ that satisfies both $r = 2\sin(\theta)$ and $r = 2 - 2\sin(\theta)$.

Assuming such a pair (r, θ) exists, then equating[11] the expressions for r gives $2\sin(\theta) = 2 - 2\sin(\theta)$ or $\sin(\theta) = \frac{1}{2}$. From this, we get $\theta = \frac{\pi}{6} + 2\pi k$ or $\theta = \frac{5\pi}{6} + 2\pi k$ for integers k.

Plugging $\theta = \frac{\pi}{6}$ into $r = 2\sin(\theta)$, we get $r = 2\sin\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{2}\right) = 1$, which is also the value we obtain when we substitute it into $r = 2 - 2\sin(\theta)$. Hence, $\left(1, \frac{\pi}{6}\right)$ is one representation for the point of intersection in the first quadrant.

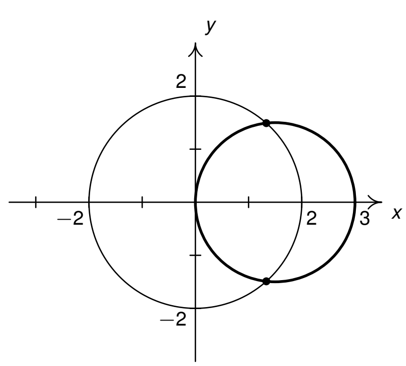
For the point of intersection in the second quadrant, we try $\theta = \frac{5\pi}{6}$. Both equations give us the point $\left(1, \frac{5\pi}{6}\right)$, so this is our answer here.

We now turn our attention to the origin. We know from Section 14.1 that the pole may be represented as $(0, \theta)$ for any angle θ. On the graph of $r = 2\sin(\theta)$, we start at the origin when $\theta = 0$ and return to it at $\theta = \pi$, and as the reader can verify, we are at the origin exactly when $\theta = \pi k$ for integers k.

On the curve $r = 2 - 2\sin(\theta)$, however, we reach the origin when $\theta = \frac{\pi}{2}$, and more generally, when $\theta = \frac{\pi}{2} + 2\pi k$ for integers k. There is no integer value of k for which $\pi k = \frac{\pi}{2} + 2\pi k$ which means while the origin is on both graphs, the point is never reached simultaneously. In any case, we have determined the three points of intersection to be $\left(1, \frac{\pi}{6}\right)$, $\left(1, \frac{5\pi}{6}\right)$ and the origin.

2. As before, we make a quick sketch of $r = 2$ and $r = 3\cos(\theta)$ to get feel for the number and location of the intersection points. The graph of $r = 2$ is a circle, centered at the origin, with a radius of 2.

   The graph of $r = 3\cos(\theta)$ is also a circle - but this one is centered at the point with rectangular coordinates $\left(\frac{3}{2}, 0\right)$ and has a radius of $\frac{3}{2}$.



$r = 2$ and $\boldsymbol{r = 3\cos(\theta)}$

[11]We are really using the technique of substitution to solve the system of equations $\left\{\begin{array}{rcl} r & = & 2\sin(\theta) \\ r & = & 2 - 2\sin(\theta) \end{array}\right.$

We have two intersection points to find, one in Quadrant I and one in Quadrant IV. Proceeding as above, we first determine if any of the intersection points P have a representation (r, θ) which satisfies both $r = 2$ and $r = 3\cos(\theta)$.

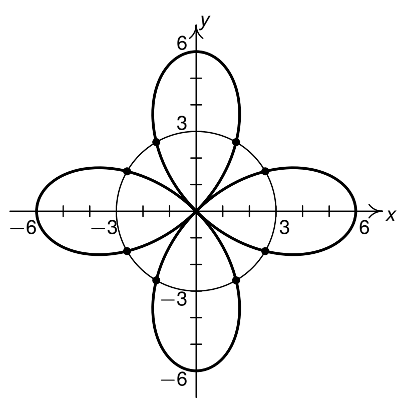
Equating $r = 2$ and $r = 3\cos(\theta)$, we get $2 = 3\cos(\theta)$, or $\cos(\theta) = \frac{2}{3}$. To solve this equation, we need the arccosine function: $\theta = \arccos\left(\frac{2}{3}\right) + 2\pi k$ or $\theta = 2\pi - \arccos\left(\frac{2}{3}\right) + 2\pi k$ for integers k.

From these solutions, we get $\left(2, \arccos\left(\frac{2}{3}\right)\right)$ as one representation for our answer in Quadrant I, and $\left(2, 2\pi - \arccos\left(\frac{2}{3}\right)\right)$ as one representation for our answer in Quadrant IV.

The reader is encouraged to check these results algebraically and geometrically.

3. Proceeding as above, we first graph $r = 3$ and $r = 6\cos(2\theta)$ to get an idea of how many intersection points to expect and where they lie.

   The graph of $r = 3$ is a circle centered at the origin with a radius of 3 and the graph of $r = 6\cos(2\theta)$ is another four-leafed rose.[12]



$r = 3$ and $\boldsymbol{r = 6\cos(2\theta)}$

   It appears as if there are eight points of intersection - two in each quadrant. We first look to see if there any points $P(r, \theta)$ with a representation that satisfies both $r = 3$ and $r = 6\cos(2\theta)$.

   Solving $6\cos(2\theta) = 3$, we get $\cos(2\theta) = \frac{1}{2}$, so $\theta = \frac{\pi}{6} + \pi k$ or $\theta = \frac{5\pi}{6} + \pi k$ for integers k. From these, we obtain four distinct points represented by $\left(3, \frac{\pi}{6}\right)$, $\left(3, \frac{5\pi}{6}\right)$, $\left(3, \frac{7\pi}{6}\right)$ and $\left(3, \frac{11\pi}{6}\right)$.

   To determine the coordinates of the remaining four points, we have to consider how the representations of the points of intersection can differ. We know from Section 14.1 that if (r, θ) and (r', θ') represent the same point and $r \neq 0$, then either $r = r'$ or $r = -r'$.

   If $r = r'$, then $\theta' = \theta + 2\pi k$, so one possibility is that an intersection point P has a representation (r, θ) which satisfies $r = 3$ and another representation $(r, \theta + 2\pi k)$ for some integer, k which satisfies

[12] See Example 14.2.2 number 3.

$r = 6\cos(2\theta)$. At this point,[13] we replace every occurrence of θ in the equation $r = 6\cos(2\theta)$ with $(\theta + 2\pi k)$ to see if, by equating the resulting expressions for r, we get any more solutions for θ.

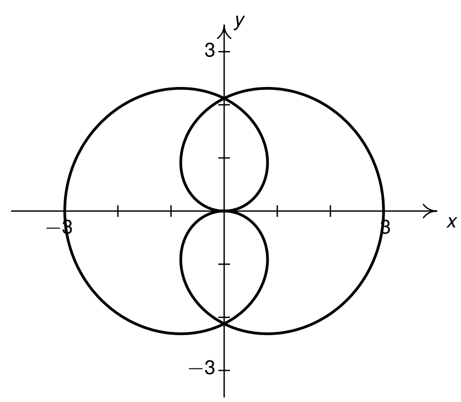
Doing so, we get $\cos(2(\theta + 2\pi k)) = \cos(2\theta + 4\pi k) = \cos(2\theta)$ for every integer k. Hence, the equation $r = 6\cos(2(\theta + 2\pi k))$ reduces to the same equation we had before, $r = 6\cos(2\theta)$, which means we get no additional solutions.

Moving on to the case where $r = -r'$, we have that $\theta' = \theta + (2k+1)\pi$ for integers k. We look to see if we can find points P which have a representation (r, θ) that satisfies $r = 3$ and another, $(-r, \theta + (2k+1)\pi)$, that satisfies $r = 6\cos(2\theta)$.

Substituting[14] $(-r)$ for r and $(\theta + (2k+1)\pi)$ for θ in $r = 6\cos(2\theta)$ gives $-r = 6\cos(2(\theta + (2k+1)\pi))$. Since $\cos(2(\theta + (2k+1)\pi)) = \cos(2\theta + (2k+1)(2\pi)) = \cos(2\theta)$ for all integers k, the equation $-r = 6\cos(2(\theta + (2k+1)\pi))$ reduces to $-r = 6\cos(2\theta)$, or $r = -6\cos(2\theta)$.

Coupling $r = -6\cos(2\theta)$ with $r = 3$ gives $-6\cos(2\theta) = 3$ or $\cos(2\theta) = -\frac{1}{2}$. Solving, we get $\theta = \frac{\pi}{3} + \pi k$ or $\theta = \frac{2\pi}{3} + \pi k$. From these solutions, we obtain[15] the remaining four intersection points with representations $\left(-3, \frac{\pi}{3}\right)$, $\left(-3, \frac{2\pi}{3}\right)$, $\left(-3, \frac{4\pi}{3}\right)$ and $\left(-3, \frac{5\pi}{3}\right)$, which check graphically.

4. As usual, we begin by graphing $r = 3\sin\left(\frac{\theta}{2}\right)$ and $r = 3\cos\left(\frac{\theta}{2}\right)$. Using the techniques presented in Example 14.2.2, we plot both functions as θ ranges from 0 to 4π to obtain the complete graph. To our surprise and/or delight, it appears as if these two equations describe the *same* curve!



$r = 3\sin\left(\frac{\theta}{2}\right)$ and $\boldsymbol{r = 3\cos\left(\frac{\theta}{2}\right)}$
appear to determine the same curve in the *xy*-plane

[13]The authors have chosen to replace θ with $\theta + 2\pi k$ in the equation $r = 6\cos(2\theta)$ for illustration purposes only. We could have just as easily chosen to do this substitution in the equation $r = 3$. Since there is no θ in $r = 3$, however, this case would reduce to the previous case instantly. The reader is encouraged to follow this latter procedure in the interests of efficiency.

[14]Again, we could have easily chosen to substitute these into $r = 3$ which would give $-r = 3$, or $r = -3$.

[15]We obtain these representations by substituting the values for θ into $r = 6\cos(2\theta)$, once again, for illustration purposes. Again, we could 'plug' these values for θ into $r = 3$ (where there is no θ) and get the list of points: $\left(3, \frac{\pi}{3}\right)$, $\left(3, \frac{2\pi}{3}\right)$, $\left(3, \frac{4\pi}{3}\right)$ and $\left(3, \frac{5\pi}{3}\right)$. While it is not true that $\left(3, \frac{\pi}{3}\right)$ represents the same point as $\left(-3, \frac{\pi}{3}\right)$, we still get the same *set* of solutions.

To verify this incredible claim,[16] we need to show that, in fact, the graphs of these two equations intersect at all points on the plane.

Suppose P has a representation (r, θ) which satisfies both $r = 3\sin\left(\frac{\theta}{2}\right)$ and $r = 3\cos\left(\frac{\theta}{2}\right)$. Equating these two expressions for r gives the equation $3\sin\left(\frac{\theta}{2}\right) = 3\cos\left(\frac{\theta}{2}\right)$. While normally we discourage dividing by a variable expression (in case it could be 0), we use the same logic here as we did in the solution to Example 14.1.3 number 1c in Section 14.1.

If $3\cos\left(\frac{\theta}{2}\right) = 0$, then $\cos\left(\frac{\theta}{2}\right) = 0$ and for the equation $3\sin\left(\frac{\theta}{2}\right) = 3\cos\left(\frac{\theta}{2}\right)$ to hold, $\sin\left(\frac{\theta}{2}\right) = 0$ as well. Since no angles have both cosine and sine equal to zero, we are safe to divide both sides of the equation $3\sin\left(\frac{\theta}{2}\right) = 3\cos\left(\frac{\theta}{2}\right)$ by $3\cos\left(\frac{\theta}{2}\right)$ to get $\tan\left(\frac{\theta}{2}\right) = 1$. Solving this equation gives $\theta = \frac{\pi}{2} + 2\pi k$ for integers k which corresponds to just *one* intersection point: $\left(\frac{3\sqrt{2}}{2}, \frac{\pi}{2}\right)$. We now investigate other representations for the intersection points.

Suppose P is an intersection point with a representation (r, θ) which satisfies $r = 3\sin\left(\frac{\theta}{2}\right)$ and a different representation $(r, \theta + 2\pi k)$ for some integer k which satisfies $r = 3\cos\left(\frac{\theta}{2}\right)$.

Substituting $(r, \theta + 2\pi k)$ into $r = 3\cos\left(\frac{\theta}{2}\right)$, we get $r = 3\cos\left(\frac{1}{2}\left[\theta + 2\pi k\right]\right) = 3\cos\left(\frac{\theta}{2} + \pi k\right)$. Using the sum formula for cosine, we expand $3\cos\left(\frac{\theta}{2} + \pi k\right) = 3\cos\left(\frac{\theta}{2}\right)\cos(\pi k) - 3\sin\left(\frac{\theta}{2}\right)\sin(\pi k)$. Since $\sin(\pi k) = 0$ for all integers k, $r = 3\cos\left(\frac{\theta}{2} + \pi k\right)$ reduces to $r = 3\cos\left(\frac{\theta}{2}\right)\cos(\pi k)$.

If k is an even integer, $\cos(\pi k) = 1$, so we get the same equation $r = 3\cos\left(\frac{\theta}{2}\right)$ as before, and hence any new solutions come from the case when k is odd.

If k is odd, $r = 3\cos\left(\frac{\theta}{2}\right)\cos(\pi k)$ reduces to $r = -3\cos\left(\frac{\theta}{2}\right)$. Coupling $r = -3\cos\left(\frac{\theta}{2}\right)$ with the equation $r = 3\sin\left(\frac{\theta}{2}\right)$ gives $3\sin\left(\frac{\theta}{2}\right) = -3\cos\left(\frac{\theta}{2}\right)$, or $\tan\left(\frac{\theta}{2}\right) = -1$. Solving, we get $\theta = -\frac{\pi}{2} + 2\pi k$ for integers k, which again produces just one intersection point: $\left(\frac{3\sqrt{2}}{2}, -\frac{\pi}{2}\right)$.

Next, we assume P has a representation (r, θ) which satisfies $r = 3\sin\left(\frac{\theta}{2}\right)$ and a representation $(-r, \theta + (2k+1)\pi)$ which satisfies $r = 3\cos\left(\frac{\theta}{2}\right)$ for some integer k.

Substituting $(-r)$ for r and $(\theta + (2k+1)\pi)$ in for θ into $r = 3\cos\left(\frac{\theta}{2}\right)$ gives $-r = 3\cos\left(\frac{1}{2}\left[\theta + (2k+1)\pi\right]\right)$ or $r = -3\cos\left(\frac{1}{2}\left[\theta + (2k+1)\pi\right]\right)$. Once again, we use the sum formula for cosine to get

$$\begin{aligned} r &= -3\cos\left(\frac{1}{2}\left[\theta + (2k+1)\pi\right]\right) \\ &= -3\cos\left(\frac{\theta}{2} + \frac{(2k+1)\pi}{2}\right) \\ &= -3\left[\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{(2k+1)\pi}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{(2k+1)\pi}{2}\right)\right] \\ &= 3\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{(2k+1)\pi}{2}\right) \end{aligned}$$

where the last equality is true since $\cos\left(\frac{(2k+1)\pi}{2}\right) = 0$.

[16]Graphing $r = 3\sin\left(\frac{\theta}{2}\right)$ and $r = 3\cos\left(\frac{\theta}{2}\right)$ in the θr-plane show that viewed as functions of r, these are two different animals.

Note when $k = 0$, $\sin\left(\frac{(2k+1)\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$, and the equation $r = -3\cos\left(\frac{1}{2}\left[\theta + (2k+1)\pi\right]\right)$ reduces to $r = 3\sin\left(\frac{\theta}{2}\right)$, which is the other equation under consideration!

What this means is that if a polar representation (r, θ) for the point P satisfies $r = 3\sin\left(\frac{\theta}{2}\right)$, then the representation $(-r, \theta + \pi)$ for P automatically satisfies $r = 3\cos\left(\frac{\theta}{2}\right)$. Hence the equations $r = 3\sin\left(\frac{\theta}{2}\right)$ and $r = 3\cos\left(\frac{\theta}{2}\right)$ determine the same set of points in the plane. □

Our work in Example 14.2.3 justifies the following.

Guidelines for Finding Points of Intersection of Graphs of Polar Equations:

To find the points of intersection of the graphs of two polar equations E_1 and E_2:

- Sketch the graphs of E_1 and E_2. Check to see if the curves intersect at the origin (pole).
- Solve for pairs (r, θ) which satisfy both E_1 and E_2.
- Substitute $(\theta + 2\pi k)$ for θ in either one of E_1 or E_2 (but not both) and solve for pairs (r, θ) which satisfy both equations. Keep in mind that k is an integer.
- Substitute $(-r)$ for r and $(\theta + (2k+1)\pi)$ for θ in either one of E_1 or E_2 (but not both) and solve for pairs (r, θ) which satisfy both equations. Keep in mind that k is an integer.

Our last example ties together graphing and points of intersection to describe regions in the plane.

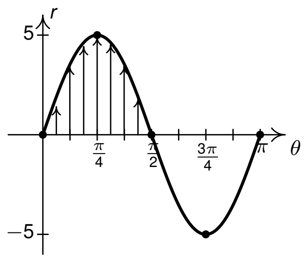
Example 14.2.4. Sketch the region in the xy-plane described by the following sets.

1. $\left\{(r, \theta) \,|\, 0 \leq r \leq 5\sin(2\theta),\; 0 \leq \theta \leq \frac{\pi}{2}\right\}$
2. $\left\{(r, \theta) \,|\, 3 \leq r \leq 6\cos(2\theta),\; 0 \leq \theta \leq \frac{\pi}{6}\right\}$
3. $\left\{(r, \theta) \,|\, 2 + 4\cos(\theta) \leq r \leq 0,\; \frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}\right\}$
4. $\left\{(r, \theta) \,|\, 0 \leq r \leq 2\sin(\theta),\; 0 \leq \theta \leq \frac{\pi}{6}\right\} \cup \left\{(r, \theta) \,|\, 0 \leq r \leq 2 - 2\sin(\theta),\; \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}\right\}$

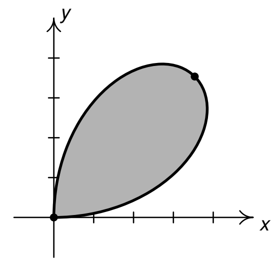
Solution. Our first step in these problems is to sketch the graphs of the polar equations involved to get a sense of the geometric situation. Since all of the equations in this example are found in either Example 14.2.2 or Example 14.2.3, most of the work is done for us.

1. We know from Example 14.2.2 number 3 that the graph of $r = 5\sin(2\theta)$ is a rose. Moreover, we know as $0 \leq \theta \leq \frac{\pi}{2}$, we trace out the 'leaf' of the rose which lies in the first quadrant.

   The inequality $0 \leq r \leq 5\sin(2\theta)$ means we want all of the points between the origin ($r = 0$) and the curve $r = 5\sin(2\theta)$ as θ runs through $\left[0, \frac{\pi}{2}\right]$. Hence, the region we seek is the leaf itself.



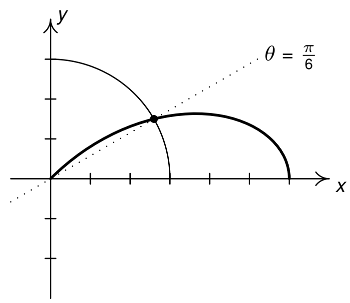
$r = 5\sin(2\theta)$



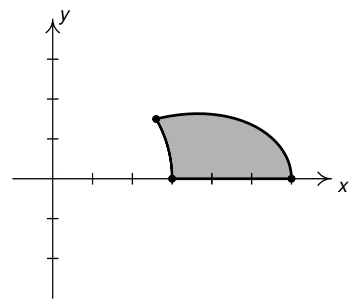
$\left\{ (r, \theta) \,|\, 0 \leq r \leq 5\sin(2\theta), 0 \leq \theta \leq \frac{\pi}{2} \right\}$

2. We know from Example 14.2.3 number 3 that $r = 3$ and $r = 6\cos(2\theta)$ intersect at $\theta = \frac{\pi}{6}$, so the region that is being described here is the set of points whose directed distance r from the origin is at least 3 but no more than $6\cos(2\theta)$ as θ runs from 0 to $\frac{\pi}{6}$.

   In other words, we are looking at the points outside or on the circle (since $r \geq 3$) but inside or on the rose (since $r \leq 6\cos(2\theta)$). We shade the region below.



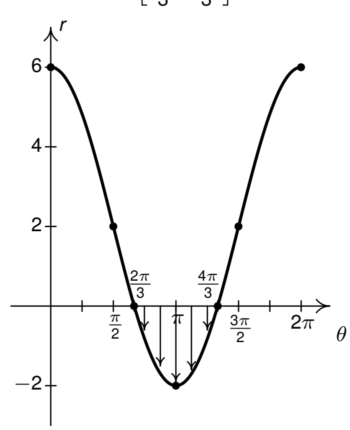
$r = 3$ and $\boldsymbol{r = 6\cos(2\theta)}$



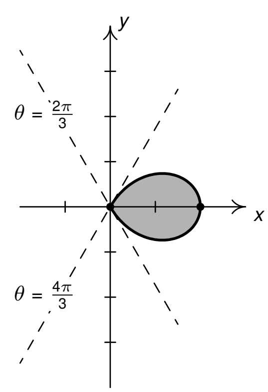
$\left\{ (r, \theta) \,|\, 3 \leq r \leq 6\cos(2\theta), 0 \leq \theta \leq \frac{\pi}{6} \right\}$

3. From Example 14.2.2 number 2, we know that the graph of $r = 2 + 4\cos(\theta)$ is a limaçon whose 'inner loop' is traced out as θ runs through the given values $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$.

   Since the values r takes on in this interval are non-positive, the inequality $2 + 4\cos(\theta) \leq r \leq 0$ makes sense, and we are looking for all of the points between the pole $r = 0$ and the limaçon as θ ranges over the interval $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$. In other words, we shade in the inner loop of the limaçon.



$r = 2 + 4\cos(\theta)$



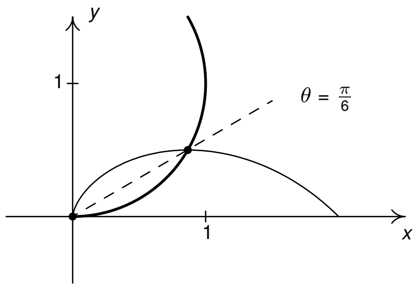
$\left\{ (r, \theta) \,|\, 2 + 4\cos(\theta) \leq r \leq 0, \frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3} \right\}$

4. We have two regions described here connected with the union symbol '$\cup$.' We shade each in turn and find our final answer by combining the two.

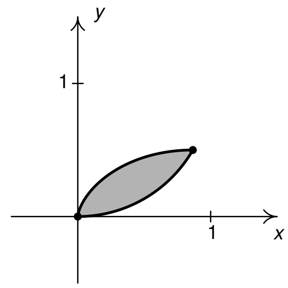
   In Example 14.2.3, number 1, we found that the curves $r = 2\sin(\theta)$ and $r = 2 - 2\sin(\theta)$ intersect when $\theta = \frac{\pi}{6}$. Hence, for the first region, $\left\{(r,\theta) \,|\, 0 \leq r \leq 2\sin(\theta), 0 \leq \theta \leq \frac{\pi}{6}\right\}$, we are shading the region between the origin ($r = 0$) out to the circle ($r = 2\sin(\theta)$) as θ ranges from 0 to $\frac{\pi}{6}$, which is the angle of intersection of the two curves.

   For the second region, $\left\{(r,\theta) \,|\, 0 \leq r \leq 2 - 2\sin(\theta), \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}\right\}$, θ picks up where it left off at $\frac{\pi}{6}$ and continues to $\frac{\pi}{2}$. In this case, however, we are shading from the origin ($r = 0$) out to the cardioid $r = 2 - 2\sin(\theta)$ which pulls into the origin at $\theta = \frac{\pi}{2}$.

   We combine these two regions to obtain our final answer.



$r = 2 - 2\sin(\theta)$ and $\boldsymbol{r = 2\sin(\theta)}$



$\left\{(r,\theta) \,|\, 0 \leq r \leq 2\sin(\theta),\ 0 \leq \theta \leq \frac{\pi}{6}\right\} \cup$
$\left\{(r,\theta) \,|\, 0 \leq r \leq 2 - 2\sin(\theta),\ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}\right\}$

$\square$

14.2.1 Exercises

In Exercises 1 - 20, plot the graph of the polar equation by hand. Carefully label your graphs.

1. Circle: $r = 6\sin(\theta)$
2. Circle: $r = 2\cos(\theta)$
3. Rose: $r = 2\sin(2\theta)$
4. Rose: $r = 4\cos(2\theta)$
5. Rose: $r = 5\sin(3\theta)$
6. Rose: $r = \cos(5\theta)$
7. Rose: $r = \sin(4\theta)$
8. Rose: $r = 3\cos(4\theta)$
9. Cardioid: $r = 3 - 3\cos(\theta)$
10. Cardioid: $r = 5 + 5\sin(\theta)$
11. Cardioid: $r = 2 + 2\cos(\theta)$
12. Cardioid: $r = 1 - \sin(\theta)$
13. Limaçon: $r = 1 - 2\cos(\theta)$
14. Limaçon: $r = 1 - 2\sin(\theta)$
15. Limaçon: $r = 2\sqrt{3} + 4\cos(\theta)$
16. Limaçon: $r = 3 - 5\cos(\theta)$
17. Limaçon: $r = 3 - 5\sin(\theta)$
18. Limaçon: $r = 2 + 7\sin(\theta)$
19. Lemniscate: $r^2 = \sin(2\theta)$
20. Lemniscate: $r^2 = 4\cos(2\theta)$

In Exercises 21 - 30, find the exact polar coordinates of the points of intersection of graphs of the polar equations. Remember to check for intersection at the pole (origin).

21. $r = 3\cos(\theta)$ and $r = 1 + \cos(\theta)$
22. $r = 1 + \sin(\theta)$ and $r = 1 - \cos(\theta)$
23. $r = 1 - 2\sin(\theta)$ and $r = 2$
24. $r = 1 - 2\cos(\theta)$ and $r = 1$
25. $r = 2\cos(\theta)$ and $r = 2\sqrt{3}\sin(\theta)$
26. $r = 3\cos(\theta)$ and $r = \sin(\theta)$
27. $r^2 = 4\cos(2\theta)$ and $r = \sqrt{2}$
28. $r^2 = 2\sin(2\theta)$ and $r = 1$
29. $r = 4\cos(2\theta)$ and $r = 2$
30. $r = 2\sin(2\theta)$ and $r = 1$

In Exercises 31 - 40, sketch the region in the *xy*-plane described by the given set.

31. $\{(r,\theta)\,|\,0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}$
32. $\{(r,\theta)\,|\,0 \leq r \leq 4\sin(\theta),\ 0 \leq \theta \leq \pi\}$
33. $\left\{(r,\theta)\,|\,0 \leq r \leq 3\cos(\theta),\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right\}$
34. $\left\{(r,\theta)\,|\,0 \leq r \leq 2\sin(2\theta),\ 0 \leq \theta \leq \frac{\pi}{2}\right\}$

35. $\left\{(r, \theta) \,|\, 0 \leq r \leq 4\cos(2\theta),\ -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\right\}$

36. $\left\{(r, \theta) \,|\, 1 \leq r \leq 1 - 2\cos(\theta),\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right\}$

37. $\left\{(r, \theta) \,|\, 1 + \cos(\theta) \leq r \leq 3\cos(\theta),\ -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}\right\}$

38. $\left\{(r, \theta) \,|\, 1 \leq r \leq \sqrt{2\sin(2\theta)},\ \frac{13\pi}{12} \leq \theta \leq \frac{17\pi}{12}\right\}$

39. $\left\{(r, \theta) \,|\, 0 \leq r \leq 2\sqrt{3}\sin(\theta),\ 0 \leq \theta \leq \frac{\pi}{6}\right\} \cup \left\{(r, \theta) \,|\, 0 \leq r \leq 2\cos(\theta),\ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}\right\}$

40. $\left\{(r, \theta) \,|\, 0 \leq r \leq 2\sin(2\theta),\ 0 \leq \theta \leq \frac{\pi}{12}\right\} \cup \left\{(r, \theta) \,|\, 0 \leq r \leq 1,\ \frac{\pi}{12} \leq \theta \leq \frac{\pi}{4}\right\}$

In Exercises 41 - 50, use set-builder notation to describe the polar region. Assume that the region contains its bounding curves.

41. The region inside the circle $r = 5$.

42. The region inside the circle $r = 5$ which lies in Quadrant III.

43. The region inside the left half of the circle $r = 6\sin(\theta)$.

44. The region inside the circle $r = 4\cos(\theta)$ which lies in Quadrant IV.

45. The region inside the top half of the cardioid $r = 3 - 3\cos(\theta)$

46. The region inside the cardioid $r = 2 - 2\sin(\theta)$ which lies in Quadrants I and IV.

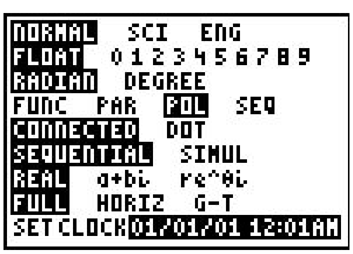
47. The inside of the petal of the rose $r = 3\cos(4\theta)$ which lies on the positive x-axis

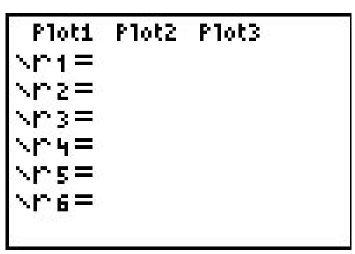
48. The region inside the circle $r = 5$ but outside the circle $r = 3$.

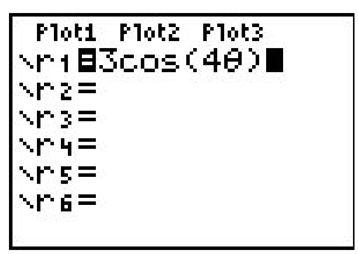
49. The region which lies inside of the circle $r = 3\cos(\theta)$ but outside of the circle $r = \sin(\theta)$

50. The region in Quadrant I which lies inside both the circle $r = 3$ as well as the rose $r = 6\sin(2\theta)$

While the authors truly believe that graphing polar curves by hand is fundamental to your understanding of the polar coordinate system, we would be derelict in our duties if we totally ignored the graphing utility.[17] Indeed, there are some important polar curves which are simply too difficult to graph by hand and that makes the calculator an important tool for your further studies in Mathematics, Science and Engineering. We now give a brief demonstration of how to use the graphing utility to plot polar curves. The first thing you must do is switch the MODE of your calculator to POL, which stands for “polar”.

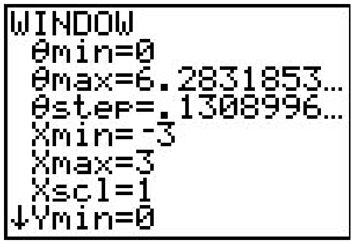


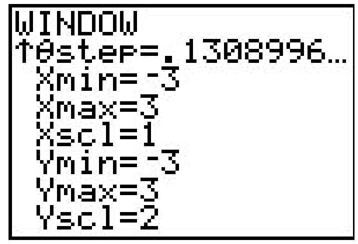




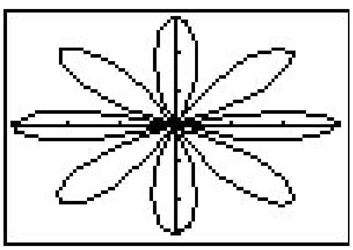
[17] As of this writing, while free online websites and apps like desmos are gaining popularity, the TI-83/84 series calculators are still in wide circulation.

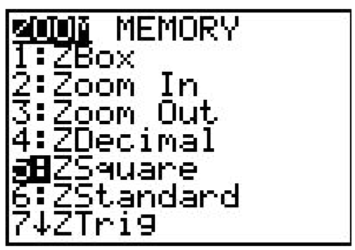
This changes the “Y=” menu as seen above in the middle. Let’s plot the polar rose given by $r = 3\cos(4\theta)$ from Exercise 8 above. We type the function into the “r=” menu as seen above on the right. We need to set the viewing window so that the curve displays properly, but when we look at the WINDOW menu, we find three extra lines.

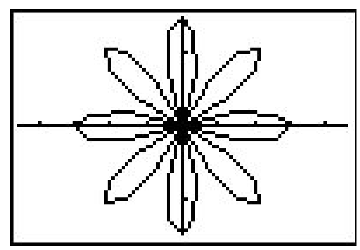




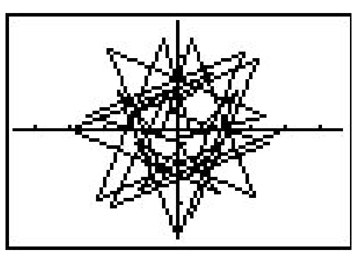
In order for the calculator to be able to plot $r = 3\cos(4\theta)$ in the xy-plane, we need to tell it not only the dimensions which x and y will assume, but we also what values of θ to use. From our previous work, we know that we need $0 \leq \theta \leq 2\pi$, so we enter the data you see above. (I’ll say more about the θ-step in just a moment.) Hitting GRAPH yields the curve below on the left which doesn’t look quite right. The issue here is that the calculator screen is 96 pixels wide but only 64 pixels tall. To get a true geometric perspective, we need to hit ZOOM SQUARE (seen below in the middle) to produce a more accurate graph which we present below on the right.







In function mode, the calculator automatically divided the interval [Xmin, Xmax] into 96 equal subintervals. In polar mode, however, we must specify how to split up the interval [θmin, θmax] using the θstep. For most graphs, a θstep of 0.1 is fine. If you make it too small then the calculator takes a long time to graph. It you make it too big, you get chunky garbage like this.



You will need to experiment with the settings in order to get a nice graph. Exercises 51 - 60 give you some curves to graph using your calculator. Note some of them have explicit bounds on θ and others do not.

51. $r = \theta$, $0 \leq \theta \leq 12\pi$

52. $r = \ln(\theta)$, $1 \leq \theta \leq 12\pi$

53. $r = e^{.1\theta}$, $0 \leq \theta \leq 12\pi$

54. $r = \theta^3 - \theta$, $-1.2 \leq \theta \leq 1.2$

55. $r = \sin(5\theta) - 3\cos(\theta)$

56. $r = \sin^3\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{3}\right)$

57. $r = \arctan(\theta)$, $-\pi \leq \theta \leq \pi$

58. $r = \dfrac{1}{1 - \cos(\theta)}$

59. $r = \dfrac{1}{2 - \cos(\theta)}$

60. $r = \dfrac{1}{2 - 3\cos(\theta)}$

61. Use a graphing utility to graph $r = a - b\sin(\theta)$ for various (positive) values of a and b. Describe the shape of the curve when $a = b$, $a < b$, and when $a > b$.

62. How many petals does the polar rose $r = \sin(2\theta)$ have? What about $r = \sin(3\theta)$, $r = \sin(4\theta)$ and $r = \sin(5\theta)$? With the help of your classmates, make a conjecture as to how many petals the polar rose $r = \sin(n\theta)$ has for any natural number n. Replace sine with cosine and repeat the investigation. How many petals does $r = \cos(n\theta)$ have for each natural number n?

Looking back through the graphs in the section, it's clear that many polar curves enjoy various forms of symmetry. However, classifying symmetry for polar curves is not as straight-forward as it was for equations back in Section 5.5. In Exercises 63 - 65, we have you and your classmates explore some of the more basic forms of symmetry seen in common polar curves.

63. Show that if f is even[18] then the graph of $r = f(\theta)$ is symmetric about the x-axis.

    (a) Show that $f(\theta) = 2 + 4\cos(\theta)$ is even and verify that the graph of $r = 2 + 4\cos(\theta)$ is indeed symmetric about the x-axis. (See Example 14.2.2 number 2.)

    (b) Show that $f(\theta) = 3\sin\left(\frac{\theta}{2}\right)$ is **not** even, yet the graph of $r = 3\sin\left(\frac{\theta}{2}\right)$ **is** symmetric about the x-axis. (See Example 14.2.3 number 4.)

64. Show that if f is odd[19] then the graph of $r = f(\theta)$ is symmetric about the origin.

    (a) Show that $f(\theta) = 5\sin(2\theta)$ is odd and verify that the graph of $r = 5\sin(2\theta)$ is indeed symmetric about the origin. (See Example 14.2.2 number 3.)

    (b) Show that $f(\theta) = 3\cos\left(\frac{\theta}{2}\right)$ is **not** odd, yet the graph of $r = 3\cos\left(\frac{\theta}{2}\right)$ **is** symmetric about the origin. (See Example 14.2.3 number 4.)

[18] Recall that this means $f(-\theta) = f(\theta)$ for θ in the domain of f.

[19] Recall that this means $f(-\theta) = -f(\theta)$ for θ in the domain of f.

65. Show that if $f(\pi - \theta) = f(\theta)$ for all θ in the domain of f then the graph of $r = f(\theta)$ is symmetric about the y-axis.

    (a) For $f(\theta) = 4 - 2\sin(\theta)$, show that $f(\pi - \theta) = f(\theta)$ and the graph of $r = 4 - 2\sin(\theta)$ is symmetric about the y-axis, as required. (See Example 14.2.2 number 1.)

    (b) For $f(\theta) = 5\sin(2\theta)$, show that $f\left(\pi - \frac{\pi}{4}\right) \neq f\left(\frac{\pi}{4}\right)$, yet the graph of $r = 5\sin(2\theta)$ **is** symmetric about the y-axis. (See Example 14.2.2 number 3.)

In Section 5.4, we discussed transformations of graphs. In Exercise 66 we have you and your classmates explore transformations of polar graphs.

66. For Exercises 66a and 66b below, let $f(\theta) = \cos(\theta)$ and $g(\theta) = 2 - \sin(\theta)$.

    (a) Using a graphing utility, compare the graph of $r = f(\theta)$ to each of the graphs of $r = f\left(\theta + \frac{\pi}{4}\right)$, $r = f\left(\theta + \frac{3\pi}{4}\right)$, $r = f\left(\theta - \frac{\pi}{4}\right)$ and $r = f\left(\theta - \frac{3\pi}{4}\right)$. Repeat this process for $g(\theta)$. In general, how do you think the graph of $r = f(\theta + \alpha)$ compares with the graph of $r = f(\theta)$?

    (b) Using a graphing utility, compare the graph of $r = f(\theta)$ to each of the graphs of $r = 2f(\theta)$, $r = \frac{1}{2}f(\theta)$, $r = -f(\theta)$ and $r = -3f(\theta)$. Repeat this process for $g(\theta)$. In general, how do you think the graph of $r = k \cdot f(\theta)$ compares with the graph of $r = f(\theta)$?

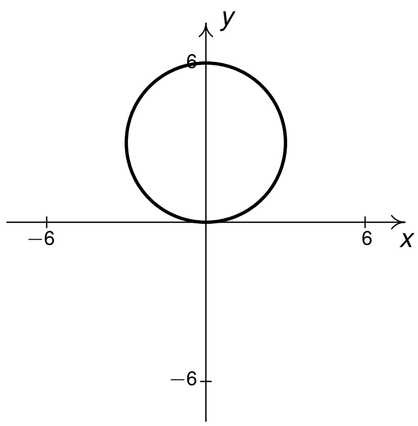
        Follow up question: does it matter if $k > 0$ or $k < 0$?

67. In light of Exercises 63 - 65, how would the graph of $r = f(-\theta)$ compare with the graph of $r = f(\theta)$ for a generic function f? What about the graphs of $r = -f(\theta)$ and $r = f(\theta)$? What about $r = f(\theta)$ and $r = f(\pi - \theta)$? Test out your conjectures using a variety of polar functions found in this section with the help of a graphing utility.

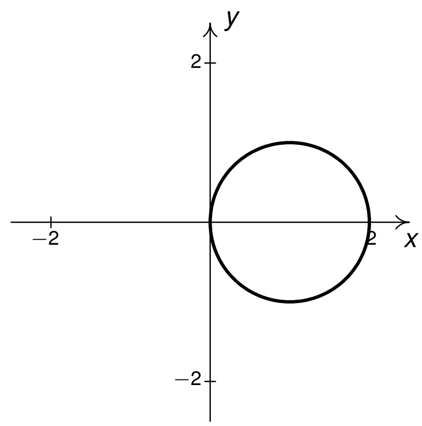
68. With the help of your classmates, research cardioid microphones.

14.2.2 Answers

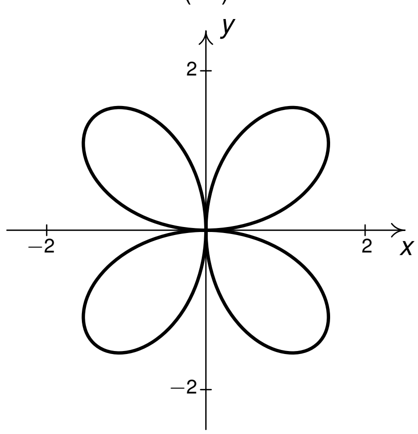
1. Circle: $r = 6\sin(\theta)$



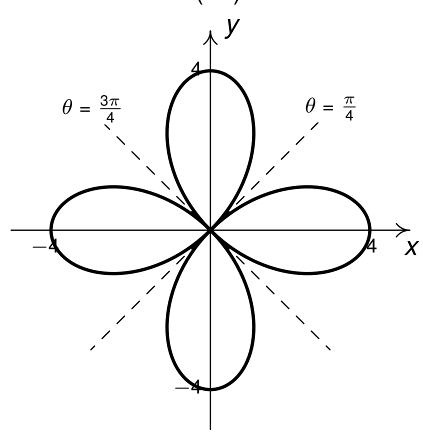
2. Circle: $r = 2\cos(\theta)$



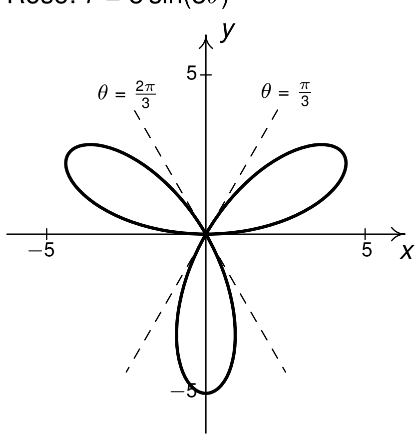
3. Rose: $r = 2\sin(2\theta)$



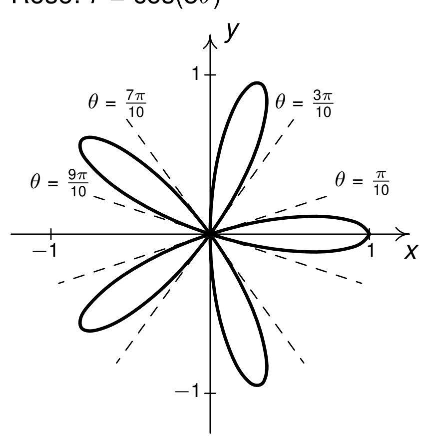
4. Rose: $r = 4\cos(2\theta)$



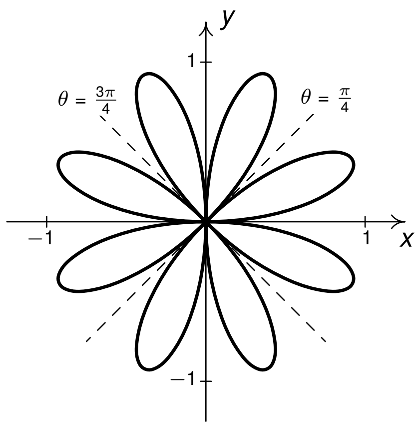
5. Rose: $r = 5\sin(3\theta)$



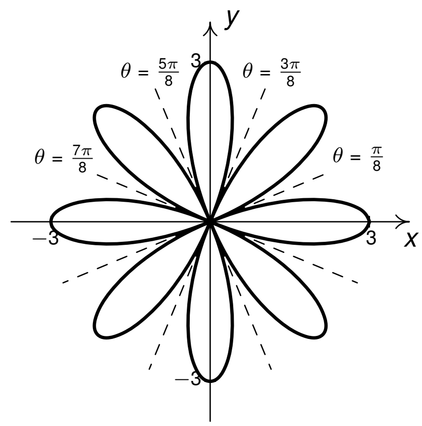
6. Rose: $r = \cos(5\theta)$



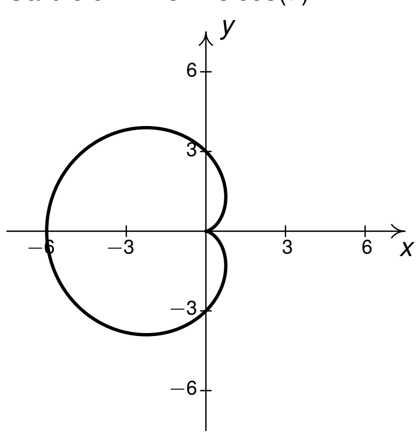
7. Rose: $r = \sin(4\theta)$



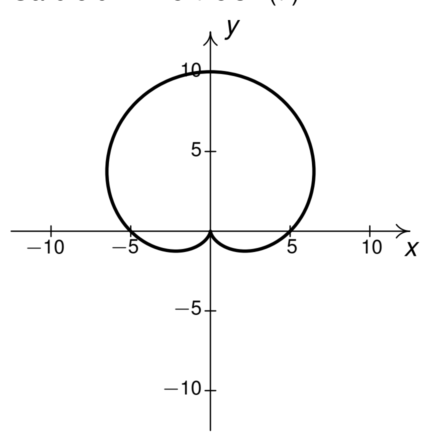
8. Rose: $r = 3\cos(4\theta)$



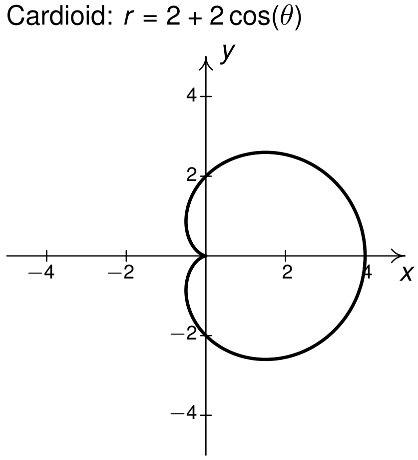
9. Cardioid: $r = 3 - 3\cos(\theta)$



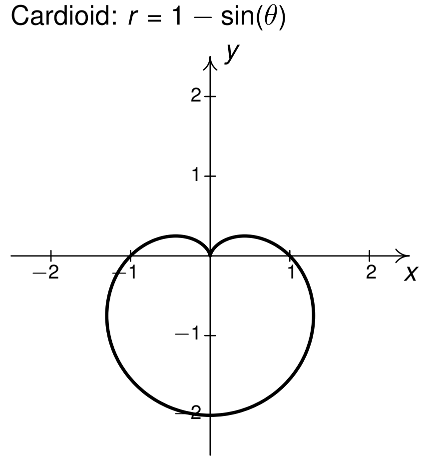
10. Cardioid: $r = 5 + 5\sin(\theta)$



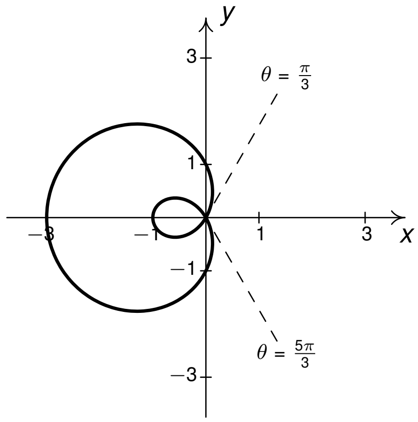
11. Cardioid: $r = 2 + 2\cos(\theta)$



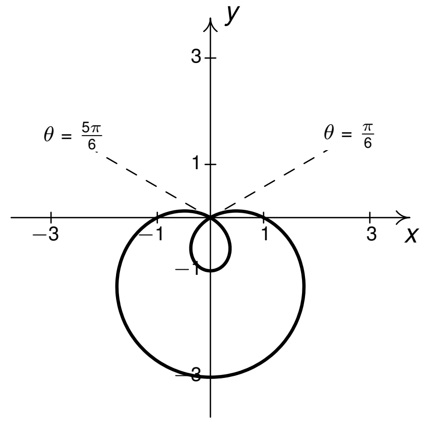
12. Cardioid: $r = 1 - \sin(\theta)$



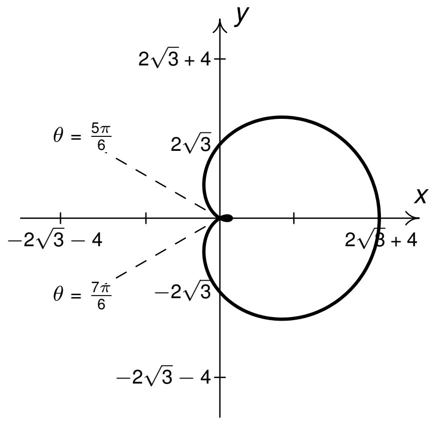
13. Limaçon: $r = 1 - 2\cos(\theta)$



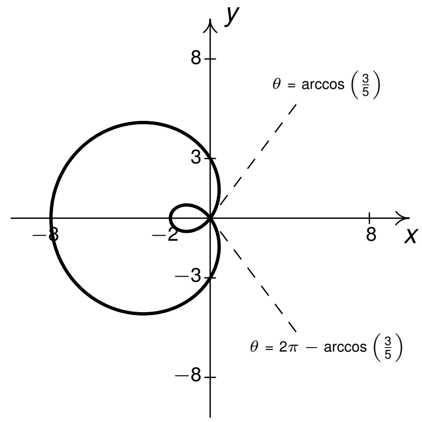
14. Limaçon: $r = 1 - 2\sin(\theta)$



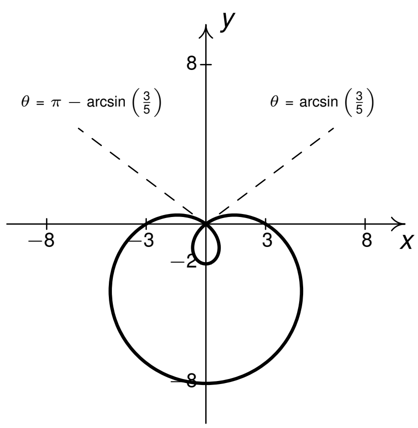
15. Limaçon: $r = 2\sqrt{3} + 4\cos(\theta)$



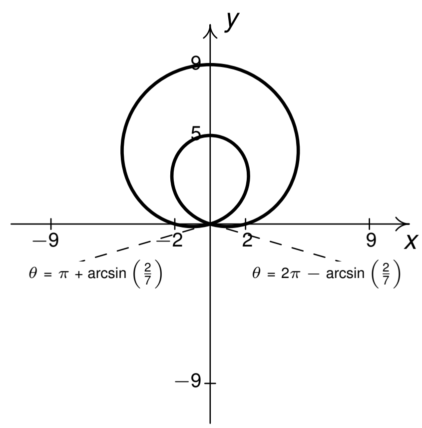
16. Limaçon: $r = 3 - 5\cos(\theta)$



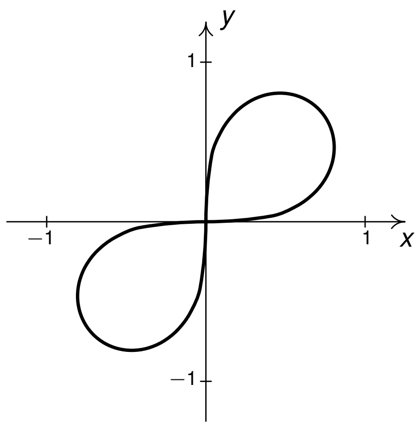
17. Limaçon: $r = 3 - 5\sin(\theta)$



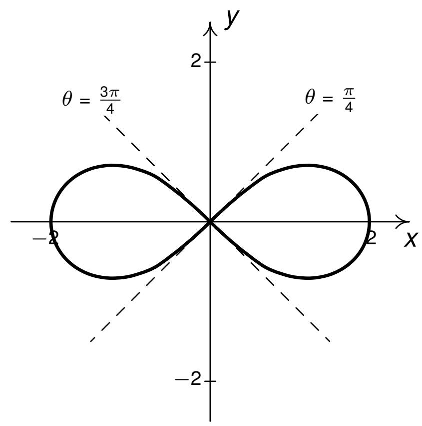
18. Limaçon: $r = 2 + 7\sin(\theta)$



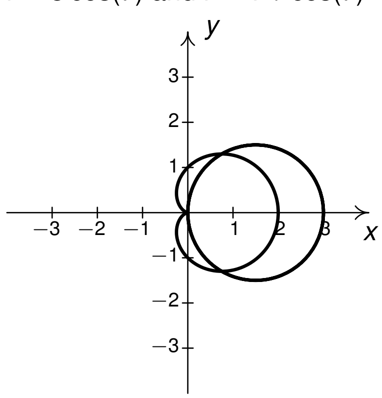
19. Lemniscate: $r^2 = \sin(2\theta)$



20. Lemniscate: $r^2 = 4\cos(2\theta)$

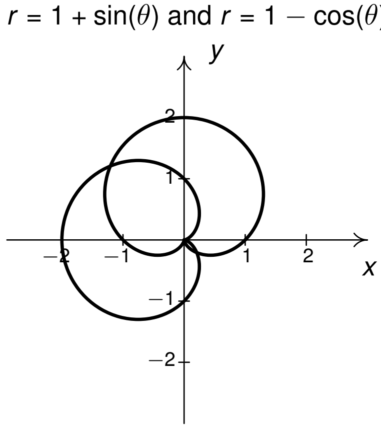


21. $r = 3\cos(\theta)$ and $r = 1 + \cos(\theta)$



$\left(\dfrac{3}{2}, \dfrac{\pi}{3}\right)$, $\left(\dfrac{3}{2}, \dfrac{5\pi}{3}\right)$, pole

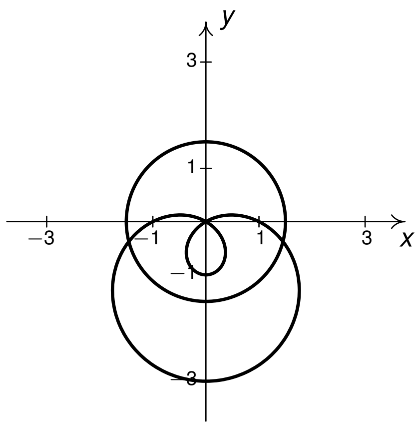
22. $r = 1 + \sin(\theta)$ and $r = 1 - \cos(\theta)$



$\left(\dfrac{2+\sqrt{2}}{2}, \dfrac{3\pi}{4}\right)$, $\left(\dfrac{2-\sqrt{2}}{2}, \dfrac{7\pi}{4}\right)$, pole

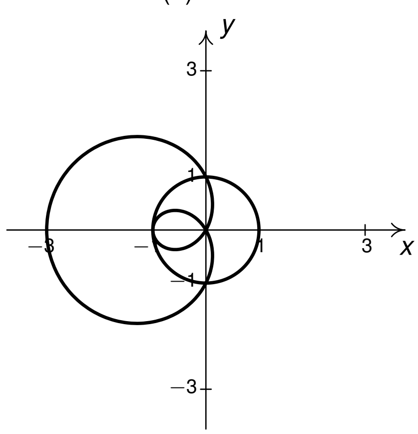
23. $r = 1 - 2\sin(\theta)$ and $r = 2$

$\left(2, \dfrac{7\pi}{6}\right)$, $\left(2, \dfrac{11\pi}{6}\right)$



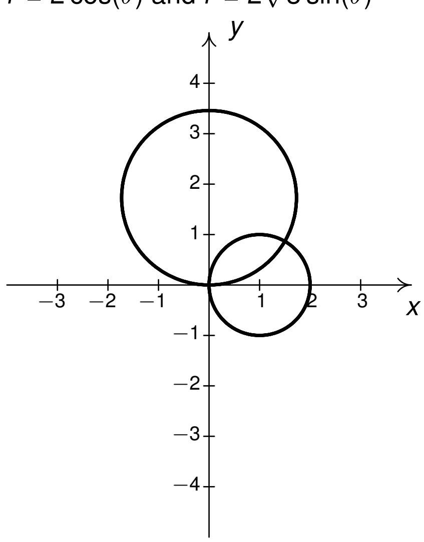
24. $r = 1 - 2\cos(\theta)$ and $r = 1$

$\left(1, \dfrac{\pi}{2}\right)$, $\left(1, \dfrac{3\pi}{2}\right)$, $(-1, 0)$

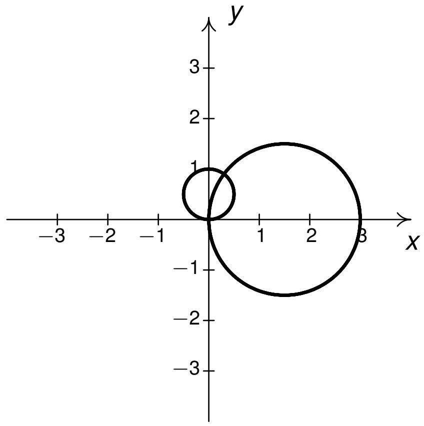


25. $r = 2\cos(\theta)$ and $r = 2\sqrt{3}\sin(\theta)$

$\left(\sqrt{3}, \dfrac{\pi}{6}\right)$, pole

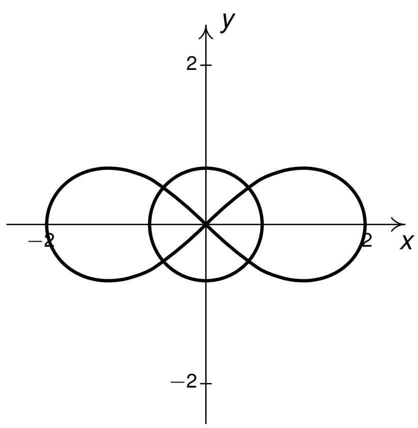


26. $r = 3\cos(\theta)$ and $r = \sin(\theta)$



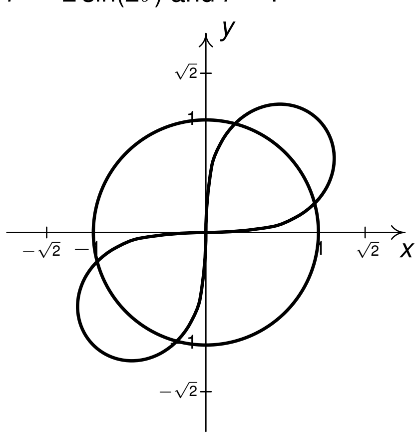
$\left(\dfrac{3\sqrt{10}}{10}, \arctan(3)\right)$, pole

27. $r^2 = 4\cos(2\theta)$ and $r = \sqrt{2}$



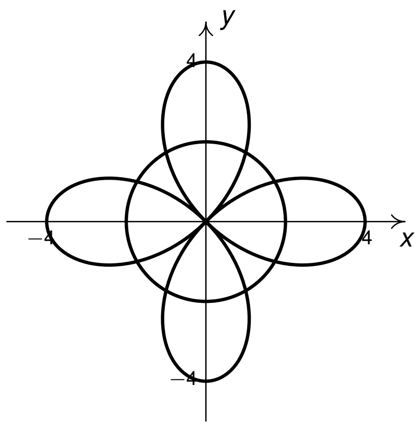
$\left(\sqrt{2}, \dfrac{\pi}{6}\right), \left(\sqrt{2}, \dfrac{5\pi}{6}\right), \left(\sqrt{2}, \dfrac{7\pi}{6}\right), \left(\sqrt{2}, \dfrac{11\pi}{6}\right)$

28. $r^2 = 2\sin(2\theta)$ and $r = 1$



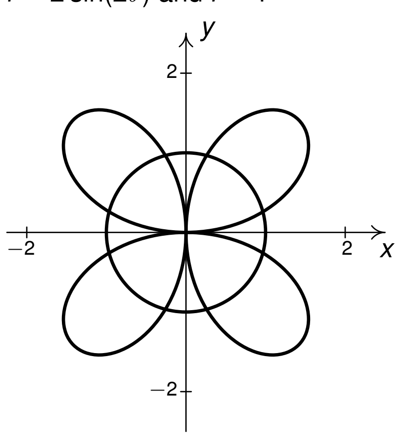
$\left(1, \dfrac{\pi}{12}\right), \left(1, \dfrac{5\pi}{12}\right), \left(1, \dfrac{13\pi}{12}\right), \left(1, \dfrac{17\pi}{12}\right)$

29. $r = 4\cos(2\theta)$ and $r = 2$



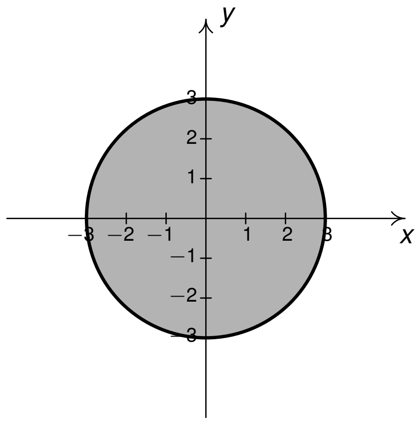
$\left(2, \dfrac{\pi}{6}\right)$, $\left(2, \dfrac{5\pi}{6}\right)$, $\left(2, \dfrac{7\pi}{6}\right)$, $\left(2, \dfrac{11\pi}{6}\right)$, $\left(-2, \dfrac{\pi}{3}\right)$, $\left(-2, \dfrac{2\pi}{3}\right)$, $\left(-2, \dfrac{4\pi}{3}\right)$, $\left(-2, \dfrac{5\pi}{3}\right)$

30. $r = 2\sin(2\theta)$ and $r = 1$

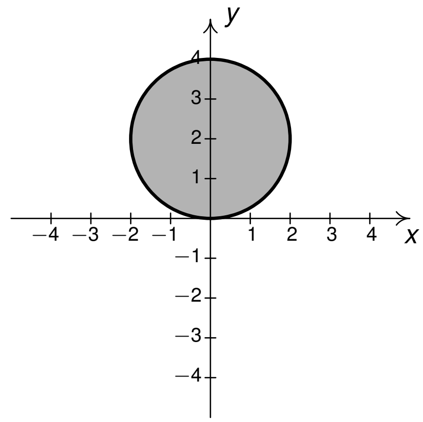


$\left(1, \dfrac{\pi}{12}\right)$, $\left(1, \dfrac{5\pi}{12}\right)$, $\left(1, \dfrac{13\pi}{12}\right)$, $\left(1, \dfrac{17\pi}{12}\right)$, $\left(-1, \dfrac{7\pi}{12}\right)$, $\left(-1, \dfrac{11\pi}{12}\right)$, $\left(-1, \dfrac{19\pi}{12}\right)$, $\left(-1, \dfrac{23\pi}{12}\right)$

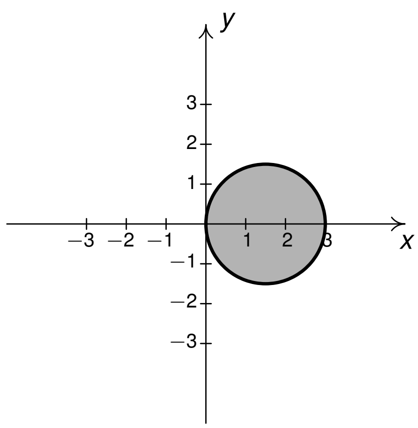
31. $\{(r, \theta) \,|\, 0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}$



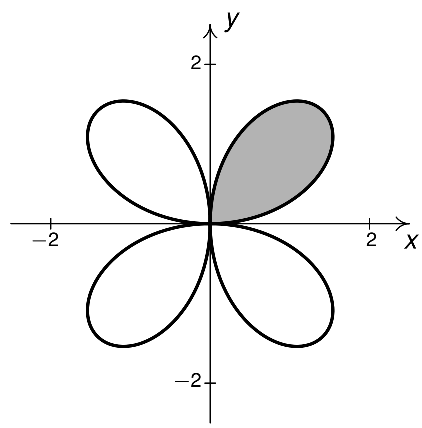
32. $\{(r, \theta) \,|\, 0 \leq r \leq 4\sin(\theta),\ 0 \leq \theta \leq \pi\}$



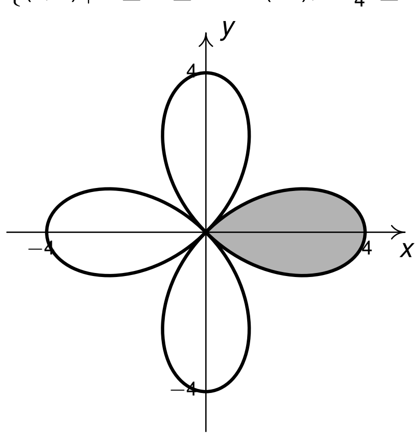
33. $\left\{(r, \theta) \,|\, 0 \leq r \leq 3\cos(\theta),\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right\}$



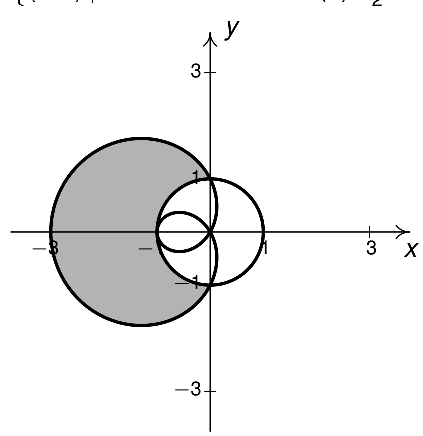
34. $\left\{(r, \theta) \,|\, 0 \leq r \leq 2\sin(2\theta),\ 0 \leq \theta \leq \frac{\pi}{2}\right\}$



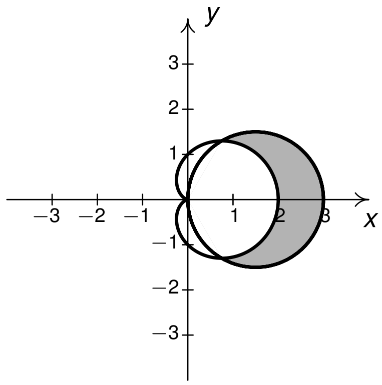
35. $\left\{(r, \theta) \,|\, 0 \leq r \leq 4\cos(2\theta),\ -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\right\}$



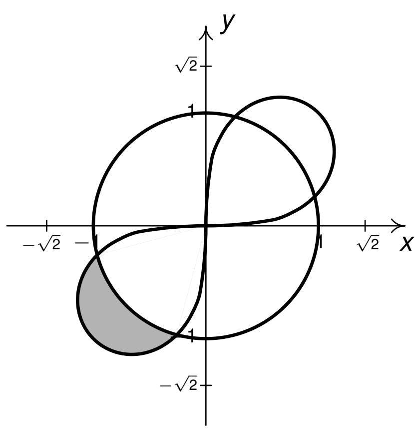
36. $\left\{(r, \theta) \,|\, 1 \leq r \leq 1 - 2\cos(\theta),\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right\}$



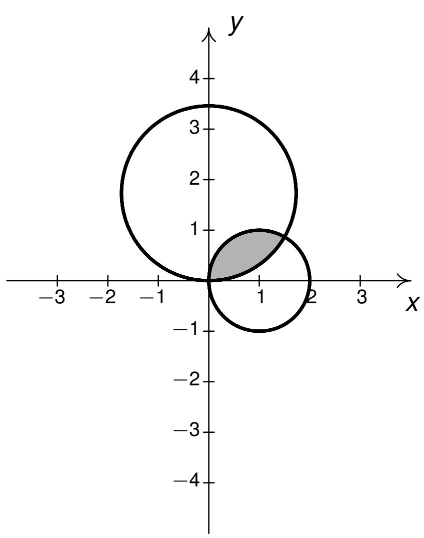
37. $\left\{(r,\theta)\,|\, 1+\cos(\theta) \leq r \leq 3\cos(\theta),\ -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}\right\}$



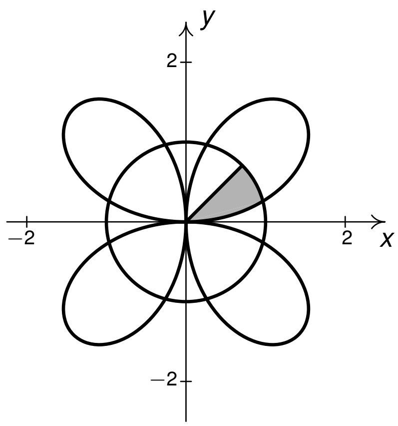
38. $\left\{(r,\theta)\,|\, 1 \leq r \leq \sqrt{2\sin(2\theta)},\ \frac{13\pi}{12} \leq \theta \leq \frac{17\pi}{12}\right\}$



39. $\left\{(r,\theta)\,|\, 0 \leq r \leq 2\sqrt{3}\sin(\theta),\ 0 \leq \theta \leq \frac{\pi}{6}\right\} \cup \left\{(r,\theta)\,|\, 0 \leq r \leq 2\cos(\theta),\ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}\right\}$



40. $\left\{(r,\theta)\,|\,0 \leq r \leq 2\sin(2\theta),\; 0 \leq \theta \leq \frac{\pi}{12}\right\} \cup \left\{(r,\theta)\,|\,0 \leq r \leq 1,\; \frac{\pi}{12} \leq \theta \leq \frac{\pi}{4}\right\}$



41. $\left\{(r,\theta)\,|\,0 \leq r \leq 5,\; 0 \leq \theta \leq 2\pi\right\}$

42. $\left\{(r,\theta)\,|\,0 \leq r \leq 5,\; \pi \leq \theta \leq \frac{3\pi}{2}\right\}$

43. $\left\{(r,\theta)\,|\,0 \leq r \leq 6\sin(\theta),\; \frac{\pi}{2} \leq \theta \leq \pi\right\}$

44. $\left\{(r,\theta)\,|\,4\cos(\theta) \leq r \leq 0,\; \frac{\pi}{2} \leq \theta \leq \pi\right\}$

45. $\{(r,\theta)\,|\,0 \leq r \leq 3 - 3\cos(\theta),\; 0 \leq \theta \leq \pi\}$

46. $\left\{(r,\theta)\,|\,0 \leq r \leq 2 - 2\sin(\theta),\; 0 \leq \theta \leq \frac{\pi}{2}\right\} \cup \left\{(r,\theta)\,|\,0 \leq r \leq 2 - 2\sin(\theta),\; \frac{3\pi}{2} \leq \theta \leq 2\pi\right\}$
or $\left\{(r,\theta)\,|\,0 \leq r \leq 2 - 2\sin(\theta),\; \frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{2}\right\}$

47. $\left\{(r,\theta)\,|\,0 \leq r \leq 3\cos(4\theta),\; 0 \leq \theta \leq \frac{\pi}{8}\right\} \cup \left\{(r,\theta)\,|\,0 \leq r \leq 3\cos(4\theta),\; \frac{15\pi}{8} \leq \theta \leq 2\pi\right\}$
or $\left\{(r,\theta)\,|\,0 \leq r \leq 3\cos(4\theta),\; -\frac{\pi}{8} \leq \theta \leq \frac{\pi}{8}\right\}$

48. $\left\{(r,\theta)\,|\,3 \leq r \leq 5,\; 0 \leq \theta \leq 2\pi\right\}$

49. $\left\{(r,\theta)\,|\,0 \leq r \leq 3\cos(\theta),\; -\frac{\pi}{2} \leq \theta \leq 0\right\} \cup \{(r,\theta)\,|\,\sin(\theta) \leq r \leq 3\cos(\theta),\; 0 \leq \theta \leq \arctan(3)\}$

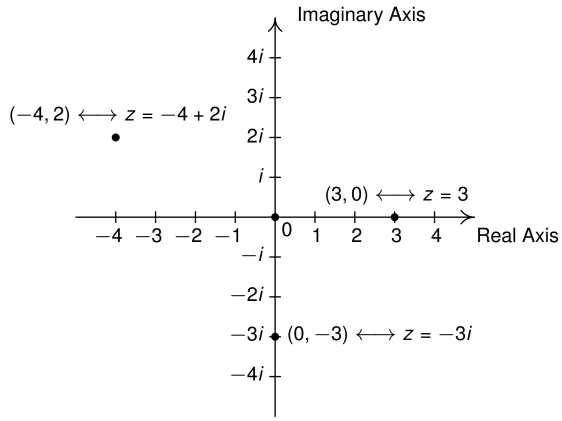
50. $\left\{(r,\theta)\,|\,0 \leq r \leq 6\sin(2\theta),\; 0 \leq \theta \leq \frac{\pi}{12}\right\} \cup \left\{(r,\theta)\,|\,0 \leq r \leq 3,\; \frac{\pi}{12} \leq \theta \leq \frac{5\pi}{12}\right\} \cup$
$\left\{(r,\theta)\,|\,0 \leq r \leq 6\sin(2\theta),\; \frac{5\pi}{12} \leq \theta \leq \frac{\pi}{2}\right\}$

14.3 The Polar Form of Complex Numbers

In this section, we return to our study of complex numbers which were first introduced in Section 2.4. Recall that a **complex number** is a number of the form $z = a + bi$ where a and b are real numbers and i is the imaginary unit defined by $i = \sqrt{-1}$.

The number a is called the **real part** of z, denoted $\text{Re}(z)$, while the real number b is called the **imaginary part** of z, denoted $\text{Im}(z)$. From Intermediate Algebra, we know that if $z = a + bi = c + di$ where a, b, c and d are real numbers, then $a = c$ and $b = d$, which means $\text{Re}(z)$ and $\text{Im}(z)$ are well-defined.[1]

To start off this section, we associate each complex number $z = a + bi$ with the point (a, b) on the Cartesian (rectangular) coordinate plane. In this case, the x-axis is relabeled as the **real axis**, which corresponds to the real number line as usual, and the y-axis is relabeled as the **imaginary axis**, which is demarcated in increments of the imaginary unit i. The plane determined by these two axes is called the **complex plane**.



The Complex Plane

Since the ordered pair (a, b) gives the *rectangular* coordinates associated with $z = a + bi$, the expression $z = a + bi$ is called the **rectangular form** of the complex number z.

We could just as easily associate z with a pair of *polar* coordinates (r, θ). Although it is not as straightforward as the definitions of $\text{Re}(z)$ and $\text{Im}(z)$, we give r and θ special names in relation to z below.

Definition 14.1. The Modulus and Argument of Complex Numbers:
Let $z = a + bi$ be a complex number with $a = \text{Re}(z)$ and $b = \text{Im}(z)$. Let (r, θ) be a polar representation of the point with rectangular coordinates (a, b) where $r \geq 0$.

- The **modulus** of z, denoted $|z|$, is defined by $|z| = r$.
- The angle θ is an **argument** of z. The set of all arguments of z is denoted $\arg(z)$.
- If $z \neq 0$ and $-\pi < \theta \leq \pi$, then θ is the **principal argument** of z, written $\theta = \text{Arg}(z)$.

---

[1] 'Well-defined' means that no matter how we express z, the number $\text{Re}(z)$ is always the same, and the number $\text{Im}(z)$ is always the same. In other words, Re and Im are *functions* of complex numbers.

Some remarks about Definition 14.1 are in order. We know from Section 14.1 that every point in the plane has infinitely many polar coordinate representations (r, θ) which means it's worth our time to make sure the quantities 'modulus', 'argument' and 'principal argument' are well-defined.

Concerning the modulus, if $z = 0$ then the point associated with z is the origin. In this case, the *only* r-value which can be used here is $r = 0$. Hence for $z = 0$, $|z| = 0$ is well-defined.

If $z \neq 0$, then the point associated with z is not the origin, and there are *two* possibilities for r: one positive and one negative. However, we stipulated $r \geq 0$ in our definition so this pins down the value of $|z|$ to one and only one number. Thus the modulus is well-defined in this case, too.[2]

Even with the requirement $r \geq 0$, there are infinitely many angles θ which can be used in a polar representation of a point (r, θ). If $z \neq 0$ then the point in question is not the origin, so all of these angles θ are coterminal. Since coterminal angles are exactly 2π radians apart, we are guaranteed that only one of them lies in the interval $(-\pi, \pi]$, and this angle is what we call the principal argument of z, $\text{Arg}(z)$.

The set $\arg(z)$ of all arguments of z can be described as $\arg(z) = \{\text{Arg}(z) + 2\pi k \mid k \text{ is an integer}\}$. Note that since $\arg(z)$ is a *set*, we will write '$\theta \in \arg(z)$' to mean 'θ is in[3] the set of arguments of z'.

If $z = 0$ then the point in question is the origin, which we know can be represented in polar coordinates as $(0, \theta)$ for *any* angle θ. In this case, we have $\arg(0) = (-\infty, \infty)$ and since there is no one value of θ which lies $(-\pi, \pi]$, we leave $\text{Arg}(0)$ undefined. It is time for an example.

Example 14.3.1. For each of the following complex numbers find $\text{Re}(z)$, $\text{Im}(z)$, $|z|$, $\arg(z)$ and $\text{Arg}(z)$. Plot each complex number z in the complex plane.

1. $z = \sqrt{3} - i$
2. $z = -2 + 4i$
3. $z = 3i$
4. $z = -117$

Solution.

1. For $z = \sqrt{3} - i = \sqrt{3} + (-1)i$, we have $\text{Re}(z) = \sqrt{3}$ and $\text{Im}(z) = -1$. To find $|z|$, $\arg(z)$ and $\text{Arg}(z)$, we need to find a polar representation (r, θ) with $r \geq 0$ for the point $P(\sqrt{3}, -1)$ associated with z.

   We know $r^2 = (\sqrt{3})^2 + (-1)^2 = 4$, so $r = \pm 2$. Since we require $r \geq 0$, we choose $r = 2$, so $|z| = 2$.

   To find a corresponding angle θ, we note that since $r > 0$ and P lies in Quadrant IV, θ must be a Quadrant IV angle. We know $\tan(\theta) = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$, so $\theta = -\frac{\pi}{6} + 2\pi k$ for integers k. Hence, $\arg(z) = \left\{-\frac{\pi}{6} + 2\pi k \mid k \text{ is an integer}\right\}$. Of these values, only $\theta = -\frac{\pi}{6}$ satisfies $-\pi < \theta \leq \pi$, hence we get $\text{Arg}(z) = -\frac{\pi}{6}$.

2. The complex number $z = -2 + 4i$ has $\text{Re}(z) = -2$, $\text{Im}(z) = 4$, and is associated with the point $P(-2, 4)$. Our next task is to find a polar representation (r, θ) for P where $r \geq 0$.

   Running through the usual calculations gives $r = 2\sqrt{5}$, so $|z| = 2\sqrt{5}$. To find θ, we get $\tan(\theta) = -2$, and since $r > 0$ and P lies in Quadrant II, we know θ is a Quadrant II angle.

[2] In case you're wondering, the use of the absolute value notation $|z|$ for modulus will be explained shortly.

[3] Recall the symbol being used here, '$\in$,' is the mathematical symbol which denotes membership in a set. See Section A.1.

We find $\theta = \pi + \arctan(-2) + 2\pi k$, or, more succinctly $\theta = \pi - \arctan(2) + 2\pi k$ for integers k. Hence $\arg(z) = \{\pi - \arctan(2) + 2\pi k \mid k$ is an integer$\}$. Only $\theta = \pi - \arctan(2)$ satisfies $-\pi < \theta \leq \pi$, so we get $\text{Arg}(z) = \pi - \arctan(2)$.

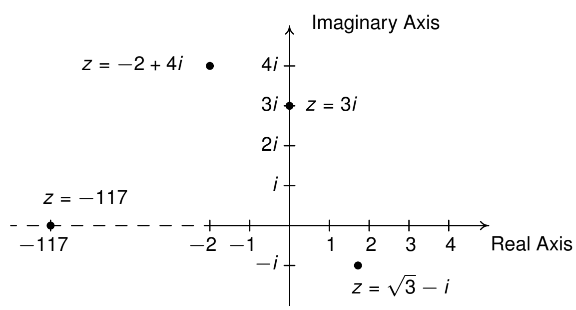
3. We rewrite $z = 3i$ as $z = 0 + 3i$ to find $\text{Re}(z) = 0$ and $\text{Im}(z) = 3$. The point in the plane which corresponds to z is $(0, 3)$ and while we could go through the usual calculations to find the required polar form of this point, we can obtain the answer 'by inspection.'

   The point $(0, 3)$ lies 3 units away from the origin on the positive y-axis. Hence, $r = |z| = 3$ and $\theta = \frac{\pi}{2} + 2\pi k$ for integers k. We get $\arg(z) = \left\{\frac{\pi}{2} + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{\pi}{2}$.

4. As in the previous problem, we write $z = -117 = -117 + 0i$ so $\text{Re}(z) = -117$ and $\text{Im}(z) = 0$. The number $z = -117$ corresponds to the point $(-117, 0)$, and this is another instance where we can determine the polar form 'by eye'.

   The point $(-117, 0)$ is 117 units away from the origin along the negative x-axis. Hence, $r = |z| = 117$ and $\theta = \pi + 2\pi = (2k + 1)\pi k$ for integers k. We have $\arg(z) = \{(2k + 1)\pi \mid k \text{ is an integer}\}$.

   Only one of these values, $\theta = \pi$, (just barely!) lies in the interval $(-\pi, \pi]$ which means and $\text{Arg}(z) = \pi$. We plot z along with the other numbers in this example below.



□

Now that we've had practice computing the modulus of a complex number, we state some properties below.

Theorem 14.3. Properties of the Modulus: Let z and w be complex numbers.

- $|z|$ is the distance from z to 0 in the complex plane
- $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$
- $|z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$
- **Product Rule:** $|zw| = |z||w|$
- **Power Rule:** $|z^n| = |z|^n$ for all natural numbers, n
- **Quotient Rule:** $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$, provided $w \neq 0$

To prove the first three properties in Theorem 14.3, suppose $z = a + bi$ where a and b are real numbers. To determine $|z|$, we find a polar representation (r, θ) with $r \geq 0$ for the point (a, b).

From Section 14.1, we know $r^2 = a^2 + b^2$ so that $r = \pm\sqrt{a^2+b^2}$. Since we require $r \geq 0$, then it must be that $r = \sqrt{a^2+b^2}$, which means $|z| = \sqrt{a^2+b^2}$. Using the distance formula, we find the distance from $(0, 0)$ to (a, b) is also $\sqrt{a^2+b^2}$, establishing the first property.[4]

For the second property, note that since $|z|$ is a distance, $|z| \geq 0$. Furthermore, $|z| = 0$ if and only if the distance from z to 0 is 0, and the latter happens if and only if $z = 0$, which is what we were asked to show.[5]

For the third property, we note that since $a = \text{Re}(z)$ and $b = \text{Im}(z)$, $z = \sqrt{a^2+b^2} = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$.

To prove the product rule, suppose $z = a + bi$ and $w = c + di$ for real numbers a, b, c and d. Then $zw = (a + bi)(c + di)$. After the usual arithmetic[6] we get $zw = (ac - bd) + (ad + bc)i$. Therefore,

$$\begin{array}{rcll}
|zw| & = & \sqrt{(ac-bd)^2 + (ad+bc)^2} & \\
& = & \sqrt{a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2} & \text{Expand} \\
& = & \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2} & \text{Rearrange terms} \\
& = & \sqrt{a^2\left(c^2+d^2\right) + b^2\left(c^2+d^2\right)} & \text{Factor} \\
& = & \sqrt{\left(a^2+b^2\right)\left(c^2+d^2\right)} & \text{Factor} \\
& = & \sqrt{a^2+b^2}\sqrt{c^2+d^2} & \text{Product Rule for Radicals} \\
& = & |z||w| & \text{Definition of } |z| \text{ and } |w|
\end{array}$$

Hence $|zw| = |z||w|$ as required.

Now that the Product Rule has been established, we use it and the Principle of Mathematical Induction[7] to prove the power rule. Let $P(n)$ be the statement $|z^n| = |z|^n$. Then $P(1)$ is true since $\left|z^1\right| = |z| = |z|^1$.

Next, assume $P(k)$ is true. That is, assume $\left|z^k\right| = |z|^k$ for some $k \geq 1$. Our job is to show that $P(k + 1)$ is true, namely $\left|z^{k+1}\right| = |z|^{k+1}$. As is customary with induction proofs, we first try to reduce the problem in such a way as to use the Induction Hypothesis.

$$\begin{array}{rcll}
\left|z^{k+1}\right| & = & \left|z^k z\right| & \text{Properties of Exponents} \\
& = & \left|z^k\right||z| & \text{Product Rule} \\
& = & |z|^k|z| & \text{Induction Hypothesis} \\
& = & |z|^{k+1} & \text{Properties of Exponents}
\end{array}$$

Hence, $P(k + 1)$ is true, which means $|z^n| = |z|^n$ is true for all natural numbers n.

---

[4] Since the absolute value $|x|$ of a real number x can be viewed as the distance from x to 0 on the number line, this first property justifies the notation $|z|$ for modulus. We leave it to the reader to show that if z is real, then the definition of modulus coincides with absolute value so the notation $|z|$ is unambiguous.

[5] This may be considered by some to be a bit of a cheat, so we work through the underlying Algebra to see this is true. We know $|z| = 0$ if and only if $\sqrt{a^2+b^2} = 0$ if and only if $a^2 + b^2 = 0$, which is true if and only if $a = b = 0$. The latter happens if and only if $z = a + bi = 0$. There.

[6] See Example A.11.1 in Section A.11 for a review of complex number arithmetic.

[7] See Section 10.3 for a review of this technique.

Like the Power Rule, the Quotient Rule can also be established with the help of the Product Rule. We assume $w \neq 0$ (so $|w| \neq 0$) and we get

$$\begin{aligned} \left|\frac{z}{w}\right| &= \left|(z)\left(\frac{1}{w}\right)\right| \\ &= |z|\left|\frac{1}{w}\right| \qquad \text{Product Rule.} \end{aligned}$$

Hence, the proof really boils down to showing $\left|\frac{1}{w}\right| = \frac{1}{|w|}$. This is left as an exercise.

Next, we characterize the argument of a complex number in terms of its real and imaginary parts.

Theorem 14.4. Properties of the Argument: Let z be a complex number.

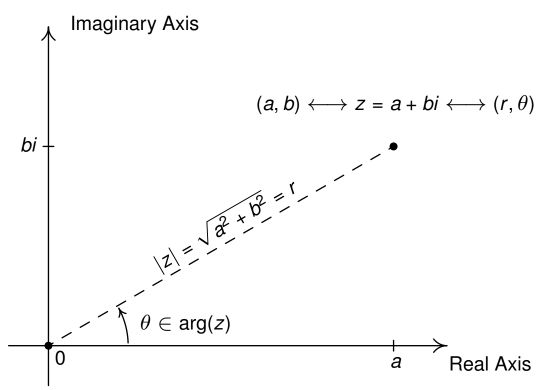
- If $\text{Re}(z) \neq 0$ and $\theta \in \arg(z)$, then $\tan(\theta) = \frac{\text{Im}(z)}{\text{Re}(z)}$.
- If $\text{Re}(z) = 0$ and $\text{Im}(z) > 0$, then $\arg(z) = \left\{\frac{\pi}{2} + 2\pi k \mid k \text{ is an integer}\right\}$.
- If $\text{Re}(z) = 0$ and $\text{Im}(z) < 0$, then $\arg(z) = \left\{-\frac{\pi}{2} + 2\pi k \mid k \text{ is an integer}\right\}$.
- If $\text{Re}(z) = \text{Im}(z) = 0$, then $z = 0$ and $\arg(z) = (-\infty, \infty)$.

To prove Theorem 14.4, suppose $z = a + bi$ for real numbers a and b. By definition, $a = \text{Re}(z)$ and $b = \text{Im}(z)$, so the point associated with z is $(a, b) = (\text{Re}(z), \text{Im}(z))$. From Section 14.1, we know that if (r, θ) is a polar representation for $(\text{Re}(z), \text{Im}(z))$, then $\tan(\theta) = \frac{\text{Im}(z)}{\text{Re}(z)}$, provided $\text{Re}(z) \neq 0$.

If $\text{Re}(z) = 0$ and $\text{Im}(z) > 0$, then z lies on the positive imaginary axis. Since we take $r > 0$, we have that θ is coterminal with $\frac{\pi}{2}$, and the result follows. If $\text{Re}(z) = 0$ and $\text{Im}(z) < 0$, then z lies on the negative imaginary axis, and a similar argument shows θ is coterminal with $-\frac{\pi}{2}$.

The last property in the theorem was already discussed in the remarks following Definition 14.1.

Our next goal is to completely marry the Geometry and the Algebra of the complex numbers. To that end, consider the figure below.



Polar coordinates, (r, θ) associated with $z = a + bi$ with $r \geq 0$.

We know from Theorem 14.1 that $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Making these substitutions for a and b gives $z = a + bi = r\cos(\theta) + r\sin(\theta)i = r[\cos(\theta) + i\sin(\theta)]$.

The expression '$\cos(\theta) + i\sin(\theta)$' is abbreviated $\text{cis}(\theta)$ so we can write $z = r\text{cis}(\theta) = |z|\text{cis}(\theta)$.

Definition 14.2. A Polar Form of a Complex Number:
Suppose z is a complex number and $\theta \in \arg(z)$. The expression:

$$|z|\text{cis}(\theta) = |z|\left[\cos(\theta) + i\sin(\theta)\right]$$

is called a polar form for z.

Since there are infinitely many choices for $\theta \in \arg(z)$, there infinitely many polar forms for z, so we used the indefinite article 'a' in Definition 14.2. It is time for an example.

Example 14.3.2.

1. Find the rectangular form of the following complex numbers. Find $\text{Re}(z)$ and $\text{Im}(z)$.

   (a) $z = 4\text{cis}\left(\frac{2\pi}{3}\right)$ (b) $z = 2\text{cis}\left(-\frac{3\pi}{4}\right)$ (c) $z = 3\text{cis}(0)$ (d) $z = \text{cis}\left(\frac{\pi}{2}\right)$

2. Use the results from Example 14.3.1 to find a polar form of the following complex numbers.

   (a) $z = \sqrt{3} - i$ (b) $z = -2 + 4i$ (c) $z = 3i$ (d) $z = -117$

Solution.

1. The key to this problem is to write out $\text{cis}(\theta)$ as $\cos(\theta) + i\sin(\theta)$.

   (a) By definition, $z = 4\text{cis}\left(\frac{2\pi}{3}\right) = 4\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$. Simplifying, we get $z = -2 + 2i\sqrt{3}$, so that $\text{Re}(z) = -2$ and $\text{Im}(z) = 2\sqrt{3}$.

   (b) Expanding, we get $z = 2\text{cis}\left(-\frac{3\pi}{4}\right) = 2\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$. Hence, $z = -\sqrt{2} - i\sqrt{2}$, so $\text{Re}(z) = -\sqrt{2} = \text{Im}(z)$.

   (c) We get $z = 3\text{cis}(0) = 3\left[\cos(0) + i\sin(0)\right] = 3$. Writing $3 = 3 + 0i$, we get $\text{Re}(z) = 3$ and $\text{Im}(z) = 0$, which makes sense seeing as 3 is a real number.

   (d) Lastly, we have $z = \text{cis}\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$. Since $i = 0 + 1i$, we get $\text{Re}(z) = 0$ and $\text{Im}(z) = 1$. Since i is called the 'imaginary unit,' these answers make perfect sense.

2. To write a polar form of a complex number z, we need two pieces of information: the modulus $|z|$ and an argument (not necessarily the principal argument) of z.

   We shamelessly mine our solution to Example 14.3.1 to find what we need.

   (a) For $z = \sqrt{3} - i$, $|z| = 2$ and $\theta = -\frac{\pi}{6}$, so $z = 2\text{cis}\left(-\frac{\pi}{6}\right)$. We can check our answer by converting it back to rectangular form to see that it simplifies to $z = \sqrt{3} - i$.

(b) For $z = -2 + 4i$, $|z| = 2\sqrt{5}$ and $\theta = \pi - \arctan(2)$. Hence, $z = 2\sqrt{5}\text{cis}(\pi - \arctan(2))$. It is a good exercise to actually show that this polar form reduces to $z = -2 + 4i$.

(c) For $z = 3i$, $|z| = 3$ and $\theta = \frac{\pi}{2}$. In this case, $z = 3\text{cis}\left(\frac{\pi}{2}\right)$. This can be checked geometrically.

Head out 3 units from 0 along the positive real axis. Rotating $\frac{\pi}{2}$ radians counter-clockwise lands you exactly 3 units above 0 on the imaginary axis at $z = 3i$.

(d) Last but not least, for $z = -117$, $|z| = 117$ and $\theta = \pi$. We get $z = 117\text{cis}(\pi)$. As with the previous problem, our answer is easily checked geometrically. □

The following theorem summarizes the advantages of working with complex numbers in polar form.

Theorem 14.5. Products, Powers and Quotients Complex Numbers in Polar Form: Suppose z and w are complex numbers with polar forms $z = |z|\text{cis}(\alpha)$ and $w = |w|\text{cis}(\beta)$. Then

- **Product Rule:** $zw = |z||w|\text{cis}(\alpha + \beta)$
- **Power Rule** (**DeMoivre's Theorem**) **:** $z^n = |z|^n\text{cis}(n\theta)$ for every natural number n
- **Quotient Rule:** $\dfrac{z}{w} = \dfrac{|z|}{|w|}\text{cis}(\alpha - \beta)$, provided $|w| \neq 0$

The proof of Theorem 14.5 requires a healthy mix of definition, arithmetic and identities. We first start with the product rule.

$$\begin{aligned} zw &= \left[|z|\text{cis}(\alpha)\right]\left[|w|\text{cis}(\beta)\right] \\ &= |z||w|\left[\cos(\alpha) + i\sin(\alpha)\right]\left[\cos(\beta) + i\sin(\beta)\right] \end{aligned}$$

We now focus on the quantity in brackets on the right hand side of the equation.

$$\begin{aligned} [\cos(\alpha) + i\sin(\alpha)][\cos(\beta) + i\sin(\beta)] &= \cos(\alpha)\cos(\beta) + i\cos(\alpha)\sin(\beta) \\ &\quad + i\sin(\alpha)\cos(\beta) + i^2\sin(\alpha)\sin(\beta) \\ &= \cos(\alpha)\cos(\beta) + i^2\sin(\alpha)\sin(\beta) && \text{Rearranging terms} \\ &\quad + i\sin(\alpha)\cos(\beta) + i\cos(\alpha)\sin(\beta) \\ &= (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) && \text{Since } i^2 = -1 \\ &\quad + i\,(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) && \text{Factor out } i \\ &= \cos(\alpha + \beta) + i\sin(\alpha + \beta) && \text{Sum identities} \\ &= \text{cis}(\alpha + \beta) && \text{Definition of `cis'} \end{aligned}$$

Putting this together with our earlier work, we get $zw = |z||w|\text{cis}(\alpha + \beta)$, as required.

Next take aim at the Power Rule, better known as DeMoivre's Theorem.[8] We proceed by induction on n. Let $P(n)$ be the sentence $z^n = |z|^n\text{cis}(n\theta)$. Then $P(1)$ is true, since $z^1 = z = |z|\text{cis}(\theta) = |z|^1\text{cis}(1 \cdot \theta)$.

[8] Compare this proof with the proof of the Power Rule in Theorem 14.3.

We now assume $P(k)$ is true, that is, we assume $z^k = |z|^k \text{cis}(k\theta)$ for some $k \geq 1$. Our goal is to show that $P(k+1)$ is true, or that $z^{k+1} = |z|^{k+1}\text{cis}((k+1)\theta)$. We have

$$\begin{array}{rcll} z^{k+1} & = & z^k z & \text{Properties of Exponents} \\ & = & \left(|z|^k \text{cis}(k\theta)\right)\left(|z|\text{cis}(\theta)\right) & \text{Induction Hypothesis} \\ & = & \left(|z|^k|z|\right)\text{cis}(k\theta+\theta) & \text{Product Rule} \\ & = & |z|^{k+1}\text{cis}((k+1)\theta) & \end{array}$$

Hence, assuming $P(k)$ is true, we have that $P(k+1)$ is true, so by the Principle of Mathematical Induction, $z^n = |z|^n \text{cis}(n\theta)$ for all natural numbers n.

The last property in Theorem 14.5 to prove is the quotient rule. Assuming $|w| \neq 0$ we have

$$\begin{array}{rcl} \dfrac{z}{w} & = & \dfrac{|z|\text{cis}(\alpha)}{|w|\text{cis}(\beta)} \\ & = & \left(\dfrac{|z|}{|w|}\right)\dfrac{\cos(\alpha)+i\sin(\alpha)}{\cos(\beta)+i\sin(\beta)} \end{array}$$

Next, we multiply both the numerator and denominator of the right hand side by $(\cos(\beta) - i\sin(\beta))$ which is the complex conjugate of $(\cos(\beta) + i\sin(\beta))$ to get

$$\frac{z}{w} = \left(\frac{|z|}{|w|}\right)\frac{\cos(\alpha)+i\sin(\alpha)}{\cos(\beta)+i\sin(\beta)} \cdot \frac{\cos(\beta)-i\sin(\beta)}{\cos(\beta)-i\sin(\beta)}$$

If we let the numerator be $N = [\cos(\alpha) + i\sin(\alpha)][\cos(\beta) - i\sin(\beta)]$ and simplify we get

$$\begin{array}{rcll} N & = & [\cos(\alpha)+i\sin(\alpha)][\cos(\beta)-i\sin(\beta)] & \\ & = & \cos(\alpha)\cos(\beta) - i\cos(\alpha)\sin(\beta) + i\sin(\alpha)\cos(\beta) - i^2\sin(\alpha)\sin(\beta) & \text{Expand} \\ & = & [\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)] + i[\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)] & \text{Rearrange and Factor} \\ & = & \cos(\alpha-\beta)+i\sin(\alpha-\beta) & \text{Difference Identities} \\ & = & \text{cis}(\alpha-\beta) & \text{Definition of `cis'} \end{array}$$

If we call the denominator D then we get

$$\begin{array}{rcll} D & = & [\cos(\beta)+i\sin(\beta)][\cos(\beta)-i\sin(\beta)] & \\ & = & \cos^2(\beta) - i\cos(\beta)\sin(\beta) + i\cos(\beta)\sin(\beta) - i^2\sin^2(\beta) & \text{Expand} \\ & = & \cos^2(\beta) - i^2\sin^2(\beta) & \text{Simplify} \\ & = & \cos^2(\beta)+\sin^2(\beta) & \text{Again, } i^2=-1 \\ & = & 1 & \text{Pythagorean Identity} \end{array}$$

Putting it all together, we get

$$\begin{aligned} \frac{z}{w} &= \left(\frac{|z|}{|w|}\right) \frac{\cos(\alpha) + i\sin(\alpha)}{\cos(\beta) + i\sin(\beta)} \cdot \frac{\cos(\beta) - i\sin(\beta)}{\cos(\beta) - i\sin(\beta)} \\ &= \left(\frac{|z|}{|w|}\right) \frac{\operatorname{cis}(\alpha - \beta)}{1} \\ &= \frac{|z|}{|w|}\operatorname{cis}(\alpha - \beta) \end{aligned}$$

and we are done. The next example makes good use of Theorem 14.5.

Example 14.3.3. Let $z = 2\sqrt{3} + 2i$ and $w = -1 + i\sqrt{3}$. Use Theorem 14.5 to find the following.

1. zw
2. w^5
3. $\dfrac{z}{w}$

Write your final answers in rectangular form.

Solution. In order to use Theorem 14.5, we need to write z and w in polar form.

For $z = 2\sqrt{3} + 2i$, we find $|z| = \sqrt{(2\sqrt{3})^2 + (2)^2} = \sqrt{16} = 4$. If $\theta \in \arg(z)$, then $\tan(\theta) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} = \frac{2}{2\sqrt{3}} = \frac{\sqrt{3}}{3}$. Since z lies in Quadrant I, we have $\theta = \frac{\pi}{6} + 2\pi k$ for integers k. Hence, $z = 4\operatorname{cis}\left(\frac{\pi}{6}\right)$.

For $w = -1 + i\sqrt{3}$, we have $|w| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$. For an argument θ of w, $\tan(\theta) = \frac{\sqrt{3}}{-1} = -\sqrt{3}$. Since w lies in Quadrant II, $\theta = \frac{2\pi}{3} + 2\pi k$ for integers k and $w = 2\operatorname{cis}\left(\frac{2\pi}{3}\right)$.

Since we now have polar forms of z and w, we can now proceed using Theorem 14.5.

1. We get $zw = \left(4\operatorname{cis}\left(\frac{\pi}{6}\right)\right)\left(2\operatorname{cis}\left(\frac{2\pi}{3}\right)\right) = 8\operatorname{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) = 8\operatorname{cis}\left(\frac{5\pi}{6}\right) = 8\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$.

   After simplifying, we get $zw = -4\sqrt{3} + 4i$.

2. We use DeMoivre's Theorem which yields $w^5 = \left[2\operatorname{cis}\left(\frac{2\pi}{3}\right)\right]^5 = 2^5\operatorname{cis}\left(5 \cdot \frac{2\pi}{3}\right) = 32\operatorname{cis}\left(\frac{10\pi}{3}\right)$.

   Since $\frac{10\pi}{3}$ is coterminal with $\frac{4\pi}{3}$, we get $w^5 = 32\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right] = -16 - 16i\sqrt{3}$.

3. Last, but not least, we have $\dfrac{z}{w} = \frac{4\operatorname{cis}\left(\frac{\pi}{6}\right)}{2\operatorname{cis}\left(\frac{2\pi}{3}\right)} = \frac{4}{2}\operatorname{cis}\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) = 2\operatorname{cis}\left(-\frac{\pi}{2}\right)$.

   Since $-\frac{\pi}{2}$ is a quadrantal angle, we can 'see' the rectangular form by moving out 2 units along the positive real axis, then rotating $\frac{\pi}{2}$ radians *clockwise* to arrive at the point 2 units below 0 on the imaginary axis. The long and short of it is that $\frac{z}{w} = -2i$. $\square$

Some remarks are in order. First, the reader may not be sold on using the polar form of complex numbers to multiply complex numbers – especially if they aren't given in polar form to begin with.

Indeed, a lot of work was needed to convert the numbers z and w in Example 14.3.3 into polar form, compute their product, and convert back to rectangular form – certainly more work than is required to multiply out $zw = (2\sqrt{3} + 2i)(-1 + i\sqrt{3})$ the old-fashioned way.

However, Theorem 14.5 pays huge dividends when computing powers of complex numbers. Consider how we computed w^5 above and compare that to using the Binomial Theorem, Theorem 10.7, to accomplish the same feat by expanding $(-1 + i\sqrt{3})^5$.

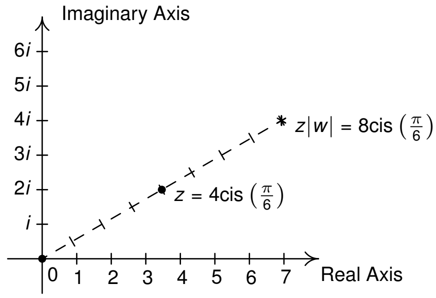
Moreover, division is tricky in the best of times, and we saved ourselves a lot of time and effort using Theorem 14.5 to find and simplify $\frac{z}{w}$ using their polar forms as opposed to starting with $\frac{2\sqrt{3}+2i}{-1+i\sqrt{3}}$, rationalizing the denominator, and so forth.

There is geometric reason for studying these polar forms and we would be derelict in our duties if we did not mention the Geometry hidden in Theorem 14.5.

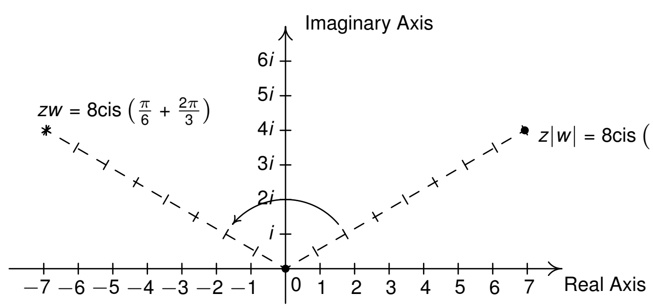
Take the product rule, for instance. If $z = |z|\text{cis}(\alpha)$ and $w = |w|\text{cis}(\beta)$, the formula $zw = |z||w|\text{cis}(\alpha + \beta)$ can be viewed geometrically as a two step process.

The multiplication of $|z|$ by $|w|$ can be interpreted as magnifying[9] the distance $|z|$ from z to 0, by the factor $|w|$. Adding the argument of w to the argument of z can be interpreted geometrically as a rotation of β radians counter-clockwise.[10]

Focusing on z and w from Example 14.3.3, we can arrive at the product zw by plotting z, doubling its distance from 0 (since $|w| = 2$), and rotating $\frac{2\pi}{3}$ radians counter-clockwise. The sequence of diagrams below attempts to describe this process geometrically.



Multiplying z by $|w| = 2$.



Rotating counter-clockwise by $\text{Arg}(w) = \frac{2\pi}{3}$ radians.

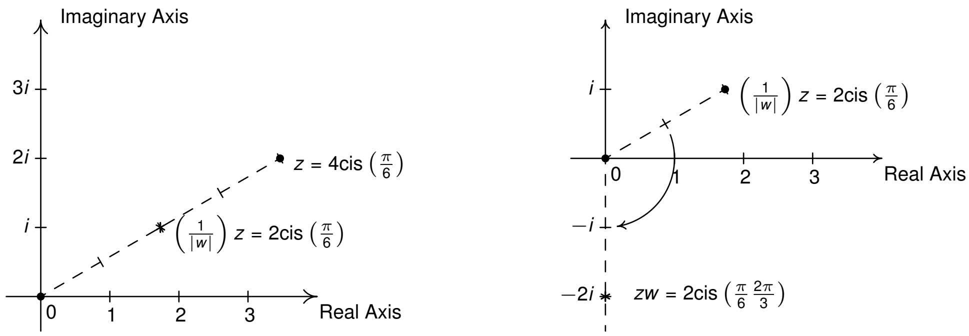
Visualizing zw for $z = 4\text{cis}\left(\frac{\pi}{6}\right)$ and $w = 2\text{cis}\left(\frac{2\pi}{3}\right)$.

We may also visualize division similarly. Here, the formula $\frac{z}{w} = \frac{|z|}{|w|}\text{cis}(\alpha - \beta)$ may be interpreted as shrinking[11] the distance from 0 to z by the factor $|w|$, followed up by a *clockwise*[12] rotation of β radians.

In the case of z and w from Example 14.3.3, we arrive at $\frac{z}{w}$ by first halving the distance from 0 to z, then rotating clockwise $\frac{2\pi}{3}$ radians as shown below.

---

[9]Assuming $|w| > 1$.
[10]Assuming $\beta > 0$.
[11]Again, assuming $|w| > 1$.
[12]Again, assuming $\beta > 0$.



Dividing z by $|w| = 2$.

Rotating clockwise by $\text{Arg}(w) = \frac{2\pi}{3}$ radians.

Visualizing $\dfrac{z}{w}$ for $z = 4\text{cis}\left(\frac{\pi}{6}\right)$ and $w = 2\text{cis}\left(\frac{2\pi}{3}\right)$.

Our last goal of the section is to reverse DeMoivre's Theorem to extract roots of complex numbers.

Definition 14.3. Let z and w be complex numbers. If there is a natural number n such that $w^n = z$, then w is an $\boldsymbol{n}^{\textbf{th}}$ **root** of z.

Unlike Definition A.8 in Section A.2, we do not specify one particular *prinicpal* n^{th} root, hence the use of the indefinite article 'an' as in 'an n^{th} root of z'. Using this definition, both 4 and -4 are square roots of 16, while $\sqrt{16}$ means the principal square root of 16 as in $\sqrt{16} = 4$.

Suppose we wish to find all complex third (cube) roots of 8. Algebraically, we are trying to solve $w^3 = 8$. We know that there is only one *real* solution to this equation, namely $w = \sqrt[3]{8} = 2$, but if we take the time to rewrite this equation as $w^3 - 8 = 0$ and factor, we get $(w-2)\left(w^2 + 2w + 4\right) = 0$.

Solving $w^2 + 2w + 4 = 0$ gives two more cube roots $w = -1 \pm i\sqrt{3}$, for a total of three cube roots of 8. Per Theorem 2.16, since the degree of $p(w) = w^3 - 8$ is three, there are three complex zeros, counting multiplicity. Since we have found three distinct zeros, we know we have found *all* of the zeros, so there are *exactly three distinct* cube roots of 8.

Let us now solve this same problem using the machinery developed in this section. To do so, we express $z = 8$ in polar form. Since $z = 8$ lies 8 units away on the positive real axis, we get $z = 8\text{cis}(0)$. If we let $w = |w|\text{cis}(\alpha)$ be a polar form of w, the equation $w^3 = 8$ becomes

$$\begin{array}{rcll} w^3 & = & 8 & \\ \left(|w|\text{cis}(\alpha)\right)^3 & = & 8\text{cis}(0) & \\ |w|^3\text{cis}(3\alpha) & = & 8\text{cis}(0) & \text{DeMoivre's Theorem} \end{array}$$

The complex number on the left hand side of the equation corresponds to the point with polar coordinates $\left(|w|^3, 3\alpha\right)$, while the complex number on the right hand side corresponds to the point with polar coordinates $(8, 0)$. Since $|w| \geq 0$, so is $|w|^3$, which means $\left(|w|^3, 3\alpha\right)$ and $(8, 0)$ are two polar representations corresponding to the same complex number, both with positive r values.

From Section 14.1, we know $|w|^3 = 8$ and $3\alpha = 0 + 2\pi k$ for integers k. Since $|w|$ is a real number, we solve $|w|^3 = 8$ by extracting the principal cube root to get $|w| = \sqrt[3]{8} = 2$.

As for α, we get $\alpha = \frac{2\pi k}{3}$ for integers k. This produces three distinct points with polar coordinates corresponding to $k = 0$, 1 and 2: specifically $(2, 0)$, $\left(2, \frac{2\pi}{3}\right)$ and $\left(2, \frac{4\pi}{3}\right)$.

The point $(2, 0)$ corresponds to the complex number $w_0 = 2\text{cis}(0)$, the point $\left(2, \frac{2\pi}{3}\right)$ corresponds to the complex number $w_1 = 2\text{cis}\left(\frac{2\pi}{3}\right)$, and the point $\left(2, \frac{4\pi}{3}\right)$ corresponds to the complex number $w_2 = 2\text{cis}\left(\frac{4\pi}{3}\right)$. Converting to rectangular form, we find $w_0 = 2$, $w_1 = -1 + i\sqrt{3}$ and $w_2 = -1 - i\sqrt{3}$.

While this process seems a tad more involved than our previous factoring approach, this procedure can be generalized to find, for example, all of the fifth roots of 32. (Try using Chapter 2 techniques on that!)

If we start with a generic complex number in polar form $z = |z|\text{cis}(\theta)$ and solve $w^n = z$ in the same manner as above, we arrive at the following theorem.

Theorem 14.6. The n^{th} roots of a Complex Number:
Let $z \neq 0$ be a complex number with polar form $z = r\text{cis}(\theta)$. For each natural number n, z has n distinct n^{th} roots, which we denote by $w_0, w_1, \ldots, w_{n-1}$, and they are given by the formula

$$w_k = \sqrt[n]{r}\text{cis}\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)$$

The proof of Theorem 14.6 breaks into to two parts: first, showing that each w_k is an n^{th} root, and second, showing that the set $\{w_k \mid k = 0, 1, \ldots, (n-1)\}$ consists of n different complex numbers.

To show w_k is an n^{th} root of z, we use DeMoivre's Theorem to show $(w_k)^n = z$.

$$\begin{array}{rcll}
(w_k)^n & = & \left(\sqrt[n]{r}\text{cis}\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)\right)^n & \\
& = & \left(\sqrt[n]{r}\right)^n \text{cis}\left(n \cdot \left[\frac{\theta}{n} + \frac{2\pi}{n}k\right]\right) & \text{DeMoivre's Theorem} \\
& = & r\text{cis}\left(\theta + 2\pi k\right) &
\end{array}$$

Since k is a whole number, $\cos(\theta + 2\pi k) = \cos(\theta)$ and $\sin(\theta + 2\pi k) = \sin(\theta)$. Hence, it follows that $\text{cis}(\theta + 2\pi k) = \text{cis}(\theta)$, so $(w_k)^n = r\text{cis}(\theta) = z$, as required.

To show that the formula in Theorem 14.6 generates n distinct numbers, we assume $n \geq 2$ (or else there is nothing to prove) and note that the modulus of each of the w_k is the same, namely $\sqrt[n]{r}$.

Therefore, the only way any two of these polar forms correspond to the same number is if their arguments are coterminal – that is, if the arguments differ by an integer multiple of 2π.

Suppose k and j are whole numbers between 0 and $(n-1)$, inclusive, with $k \neq j$. Since k and j are different, let's assume for the sake of argument that $k > j$. Then $\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right) - \left(\frac{\theta}{n} + \frac{2\pi}{n}j\right) = 2\pi\left(\frac{k-j}{n}\right)$.

For $2\pi\left(\frac{k-j}{n}\right)$ to be an integer multiple of 2π, $(k-j)$ must be a multiple of n. But because of the restrictions on k and j, $0 < k - j \leq n - 1$. (Think this through.) Hence, $(k-j)$ is a positive number less than n, so it cannot be a multiple of n.

As a result, w_k and w_j are different complex numbers, and we are done. By Theorem 2.16, we know there at most n distinct solutions to $w^n = z$, and we have just found all n of them.

We illustrate Theorem 14.6 in the next example.

Example 14.3.4. Use Theorem 14.6 to find the following:

1. both square roots of $z = -2 + 2i\sqrt{3}$

2. the four fourth roots of $z = -16$

3. the three cube roots of $z = \sqrt{2} + i\sqrt{2}$

4. the five fifth roots of $z = 1$.

Solution.

1. We start by writing $z = -2 + 2i\sqrt{3}$ in polar form as $z = 4\text{cis}\left(\frac{2\pi}{3}\right)$. Since we are looking for *square* roots, $n = 2$. In keeping with the notation used in Theorem 14.6 we will call these roots w_0 and w_1, in keeping with the notation suggested there.

   Identifying $r = 4$, $\theta = \frac{2\pi}{3}$, Theorem 14.6 gives one root as $w_0 = \sqrt{4}\text{cis}\left(\frac{(2\pi/3)}{2} + \frac{2\pi}{2}(0)\right) = 2\text{cis}\left(\frac{\pi}{3}\right)$ and the other root as $w_1 = \sqrt{4}\text{cis}\left(\frac{(2\pi/3)}{2} + \frac{2\pi}{2}(1)\right) = 2\text{cis}\left(\frac{4\pi}{3}\right)$.

   Though not asked to do so, we can easily convert each of w_0 and w_1 to rectangular form: $w_0 = 1+i\sqrt{3}$ and $w_1 = -1 - i\sqrt{3}$. We can check our answers by showing $w_0^2 = -2 + 2i\sqrt{3}$ and $w_1^2 = -2 + 2i\sqrt{3}$.

2. Proceeding as above, we begin by converting z to polar form: $z = -16 = 16\text{cis}(\pi)$. Here, $n = 4$, so Theorem 14.6 guarantees us *four* fourth roots.

   Identifying $r = 16$, $\theta = \pi$ and $n = 4$, Theorem 14.6 gives us: $w_0 = \sqrt[4]{16}\text{cis}\left(\frac{\pi}{4} + \frac{2\pi}{4}(0)\right) = 2\text{cis}\left(\frac{\pi}{4}\right)$, $w_1 = \sqrt[4]{16}\text{cis}\left(\frac{\pi}{4} + \frac{2\pi}{4}(1)\right) = 2\text{cis}\left(\frac{3\pi}{4}\right)$, $w_2 = \sqrt[4]{16}\text{cis}\left(\frac{\pi}{4} + \frac{2\pi}{4}(2)\right) = 2\text{cis}\left(\frac{5\pi}{4}\right)$ and last, but not least, $w_3 = \sqrt[4]{16}\text{cis}\left(\frac{\pi}{4} + \frac{2\pi}{4}(3)\right) = 2\text{cis}\left(\frac{7\pi}{4}\right)$.

   Once agin, we can conveniently convert our answers to rectangular form. We get: $w_0 = \sqrt{2} + i\sqrt{2}$, $w_1 = -\sqrt{2} + i\sqrt{2}$, $w_2 = -\sqrt{2} - i\sqrt{2}$ and $w_3 = \sqrt{2} - i\sqrt{2}$. We invite the reader to check our answers algebraically by showing $w_0^4 = w_1^4 = w_2^4 = w_3^4 = -16$.

3. For $z = \sqrt{2} + i\sqrt{2}$, we have $z = 2\text{cis}\left(\frac{\pi}{4}\right)$. With $r = 2$, $\theta = \frac{\pi}{4}$ and $n = 3$ the usual computations yield $w_0 = \sqrt[3]{2}\text{cis}\left(\frac{\pi}{12}\right)$, $w_1 = \sqrt[3]{2}\text{cis}\left(\frac{9\pi}{12}\right) = \sqrt[3]{2}\text{cis}\left(\frac{3\pi}{4}\right)$ and $w_2 = \sqrt[3]{2}\text{cis}\left(\frac{17\pi}{12}\right)$.

   To convert our answers to rectangular form requires the use of either the Sum and Difference Identities in Theorem 12.8 or the Half-Angle Identities in Theorem 12.11 to evaluate w_0 and w_2. Since we are not explicitly told to do so, we leave this as a good, but messy, exercise.

4. To find the five fifth roots of 1, we write $1 = 1\text{cis}(0)$. We have $r = 1$, $\theta = 0$ and $n = 5$. Since $\sqrt[5]{1} = 1$, the roots are $w_0 = \text{cis}(0) = 1$, $w_1 = \text{cis}\left(\frac{2\pi}{5}\right)$, $w_2 = \text{cis}\left(\frac{4\pi}{5}\right)$, $w_3 = \text{cis}\left(\frac{6\pi}{5}\right)$ and $w_4 = \text{cis}\left(\frac{8\pi}{5}\right)$.

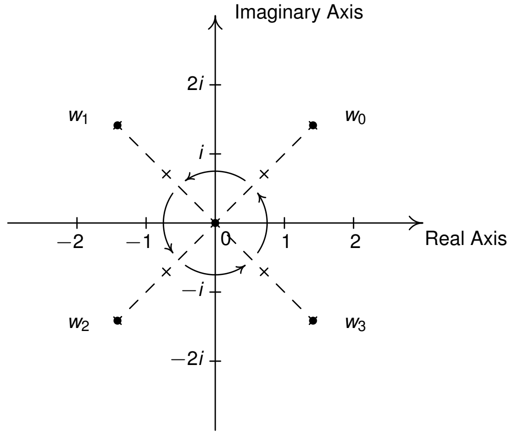
   The situation here is even graver than in the previous example, since we have not developed any identities to help us determine the cosine or sine of $\frac{2\pi}{5}$. At this stage, we could approximate our answers using a calculator, and we leave this as an exercise. $\square$

Having done some computations with Theorem 14.6, it's time to take a step back to look at things geometrically.

Essentially, Theorem 14.6 says that to find the n^{th} roots of a complex number, we first take the n^{th} root of the modulus and divide the argument by n. This gives the first root w_0.

Each successive root is found by adding $\frac{2\pi}{n}$ to the argument, which amounts to rotating w_0 by $\frac{2\pi}{n}$ radians. The result of these actions produces n roots, spaced equally around the complex plane.

As an example of this, we plot our answers to number 2 in Example 14.3.4 below.



The four fourth roots of $z = -16$ equally spaced $\frac{2\pi}{4} = \frac{\pi}{2}$ around the plane.

We have only glimpsed at the beauty of the complex numbers in this section. The complex plane is without a doubt one of the most important mathematical constructs ever devised. Coupled with Calculus, it is the venue for incredibly important Science and Engineering applications.[13]

[13]For more on this, see the beautifully written epilogue to Section 2.4 found on page 208.

14.3.1 Exercises

In Exercises 1 - 20, find a polar representation for the complex number z. Identify $\text{Re}(z)$, $\text{Im}(z)$, $|z|$, $\arg(z)$ and $\text{Arg}(z)$.

1. $z = 9 + 9i$
2. $z = 5 + 5i\sqrt{3}$
3. $z = 6i$
4. $z = -3\sqrt{2} + 3i\sqrt{2}$
5. $z = -6\sqrt{3} + 6i$
6. $z = -2$
7. $z = -\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$
8. $z = -3 - 3i$
9. $z = -5i$
10. $z = 2\sqrt{2} - 2i\sqrt{2}$
11. $z = 6$
12. $z = i\sqrt[3]{7}$
13. $z = 3 + 4i$
14. $z = \sqrt{2} + i$
15. $z = -7 + 24i$
16. $z = -2 + 6i$
17. $z = -12 - 5i$
18. $z = -5 - 2i$
19. $z = 4 - 2i$
20. $z = 1 - 3i$

In Exercises 21 - 40, find the rectangular form of the given complex number. Use whatever identities are necessary to find the exact values.

21. $z = 6\text{cis}(0)$
22. $z = 2\text{cis}\left(\dfrac{\pi}{6}\right)$
23. $z = 7\sqrt{2}\text{cis}\left(\dfrac{\pi}{4}\right)$
24. $z = 3\text{cis}\left(\dfrac{\pi}{2}\right)$
25. $z = 4\text{cis}\left(\dfrac{2\pi}{3}\right)$
26. $z = \sqrt{6}\text{cis}\left(\dfrac{3\pi}{4}\right)$
27. $z = 9\text{cis}(\pi)$
28. $z = 3\text{cis}\left(\dfrac{4\pi}{3}\right)$
29. $z = 7\text{cis}\left(-\dfrac{3\pi}{4}\right)$
30. $z = \sqrt{13}\text{cis}\left(\dfrac{3\pi}{2}\right)$
31. $z = \dfrac{1}{2}\text{cis}\left(\dfrac{7\pi}{4}\right)$
32. $z = 12\text{cis}\left(-\dfrac{\pi}{3}\right)$
33. $z = 8\text{cis}\left(\dfrac{\pi}{12}\right)$
34. $z = 2\text{cis}\left(\dfrac{7\pi}{8}\right)$
35. $z = 5\text{cis}\left(\arctan\left(\dfrac{4}{3}\right)\right)$
36. $z = \sqrt{10}\text{cis}\left(\arctan\left(\dfrac{1}{3}\right)\right)$
37. $z = 15\text{cis}(\arctan(-2))$
38. $z = \sqrt{3}\left(\arctan\left(-\sqrt{2}\right)\right)$
39. $z = 50\text{cis}\left(\pi - \arctan\left(\dfrac{7}{24}\right)\right)$
40. $z = \dfrac{1}{2}\text{cis}\left(\pi + \arctan\left(\dfrac{5}{12}\right)\right)$

For Exercises 41 - 52, use $z = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$ and $w = 3\sqrt{2} - 3i\sqrt{2}$ to compute the quantity. Express your answers in polar form using the principal argument.

41. zw
42. $\dfrac{z}{w}$
43. $\dfrac{w}{z}$
44. z^4
45. w^3
46. z^5w^2
47. z^3w^2
48. $\dfrac{z^2}{w}$

49. $\dfrac{w}{z^2}$

50. $\dfrac{z^3}{w^2}$

51. $\dfrac{w^2}{z^3}$

52. $\left(\dfrac{w}{z}\right)^6$

In Exercises 53 - 64, use DeMoivre's Theorem to find the indicated power of the given complex number. Express your final answers in rectangular form.

53. $\left(-2+2i\sqrt{3}\right)^3$

54. $(-\sqrt{3}-i)^3$

55. $(-3+3i)^4$

56. $(\sqrt{3}+i)^4$

57. $\left(\dfrac{5}{2}+\dfrac{5}{2}i\right)^3$

58. $\left(-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)^6$

59. $\left(\dfrac{3}{2}-\dfrac{3}{2}i\right)^3$

60. $\left(\dfrac{\sqrt{3}}{3}-\dfrac{1}{3}i\right)^4$

61. $\left(\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i\right)^4$

62. $(2+2i)^5$

63. $(\sqrt{3}-i)^5$

64. $(1-i)^8$

In Exercises 65 - 76, find the indicated complex roots. Express your answers in polar form and then convert them into rectangular form.

65. the two square roots of $z = 4i$

66. the two square roots of $z = -25i$

67. the two square roots of $z = 1 + i\sqrt{3}$

68. the two square roots of $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

69. the three cube roots of $z = 64$

70. the three cube roots of $z = -125$

71. the three cube roots of $z = i$

72. the three cube roots of $z = -8i$

73. the four fourth roots of $z = 16$

74. the four fourth roots of $z = -81$

75. the six sixth roots of $z = 64$

76. the six sixth roots of $z = -729$

77. Use the Sum and Difference Identities in Theorem 12.8 or the Half Angle Identities in Theorem 12.11 to convert the three cube roots of $z = \sqrt{2} + i\sqrt{2}$ we found in Example 14.3.4, number 3 from polar form to rectangular form.

78. Use a calculator to approximate the rectangular form of the five fifth roots of 1 we found in Example 14.3.4, number 4.

79. According to Theorem 2.18 in Section 2.4, the polynomial $p(x) = x^4 + 4$ can be factored into the product linear and irreducible quadratic factors. In Exercise 22 in Section 9.7, we showed you how to factor this polynomial into the product of two irreducible quadratic factors using a system of non-linear equations. Now that we can compute the complex fourth roots of -4 directly using Theorem 14.6, we can apply the Complex Factorization Theorem, Theorem 2.16, to obtain the linear factorization $p(x) = (x-(1+i))(x-(1-i))(x-(-1+i))(x-(-1-i))$. By multiplying the first two factors together and then the second two factors together, thus pairing up the complex conjugate pairs of zeros Theorem 2.17 told us we'd get, we have that $p(x) = (x^2-2x+2)(x^2+2x+2)$. Use the 12 complex 12$^{\text{th}}$ roots of 4096 to factor $p(x) = x^{12} - 4096$ into a product of linear and irreducible quadratic factors.

80. Use Exercise 30 from Section 13.4 to show the the Triangle Inequality $|z+w| \leq |z|+|w|$ holds for all complex numbers z and w as well. Identify the complex number $z = a+bi$ with the vector $u = \langle a, b \rangle$ and identify the complex number $w = c+di$ with the vector $v = \langle c, d \rangle$ and just follow your nose!

81. Complete the proof of Theorem 14.3 by showing that if $w \neq 0$ than $\left|\frac{1}{w}\right| = \frac{1}{|w|}$.

82. Recall from Section 2.4 that given a complex number $z = a+bi$ its complex conjugate, denoted $\overline{z}$, is given by $\overline{z} = a - bi$.

    (a) Prove that $|\overline{z}| = |z|$.

    (b) Prove that $|z| = \sqrt{z\overline{z}}$

    (c) Show that $\text{Re}(z) = \dfrac{z+\overline{z}}{2}$ and $\text{Im}(z) = \dfrac{z-\overline{z}}{2i}$

    (d) Show that if $\theta \in \arg(z)$ then $-\theta \in \arg(\overline{z})$. Interpret this result geometrically.

    (e) Is it always true that $\text{Arg}(\overline{z}) = -\text{Arg}(z)$?

83. Given a natural number $n \geq 2$, the n complex n^{th} roots of $z = 1$ are called the **n^{th} Roots of Unity**. In the following exercises, assume that n is a fixed, but arbitrary, natural number such that $n \geq 2$.

    (a) Show that $w = 1$ is an n^{th} root of unity.

    (b) Show that if both w_j and w_k are n^{th} roots of unity then so is their product $w_j w_k$.

    (c) Show that if w_j is an n^{th} root of unity then there is an n^{th} root of unity $w_{j'}$ so that $w_j w_{j'} = 1$.

    HINT: If $w_j = \text{cis}(\theta)$ let $w_{j'} = \text{cis}(2\pi - \theta)$. Show $w_{j'} = \text{cis}(2\pi - \theta)$ is indeed an n^{th} root of unity.

84. Another way to express the polar form of a complex number is to use the exponential function. For real numbers t, Euler's Formula defines $e^{it} = \cos(t) + i\sin(t)$.

    (a) Use Theorem 14.5 to show that:

        i. $e^{ix}e^{iy} = e^{i(x+y)}$ for all real numbers x and y.

        ii. $\left(e^{ix}\right)^n = e^{i(nx)}$ for any real number x and any natural number n.

        iii. $\dfrac{e^{ix}}{e^{iy}} = e^{i(x-y)}$ for all real numbers x and y.

    (b) If $z = r\text{cis}(\theta)$ is the polar form of z, show that $z = re^{it}$ where $\theta = t$ radians.

    (c) Show that $e^{i\pi} + 1 = 0$. (This famous equation relates the five most important constants in all of Mathematics with the three most fundamental operations in Mathematics.)

    (d) Show that $\cos(t) = \dfrac{e^{it}+e^{-it}}{2}$ and that $\sin(t) = \dfrac{e^{it}-e^{-it}}{2i}$ for all real numbers t.

14.3.2 Answers

1. $z = 9 + 9i = 9\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$, $\text{Re}(z) = 9$, $\text{Im}(z) = 9$, $|z| = 9\sqrt{2}$
   $\arg(z) = \left\{\frac{\pi}{4} + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{\pi}{4}$.

2. $z = 5 + 5i\sqrt{3} = 10\text{cis}\left(\frac{\pi}{3}\right)$, $\text{Re}(z) = 5$, $\text{Im}(z) = 5\sqrt{3}$, $|z| = 10$
   $\arg(z) = \left\{\frac{\pi}{3} + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{\pi}{3}$.

3. $z = 6i = 6\text{cis}\left(\frac{\pi}{2}\right)$, $\text{Re}(z) = 0$, $\text{Im}(z) = 6$, $|z| = 6$
   $\arg(z) = \left\{\frac{\pi}{2} + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{\pi}{2}$.

4. $z = -3\sqrt{2} + 3i\sqrt{2} = 6\text{cis}\left(\frac{3\pi}{4}\right)$, $\text{Re}(z) = -3\sqrt{2}$, $\text{Im}(z) = 3\sqrt{2}$, $|z| = 6$
   $\arg(z) = \left\{\frac{3\pi}{4} + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{3\pi}{4}$.

5. $z = -6\sqrt{3} + 6i = 12\text{cis}\left(\frac{5\pi}{6}\right)$, $\text{Re}(z) = -6\sqrt{3}$, $\text{Im}(z) = 6$, $|z| = 12$
   $\arg(z) = \left\{\frac{5\pi}{6} + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{5\pi}{6}$.

6. $z = -2 = 2\text{cis}(\pi)$, $\text{Re}(z) = -2$, $\text{Im}(z) = 0$, $|z| = 2$
   $\arg(z) = \{(2k+1)\pi \mid k \text{ is an integer}\}$ and $\text{Arg}(z) = \pi$.

7. $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i = \text{cis}\left(\frac{7\pi}{6}\right)$, $\text{Re}(z) = -\frac{\sqrt{3}}{2}$, $\text{Im}(z) = -\frac{1}{2}$, $|z| = 1$
   $\arg(z) = \left\{\frac{7\pi}{6} + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = -\frac{5\pi}{6}$.

8. $z = -3 - 3i = 3\sqrt{2}\text{cis}\left(\frac{5\pi}{4}\right)$, $\text{Re}(z) = -3$, $\text{Im}(z) = -3$, $|z| = 3\sqrt{2}$
   $\arg(z) = \left\{\frac{5\pi}{4} + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = -\frac{3\pi}{4}$.

9. $z = -5i = 5\text{cis}\left(\frac{3\pi}{2}\right)$, $\text{Re}(z) = 0$, $\text{Im}(z) = -5$, $|z| = 5$
   $\arg(z) = \left\{\frac{3\pi}{2} + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = -\frac{\pi}{2}$.

10. $z = 2\sqrt{2} - 2i\sqrt{2} = 4\text{cis}\left(\frac{7\pi}{4}\right)$, $\text{Re}(z) = 2\sqrt{2}$, $\text{Im}(z) = -2\sqrt{2}$, $|z| = 4$
    $\arg(z) = \left\{\frac{7\pi}{4} + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = -\frac{\pi}{4}$.

11. $z = 6 = 6\text{cis}(0)$, $\text{Re}(z) = 6$, $\text{Im}(z) = 0$, $|z| = 6$
    $\arg(z) = \{2\pi k \mid k \text{ is an integer}\}$ and $\text{Arg}(z) = 0$.

12. $z = i\sqrt[3]{7} = \sqrt[3]{7}\text{cis}\left(\frac{\pi}{2}\right)$, $\text{Re}(z) = 0$, $\text{Im}(z) = \sqrt[3]{7}$, $|z| = \sqrt[3]{7}$
    $\arg(z) = \left\{\frac{\pi}{2} + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{\pi}{2}$.

13. $z = 3 + 4i = 5\text{cis}\left(\arctan\left(\frac{4}{3}\right)\right)$, $\text{Re}(z) = 3$, $\text{Im}(z) = 4$, $|z| = 5$
    $\arg(z) = \left\{\arctan\left(\frac{4}{3}\right) + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \arctan\left(\frac{4}{3}\right)$.

14. $z = \sqrt{2} + i = \sqrt{3}\text{cis}\left(\arctan\left(\frac{\sqrt{2}}{2}\right)\right)$, $\text{Re}(z) = \sqrt{2}$, $\text{Im}(z) = 1$, $|z| = \sqrt{3}$
$\arg(z) = \left\{\arctan\left(\frac{\sqrt{2}}{2}\right) + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \arctan\left(\frac{\sqrt{2}}{2}\right)$.

15. $z = -7 + 24i = 25\text{cis}\left(\pi - \arctan\left(\frac{24}{7}\right)\right)$, $\text{Re}(z) = -7$, $\text{Im}(z) = 24$, $|z| = 25$
$\arg(z) = \left\{\pi - \arctan\left(\frac{24}{7}\right) + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \pi - \arctan\left(\frac{24}{7}\right)$.

16. $z = -2 + 6i = 2\sqrt{10}\text{cis}\left(\pi - \arctan(3)\right)$, $\text{Re}(z) = -2$, $\text{Im}(z) = 6$, $|z| = 2\sqrt{10}$
$\arg(z) = \{\pi - \arctan(3) + 2\pi k \mid k \text{ is an integer}\}$ and $\text{Arg}(z) = \pi - \arctan(3)$.

17. $z = -12 - 5i = 13\text{cis}\left(\pi + \arctan\left(\frac{5}{12}\right)\right)$, $\text{Re}(z) = -12$, $\text{Im}(z) = -5$, $|z| = 13$
$\arg(z) = \left\{\pi + \arctan\left(\frac{5}{12}\right) + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \arctan\left(\frac{5}{12}\right) - \pi$.

18. $z = -5 - 2i = \sqrt{29}\text{cis}\left(\pi + \arctan\left(\frac{2}{5}\right)\right)$, $\text{Re}(z) = -5$, $\text{Im}(z) = -2$, $|z| = \sqrt{29}$
$\arg(z) = \left\{\pi + \arctan\left(\frac{2}{5}\right) + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \arctan\left(\frac{2}{5}\right) - \pi$.

19. $z = 4 - 2i = 2\sqrt{5}\text{cis}\left(\arctan\left(-\frac{1}{2}\right)\right)$, $\text{Re}(z) = 4$, $\text{Im}(z) = -2$, $|z| = 2\sqrt{5}$
$\arg(z) = \left\{\arctan\left(-\frac{1}{2}\right) + 2\pi k \mid k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \arctan\left(-\frac{1}{2}\right) = -\arctan\left(\frac{1}{2}\right)$.

20. $z = 1 - 3i = \sqrt{10}\text{cis}\left(\arctan(-3)\right)$, $\text{Re}(z) = 1$, $\text{Im}(z) = -3$, $|z| = \sqrt{10}$
$\arg(z) = \{\arctan(-3) + 2\pi k \mid k \text{ is an integer}\}$ and $\text{Arg}(z) = \arctan(-3) = -\arctan(3)$.

21. $z = 6\text{cis}(0) = 6$

22. $z = 2\text{cis}\left(\frac{\pi}{6}\right) = \sqrt{3} + i$

23. $z = 7\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right) = 7 + 7i$

24. $z = 3\text{cis}\left(\frac{\pi}{2}\right) = 3i$

25. $z = 4\text{cis}\left(\frac{2\pi}{3}\right) = -2 + 2i\sqrt{3}$

26. $z = \sqrt{6}\text{cis}\left(\frac{3\pi}{4}\right) = -\sqrt{3} + i\sqrt{3}$

27. $z = 9\text{cis}(\pi) = -9$

28. $z = 3\text{cis}\left(\frac{4\pi}{3}\right) = -\frac{3}{2} - \frac{3i\sqrt{3}}{2}$

29. $z = 7\text{cis}\left(-\frac{3\pi}{4}\right) = -\frac{7\sqrt{2}}{2} - \frac{7\sqrt{2}}{2}i$

30. $z = \sqrt{13}\text{cis}\left(\frac{3\pi}{2}\right) = -i\sqrt{13}$

31. $z = \frac{1}{2}\text{cis}\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4}$

32. $z = 12\text{cis}\left(-\frac{\pi}{3}\right) = 6 - 6i\sqrt{3}$

33. $z = 8\text{cis}\left(\frac{\pi}{12}\right) = 4\sqrt{2+\sqrt{3}} + 4i\sqrt{2-\sqrt{3}}$

34. $z = 2\text{cis}\left(\frac{7\pi}{8}\right) = -\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}$

35. $z = 5\text{cis}\left(\arctan\left(\frac{4}{3}\right)\right) = 3 + 4i$

36. $z = \sqrt{10}\text{cis}\left(\arctan\left(\frac{1}{3}\right)\right) = 3 + i$

37. $z = 15\text{cis}\left(\arctan(-2)\right) = 3\sqrt{5} - 6i\sqrt{5}$

38. $z = \sqrt{3}\text{cis}\left(\arctan\left(-\sqrt{2}\right)\right) = 1 - i\sqrt{2}$

39. $z = 50\text{cis}\left(\pi - \arctan\left(\frac{7}{24}\right)\right) = -48 + 14i$

40. $z = \frac{1}{2}\text{cis}\left(\pi + \arctan\left(\frac{5}{12}\right)\right) = -\frac{6}{13} - \frac{5i}{26}$

In Exercises 41 - 52, we have that $z = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i = 3\text{cis}\left(\frac{5\pi}{6}\right)$ and $w = 3\sqrt{2} - 3i\sqrt{2} = 6\text{cis}\left(-\frac{\pi}{4}\right)$ so we get the following.

41. $zw = 18\text{cis}\left(\frac{7\pi}{12}\right)$
42. $\frac{z}{w} = \frac{1}{2}\text{cis}\left(-\frac{11\pi}{12}\right)$
43. $\frac{w}{z} = 2\text{cis}\left(\frac{11\pi}{12}\right)$
44. $z^4 = 81\text{cis}\left(-\frac{2\pi}{3}\right)$
45. $w^3 = 216\text{cis}\left(-\frac{3\pi}{4}\right)$
46. $z^5w^2 = 8748\text{cis}\left(-\frac{\pi}{3}\right)$
47. $z^3w^2 = 972\text{cis}(0)$
48. $\frac{z^2}{w} = \frac{3}{2}\text{cis}\left(-\frac{\pi}{12}\right)$
49. $\frac{w}{z^2} = \frac{2}{3}\text{cis}\left(\frac{\pi}{12}\right)$
50. $\frac{z^3}{w^2} = \frac{3}{4}\text{cis}(\pi)$
51. $\frac{w^2}{z^3} = \frac{4}{3}\text{cis}(\pi)$
52. $\left(\frac{w}{z}\right)^6 = 64\text{cis}\left(-\frac{\pi}{2}\right)$
53. $\left(-2 + 2i\sqrt{3}\right)^3 = 64$
54. $(-\sqrt{3} - i)^3 = -8i$
55. $(-3 + 3i)^4 = -324$
56. $(\sqrt{3} + i)^4 = -8 + 8i\sqrt{3}$
57. $\left(\frac{5}{2} + \frac{5}{2}i\right)^3 = -\frac{125}{4} + \frac{125}{4}i$
58. $\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^6 = 1$
59. $\left(\frac{3}{2} - \frac{3}{2}i\right)^3 = -\frac{27}{4} - \frac{27}{4}i$
60. $\left(\frac{\sqrt{3}}{3} - \frac{1}{3}i\right)^4 = -\frac{8}{81} - \frac{8i\sqrt{3}}{81}$
61. $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^4 = -1$
62. $(2 + 2i)^5 = -128 - 128i$
63. $(\sqrt{3} - i)^5 = -16\sqrt{3} - 16i$
64. $(1 - i)^8 = 16$
65. Since $z = 4i = 4\text{cis}\left(\frac{\pi}{2}\right)$ we have

    $w_0 = 2\text{cis}\left(\frac{\pi}{4}\right) = \sqrt{2} + i\sqrt{2}$ $\quad$ $w_1 = 2\text{cis}\left(\frac{5\pi}{4}\right) = -\sqrt{2} - i\sqrt{2}$

66. Since $z = -25i = 25\text{cis}\left(\frac{3\pi}{2}\right)$ we have

    $w_0 = 5\text{cis}\left(\frac{3\pi}{4}\right) = -\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$ $\quad$ $w_1 = 5\text{cis}\left(\frac{7\pi}{4}\right) = \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$

67. Since $z = 1 + i\sqrt{3} = 2\text{cis}\left(\frac{\pi}{3}\right)$ we have

    $w_0 = \sqrt{2}\text{cis}\left(\frac{\pi}{6}\right) = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$ $\quad$ $w_1 = \sqrt{2}\text{cis}\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$

68. Since $z = \frac{5}{2} - \frac{5\sqrt{3}}{2}i = 5\text{cis}\left(\frac{5\pi}{3}\right)$ we have

    $w_0 = \sqrt{5}\text{cis}\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{15}}{2} + \frac{\sqrt{5}}{2}i$ $\quad$ $w_1 = \sqrt{5}\text{cis}\left(\frac{11\pi}{6}\right) = \frac{\sqrt{15}}{2} - \frac{\sqrt{5}}{2}i$

69. Since $z = 64 = 64\text{cis}(0)$ we have

    $w_0 = 4\text{cis}(0) = 4$ $\quad$ $w_1 = 4\text{cis}\left(\frac{2\pi}{3}\right) = -2 + 2i\sqrt{3}$ $\quad$ $w_2 = 4\text{cis}\left(\frac{4\pi}{3}\right) = -2 - 2i\sqrt{3}$

70. Since $z = -125 = 125\text{cis}(\pi)$ we have

$w_0 = 5\text{cis}\left(\frac{\pi}{3}\right) = \frac{5}{2} + \frac{5\sqrt{3}}{2}i$ $\quad w_1 = 5\text{cis}(\pi) = -5$ $\quad w_2 = 5\text{cis}\left(\frac{5\pi}{3}\right) = \frac{5}{2} - \frac{5\sqrt{3}}{2}i$

71. Since $z = i = \text{cis}\left(\frac{\pi}{2}\right)$ we have

$w_0 = \text{cis}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ $\quad w_1 = \text{cis}\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ $\quad w_2 = \text{cis}\left(\frac{3\pi}{2}\right) = -i$

72. Since $z = -8i = 8\text{cis}\left(\frac{3\pi}{2}\right)$ we have

$w_0 = 2\text{cis}\left(\frac{\pi}{2}\right) = 2i$ $\quad w_1 = 2\text{cis}\left(\frac{7\pi}{6}\right) = -\sqrt{3} - i$ $\quad w_2 = \text{cis}\left(\frac{11\pi}{6}\right) = \sqrt{3} - i$

73. Since $z = 16 = 16\text{cis}(0)$ we have

$w_0 = 2\text{cis}(0) = 2$ $\quad w_1 = 2\text{cis}\left(\frac{\pi}{2}\right) = 2i$

$w_2 = 2\text{cis}(\pi) = -2$ $\quad w_3 = 2\text{cis}\left(\frac{3\pi}{2}\right) = -2i$

74. Since $z = -81 = 81\text{cis}(\pi)$ we have

$w_0 = 3\text{cis}\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$ $\quad w_1 = 3\text{cis}\left(\frac{3\pi}{4}\right) = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

$w_2 = 3\text{cis}\left(\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$ $\quad w_3 = 3\text{cis}\left(\frac{7\pi}{4}\right) = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$

75. Since $z = 64 = 64\text{cis}(0)$ we have

$w_0 = 2\text{cis}(0) = 2$ $\quad w_1 = 2\text{cis}\left(\frac{\pi}{3}\right) = 1 + \sqrt{3}i$ $\quad w_2 = 2\text{cis}\left(\frac{2\pi}{3}\right) = -1 + \sqrt{3}i$

$w_3 = 2\text{cis}(\pi) = -2$ $\quad w_4 = 2\text{cis}\left(-\frac{2\pi}{3}\right) = -1 - \sqrt{3}i$ $\quad w_5 = 2\text{cis}\left(-\frac{\pi}{3}\right) = 1 - \sqrt{3}i$

76. Since $z = -729 = 729\text{cis}(\pi)$ we have

$w_0 = 3\text{cis}\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$ $\quad w_1 = 3\text{cis}\left(\frac{\pi}{2}\right) = 3i$ $\quad w_2 = 3\text{cis}\left(\frac{5\pi}{6}\right) = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$

$w_3 = 3\text{cis}\left(\frac{7\pi}{6}\right) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$ $\quad w_4 = 3\text{cis}\left(-\frac{3\pi}{2}\right) = -3i$ $\quad w_5 = 3\text{cis}\left(-\frac{11\pi}{6}\right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$

77. Note: In the answers for w_0 and w_2 the first rectangular form comes from applying the appropriate Sum or Difference Identity ($\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ and $\frac{17\pi}{12} = \frac{2\pi}{3} + \frac{3\pi}{4}$, respectively) and the second comes from using the Half-Angle Identities.

$w_0 = \sqrt[3]{2}\text{cis}\left(\frac{\pi}{12}\right) = \sqrt[3]{2}\left(\frac{\sqrt{6}+\sqrt{2}}{4} + i\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right) = \sqrt[3]{2}\left(\frac{\sqrt{2+\sqrt{3}}}{2} + i\frac{\sqrt{2-\sqrt{3}}}{2}\right)$

$w_1 = \sqrt[3]{2}\text{cis}\left(\frac{3\pi}{4}\right) = \sqrt[3]{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

$w_2 = \sqrt[3]{2}\text{cis}\left(\frac{17\pi}{12}\right) = \sqrt[3]{2}\left(\frac{\sqrt{2}-\sqrt{6}}{4} + i\left(\frac{-\sqrt{2}-\sqrt{6}}{4}\right)\right) = \sqrt[3]{2}\left(\frac{\sqrt{2-\sqrt{3}}}{2} + i\frac{\sqrt{2+\sqrt{3}}}{2}\right)$

78. $w_0 = \text{cis}(0) = 1$
$w_1 = \text{cis}\left(\frac{2\pi}{5}\right) \approx 0.309 + 0.951i$
$w_2 = \text{cis}\left(\frac{4\pi}{5}\right) \approx -0.809 + 0.588i$
$w_3 = \text{cis}\left(\frac{6\pi}{5}\right) \approx -0.809 - 0.588i$
$w_4 = \text{cis}\left(\frac{8\pi}{5}\right) \approx 0.309 - 0.951i$

79. $p(x) = x^{12} - 4096 = (x-2)(x+2)(x^2+4)(x^2-2x+4)(x^2+2x+4)(x^2-2\sqrt{3}x+4)(x^2+2\sqrt{3}+4)$

14.4 The Polar Form of the Conic Sections

In this section, we revisit our friends the Conic Sections which we began studying in Chapter 8. Our first task is to formalize the notion of rotating axes so this subsection is actually a follow-up to Example 9.3.3 in Section 9.3. In that example, we saw that the graph of $y = \frac{2}{x}$ is actually a hyperbola.

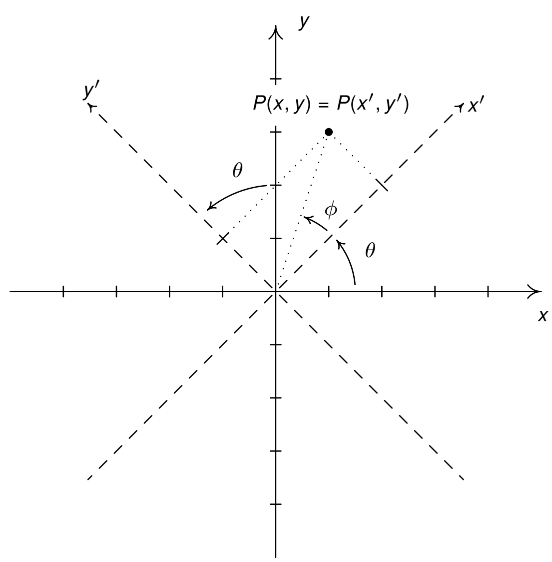
More specifically, the graph of $y = \frac{1}{x}$ is the hyperbola obtained by rotating the graph of $x^2 - y^2 = 4$ counter-clockwise through a 45° angle. Armed with polar coordinates, we can generalize the process of rotating axes as shown below.

14.4.1 Rotation of Axes

Consider the x- and y-axes below along with the dashed x'- and y'-axes obtained by rotating the x- and y-axes counter-clockwise through an angle θ and consider the point $P(x, y)$. The coordinates (x, y) are rectangular coordinates and are based on the x- and y-axes.

Suppose we wished to find rectangular coordinates based on the x'- and y'-axes. That is, we wish to determine $P(x', y')$. While this seems like a formidable challenge, it is nearly trivial if we use polar coordinates.

Consider the angle ϕ whose initial side is the positive x'-axis and whose terminal side contains the point P. We relate $P(x, y)$ and $P(x', y')$ by converting them to polar coordinates.



Converting $P(x, y)$ to polar coordinates with $r > 0$ yields $x = r\cos(\theta + \phi)$ and $y = r\sin(\theta + \phi)$. To convert the point $P(x', y')$ into polar coordinates, we first match the polar axis with the positive x'-axis, choose the same $r > 0$ (since the origin is the same in both systems) and get $x' = r\cos(\phi)$ and $y' = r\sin(\phi)$.

Using the sum formulas for sine and cosine, we have

$$\begin{aligned} x &= r\cos(\theta+\phi) \\ &= r\cos(\theta)\cos(\phi) - r\sin(\theta)\sin(\phi) && \text{Sum formula for cosine} \\ &= (r\cos(\phi))\cos(\theta) - (r\sin(\phi))\sin(\theta) \\ &= x'\cos(\theta) - y'\sin(\theta) && \text{Since } x' = r\cos(\phi) \text{ and } y' = r\sin(\phi) \end{aligned}$$

Similarly, using the sum formula for sine we get $y = x'\sin(\theta) + y'\cos(\theta)$. These equations enable us to easily convert points with $x'y'$-coordinates back into xy-coordinates. They also enable us to easily convert equations in the variables x and y into equations in the variables in terms of x' and y'.[1]

If we want equations which enable us to convert points with xy-coordinates into $x'y'$-coordinates, we need to solve the system

$$\left\{ \begin{array}{rcl} x'\cos(\theta) - y'\sin(\theta) &=& x \\ x'\sin(\theta) + y'\cos(\theta) &=& y \end{array} \right.$$

for x' and y'. Perhaps the cleanest way[2] to solve this system is to write it as a matrix equation. Using the machinery developed in Section 9.4, we write the above system as the matrix equation $AX' = X$ where

$$A = \left[\begin{array}{rr} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array} \right], \quad X' = \left[\begin{array}{c} x' \\ y' \end{array} \right], \quad X = \left[\begin{array}{c} x \\ y \end{array} \right]$$

Since $\det(A) = (\cos(\theta))(\cos(\theta)) - (-\sin(\theta))(\sin(\theta)) = \cos^2(\theta) + \sin^2(\theta) = 1$, the determinant of A is not zero so A is invertible and $X' = A^{-1}X$. Using the formula given in Equation 9.2 with $\det(A) = 1$, we find

$$A^{-1} = \left[\begin{array}{rr} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{array} \right]$$

so that

$$\begin{aligned} X' &= A^{-1}X \\ \left[\begin{array}{c} x' \\ y' \end{array} \right] &= \left[\begin{array}{rr} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{array} \right] \left[\begin{array}{c} x \\ y \end{array} \right] \\ \left[\begin{array}{c} x' \\ y' \end{array} \right] &= \left[\begin{array}{c} x\cos(\theta) + y\sin(\theta) \\ -x\sin(\theta) + y\cos(\theta) \end{array} \right] \end{aligned}$$

From which we get $x' = x\cos(\theta) + y\sin(\theta)$ and $y' = -x\sin(\theta) + y\cos(\theta)$. To summarize,

Theorem 14.7. Rotation of Axes: Suppose the positive x and y axes are rotated counter-clockwise through an angle θ to produce the axes x' and y', respectively. Then the coordinates $P(x, y)$ and $P(x', y')$ are related by the following systems of equations

$$\left\{ \begin{array}{rcl} x &=& x'\cos(\theta) - y'\sin(\theta) \\ y &=& x'\sin(\theta) + y'\cos(\theta) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} x' &=& x\cos(\theta) + y\sin(\theta) \\ y' &=& -x\sin(\theta) + y\cos(\theta) \end{array} \right.$$

[1] Just like in Section 14.1, the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$ make it easy to convert *points* from polar coordinates into rectangular coordinates, and they make it easy to convert *equations* from rectangular coordinates into polar coordinates.

[2] We could, of course, interchange the roles of x and x', y and y' and replace ϕ with $-\phi$ to get x' and y' in terms of x and y, but that seems like cheating. The matrix A introduced here is revisited in the Exercises.

We put the formulas in Theorem 14.7 to good use in the following example.

Example 14.4.1. Suppose the x- and y- axes are both rotated counter-clockwise through the angle $\theta = \frac{\pi}{3}$ to produce the x'- and y'- axes, respectively.

1. Let $P(x, y) = (2, -4)$ and find $P(x', y')$. Check your answer algebraically and graphically.

2. Convert the equation $21x^2 + 10xy\sqrt{3} + 31y^2 = 144$ to an equation in x' and y' and graph.

Solution.

1. If $P(x, y) = (2, -4)$ then $x = 2$ and $y = -4$. Using these values for x and y along with $\theta = \frac{\pi}{3}$, Theorem 14.7 gives $x' = x\cos(\theta) + y\sin(\theta) = 2\cos\left(\frac{\pi}{3}\right) + (-4)\sin\left(\frac{\pi}{3}\right)$ which simplifies to $x' = 1 - 2\sqrt{3}$.

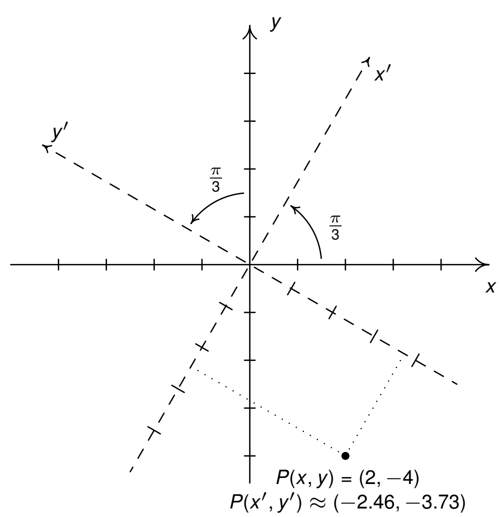
   Similarly, $y' = -x\sin(\theta) + y\cos(\theta) = (-2)\sin\left(\frac{\pi}{3}\right) + (-4)\cos\left(\frac{\pi}{3}\right)$ which gives $y' = -\sqrt{3} - 2 = -2 - \sqrt{3}$. Hence $P(x', y') = \left(1 - 2\sqrt{3}, -2 - \sqrt{3}\right)$.

   To check our answer algebraically, we convert $P(x', y') = \left(1 - 2\sqrt{3}, -2 - \sqrt{3}\right)$ back into x and y coordinates using the formulas in Theorem 14.7. We get

$$\begin{aligned} x &= x'\cos(\theta) - y'\sin(\theta) \\ &= (1 - 2\sqrt{3})\cos\left(\frac{\pi}{3}\right) - (-2 - \sqrt{3})\sin\left(\frac{\pi}{3}\right) \\ &= \left(\frac{1}{2} - \sqrt{3}\right) - \left(-\sqrt{3} - \frac{3}{2}\right) \\ &= 2 \end{aligned}$$

   Similarly, using $y = x'\sin(\theta) + y'\cos(\theta)$, we obtain $y = -4$ as required.

   To check our answer graphically, we sketch in the x'-axis and y'-axis to see if the new coordinates $P(x', y') = \left(1 - 2\sqrt{3}, -2 - \sqrt{3}\right) \approx (-2.46, -3.73)$ seem reasonable. Our graph is below.



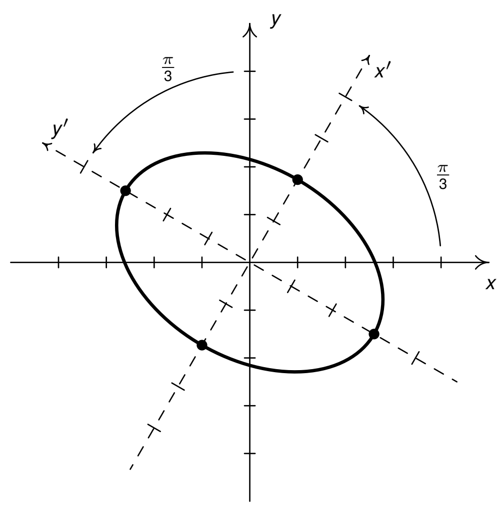
2. To convert the equation $21x^2 + 10xy\sqrt{3} + 31y^2 = 144$ to an equation in the variables x' and y', we substitute $x = x'\cos\left(\frac{\pi}{3}\right) - y'\sin\left(\frac{\pi}{3}\right) = \frac{x'}{2} - \frac{y'\sqrt{3}}{2}$ and $y = x'\sin\left(\frac{\pi}{3}\right) + y'\cos\left(\frac{\pi}{3}\right) = \frac{x'\sqrt{3}}{2} + \frac{y'}{2}$.

   While this is by no means a trivial task, it is nothing more than a hefty dose of Intermediate Algebra. While we leave most of the details to the reader, a good starting point is to verify:

$$x^2 = \frac{(x')^2}{4} - \frac{x'y'\sqrt{3}}{2} + \frac{3(y')^2}{4}, \quad xy = \frac{(x')^2\sqrt{3}}{4} - \frac{x'y'}{2} - \frac{(y')^2\sqrt{3}}{4}, \quad y^2 = \frac{3(x')^2}{4} + \frac{x'y'\sqrt{3}}{2} + \frac{(y')^2}{4}$$

   To our surprise and delight, the equation $21x^2 + 10xy\sqrt{3} + 31y^2 = 144$ in xy-coordinates reduces to $36(x')^2 + 16(y')^2 = 144$, or $\frac{(x')^2}{4} + \frac{(y')^2}{9} = 1$ in $x'y'$-coordinates.

   That is, the curve is an ellipse centered at $(0, 0)$ with vertices along the y'-axis with ($x'y'$-coordinates) $(0, \pm 3)$ and whose minor axis has endpoints with ($x'y'$-coordinates) $(\pm 2, 0)$ as seen below.



$21x^2 + 10xy\sqrt{3} + 31y^2 = 144$

□

Thanks to the elimination of the xy' term from the equation $21x^2+10xy\sqrt{3}+31y^2 = 144$ in Example 14.4.1 number 2, we were able to graph the equation on the $x'y'$-plane using what we know from Chapter 8.

It is natural to wonder if, given an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, with $B \neq 0$, is there an angle θ so that if we rotate the x and y-axes counter-clockwise through that angle θ, the equation in the rotated variables x' and y' contains no $x'y'$ term.

To find out, we make the usual substitutions $x = x'\cos(\theta) - y'\sin(\theta)$ and $y = x'\sin(\theta) + y'\cos(\theta)$ into the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ and set the coefficient of the $x'y'$ term equal to 0.

Terms containing $x'y'$ in this expression will come from the first three terms of the equation: Ax^2, Bxy and Cy^2. We leave it to the reader to verify that

$$\begin{aligned}
x^2 &= (x')^2\cos^2(\theta) - 2x'y'\cos(\theta)\sin(\theta) + (y')^2\sin(\theta) \\
xy &= (x')^2\cos(\theta)\sin(\theta) + x'y'\left(\cos^2(\theta) - \sin^2(\theta)\right) - (y')^2\cos(\theta)\sin(\theta) \\
y^2 &= (x')^2\sin^2(\theta) + 2x'y'\cos(\theta)\sin(\theta) + (y')^2\cos^2(\theta)
\end{aligned}$$

The contribution to the $x'y'$-term from Ax^2 is $-2A\cos(\theta)\sin(\theta)$, from Bxy it is $B\left(\cos^2(\theta) - \sin^2(\theta)\right)$, and from Cy^2 it is $2C\cos(\theta)\sin(\theta)$. Equating the $x'y'$-term to 0, we get

$$\begin{aligned}
-2A\cos(\theta)\sin(\theta) + B\left(\cos^2(\theta) - \sin^2(\theta)\right) + 2C\cos(\theta)\sin(\theta) &= 0 \\
-A\sin(2\theta) + B\cos(2\theta) + C\sin(2\theta) &= 0 \quad \text{Double Angle Identities}
\end{aligned}$$

From this, we get $B\cos(2\theta) = (A - C)\sin(2\theta)$. Our goal is to solve for θ in terms of A, B and C. Since we are assuming $B \neq 0$, we can divide both sides of this equation by B. To solve for θ we would like to divide both sides of the equation by $\sin(2\theta)$, provided of course that we have assurances that $\sin(2\theta) \neq 0$. If $\sin(2\theta) = 0$, then we would have $B\cos(2\theta) = 0$, and since $B \neq 0$, this would force $\cos(2\theta) = 0$. Since no angle θ can have both $\sin(2\theta) = 0$ and $\cos(2\theta) = 0$, we can safely assume[3] $\sin(2\theta) \neq 0$. Hence, we get $\frac{\cos(2\theta)}{\sin(2\theta)} = \frac{A-C}{B}$, or $\cot(2\theta) = \frac{A-C}{B}$. We have just proved the following theorem.

Theorem 14.8. The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ with $B \neq 0$ can be transformed into an equation in variables x' and y' without any $x'y'$ terms by rotating the x- and y- axes counter-clockwise through an angle θ which satisfies $\cot(2\theta) = \frac{A-C}{B}$.

We put Theorem 14.8 to good use in the following example.

Example 14.4.2. Graph the following equations.

1. $5x^2 + 26xy + 5y^2 - 16x\sqrt{2} + 16y\sqrt{2} - 104 = 0$
2. $16x^2 + 24xy + 9y^2 + 15x - 20y = 0$

Solution.

1. Since the equation $5x^2 + 26xy + 5y^2 - 16x\sqrt{2} + 16y\sqrt{2} - 104 = 0$ is already given to us in the form required by Theorem 14.8, we identify $A = 5$, $B = 26$ and $C = 5$ so that $\cot(2\theta) = \frac{A-C}{B} = \frac{5-5}{26} = 0$.

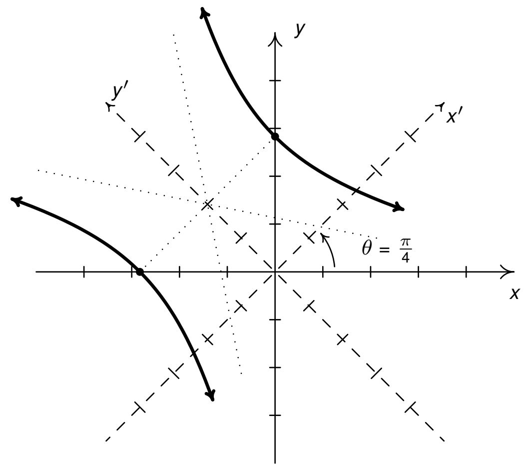
   This means $\cot(2\theta) = 0$ which gives $\theta = \frac{\pi}{4} + \frac{\pi}{2}k$ for integers k. We choose $\theta = \frac{\pi}{4}$ so that our rotation equations are $x = \frac{x'\sqrt{2}}{2} - \frac{y'\sqrt{2}}{2}$ and $y = \frac{x'\sqrt{2}}{2} + \frac{y'\sqrt{2}}{2}$. The reader should verify that

   $$x^2 = \frac{(x')^2}{2} - x'y' + \frac{(y')^2}{2}, \quad xy = \frac{(x')^2}{2} - \frac{(y')^2}{2}, \quad y^2 = \frac{(x')^2}{2} + x'y' + \frac{(y')^2}{2}$$

   Making the other substitutions, we get that $5x^2 + 26xy + 5y^2 - 16x\sqrt{2} + 16y\sqrt{2} - 104 = 0$ reduces to $18(x')^2 - 8(y')^2 + 32y' - 104 = 0$, or $\frac{(x')^2}{4} - \frac{(y'-2)^2}{9} = 1$.

   Hence, we have a hyperbola centered at the $x'y'$-coordinates $(0, 2)$ opening in the x' direction with vertices $(\pm 2, 2)$ (in $x'y'$-coordinates) and asymptotes $y' = \pm\frac{3}{2}x' + 2$. We graph this equation below.

[3]The reader is invited to think about the case $\sin(2\theta) = 0$ geometrically. What happens to the axes in this case?



$$5x^2 + 26xy + 5y^2 - 16x\sqrt{2} + 16y\sqrt{2} - 104 = 0$$

2. From $16x^2 + 24xy + 9y^2 + 15x - 20y = 0$, we get $A = 16$, $B = 24$ and $C = 9$ so that $\cot(2\theta) = \frac{7}{24}$. Since this isn't one of the values of the common angles, we will need to use inverse functions.

   Ultimately, we need to find $\cos(\theta)$ and $\sin(\theta)$, which means we have two options. If we use the arccotangent function immediately, after the usual calculations we get $\theta = \frac{1}{2}\text{arccot}\left(\frac{7}{24}\right)$. To get $\cos(\theta)$ and $\sin(\theta)$ from this, we would need to use half angle identities.

   Alternatively, we can start with $\cot(2\theta) = \frac{7}{24}$, use a double angle identity, and then go after $\cos(\theta)$ and $\sin(\theta)$. We adopt the second approach.

   From $\cot(2\theta) = \frac{7}{24}$, we have $\tan(2\theta) = \frac{24}{7}$. Using the double angle identity for tangent, we have $\frac{2\tan(\theta)}{1-\tan^2(\theta)} = \frac{24}{7}$, which gives $24\tan^2(\theta) + 14\tan(\theta) - 24 = 0$.

   Factoring, we get $2(3\tan(\theta) + 4)(4\tan(\theta) - 3) = 0$ which gives $\tan(\theta) = -\frac{4}{3}$ or $\tan(\theta) = \frac{3}{4}$. While either of these values of $\tan(\theta)$ satisfies the equation $\cot(2\theta) = \frac{7}{24}$, we choose $\tan(\theta) = \frac{3}{4}$, since this produces an acute angle,[4] $\theta = \arctan\left(\frac{3}{4}\right)$.

   To find the rotation equations, we need $\cos(\theta) = \cos\left(\arctan\left(\frac{3}{4}\right)\right)$ and $\sin(\theta) = \sin\left(\arctan\left(\frac{3}{4}\right)\right)$. Using the techniques developed in Section 12.3 we get $\cos(\theta) = \frac{4}{5}$ and $\sin(\theta) = \frac{3}{5}$.

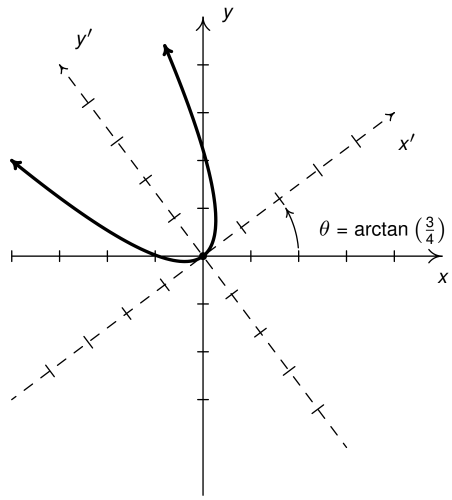
   Our rotation equations are $x = x'\cos(\theta) - y'\sin(\theta) = \frac{4x'}{5} - \frac{3y'}{5}$ and $y = x'\sin(\theta) + y'\cos(\theta) = \frac{3x'}{5} + \frac{4y'}{5}$.

   As usual, we now substitute these quantities into $16x^2 + 24xy + 9y^2 + 15x - 20y = 0$ and simplify. As a first step, the reader can verify

$$x^2 = \frac{16(x')^2}{25} - \frac{24x'y'}{25} + \frac{9(y')^2}{25}, \quad xy = \frac{12(x')^2}{25} + \frac{7x'y'}{25} - \frac{12(y')^2}{25}, \quad y^2 = \frac{9(x')^2}{25} + \frac{24x'y'}{25} + \frac{16(y')^2}{25}$$

[4]As usual, there are infinitely many solutions to $\tan(\theta) = \frac{3}{4}$. We choose the acute angle $\theta = \arctan\left(\frac{3}{4}\right)$. The reader is encouraged to think about why there is always at least one acute answer to $\cot(2\theta) = \frac{A-C}{B}$ and what this means geometrically in terms of what we are trying to accomplish by rotating the axes. The reader is also encouraged to keep a sharp lookout for the angles which satisfy $\tan(\theta) = -\frac{4}{3}$ in our final graph. (Hint: $\left(\frac{3}{4}\right)\left(-\frac{4}{3}\right) = -1$.)

Once the dust settles, we get $25(x')^2 - 25y' = 0$, or $y' = (x')^2$, whose graph is a parabola opening along the positive y'-axis with vertex $(0, 0)$. We graph this equation below.



$16x^2 + 24xy + 9y^2 + 15x - 20y = 0$

□

Note that even though the coefficients of x^2 and y^2 were both positive numbers in parts 1 and 2 of Example 14.4.2, the graph in part 1 turned out to be a hyperbola and the graph in part 2 worked out to be a parabola. Whereas in Chapter 8, we could easily pick out which conic section we were dealing with based on the presence (or absence) of quadratic terms and their coefficients, Example 14.4.2 demonstrates the situation is much more complicated when an xy term is present.

Nevertheless, it is possible to determine which conic section we have by looking at a special, familiar *combination* of the coefficients of the quadratic terms. We have the following theorem.

Theorem 14.9.
Suppose the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ describes a non-degenerate conic section.[a]

- If $B^2 - 4AC > 0$ then the graph of the equation is a hyperbola.
- If $B^2 - 4AC = 0$ then the graph of the equation is a parabola.
- If $B^2 - 4AC < 0$ then the graph of the equation is an ellipse or circle.

[a]Recall that this means its graph is either a circle, parabola, ellipse or hyperbola. See page 675.

As you may expect, the quantity $B^2 - 4AC$ mentioned in Theorem 14.9 is called the **discriminant** of the conic section. While we will not attempt to explain the deep Mathematics which produces this ‘coincidence’, we will at least work through the proof of Theorem 14.9 mechanically to show that it is true.[5]

First note that if the coefficient $B = 0$ in the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, Theorem 14.9 reduces to the result presented in Exercise 40 in Section 8.5.

[5]We hope that someday you get to see *why* this works the way it does.

Hence, we proceed under the assumption that $B \neq 0$. We rotate the xy-axes counter-clockwise through an angle θ which satisfies $\cot(2\theta) = \frac{A-C}{B}$ to produce an equation with no $x'y'$-term in accordance with Theorem 14.8: $A'(x')^2 + C(y')^2 + Dx' + Ey' + F' = 0$.

In this form, we can invoke Exercise 40 in Section 8.5 once more using the product $A'C'$. Our goal is to find the product $A'C'$ in terms of the coefficients A, B and C in the original equation.

We substitute $x = x'\cos(\theta) - y'\sin(\theta)$ $y = x'\sin(\theta) + y'\cos(\theta)$ into $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. After gathering like terms, the coefficient A' on $(x')^2$ and the coefficient C' on $(y')^2$ are

$$\begin{aligned} A' &= A\cos^2(\theta) + B\cos(\theta)\sin(\theta) + C\sin^2(\theta) \\ C' &= A\sin^2(\theta) - B\cos(\theta)\sin(\theta) + C\cos^2(\theta) \end{aligned}$$

In order to make use of the condition $\cot(2\theta) = \frac{A-C}{B}$, we rewrite our formulas for A' and C' using the power reduction formulas. After some regrouping, we get

$$\begin{aligned} 2A' &= [(A+C) + (A-C)\cos(2\theta)] + B\sin(2\theta) \\ 2C' &= [(A+C) - (A-C)\cos(2\theta)] - B\sin(2\theta) \end{aligned}$$

Next, we try to make sense of the product

$$(2A')(2C') = \{[(A+C) + (A-C)\cos(2\theta)] + B\sin(2\theta)\}\{[(A+C) - (A-C)\cos(2\theta)] - B\sin(2\theta)\}$$

We break this product into pieces. First, we use the difference of squares to multiply the 'first' quantities in each factor to get

$$[(A+C) + (A-C)\cos(2\theta)][(A+C) - (A-C)\cos(2\theta)] = (A+C)^2 - (A-C)^2\cos^2(2\theta)$$

Next, we add the product of the 'outer' and 'inner' quantities in each factor to get

$$\begin{aligned} &-B\sin(2\theta)[(A+C) + (A-C)\cos(2\theta)] \\ &+B\sin(2\theta)[(A+C) - (A-C)\cos(2\theta)] = -2B(A-C)\cos(2\theta)\sin(2\theta) \end{aligned}$$

The product of the 'last' quantity in each factor is $(B\sin(2\theta))(-B\sin(2\theta)) = -B^2\sin^2(2\theta)$.

Putting all of this together yields

$$4A'C' = (A+C)^2 - (A-C)^2\cos^2(2\theta) - 2B(A-C)\cos(2\theta)\sin(2\theta) - B^2\sin^2(2\theta)$$

From $\cot(2\theta) = \frac{A-C}{B}$, we get $\frac{\cos(2\theta)}{\sin(2\theta)} = \frac{A-C}{B}$, or $(A-C)\sin(2\theta) = B\cos(2\theta)$.

Using this substitution twice along with the Pythagorean Identity $\cos^2(2\theta) = 1 - \sin^2(2\theta)$ we get:

$$\begin{aligned}
4A'C' &= (A+C)^2 - (A-C)^2\cos^2(2\theta) - 2B(A-C)\cos(2\theta)\sin(2\theta) - B^2\sin^2(2\theta) \\
&= (A+C)^2 - (A-C)^2\left[1-\sin^2(2\theta)\right] - 2B\cos(2\theta)B\cos(2\theta) - B^2\sin^2(2\theta) \\
&= (A+C)^2 - (A-C)^2 + (A-C)^2\sin^2(2\theta) - 2B^2\cos^2(2\theta) - B^2\sin^2(2\theta) \\
&= (A+C)^2 - (A-C)^2 + [(A-C)\sin(2\theta)]^2 - 2B^2\cos^2(2\theta) - B^2\sin^2(2\theta) \\
&= (A+C)^2 - (A-C)^2 + [B\cos(2\theta)]^2 - 2B^2\cos^2(2\theta) - B^2\sin^2(2\theta) \\
&= (A+C)^2 - (A-C)^2 + B^2\cos^2(2\theta) - 2B^2\cos^2(2\theta) - B^2\sin^2(2\theta) \\
&= (A+C)^2 - (A-C)^2 - B^2\cos^2(2\theta) - B^2\sin^2(2\theta) \\
&= (A+C)^2 - (A-C)^2 - B^2\left[\cos^2(2\theta)+\sin^2(2\theta)\right] \\
&= (A+C)^2 - (A-C)^2 - B^2(1) \\
&= \left(A^2+2AC+C^2\right) - \left(A^2-2AC+C^2\right) - B^2 \\
&= 4AC - B^2
\end{aligned}$$

Hence, $B^2 - 4AC = -4A'C'$, so the quantity $B^2 - 4AC$ has the opposite sign of $A'C'$. The result now follows by applying Exercise 40 in Section 8.5.

Example 14.4.3. Use Theorem 14.9 to classify the graphs of the following non-degenerate conics.

1. $21x^2 + 10xy\sqrt{3} + 31y^2 = 144$
2. $5x^2 + 26xy + 5y^2 - 16x\sqrt{2} + 16y\sqrt{2} - 104 = 0$
3. $16x^2 + 24xy + 9y^2 + 15x - 20y = 0$

Solution. This is a straightforward application of Theorem 14.9.

1. We have $A = 21$, $B = 10\sqrt{3}$ and $C = 31$ so $B^2 - 4AC = (10\sqrt{3})^2 - 4(21)(31) = -2304 < 0$. Theorem 14.9 predicts the graph is an ellipse, which checks with our work from Example 14.4.1 number 2.
2. Here, $A = 5$, $B = 26$ and $C = 5$, so $B^2 - 4AC = 26^2 - 4(5)(5) = 576 > 0$. Theorem 14.9 classifies the graph as a hyperbola, which matches our answer to Example 14.4.2 number 1.
3. Finally, we have $A = 16$, $B = 24$ and $C = 9$ which gives $24^2 - 4(16)(9) = 0$. Theorem 14.9 tells us that the graph is a parabola, matching our result from Example 14.4.2 number 2. □

14.4.2 The Polar Form of Conics

Here, we revisit the conic sections from a more unified perspective starting with a 'new' definition below.

Definition 14.4. Given a fixed line L, a point F not on L, and a positive number e, a conic section is the set of all points P such that

$$\frac{\text{the distance from } P \text{ to } F}{\text{the distance from } P \text{ to } L} = e$$

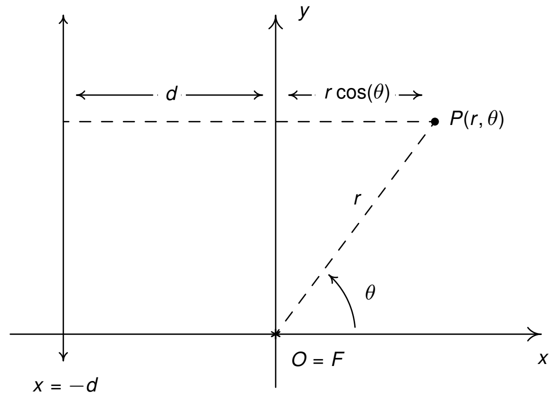
The line L is called the **directrix** of the conic section, the point F is called a **focus** of the conic section, and the constant e is called the **eccentricity** of the conic section.

We have seen the notions of focus and directrix before in the definition of a parabola, Definition 8.1. There, a parabola is defined as the set of points equidistant from the focus and directrix, giving an eccentricity $e = 1$ according to Definition 14.4.

We have also seen the concept of eccentricity before. It was introduced for ellipses in Definition 8.5 in Section 8.4, and later extended to hyperbolas in Exercise 37 in Section 8.5. There, e was also defined as a ratio of distances, though in these cases the distances involved were measurements from the center to a focus and from the center to a vertex.

One way to reconcile the 'old' ideas of focus, directrix and eccentricity with the 'new' ones presented in Definition 14.4 is to derive equations for the conic sections using Definition 14.4 and compare these parameters with what we know from Chapter 8.

We begin by assuming the conic section has eccentricity e, a focus F at the origin and that the directrix is the vertical line $x = -d$ as in the figure below.



Using a polar coordinate representation $P(r, \theta)$ for a point on the conic with $r > 0$, we get

$$e = \frac{\text{the distance from } P \text{ to } F}{\text{the distance from } P \text{ to } L} = \frac{r}{d + r\cos(\theta)}$$

so that $r = e(d + r\cos(\theta))$. Solving this equation for r, yields

$$r = \frac{ed}{1 - e\cos(\theta)}$$

At this point, we convert the equation $r = e(d + r\cos(\theta))$ back into a rectangular equation in the variables x and y. If $e > 0$, but $e \neq 1$, the usual conversion process outlined in Section 14.1 gives[6]

$$\left(\frac{(1-e^2)^2}{e^2d^2}\right)\left(x - \frac{e^2d}{1-e^2}\right)^2 + \left(\frac{1-e^2}{e^2d^2}\right)y^2 = 1$$

If $0 < e < 1$, then $0 < 1 - e^2 < 1$ and, hence, $(1 - e^2)^2 < 1 - e^2$. We leave it to the reader to show that this means we have the equation of an ellipse centered at $\left(\frac{e^2d}{1-e^2}, 0\right)$ with major axis along the x-axis.

[6]Turn $r = e(d + r\cos(\theta))$ into $r = e(d + x)$ and square both sides to get $r^2 = e^2(d + x)^2$. Replace r^2 with $x^2 + y^2$, expand $(d + x)^2$, combine like terms, complete the square on x and clean things up.

Using the notation from Section 8.4, we have $a^2 = \frac{e^2d^2}{\left(1-e^2\right)^2}$ and $b^2 = \frac{e^2d^2}{1-e^2}$, so the major axis has length $\frac{2ed}{1-e^2}$ and the minor axis has length $\frac{2ed}{\sqrt{1-e^2}}$.

Moreover, we find that one focus is $(0,0)$ and working through the formula given in Definition 8.5 gives the eccentricity to be e, as required.

If $e > 1$, then $1 - e^2 < 0$ but $(1-e^2)^2 > 0$ so the equation generates a hyperbola with center $\left(\frac{e^2d}{1-e^2}, 0\right)$ whose transverse axis lies along the x-axis.

Since such hyperbolas have the form $\frac{(x-h)^2}{a^2} - \frac{y^2}{b^2} = 1$, we need to take the *opposite* reciprocal of the coefficient of y^2 to find b^2.

Doing this, we obtain[7] $a^2 = \frac{e^2d^2}{\left(1-e^2\right)^2} = \frac{e^2d^2}{\left(e^2-1\right)^2}$ and $b^2 = -\frac{e^2d^2}{1-e^2} = \frac{e^2d^2}{e^2-1}$, so the transverse axis has length $\frac{2ed}{e^2-1}$ and the conjugate axis has length $\frac{2ed}{\sqrt{e^2-1}}$.

Additionally, we verify that one focus is at $(0,0)$, and the formula given in Exercise 37 in Section 8.5 gives the eccentricity is e in this case as well.

If $e = 1$, the equation $r = \frac{ed}{1-e\cos(\theta)}$ reduces to $r = \frac{d}{1-\cos(\theta)}$ which translates to $y^2 = 2d\left(x + \frac{d}{2}\right)$.

The equation $y^2 = 2d\left(x + \frac{d}{2}\right)$ describes a parabola with vertex $\left(-\frac{d}{2}, 0\right)$ opening to the right.

In the language of Section 8.2, $4p = 2d$ so $p = \frac{d}{2}$, the focus is $(0,0)$, the focal diameter is $2d$ and the directrix is $x = -d$, as required.

Hence, we have shown that in all cases, our 'new' understanding of 'conic section', 'focus', 'eccentricity' and 'directrix' as presented in Definition 14.4 correspond with the 'old' definitions given in Chapter 8.

Before we summarize our findings, we note that in order to arrive at our general equation of a conic $r = \frac{ed}{1-e\cos(\theta)}$, we assumed that the directrix was the line $x = -d$ for $d > 0$.

We could have just as easily chosen the directrix to be $x = d$, $y = -d$ or $y = d$. As the reader can verify, in these cases we obtain the forms $r = \frac{ed}{1+e\cos(\theta)}$, $r = \frac{ed}{1-e\sin(\theta)}$ and $r = \frac{ed}{1+e\sin(\theta)}$, respectively.

The key thing to remember is that in any of these cases, the directrix is always perpendicular to the major axis of an ellipse and it is always perpendicular to the transverse axis of the hyperbola.

For parabolas, knowing the focus is $(0,0)$ and the directrix also tells us which way the parabola opens.

We have established the following theorem.

[7] Since $1 - e^2 < 0$ here, we rewrite $\left(1-e^2\right)^2 = \left(e^2-1\right)^2$ to help simplify things later on.

Theorem 14.10. Suppose e and d are positive numbers. Then

- the graph of $r = \dfrac{ed}{1 - e\cos(\theta)}$ is the graph of a conic section with directrix $x = -d$.
- the graph of $r = \dfrac{ed}{1 + e\cos(\theta)}$ is the graph of a conic section with directrix $x = d$.
- the graph of $r = \dfrac{ed}{1 - e\sin(\theta)}$ is the graph of a conic section with directrix $y = -d$.
- the graph of $r = \dfrac{ed}{1 + e\sin(\theta)}$ is the graph of a conic section with directrix $y = d$.

In each case above, $(0, 0)$ is a focus of the conic and the number e is the eccentricity of the conic.

- If $0 < e < 1$, the graph is an ellipse. The quantities $\dfrac{2ed}{1 - e^2}$ and $\dfrac{2ed}{\sqrt{1 - e^2}}$ are the lengths of the major and minor axes, respectively.
- If $e = 1$, the graph is a parabola whose focal diameter is $2d$.
- If $e > 1$, the graph is a hyperbola. The quantities $\dfrac{2ed}{e^2 - 1}$ and $\dfrac{2ed}{\sqrt{e^2 - 1}}$ are the lengths of the transverse and conjugate axes, respectively.

We test out Theorem 14.10 in the next example.

Example 14.4.4. Sketch the graphs of the following equations.

1. $r = \dfrac{4}{1 - \sin(\theta)}$
2. $r = \dfrac{12}{3 - \cos(\theta)}$
3. $r = \dfrac{6}{1 + 2\sin(\theta)}$

Solution.

1. From $r = \frac{4}{1-\sin(\theta)}$, we first note $e = 1$ which means we have a parabola on our hands.

   Since $ed = 4$, we have $d = 4$ and given the form of the equation, the directrix at $y = -4$.

   Since the focus is at $(0, 0)$, we know that the vertex is located at the point (in rectangular coordinates) $(0, -2)$ and must open upwards.

   With $d = 4$, we have a focal diameter of $2d = 8$, so the parabola contains the points $(\pm 4, 0)$.

   Putting all this together, we graph $r = \frac{4}{1-\sin(\theta)}$ below on the left.

2. We first rewrite $r = \frac{12}{3-\cos(\theta)}$ in the form found in Theorem 14.10, namely $r = \frac{4}{1-(1/3)\cos(\theta)}$.

   Since $e = \frac{1}{3}$ satisfies $0 < e < 1$, we know that the graph of this equation is an ellipse.

Since $ed = 4$, we have $d = 12$ and, based on the form of the equation, the directrix is $x = -12$.

Hence, the ellipse has its major axis along the x-axis, which means we can find the vertices of the ellipse by finding where the ellipse intersects the x-axis.

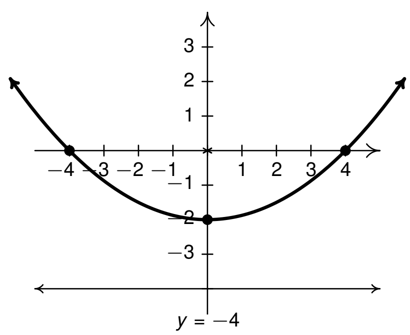
Since $r(0) = 6$ and $r(\pi) = 3$, our vertices are the rectangular points $(-3, 0)$ and $(6, 0)$.

The center of the ellipse is the midpoint of the vertices, which in this case is $\left(\frac{3}{2}, 0\right)$.[8]

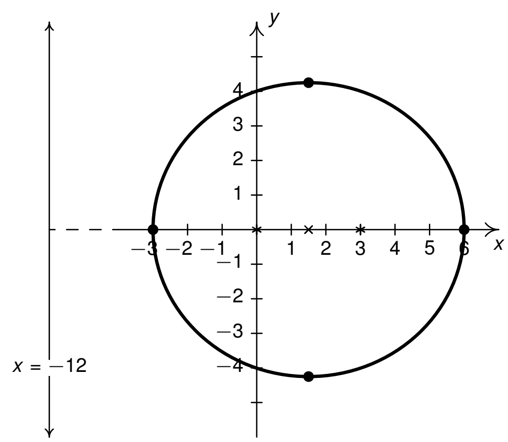
We know one focus is $(0, 0)$, which is $\frac{3}{2}$ from the center $\left(\frac{3}{2}, 0\right)$ and this allows us to find the other focus $(3, 0)$, even though we are not asked to do so.

Finally, we know from Theorem 14.10 that the length of the minor axis is $\frac{2ed}{\sqrt{1-e^2}} = \frac{4}{\sqrt{1-(1/3)^2}} = 6\sqrt{3}$ which means the endpoints of the minor axis are $\left(\frac{3}{2}, \pm 3\sqrt{2}\right)$.

We now have everything we need to graph $r = \frac{12}{3-\cos(\theta)}$ below on the right.



$r = \frac{4}{1-\sin(\theta)}$



$r = \frac{12}{3-\cos(\theta)}$

3. From $r = \frac{6}{1+2\sin(\theta)}$ we get $e = 2 > 1$ so the graph is a hyperbola.

   Since $ed = 6$, we get $d = 3$, and from the form of the equation, we know the directrix is $y = 3$.

   Hence, the transverse axis of the hyperbola lies along the y-axis, so we can find the vertices by looking where the hyperbola intersects the y-axis.

   We find $r\left(\frac{\pi}{2}\right) = 2$ and $r\left(\frac{3\pi}{2}\right) = -6$. These two points correspond to the rectangular points $(0, 2)$ and $(0, 6)$ which puts the center of the hyperbola at $(0, 4)$.

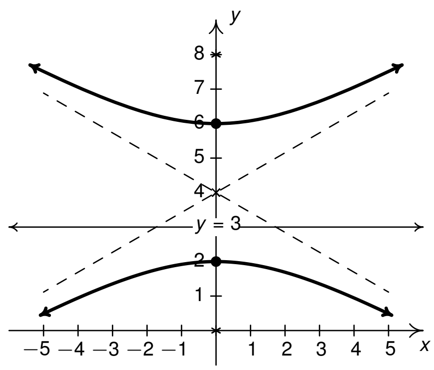
   Since one focus is at $(0, 0)$, 4 units away from the center, we know the other focus is at $(0, 8)$.

   According to Theorem 14.10, the conjugate axis has a length of $\frac{2ed}{\sqrt{e^2-1}} = \frac{(2)(6)}{\sqrt{2^2-1}} = 4\sqrt{3}$. This together with the location of the vertices give the slopes of the asymptotes as: $\pm\frac{2}{2\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$.

[8]As a quick check, we have from Theorem 14.10 the major axis should have length $\frac{2ed}{1-e^2} = \frac{(2)(4)}{1-(1/3)^2} = 9$.

Since the center of the hyperbola is $(0,4)$, the asymptotes are $y = \pm\frac{\sqrt{3}}{3}x + 4$.

Using all of our work, we graph the hyperbola below.



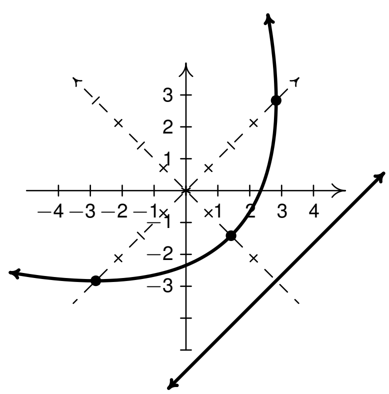
$r = \frac{6}{1+2\sin(\theta)}$

□

In light of Section 14.4.1, the reader may wonder what the rotated form of the conic sections would look like in polar form.

We know from Exercise 66 in Section 14.2 that replacing θ with $(\theta - \phi)$ in an expression $r = f(\theta)$ rotates the graph of $r = f(\theta)$ counter-clockwise by an angle ϕ.

For instance, to graph $r = \frac{4}{1-\sin\left(\theta-\frac{\pi}{4}\right)}$ all we need to do is rotate the graph of $r = \frac{4}{1-\sin(\theta)}$, which we obtained in Example 14.4.4 number 1, counter-clockwise by $\frac{\pi}{4}$ radians, as shown below.



$r = \frac{4}{1-\sin\left(\theta-\frac{\pi}{4}\right)}$

Using rotations, we can greatly simplify the form of the conic sections presented in Theorem 14.10, since any three of the forms given there can be obtained from the fourth by rotating through some multiple of $\frac{\pi}{2}$.

Moreover, since rotations do not affect lengths, all of the formulas for lengths Theorem 14.10 remain intact.

The formula in Theorem 14.11 below captures all the conic sections that have a focus at $(0,0)$. It also includes circles centered at the origin by extending the concept of eccentricity to include $e = 0$.

While substituting $e = 0$ into the equation given in Theorem 14.11 quickly reduces to a circle centered at the origin, the reader is best advised to think about this idea in light of Definition 8.5 in Section 8.4.

Theorem 14.11. Given constants $\ell > 0$, $e \geq 0$ and ϕ, the graph of the equation

$$r = \frac{\ell}{1 - e\cos(\theta - \phi)}$$

is a conic section with eccentricity e and one focus at $(0,0)$.

- If $e = 0$, the graph is a circle centered at $(0,0)$ with radius ℓ.
- If $e \neq 0$, then the conic has a focus at $(0,0)$.

  Defining $d = \dfrac{\ell}{e}$, the directrix contains the point with polar coordinates $(-d, \phi)$.

  - If $0 < e < 1$, the graph is an ellipse. The quantities $\dfrac{2ed}{1-e^2}$ and $\dfrac{2ed}{\sqrt{1-e^2}}$ are the lengths of the major and minor axes, respectively.
  - If $e = 1$, the graph is a parabola whose focal diameter is $2d$.
  - If $e > 1$, the graph is a hyperbola. The quantities $\dfrac{2ed}{e^2-1}$ and $\dfrac{2ed}{\sqrt{e^2-1}}$ are the lengths of the transverse and conjugate axes, respectively.

14.4.3 Exercises

Graph the following equations.

1. $x^2 + 2xy + y^2 - x\sqrt{2} + y\sqrt{2} - 6 = 0$
2. $7x^2 - 4xy\sqrt{3} + 3y^2 - 2x - 2y\sqrt{3} - 5 = 0$
3. $5x^2 + 6xy + 5y^2 - 4\sqrt{2}x + 4\sqrt{2}y = 0$
4. $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y - 16 = 0$
5. $13x^2 - 34xy\sqrt{3} + 47y^2 - 64 = 0$
6. $x^2 - 2\sqrt{3}xy - y^2 + 8 = 0$
7. $x^2 - 4xy + 4y^2 - 2x\sqrt{5} - y\sqrt{5} = 0$
8. $8x^2 + 12xy + 17y^2 - 20 = 0$

Graph the following equations.

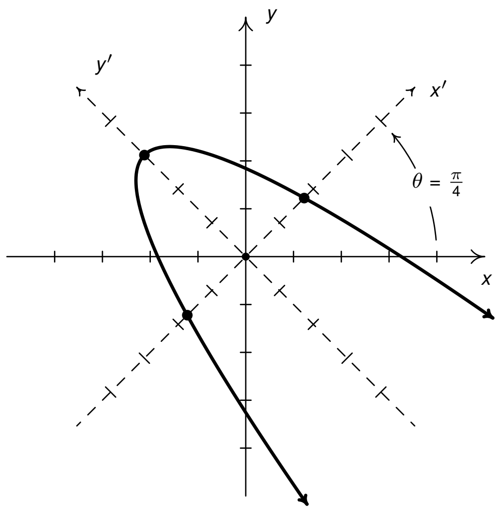
9. $r = \dfrac{2}{1 - \cos(\theta)}$
10. $r = \dfrac{3}{2 + \sin(\theta)}$
11. $r = \dfrac{3}{2 - \cos(\theta)}$
12. $r = \dfrac{2}{1 + \sin(\theta)}$
13. $r = \dfrac{4}{1 + 3\cos(\theta)}$
14. $r = \dfrac{2}{1 - 2\sin(\theta)}$
15. $r = \dfrac{2}{1 + \sin(\theta - \frac{\pi}{3})}$
16. $r = \dfrac{6}{3 - \cos\left(\theta + \frac{\pi}{4}\right)}$

The matrix $A(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ is called a **rotation matrix**.
We've seen this matrix most recently used in the proof of Theorem 14.7.

17. Show the matrix from Example 9.3.3 in Section 9.3 is none other than $A\left(\frac{\pi}{4}\right)$.
18. Discuss with your classmates how to use $A(\theta)$ to rotate points in the plane.
19. Using the even / odd identities for cosine and sine, show $A(\theta)^{-1} = A(-\theta)$. Interpret this geometrically.

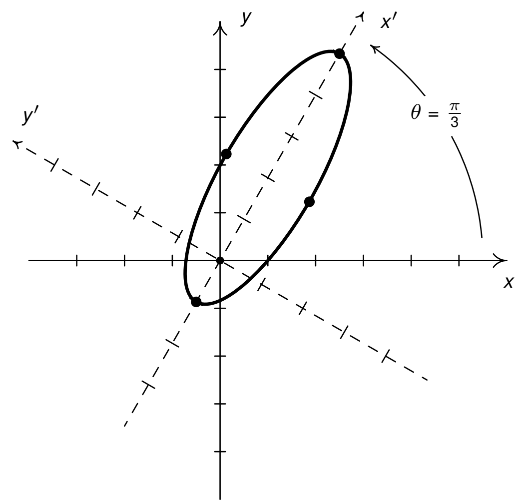
14.4.4 Answers

1. $x^2 + 2xy + y^2 - x\sqrt{2} + y\sqrt{2} - 6 = 0$
   becomes $(x')^2 = -(y' - 3)$ after rotating
   counter-clockwise through $\theta = \frac{\pi}{4}$.



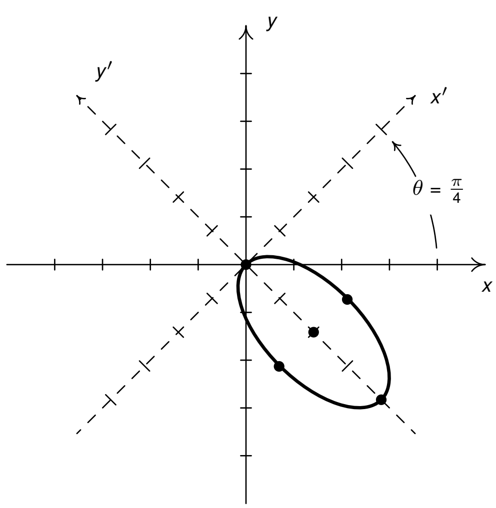
$x^2 + 2xy + y^2 - x\sqrt{2} + y\sqrt{2} - 6 = 0$

2. $7x^2 - 4xy\sqrt{3} + 3y^2 - 2x - 2y\sqrt{3} - 5 = 0$
   becomes $\frac{(x'-2)^2}{9} + (y')^2 = 1$ after rotating
   counter-clockwise through $\theta = \frac{\pi}{3}$



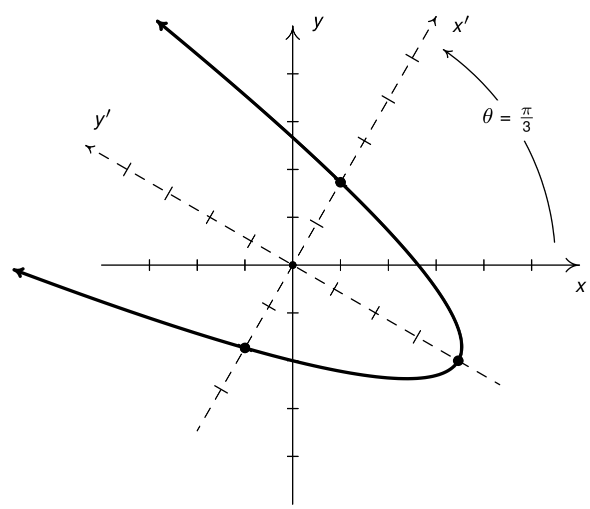
$7x^2 - 4xy\sqrt{3} + 3y^2 - 2x - 2y\sqrt{3} - 5 = 0$

3. $5x^2 + 6xy + 5y^2 - 4\sqrt{2}x + 4\sqrt{2}y = 0$
   becomes $(x')^2 + \frac{(y'+2)^2}{4} = 1$ after rotating
   counter-clockwise through $\theta = \frac{\pi}{4}$.



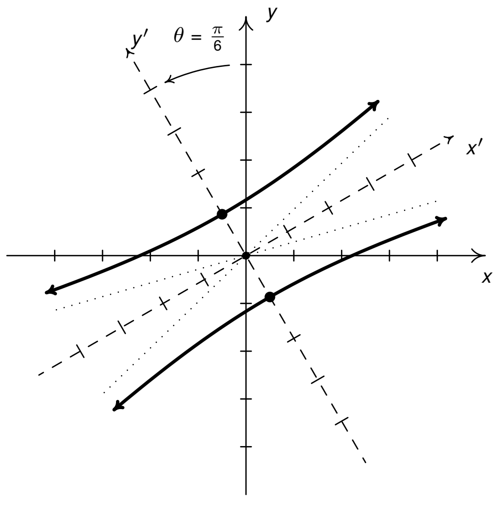
$5x^2 + 6xy + 5y^2 - 4\sqrt{2}x + 4\sqrt{2}y = 0$

4. $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y - 16 = 0$
   becomes$(x')^2 = y' + 4$ after rotating
   counter-clockwise through $\theta = \frac{\pi}{3}$



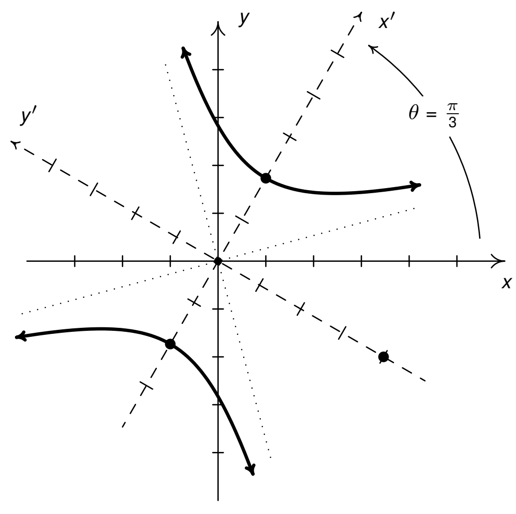
$x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y - 16 = 0$

5. $13x^2 - 34xy\sqrt{3} + 47y^2 - 64 = 0$
becomes $(y')^2 - \frac{(x')^2}{16} = 1$ after rotating
counter-clockwise through $\theta = \frac{\pi}{6}$.



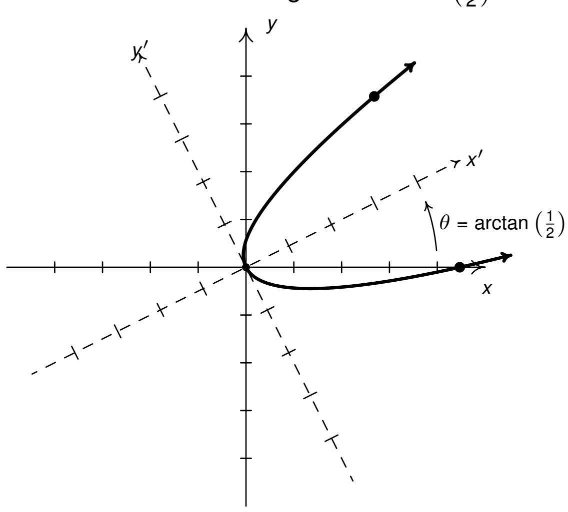
$13x^2 - 34xy\sqrt{3} + 47y^2 - 64 = 0$

6. $x^2 - 2\sqrt{3}xy - y^2 + 8 = 0$
becomes $\frac{(x')^2}{4} - \frac{(y')^2}{4} = 1$ after rotating
counter-clockwise through $\theta = \frac{\pi}{3}$



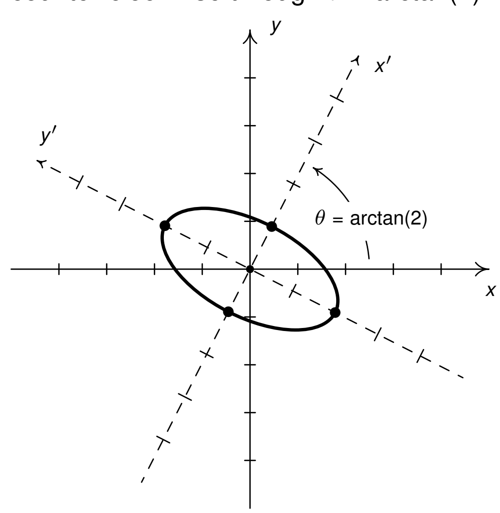
$x^2 - 2\sqrt{3}xy - y^2 + 8 = 0$

7. $x^2 - 4xy + 4y^2 - 2x\sqrt{5} - y\sqrt{5} = 0$
becomes $(y')^2 = x$ after rotating
counter-clockwise through $\theta = \arctan\left(\frac{1}{2}\right)$.



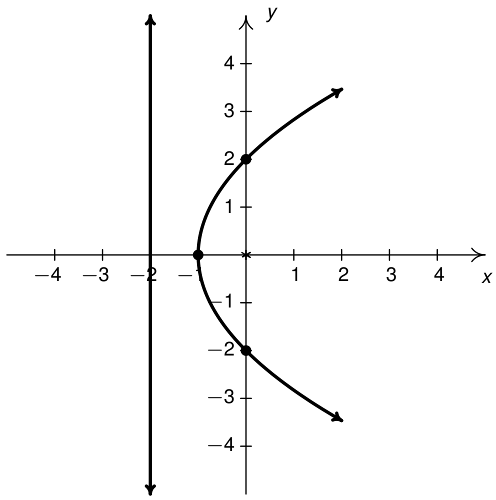
$x^2 - 4xy + 4y^2 - 2x\sqrt{5} - y\sqrt{5} = 0$

8. $8x^2 + 12xy + 17y^2 - 20 = 0$
becomes $(x')^2 + \frac{(y')^2}{4} = 1$ after rotating
counter-clockwise through $\theta = \arctan(2)$

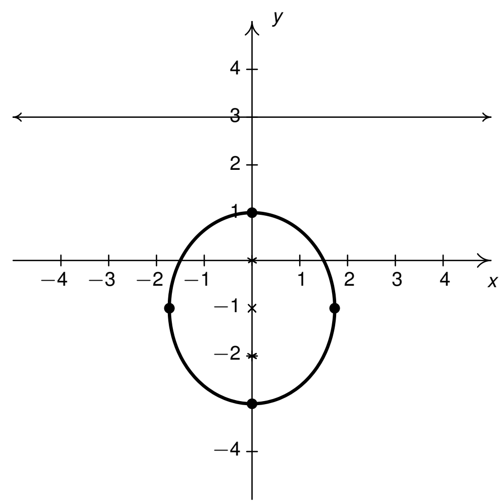


$8x^2 + 12xy + 17y^2 - 20 = 0$

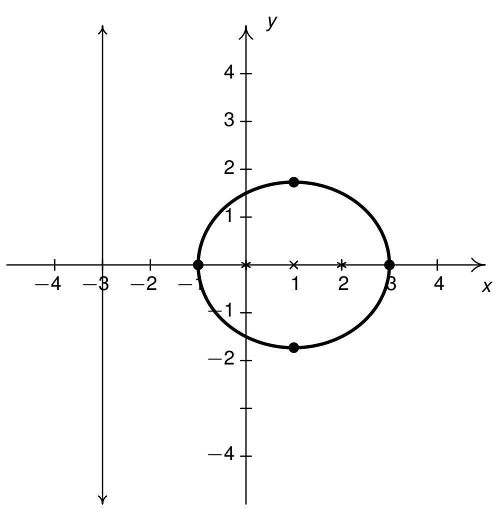
9. $r = \frac{2}{1-\cos(\theta)}$ is a parabola
directrix $x = -2$, vertex $(-1, 0)$
focus $(0, 0)$, focal diameter 4



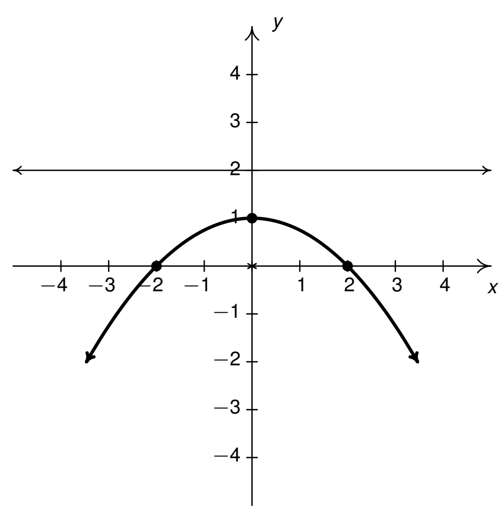
10. $r = \frac{3}{2+\sin(\theta)} = \frac{\frac{3}{2}}{1+\frac{1}{2}\sin(\theta)}$ is an ellipse
directrix $y = 3$, vertices $(0, 1)$, $(0, -3)$
center $(0, -2)$, foci $(0, 0)$, $(0, -2)$
minor axis length $2\sqrt{3}$



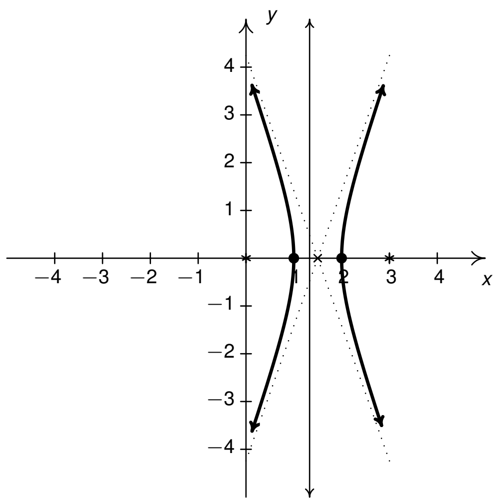
11. $r = \frac{3}{2-\cos(\theta)} = \frac{\frac{3}{2}}{1-\frac{1}{2}\cos(\theta)}$ is an ellipse
directrix $x = -3$, vertices $(-1, 0)$, $(3, 0)$
center $(1, 0)$, foci $(0, 0)$, $(2, 0)$
minor axis length $2\sqrt{3}$



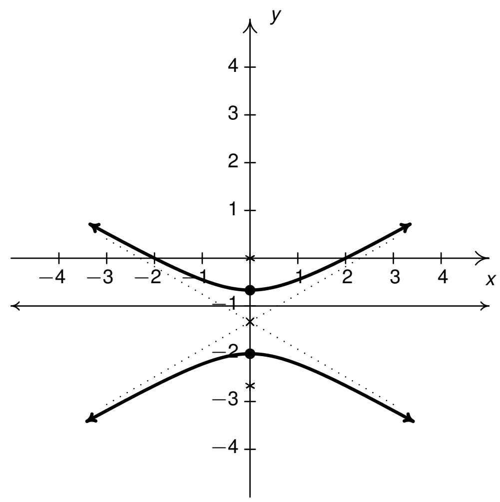
12. $r = \frac{2}{1+\sin(\theta)}$ is a parabola
directrix $y = 2$, vertex $(0, 1)$
focus $(0, 0)$, focal diameter 4



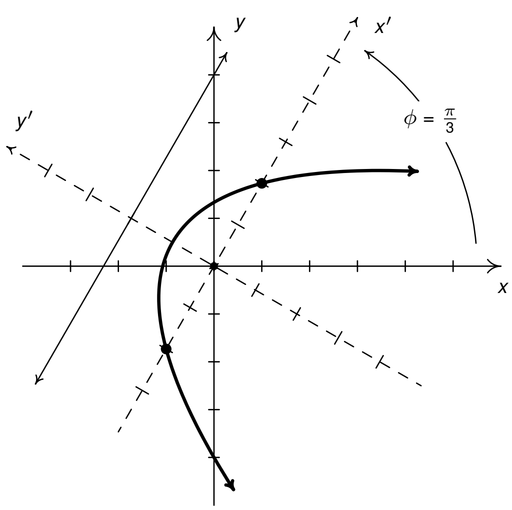
13. $r = \frac{4}{1+3\cos(\theta)}$ is a hyperbola
directrix $x = \frac{4}{3}$, vertices $(1, 0)$, $(2, 0)$
center $\left(\frac{3}{2}, 0\right)$, foci $(0, 0)$, $(3, 0)$
conjugate axis length $2\sqrt{2}$



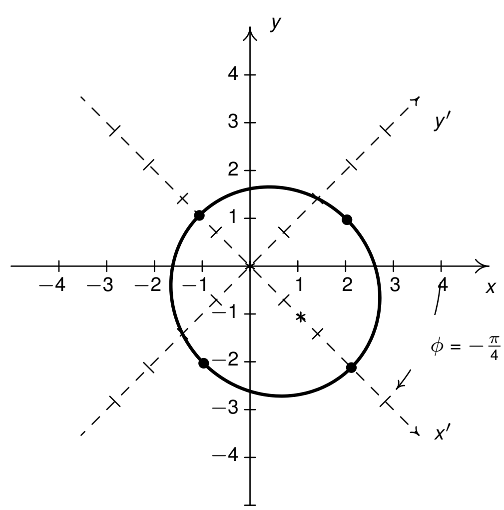
14. $r = \frac{2}{1-2\sin(\theta)}$ is a hyperbola
directrix $y = -1$, vertices $\left(0, -\frac{2}{3}\right)$, $(0, -2)$
center $\left(0, -\frac{4}{3}\right)$, foci $(0, 0)$, $\left(0, -\frac{8}{3}\right)$
conjugate axis length $\frac{2\sqrt{3}}{3}$



15. $r = \frac{2}{1+\sin\left(\theta-\frac{\pi}{3}\right)}$ is
the parabola $r = \frac{2}{1+\sin(\theta)}$
rotated through $\phi = \frac{\pi}{3}$



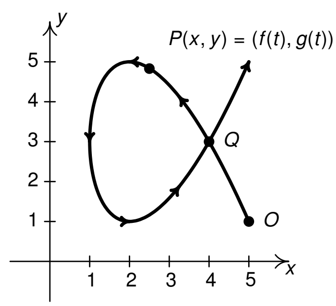
16. $r = \frac{6}{3-\cos\left(\theta+\frac{\pi}{4}\right)}$ is the ellipse
$r = \frac{6}{3-\cos(\theta)} = \frac{2}{1-\frac{1}{3}\cos(\theta)}$
rotated through $\phi = -\frac{\pi}{4}$



14.5 Parametric Equations

As we have seen in Exercises 43 - 46 in Section 5.5, Chapter 8 and most recently in Section 14.2, there are scores of interesting curves which, when plotted in the xy-plane, neither represent y as a function of x nor x as a function of y.

In this section, we present a new concept which allows us to use functions to study these kinds of curves. To motivate the idea, we imagine a bug crawling across a table top starting at the point O and tracing out a curve C in the plane, as shown below.



The curve C does not represent y as a function of x because it fails the Vertical Line Test and it does not represent x as a function of y because it fails the Horizontal Line Test.

However, since the bug can be in only one *place* $P(x, y)$ at any given time t, we can define the x-coordinate of P as a function of t and the y-coordinate of P as a (usually, but not necessarily) different function of t. Traditionally, $f(t)$ is used for x and $g(t)$ is used for y.

The independent variable t in this case is called a **parameter** and the system of equations

$$\begin{cases} x &= f(t) \\ y &= g(t) \end{cases}$$

is called a **system of parametric equations** or a **parametrization** of the curve C.[1]

The parametrization of C endows it with an *orientation* and the arrows on C indicate motion in the direction of increasing values of t.

In this case, our bug starts at the point O, travels upwards to the left, then loops back around to cross its path[2] at the point Q and finally heads off into the first quadrant.

It is important to note that the curve itself is a set of points and as such is devoid of any orientation. The parametrization determines the orientation and as we shall see, different parametrizations can determine different orientations.

If all of this seems hauntingly familiar, it should. By definition, the system of equations $\{x = \cos(t),\ y = \sin(t)$ parametrizes the Unit Circle, giving it a counter-clockwise orientation.

[1]Note the use of the indefinite article 'a'. As we shall see, there are infinitely many different parametric representations for any given curve.

[2]Here, the bug reaches the point Q at two different times. While this does not contradict our claim that $f(t)$ and $g(t)$ are *functions* of t, it shows that neither f nor g can be one-to-one. (Think about this before reading on.)

More generally, the equations of circular motion $\{x = r\cos(\omega t),\ y = r\sin(\omega t)$ developed on page 936 in Section 11.2.1 are parametric equations which trace out a circle of radius r centered at the origin.

If $\omega > 0$, the orientation is counter-clockwise; if $\omega < 0$, the orientation is clockwise. The angular frequency ω determines 'how fast' the object moves around the circle.

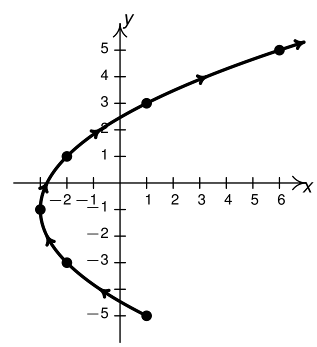
In particular, the equations $\left\{x = 2960\cos\left(\frac{\pi}{12}t\right),\ y = 2960\sin\left(\frac{\pi}{12}t\right)\right.$ that model the motion of Lakeland Community College as the earth rotates (see Example 11.2.6 in Section 11.2.1) parameterize a circle of radius 2960 with a counter-clockwise rotation which completes one revolution as t runs through the interval $[0, 24)$. It is time for another example.

Example 14.5.1. Sketch the curve described by $\left\{\begin{array}{rcl} x & = & t^2 - 3 \\ y & = & 2t - 1 \end{array}\right.$ for $t \geq -2$.

Solution. We follow the same procedure here as we have time and time again when asked to graph anything new – choose values for t, then plot and connect the corresponding points.

Since we are told $t \geq -2$, we start there and as we plot successive points, we draw an arrow to indicate the direction of the path for increasing values of t.

t	$x(t)$	$y(t)$	$(x(t), y(t))$
-2	1	-5	$(1, -5)$
-1	-2	-3	$(-2, -3)$
0	-3	-1	$(-3, -1)$
1	-2	1	$(-2, 1)$
2	1	3	$(1, 3)$
3	6	5	$(6, 5)$



□

The curve sketched out in Example 14.5.1 certainly looks like a parabola, and the presence of the t^2 term in the equation $x = t^2 - 3$ reinforces this hunch.

Since the parametric equations $\{x = t^2 - 3,\ y = 2t - 1$ given to describe this curve are a *system* of equations, we can use the technique of substitution as described in Section 9.7 to eliminate the parameter t and get an equation involving just x and y.

To do so, we choose to solve the equation $y = 2t - 1$ for t to get $t = \frac{y+1}{2}$. Substituting this into the equation $x = t^2 - 3$ yields $x = \left(\frac{y+1}{2}\right)^2 - 3$ or, after some rearrangement, $(y + 1)^2 = 4(x + 3)$.

Thinking back to Section 8.2, we see that the graph of this equation is a parabola with vertex $(-3, -1)$ which opens to the right, as required.

Technically speaking, the equation $(y + 1)^2 = 4(x + 3)$ describes the *entire* parabola, while the parametric equations $\{x = t^2 - 3,\ y = 2t - 1$ for $t \geq -2$ describe only a *portion* of the parabola.

In this case,[3] we can remedy this situation by restricting the bounds on y. Since the portion of the parabola we want is exactly the part where $y \geq -5$, the equation $(y + 1)^2 = 4(x + 3)$ coupled with the restriction

[3]We will have an example shortly where no matter how we restrict x and y, we can never accurately describe the curve once we've eliminated the parameter.

$y \geq -5$ describes the same curve as the given parametric equations. The one piece of information we can never recover after eliminating the parameter, however, is the orientation of the curve.

Eliminating the parameter and obtaining an equation in terms of x and y, whenever possible, can be a great help in graphing curves determined by parametric equations.

If the system of parametric equations contains algebraic functions, as was the case in Example 14.5.1, then the usual techniques of substitution and elimination as learned in Section 9.7 can be applied to the system $\{x = f(t),\ y = g(t)$ to eliminate the parameter.

If, on the other hand, the parametrization involves the trigonometric functions, the strategy changes slightly. In this case, it is often best to solve for the trigonometric functions and relate them using an identity.

We demonstrate these techniques in the following example.

Example 14.5.2. Sketch the curves described by the following parametric equations.

1. $\left\{ \begin{array}{rcl} x & = & t^3 \\ y & = & 2t^2 \end{array} \right.$ for $-1 \leq t \leq 1$

2. $\left\{ \begin{array}{rcl} x & = & e^{-t} \\ y & = & e^{-2t} \end{array} \right.$ for $t \geq 0$

3. $\left\{ \begin{array}{rcl} x & = & \sin(t) \\ y & = & \csc(t) \end{array} \right.$ for $0 < t < \pi$

4. $\left\{ \begin{array}{rcl} x & = & 1 + 3\cos(t) \\ y & = & 2\sin(t) \end{array} \right.$ for $0 \leq t \leq \frac{3\pi}{2}$

Solution.

1. To get a feel for the curve described by the system $\{x = t^3,\ y = 2t^2$ we first sketch the graphs of $x = t^3$ and $y = 2t^2$ over the interval $[-1, 1]$ below on the left in the middle, respectively.

   We note that as t takes on values in the interval $[-1, 1]$, $x = t^3$ ranges between -1 and 1, and $y = 2t^2$ ranges between 0 and 2. This means that all of the action is happening on a portion of the plane, namely $\{(x, y) \,|\, -1 \leq x \leq 1,\ 0 \leq y \leq 2\}$.

   Next, we plot a few points to get a sense of the position and orientation of the curve. Certainly, $t = -1$ and $t = 1$ are good values to pick since these are the extreme values of t. We also choose $t = 0$, since that corresponds to a (local) minimum[4] on the graph of $y = 2t^2$. Plugging in $t = -1$ gives the point $(-1, 2)$, $t = 0$ gives $(0, 0)$ and $t = 1$ gives $(1, 2)$.

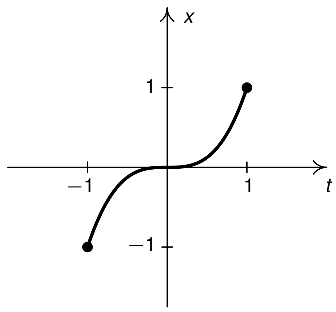
   More generally, we see that $x = t^3$ is *increasing* over the entire interval $[-1, 1]$ whereas $y = 2t^2$ is *decreasing* over the interval $[-1, 0]$ and then *increasing* over $[0, 1]$.

   Geometrically, this means that in order to trace out the path described by the parametric equations, we start at $(-1, 2)$ (where $t = -1$), then move to the right (since x is increasing) and down (since y is decreasing) to $(0, 0)$ (where $t = 0$).

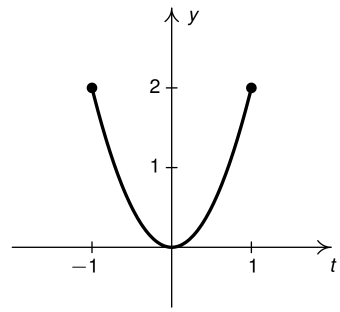
   We continue to move to the right (since x is still increasing) but now move upwards (since y is now increasing) until we reach $(1, 2)$ (where $t = 1$).

[4] You should review Definitions 1.7 and 2.7 if you've forgotten what 'increasing', 'decreasing' and 'local minimum' mean.

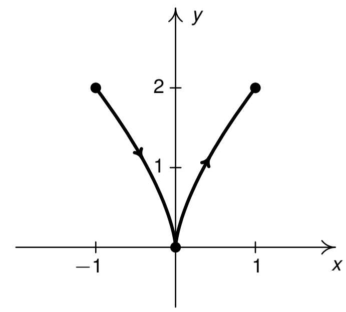
Finally, to get a good sense of the shape of the curve, we eliminate the parameter. Solving $x = t^3$ for t, we get $t = \sqrt[3]{x}$. Substituting this into $y = 2t^2$ gives $y = 2(\sqrt[3]{x})^2 = 2x^{2/3}$. Our experience in Section 4.2 yields the graph of our final answer below on the right.



$x = t^3$, $-1 \le t \le 1$



$y = 2t^2$, $-1 \le t \le 1$



$\{x = t^3,\ y = 2t^2,\ -1 \le t \le 1$

2. For the system $\{x = 2e^{-t},\ y = e^{-2t}$ for $t \geq 0$, we proceed as in the previous example and graph $x = 2e^{-t}$ and $y = e^{-2t}$ over the interval $[0, \infty)$ below on the left and in the middle, respectively.

   We find that the range of x in this case is $(0, 2]$ and the range of y is $(0, 1]$, so our graph will reside in a portion of Quadrant I: $\{(x, y) \,|\, 0 < x \leq 2,\ 0 < y \leq 1\}$.

   Next, we plug in some friendly values of t to get a sense of the orientation of the curve. Since t lies in the exponent here, 'friendly' values of t involve natural logarithms. Starting with $t = \ln(1) = 0$ we get[5] $(2, 1)$, for $t = \ln(2)$ we get $\left(1, \frac{1}{4}\right)$ and for $t = \ln(3)$ we get $\left(\frac{2}{3}, \frac{1}{9}\right)$.

   Since t is ranging over the unbounded interval $[0, \infty)$, we take the time to analyze the end behavior of both x and y. We find $\displaystyle\lim_{t\to\infty} x(t) = \lim_{t\to\infty} 2e^{-t} = 0$ and $\displaystyle\lim_{t\to\infty} y(t) = \lim_{t\to\infty} e^{-2t} = 0$ This means the graph of $\{x = 2e^{-t},\ y = e^{-2t}$ approaches the point $(0, 0)$.

   Since both $x = 2e^{-t}$ and $y = e^{-2t}$ are always decreasing for $t \geq 0$, we know that our final graph will start at $(2, 1)$ (where $t = 0$), and move consistently to the left (since x is decreasing) and down (since y is decreasing) to approach the origin.

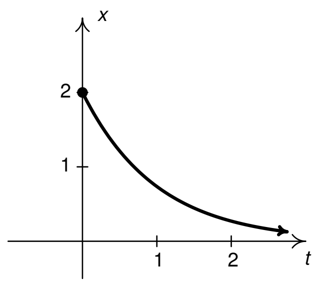
   To eliminate the parameter, one way to proceed is to solve $x = 2e^{-t}$ for t to get $t = -\ln\left(\frac{x}{2}\right)$. Substituting this for t in $y = e^{-2t}$ gives $y = e^{-2(-\ln(x/2))} = e^{2\ln(x/2)} = e^{\ln(x/2)^2} = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$.

   Alternatively, we could recognize that $y = e^{-2t} = \left(e^{-t}\right)^2$, and since $x = 2e^{-t}$ means $e^{-t} = \frac{x}{2}$, we get $y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$ this way as well.

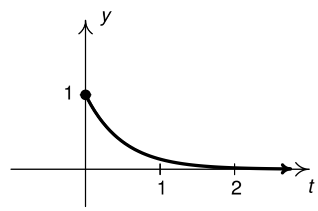
   Either way, the graph of $\{x = 2e^{-t},\ y = e^{-2t}$ for $t \geq 0$ is a portion of the parabola $y = \frac{x^2}{4}$ which starts at the point $(2, 1)$ and heads towards, but never reaches,[6] $(0, 0)$ as seen below on the right.

[5] The reader is encouraged to review Sections 7.2 and 7.3 as needed.

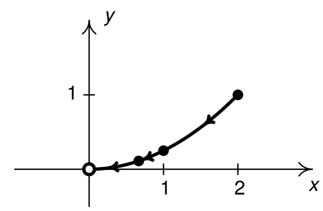
[6] Note the open circle at the origin. See our discussion about holes in graphs in Example 1.1.6 in Section 1.1.



$x = 2e^{-t}$, $t \geq 0$



$y = e^{-2t}$, $t \geq 0$



$\{x = 2e^{-t},\ y = e^{-2t},\ t \geq 0$

3. For the system $\{x = \sin(t),\ y = \csc(t)$ for $0 < t < \pi$, we start by graphing $x = \sin(t)$ and $y = \csc(t)$ over the interval $(0, \pi)$ below on the left and in the middle, respectively.

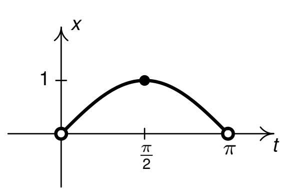
   We find that the range of x is $(0, 1]$ while the range of y is $[1, \infty)$ which means our graph will lie in the first quadrant.

   Plotting a few friendly points, we see that $t = \frac{\pi}{6}$ gives the point $\left(\frac{1}{2}, 2\right)$, $t = \frac{\pi}{2}$ gives $(1, 1)$ and $t = \frac{5\pi}{6}$ returns us to $\left(\frac{1}{2}, 2\right)$.

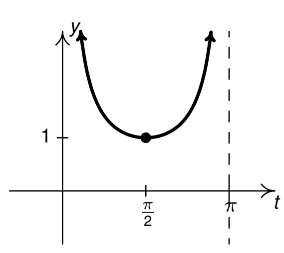
   Since $t = 0$ and $t = \pi$ aren't included in the domain for t, (because $y = \csc(t)$ is undefined at these t-values), we analyze the behavior of the system as t approaches 0 and π.

   We have $\displaystyle\lim_{t \to 0^+} x(t) = \lim_{t \to 0^+} \sin(t) = 0$ and $\displaystyle\lim_{t \to \pi^-} x(t) = \lim_{t \to \pi^-} \sin(t) = 0$. Also, $\displaystyle\lim_{t \to 0^+} y(t) = \lim_{t \to 0^+} \csc(t) = \infty$ and $\displaystyle\lim_{t \to \pi^-} y(t) = \lim_{t \to \pi^-} \csc(t) = \infty$. Piecing all of this information together, we get that for t near 0, we have points with very small positive x-values, but very large positive y-values.

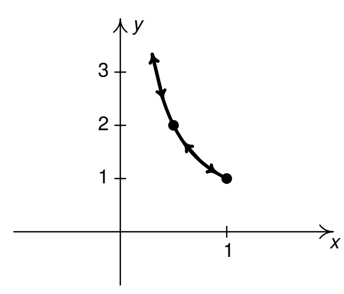
   As t ranges through the interval $\left(0, \frac{\pi}{2}\right]$, $x = \sin(t)$ is increasing and $y = \csc(t)$ is decreasing. This means that we are moving to the right and downwards, through $\left(\frac{1}{2}, 2\right)$ when $t = \frac{\pi}{6}$ to $(1, 1)$ when $t = \frac{\pi}{2}$. Once $t = \frac{\pi}{2}$, the orientation reverses, and we start to head to the left, since $x = \sin(t)$ is now decreasing, and up, since $y = \csc(t)$ is now increasing. We pass back through $\left(\frac{1}{2}, 2\right)$ when $t = \frac{5\pi}{6}$ back to the points with small positive x-coordinates and large positive y-coordinates.



$x = \sin(t)$, $0 < t < \pi$



$y = \csc(t)$, $0 < t < \pi$



$\{x = \sin(t),\ y = \csc(t)\,,\ 0 < t < \pi$

   To better explain this behavior, we eliminate the parameter. Using a reciprocal identity, we write $y = \csc(t) = \frac{1}{\sin(t)}$. Since $x = \sin(t)$, the curve traced out by this parametrization is a portion of the graph of $y = \frac{1}{x}$. We now can explain the unusual behavior as $t \to 0^+$ and $t \to \pi^-$ – for these values of t, we are hugging the vertical asymptote $x = 0$ of the graph of $y = \frac{1}{x}$.

We see that the parametrization given above traces out the portion of $y = \frac{1}{x}$ for $0 < x \leq 1$ *twice* as t runs through the interval $(0, \pi)$ as indicated above on the right.

4. Proceeding as above, we set about graphing $\{x = 1 + 3\cos(t),\ y = 2\sin(t)$ for $0 \leq t \leq \frac{3\pi}{2}$ by first graphing $x = 1 + 3\cos(t)$ and $y = 2\sin(t)$ on the interval $\left[0, \frac{3\pi}{2}\right]$ below on the left and middle, respectivelt.

   We see that x ranges from -2 to 4 and y ranges from -2 to 2. Hence our graph will reside in the region $\{(x, y) \mid -2 \leq x \leq 4,\ -2 \leq y \leq 2\}$.

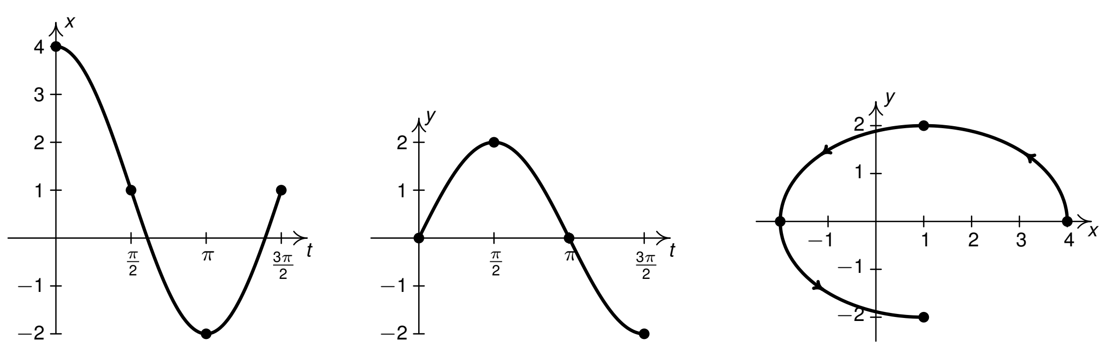
   Plugging in $t = 0$, $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$ gives the points $(4, 0)$, $(1, 2)$, $(-2, 0)$ and $(1, -2)$, respectively.

   As t ranges from 0 to $\frac{\pi}{2}$, $x = 1 + 3\cos(t)$ is decreasing, while $y = 2\sin(t)$ is increasing. This means that we start tracing out our answer at $(4, 0)$ and continue moving to the left and upwards towards $(1, 2)$. For $\frac{\pi}{2} \leq t \leq \pi$, x is decreasing, as is y, so the motion is still right to left, but now is downwards from $(1, 2)$ to $(-2, 0)$. On the interval $\left[\pi, \frac{3\pi}{2}\right]$, x begins to increase, while y continues to decrease. Hence, the motion becomes left to right but continues downwards, connecting $(-2, 0)$ to $(1, -2)$.

   To eliminate the parameter here, we note that the trigonometric functions involved, namely $\cos(t)$ and $\sin(t)$, are related by the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$. Hence, we solve $x = 1 + 3\cos(t)$ for $\cos(t)$ to get $\cos(t) = \frac{x-1}{3}$, and we solve $y = 2\sin(t)$ for $\sin(t)$ to get $\sin(t) = \frac{y}{2}$.

   Substituting these expressions into $\cos^2(t) + \sin^2(t) = 1$ gives $\left(\frac{x-1}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$, or $\frac{(x-1)^2}{9} + \frac{y^2}{4} = 1$.

   From Section 8.4, we know that the graph of this equation is an ellipse centered at $(1, 0)$ with vertices at $(-2, 0)$ and $(4, 0)$ with a minor axis of length 4. Our parametric equations here are tracing out three-quarters of this ellipse, in a counter-clockwise direction.



$x = 1 + 3\cos(t),\ 0 \leq t \leq \frac{3\pi}{2}$ $y = 2\sin(t),\ 0 \leq t \leq \frac{3\pi}{2}$ $\{x = 1 + 3\cos(t),\ y = 2\sin(t),\ 0 \leq t \leq \frac{3\pi}{2}$

□

Now that we have had some good practice sketching the graphs of parametric equations, we turn to the problem of finding parametric representations of curves. We start with the following.

Parametrizations of Common Curves

- The graph of $y = f(x)$ as x runs through some interval I is parametrized by:

  $\{x = t,\ y = f(t)$ as t runs through I.

- The graph of $x = g(y)$ as y runs through some interval I is parametrized by:

  $\{x = g(t),\ y = t$ as t runs through I.

- The graph of a directed line segment from (x_0, y_0) to (x_1, y_1) is parametrized by:

  $\{x = x_0 + (x_1 - x_0)t,\ y = y_0 + (y_1 - y_0)t$ for $0 \leq t \leq 1$.

- The graph of a circle or ellipse $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ where $a, b > 0$ is parametrized by:

  $\{x = h + a\cos(t),\ y = k + b\sin(t)$ for $0 \leq t < 2\pi$.

  NOTE: This will impart a *counter-clockwise* orientation.

The reader is encouraged to verify the above formulas by eliminating the parameter and, when indicated, checking the orientation. We put these formulas to good use in the following example.

Example 14.5.3. Find a parametrization for each of the following curves and check your answers.

1. $y = x^2$ from $x = -3$ to $x = 2$
2. $y = f^{-1}(x)$ where $f(x) = x^5 + 2x + 1$
3. The line segment which starts at $(2, -3)$ and ends at $(1, 5)$
4. The circle $x^2 + 2x + y^2 - 4y = 4$
5. The left half of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Solution.

1. Since $y = x^2$ is written in the form $y = f(x)$, we let $x = t$ and $y = f(t) = t^2$. Since $x = t$, the bounds on t match precisely the bounds on x we get $\{x = t,\ y = t^2$ for $-3 \leq t \leq 2$.

   The check is almost trivial; with $x = t$ we have $y = t^2 = x^2$ as $t = x$ runs from -3 to 2.

2. We are told to parametrize $y = f^{-1}(x)$ for $f(x) = x^5 + 2x + 1$ so it is safe to assume that f is one-to-one. (Otherwise, f^{-1} would not exist.) To find a formula $y = f^{-1}(x)$, we follow the procedure outlined on page 486 – we start with the equation $y = f(x)$, interchange x and y and solve for y.

   Doing so gives us the equation $x = y^5 + 2y + 1$. While we could attempt to solve this equation for y to get an *explicit* formula for $f^{-1}(x)$, we don't need to. We can parametrize the *implicit* function[7] $x = f(y) = y^5 + 2y + 1$ by setting $y = t$ so that $x = t^5 + 2t + 1$.

[7] See the discussion preceding Example 5.5.4 in Section 5.5 for a review of this concept.

We know from Section 2.1 that since $f(x) = x^5 + 2x + 1$ is an odd-degree polynomial, the range of $y = f(x) = x^5+2x+1$ is $(-\infty, \infty)$. Hence, in order to trace out the entire graph of $x = f(y) = y^5+2y+1$, we need to let y run through all real numbers.

Hence, our final answer to this problem is $\begin{cases} x = t^5 + 2t + 1, \ y = t \end{cases}$ for $-\infty < t < \infty$. As in the previous problem, our solution is trivial to check.[8]

3. To parametrize line segment which starts at $(2, -3)$ and ends at $(1, 5)$, we make use of the formulas $x = x_0 + (x_1 - x_0)t$ and $y = y_0 + (y_1 - y_0)t$ for $0 \leq t \leq 1$. While these equations at first glance are quite a handful,[9] they can be summarized as 'starting point + (displacement)t'.

   To find the equation for x, we have that the line segment *starts* at $x = 2$ and *ends* at $x = 1$. This means the *displacement* in the x-direction is $\Delta x = (1 - 2) = -1$. Hence, the equation for x is $x = 2 + (-1)t = 2 - t$.

   Similarly for y, we note that the line segment starts at $y = -3$ and ends at $y = 5$. Hence, the displacement in the y-direction is $\Delta y = (5 - (-3)) = 8$, so we get $y = -3 + 8t$.

   Putting together our answers for x and y, we get $\begin{cases} x = 2 - t, \ y = -3 + 8t \end{cases}$ for $0 \leq t \leq 1$.

   To check, we can solve $x = 2 - t$ for t to get $t = 2 - x$. Substituting this into $y = -3 + 8t$ gives $y = -3 + 8t = -3 + 8(2 - x)$, or $y = -8x + 13$. We know this is the graph of a line, so all we need to check is that it starts and stops at the correct points.

   When $t = 0$, $x = 2-t = 2$, and when $t = 1$, $x = 2-t = 1$. Plugging in $x = 2$ gives $y = -8(2)+13 = -3$, for an initial point of $(2, -3)$. When $x = 1$, $y = -8(1)+13 = 5$ for an ending point of $(1, 5)$, as required.

4. In order to use the formulas above to parametrize the circle $x^2 + 2x + y^2 - 4y = 4$, we first need to put the equation into the correct form.

   After completing the squares, we get $(x + 1)^2 + (y - 2)^2 = 9$, or $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{9} = 1$.

   Once again, the formulas $x = h + a\cos(t)$ and $y = k + b\sin(t)$ can be a challenge to memorize, but they come from the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$, so we can always use the identity to get our parametrization instead of relying on memorizing a formula.

   In the equation $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{9} = 1$, we identify $\cos(t) = \frac{x+1}{3}$ and $\sin(t) = \frac{y-2}{3}$. Rearranging these last two equations, we get $x = -1 + 3\cos(t)$ and $y = 2 + 3\sin(t)$.

   In order to complete one revolution around the circle, we let t range through the interval $[0, 2\pi)$, so our final answer $\begin{cases} x = -1 + 3\cos(t), \ y = 2 + 3\sin(t) \end{cases}$ for $0 \leq t < 2\pi$.

   To check our answer, we could eliminate the parameter by solving $x = -1 + 3\cos(t)$ for $\cos(t)$ and $y = 2 + 3\sin(t)$ for $\sin(t)$, invoking a Pythagorean Identity, and then manipulating the resulting equation in x and y into the original equation $x^2 + 2x + y^2 - 4y = 4$.

[8] Provided you followed the inverse function theory, of course.

[9] Compare and contrast this with Exercise 65 in Section 13.3.

Instead, we opt for a more direct approach. We substitute $x = -1 + 3\cos(t)$ and $y = 2 + 3\sin(t)$ into the equation $x^2 + 2x + y^2 - 4y = 4$ and show that the latter is satisfied for all t such that $0 \leq t < 2\pi$.

$$\begin{array}{rcl} x^2 + 2x + y^2 - 4y & = & 4 \\ (-1 + 3\cos(t))^2 + 2(-1 + 3\cos(t)) + (2 + 3\sin(t))^2 - 4(2 + 3\sin(t)) & \stackrel{?}{=} & 4 \\ 1 - 6\cos(t) + 9\cos^2(t) - 2 + 6\cos(t) + 4 + 12\sin(t) + 9\sin^2(t) - 8 - 12\sin(t) & \stackrel{?}{=} & 4 \\ 9\cos^2(t) + 9\sin^2(t) - 5 & \stackrel{?}{=} & 4 \\ 9\left(\cos^2(t) + \sin^2(t)\right) - 5 & \stackrel{?}{=} & 4 \\ 9\,(1) - 5 & \stackrel{?}{=} & 4 \\ 4 & \stackrel{\checkmark}{=} & 4 \end{array}$$

Now that we know the parametric equations give us points on the circle, we can go through the usual analysis as demonstrated in Example 14.5.2 to show that the entire circle is covered as t ranges through the interval $[0, 2\pi)$.

5. In the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$, we can either use the formulas above or think back to the Pythagorean Identity to get $x = 2\cos(t)$ and $y = 3\sin(t)$.

   The normal range on the parameter in this case is $0 \leq t < 2\pi$, but since we are interested in only the left half of the ellipse, we restrict t to the values which correspond to Quadrant II and Quadrant III angles, namely $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$. Hence, our final answer is $\left\{x = 2\cos(t),\ y = 3\sin(t)\ \text{for}\ \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}\right.$.

   Substituting $x = 2\cos(t)$ and $y = 3\sin(t)$ into $\frac{x^2}{4} + \frac{y^2}{9} = 1$ gives $\frac{4\cos^2(t)}{4} + \frac{9\sin^2(t)}{9} = 1$, which reduces to the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$. This proves the points generated by the parametric equations $\left\{x = 2\cos(t),\ y = 3\sin(t)\right.$ lie on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

   Employing the techniques demonstrated in Example 14.5.2, we find that the restriction $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$ generates the left half of the ellipse, as required. □

We note that the formulas given on page 1297 offer only *one* of literally *infinitely* many ways to parametrize the common curves listed there. At times, the formulas offered there need to be altered to suit the situation.

Adjusting Parametric Equations

- **Reversing Orientation:**

  Replacing every occurrence of t with $-t$ in a parametric description for a curve (including any inequalities which describe the bounds on t) reverses the orientation of the curve.

- **Shift of Parameter:**

  Replacing every occurrence of t with $(t - c)$ in a parametric description for a curve (including any inequalities which describe the bounds on t) shifts the start of the parameter t ahead by c units.

We demonstrate these techniques in the following example.

Example 14.5.4. Find a parametrization for the following curves.

1. The curve which starts at $(2,4)$ and follows the parabola $y = x^2$ to end at $(-1,1)$. Shift the parameter so that the path starts at $t = 0$.

2. The two part path which starts at $(0,0)$, travels along a line to $(3,4)$, then travels along a line to $(5,0)$.

3. The Unit Circle, oriented clockwise, with $t = 0$ corresponding to $(0,-1)$.

Solution.

1. We can parametrize $y = x^2$ from $x = -1$ to $x = 2$ using the formula given on Page 1297 as $\left\{ x = t, \ y = t^2 \right.$ for $-1 \leq t \leq 2$. This parametrization, however, starts at $(-1,1)$ and ends at $(2,4)$. Hence, we need to reverse the orientation.

   To this end, we replace every occurrence of t with $-t$: $\left\{ x = -t, \ y = (-t)^2 \right.$ for $-1 \leq -t \leq 2$. After simplifying, we get $\left\{ x = -t, \ y = t^2 \right.$ for $-2 \leq t \leq 1$.

   We would like t to begin at $t = 0$ instead of $t = -2$. The problem here is that the parametrization we have starts 2 units 'too soon', so we need to introduce a 'time delay' of 2.

   Replacing every occurrence of t with $(t-2)$ gives $\left\{ x = -(t-2), \ y = (t-2)^2 \right.$ for $-2 \leq t-2 \leq 1$. Simplifying yields $\left\{ x = 2-t, \ y = t^2 - 4t + 4 \right.$ for $0 \leq t \leq 3$.

   We leave it to the reader to verify this system traces $y = x^2$ starting with $(2,4)$ and ending at $(-1,1)$.

2. Again, when parameterizing line segments, we think: 'starting point + (displacement)t'. For the first part of the path, we get $\{ x = 3t, \ y = 4t$ for $0 \leq t \leq 1$, and for the second part we get $\{ x = 3 + 2t, \ y = 4 - 4t$ for $0 \leq t \leq 1$.

   Since the first parametrization leaves off at $t = 1$, we shift the parameter in the second part so it starts at $t = 1$. Our current description of the second part starts at $t = 0$, so we need to introduce a 'time delay' of 1 unit to the second set of parametric equations.

   Replacing t with $(t-1)$ in the second set of equations gives $\{ x = 3 + 2(t-1), \ y = 4 - 4(t-1)$ for $0 \leq t - 1 \leq 1$. Simplifying yields $\{ x = 1 + 2t, \ y = 8 - 4t$ for $1 \leq t \leq 2$. Hence, we may parametrize the path as $\{ x = f(t), \ y = g(t)$ for $0 \leq t \leq 2$ where

   $$f(t) = \left\{ \begin{array}{rl} 3t, & \text{for } 0 \leq t \leq 1 \\ 1 + 2t, & \text{for } 1 \leq t \leq 2 \end{array} \right. \quad \text{and} \quad g(t) = \left\{ \begin{array}{rl} 4t, & \text{for } 0 \leq t \leq 1 \\ 8 - 4t, & \text{for } 1 \leq t \leq 2 \end{array} \right.$$

   Again, we encourage the reader to check our solution.

3. We know that $\{ x = \cos(t), \ y = \sin(t)$ for $0 \leq t < 2\pi$ gives a *counter-clockwise* parametrization of the Unit Circle with $t = 0$ corresponding to $(1,0)$, so our first task is to reverse orientation.

   Replacing t with $-t$ gives $\{ x = \cos(-t), \ y = \sin(-t)$ for $0 \leq -t < 2\pi$. Using the Even/Odd Identities, we simplify: $\{ x = \cos(t), \ y = -\sin(t)$ for $-2\pi < t \leq 0$. This parametrization gives a clockwise orientation, but $t = 0$ still corresponds to the point $(1,0)$; the point $(0,-1)$ is reached when $t = -\frac{3\pi}{2}$.

Our strategy is to first get the parametrization to 'start' at the point $(0, -1)$ and then shift the parameter accordingly so the 'start' coincides with $t = 0$.

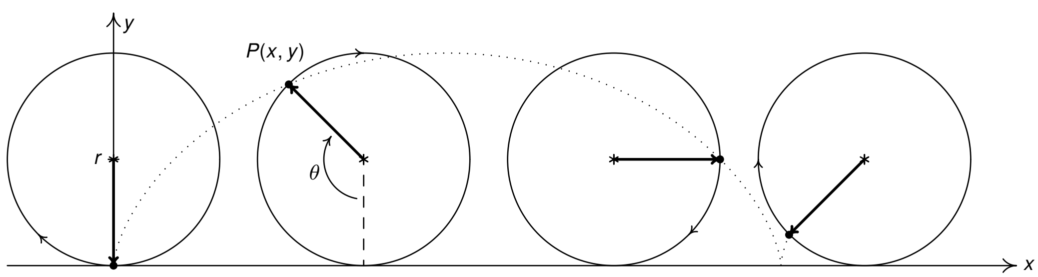
We know that any interval of length 2π will parametrize the entire circle, so we keep the equations $\{x = \cos(t),\; y = -\sin(t)$, but start the parameter t at $-\frac{3\pi}{2}$, and find the upper bound by adding 2π so $-\frac{3\pi}{2} \leq t < \frac{\pi}{2}$. We leave it to the reader to verify that $\{x = \cos(t),\; y = -\sin(t)$ for $-\frac{3\pi}{2} \leq t < \frac{\pi}{2}$ traces out the Unit Circle clockwise starting at the point $(0, -1)$.

We now shift the parameter by introducing a 'time delay' of $\frac{3\pi}{2}$ units by replacing every occurrence of t with $\left(t - \frac{3\pi}{2}\right)$. We get $\left\{x = \cos\left(t - \frac{3\pi}{2}\right),\; y = -\sin\left(t - \frac{3\pi}{2}\right)\right.$ for $-\frac{3\pi}{2} \leq t - \frac{3\pi}{2} < \frac{\pi}{2}$. This simplifies courtesy of the Sum/Difference Formulas to $\{x = -\sin(t),\; y = -\cos(t)$ for $0 \leq t < 2\pi$.

We leave the check of our solution to the reader. □

We put our answer to Example 14.5.4 number 3 to good use to derive the equation of a cycloid.

Suppose a circle of radius r rolls along the positive x-axis at a constant velocity v as pictured below. Let θ be the angle in radians which measures the amount of clockwise rotation experienced by the radius highlighted in the figure.



Our goal is to find parametric equations for the coordinates of the point $P(x, y)$ in terms of θ. From our work in Example 14.5.4 number 3, we know that clockwise motion along the Unit Circle starting at the point $(0, -1)$ can be modeled by the equations $\{x = -\sin(\theta),\; y = -\cos(\theta)$ for $0 \leq \theta < 2\pi$. (We have renamed the parameter 'θ' to match the context of this problem.)

To model this motion on a circle of radius r, all we need to do[10] is multiply both x and y by the factor r which yields $\{x = -r\sin(\theta),\; y = -r\cos(\theta)$.

Next, we adjust for the fact that the circle isn't stationary with center $(0, 0)$, but rather, is rolling along the positive x-axis. Since the velocity v is constant, we know that at time t, the center of the circle has traveled a distance vt down the positive x-axis. Furthermore, since the radius of the circle is r and the circle isn't moving vertically, we know that the center of the circle is always r units above the x-axis. Putting these two facts together, we have that at time t, the center of the circle is at the point (vt, r).

From Section 11.1.1, we know $v = \frac{r\theta}{t}$, or $vt = r\theta$. Hence, the center of the circle, in terms of the parameter θ, is $(r\theta, r)$. As a result, we need to modify the equations $\{x = -r\sin(\theta),\; y = -r\cos(\theta)$ by shifting the

[10] If we replace x with $\frac{x}{r}$ and y with $\frac{y}{r}$ in the equation for the Unit Circle $x^2 + y^2 = 1$, we obtain $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ which reduces to $x^2 + y^2 = r^2$. In the language of Section 5.4, we are stretching the graph by a factor of r in both the x- and y-directions. Hence, we multiply both the x- and y-coordinates of points on the graph by r.

x-coordinate to the right $r\theta$ units (by adding $r\theta$ to the expression for x) and the y-coordinate up r units[11] (by adding r to the expression for y).

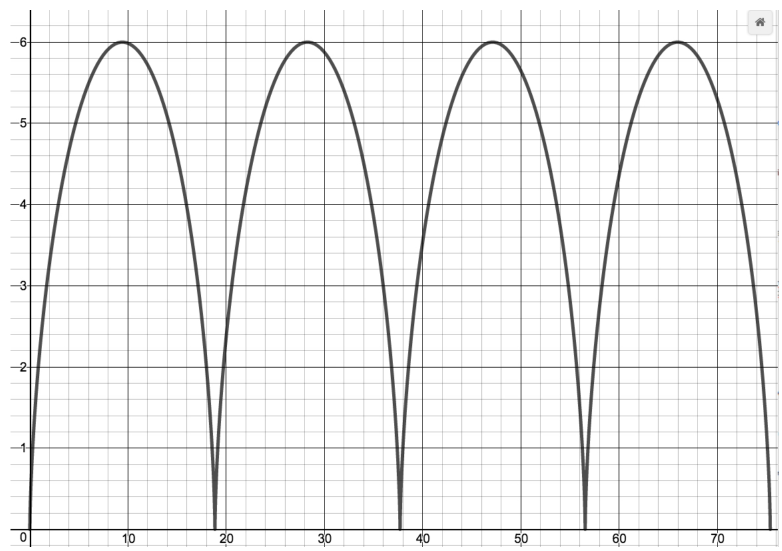
We get $\{x = -r\sin(\theta) + r\theta,\ y = -r\cos(\theta) + r$, which can be written as $\{x = r(\theta - \sin(\theta)),\ y = r(1 - \cos(\theta))$. Since the motion starts at $\theta = 0$ and proceeds indefinitely, we set $\theta \geq 0$.

We end the section by using technology to graph a cycloid.

Example 14.5.5. Find the parametric equations of a cycloid which results from a circle of radius 3 rolling down the positive x-axis as described above. Graph your answer using a graphing utility.

Solution. We have $r = 3$ which gives the equations $\{x = 3(t - \sin(t)),\ y = 3(1 - \cos(t))$ for $t \geq 0$. (Here we have returned to the convention of using t as the parameter.)

Sketching the cycloid by hand is a wonderful exercise in Calculus, but for the purposes of this book, we use a graphing utility. Below is the graph of this cycloid created by desmos.



We see the equations create a series of 'arches' and can (partially) verify the reasonableness the graph by finding the x-intercepts. To do this, we set $y = 3(1 - \cos(t)) = 0$, which amounts to solving $\cos(t) = 1$.

We get $t = 2\pi k$ and since $t \geq 0$, k can be any *nonnegative* integer. Substituting a few of these values for t, $t = 0$, $t = 2\pi$, $t = 4\pi$, and $t = 6\pi$ into the equations $x = 3(t - \sin(t))$ and $y = 3(1 - \cos(t))$ we obtain the points $(0, 0)$, $(6\pi, 0) \approx (18.85, 0)$, $(12\pi, 0) \approx (37.70, 0)$ and $(18\pi, 0) \approx (56.55, 0)$, which match the graph. In general, the x-intercepts are $(6\pi k, 0)$ for nonnegative integers k. We leave the details to the reader.

We note it is also possible to analytically determine the (local) maximums of the graph using the techniques demonstrated in Example 14.5.2 by analyzing $y = 3(1 - \cos(t))$. The maximums occur when $t = (2k + 1)\pi k$ where k is a nonnegative integer, which isn't too surprising just looking at the problem from a symmetry perspective. Substituting these values for t into our equations for x and y produce points of the form $(3(2k + 1)\pi, 6)$. We leave the details to the reader. $\square$

[11] Does this seem familiar? See Example 11.3.4 in Section 11.3.1.

14.5.1 Exercises

In Exercises 1 - 20, plot the set of parametric equations by hand. Be sure to indicate the orientation imparted on the curve by the parametrization.

1. $\begin{cases} x = 4t-3 \\ y = 6t-2 \end{cases}$ for $0 \leq t \leq 1$

2. $\begin{cases} x = 4t-1 \\ y = 3-4t \end{cases}$ for $0 \leq t \leq 1$

3. $\begin{cases} x = 2t \\ y = t^2 \end{cases}$ for $-1 \leq t \leq 2$

4. $\begin{cases} x = t-1 \\ y = 3+2t-t^2 \end{cases}$ for $0 \leq t \leq 3$

5. $\begin{cases} x = t^2+2t+1 \\ y = t+1 \end{cases}$ for $t \leq 1$

6. $\begin{cases} x = \frac{1}{9}\left(18-t^2\right) \\ y = \frac{1}{3}t \end{cases}$ for $t \geq -3$

7. $\begin{cases} x = t \\ y = t^3 \end{cases}$ for $-\infty < t < \infty$

8. $\begin{cases} x = t^3 \\ y = t \end{cases}$ for $-\infty < t < \infty$

9. $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$ for $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$

10. $\begin{cases} x = 3\cos(t) \\ y = 3\sin(t) \end{cases}$ for $0 \leq t \leq \pi$

11. $\begin{cases} x = -1+3\cos(t) \\ y = 4\sin(t) \end{cases}$ for $0 \leq t \leq 2\pi$

12. $\begin{cases} x = 3\cos(t) \\ y = 2\sin(t)+1 \end{cases}$ for $\dfrac{\pi}{2} \leq t \leq 2\pi$

13. $\begin{cases} x = 2\cos(t) \\ y = \sec(t) \end{cases}$ for $0 \leq t < \dfrac{\pi}{2}$

14. $\begin{cases} x = 2\tan(t) \\ y = \cot(t) \end{cases}$ for $0 < t < \dfrac{\pi}{2}$

15. $\begin{cases} x = \sec(t) \\ y = \tan(t) \end{cases}$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

16. $\begin{cases} x = \sec(t) \\ y = \tan(t) \end{cases}$ for $\dfrac{\pi}{2} < t < \dfrac{3\pi}{2}$

17. $\begin{cases} x = \tan(t) \\ y = 2\sec(t) \end{cases}$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

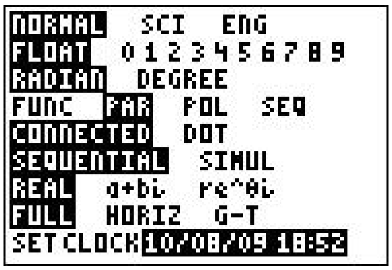
18. $\begin{cases} x = \tan(t) \\ y = 2\sec(t) \end{cases}$ for $\dfrac{\pi}{2} < t < \dfrac{3\pi}{2}$

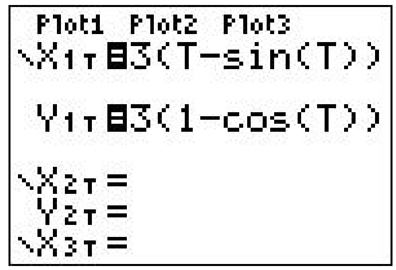
19. $\begin{cases} x = \cos(t) \\ y = t \end{cases}$ for $0 \leq t \leq \pi$

20. $\begin{cases} x = \sin(t) \\ y = t \end{cases}$ for $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$

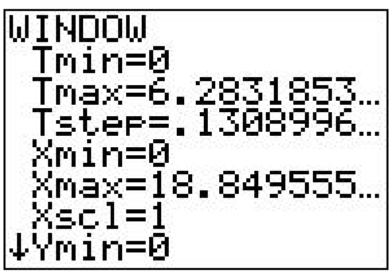
In the same way (and for the same reason) we took the time on page 1235 in Section 14.2 to show how to graph polar equations using a graphing calculator, we take a few moments here to explain how to graph a system of parametric equations using a calculator. Our task is to graph the cycloid from Example 14.5.5, $\{x = 3(t - \sin(t)),\ y = 3(1 - \cos(t))$ for $t \geq 0$ using a graphing calculator.

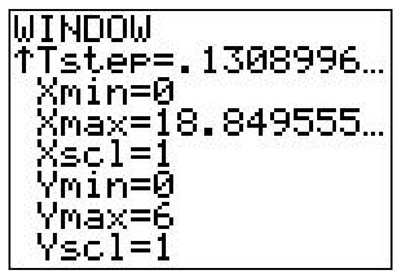
We first must ensure that the calculator is in 'Parametric Mode' and 'radian mode' when we enter the equations and advance to the 'Window' screen.



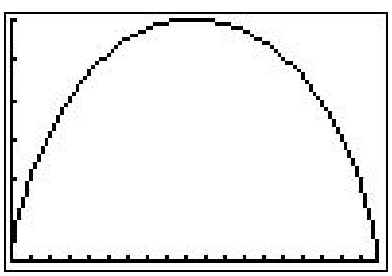


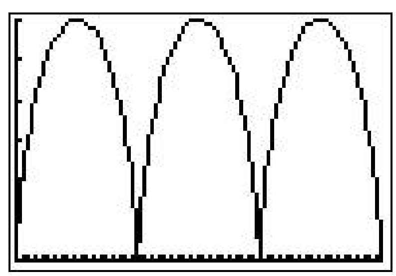
Our next step is to find appropriate bounds on the parameter, t, as well as for x and y. We know that one full revolution of the circle occurs over the interval $0 \leq t < 2\pi$, so it seems reasonable to keep these as our bounds on t. The 'Tstep' seems reasonably small – too large a value here can lead to incorrect graphs.[12] We know from our derivation of the equations of the cycloid that the center of the generating circle has coordinates $(r\theta, r) = (3t, 3)$. Since t ranges between 0 and 2π, we set x to range between 0 and 6π. The values of y go from the bottom of the circle to the top, so y ranges between 0 and 6.





Below we graph the cycloid with these settings, and then extend t to range from 0 to 6π which forces x to range from 0 to 18π yielding three arches of the cycloid.[13]





[12] Again, see page 1235 in Section 14.2.

[13] It is instructive to note that keeping the y settings between 0 and 6 skews the aspect ratio of the cycloid. Using the 'Zoom Square' feature on the graphing calculator gives a true geometric perspective of the three arches.

In Exercises 21 - 24, plot the set of parametric equations with the help of a graphing utility. Be sure to indicate the orientation imparted on the curve by the parametrization.

21. $\begin{cases} x = t^3 - 3t \\ y = t^2 - 4 \end{cases}$ for $-2 \leq t \leq 2$

22. $\begin{cases} x = 4\cos^3(t) \\ y = 4\sin^3(t) \end{cases}$ for $0 \leq t \leq 2\pi$

23. $\begin{cases} x = e^t + e^{-t} \\ y = e^t - e^{-t} \end{cases}$ for $-2 \leq t \leq 2$

24. $\begin{cases} x = \cos(3t) \\ y = \sin(4t) \end{cases}$ for $0 \leq t \leq 2\pi$

In Exercises 25 - 39, find a parametric description for the given oriented curve.

25. the directed line segment from $(3, -5)$ to $(-2, 2)$

26. the directed line segment from $(-2, -1)$ to $(3, -4)$

27. the curve $y = 4 - x^2$ from $(-2, 0)$ to $(2, 0)$.

28. the curve $y = 4 - x^2$ from $(-2, 0)$ to $(2, 0)$
(Shift the parameter so $t = 0$ corresponds to $(-2, 0)$.)

29. the curve $x = y^2 - 9$ from $(-5, -2)$ to $(0, 3)$.

30. the curve $x = y^2 - 9$ from $(0, 3)$ to $(-5, -2)$.
(Shift the parameter so $t = 0$ corresponds to $(0, 3)$.)

31. the circle $x^2 + y^2 = 25$, oriented counter-clockwise

32. the circle $(x - 1)^2 + y^2 = 4$, oriented counter-clockwise

33. the circle $x^2 + y^2 - 6y = 0$, oriented counter-clockwise

34. the circle $x^2 + y^2 - 6y = 0$, oriented *clockwise*
(Shift the parameter so t begins at 0.)

35. the circle $(x - 3)^2 + (y + 1)^2 = 117$, oriented counter-clockwise

36. the ellipse $(x - 1)^2 + 9y^2 = 9$, oriented counter-clockwise

37. the ellipse $9x^2 + 4y^2 + 24y = 0$, oriented counter-clockwise

38. the ellipse $9x^2 + 4y^2 + 24y = 0$, oriented clockwise
(Shift the parameter so $t = 0$ corresponds to $(0, 0)$.)

39. the triangle with vertices $(0, 0)$, $(3, 0)$, $(0, 4)$, oriented counter-clockwise
(Shift the parameter so $t = 0$ corresponds to $(0, 0)$.)

40. Use parametric equations and a graphing utility to graph the inverse of $f(x) = x^3 + 3x - 4$.

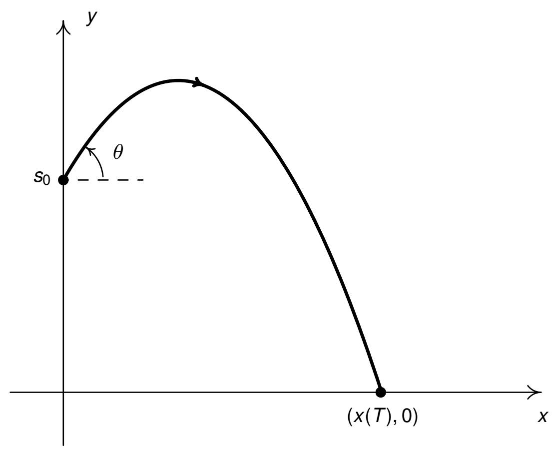
41. Every polar curve $r = f(\theta)$ can be translated to a system of parametric equations with parameter θ by $\{x = r\cos(\theta) = f(\theta)\cos(\theta),\ y = r\sin(\theta) = f(\theta)\sin(\theta)$. Convert $r = 6\cos(2\theta)$ to a system of parametric equations. Check your answer by graphing $r = 6\cos(2\theta)$ by hand using the techniques presented in Section 14.2 and then graphing the parametric equations you found using a graphing utility.

42. Use your results from Exercises 26 and 27 in Section 11.3.1 to find the parametric equations which model a passenger's position as they ride the London Eye.

Suppose an object, called a projectile, is launched into the air. Ignoring everything except the force gravity, the path of the projectile is given by[14]

$$\begin{cases} x = v_0\cos(\theta)\, t \\ y = -\dfrac{1}{2}gt^2 + v_0\sin(\theta)\, t + s_0 \end{cases} \quad \text{for } 0 \leq t \leq T$$

where v_0 is the initial speed of the object, θ is the angle from the horizontal at which the projectile is launched,[15] g is the acceleration due to gravity, s_0 is the initial height of the projectile above the ground and T is the time when the object returns to the ground. (See the figure below.)



43. Carl's friend Jason competes in Highland Games Competitions across the country. In one event, the 'hammer throw', he throws a 56 pound weight for distance. If the weight is released 6 feet above the ground at an angle of 42° with respect to the horizontal with an initial speed of 33 feet per second, find the parametric equations for the flight of the hammer. (Here, use $g = 32\frac{\text{ft.}}{s^2}$.) When will the hammer hit the ground? How far away will it hit the ground? Check your answer using a graphing utility.

44. Eliminate the parameter in the equations for projectile motion to show that the path of the projectile follows the curve

$$y = -\frac{g\sec^2(\theta)}{2v_0^2}x^2 + \tan(\theta)x + s_0$$

[14]A nice mix of vectors and Calculus are needed to derive this.

[15]We've seen this before. It's the angle of elevation which was defined on page 1510.

Use the vertex formula (Equation 1.2) to show the maximum height of the projectile is

$$y = \frac{v_0^2 \sin^2(\theta)}{2g} + s_0 \quad \text{when} \quad x = \frac{v_0^2 \sin(2\theta)}{2g}$$

45. In another event, the 'sheaf toss', Jason throws a 20 pound weight for height. If the weight is released 5 feet above the ground at an angle of 85° with respect to the horizontal and the sheaf reaches a maximum height of 31.5 feet, use your results from part 44 to determine how fast the sheaf was launched into the air. (Once again, use $g = 32 \frac{\text{ft.}}{s^2}$.)

46. Suppose $\theta = \frac{\pi}{2}$. (The projectile was launched vertically.) Simplify the general parametric formula given for $y(t)$ above using $g = 9.8 \frac{m}{s^2}$ and compare that to the formula for $s(t)$ given in Exercise 49 in Section 1.4. What is $x(t)$ in this case?

47. If f and g are functions, explain why the function $\vec{r}(t) = \langle f(t), g(t) \rangle$ is a function. The function $\vec{r}$ is called a **vector-valued function** since it matches real number inputs, t, with vector outputs, $\vec{r}(t)$. Explain why when the vectors $\vec{r}(t)$ are plotted in standard position, their terminal points trace out the curve described parametrically by the system of equations: $\{x = f(t)\ y = g(t)$ (In Calculus, you will see systems of parametric equations 'packaged' together using vectors.)

In Exercises 48 - 52, we explore the **hyperbolic cosine** function, denoted $\cosh(t)$, and the **hyperbolic sine** function, denoted $\sinh(t)$, defined below:

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \quad \text{and} \quad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

48. Using a graphing utility as needed, verify the following:

    (a) the domain of $\cosh(t)$ is $(-\infty, \infty)$ and the range of $\cosh(t)$ is $[1, \infty)$.

    (b) the domain and range of $\sinh(t)$ are both $(-\infty, \infty)$.

49. Show that $\{x(t) = \cosh(t),\ y(t) = \sinh(t)$ parametrize the right half of the 'unit' hyperbola $x^2 - y^2 = 1$. (Hence the use of the adjective 'hyperbolic.')

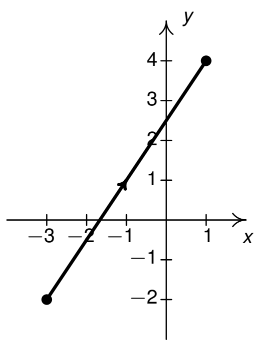
50. Compare and contrast the definitions of $\cosh(t)$ and $\sinh(t)$ to the formulas for $\cos(t)$ and $\sin(t)$ given in Exercise 84d in Section 14.3.

51. Four other hyperbolic functions are waiting to be defined: the hyperbolic secant $\operatorname{sech}(t)$, the hyperbolic cosecant $\operatorname{csch}(t)$, the hyperbolic tangent $\tanh(t)$ and the hyperbolic cotangent $\coth(t)$. Define these functions in terms of $\cosh(t)$ and $\sinh(t)$, then convert them to formulas involving e^t and e^{-t}. Consult a suitable reference (a Calculus book, or this entry on the hyperbolic functions) and spend some time reliving the thrills of trigonometry with these 'hyperbolic' functions.

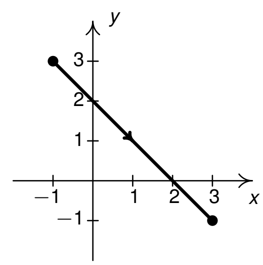
52. If these functions look familiar, they should. Enjoy some nostalgia and revisit Exercise 37 in Section 7.6, Exercise 55 in Section 7.4 and the answer to Exercise 44 in Section 7.5.

14.5.2 Answers

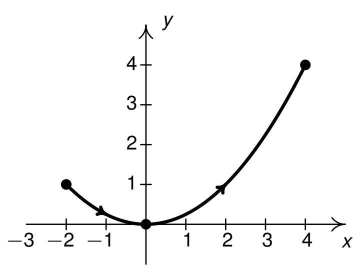
1. $\begin{cases} x = 4t - 3 \\ y = 6t - 2 \end{cases}$ for $0 \leq t \leq 1$



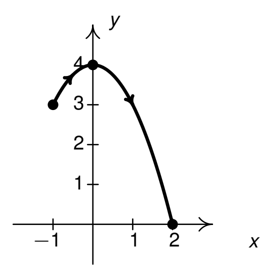
2. $\begin{cases} x = 4t - 1 \\ y = 3 - 4t \end{cases}$ for $0 \leq t \leq 1$



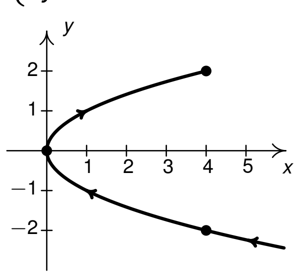
3. $\begin{cases} x = 2t \\ y = t^2 \end{cases}$ for $-1 \leq t \leq 2$



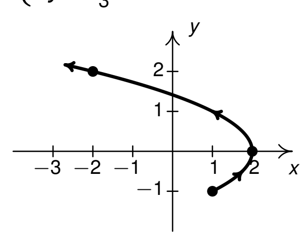
4. $\begin{cases} x = t - 1 \\ y = 3 + 2t - t^2 \end{cases}$ for $0 \leq t \leq 3$



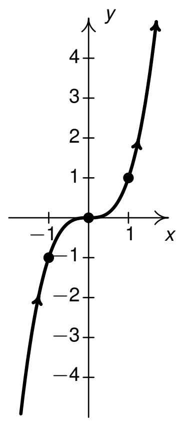
5. $\begin{cases} x = t^2 + 2t + 1 \\ y = t + 1 \end{cases}$ for $t \leq 1$



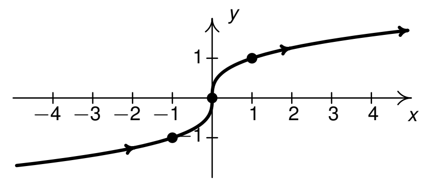
6. $\begin{cases} x = \frac{1}{9}\left(18 - t^2\right) \\ y = \frac{1}{3}t \end{cases}$ for $t \geq -3$



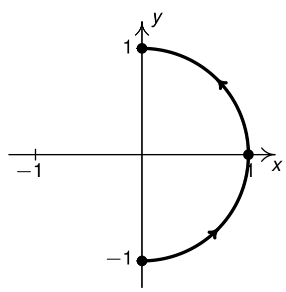
7. $\begin{cases} x = t \\ y = t^3 \end{cases}$ for $-\infty < t < \infty$



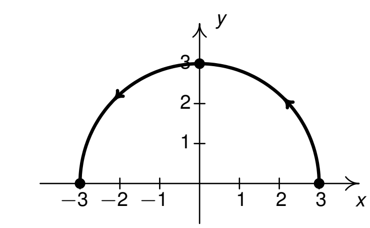
8. $\begin{cases} x = t^3 \\ y = t \end{cases}$ for $-\infty < t < \infty$



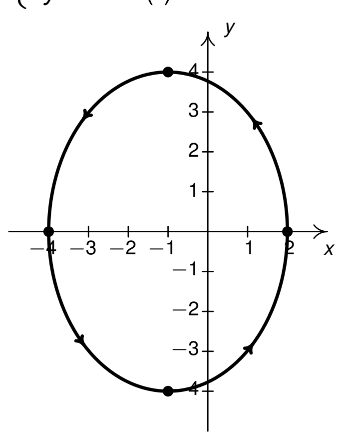
9. $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$ for $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$



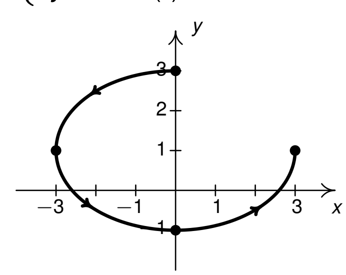
10. $\begin{cases} x = 3\cos(t) \\ y = 3\sin(t) \end{cases}$ for $0 \leq t \leq \pi$



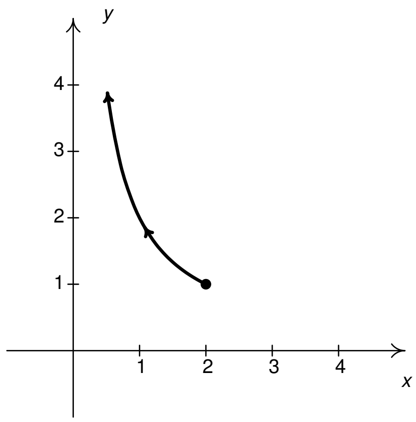
11. $\begin{cases} x = -1 + 3\cos(t) \\ y = 4\sin(t) \end{cases}$ for $0 \leq t \leq 2\pi$



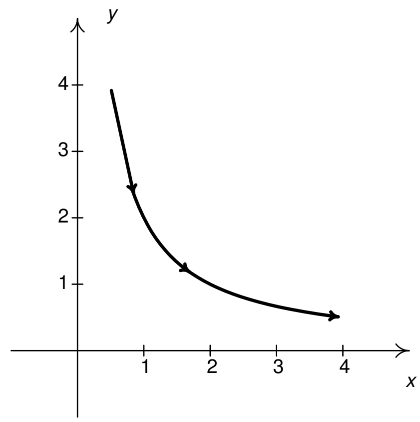
12. $\begin{cases} x = 3\cos(t) \\ y = 2\sin(t) + 1 \end{cases}$ for $\dfrac{\pi}{2} \leq t \leq 2\pi$



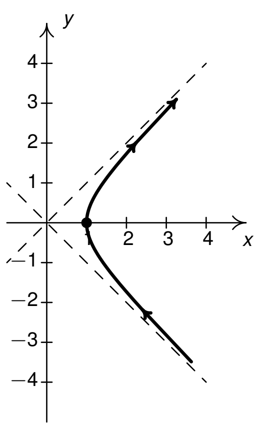
13. $\begin{cases} x = 2\cos(t) \\ y = \sec(t) \end{cases}$ for $0 \leq t < \dfrac{\pi}{2}$



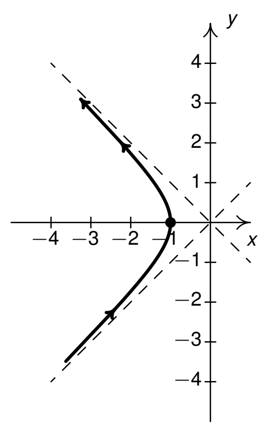
14. $\begin{cases} x = 2\tan(t) \\ y = \cot(t) \end{cases}$ for $0 < t < \dfrac{\pi}{2}$



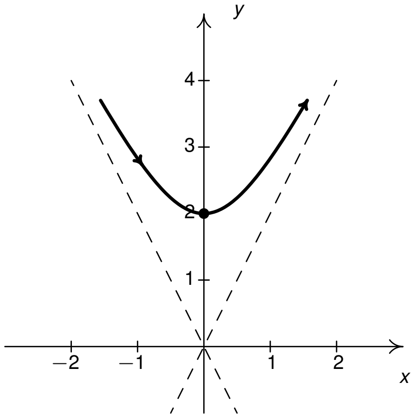
15. $\begin{cases} x = \sec(t) \\ y = \tan(t) \end{cases}$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$



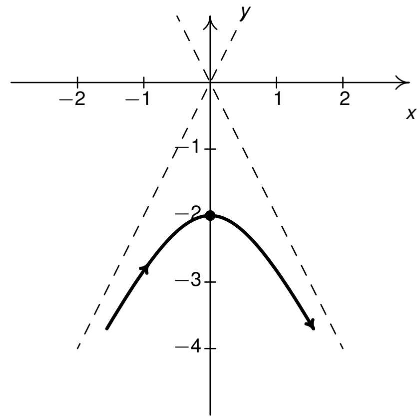
16. $\begin{cases} x = \sec(t) \\ y = \tan(t) \end{cases}$ for $\dfrac{\pi}{2} < t < \dfrac{3\pi}{2}$



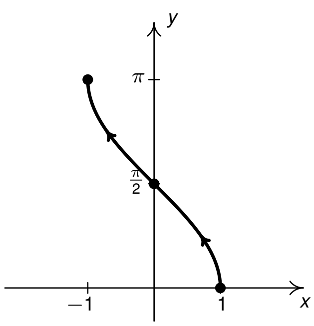
17. $\begin{cases} x = \tan(t) \\ y = 2\sec(t) \end{cases}$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$



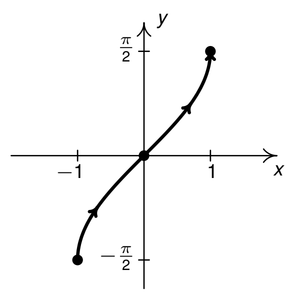
18. $\begin{cases} x = \tan(t) \\ y = 2\sec(t) \end{cases}$ for $\dfrac{\pi}{2} < t < \dfrac{3\pi}{2}$



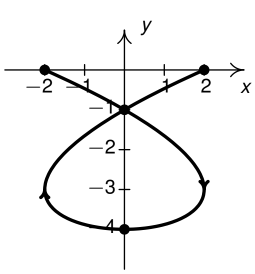
19. $\begin{cases} x = \cos(t) \\ y = t \end{cases}$ for $0 < t < \pi$



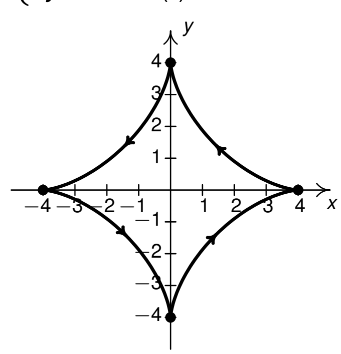
20. $\begin{cases} x = \sin(t) \\ y = t \end{cases}$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$



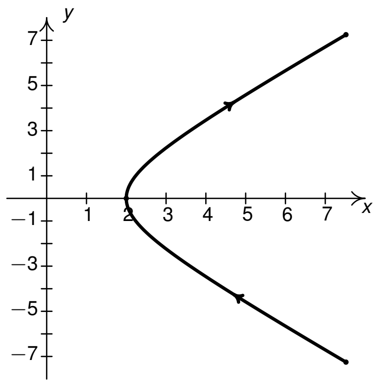
21. $\begin{cases} x = t^3 - 3t \\ y = t^2 - 4 \end{cases}$ for $-2 \le t \le 2$



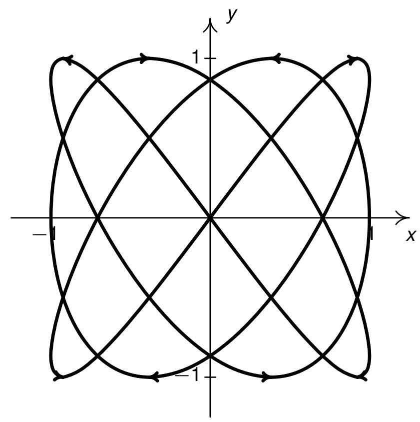
22. $\begin{cases} x = 4\cos^3(t) \\ y = 4\sin^3(t) \end{cases}$ for $0 \le t \le 2\pi$



23. $\begin{cases} x = e^t + e^{-t} \\ y = e^t - e^{-t} \end{cases}$ for $-2 \leq t \leq 2$



24. $\begin{cases} x = \cos(3t) \\ y = \sin(4t) \end{cases}$ for $0 \leq t \leq 2\pi$



25. $\begin{cases} x = 3 - 5t \\ y = -5 + 7t \end{cases}$ for $0 \leq t \leq 1$

26. $\begin{cases} x = 5t - 2 \\ y = -1 - 3t \end{cases}$ for $0 \leq t \leq 1$

27. $\begin{cases} x = t \\ y = 4 - t^2 \end{cases}$ for $-2 \leq t \leq 2$

28. $\begin{cases} x = t - 2 \\ y = 4t - t^2 \end{cases}$ for $0 \leq t \leq 4$

29. $\begin{cases} x = t^2 - 9 \\ y = t \end{cases}$ for $-2 \leq t \leq 3$

30. $\begin{cases} x = t^2 - 6t \\ y = 3 - t \end{cases}$ for $0 \leq t \leq 5$

31. $\begin{cases} x = 5\cos(t) \\ y = 5\sin(t) \end{cases}$ for $0 \leq t < 2\pi$

32. $\begin{cases} x = 1 + 2\cos(t) \\ y = 2\sin(t) \end{cases}$ for $0 \leq t < 2\pi$

33. $\begin{cases} x = 3\cos(t) \\ y = 3 + 3\sin(t) \end{cases}$ for $0 \leq t < 2\pi$

34. $\begin{cases} x = 3\cos(t) \\ y = 3 - 3\sin(t) \end{cases}$ for $0 \leq t < 2\pi$

35. $\begin{cases} x = 3 + \sqrt{117}\cos(t) \\ y = -1 + \sqrt{117}\sin(t) \end{cases}$ for $0 \leq t < 2\pi$

36. $\begin{cases} x = 1 + 3\cos(t) \\ y = \sin(t) \end{cases}$ for $0 \leq t < 2\pi$

37. $\begin{cases} x = 2\cos(t) \\ y = 3\sin(t) - 3 \end{cases}$ for $0 \leq t < 2\pi$

38. $\begin{cases} x = 2\cos\left(t - \dfrac{\pi}{2}\right) = 2\sin(t) \\ y = -3 - 3\sin\left(t - \dfrac{\pi}{2}\right) = -3 + 3\cos(t) \end{cases}$ for $0 \leq t < 2\pi$

39. $\{x(t),\ y(t)$ where:

$$x(t) = \begin{cases} 3t, & 0 \leq t \leq 1 \\ 6 - 3t, & 1 \leq t \leq 2 \\ 0, & 2 \leq t \leq 3 \end{cases} \qquad y(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ 4t - 4, & 1 \leq t \leq 2 \\ 12 - 4t, & 2 \leq t \leq 3 \end{cases}$$

40. The parametric equations for the inverse are $\begin{cases} x = t^3 + 3t - 4 \\ y = t \end{cases}$ for $-\infty < t < \infty$

41. $r = 6\cos(2\theta)$ translates to $\begin{cases} x = 6\cos(2\theta)\cos(\theta) \\ y = 6\cos(2\theta)\sin(\theta) \end{cases}$ for $0 \leq \theta < 2\pi$.

42. The parametric equations which describe the locations of passengers on the London Eye are

$$\begin{cases} x = 67.5\cos\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) = 67.5\sin\left(\frac{\pi}{15}t\right) \\ y = 67.5\sin\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) + 67.5 = 67.5 - 67.5\cos\left(\frac{\pi}{15}t\right) \end{cases} \text{ for } -\infty < t < \infty$$

43. The parametric equations for the hammer throw are $\begin{cases} x = 33\cos(42^\circ)t \\ y = -16t^2 + 33\sin(42^\circ)t + 6 \end{cases}$ for $t \geq 0$.

To find when the hammer hits the ground, we solve $y(t) = 0$ and get $t \approx -0.23$ or 1.61. Since $t \geq 0$, the hammer hits the ground after approximately $t = 1.61$ seconds after it was launched into the air.

To find how far away the hammer hits the ground, we find $x(1.61) \approx 39.48$ feet from where it was thrown into the air.

45. We solve $y = \dfrac{v_0^2\sin^2(\theta)}{2g} + s_0 = \dfrac{v_0^2\sin^2(85^\circ)}{2(32)} + 5 = 31.5$ to get $v_0 = \pm 41.34$.

The initial speed of the sheaf was approximately 41.34 feet per second.

Appendix A

Algebra Review

One purpose of this Algebra Review Appendix is to support a "co-requisite" approach to teaching College Algebra or Precalculus.[1] Our goal is to provide instructors with supplemental material linked to the main textbook that can be used to support students who have minor gaps in their pre-college mathematical backgrounds. To that end, we have written a collection of somewhat independent sections designed to review the concepts, skills and vocabulary that we believe are prerequisite to a rigorous, college-level Precalculus course. This review is not designed to teach the material to students who have never seen it before so the presentation is more succinct and the exercise sets are shorter than those usually found in an Intermediate Algebra or high school Algebra II text. Some of this material (like adding fractions and plotting points) is used throughout the text but, where appropriate, we have referenced specific sections of the main body of the Precalculus text in an effort to assist faculty who would like to assign the Appendix as "just in time" review reading to their students. An outline of the chapter with short descriptions of each section is given below:

Section A.1 (Basic Set Theory and Interval Notation) contains a brief summary of the set theory terminology used throughout the text including sets of real numbers and interval notation.

Section A.2 (Real Number Arithmetic) lists the properties of real number arithmetic.[2]

Section A.3 (The Cartesian Plane) discusses the basic notions of plotting points in the plane, reflections and symmetry. We then develop the Distance Formula and the Midpoint Formula.

Section A.4 (Linear Equations and Inequalities) focuses on solving linear equations and linear inequalities from a strictly algebraic perspective. The geometry of graphing lines in the plane is deferred until Section A.5 (Graphing Lines).

Section A.5 (Graphing Lines) starts by defining the slope of a line between two points and then develops the point-slope form and the slope-intercept form of equations for lines. Horizontal and vertical lines are

[1]Please read The New Preface for details as to why we decided to organize our book in order to support this pedagogy. Also, to us Precalculus = College Algebra + College Trigonometry without the formalization of limits. This distinction is not universally agreed upon so we felt the need to point it out.

[2]You know, the stuff students mess up all of the time like fractions and negative signs. The collection is close to exhaustive and is definitely exhausting!

discussed as are parallel and perpendicular lines. This material is required for Section 1.2 (Constant and Linear Functions).

Section A.6 (Systems of Two Linear Equations in Two Unknowns) is a review of the basic terminology and techniques related to solving systems of two linear equations that each have the same two variables in them. We start Chapter 9 (Systems of Equations and Matrices) assuming students know these techniques.

Section A.7 (Absolute Value Equations and Inequalities) begins with a definition of absolute value as a distance. Fundamental properties of absolute value are listed and then basic equations and inequalities involving absolute value are solved using the 'distance definition' and those properties. Absolute value is revisited in much greater depth in Section 1.3 (Absolute Value Functions).

Section A.8 (Polynomial Arithmetic) covers the addition, subtraction, multiplication and division of polynomials as well as the vocabulary which is used extensively when the graphs of polynomials are studied in Chapter 2 (Polynomials).

Section A.9 (Basic Factoring Techniques) contains pretty much what it says: basic factoring techniques and how to solve equations using those techniques along with the Zero Product Property of Real Numbers.

Section A.10 (Quadratic Equations) discusses solving quadratic equations using the technique of 'completing the square' and by using the Quadratic Formula. Equations that are 'quadratic in form' are also discussed. This material is revisited in Section 1.4 (Quadratic Functions).

Section A.11 (Complex Numbers) covers the basic arithmetic of complex numbers and the solving of quadratic equations with complex solutions. It's required for Section 2.4 (Complex Zeros and the Fundamental Theorem of Algebra).

Section A.12 (Rational Expressions and Equations) starts with the basic arithmetic of rational expressions and the simplifying of compound fractions. Solving equations by clearing denominators and the handling of negative integer exponents are presented but the graphing of rational functions is deferred until Chapter 3 (Rational Functions).

Section A.13 (Radicals and Equations) covers simplifying radicals as well as the solving of basic equations involving radicals. Students should be familiar with this material before starting Chapter 4 (Root and Radical Functions).

Section A.14 (Variation) looks at a variety of equations from Science and Engineering. It's a self-contained section that can be covered at any time.

A.1 Basic Set Theory and Interval Notation

A.1.1 Some Basic Set Theory Notions

We begin this section with the definition of a concept that is central to all of Mathematics.

Definition A.1. A **set** is a well-defined collection of objects which are called the elements of the set. Here, 'well-defined' means that it is possible to determine if something belongs to the collection or not, without prejudice.

For example, the collection of letters that make up the word "smolko" is well-defined and is a set, but the collection of the worst Math teachers in the world is **not** well-defined and therefore is **not** a set.[1]

In general, there are three ways to describe sets and those methods are listed below.

Ways to Describe Sets

1. **The Verbal Method:** Use a sentence to describe the elements the set.

2. **The Roster Method:** Begin with a left brace '$\{$', list each element of the set *only once* and then end with a right brace '$\}$'.

3. **The Set-Builder Method:** A combination of the verbal and roster methods using a "dummy variable" such as x and conditions on that variable.

Let S be the set described *verbally* as the set of letters that make up the word "smolko". A *roster* description of S is $\{s, m, o, l, k\}$. Note that we listed 'o' only once, even though it appears twice in the word "smolko". Also, the order of the elements doesn't matter, so $\{k, l, m, o, s\}$ is also a roster description of S. A *set-builder* description of S is: $\{x \mid x \text{ is a letter in the word "smolko"}\}$. The way to read this is 'The set of elements x <u>such that</u> x is a letter in the word "smolko".' In each of the above cases, we may use the familiar equals sign '=' and write $S = \{s, m, o, l, k\}$ or $S = \{x \mid x \text{ is a letter in the word "smolko"}\}$.

Notice that m is in S but many other letters, such as q, are not in S. We express these ideas of set inclusion and exclusion mathematically using the symbols $m \in S$ (read 'm <u>is in</u> S') and $q \notin S$ (read 'q <u>is not in</u> S'). More precisely, we have the following.

Definition A.2. Let A be a set.

- If x is an element of A then we write $x \in A$ which is read 'x is in A'.

- If x is *not* an element of A then we write $x \notin A$ which is read 'x is not in A'.

Now let's consider the set $C = \{x \mid x \text{ is a consonant in the word "smolko"}\}$. A roster description of C is $C = \{s, m, l, k\}$. Note that by construction, every element of C is also in S. We express this relationship by stating that the set C is a **subset** of the set S, which is written in symbols as $C \subseteq S$. The more formal definition is given at the top of the next page.

[1] For a more thought-provoking example, consider the collection of all things that do not contain themselves - this leads to the famous paradox known as Russell's Paradox.

Definition A.3. Given sets A and B, we say that the set A is a **subset** of the set B and write '$A \subseteq B$' if every element in A is also an element of B.

In our previous example, $C \subseteq S$ yet not vice-versa since $o \in S$ but $o \notin C$. Additionally, the set of vowels $V = \{a, e, i, o, u\}$, while it does have an element in common with S, is not a subset of S. (As an added note, S is not a subset of V, either.) We could, however, *build* a set which contains both S and V as subsets by gathering all of the elements in both S and V together into a single set, say $U = \{s, m, o, l, k, a, e, i, u\}$. Then $S \subseteq U$ and $V \subseteq U$. The set U we have built is called the **union** of the sets S and V and is denoted $S \cup V$. Furthermore, S and V aren't completely *different* sets since they both contain the letter 'o.' The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of S and V is $\{o\}$, written $S \cap V = \{o\}$. We formalize these ideas below.

Definition A.4. Suppose A and B are sets.

- The **intersection** of A and B is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The **union** of A and B is $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both)}\}$

The key words in Definition A.4 to focus on are the conjunctions: 'intersection' corresponds to 'and' meaning the elements have to be in *both* sets to be in the intersection, whereas 'union' corresponds to 'or' meaning the elements have to be in one set, or the other set (or both). Please note that this mathematical use of the word 'or' differs than how we use 'or' in spoken English. In Math, we use the *inclusive or* which allows for the element to be in both sets. At a restaurant if you're asked "Do you want fries or a salad?" you must pick one and only one. This is known as the *exclusive or* and it plays a role in other Math classes. For our purposes it is good enough to say that for an element to belong to the union of two sets it must belong to *at least one* of them.

Returning to the sets C and V above, $C \cup V = \{s, m, l, k, a, e, i, o, u\}$.[2] Their intersection, however, creates a bit of notational awkwardness since C and V have no elements in common. While we could write $C \cap V = \{\}$, this sort of thing happens often enough that we give the set with no elements a name.

Definition A.5. The **Empty Set** is the set which contains no elements and is denoted $\emptyset$. That is,

$$\emptyset = \{\} = \{x \mid x \neq x\}.$$

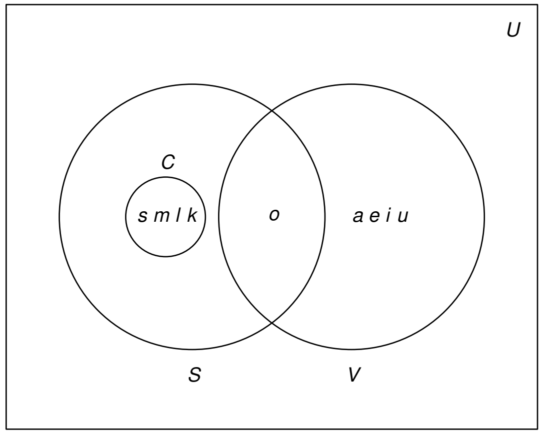
As promised, the empty set is the set containing no elements since no matter what 'x' is, '$x = x$.' Like the number '0,' the empty set plays a vital role in mathematics.[3] We introduce it here more as a symbol of convenience as opposed to a contrivance[4] because saying that $C \cap V = \emptyset$ is unambiguous whereas $\{\}$ looks like a typographical error.

A nice way to visualize the relationships between sets and set operations is to draw a **Venn Diagram**. A Venn Diagram for the sets S, C and V is drawn at the top of the next page.

[2]Which just so happens to be the same set as $S \cup V$.

[3]Sadly, the full extent of the empty set's role will not be explored in this text.

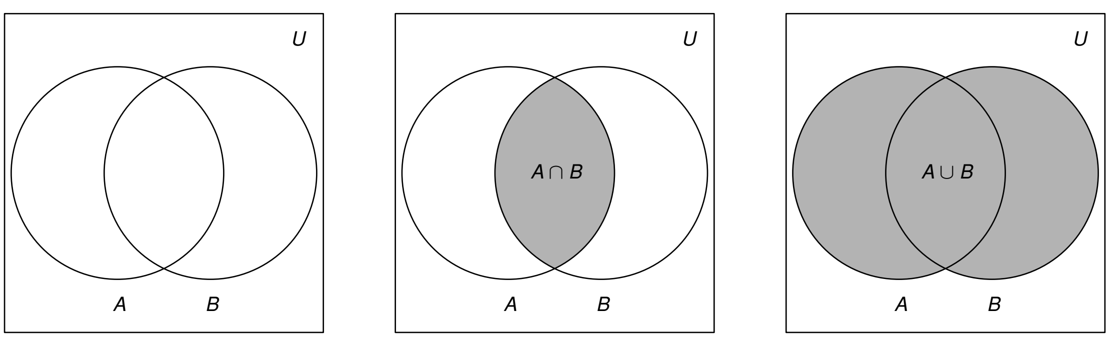
[4]Actually, the empty set can be used to generate numbers - mathematicians can create something from nothing!



A Venn Diagram for C, S and V.

In the Venn Diagram above we have three circles - one for each of the sets C, S and V. We visualize the area enclosed by each of these circles as the elements of each set. Here, we've spelled out the elements for definitiveness. Notice that the circle representing the set C is completely inside the circle representing S. This is a geometric way of showing that $C \subseteq S$. Also, notice that the circles representing S and V overlap on the letter 'o'. This common region is how we visualize $S \cap V$. Notice that since $C \cap V = \emptyset$, the circles which represent C and V have no overlap whatsoever.

All of these circles lie in a rectangle labeled U for the 'universal' set. A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take $U = S \cup V$ or U as the set of letters in the entire alphabet. The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is 'no' and we refer you once again to Russell's Paradox. The usual triptych of Venn Diagrams indicating generic sets A and B along with $A \cap B$ and $A \cup B$ is given below.



Sets A and B. $A \cap B$ is shaded. $A \cup B$ is shaded.

The one major limitation of Venn Diagrams is that they become unwieldy if more than four sets need to be drawn simultaneously within the same universal set. This idea is explored in the Exercises.

A.1.2 Sets of Real Numbers

The playground for most of this text is the set of **Real Numbers**. Much of the "real world" can be quantified using real numbers: the temperature at a given time, the revenue generated by selling a certain number of products and the maximum population of Sasquatch which can inhabit a particular region are just three basic examples. A succinct, but nonetheless incomplete[5] definition of a real number is given below.

Definition A.6. A **real number** is any number which possesses a decimal representation. The set of real numbers is denoted by the character $\mathbb{R}$.

Certain subsets of the real numbers are worthy of note and are listed below. In fact, in more advanced texts,[6] the real numbers are *constructed* from some of these subsets.

Special Subsets of Real Numbers

1. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$ The periods of ellipsis '...' here indicate that the natural numbers contain 1, 2, 3 'and so forth'.

2. The **Whole Numbers**: $\mathbb{W} = \{0, 1, 2, \ldots\}$.

3. The **Integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.[a]

4. The **Rational Numbers**: $\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ where } b \neq 0 \right\}$. Rational numbers are the ratios of integers where the denominator is not zero. It turns out that another way to describe the rational numbers[b] is:

$$\mathbb{Q} = \{x \,|\, x \text{ possesses a repeating or terminating decimal representation}\}$$

5. The **Irrational Numbers**: $\mathbb{P} = \{x \,|\, x \in \mathbb{R} \text{ but } x \notin \mathbb{Q}\}$.[c] That is, an irrational number is a real number which isn't rational. Said differently,

$$\mathbb{P} = \{x \,|\, x \text{ possesses a decimal representation which neither repeats nor terminates}\}$$

[a] The symbol $\pm$ is read 'plus or minus' and it is a shorthand notation which appears throughout the text. Just remember that $x = \pm 3$ means $x = 3$ or $x = -3$.

[b] See Section 10.2.

[c] Examples here include number π (See Section B.1), $\sqrt{2}$ and 0.101001000100001 ....

Note that every natural number is a whole number which, in turn, is an integer. Each integer is a rational number (take $b = 1$ in the above definition for $\mathbb{Q}$) and since every rational number is a real number[7] the sets $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are nested like Matryoshka dolls. More formally, these sets form a subset chain: $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. The reader is encouraged to sketch a Venn Diagram depicting $\mathbb{R}$ and all of the subsets mentioned above.

It is time to put all of this together in an example.

[5] Math pun intended!

[6] See, for instance, Landau's Foundations of Analysis.

[7] Thanks to long division!

Example A.1.1.

1. Write a roster description for $P = \{2^n \,|\, n \in \mathbb{N}\}$ and $E = \{2n \,|\, n \in \mathbb{Z}\}$.
2. Write a verbal description for $S = \{x^2 \,|\, x \in \mathbb{R}\}$.
3. Let $A = \left\{-117, \frac{4}{5}, 0.20\overline{2002}, 0.202002000200002\ldots\right\}$.
   (a) Which elements of A are natural numbers? Rational numbers? Real numbers?
   (b) Find $A \cap \mathbb{W}$, $A \cap \mathbb{Z}$ and $A \cap \mathbb{P}$.
4. What is another name for $\mathbb{N} \cup \mathbb{Q}$? What about $\mathbb{Q} \cup \mathbb{P}$?

Solution.

1. To find roster descriptions for each of these sets, we need to list their elements. Starting with the set $P = \{2^n \,|\, n \in \mathbb{N}\}$, we substitute natural number values n into the formula 2^n. For $n = 1$ we get $2^1 = 2$, for $n = 2$ we get $2^2 = 4$, for $n = 3$ we get $2^3 = 8$ and for $n = 4$ we get $2^4 = 16$. Hence P describes the powers of 2, so a roster description for P is $P = \{2, 4, 8, 16, \ldots\}$ where the '...' indicates the that pattern continues.[8]

   Proceeding in the same way, we generate elements in $E = \{2n \,|\, n \in \mathbb{Z}\}$ by plugging in integer values of n into the formula $2n$. Starting with $n = 0$ we obtain $2(0) = 0$. For $n = 1$ we get $2(1) = 2$, for $n = -1$ we get $2(-1) = -2$ for $n = 2$, we get $2(2) = 4$ and for $n = -2$ we get $2(-2) = -4$. As n moves through the integers, $2n$ produces all of the *even* integers.[9] A roster description for E is $E = \{0, \pm 2, \pm 4, \ldots\}$.

2. One way to verbally describe S is to say that S is the 'set of all squares of real numbers'. While this isn't incorrect, we'd like to take this opportunity to delve a little deeper.[10] What makes the set $S = \{x^2 \,|\, x \in \mathbb{R}\}$ a little trickier to wrangle than the sets P or E above is that the dummy variable here, x, runs through all *real* numbers. Unlike the natural numbers or the integers, the real numbers cannot be listed in any methodical way.[11] Nevertheless, we can select some real numbers, square them and get a sense of what kind of numbers lie in S. For $x = -2$, $x^2 = (-2)^2 = 4$ so 4 is in S, as are $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and $(\sqrt{117})^2 = 117$. Even things like $(-\pi)^2$ and $(0.101001000100001\ldots)^2$ are in S.

   So suppose $s \in S$. What can be said about s? We know there is some real number x so that $s = x^2$. Since $x^2 \geq 0$ for any real number x, we know $s \geq 0$. This tells us that everything in S is a non-negative real number.[12] This begs the question: are <u>all</u> of the non-negative real numbers in S? Suppose n is a non-negative real number, that is, $n \geq 0$. If n were in S, there would be a real number

[8]This isn't the most *precise* way to describe this set - it's always dangerous to use '...' since we assume that the pattern is clearly demonstrated and thus made evident to the reader. Formulas are more precise because the pattern is clear.

[9]This shouldn't be too surprising, since an even integer is *defined* to be an integer multiple of 2.

[10]Think of this as an opportunity to stop and smell the mathematical roses.

[11]This is a nontrivial statement. Interested readers are directed to a discussion of Cantor's Diagonal Argument.

[12]This means S is a subset of the non-negative real numbers.

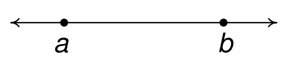
x so that $x^2 = n$. As you may recall, we can solve $x^2 = n$ by 'extracting square roots': $x = \pm\sqrt{n}$. Since $n \geq 0$, $\sqrt{n}$ is a real number.[13] Moreover, $(\sqrt{n})^2 = n$ so n is the square of a real number which means $n \in S$. Hence, S is the set of non-negative real numbers.

3. (a) The set A contains no natural numbers.[14] Clearly $\frac{4}{5}$ is a rational number as is -117 (which can be written as $\frac{-117}{1}$). It's the last two numbers listed in A, $0.20\overline{2002}$ and $0.202002000200002\ldots$, that warrant some discussion. First, recall that the 'line' over the digits 2002 in $0.20\overline{2002}$ (called the vinculum) indicates that these digits repeat, so it is a rational number.[15] As for the number $0.202002000200002\ldots$, the '...' indicates the pattern of adding an extra '0' followed by a '2' is what defines this real number. Despite the fact there is a *pattern* to this decimal, this decimal is *not repeating*, so it is not a rational number - it is, in fact, an irrational number. All of the elements of A are real numbers, since all of them can be expressed as decimals (remember that $\frac{4}{5} = 0.8$).

   (b) The set $A \cap \mathbb{W} = \{x \,|\, x \in A \text{ and } x \in \mathbb{W}\}$ is another way of saying we are looking for the set of numbers in A which are whole numbers. Since A contains no whole numbers, $A \cap \mathbb{W} = \emptyset$. Similarly, $A \cap \mathbb{Z}$ is looking for the set of numbers in A which are integers. Since -117 is the only integer in A, $A \cap \mathbb{Z} = \{-117\}$. For the set $A \cap \mathbb{P}$, as discussed in part (a), the number $0.202002000200002\ldots$ is irrational, so $A \cap \mathbb{P} = \{0.202002000200002\ldots\}$.

4. The set $\mathbb{N} \cup \mathbb{Q} = \{x \,|\, x \in \mathbb{N} \text{ or } x \in \mathbb{Q}\}$ is the union of the set of natural numbers with the set of rational numbers. Since every natural number is a rational number, $\mathbb{N}$ doesn't contribute any new elements to $\mathbb{Q}$, so $\mathbb{N} \cup \mathbb{Q} = \mathbb{Q}$.[16] For the set $\mathbb{Q} \cup \mathbb{P}$, we note that every real number is either rational or not, hence $\mathbb{Q} \cup \mathbb{P} = \mathbb{R}$, pretty much by the definition of the set $\mathbb{P}$. □

As you may recall, we often visualize the set of real numbers $\mathbb{R}$ as a line where each point on the line corresponds to one and only one real number. Given two different real numbers a and b, we write $a < b$ if a is located to the left of b on the number line, as shown below.



The real number line with two numbers a and b where $a < b$.

While this notion seems innocuous, it is worth pointing out that this convention is rooted in two deep properties of real numbers. The first property is that $\mathbb{R}$ is complete. This means that there are no 'holes' or 'gaps' in the real number line.[17] Another way to think about this is that if you choose any two distinct (different) real numbers, and look between them, you'll find a solid line segment (or interval) consisting of infinitely many real numbers. The next result tells us what types of numbers we can expect to find.

[13] This is called the 'square root closed property' of the non-negative real numbers.

[14] Carl was tempted to include $0.\overline{9}$ in the set A, but thought better of it. See Section 10.2 for details.

[15] So $0.20\overline{2002} = 0.20200220022002\ldots$.

[16] In fact, anytime $A \subseteq B$, $A \cup B = B$ and vice-versa. See the exercises.

[17] Alas, this intuitive feel for what it means to be 'complete' is as good as it gets at this level. Completeness is given a much more precise meaning later in courses like Analysis and Topology.

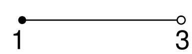
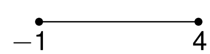
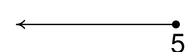
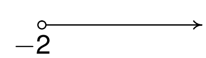
> **Density Property of $\mathbb{Q}$ and $\mathbb{P}$ in $\mathbb{R}$**
> Between any two distinct real numbers, there is at least one rational number and one irrational number. It then follows that between any two distinct real numbers there will be infinitely many rational and infinitely many irrational numbers.

The root word 'dense' here communicates the idea that rationals and irrationals are 'thoroughly mixed' into $\mathbb{R}$. The reader is encouraged to think about how one would find both a rational and an irrational number between, say, 0.9999 and 1. Once you've done that, try doing the same thing for the numbers $0.\overline{9}$ and 1. ('Try' is the operative word, here.[18])

The second property $\mathbb{R}$ possesses that lets us view it as a line is that the set is totally ordered. This means that given any two real numbers a and b, either $a < b$, $a > b$ or $a = b$ which allows us to arrange the numbers from least (left) to greatest (right). This property is given below.

> **Law of Trichotomy**
> If a and b are real numbers then exactly one of the following statements is true:
> $a < b$ $\qquad$ $a > b$ $\qquad$ $a = b$

Segments of the real number line are called **intervals**. They play a huge role not only in this text but also in the Calculus curriculum so we need a concise way to describe them. We start by examining a few examples of the **interval notation** associated with some specific sets of numbers.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid 1 \leq x < 3\}$	$[1, 3)$	
$\{x \mid -1 \leq x \leq 4\}$	$[-1, 4]$	
$\{x \mid x \leq 5\}$	$(-\infty, 5]$	
$\{x \mid x > -2\}$	$(-2, \infty)$	

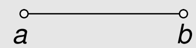
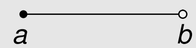
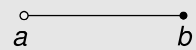
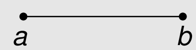
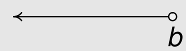
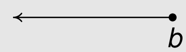
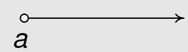
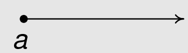
As you can glean from the table, for intervals with finite endpoints we start by writing 'left endpoint, right endpoint'. We use square brackets, '[' or ']', if the endpoint is included in the interval. This corresponds to a 'filled-in' or 'closed' dot on the number line to indicate that the number is included in the set. Otherwise, we use parentheses, '(' or ')' that correspond to an 'open' circle which indicates that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbol $-\infty$ to indicate that the interval extends indefinitely to the left and the symbol ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use the appropriate arrow to indicate that the interval extends indefinitely in one or both directions. We summarize all of the possible cases in one convenient table below.[19]

[18] Again, see Section 10.2 for details.

[19] The importance of understanding interval notation in this book and also in Calculus cannot be overstated so please do yourself a favor and memorize this chart.

Interval Notation

Let a and b be real numbers with $a < b$.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \leq x < b\}$	$[a, b)$	
$\{x \mid a < x \leq b\}$	$(a, b]$	
$\{x \mid a \leq x \leq b\}$	$[a, b]$	
$\{x \mid x < b\}$	$(-\infty, b)$	
$\{x \mid x \leq b\}$	$(-\infty, b]$	
$\{x \mid x > a\}$	(a, ∞)	
$\{x \mid x \geq a\}$	$[a, \infty)$	
$\mathbb{R}$	$(-\infty, \infty)$	

Intervals of the forms $(a, b), (-\infty, b)$ and (a, ∞) are said to be **open** intervals. Those of the forms $[a, b], (-\infty, b]$ and $[a, \infty)$ are said to be **closed** intervals.

Unfortunately, the words 'open' and 'closed' are not antonyms here because the empty set $\emptyset$ and the set $(-\infty, \infty)$ are simultaneously open and closed[20] while the intervals $(a, b]$ and $[a, b)$ are neither open nor closed. The inclusion or exclusion of an endpoint might seem like a terribly small thing to fuss about but these sorts of technicalities in the language become important in Calculus so we feel the need to put this material in the Precalculus book.

We close this section with an example that ties together some of the concepts presented earlier. Specifically, we demonstrate how to use interval notation along with the concepts of union and intersection to describe a variety of sets on the real number line. In many sections of the text to come you will need to be fluent with this notation so take the time to study it deeply now.

[20]You don't need to worry about that fact until you take an advanced course in Topology.

Example A.1.2.

1. Express the following sets of numbers using interval notation.

   (a) $\{x \,|\, x \leq -2 \text{ or } x \geq 2\}$

   (b) $\{x \,|\, x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\}$

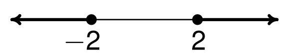
   (c) $\{x \,|\, x \neq \pm 3\}$

   (d) $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\}$

2. Let $A = [-5, 3)$ and $B = (1, \infty)$. Find $A \cap B$ and $A \cup B$.

Solution.

1. (a) The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality $x \leq -2$ corresponds to the interval $(-\infty, -2]$ and the inequality $x \geq 2$ corresponds to the interval $[2, \infty)$. The 'or' in $\{x \,|\, x \leq -2 \text{ or } x \geq 2\}$ tells us that we are looking for the union of these two intervals, so our answer is $(-\infty, -2] \cup [2, \infty)$.

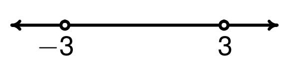


$(-\infty, -2] \cup [2, \infty)$

   (b) For the set $\{x \,|\, x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\}$, we need the real numbers less than (to the left of) $\sqrt{3}$ that are simultaneously greater than (to the right of) $-\frac{8}{5}$, including $-\frac{8}{5}$ but excluding $\sqrt{3}$. This yields $\{x \,|\, x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\} = \left[-\frac{8}{5}, \sqrt{3}\right)$.

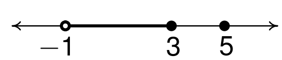
$\left[-\frac{8}{5}, \sqrt{3}\right)$

   (c) For the set $\{x \,|\, x \neq \pm 3\}$, we proceed as before and exclude both $x = 3$ and $x = -3$ from our set. (Refer back to page 1320 for a discussion about $x = \pm 3$) This breaks the number line into *three* intervals, $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$. Since the set describes real numbers which come from the first, second *or* third interval, we have $\{x \,|\, x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.



$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

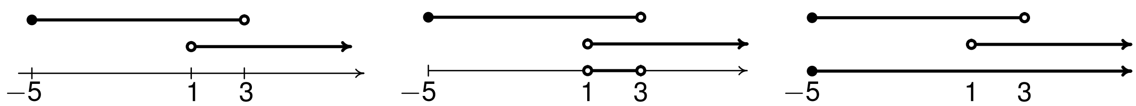
   (d) Graphing the set $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\}$ yields the interval $(-1, 3]$ along with the single number 5. While we *could* express this single point as $[5, 5]$, it is customary to write a single point as a 'singleton set', so in our case we have the set $\{5\}$. This means that our final answer is written $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\} = (-1, 3] \cup \{5\}$.



$(-1, 3] \cup \{5\}$

2. We start by graphing $A = [-5, 3)$ and $B = (1, \infty)$ on the number line. To find $A \cap B$, we need to find the numbers common to both A and B; in other words, we need to find the overlap of the two intervals. Clearly, everything between 1 and 3 is in both A and B. However, since 1 is in A but not in B, 1 is not in the intersection. Similarly, since 3 is in B but not in A, it isn't in the intersection either. Hence, $A \cap B = (1, 3)$.

   To find $A \cup B$, we need to find the numbers in at least one of A or B. Graphically, we shade A and B along with it. Notice here that even though 1 isn't in B, it is in A, so it's in the union along with all of the other elements of A between -5 and 1. A similar argument goes for the inclusion of 3 in the union. The result of shading both A and B together gives us $A \cup B = [-5, \infty)$.



$A = [-5, 3)$, $B = (1, \infty)$ $\quad$ $A \cap B = (1, 3)$ $\quad$ $A \cup B = [-5, \infty)$ $\square$

A.1.3 Exercises

1. Find a verbal description for $O = \{2n - 1 \mid n \in \mathbb{N}\}$
2. Find a roster description for $X = \{z^2 \mid z \in \mathbb{Z}\}$
3. Let $A = \left\{-3, -1.02, -\dfrac{3}{5}, 0.57, 1.\overline{23}, \sqrt{3}, 5.2020020002\ldots, \dfrac{20}{10}, 117\right\}$

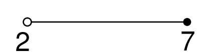
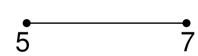
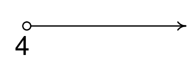
   (a) List the elements of A which are natural numbers.

   (b) List the elements of A which are irrational numbers.

   (c) Find $A \cap \mathbb{Z}$

   (d) Find $A \cap \mathbb{Q}$

4. Fill in the chart below.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$		
	$[0, 3)$	
		
$\{x \mid -5 < x \leq 0\}$		
	$(-3, 3)$	
		
$\{x \mid x \leq 3\}$		
	$(-\infty, 9)$	
		
$\{x \mid x \geq -3\}$		

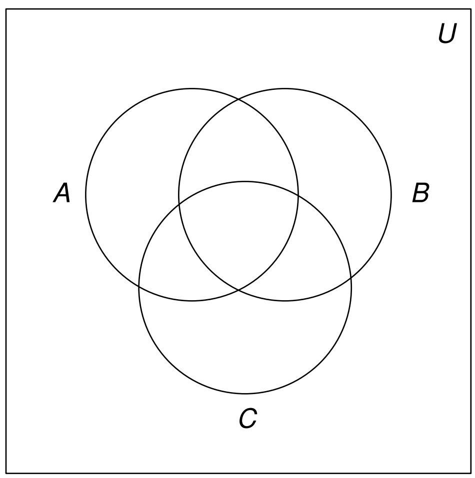
In Exercises 5 - 10, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

5. $(-1, 5] \cap [0, 8)$
6. $(-1, 1) \cup [0, 6]$
7. $(-\infty, 4] \cap (0, \infty)$
8. $(-\infty, 0) \cap [1, 5]$
9. $(-\infty, 0) \cup [1, 5]$
10. $(-\infty, 5] \cap [5, 8)$

In Exercises 11 - 22, write the set using interval notation.

11. $\{x \,|\, x \neq 5\}$
12. $\{x \,|\, x \neq -1\}$
13. $\{x \,|\, x \neq -3,\ 4\}$
14. $\{x \,|\, x \neq 0,\ 2\}$
15. $\{x \,|\, x \neq 2,\ -2\}$
16. $\{x \,|\, x \neq 0,\ \pm 4\}$
17. $\{x \,|\, x \leq -1 \text{ or } x \geq 1\}$
18. $\{x \,|\, x < 3 \text{ and } x \geq 2\}$
19. $\{x \,|\, x \leq -3 \text{ or } x > 0\}$
20. $\{x \,|\, x \leq 2 \text{ and } x > 3\}$
21. $\{x \,|\, x > 2 \text{ or } x = \pm 1\}$
22. $\{x \,|\, 3 < x < 13 \text{ and } x \neq 4\}$

For Exercises 23 - 28, use the blank Venn Diagram below with A, B, and C in it as a guide to help you shade the following sets.



23. $A \cup C$
24. $B \cap C$
25. $(A \cup B) \cup C$
26. $(A \cap B) \cap C$
27. $A \cap (B \cup C)$
28. $(A \cap B) \cup (A \cap C)$

29. Explain how your answers to problems 27 and 28 show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Phrased differently, this shows 'intersection *distributes* over union.' Discuss with your classmates if 'union' distributes over 'intersection.' Use a Venn Diagram to support your answer.

30. Show that $A \subseteq B$ if and only if $A \cup B = B$.

31. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8, 10\}$, $C = \{1, 6, 9\}$ and $D = \{2, 7, 10\}$. Draw one Venn Diagram that shows all four of these sets. What sort of difficulties do you encounter?

A.1.4 Answers

1. O is the odd natural numbers.

2. $X = \{0, 1, 4, 9, 16, ...\}$

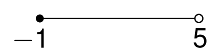
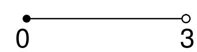
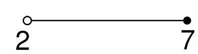
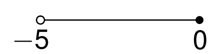
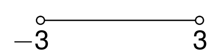
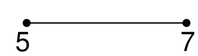
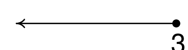
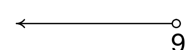
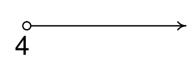
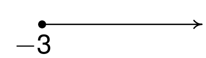
3. (a) $\frac{20}{10} = 2$ and 117

   (b) $\sqrt{3}$ and 5.2020020002

   (c) $\left\{-3, \frac{20}{10}, 117\right\}$

   (d) $\left\{-3, -1.02, -\frac{3}{5}, 0.57, 1.\overline{23}, \frac{20}{10}, 117\right\}$

4.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$	$[-1, 5)$	
$\{x \mid 0 \leq x < 3\}$	$[0, 3)$	
$\{x \mid 2 < x \leq 7\}$	$(2, 7]$	
$\{x \mid -5 < x \leq 0\}$	$(-5, 0]$	
$\{x \mid -3 < x < 3\}$	$(-3, 3)$	
$\{x \mid 5 \leq x \leq 7\}$	$[5, 7]$	
$\{x \mid x \leq 3\}$	$(-\infty, 3]$	
$\{x \mid x < 9\}$	$(-\infty, 9)$	
$\{x \mid x > 4\}$	$(4, \infty)$	
$\{x \mid x \geq -3\}$	$[-3, \infty)$	

5. $(-1,5]\cap[0,8)=[0,5]$

6. $(-1,1)\cup[0,6]=(-1,6]$

7. $(-\infty,4]\cap(0,\infty)=(0,4]$

8. $(-\infty,0)\cap[1,5]=\emptyset$

9. $(-\infty,0)\cup[1,5]=(-\infty,0)\cup[1,5]$

10. $(-\infty,5]\cap[5,8)=\{5\}$

11. $(-\infty,5)\cup(5,\infty)$

12. $(-\infty,-1)\cup(-1,\infty)$

13. $(-\infty,-3)\cup(-3,4)\cup(4,\infty)$

14. $(-\infty,0)\cup(0,2)\cup(2,\infty)$

15. $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

16. $(-\infty,-4)\cup(-4,0)\cup(0,4)\cup(4,\infty)$

17. $(-\infty,-1]\cup[1,\infty)$

18. $[2,3)$

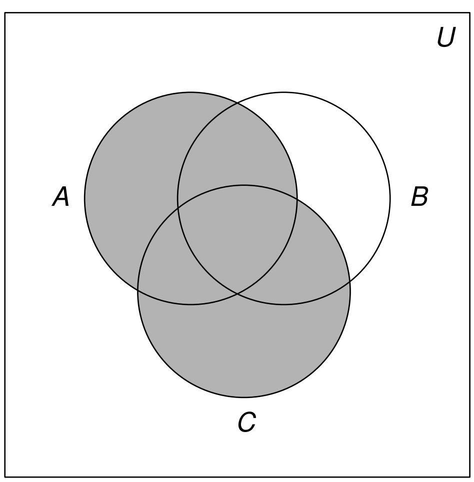
19. $(-\infty,-3]\cup(0,\infty)$

20. $\emptyset$

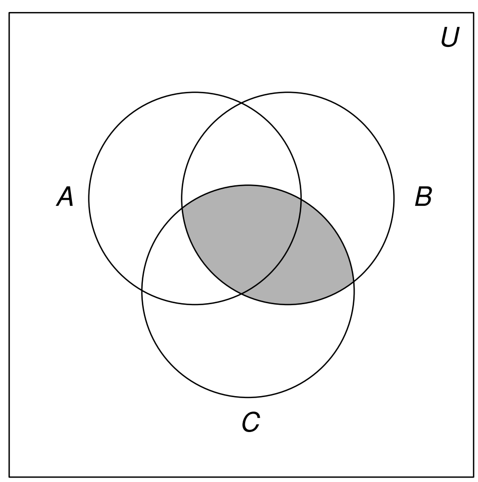
21. $\{-1\}\cup\{1\}\cup(2,\infty)$

22. $(3,4)\cup(4,13)$

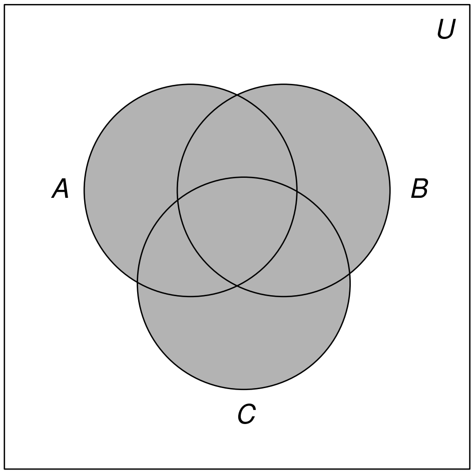
23. $A\cup C$



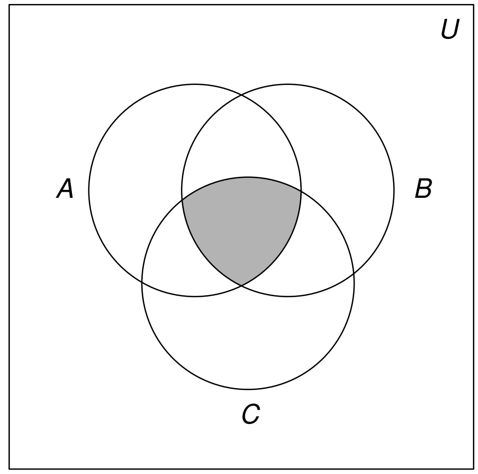
24. $B\cap C$



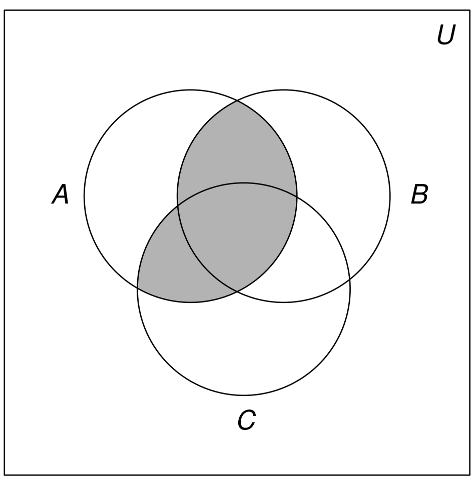
25. $(A \cup B) \cup C$



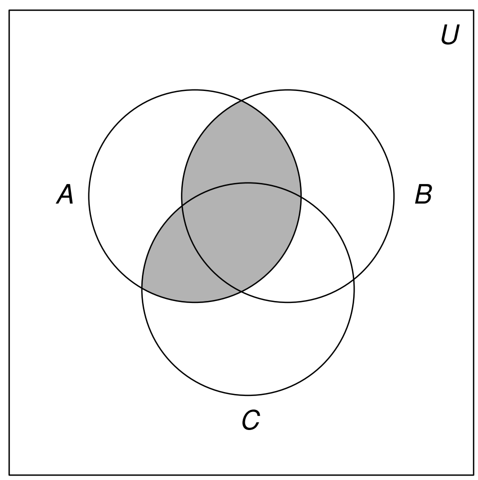
26. $(A \cap B) \cap C$



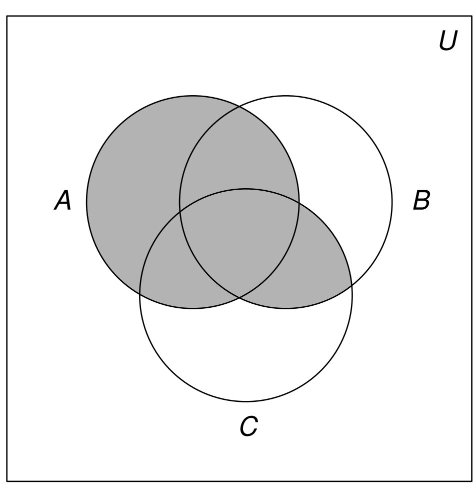
27. $A \cap (B \cup C)$



28. $(A \cap B) \cup (A \cap C)$



29. Yes, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.



A.2 Real Number Arithmetic

In this section we list the properties of real number arithmetic. This is meant to be a succinct, targeted review so we'll resist the temptation to wax poetic about these axioms and their subtleties and refer the interested reader to a more formal course in Abstract Algebra. There are two primary operations one can perform with real numbers: addition and multiplication. We'll start with the properties of addition.

Properties of Real Number Addition

- **Closure:** For all real numbers a and b, $a + b$ is also a real number.
- **Commutativity:** For all real numbers a and b, $a + b = b + a$.
- **Associativity:** For all real numbers a, b and c, $a + (b + c) = (a + b) + c$.
- **Identity:** There is a real number '0' so that for all real numbers a, $a + 0 = a$.
- **Inverse:** For all real numbers a, there is a real number $-a$ such that $a + (-a) = 0$.
- **Definition of Subtraction:** For all real numbers a and b, $a - b = a + (-b)$.

Next, we give real number multiplication a similar treatment. Recall that we may denote the product of two real numbers a and b a variety of ways: ab, $a \cdot b$, $a(b)$, $(a)(b)$ and so on. We'll refrain from using $a \times b$ for real number multiplication in this text with one notable exception in Definition A.7.

Properties of Real Number Multiplication

- **Closure:** For all real numbers a and b, ab is also a real number.
- **Commutativity:** For all real numbers a and b, $ab = ba$.
- **Associativity:** For all real numbers a, b and c, $a(bc) = (ab)c$.
- **Identity:** There is a real number '1' so that for all real numbers a, $a \cdot 1 = a$.
- **Inverse:** For all real numbers $a \neq 0$, there is a real number $\frac{1}{a}$ such that $a\left(\frac{1}{a}\right) = 1$.
- **Definition of Division:** For all real numbers a and $b \neq 0$, $a \div b = \frac{a}{b} = a\left(\frac{1}{b}\right)$.

While most students and some faculty tend to skip over these properties or give them a cursory glance at best,[1] it is important to realize that the properties stated above are what drive the symbolic manipulation in all of Algebra. When listing a tally of more than two numbers, $1+2+3$ for example, we don't need to specify the order in which those numbers are added. Notice though, try as we might, we can add only two numbers at a time and it is the associative property of addition which assures us that we could organize this sum as $(1+2)+3$ or $1+(2+3)$. This brings up a note about 'grouping symbols'. Recall that parentheses and brackets are used in order to specify which operations are to be performed first. In the absence of such grouping

[1] Not unlike how Carl approached all the Elven poetry in The Lord of the Rings.

symbols, multiplication (and hence division) is given priority over addition (and hence subtraction). For example, $1 + 2 \cdot 3 = 1 + 6 = 7$, but $(1 + 2) \cdot 3 = 3 \cdot 3 = 9$. As you may recall, we can 'distribute' the 3 across the addition if we really wanted to do the multiplication first: $(1 + 2) \cdot 3 = 1 \cdot 3 + 2 \cdot 3 = 3 + 6 = 9$. More generally, we have the following.

The Distributive Property and Factoring

For all real numbers a, b and c:

- **Distributive Property:** $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.
- **Factoring:**[a] $ab + ac = a(b + c)$ and $ac + bc = (a + b)c$.

[a]Or, as Carl calls it, 'reading the Distributive Property from right to left.'

It is worth pointing out that we didn't really need to list the Distributive Property both for $a(b+c)$ (distributing from the left) and $(a+b)c$ (distributing from the right), since the commutative property of multiplication gives us one from the other. Also, 'factoring' is really the same equation as the distributive property, just read from right to left. These are the first of many redundancies in this section, and they exist in this review section for one reason only - in our experience, many students *see* these things differently so we will list them as such.

It is hard to overstate the importance of the Distributive Property. For example, in the expression $5(2 + x)$, without knowing the value of x, we cannot perform the addition inside the parentheses first; we must rely on the distributive property here to get $5(2 + x) = 5 \cdot 2 + 5 \cdot x = 10 + 5x$. The Distributive Property is also responsible for combining 'like terms'. Why is $3x + 2x = 5x$? Because $3x + 2x = (3 + 2)x = 5x$.

We continue our review with summaries of other properties of arithmetic, each of which can be derived from the properties listed above. First up are properties of the additive identity 0.

Properties of Zero

Suppose a and b are real numbers.

- **Zero Product Property:** $ab = 0$ if and only if $a = 0$ or $b = 0$ (or both)

  Note: This not only says that $0 \cdot a = 0$ for any real number a, it also says that the *only* way to get an answer of '0' when multiplying two real numbers is to have one (or both) of the numbers be '0' in the first place.

- **Zeros in Fractions:** If $a \neq 0$, $\dfrac{0}{a} = 0 \cdot \left(\dfrac{1}{a}\right) = 0$.

  Note: The quantity $\dfrac{a}{0}$ is undefined.[a]

[a]The expression $\frac{0}{0}$ is technically an 'indeterminate form' as opposed to being strictly 'undefined' meaning that with Calculus we can make some sense of it in certain situations. We'll talk more about this in Chapter 3.

The Zero Product Property drives most of the equation solving algorithms in Algebra because it allows us to take complicated equations and reduce them to simpler ones. For example, you may recall that one way to solve $x^2 + x - 6 = 0$ is by factoring[2] the left hand side of this equation to get $(x-2)(x+3) = 0$. From here, we apply the Zero Product Property and set each factor equal to zero. This yields $x - 2 = 0$ or $x + 3 = 0$ so $x = 2$ or $x = -3$. This application to solving equations leads, in turn, to some deep and profound structure theorems in Chapter 2.

Next up is a review of the arithmetic of 'negatives'. On page 1332 we first introduced the dash which we all recognize as the 'negative' symbol in terms of the additive inverse. For example, the number -3 (read 'negative 3') is defined so that $3 + (-3) = 0$. We then defined subtraction using the concept of the additive inverse again so that, for example, $5-3 = 5+(-3)$. In this text we do not distinguish typographically between the dashes in the expressions '$5 - 3$' and '-3' even though they are mathematically quite different.[3] In the expression '$5 - 3$,' the dash is a *binary* operation (that is, an operation requiring *two* numbers) whereas in '-3', the dash is a *unary* operation (that is, an operation requiring only one number). You might ask, 'Who cares?' Your calculator does - that's who! In the text we can write $-3 - 3 = -6$ but that will not work in your calculator. Instead you'd need to type $^{-}3 - 3$ to get -6 where the first dash comes from the '$+/-$' key and the second dash comes from the subtraction key.

Properties of Negatives

Given real numbers a and b we have the following.

- **Additive Inverse Properties:** $-a = (-1)a$ and $-(-a) = a$
- **Products of Negatives:** $(-a)(-b) = ab$.
- **Negatives and Products:** $-ab = -(ab) = (-a)b = a(-b)$.
- **Negatives and Fractions:** If b is nonzero, $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$ and $\dfrac{-a}{-b} = \dfrac{a}{b}$.
- **'Distributing' Negatives:** $-(a + b) = -a - b$ and $-(a - b) = -a + b = b - a$.
- **'Factoring' Negatives:**[a] $-a - b = -(a + b)$ and $b - a = -(a - b)$.

[a]Or, as Carl calls it, reading 'Distributing' Negatives from right to left.

An important point here is that when we 'distribute' negatives, we do so across addition or subtraction only. This is because we are really distributing a factor of -1 across each of these terms: $-(a + b) = (-1)(a+b) = (-1)(a) + (-1)(b) = (-a) + (-b) = -a - b$. Negatives do not 'distribute' across multiplication: $-(2 \cdot 3) \neq (-2) \cdot (-3)$. Instead, $-(2 \cdot 3) = (-2) \cdot (3) = (2) \cdot (-3) = -6$.

The same sort of thing goes for fractions: $-\frac{3}{5}$ can be written as $\frac{-3}{5}$ or $\frac{3}{-5}$, but not $\frac{-3}{-5}$.

[2]Don't worry. We'll review this in due course. And, yes, this is our old friend the Distributive Property!

[3]We're not just being lazy here. We looked at many of the big publishers' Precalculus books and none of them use different dashes, either.

Speaking of fractions, we now review their arithmetic.

Properties of Fractions

Suppose a, b, c and d are real numbers. Assume them to be nonzero whenever necessary; for example, when they appear in a denominator.

- **Identity Properties:** $a = \frac{a}{1}$ and $\frac{a}{a} = 1$.
- **Fraction Equality:** $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
- **Multiplication of Fractions:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. In particular: $\frac{a}{b} \cdot c = \frac{a}{b} \cdot \frac{c}{1} = \frac{ac}{b}$

  Note: A common denominator is **not** required to **multiply** fractions!
- **Division**[a] **of Fractions:** $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

  In particular: $1 \div \frac{a}{b} = \frac{b}{a}$ and $\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$

  Note: A common denominator is **not** required to **divide** fractions!
- **Addition and Subtraction of Fractions:** $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$.

  Note: A common denominator **is** required to **add or subtract** fractions!
- **Equivalent Fractions:** $\frac{a}{b} = \frac{ad}{bd}$, since $\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{d}{d} = \frac{ad}{bd}$

  Note: The *only* way to change the denominator is to multiply both it and the numerator by the same nonzero value because we are, in essence, multiplying the fraction by 1.
- **'Reducing'**[b] **Fractions:** $\frac{a\cancel{d}}{b\cancel{d}} = \frac{a}{b}$, since $\frac{ad}{bd} = \frac{a}{b} \cdot \frac{d}{d} = \frac{a}{b} \cdot 1 = \frac{a}{b}$.

  In particular, $\frac{ab}{b} = a$ since $\frac{ab}{b} = \frac{ab}{1 \cdot b} = \frac{a\cancel{b}}{1 \cdot \cancel{b}} = \frac{a}{1} = a$ and $\frac{b-a}{a-b} = \frac{(-1)\cancel{(a-b)}}{\cancel{(a-b)}} = -1$.

  Note: We may only cancel common **factors** from both numerator and denominator.

[a] The old 'invert and multiply' or 'fraction gymnastics' play.

[b] Or 'Canceling' Common Factors - this is really just reading the previous property 'from right to left'.

Students make so many mistakes with fractions that we feel it is necessary to pause the narrative for a moment and offer you the following examples. Please take the time to read these carefully. In the main body of the text we will skip many of the steps shown here and it is your responsibility to understand the arithmetic behind the computations we use throughout the text. We deliberately limited these examples to "nice" numbers (meaning that the numerators and denominators of the fractions are small integers) and will discuss more complicated matters later. In the upcoming example, we will make use of the Fundamental Theorem of Arithmetic which essentially says that every natural number has a unique prime factorization. Thus 'lowest terms' is clearly defined when reducing the fractions you're about to see.

Example A.2.1. Perform the indicated operations and simplify. By 'simplify' here, we mean to have the final answer written in the form $\frac{a}{b}$ where a and b are integers which have no common factors. Said another way, we want $\frac{a}{b}$ in 'lowest terms'.

1. $\dfrac{1}{4}+\dfrac{6}{7}$
2. $\dfrac{5}{12}-\left(\dfrac{47}{30}-\dfrac{7}{3}\right)$
3. $\dfrac{\dfrac{7}{3-5}-\dfrac{7}{3-5.21}}{5-5.21}$
4. $\dfrac{\dfrac{12}{5}-\dfrac{7}{24}}{1+\left(\dfrac{12}{5}\right)\left(\dfrac{7}{24}\right)}$
5. $\dfrac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)}$
6. $\left(\dfrac{3}{5}\right)\left(\dfrac{5}{13}\right)-\left(\dfrac{4}{5}\right)\left(-\dfrac{12}{13}\right)$

Solution.

1. It may seem silly to start with an example this basic but experience has taught us not to take much for granted. We start by finding the lowest common denominator and then we rewrite the fractions using that new denominator. Since 4 and 7 are **relatively prime**, meaning they have no factors in common, the lowest common denominator is $4 \cdot 7 = 28$.

$$\begin{aligned}
\frac{1}{4}+\frac{6}{7} &= \frac{1}{4}\cdot\frac{7}{7}+\frac{6}{7}\cdot\frac{4}{4} && \text{Equivalent Fractions}\\
&= \frac{7}{28}+\frac{24}{28} && \text{Multiplication of Fractions}\\
&= \frac{31}{28} && \text{Addition of Fractions}
\end{aligned}$$

The result is in lowest terms because 31 and 28 are relatively prime so we're done.

2. We could begin with the subtraction in parentheses, namely $\frac{47}{30}-\frac{7}{3}$, and then subtract that result from $\frac{5}{12}$. It's easier, however, to first distribute the negative across the quantity in parentheses and then use the Associative Property to perform all of the addition and subtraction in one step.[4] The lowest common denominator[5] for all three fractions is 60.

$$\begin{aligned}
\frac{5}{12}-\left(\frac{47}{30}-\frac{7}{3}\right) &= \frac{5}{12}-\frac{47}{30}+\frac{7}{3} && \text{Distribute the Negative}\\
&= \frac{5}{12}\cdot\frac{5}{5}-\frac{47}{30}\cdot\frac{2}{2}+\frac{7}{3}\cdot\frac{20}{20} && \text{Equivalent Fractions}\\
&= \frac{25}{60}-\frac{94}{60}+\frac{140}{60} && \text{Multiplication of Fractions}\\
&= \frac{71}{60} && \text{Addition and Subtraction of Fractions}
\end{aligned}$$

The numerator and denominator are relatively prime so the fraction is in lowest terms and we have our final answer.

[4]See the remark on page 1332 about how we add $1+2+3$.

[5]We could have used $12 \cdot 30 \cdot 3 = 1080$ as our common denominator but then the numerators would become unnecessarily large. It's best to use the *lowest* common denominator.

3. What we are asked to simplify in this problem is known as a 'complex' or 'compound' fraction. Simply put, we have fractions within a fraction.[6] The longest division line[7] acts as a grouping symbol, quite literally dividing the compound fraction into a numerator (containing fractions) and a denominator (which in this case does not contain fractions). The first step to simplifying a compound fraction like this one is to see if you can simplify the little fractions inside it. To that end, we clean up the fractions in the numerator as follows.

$$\begin{aligned}
\frac{\frac{7}{3-5}-\frac{7}{3-5.21}}{5-5.21} &= \frac{\frac{7}{-2}-\frac{7}{-2.21}}{-0.21} \\
&= \frac{-\left(-\frac{7}{2}+\frac{7}{2.21}\right)}{0.21} && \text{Properties of Negatives} \\
&= \frac{\frac{7}{2}-\frac{7}{2.21}}{0.21} && \text{Distribute the Negative}
\end{aligned}$$

We are left with a compound fraction with decimals. We could replace 2.21 with $\frac{221}{100}$ but that would make a mess.[8] It's better in this case to eliminate the decimal by multiplying the numerator and denominator of the fraction with the decimal in it by 100 (since $2.21 \cdot 100 = 221$ is an integer) as shown below.

$$\frac{\frac{7}{2}-\frac{7}{2.21}}{0.21} = \frac{\frac{7}{2}-\frac{7\cdot 100}{2.21\cdot 100}}{0.21} = \frac{\frac{7}{2}-\frac{700}{221}}{0.21}$$

We now perform the subtraction in the numerator and replace 0.21 with $\frac{21}{100}$ in the denominator. This will leave us with one fraction divided by another fraction. We finish by performing the 'division by a fraction is multiplication by the reciprocal' trick and then cancel any factors that we can.

$$\begin{aligned}
\frac{\frac{7}{2}-\frac{700}{221}}{0.21} &= \frac{\frac{7}{2}\cdot\frac{221}{221}-\frac{700}{221}\cdot\frac{2}{2}}{\frac{21}{100}} = \frac{\frac{1547}{442}-\frac{1400}{442}}{\frac{21}{100}} \\
&= \frac{\frac{147}{442}}{\frac{21}{100}} = \frac{147}{442}\cdot\frac{100}{21} = \frac{14700}{9282} = \frac{350}{221}
\end{aligned}$$

The last step comes from the factorizations $14700 = 42 \cdot 350$ and $9282 = 42 \cdot 221$.

4. We are given another compound fraction to simplify and this time both the numerator and denominator contain fractions. As before, the longest division line acts as a grouping symbol to separate the

[6] Fractionception, perhaps?
[7] Also called a 'vinculum'.
[8] Try it if you don't believe us.

numerator from the denominator.

$$\frac{\frac{12}{5}-\frac{7}{24}}{1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)} = \frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)}$$

Hence, one way to proceed is as before: simplify the numerator and the denominator then perform the 'division by a fraction is the multiplication by the reciprocal' trick. While there is nothing wrong with this approach, we'll use our Equivalent Fractions property to rid ourselves of the 'compound' nature of this fraction straight away. The idea is to multiply both the numerator and denominator by the lowest common denominator of each of the 'smaller' fractions - in this case, $24 \cdot 5 = 120$.

$$\begin{aligned}
\frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5}-\frac{7}{24}\right)\cdot 120}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)\cdot 120} && \text{Equivalent Fractions} \\
&= \frac{\left(\frac{12}{5}\right)(120)-\left(\frac{7}{24}\right)(120)}{(1)(120)+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)(120)} && \text{Distributive Property} \\
&= \frac{\frac{12\cdot 120}{5}-\frac{7\cdot 120}{24}}{120+\frac{12\cdot 7\cdot 120}{5\cdot 24}} && \text{Multiply fractions} \\
&= \frac{\frac{12\cdot 24\cdot \cancel{5}}{\cancel{5}}-\frac{7\cdot 5\cdot \cancel{24}}{\cancel{24}}}{120+\frac{12\cdot 7\cdot \cancel{5}\cdot \cancel{24}}{\cancel{5}\cdot \cancel{24}}} && \text{Factor and cancel} \\
&= \frac{(12\cdot 24)-(7\cdot 5)}{120+(12\cdot 7)} \\
&= \frac{288-35}{120+84} \\
&= \frac{253}{204}
\end{aligned}$$

Since $253 = 11 \cdot 23$ and $204 = 2 \cdot 2 \cdot 3 \cdot 17$ have no common factors our result is in lowest terms which means we are done.

5. This fraction may look simpler than the one before it, but the negative signs and parentheses mean that we shouldn't get complacent. Again we note that the division line here acts as a grouping symbol. That is,

$$\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} = \frac{((2(2)+1)(-3-(-3))-5(4-7))}{(4-2(3))}$$

This means that we should simplify the numerator and denominator first, then perform the division last. We tend to what's in parentheses first, giving multiplication priority over addition and subtraction.

$$\begin{array}{rcll}
\dfrac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} & = & \dfrac{(4+1)(-3+3)-5(-3)}{4-6} & \\
& = & \dfrac{(5)(0)+15}{-2} & \\
& = & \dfrac{15}{-2} & \\
& = & -\dfrac{15}{2} & \text{Properties of Negatives}
\end{array}$$

Since $15 = 3 \cdot 5$ and 2 have no common factors, we are done.

6. In this problem, we have multiplication and subtraction. Multiplication takes precedence so we perform it first. Recall that to multiply fractions, we do *not* need to obtain common denominators; rather, we multiply the corresponding numerators together along with the corresponding denominators. Like the previous example, we have parentheses and negative signs for added fun!

$$\begin{array}{rcll}
\left(\dfrac{3}{5}\right)\left(\dfrac{5}{13}\right) - \left(\dfrac{4}{5}\right)\left(-\dfrac{12}{13}\right) & = & \dfrac{3\cdot 5}{5\cdot 13} - \dfrac{4\cdot(-12)}{5\cdot 13} & \text{Multiply fractions} \\
& = & \dfrac{15}{65} - \dfrac{-48}{65} & \\
& = & \dfrac{15}{65} + \dfrac{48}{65} & \text{Properties of Negatives} \\
& = & \dfrac{15+48}{65} & \text{Add numerators} \\
& = & \dfrac{63}{65} &
\end{array}$$

Since $64 = 3 \cdot 3 \cdot 7$ and $65 = 5 \cdot 13$ have no common factors, our answer $\frac{63}{65}$ is in lowest terms and we are done. □

Of the issues discussed in the previous set of examples none causes students more trouble than simplifying compound fractions. We presented two different methods for simplifying them: one in which we simplified the overall numerator and denominator and then performed the division and one in which we removed the compound nature of the fraction at the very beginning. We encourage the reader to go back and use both methods on each of the compound fractions presented. Keep in mind that when a compound fraction is encountered in the rest of the text it will usually be simplified using only one method and we may not choose your favorite method. Feel free to use the other one in your notes.

Next, we review exponents and their properties. Recall that $2 \cdot 2 \cdot 2$ can be written as 2^3 because exponential notation expresses repeated multiplication. In the expression 2^3, 2 is called the **base** and 3 is called the **exponent**. In order to generalize exponents from natural numbers to the integers, and eventually to rational and real numbers, it is helpful to think of the exponent as a count of the number of factors of the base we are multiplying by 1. For instance,

$$2^3 = 1 \cdot (\text{three factors of two}) = 1 \cdot (2 \cdot 2 \cdot 2) = 8.$$

From this, it makes sense that

$$2^0 = 1 \cdot (\text{zero factors of two}) = 1.$$

What about 2^{-3}? The '$-$' in the exponent indicates that we are 'taking away' three factors of two, essentially dividing by three factors of two. So,

$$2^{-3} = 1 \div (\text{three factors of two}) = 1 \div (2 \cdot 2 \cdot 2) = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8}.$$

We summarize the properties of integer exponents below.

Properties of Integer Exponents

Suppose a and b are nonzero real numbers and n and m are integers.

- **Product Rules:** $(ab)^n = a^n b^n$ and $a^n a^m = a^{n+m}$.
- **Quotient Rules:** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ and $\frac{a^n}{a^m} = a^{n-m}$.
- **Power Rule:** $(a^n)^m = a^{nm}$.
- **Negatives in Exponents:** $a^{-n} = \frac{1}{a^n}$.

  In particular, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.
- **Zero Powers:** $a^0 = 1$.

  Note: The expression 0^0 is an indeterminate form.[a]
- **Powers of Zero:** For any *natural* number n, $0^n = 0$.

  Note: The expression 0^n for integers $n \leq 0$ is not defined.

[a]See the comment regarding '$\frac{0}{0}$' on page 1333.

While it is important the state the Properties of Exponents, it is also equally important to take a moment to discuss one of the most common errors in Algebra. It is true that $(ab)^2 = a^2b^2$ (which some students refer to as 'distributing' the exponent to each factor) but you cannot do this sort of thing with addition. That is, in general, $(a + b)^2 \neq a^2 + b^2$. (For example, take $a = 3$ and $b = 4$.) The same goes for any other powers.

With exponents now in the mix, we can now state the Order of Operations Agreement.

Order of Operations Agreement

When evaluating an expression involving real numbers:

1. Evaluate any expressions in **p**arentheses (or other grouping symbols).
2. Evaluate **e**xponents.
3. Evaluate **m**ultiplication and **d**ivision as you read from left to right.
4. Evaluate **a**ddition and **s**ubtraction as you read from left to right.

We note that there are many useful mnemonic devices for remembering the order of operations.[a]

[a] Our favorite is '**P**lease **e**ntertain **m**y **d**ear **a**uld **S**asquatch.'

For example, $2+3\cdot 4^2 = 2+3\cdot 16 = 2+48 = 50$. Where students get into trouble is with things like -3^2. If we think of this as $0-3^2$, then it is clear that we evaluate the exponent first: $-3^2 = 0-3^2 = 0-9 = -9$. In general, we interpret $-a^n = -\left(a^n\right)$. If we want the 'negative' to also be raised to a power, we must write $(-a)^n$ instead. To summarize, $-3^2 = -9$ but $(-3)^2 = 9$.

Of course, many of the 'properties' we've stated in this section can be viewed as ways to circumvent the order of operations. We've already seen how the distributive property allows us to simplify $5(2+x)$ by performing the indicated multiplication **before** the addition that's in parentheses. Similarly, consider trying to evaluate $2^{30172}\cdot 2^{-30169}$. The Order of Operations Agreement demands that the exponents be dealt with first, however, trying to compute 2^{30172} is a challenge, even for a calculator. One of the Product Rules of Exponents, however, allow us to rewrite this product, essentially performing the multiplication first, to get: $2^{30172-30169} = 2^3 = 8$.

Let's take a break and enjoy another example.

Example A.2.2. Perform the indicated operations and simplify.

1. $\dfrac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2}$
2. $12(-5)(-5+3)^{-4}+6(-5)^2(-4)(-5+3)^{-5}$
3. $\dfrac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)}$
4. $\dfrac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}}$

Solution.

1. We begin working inside the parentheses then deal with the exponents before working through the other operations. As we saw in Example A.2.1, the division here acts as a grouping symbol, so we

save the division to the end.

$$\begin{aligned} \frac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2} &= \frac{(2)(8)-(4)^2}{(2)^2} = \frac{(2)(8)-16}{4} \\ &= \frac{16-16}{4} = \frac{0}{4} = 0 \end{aligned}$$

2. As before, we simplify what's in the parentheses first, then work our way through the exponents, multiplication, and finally, the addition.

$$\begin{aligned} 12(-5)(-5+3)^{-4}+6(-5)^2(-4)(-5+3)^{-5} &= 12(-5)(-2)^{-4}+6(-5)^2(-4)(-2)^{-5} \\ &= 12(-5)\left(\frac{1}{(-2)^4}\right)+6(-5)^2(-4)\left(\frac{1}{(-2)^5}\right) \\ &= 12(-5)\left(\frac{1}{16}\right)+6(25)(-4)\left(\frac{1}{-32}\right) \\ &= (-60)\left(\frac{1}{16}\right)+(-600)\left(\frac{1}{-32}\right) \\ &= \frac{-60}{16}+\left(\frac{-600}{-32}\right) \\ &= \frac{-15\cdot\cancel{4}}{4\cdot\cancel{4}}+\frac{-75\cdot\cancel{8}}{-4\cdot\cancel{8}} \\ &= \frac{-15}{4}+\frac{-75}{-4} \\ &= \frac{-15}{4}+\frac{75}{4} \\ &= \frac{-15+75}{4} \\ &= \frac{60}{4} \\ &= 15 \end{aligned}$$

3. The Order of Operations Agreement mandates that we work within each set of parentheses first, giving precedence to the exponents, then the multiplication, and, finally the division. The trouble with this approach is that the exponents are so large that computation becomes a trifle unwieldy. What we observe, however, is that the bases of the exponential expressions, 3 and 4, occur in both the numerator and denominator of the compound fraction. This gives us hope that we can use some of the Properties of Exponents (the Quotient Rule, in particular) to help us out. Our first step here is to invert and multiply. We see immediately that the 5's cancel after which we group the powers of 3

together and the powers of 4 together and apply the properties of exponents.

$$
\begin{aligned}
\frac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)} &= \frac{5\cdot 3^{51}}{4^{36}}\cdot\frac{4^{34}}{5\cdot 3^{49}} = \frac{\cancel{5}\cdot 3^{51}\cdot 4^{34}}{\cancel{5}\cdot 3^{49}\cdot 4^{36}} = \frac{3^{51}}{3^{49}}\cdot\frac{4^{34}}{4^{36}} \\
&= 3^{51-49}\cdot 4^{34-36} = 3^2\cdot 4^{-2} = 3^2\cdot\left(\frac{1}{4^2}\right) \\
&= 9\cdot\left(\frac{1}{16}\right) = \frac{9}{16}
\end{aligned}
$$

4. We have yet another instance of a compound fraction so our first order of business is to rid ourselves of the compound nature of the fraction like we did in Example A.2.1. To do this, however, we need to tend to the exponents first so that we can determine what common denominator is needed to simplify the fraction.

$$
\begin{aligned}
\frac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}} &= \frac{2\left(\dfrac{12}{5}\right)}{1-\left(\dfrac{12}{5}\right)^{2}} = \frac{\left(\dfrac{24}{5}\right)}{1-\left(\dfrac{12^2}{5^2}\right)} = \frac{\left(\dfrac{24}{5}\right)}{1-\left(\dfrac{144}{25}\right)} \\
&= \frac{\left(\dfrac{24}{5}\right)\cdot 25}{\left(1-\dfrac{144}{25}\right)\cdot 25} = \frac{\left(\dfrac{24\cdot 5\cdot\cancel{5}}{\cancel{5}}\right)}{\left(1\cdot 25-\dfrac{144\cdot\cancel{25}}{\cancel{25}}\right)} = \frac{120}{25-144} \\
&= \frac{120}{-119} = -\frac{120}{119}
\end{aligned}
$$

Since 120 and 119 have no common factors, we are done. □

One of the places where the properties of exponents play an important role is in the use of **Scientific Notation**. The basis for scientific notation is that since we use decimals (base ten numerals) to represent real numbers, we can adjust where the decimal point lies by multiplying by an appropriate power of 10. This allows scientists and engineers to focus in on the 'significant' digits[9] of a number - the nonzero values - and adjust for the decimal places later. For instance, $-621 = -6.21 \times 10^2$ and $0.023 = 2.3 \times 10^{-2}$. Notice here that we revert to using the familiar '$\times$' to indicate multiplication.[10] In general, we arrange the real number so exactly one non-zero digit appears to the left of the decimal point. We make this idea precise in the following:

Definition A.7. A real number is written in **Scientific Notation** if it has the form $\pm n.d_1 d_2 \ldots \times 10^k$ where n is a natural number, d_1, d_2, etc., are whole numbers, and k is an integer.

[9] Awesome pun!

[10] This is the 'notable exception' we alluded to earlier.

On calculators, scientific notation may appear using an 'E' or 'EE' as opposed to the $\times$ symbol. For instance, while we will write 6.02×10^{23} in the text, the calculator may display 6.02 E 23 or 6.02 EE 23.

Example A.2.3. Perform the indicated operations and simplify. Write your final answer in scientific notation, rounded to two decimal places.

1. $\dfrac{\left(6.626 \times 10^{-34}\right)\left(3.14 \times 10^{9}\right)}{1.78 \times 10^{23}}$

2. $\left(2.13 \times 10^{53}\right)^{100}$

Solution.

1. As mentioned earlier, the point of scientific notation is to separate out the 'significant' parts of a calculation and deal with the powers of 10 later. In that spirit, we separate out the powers of 10 in both the numerator and the denominator and proceed as follows

$$\begin{aligned}
\frac{\left(6.626 \times 10^{-34}\right)\left(3.14 \times 10^{9}\right)}{1.78 \times 10^{23}} &= \frac{(6.626)(3.14)}{1.78} \cdot \frac{10^{-34} \cdot 10^{9}}{10^{23}} \\
&= \frac{20.80564}{1.78} \cdot \frac{10^{-34+9}}{10^{23}} \\
&= 11.685\ldots \cdot \frac{10^{-25}}{10^{23}} \\
&= 11.685\ldots \times 10^{-25-23} \\
&= 11.685\ldots \times 10^{-48}
\end{aligned}$$

We are asked to write our final answer in scientific notation, rounded to two decimal places. To do this, we note that $11.685\ldots = 1.1685\ldots \times 10^{1}$, so

$$11.685\ldots \times 10^{-48} = 1.1685\ldots \times 10^{1} \times 10^{-48} = 1.1685\ldots \times 10^{1-48} = 1.1685\ldots \times 10^{-47}$$

Our final answer, rounded to two decimal places, is 1.17×10^{-47}.

We could have done that whole computation on a calculator so why did we bother doing any of this by hand in the first place? The answer lies in the next example.

2. If you try to compute $\left(2.13 \times 10^{53}\right)^{100}$ using most hand-held calculators, you'll most likely get an 'overflow' error. It is possible, however, to use the calculator in combination with the properties of exponents to compute this number. Using properties of exponents, we get:

$$\begin{aligned}
\left(2.13 \times 10^{53}\right)^{100} &= (2.13)^{100}\left(10^{53}\right)^{100} \\
&= \left(6.885\ldots \times 10^{32}\right)\left(10^{53\times 100}\right) \\
&= \left(6.885\ldots \times 10^{32}\right)\left(10^{5300}\right) \\
&= 6.885\ldots \times 10^{32} \cdot 10^{5300} \\
&= 6.885\ldots \times 10^{5332}
\end{aligned}$$

To two decimal places our answer is 6.88×10^{5332}. $\square$

We close our review of real number arithmetic with a discussion of roots and radical notation. Just as subtraction and division were defined in terms of the inverse of addition and multiplication, respectively, we define roots by undoing natural number exponents.

Definition A.8. Let a be a real number and let n be a natural number. If n is odd, then the **principal n^{th} root** of a (denoted $\sqrt[n]{a}$) is the unique real number satisfying $\left(\sqrt[n]{a}\right)^n = a$. If n is even, $\sqrt[n]{a}$ is defined similarly provided $a \geq 0$ and $\sqrt[n]{a} \geq 0$. The number n is called the **index** of the root and the number a is called the **radicand**. For $n = 2$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$.

The reasons for the added stipulations for even-indexed roots in Definition A.8 can be found in the Properties of Negatives. First, for all real numbers, $x^{\text{even power}} \geq 0$, which means it is never negative. Thus if a is a *negative* real number, there are no real numbers x with $x^{\text{even power}} = a$. This is why if n is even, $\sqrt[n]{a}$ only exists if $a \geq 0$. The second restriction for even-indexed roots is that $\sqrt[n]{a} \geq 0$. This comes from the fact that $x^{\text{even power}} = (-x)^{\text{even power}}$, and we require $\sqrt[n]{a}$ to have just one value. So even though $2^4 = 16$ and $(-2)^4 = 16$, we require $\sqrt[4]{16} = 2$ and ignore -2.

Dealing with odd powers is much easier. For example, $x^3 = -8$ has one and only one real solution, namely $x = -2$, which means not only does $\sqrt[3]{-8}$ exist, there is only one choice, namely $\sqrt[3]{-8} = -2$. Of course, when it comes to solving $x^{5213} = -117$, it's not so clear that there is one and only one real solution, let alone that the solution is $\sqrt[5213]{-117}$. Such pills are easier to swallow once we've thought a bit about such equations graphically,[11] and ultimately, these things come from the completeness property of the real numbers mentioned earlier.

We list properties of radicals below as a 'theorem' as opposed to a definition since they can be justified using the properties of exponents.

Theorem A.1. Properties of Radicals: Let a and b be real numbers and let m and n be natural numbers. If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

- **Product Rule:** $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
- **Quotient Rule:** $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, provided $b \neq 0$.
- **Power Rule:** $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

The proof of Theorem A.1 is based on the definition of the principal n^{th} root and the Properties of Exponents. To establish the product rule, consider the following. If n is odd, then by definition $\sqrt[n]{ab}$ is the unique real number such that $(\sqrt[n]{ab})^n = ab$. Given that $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$ as well, it must be the case that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. If n is even, then $\sqrt[n]{ab}$ is the unique non-negative real number such that $(\sqrt[n]{ab})^n = ab$. Note that since n is even, $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are also non-negative thus $\sqrt[n]{a}\sqrt[n]{b} \geq 0$ as well. Proceeding as above, we find that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{a}$ is a real number to start with.[12] We leave that as an exercise as well.

[11] See Chapter 2.

[12] Otherwise we'd run into an interesting paradox. See Section A.11.

We pause here to point out one of the most common errors students make when working with radicals. Obviously $\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{9+16} = \sqrt{25} = 5$. Thus we can clearly see that $5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ because we all know that $5 \neq 7$. The authors urge you to never consider 'distributing' roots or exponents. It's wrong and no good will come of it because in general $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.

Since radicals have properties inherited from exponents, they are often written as such. We define rational exponents in terms of radicals in the box below.

Definition A.9. Let a be a real number, let m be an integer and let n be a natural number.

- $a^{\frac{1}{n}} = \sqrt[n]{a}$ whenever $\sqrt[n]{a}$ is a real number.[a]
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ whenever $\sqrt[n]{a}$ is a real number.

[a] If n is even we need $a \geq 0$.

It would make life really nice if the rational exponents defined in Definition A.9 had all of the same properties that integer exponents have as listed on page 1340 - but they don't. Why not? Let's look at an example to see what goes wrong. Consider the Product Rule which says that $(ab)^n = a^n b^n$ and let $a = -16$, $b = -81$ and $n = \frac{1}{4}$. Plugging the values into the Product Rule yields the equation $((-16)(-81))^{1/4} = (-16)^{1/4}(-81)^{1/4}$. The left side of this equation is $1296^{1/4}$ which equals 6 but the right side is undefined because neither root is a real number. Would it help if, when it comes to even roots (as signified by even denominators in the fractional exponents), we ensure that everything they apply to is non-negative? That works for some of the rules - we leave it as an exercise to see which ones - but does not work for the Power Rule.

Consider the expression $\left(a^{2/3}\right)^{3/2}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(a^{2/3}\right)^{3/2} = a^{\frac{2}{3}\cdot\frac{3}{2}} = a^1 = a$. However, if we substitute $a = -1$ and apply Definition A.9, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. Thus in this case we have $\left(a^{2/3}\right)^{3/2} \neq a$ even though all of the roots were defined. It is true, however, that $\left(a^{3/2}\right)^{2/3} = a$ and we leave this for the reader to show. The moral of the story is that when simplifying powers of rational exponents where the base is negative or worse, unknown, it's usually best to rewrite them as radicals.[13]

Example A.2.4. Perform the indicated operations and simplify.

1. $\dfrac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$
2. $\dfrac{2\left(\dfrac{\sqrt{3}}{3}\right)}{1 - \left(\dfrac{\sqrt{3}}{3}\right)^2}$

[13] Much to Jeff's chagrin. He's fairly traditional and therefore doesn't care much for radicals.

3. $(\sqrt[3]{-2} - \sqrt[3]{-54})^2$

4. $2\left(\dfrac{9}{4} - 3\right)^{1/3} + 2\left(\dfrac{9}{4}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{9}{4} - 3\right)^{-2/3}$

Solution.

1. We begin in the numerator and note that the radical here acts a grouping symbol,[14] so our first order of business is to simplify the radicand.

$$\begin{aligned}
\frac{-(-4)-\sqrt{(-4)^2-4(2)(-3)}}{2(2)} &= \frac{-(-4)-\sqrt{16-4(2)(-3)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16-4(-6)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16-(-24)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16+24}}{2(2)} \\
&= \frac{-(-4)-\sqrt{40}}{2(2)}
\end{aligned}$$

As you may recall, 40 can be factored using a perfect square as $40 = 4 \cdot 10$ so we use the product rule of radicals to write $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4}\sqrt{10} = 2\sqrt{10}$. This lets us factor a '2' out of both terms in the numerator, eventually allowing us to cancel it with a factor of 2 in the denominator.

$$\begin{aligned}
\frac{-(-4)-\sqrt{40}}{2(2)} &= \frac{-(-4)-2\sqrt{10}}{2(2)} &= \frac{4-2\sqrt{10}}{2(2)} \\
&= \frac{2\cdot 2-2\sqrt{10}}{2(2)} &= \frac{2(2-\sqrt{10})}{2(2)} \\
&= \frac{\cancel{2}(2-\sqrt{10})}{\cancel{2}(2)} &= \frac{2-\sqrt{10}}{2}
\end{aligned}$$

Since the numerator and denominator have no more common factors,[15] we are done.

2. Once again we have a compound fraction, so we first simplify the exponent in the denominator to see which factor we'll need to multiply by in order to clean up the fraction.

[14]The line extending horizontally from the square root symbol $\sqrt{}$ is, you guessed it, another vinculum.

[15]Do you see why we aren't 'canceling' the remaining 2's?

$$\begin{aligned}
\frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2} &= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{(\sqrt{3})^2}{3^2}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{3}{9}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1\cdot\cancel{3}}{3\cdot\cancel{3}}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1}{3}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)\cdot 3}{\left(1-\left(\frac{1}{3}\right)\right)\cdot 3} &&= \frac{\frac{2\cdot\sqrt{3}\cdot\cancel{3}}{\cancel{3}}}{1\cdot 3-\frac{1\cdot\cancel{3}}{\cancel{3}}} \\
&= \frac{2\sqrt{3}}{3-1} &&= \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}
\end{aligned}$$

3. Working inside the parentheses, we first encounter $\sqrt[3]{-2}$. While the -2 isn't a perfect cube,[16] we may think of $-2 = (-1)(2)$. Since $(-1)^3 = -1$, which *is* a perfect cube, we may write $\sqrt[3]{-2} = \sqrt[3]{(-1)(2)} = \sqrt[3]{-1}\sqrt[3]{2} = -\sqrt[3]{2}$. When it comes to $\sqrt[3]{54}$, we may write it as $\sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$. So,

$$\sqrt[3]{-2} - \sqrt[3]{-54} = -\sqrt[3]{2} - (-3\sqrt[3]{2}) = -\sqrt[3]{2} + 3\sqrt[3]{2}.$$

At this stage, we can simplify $-\sqrt[3]{2}+3\sqrt[3]{2} = 2\sqrt[3]{2}$. You may remember this as being called 'combining like radicals,' but it is in fact just another application of the distributive property:

$$-\sqrt[3]{2} + 3\sqrt[3]{2} = (-1)\sqrt[3]{2} + 3\sqrt[3]{2} = (-1+3)\sqrt[3]{2} = 2\sqrt[3]{2}.$$

Putting all this together, we get:

$$\begin{aligned}
(\sqrt[3]{-2} - \sqrt[3]{-54})^2 &= (-\sqrt[3]{2} + 3\sqrt[3]{2})^2 &&= (2\sqrt[3]{2})^2 \\
&= 2^2(\sqrt[3]{2})^2 = 4\sqrt[3]{2^2} &&= 4\sqrt[3]{4}
\end{aligned}$$

There are no perfect integer cubes which are factors of 4 (apart from 1, of course), so we are done.

[16] Of an integer, that is!

4. We start working in the parentheses and get a common denominator to subtract the fractions:

$$\frac{9}{4} - 3 = \frac{9}{4} - \frac{3 \cdot 4}{1 \cdot 4} = \frac{9}{4} - \frac{12}{4} = \frac{-3}{4}$$

The denominators in the fractional exponents are odd, so we can proceed by using the properties of exponents:

$$\begin{aligned}
2\left(\frac{9}{4} - 3\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4} - 3\right)^{-2/3} &= 2\left(\frac{-3}{4}\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{-3}{4}\right)^{-2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{4}{-3}\right)^{2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{(4)^{2/3}}{(-3)^{2/3}}\right) \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{2 \cdot 9 \cdot 1 \cdot 4^{2/3}}{4 \cdot 3 \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{\cancel{2} \cdot 3 \cdot \cancel{3} \cdot 4^{2/3}}{2 \cdot \cancel{2} \cdot \cancel{3} \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}}
\end{aligned}$$

At this point, we could start looking for common denominators but it turns out that these fractions reduce even further. Since $4 = 2^2$, $4^{1/3} = (2^2)^{1/3} = 2^{2/3}$. Similarly, $4^{2/3} = (2^2)^{2/3} = 2^{4/3}$. The expressions $(-3)^{1/3}$ and $(-3)^{2/3}$ contain negative bases so we proceed with caution and convert them back to radical notation to get: $(-3)^{1/3} = \sqrt[3]{-3} = -\sqrt[3]{3} = -3^{1/3}$ and $(-3)^{2/3} = (\sqrt[3]{-3})^2 = (-\sqrt[3]{3})^2 = (\sqrt[3]{3})^2 = 3^{2/3}$. Hence:

$$\begin{aligned}
\frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}} &= \frac{2 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3 \cdot 2^{4/3}}{2 \cdot 3^{2/3}} \\
&= \frac{2^1 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3^1 \cdot 2^{4/3}}{2^1 \cdot 3^{2/3}} \\
&= 2^{1-2/3} \cdot (-3^{1/3}) + 3^{1-2/3} \cdot 2^{4/3-1} \\
&= 2^{1/3} \cdot (-3^{1/3}) + 3^{1/3} \cdot 2^{1/3} \\
&= -2^{1/3} \cdot 3^{1/3} + 3^{1/3} \cdot 2^{1/3} \\
&= 0
\end{aligned}$$

□

We close this section with a note about simplifying. In the preceding examples we used "nice" numbers because we wanted to show as many properties as we could per example. This then begs the question "What happens when the numbers are *not* nice?" Unfortunately, the answer is "Not much simplifying can be done." Take, for example,

$$\frac{\sqrt{7}}{\pi} - \frac{3}{\pi^2} + \frac{4}{\sqrt{11}} = \frac{\pi\sqrt{77} - 3\sqrt{11} + 4\pi^2}{\pi^2\sqrt{11}}$$

Sadly, that's as good as it gets.

A.2.1 Exercises

In Exercises 1 - 33, perform the indicated operations and simplify.

1. $5 - 2 + 3$
2. $5 - (2 + 3)$
3. $\dfrac{2}{3} - \dfrac{4}{7}$
4. $\dfrac{3}{8} + \dfrac{5}{12}$
5. $\dfrac{5-3}{-2-4}$
6. $\dfrac{2(-3)}{3-(-3)}$
7. $\dfrac{2(3)-(4-1)}{2^2+1}$
8. $\dfrac{4-5.8}{2-2.1}$
9. $\dfrac{1-2(-3)}{5(-3)+7}$
10. $\dfrac{5(3)-7}{2(3)^2-3(3)-9}$
11. $\dfrac{2((-1)^2-1)}{((-1)^2+1)^2}$
12. $\dfrac{(-2)^2-(-2)-6}{(-2)^2-4}$
13. $\dfrac{3-\frac{4}{9}}{-2-(-3)}$
14. $\dfrac{\frac{2}{3}-\frac{4}{5}}{4-\frac{7}{10}}$
15. $\dfrac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2}$
16. $\dfrac{1-\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}{1+\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}$
17. $\left(\dfrac{2}{3}\right)^{-5}$
18. $3^{-1} - 4^{-2}$
19. $\dfrac{1+2^{-3}}{3-4^{-1}}$
20. $\dfrac{3\cdot 5^{100}}{12\cdot 5^{98}}$
21. $\sqrt{3^2+4^2}$
22. $\sqrt{12} - \sqrt{75}$
23. $(-8)^{2/3} - 9^{-3/2}$
24. $\left(-\frac{32}{9}\right)^{-3/5}$
25. $\sqrt{(3-4)^2+(5-2)^2}$
26. $\sqrt{(2-(-1))^2+\left(\frac{1}{2}-3\right)^2}$
27. $\sqrt{(\sqrt{5}-2\sqrt{5})^2+(\sqrt{18}-\sqrt{8})^2}$
28. $\dfrac{-12+\sqrt{18}}{21}$
29. $\dfrac{-2-\sqrt{(2)^2-4(3)(-1)}}{2(3)}$
30. $\dfrac{-(-4)+\sqrt{(-4)^2-4(1)(-1)}}{2(1)}$
31. $2(-5)(-5+1)^{-1} + (-5)^2(-1)(-5+1)^{-2}$
32. $3\sqrt{2(4)+1} + 3(4)\left(\frac{1}{2}\right)(2(4)+1)^{-1/2}(2)$
33. $2(-7)\sqrt[3]{1-(-7)} + (-7)^2\left(\frac{1}{3}\right)(1-(-7))^{-2/3}(-1)$
34. With the help of your calculator, find $(3.14\times 10^{87})^{117}$. Write your final answer, using scientific notation, rounded to two decimal places. (See Example A.2.3.)
35. Prove the Quotient Rule and Power Rule stated in Theorem A.1.
36. Discuss with your classmates how you might attempt to simplify the following.

    (a) $\sqrt{\dfrac{1-\sqrt{2}}{1+\sqrt{2}}}$

    (b) $\sqrt[5]{3} - \sqrt[3]{5}$

    (c) $\dfrac{\pi+7}{\pi}$

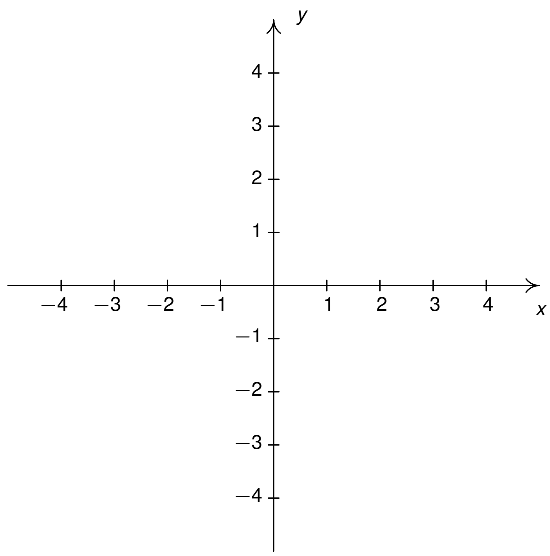
A.2.2 Answers

1. 6
2. 0
3. $\frac{2}{21}$
4. $\frac{19}{24}$
5. $-\frac{1}{3}$
6. -1
7. $\frac{3}{5}$
8. 18
9. $-\frac{7}{8}$
10. Undefined.
11. 0
12. Undefined.
13. $\frac{23}{9}$
14. $-\frac{4}{99}$
15. $-\frac{24}{7}$
16. 0
17. $\frac{243}{32}$
18. $\frac{13}{48}$
19. $\frac{9}{22}$
20. $\frac{25}{4}$
21. 5
22. $-3\sqrt{3}$
23. $\frac{107}{27}$
24. $-\frac{3\sqrt[5]{3}}{8} = -\frac{3^{6/5}}{8}$
25. $\sqrt{10}$
26. $\frac{\sqrt{61}}{2}$
27. $\sqrt{7}$
28. $\frac{-4+\sqrt{2}}{7}$
29. -1
30. $2+\sqrt{5}$
31. $\frac{15}{16}$
32. 13
33. $-\frac{385}{12}$
34. 1.38×10^{10237}

A.3 The Cartesian Plane

A.3.1 The Cartesian Coordinate Plane

In order to visualize the pure excitement that is Precalculus, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**.[1] Imagine two real number lines crossing at a right angle at 0 as drawn below.



The horizontal number line is usually called the ***x*-axis** while the vertical number line is usually called the ***y*-axis**. As with things in the 'real' world, however, it's best not to get too caught up with labels. Think of x and y as generic label placeholders, in much the same way as the variables x and y are placeholders for real numbers. The letters we choose to identify with the axes depend on the context. For example, if we were plotting the relationship between time and the number of Sasquatch sightings, we might label the horizontal axis as the t-axis (for 'time') and the vertical axis the N-axis (for 'number' of sightings.) As with the usual number line, we imagine these axes extending off indefinitely in both directions.[2] Having two number lines allows us to locate the positions of points off of the number lines as well as points on the lines themselves.

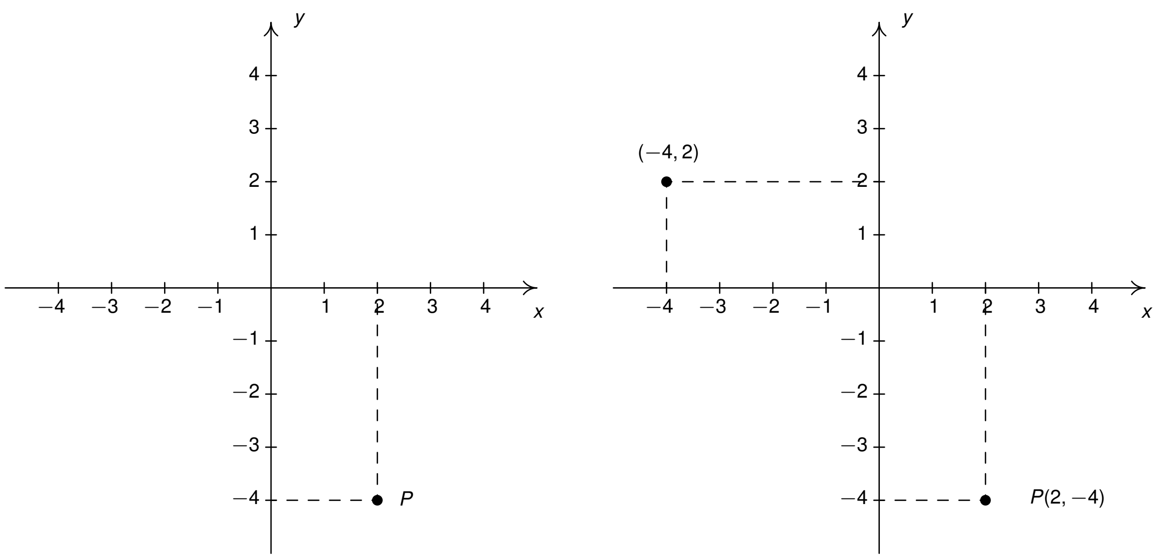
For example, consider the point P on the next page. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the x-axis to P and extending a horizontal line from the y-axis to P. This process is sometimes called 'projecting' the point P to the x- (respectively y-) axis. We then describe the point P using the **ordered pair** $(2, -4)$. The first number in the ordered pair is called the **abscissa** or ***x*-coordinate** and the second is called the **ordinate** or ***y*-coordinate**. Again, the names of the coordinates

[1] So named in honor of René Descartes.

[2] Usually extending off towards infinity is indicated by arrows, but here, the arrows are used to indicate the *direction* of increasing values of x and y.

can vary depending on the context of the application. If, as in the previous paragraph, the horizontal axis represented time and the vertical axis represented the number of Sasquatch sightings, the first coordinate would be called the t-coordinate and the second coordinate would be the N-coordinate. What's important is that we maintain the convention that the abscissa (first coordinate) always corresponds to the horizontal position, while the ordinate (second coordinate) always corresponds to the vertical position. Taken together, the ordered pair $(2, -4)$ comprise the **Cartesian coordinates**[3] of the point P.

In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of 'the point $(2, -4)$'. We can think of $(2, -4)$ as instructions on how to reach P from the **origin** $(0, 0)$ by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important, as are the signs of the numbers in the pair. If we wish to plot the point $(-4, 2)$, we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.



When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs (x, y) as x and y take values from the real numbers. Below is a summary of some basic, but nonetheless important, facts about Cartesian coordinates.

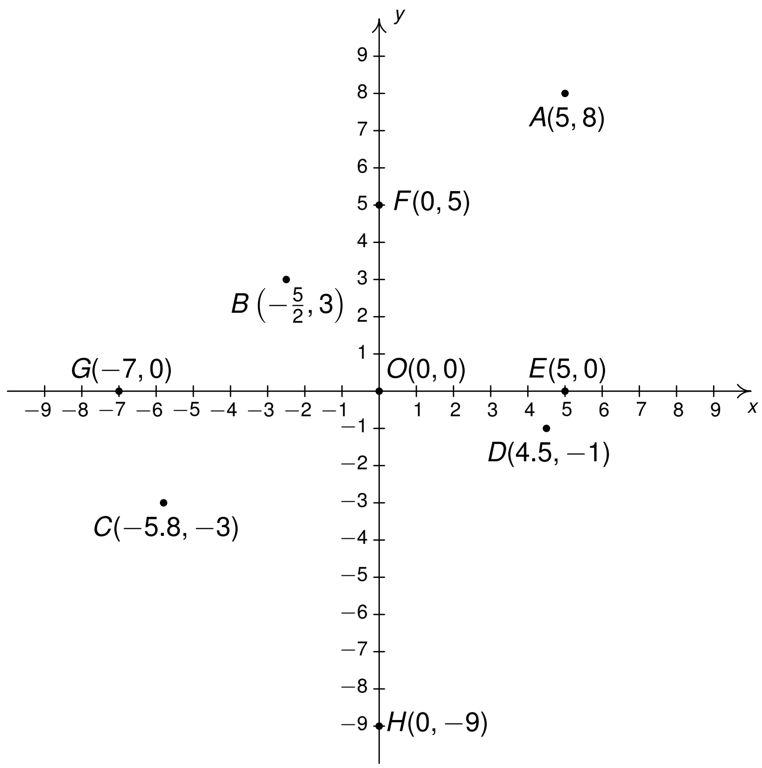
Important Facts about the Cartesian Coordinate Plane

- (a, b) and (c, d) represent the same point in the plane if and only if $a = c$ and $b = d$.
- (x, y) lies on the x-axis if and only if $y = 0$.
- (x, y) lies on the y-axis if and only if $x = 0$.
- The origin is the point $(0, 0)$. It is the only point common to both axes.

[3]Also called the 'rectangular coordinates' of P – see Section 14.1 for more details.

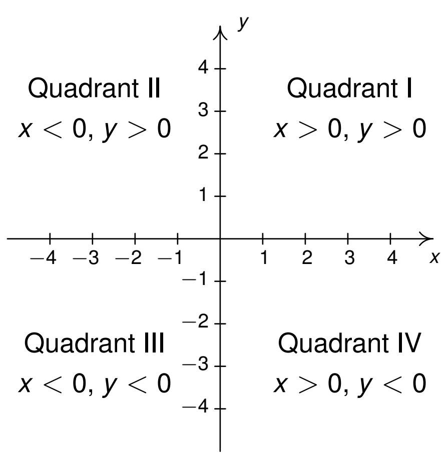
Example A.3.1. Plot the following points: $A(5, 8)$, $B\left(-\frac{5}{2}, 3\right)$, $C(-5.8, -3)$, $D(4.5, -1)$, $E(5, 0)$, $F(0, 5)$, $G(-7, 0)$, $H(0, -9)$, $O(0, 0)$.(The letter O is almost always reserved for the origin.)

Solution. To plot these points, we start at the origin and move to the right if the x-coordinate is positive; to the left if it is negative. Next, we move up if the y-coordinate is positive or down if it is negative. If the x-coordinate is 0, we start at the origin and move along the y-axis only. If the y-coordinate is 0 we move along the x-axis only.



□

The axes divide the plane into four regions called **quadrants**. They are labeled with Roman numerals and proceed counterclockwise around the plane:



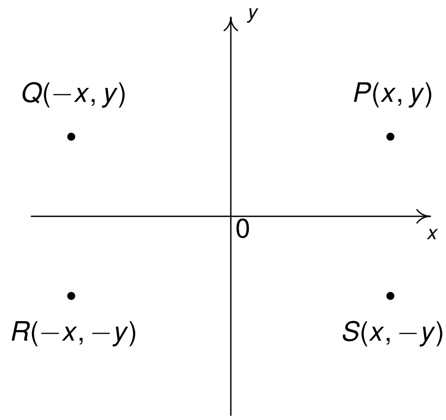
For example, $(1, 2)$ lies in Quadrant **I**, $(-1, 2)$ in Quadrant **II**, $(-1, -2)$ in Quadrant **III** and $(1, -2)$ in Quadrant **IV**. If a point other than the origin happens to lie on the axes, we typically refer to that point as lying on the positive or negative x-axis (if $y = 0$) or on the positive or negative y-axis (if $x = 0$). For example, $(0, 4)$ lies on the positive y-axis whereas $(-117, 0)$ lies on the negative x-axis. Such points do not belong to any of the four quadrants.

One of the most important concepts in all of Mathematics is **symmetry**.[4] There are many types of symmetry in Mathematics, but three of them can be discussed easily using Cartesian Coordinates.

Definition A.10. Two points (a, b) and (c, d) in the plane are said to be

- **symmetric about the *x*-axis** if $a = c$ and $b = -d$
- **symmetric about the *y*-axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

Schematically,



In the above figure, P and S are symmetric about the x-axis, as are Q and R; P and Q are symmetric about the y-axis, as are R and S; and P and R are symmetric about the origin, as are Q and S.

Example A.3.2. Let P be the point $(-2, 3)$. Find the points which are symmetric to P about the:

1. x-axis
2. y-axis
3. origin

Check your answer by plotting the points.

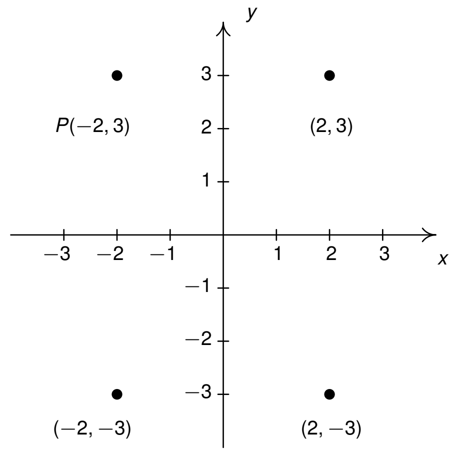
Solution. The figure after Definition A.10 gives us a good way to think about finding symmetric points in terms of taking the opposites of the x- and/or y-coordinates of $P(-2, 3)$.

1. To find the point symmetric about the x-axis, we replace the y-coordinate of 3 with its opposite -3 to get $(-2, -3)$.

[4] According to Carl. Jeff thinks symmetry is overrated.

2. To find the point symmetric about the y-axis, we replace the x-coordinate of -2 with its opposite $-(-2) = 2$ to get $(2, 3)$.

3. To find the point symmetric about the origin, we replace both the x- and y-coordinates with their opposites to get $(2, -3)$.



□

One way to visualize the processes in the previous example is with the concept of a **reflection**. If we start with our point $(-2, 3)$ and pretend that the x-axis is a mirror, then the reflection of $(-2, 3)$ across the x-axis would lie at $(-2, -3)$. If we pretend that the y-axis is a mirror, the reflection of $(-2, 3)$ across that axis would be $(2, 3)$. If we reflect across the x-axis and then the y-axis, we would go from $(-2, 3)$ to $(-2, -3)$ then to $(2, -3)$, and so we would end up at the point symmetric to $(-2, 3)$ about the origin. We summarize and generalize this process below.

Reflections

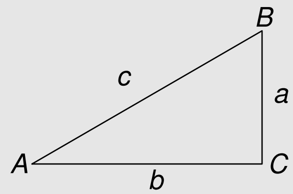
To reflect a point (x, y) about the:

- x-axis, replace y with $-y$.
- y-axis, replace x with $-x$.
- origin, replace x with $-x$ and y with $-y$.

A.3.2 Distance in the Plane

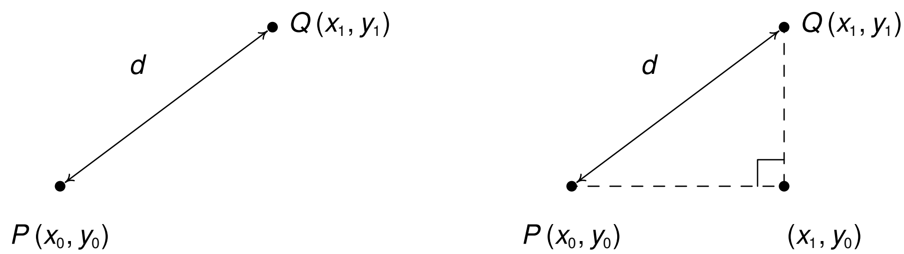
Another fundamental concept in Geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Before we can do that, we need to state what we believe is the most important theorem in all of Geometry: The Pythagorean Theorem.

Theorem A.2. The Pythagorean Theorem: The triangle ABC shown below is a right triangle if and only if $a^2 + b^2 = c^2$



A proof of this theorem will be given in Section **??**. The theorem actually says two different things. If we know that $a^2 + b^2 = c^2$ then the angle C must be a right angle. If we know geometrically that C is already a right angle then we have that $a^2 + b^2 = c^2$. We need the latter statement in the discussion which follows.

Suppose we have two points, $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$, in the plane. By the **distance** d between P and Q, we mean the length of the line segment joining P with Q. (Remember, given any two distinct points in the plane, there is a unique line containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation below on the left.



With a little more imagination, we can envision a right triangle whose hypotenuse has length d as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are $|x_1 - x_0|$ and $|y_1 - y_0|$ so the Pythagorean Theorem gives us

$$|x_1 - x_0|^2 + |y_1 - y_0|^2 = d^2$$
$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = d^2$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

Equation A.1. The Distance Formula: The distance d between the points $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

A couple of remarks about Equation A.1 are in order. First, it is not always the case that the points P and Q lend themselves to constructing such a triangle. If the points P and Q are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader in Exercise 12 to verify Equation A.1 for these cases. Second, distance is a 'length'. So, technically, the number we obtain from the distance formula has some attached units of length. In this text, we'll adopt the convention that the phrase 'units' refers to some generic units of length.[5] Our next example gives us an opportunity to test drive the distance formula as well as brush up on some arithmetic and prerequisite algebra.

Example A.3.3. Find and simplify the distance between the following sets of points:

1. $P(-2,3)$ and $Q(1,-3)$
2. $R\left(\frac{1}{2},\frac{2}{3}\right)$ and $S\left(\frac{3}{4},\frac{1}{5}\right)$
3. $T(\sqrt{3},-\sqrt{20})$ and $V(\sqrt{12},\sqrt{5})$
4. $O(0,0)$ and $P(x,y)$.

Solution. In each case, we apply the distance formula, Equation A.1 with the first point listed taken as (x_0,y_0) and the second point taken as (x_1,y_1).[6]

1. With $(-2,3)=(x_0,y_0)$ and $(1,-3)=(x_1,y_1)$, we get

$$\begin{array}{rcll} d & = & \sqrt{(x_1-x_0)^2+(y_1-y_0)^2} & \\ & = & \sqrt{(1-(-2))^2+(-3-3)^2} & \\ & = & \sqrt{9+36} & \\ & = & \sqrt{45} & \\ & = & \sqrt{9\cdot 5} & \\ & = & \sqrt{9}\sqrt{5} & \text{For nonnegative numbers, } \sqrt{ab}=\sqrt{a}\sqrt{b}. \\ & = & 3\sqrt{5} & \end{array}$$

So the distance is $3\sqrt{5}$ units.

2. With $\left(\frac{1}{2},\frac{2}{3}\right)=(x_0,y_0)$ and $\left(\frac{3}{4},\frac{1}{5}\right)=(x_1,y_1)$, we get

$$\begin{array}{rcll} d & = & \sqrt{(x_1-x_0)^2+(y_1-y_0)^2} & \\ & = & \sqrt{\left(\frac{3}{4}-\frac{1}{2}\right)^2+\left(\frac{1}{5}-\frac{2}{3}\right)^2} & \text{Get common denominators to add and subtract fractions.} \\ & = & \sqrt{\left(\frac{1}{4}\right)^2+\left(-\frac{7}{15}\right)^2} & \\ & = & \sqrt{\frac{1}{16}+\frac{49}{225}} & \text{Since } \left(\frac{a}{b}\right)^2=\frac{a^2}{b^2},\ b\neq 0. \\ & = & \sqrt{\frac{1009}{3600}} & \\ & = & \frac{\sqrt{1009}}{\sqrt{3600}} & \\ & = & \frac{\sqrt{1009}}{60} & \text{For nonnegative numbers, } \sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}},\ b\neq 0. \end{array}$$

So the distance is $\frac{\sqrt{1009}}{60}$ units.

[5] As a result, we'll measure area with 'square units,' or units^2 and volume with 'cubic units,' or units^3.

[6] This choice is completely arbitrary. The reader is encouraged to work these examples taking the first point listed as (x_1,y_1) and the second point listed as (x_0,y_0) and verifying the distance works out to be the same. Can you see why the order of the subtraction in Equation A.1 ultimately doesn't matter?

3. With $(\sqrt{3}, -\sqrt{20}) = (x_0, y_0)$ and $(\sqrt{12}, \sqrt{5}) = (x_1, y_1)$, we get

$$\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
&= \sqrt{\left(\sqrt{12} - \sqrt{3}\right)^2 + \left(\sqrt{5} - (-\sqrt{20})\right)^2} \\
&= \sqrt{\left(2\sqrt{3} - \sqrt{3}\right)^2 + \left(\sqrt{5} + 2\sqrt{5}\right)^2} && \text{Simplify the radicals to get like terms.} \\
&= \sqrt{\left(\sqrt{3}\right)^2 + \left(3\sqrt{5}\right)^2} \\
&= \sqrt{3 + 9 \cdot 5} && \text{Since } (\sqrt{a})^2 = a \text{ and } (b\sqrt{a})^2 = b^2(\sqrt{a})^2. \\
&= \sqrt{48} \\
&= 4\sqrt{3}
\end{aligned}$$

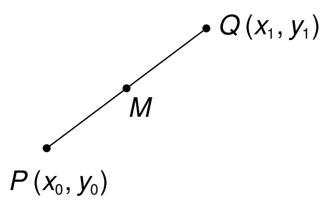
So the distance is $4\sqrt{3}$ units.

4. With $(0, 0) = (x_0, y_0)$ and $(x, y) = (x_1, y_1)$, we get

$$\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
&= \sqrt{(x - 0)^2 + (y - 0)^2} \\
&= \sqrt{x^2 + y^2}
\end{aligned}$$

As tempting as it may look, $\sqrt{x^2 + y^2}$ does not, in general, reduce to $x + y$ or even $|x| + |y|$. So, in this case, the best we can do is state that the distance is $\sqrt{x^2 + y^2}$ units. □

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points, $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$, the **midpoint** M of P and Q is defined to be the point on the line segment connecting P and Q whose distance from P is equal to its distance from Q.



If we think of reaching M by going 'halfway over' and 'halfway up' we get the following formula.

Equation A.2. The Midpoint Formula: The midpoint M of the line segment connecting $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$M = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right)$$

If we let d denote the distance between P and Q, we leave it as Exercise 13 to show that the distance between P and M is $d/2$ which is the same as the distance between M and Q. This suffices to show that Equation A.2 gives the coordinates of the midpoint.

Example A.3.4. Find the midpoint of the line segment connecting the following pairs of points:

1. $P(-2,3)$ and $Q(1,-3)$
2. $R\left(\frac{1}{2},\frac{2}{3}\right)$ and $S\left(\frac{3}{4},\frac{1}{5}\right)$
3. $T(\sqrt{3},-\sqrt{20})$ and $V(\sqrt{12},\sqrt{5})$
4. $O(0,0)$ and $P(x,y)$.

Solution. As with Example A.3.3, in each case, we apply the midpoint formula, Equation A.2 with the first point listed taken as (x_0,y_0) and the second point taken as (x_1,y_1).[7] We also note that midpoints are *points*, which means all of our answers should be *ordered pairs*.

1. With $(-2,3) = (x_0,y_0)$ and $(1,-3) = (x_1,y_1)$, we get

$$\begin{aligned} M &= \left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right) \\ &= \left(\frac{(-2)+1}{2}, \frac{3+(-3)}{2}\right) = \left(-\frac{1}{2}, \frac{0}{2}\right) \\ &= \left(-\frac{1}{2}, 0\right) \end{aligned}$$

   The midpoint is $\left(-\frac{1}{2},0\right)$.

2. With $\left(\frac{1}{2},\frac{2}{3}\right) = (x_0,y_0)$ and $\left(\frac{3}{4},\frac{1}{5}\right) = (x_1,y_1)$, we get

$$\begin{aligned} M &= \left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right) \\ &= \left(\frac{\frac{1}{2}+\frac{3}{4}}{2}, \frac{\frac{2}{3}+\frac{1}{5}}{2}\right) \\ &= \left(\frac{\left(\frac{1}{2}+\frac{3}{4}\right)\cdot 4}{2\cdot 4}, \frac{\left(\frac{2}{3}+\frac{1}{5}\right)\cdot 15}{2\cdot 15}\right) \quad \text{Simplify compound fractions.} \\ &= \left(\frac{5}{8}, \frac{13}{30}\right) \end{aligned}$$

   The midpoint is $\left(\frac{5}{8},\frac{13}{30}\right)$.

[7] As in Example A.3.3, this choice is also completely arbitrary. The reader is encouraged to work these examples taking the first point listed as (x_1,y_1) and the second point listed as (x_0,y_0) and verifying the midpoint works out to be the same. Can you see why the order of the points in Equation A.2 doesn't matter?

3. With $(\sqrt{3}, -\sqrt{20}) = (x_0, y_0)$ and $(\sqrt{12}, \sqrt{5}) = (x_1, y_1)$, we get

$$\begin{aligned} M &= \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right) \\ &= \left(\frac{\sqrt{3} + \sqrt{12}}{2}, \frac{-\sqrt{20} + \sqrt{5}}{2}\right) \\ &= \left(\frac{\sqrt{3} + 2\sqrt{3}}{2}, \frac{-2\sqrt{5} + \sqrt{5}}{2}\right) && \text{Simplify radicals to get like terms.} \\ &= \left(\frac{3\sqrt{3}}{2}, -\frac{\sqrt{5}}{2}\right) \end{aligned}$$

The midpoint is $\left(\frac{3\sqrt{3}}{2}, -\frac{\sqrt{5}}{2}\right)$.

4. With $(0, 0) = (x_0, y_0)$ and $(x, y) = (x_1, y_1)$, we get

$$\begin{aligned} M &= \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right) \\ &= \left(\frac{x + 0}{2}, \frac{y + 0}{2}\right) \\ &= \left(\frac{x}{2}, \frac{y}{2}\right) \end{aligned}$$

The midpoint is $\left(\frac{x}{2}, \frac{y}{2}\right)$. □

We close with a more abstract application of the Midpoint Formula. We will expand upon this example in Example A.5.5 in Section A.5.

Example A.3.5. If $a \neq b$, show that the line $y = x$ equally divides the line segment with endpoints (a, b) and (b, a).

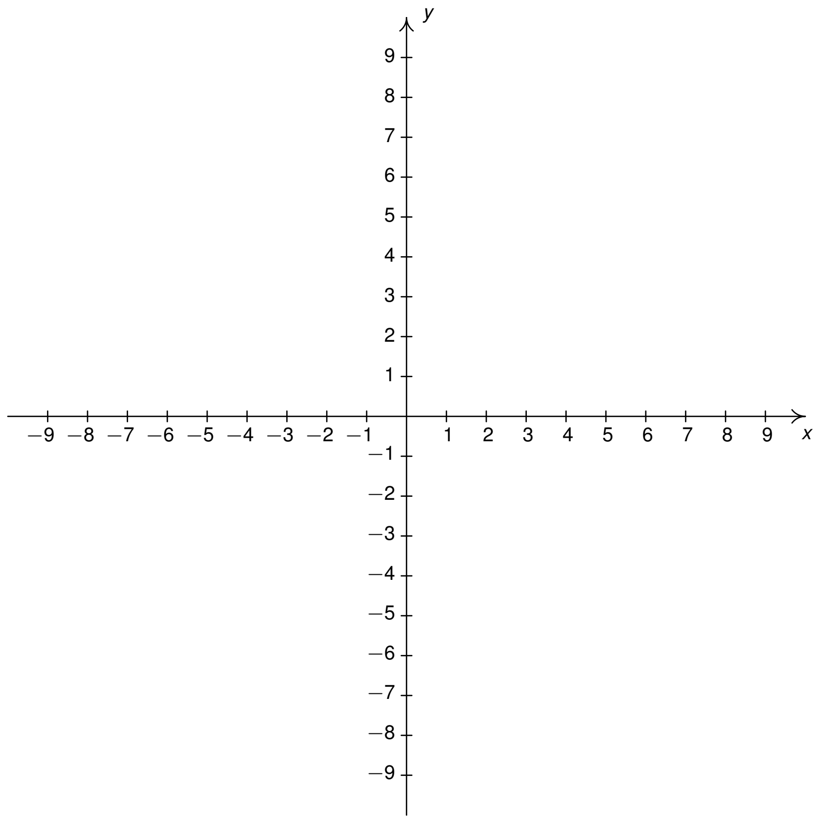
Solution. To prove the claim, we use Equation A.2 to find the midpoint

$$\begin{aligned} M &= \left(\frac{a + b}{2}, \frac{b + a}{2}\right) \\ &= \left(\frac{a + b}{2}, \frac{a + b}{2}\right) \end{aligned}$$

Since the x and y coordinates of this point are the same, we find that the midpoint lies on the line $y = x$, as required. □

A.3.3 Exercises

1. Plot and label the points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$ and $H(7,5)$ in the Cartesian Coordinate Plane given below.



2. For each point given in Exercise 1 above
   - Identify the quadrant or axis in/on which the point lies.
   - Find the point symmetric to the given point about the x-axis.
   - Find the point symmetric to the given point about the y-axis.
   - Find the point symmetric to the given point about the origin.

In Exercises 3 - 10, find the distance d between the points and the midpoint M of the line segment which connects them.

3. $(1,2)$, $(-3,5)$

4. $(3,-10)$, $(-1,2)$

5. $\left(\frac{1}{2}, 4\right), \left(\frac{3}{2}, -1\right)$

6. $\left(-\frac{2}{3}, \frac{3}{2}\right), \left(\frac{7}{3}, 2\right)$

7. $\left(\frac{24}{5}, \frac{6}{5}\right), \left(-\frac{11}{5}, -\frac{19}{5}\right)$.

8. $\left(\sqrt{2}, \sqrt{3}\right), \left(-\sqrt{8}, -\sqrt{12}\right)$

9. $\left(2\sqrt{45}, \sqrt{12}\right), \left(\sqrt{20}, \sqrt{27}\right)$.

10. $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

11. Let's assume that we are standing at the origin and the positive y-axis points due North while the positive x-axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?

12. Verify the Distance Formula A.1 for the cases when:

    (a) The points are arranged vertically. (Hint: Use $P(a, y_0)$ and $Q(a, y_1)$.)

    (b) The points are arranged horizontally. (Hint: Use $P(x_0, b)$ and $Q(x_1, b)$.)

    (c) The points are actually the same point. (You shouldn't need a hint for this one.)

13. Verify the Midpoint Formula by showing the distance between $P(x_1, y_1)$ and M and the distance between M and $Q(x_2, y_2)$ are both half of the distance between P and Q.

14. Show that the points A, B and C below are the vertices of a right triangle.

    (a) $A(-3, 2)$, $B(-6, 4)$, and $C(1, 8)$

    (b) $A(-3, 1)$, $B(4, 0)$ and $C(0, -3)$

15. Find a point $D(x, y)$ such that the points $A(-3, 1)$. $B(4, 0)$, $C(0, -3)$ and D are the corners of a square. Justify your answer.

16. Suppose the distance between $C(h, k)$ and $P(x, y)$ is r. Use the distance formula to show

$$(x - h)^2 + (y - k)^2 = r^2$$

    We will see this formula (and its cousins) in Chapter 8.

17. Let $P(x, y)$ be a point in the plane and let Q be the result of reflecting P about the x-axis, y-axis, or origin. Show the distance from the origin to P is the same as the distance from the origin to Q.

18. Let $O(0, 0)$ (that is, O is the origin), $P(-2, 1)$, $Q(-4, 2)$, and $R(6, -3)$.

    (a) Find the distance from O to P and from O to Q. What do you notice?

    (b) Find the distance from O to P and from O to R. What do you notice?

(c) For a generic point $P(x, y)$, let $Q(kx, ky)$ be the point obtained from P by multiplying both the x and y coordinates of P by the same number, k. Show the distance from O to Q is exactly $|k|$ times the distance from O to P. Explain what these results mean geometrically. (We'll revisit this in Theorem 13.8 in Section 13.3.)

19. In this exercise, we explore some of the properties of distance. For brevity, we'll adopt the notation '$d(P, Q)$' to denote the distance between points P and Q.

    (a) (Non-negative Property) Explain why $d(P, Q) \geq 0$ for any two points in the plane.

    (b) (Symmetric Property) Explain why $d(P, Q) = d(Q, P)$ for any two points in the plane.

    (c) (Identity Property) Show that $d(P, Q) = 0$ if and only if P and Q are the same point.

    NOTE: The phrase 'if and only if' means you need to show two things:

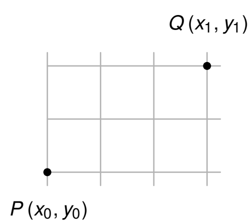
    - If P and Q are the same point, then $d(P, Q) = 0$.
    - If $d(P, Q) = 0$, then P and Q are the same point.

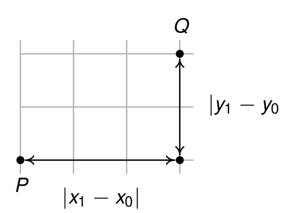
    (d) (Triangle Inequality) The Triangle Inequality says that for any triangle, the sum of the lengths of two sides of a triangle always exceeds the length of the third. Use the Triangle Inequality to show that for any three points P, Q, and R,

    $$d(P, R) \leq d(P, Q) + d(Q, R)$$

    Under what conditions does $d(P, R) = d(P, Q) + d(Q, R)$?

20. (Another way to measure distance.) In this text, we defined the distance between two points as the length of the line segment connecting the two points. Depending on the situation, however, there may be better ways to describe how far one location is from another. Consider the situation below on the left. Suppose P and Q are locations on a city grid, and a taxi is hailed at point P to travel to point Q. In this situation, diagonal movement is impossible,[8] so the taxi is limited to traveling horizontally and vertically.





From the diagram, we see the horizontal distance is $|x_1 - x_0|$ and the vertical distance is $|y_1 - y_0|$, so the total distance the taxi needs to travel to get from P to Q is given by:

$$d_T = |x_1 - x_0| + |y_1 - y_0|$$

We call d_T the 'taxi distance' from P to Q.

[8] Maybe 'discouraged' or 'difficult' would be better word choices.

(a) Let $P(-2,3)$ and $Q(4,2)$. Find the distance, d from P to Q and the taxi distance, d_T from P to Q. Repeat this exercise with several points of your own choosing. Which is larger, d or d_T?

(b) Using the notation of Exercise 19, show that $d(P,Q) \leq d_T(P,Q)$ for any two points P and Q in the plane. (The Triangle Inequality is useful once again here.) Under what conditions is $d(P,Q) = d_T(P,Q)$?

(c) Repeat Exercise 19 with the taxi distance, d_T. (You may need to skip ahead to Exercise 55 in Section 1.3 to verify the Triangle Inequality piece.)

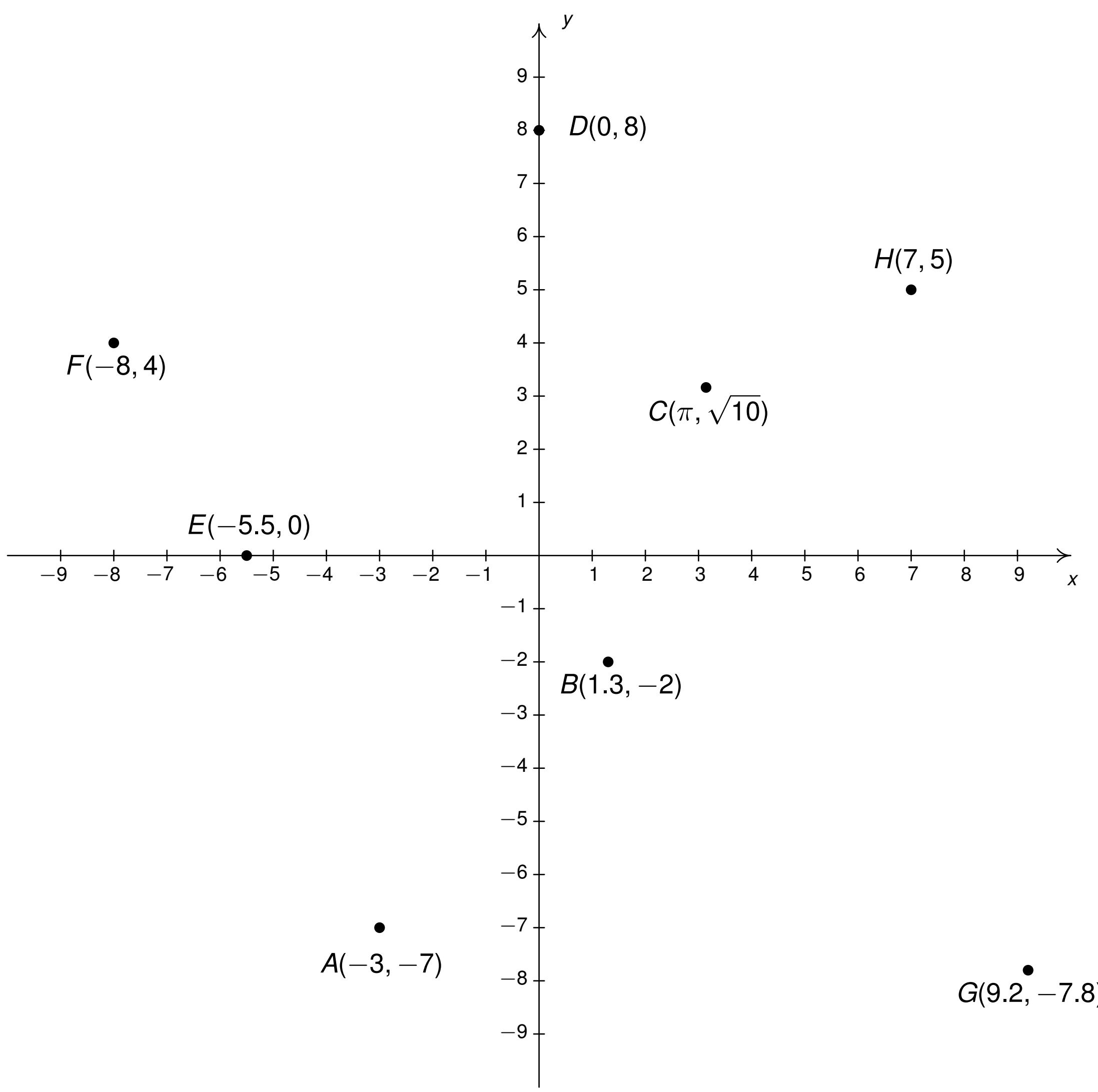
(d) Think about ways to define a 'midpoint' using the taxi distance. What would your formula be? To help you get started, play around with the origin $(0,0)$ as one point and the point $(4,2)$ as the other.

21. The world is not flat.[9] Thus the Cartesian Plane cannot possibly be the end of the story. Discuss with your classmates how you would extend Cartesian Coordinates to represent the three dimensional world. What would the Distance and Midpoint formulas look like, assuming those concepts make sense at all?

[9]There are those who disagree with this statement. Look them up on the Internet some time when you're bored.

A.3.4 Answers

1. The required points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$, and $H(7,5)$ are plotted in the Cartesian Coordinate Plane below.



2. (a) The point $A(-3,-7)$ is
   - in Quadrant III
   - symmetric about x-axis with $(-3,7)$
   - symmetric about y-axis with $(3,-7)$
   - symmetric about origin with $(3,7)$

   (b) The point $B(1.3,-2)$ is
   - in Quadrant IV
   - symmetric about x-axis with $(1.3,2)$
   - symmetric about y-axis with $(-1.3,-2)$
   - symmetric about origin with $(-1.3,2)$

   (c) The point $C(\pi,\sqrt{10})$ is
   - in Quadrant I
   - symmetric about x-axis with $(\pi,-\sqrt{10})$
   - symmetric about y-axis with $(-\pi,\sqrt{10})$
   - symmetric about origin with $(-\pi,-\sqrt{10})$

   (d) The point $D(0,8)$ is
   - on the positive y-axis
   - symmetric about x-axis with $(0,-8)$
   - symmetric about y-axis with $(0,8)$
   - symmetric about origin with $(0,-8)$

   (e) The point $E(-5.5,0)$ is
   - on the negative x-axis
   - symmetric about x-axis with $(-5.5,0)$
   - symmetric about y-axis with $(5.5,0)$
   - symmetric about origin with $(5.5,0)$

   (f) The point $F(-8,4)$ is
   - in Quadrant II
   - symmetric about x-axis with $(-8,-4)$
   - symmetric about y-axis with $(8,4)$
   - symmetric about origin with $(8,-4)$

   (g) The point $G(9.2,-7.8)$ is
   - in Quadrant IV
   - symmetric about x-axis with $(9.2,7.8)$
   - symmetric about y-axis with $(-9.2,-7.8)$
   - symmetric about origin with $(-9.2,7.8)$

   (h) The point $H(7,5)$ is
   - in Quadrant I
   - symmetric about x-axis with $(7,-5)$
   - symmetric about y-axis with $(-7,5)$
   - symmetric about origin with $(-7,-5)$

3. $d = 5$ units, $M = \left(-1, \frac{7}{2}\right)$

4. $d = 4\sqrt{10}$ units, $M = (1,-4)$

5. $d = \sqrt{26}$ units, $M = \left(1, \frac{3}{2}\right)$

6. $d = \frac{\sqrt{37}}{2}$ units, $M = \left(\frac{5}{6}, \frac{7}{4}\right)$

7. $d = \sqrt{74}$ units, $M = \left(\frac{13}{10}, -\frac{13}{10}\right)$

8. $d = 3\sqrt{5}$ units, $M = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}\right)$

9. $d = \sqrt{83}$ units, $M = \left(4\sqrt{5}, \frac{5\sqrt{3}}{2}\right)$

10. $d = 2$ units, $M = (0,0)$

11. $(-3,-4)$, 5 miles, $(4,-4)$

14. (a) The distance from A to B is $|AB| = \sqrt{13}$, the distance from A to C is $|AC| = \sqrt{52}$, and the distance from B to C is $|BC| = \sqrt{65}$. Since $\left(\sqrt{13}\right)^2 + \left(\sqrt{52}\right)^2 = \left(\sqrt{65}\right)^2$, we are guaranteed by the converse of the Pythagorean Theorem that the triangle is a right triangle.

    (b) Show that $|AC|^2 + |BC|^2 = |AB|^2$

A.4 Linear Equations and Inequalities

In the introduction to this chapter we said that we were going to review "the concepts, skills and vocabulary we believe are prerequisite to a rigorous, college-level Precalculus course." So far, we've presented a lot of vocabulary and concepts but we haven't done much to refresh the skills needed to survive in the Precalculus wilderness. Thus over the course of the next few sections we will focus our review on the Algebra skills needed to solve basic equations and inequalities, with one brief detour in Section A.5 where we discuss graphing lines in the plane. In general, equations and inequalities fall into one of three categories: conditional, identity or contradiction, depending on the nature of their solutions. A **conditional** equation or inequality is true for only *certain* real numbers. For example, $2x + 1 = 7$ is true precisely when $x = 3$, and $w - 3 \leq 4$ is true precisely when $w \leq 7$. An **identity** is an equation or inequality that is true for *all* real numbers. For example, $2x - 3 = 1 + x - 4 + x$ or $2t \leq 2t + 3$. A **contradiction** is an equation or inequality that is *never* true. Examples here include $3x - 4 = 3x + 7$ and $a - 1 > a + 3$.

As you may recall, solving an equation or inequality means finding all of the values of the variable, if any exist, which make the given equation or inequality true. This often requires us to manipulate the given equation or inequality from its given form to an easier form. For example, if we're asked to solve $3 - 2(x - 3) = 7x + 3(x + 1)$, we get $x = \frac{1}{2}$, but not without a fair amount of algebraic manipulation. In order to obtain the correct answer(s), however, we need to make sure that whatever maneuvers we apply are reversible in order to guarantee that we maintain a chain of **equivalent** equations or inequalities. Two equations or inequalities are called **equivalent** if they have the same solutions. We summarize these 'legal moves' in the box below.

Procedures which Generate Equivalent Equations

- Add (or subtract) the same real number to (from) both sides of the equation.
- Multiply (or divide) both sides of the equation by the same **nonzero** real number.[a]

Procedures which Generate Equivalent Inequalities

- Add (or subtract) the same real number to (from) both sides of the equation.
- Multiply (or divide) both sides of the equation by the same **positive** real number.[b]

[a] Multiplying both sides of an equation by 0 collapses the equation to $0 = 0$, which doesn't do anybody any good.

[b] Remember that if you multiply both sides of an inequality by a negative real number, the inequality sign is reversed: $3 \leq 4$, but $(-2)(3) \geq (-2)(4)$.

A.4.1 Linear Equations

The first equations we wish to review are **linear** equations as defined below.

Definition A.11. An equation is said to be **linear** in a variable x if it can be written in the form $ax = b$ where a and b are expressions which do not involve x and $a \neq 0$.

One key point about Definition A.11 is that the exponent on the unknown 'x' in the equation is 1, that is $x = x^1$. Our main strategy for solving linear equations is summarized below.

Strategy for Solving Linear Equations

In order to solve an equation which is linear in a given variable, say x:

1. Isolate all of the terms containing x on one side of the equation, putting all of the terms not containing x on the other side of the equation.

2. Factor out the x and divide both sides of the equation by its coefficient.

We illustrate this process with a collection of examples below.

Example A.4.1. Solve the following equations for the indicated variable. Check your answer.

1. Solve for x: $3x - 6 = 7x + 4$
2. Solve for t: $3 - 1.7t = \dfrac{t}{4}$
3. Solve for a: $\dfrac{1}{18}(7 - 4a) + 2 = \dfrac{a}{3} - \dfrac{4 - a}{12}$
4. Solve for y: $8y\sqrt{3} + 1 = 7 - \sqrt{12}(5 - y)$
5. Solve for x: $\dfrac{3x - 1}{2} = x\sqrt{50} + 4$
6. Solve for y: $x(4 - y) = 8y$

Solution.

1. The variable we are asked to solve for is x so our first move is to gather all of the terms involving x on one side and put the remaining terms on the other.[1]

$$\begin{array}{rclr}
3x - 6 & = & 7x + 4 & \\
(3x - 6) - 7x + 6 & = & (7x + 4) - 7x + 6 & \text{Subtract } 7x\text{, add } 6 \\
3x - 7x - 6 + 6 & = & 7x - 7x + 4 + 6 & \text{Rearrange terms} \\
-4x & = & 10 & 3x - 7x = (3 - 7)x = -4x \\
\dfrac{-4x}{-4} & = & \dfrac{10}{-4} & \text{Divide by the coefficient of } x \\
x & = & -\dfrac{5}{2} & \text{Reduce to lowest terms}
\end{array}$$

To check our answer, we substitute $x = -\frac{5}{2}$ into each side of the **orginial** equation to see the equation is satisfied. Sure enough, $3\left(-\frac{5}{2}\right) - 6 = -\frac{27}{2}$ and $7\left(-\frac{5}{2}\right) + 4 = -\frac{27}{2}$.

[1] In the margin notes, when we speak of operations, e.g.,'Subtract $7x$,' we mean to subtract $7x$ from **both** sides of the equation. The 'from both sides of the equation' is omitted in the interest of spacing.

2. In our next example, the unknown is t and we not only have a fraction but also a decimal to wrangle. Fortunately, with equations we can multiply both sides to rid us of these computational obstacles:

$$\begin{array}{rcll}
3-1.7t & = & \dfrac{t}{4} & \\
40(3-1.7t) & = & 40\left(\dfrac{t}{4}\right) & \text{Multiply by 40} \\
40(3)-40(1.7t) & = & \dfrac{40t}{4} & \text{Distribute} \\
120-68t & = & 10t & \\
(120-68t)+68t & = & 10t+68t & \text{Add } 68t \text{ to both sides} \\
120 & = & 78t & 68t+10t=(68+10)t=78t \\
\dfrac{120}{78} & = & \dfrac{78t}{78} & \text{Divide by the coefficient of } t \\
\dfrac{120}{78} & = & t & \\
\dfrac{20}{13} & = & t & \text{Reduce to lowest terms}
\end{array}$$

To check, we again substitute $t = \frac{20}{13}$ into each side of the original equation. We find that $3-1.7\left(\frac{20}{13}\right) = 3-\left(\frac{17}{10}\right)\left(\frac{20}{13}\right) = \frac{5}{13}$ and $\frac{(20/13)}{4} = \frac{20}{13}\cdot\frac{1}{4} = \frac{5}{13}$ as well.

3. To solve this next equation, we begin once again by clearing fractions. The least common denominator here is 36:

$$\begin{array}{rcll}
\dfrac{1}{18}(7-4a)+2 & = & \dfrac{a}{3}-\dfrac{4-a}{12} & \\
36\left(\dfrac{1}{18}(7-4a)+2\right) & = & 36\left(\dfrac{a}{3}-\dfrac{4-a}{12}\right) & \text{Multiply by 36} \\
\dfrac{36}{18}(7-4a)+(36)(2) & = & \dfrac{36a}{3}-\dfrac{36(4-a)}{12} & \text{Distribute} \\
2(7-4a)+72 & = & 12a-3(4-a) & \text{Distribute} \\
14-8a+72 & = & 12a-12+3a & \\
86-8a & = & 15a-12 & 12a+3a=(12+3)a=15a \\
(86-8a)+8a+12 & = & (15a-12)+8a+12 & \text{Add } 8a \text{ and } 12 \\
86+12-8a+8a & = & 15a+8a-12+12 & \text{Rearrange terms} \\
98 & = & 23a & 15a+8a=(15+8)a=23a \\
\dfrac{98}{23} & = & \dfrac{23a}{23} & \text{Divide by the coefficient of } a \\
\dfrac{98}{23} & = & a &
\end{array}$$

The check, as usual, involves substituting $a = \frac{98}{23}$ into both sides of the original equation. The reader is encouraged to work through the (admittedly messy) arithmetic. Both sides work out to $\frac{199}{138}$.

4. The square roots may dishearten you but we treat them just like the real numbers they are. Our strategy is the same: get everything with the variable (in this case y) on one side, put everything else on the other and divide by the coefficient of the variable. We've added a few steps to the narrative that we would ordinarily omit just to help you see that this equation is indeed linear.

$$\begin{array}{rcll}
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5-y) & \\
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5)+\sqrt{12}y & \text{Distribute} \\
8y\sqrt{3}+1 & = & 7-(2\sqrt{3})5+(2\sqrt{3})y & \sqrt{12}=\sqrt{4\cdot 3}=2\sqrt{3} \\
8y\sqrt{3}+1 & = & 7-10\sqrt{3}+2y\sqrt{3} & \\
(8y\sqrt{3}+1)-1-2y\sqrt{3} & = & (7-10\sqrt{3}+2y\sqrt{3})-1-2y\sqrt{3} & \text{Subtract 1 and } 2y\sqrt{3} \\
8y\sqrt{3}-2y\sqrt{3}+1-1 & = & 7-1-10\sqrt{3}+2y\sqrt{3}-2y\sqrt{3} & \text{Rearrange terms} \\
(8\sqrt{3}-2\sqrt{3})y & = & 6-10\sqrt{3} & \\
6y\sqrt{3} & = & 6-10\sqrt{3} & \text{See note below} \\
\dfrac{6y\sqrt{3}}{6\sqrt{3}} & = & \dfrac{6-10\sqrt{3}}{6\sqrt{3}} & \text{Divide } 6\sqrt{3} \\
y & = & \dfrac{2\cdot\sqrt{3}\cdot\sqrt{3}-2\cdot 5\cdot\sqrt{3}}{2\cdot 3\cdot\sqrt{3}} & \\
y & = & \dfrac{\cancel{2}\cancel{\sqrt{3}}(\sqrt{3}-5)}{\cancel{2}\cdot 3\cdot\cancel{\sqrt{3}}} & \text{Factor and cancel} \\
y & = & \dfrac{\sqrt{3}-5}{3} &
\end{array}$$

In the list of computations above we marked the row $6y\sqrt{3}=6-10\sqrt{3}$ with a note. That's because we wanted to draw your attention to this line without breaking the flow of the manipulations. The equation $6y\sqrt{3}=6-10\sqrt{3}$ is in fact linear according to Definition A.11: the variable is y, the value of A is $6\sqrt{3}$ and $B=6-10\sqrt{3}$. Checking the solution, while not trivial, is good mental exercise. Each side works out to be $\frac{27-40\sqrt{3}}{3}$.

5. Proceeding as before, we simplify radicals and clear denominators. Once we gather all of the terms containing x on one side and move the other terms to the other, we factor out x to identify its

coefficient then divide to get our answer.

$$\begin{array}{rcll}
\dfrac{3x-1}{2} & = & x\sqrt{50}+4 & \\
\dfrac{3x-1}{2} & = & 5x\sqrt{2}+4 & \sqrt{50}=\sqrt{25\cdot 2} \\
2\left(\dfrac{3x-1}{2}\right) & = & 2\left(5x\sqrt{2}+4\right) & \text{Multiply by } 2 \\
\dfrac{\cancel{2}\cdot(3x-1)}{\cancel{2}} & = & 2(5x\sqrt{2})+2\cdot 4 & \text{Distribute} \\
3x-1 & = & 10x\sqrt{2}+8 & \\
(3x-1)-10x\sqrt{2}+1 & = & (10x\sqrt{2}+8)-10x\sqrt{2}+1 & \text{Subtract } 10x\sqrt{2}\text{, add } 1 \\
3x-10x\sqrt{2}-1+1 & = & 10x\sqrt{2}-10x\sqrt{2}+8+1 & \text{Rearrange terms} \\
3x-10x\sqrt{2} & = & 9 & \\
(3-10\sqrt{2})x & = & 9 & \text{Factor} \\
\dfrac{(3-10\sqrt{2})x}{3-10\sqrt{2}} & = & \dfrac{9}{3-10\sqrt{2}} & \text{Divide by the coefficient of } x \\
x & = & \dfrac{9}{3-10\sqrt{2}} &
\end{array}$$

The reader is encouraged to check this solution - it isn't as bad as it looks if you're careful! Each side works out to be $\dfrac{12+5\sqrt{2}}{3-10\sqrt{2}}$.

6. If we were instructed to solve our last equation for x, we'd be done in one step: divide both sides by $(4-y)$ - assuming $4-y\neq 0$, that is. Alas, we are instructed to solve for y, which means we have some more work to do.

$$\begin{array}{rcll}
x(4-y) & = & 8y & \\
4x-xy & = & 8y & \text{Distribute} \\
(4x-xy)+xy & = & 8y+xy & \text{Add } xy \\
4x & = & (8+x)y & \text{Factor}
\end{array}$$

In order to finish the problem, we need to divide both sides of the equation by the coefficient of y which in this case is $8+x$. This expression contains a variable so we need to stipulate that we may perform this division only if $8+x\neq 0$, or, in other words, $x\neq -8$. Hence, we write our solution as:

$$y=\frac{4x}{8+x}, \quad \text{provided } x\neq -8$$

What happens if $x=-8$? Substituting $x=-8$ into the original equation gives $(-8)(4-y)=8y$ or $-32+8y=8y$. This reduces to $-32=0$, which is a contradiction. This means there is no solution when $x=-8$, so we've covered all the bases. Checking our answer requires some Algebra we haven't reviewed yet in this text, but the necessary skills *should* be lurking somewhere in the mathematical mists of your mind. The adventurous reader is invited to plug $y=\frac{4x}{8+x}$ into the original equation and show that both sides work out to $\frac{32x}{x+8}$. $\square$

A.4.2 Linear Inequalities

We now turn our attention to linear inequalities. Unlike linear equations which admit at most one solution, the solutions to linear inequalities are generally intervals of real numbers. While the solution strategy for solving linear inequalities is the same as with solving linear equations, we need to remind ourselves that, should we decide to multiply or divide both sides of an inequality by a **negative** number, we need to reverse the direction of the inequality. (See the footnote in the box on page 1369.) In the example below, we work not only some 'simple' linear inequalities in the sense there is only one inequality present, but also some 'compound' linear inequalities which require us to revisit the notions of intersection and union.

Example A.4.2. Solve the following inequalities for the indicated variable.

1. Solve for x: $\dfrac{7-8x}{2} \geq 4x+1$
2. Solve for y: $\dfrac{3}{4} \leq \dfrac{7-y}{2} < 6$
3. Solve for t: $2t-1 \leq 4-t < 6t+1$
4. Solve for x: $5+\sqrt{7}x \leq 4x+1 \leq 8$
5. Solve for w: $2.1-0.01w \leq -3$ or $2.1-0.01w \geq 3$

Solution.

1. We begin by clearing denominators and gathering all of the terms containing x to one side of the inequality and putting the remaining terms on the other.

$$\begin{array}{rcll}
\dfrac{7-8x}{2} & \geq & 4x+1 & \\
2\left(\dfrac{7-8x}{2}\right) & \geq & 2(4x+1) & \text{Multiply by 2} \\
\dfrac{\cancel{2}(7-8x)}{\cancel{2}} & \geq & 2(4x)+2(1) & \text{Distribute} \\
7-8x & \geq & 8x+2 & \\
(7-8x)+8x-2 & \geq & 8x+2+8x-2 & \text{Add } 8x\text{, subtract 2} \\
7-2-8x+8x & \geq & 8x+8x+2-2 & \text{Rearrange terms} \\
5 & \geq & 16x & 8x+8x=(8+8)x=16x \\
\dfrac{5}{16} & \geq & \dfrac{16x}{16} & \text{Divide by the coefficient of } x \\
\dfrac{5}{16} & \geq & x &
\end{array}$$

We get $\frac{5}{16} \geq x$ or, said differently, $x \leq \frac{5}{16}$. We express this set[2] of real numbers as $\left(-\infty, \frac{5}{16}\right]$. Though not required to do so, we could partially check our answer by substituting $x = \frac{5}{16}$ and a few other values in our solution set ($x = 0$, for instance) to make sure the inequality holds. (It also isn't a bad idea to choose an $x > \frac{5}{16}$, say $x = 1$, to see that the inequality *doesn't* hold there.) The

[2]Using set-builder notation, our 'set' of solutions here is $\{x \,|\, x \leq \frac{5}{16}\}$.

only real way to actually show that our answer works for *all* values in our solution set is to start with $x \leq \frac{5}{16}$ and reverse all of the steps in our solution procedure to prove it is equivalent to our original inequality.

2. We have our first example of a 'compound' inequality. The solutions to
$$\frac{3}{4} \leq \frac{7-y}{2} < 6$$
must satisfy
$$\frac{3}{4} \leq \frac{7-y}{2} \qquad \underline{\text{and}} \qquad \frac{7-y}{2} < 6$$
One approach is to solve each of these inequalities separately, then intersect their solution sets. While this method works (and will be used later for more complicated problems), our variable y appears only in the middle expression so we can proceed by working both inequalities at once:
$$\begin{array}{rccr}
\frac{3}{4} \leq & \frac{7-y}{2} & < 6 & \\
4\left(\frac{3}{4}\right) \leq & 4\left(\frac{7-y}{2}\right) & < 4(6) & \text{Multiply by 4} \\
\frac{\cancel{4}\cdot 3}{\cancel{4}} \leq & \frac{\cancelto{2}{4}(7-y)}{\cancel{2}} & < 24 & \\
3 \leq & 2(7-y) & < 24 & \\
3 \leq & 2(7)-2y & < 24 & \text{Distrbute} \\
3 \leq & 14-2y & < 24 & \\
3-14 \leq & (14-2y)-14 & < 24-14 & \text{Subtract 14} \\
-11 \leq & -2y & < 10 & \\
\frac{-11}{-2} \geq & \frac{-2y}{-2} & > \frac{10}{-2} & \text{Divide by the coefficient of } y \\
 & & & \text{Reverse inequalities} \\
\frac{11}{2} \geq & y & > -5 &
\end{array}$$
Our final answer is $\frac{11}{2} \geq y > -5$, or, said differently, $-5 < y \leq \frac{11}{2}$. In interval notation, this is $\left(-5, \frac{11}{2}\right]$. We could check the reasonableness of our answer as before, and the reader is encouraged to do so.

3. We have another compound inequality and what distinguishes this one from our previous example is that 't' appears on both sides of both inequalities. In this case, we need to create two separate inequalities and find all of the real numbers t which satisfy both $2t-1 \leq 4-t$ *and* $4-t < 6t+1$. The first inequality, $2t-1 \leq 4-t$, reduces to $3t \leq 5$ or $t \leq \frac{5}{3}$. The second inequality, $4-t < 6t+1$, becomes $3 < 7t$ which reduces to $t > \frac{3}{7}$. Thus our solution is all real numbers t with $t \leq \frac{5}{3}$ *and* $t > \frac{3}{7}$, or, writing this as a compound inequality, $\frac{3}{7} < t \leq \frac{5}{3}$. Using interval notation,[3] we express our solution as $\left(\frac{3}{7}, \frac{5}{3}\right]$.

[3] If we intersect the solution sets of the two individual inequalities, we get the answer, too: $\left(-\infty, \frac{5}{3}\right] \cap \left(\frac{3}{7}, \infty\right) = \left(\frac{3}{7}, \frac{5}{3}\right]$.

4. As before, with this inequality we have no choice but to solve each inequality individually and intersect the solution sets. Starting with the leftmost inequality, we first note that the in the term $\sqrt{7}x$, the vinculum of the square root extends over the 7 only, meaning the x is not part of the radicand. In order to avoid confusion, we will write $\sqrt{7}x$ as $x\sqrt{7}$.

$$\begin{array}{rcll} 5 + x\sqrt{7} & \leq & 4x + 1 & \\ (5 + x\sqrt{7}) - 4x - 5 & \leq & (4x + 1) - 4x - 5 & \text{Subtract } 4x \text{ and } 5 \\ x\sqrt{7} - 4x + 5 - 5 & \leq & 4x - 4x + 1 - 5 & \text{Rearrange terms} \\ x(\sqrt{7} - 4) & \leq & -4 & \text{Factor} \end{array}$$

At this point, we need to exercise a bit of caution because the number $\sqrt{7} - 4$ is negative.[4] When we divide by it the inequality reverses:

$$\begin{array}{rcll} x(\sqrt{7} - 4) & \leq & -4 & \\ \dfrac{x(\sqrt{7} - 4)}{\sqrt{7} - 4} & \geq & \dfrac{-4}{\sqrt{7} - 4} & \begin{array}{r}\text{Divide by the coefficient of } x \\ \text{Reverse inequalities}\end{array} \\ x & \geq & \dfrac{-4}{\sqrt{7} - 4} & \\ x & \geq & \dfrac{-4}{-(4 - \sqrt{7})} & \\ x & \geq & \dfrac{4}{4 - \sqrt{7}} & \end{array}$$

We're only half done because we still have the rightmost inequality to solve. Fortunately, that one seems rather mundane: $4x + 1 \leq 8$ reduces to $x \leq \frac{7}{4}$ without too much incident. Our solution is $x \geq \frac{4}{4-\sqrt{7}}$ *and* $x \leq \frac{7}{4}$. We may be tempted to write $\frac{4}{4-\sqrt{7}} \leq x \leq \frac{7}{4}$ and call it a day but that would be nonsense! To see why, notice that $\sqrt{7}$ is between 2 and 3 so $\frac{4}{4-\sqrt{7}}$ is between $\frac{4}{4-2} = 2$ and $\frac{4}{4-3} = 4$. In particular, we get $\frac{4}{4-\sqrt{7}} > 2$. On the other hand, $\frac{7}{4} < 2$. This means that our 'solutions' have to be simultaneously greater than 2 AND less than 2 which is impossible. Therefore, this compound inequality has no solution, which means we did all that work for nothing.[5]

5. Our last example is yet another compound inequality but here, instead of the two inequalities being connected with the conjunction '*and*', they are connected with '*or*', which indicates that we need to find the *union* of the results of each. Starting with $2.1 - 0.01w \leq -3$, we get $-0.01w \leq -5.1$, which gives[6] $w \geq 510$. The second inequality, $2.1 - 0.01w \geq 3$, becomes $-0.01w \geq 0.9$, which reduces to $w \leq -90$. Our solution set consists of all real numbers w with $w \geq 510$ *or* $w \leq -90$. In interval notation, this is $(-\infty, -90] \cup [510, \infty)$. $\square$

[4] Since $4 < 7 < 9$, it stands to reason that $\sqrt{4} < \sqrt{7} < \sqrt{9}$ so $2 < \sqrt{7} < 3$.

[5] Much like how people walking on treadmills get nowhere. Math is the endurance cardio of the brain, folks!

[6] Don't forget to flip the inequality!

A.4.3 Exercises

In Exercises 1 - 9, solve the given linear equation and check your answer.

1. $3x - 4 = 2 - 4(x - 3)$
2. $\dfrac{3 - 2t}{4} = 7t + 1$
3. $\dfrac{2(w - 3)}{5} = \dfrac{4}{15} - \dfrac{3w + 1}{9}$
4. $-0.02y + 1000 = 0$
5. $\dfrac{49w - 14}{7} = 3w - (2 - 4w)$
6. $7 - (4 - x) = \dfrac{2x - 3}{2}$
7. $3t\sqrt{7} + 5 = 0$
8. $\sqrt{50}y = \dfrac{6 - \sqrt{8}y}{3}$
9. $4 - (2x + 1) = \dfrac{x\sqrt{7}}{9}$

In equations 10 - 27, solve each equation for the indicated variable.

10. Solve for y: $3x + 2y = 4$
11. Solve for x: $3x + 2y = 4$
12. Solve for C: $F = \dfrac{9}{5}C + 32$
13. Solve for x: $p = -2.5x + 15$
14. Solve for x: $C = 200x + 1000$
15. Solve for y: $x = 4(y + 1) + 3$
16. Solve for w: $vw - 1 = 3v$
17. Solve for v: $vw - 1 = 3v$
18. Solve for y: $x(y - 3) = 2y + 1$
19. Solve for π: $C = 2\pi r$
20. Solve for V: $PV = nRT$
21. Solve for R: $PV = nRT$
22. Solve for g: $E = mgh$
23. Solve for m: $E = \dfrac{1}{2}mv^2$

In Exercises 24 - 27, the subscripts on the variables have no intrinsic mathematical meaning; they're just used to distinguish one variable from another. In other words, treat 'P_1' and 'P_2' as two different variables as you would 'x' and 'y.' (The same goes for 'x' and 'x_0,' etc.)

24. Solve for V_2: $P_1V_1 = P_2V_2$
25. Solve for t: $x = x_0 + at$
26. Solve for x: $y - y_0 = m(x - x_0)$
27. Solve for T_1: $q = mc(T_2 - T_1)$
28. With the help of your classmates, find values for c so that the equation: $2x - 5c = 1 - c(x + 2)$

    (a) has $x = 42$ as a solution.

    (b) has no solution (that is, the equation is a contradiction.)

    Is it possible to find a value of c so the equation is an identity? Explain.

In Exercises 29 - 46, solve the given inequality. Write your answer using interval notation.

29. $3 - 4x \geq 0$

30. $2t - 1 < 3 - (4t - 3)$

31. $\dfrac{7 - y}{4} \geq 3y + 1$

32. $0.05R + 1.2 > 0.8 - 0.25R$

33. $7 - (2 - x) \leq x + 3$

34. $\dfrac{10m + 1}{5} \geq 2m - \dfrac{1}{2}$

35. $x\sqrt{12} - \sqrt{3} > \sqrt{3}x + \sqrt{27}$

36. $2t - 7 \leq \sqrt[3]{18}t$

37. $117y \geq y\sqrt{2} - 7y\sqrt[4]{8}$

38. $-\dfrac{1}{2} \leq 5x - 3 \leq \dfrac{1}{2}$

39. $-\dfrac{3}{2} \leq \dfrac{4 - 2t}{10} < \dfrac{7}{6}$

40. $-0.1 \leq \dfrac{5 - x}{3} - 2 < 0.1$

41. $2y \leq 3 - y < 7$

42. $3x \geq 4 - x \geq 3$

43. $6 - 5t > \dfrac{4t}{3} \geq t - 2$

44. $2x + 1 \leq -1$ or $2x + 1 \geq 1$

45. $4 - x \leq 0$ or $2x + 7 < x$

46. $\dfrac{5 - 2x}{3} > x$ or $2x + 5 \geq 1$

A.4.4 Answers

1. $x = \frac{18}{7}$

2. $t = -\frac{1}{30}$

3. $w = \frac{61}{33}$

4. $y = 50000$

5. All real numbers.

6. No solution.

7. $t = -\frac{5}{3\sqrt{7}} = -\frac{5\sqrt{7}}{21}$

8. $y = \frac{6}{17\sqrt{2}} = \frac{3\sqrt{2}}{17}$

9. $x = \frac{27}{18+\sqrt{7}}$

10. $y = \frac{4-3x}{2}$ or $y = -\frac{3}{2}x + 2$

11. $x = \frac{4-2y}{3}$ or $x = -\frac{2}{3}y + \frac{4}{3}$

12. $C = \frac{5}{9}(F-32)$ or $C = \frac{5}{9}F - \frac{160}{9}$

13. $x = \frac{p-15}{-2.5} = \frac{15-p}{2.5}$ or $x = -\frac{2}{5}p + 6$.

14. $x = \frac{C-1000}{200}$ or $x = \frac{1}{200}C - 5$

15. $y = \frac{x-7}{4}$ or $y = \frac{1}{4}x - \frac{7}{4}$

16. $w = \frac{3v+1}{v}$, provided $v \neq 0$.

17. $v = \frac{1}{w-3}$, provided $w \neq 3$.

18. $y = \frac{3x+1}{x-2}$, provided $x \neq 2$.

19. $\pi = \frac{C}{2r}$, provided $r \neq 0$.

20. $V = \frac{nRT}{P}$, provided $P \neq 0$.

21. $R = \frac{PV}{nT}$, provided $n \neq 0$, $T \neq 0$.

22. $g = \frac{E}{mh}$, provided $m \neq 0$, $h \neq 0$.

23. $m = \frac{2E}{v^2}$, provided $v^2 \neq 0$ (so $v \neq 0$).

24. $V_2 = \frac{P_1 V_1}{P_2}$, provided $P_2 \neq 0$.

25. $t = \frac{x - x_0}{a}$, provided $a \neq 0$.

26. $x = \frac{y - y_0 + mx_0}{m}$ or $x = x_0 + \frac{y - y_0}{m}$, provided $m \neq 0$.

27. $T_1 = \frac{mcT_2 - q}{mc}$ or $T_1 = T_2 - \frac{q}{mc}$, provided $m \neq 0$, $c \neq 0$.

29. $\left(-\infty, \frac{3}{4}\right]$

30. $\left(-\infty, \frac{7}{6}\right)$

31. $\left(-\infty, \frac{3}{13}\right]$

32. $\left(-\frac{4}{3}, \infty\right)$

33. No solution.

34. $(-\infty, \infty)$

35. $(4, \infty)$

36. $\left[\frac{7}{2-\sqrt[3]{18}}, \infty\right)$

37. $[0, \infty)$

38. $\left[\frac{1}{2}, \frac{7}{10}\right]$

39. $\left(-\frac{23}{6}, \frac{19}{2}\right]$

40. $\left(-\frac{13}{10}, -\frac{7}{10}\right]$

41. $(-4, 1]$

42. $\{1\} = [1, 1]$

43. $\left[-6, \frac{18}{19}\right)$

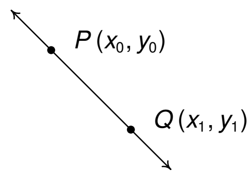
44. $(-\infty, -1] \cup [0, \infty)$

45. $(-\infty, -7) \cup [4, \infty)$

46. $(-\infty, \infty)$

A.5 Graphing Lines

In Section A.3.2, we concerned ourselves with the finite line segment between two points P and Q. Specifically, we found its length (the distance between P and Q) and its midpoint. In this section, our focus will be on the *entire* line, and ways to describe it algebraically. Consider the generic situation below.



To give a sense of the 'steepness' of the line, we recall that we can compute the **slope** of the line as follows. (Read the character Δ as 'change in'.)

Equation A.3. The **slope** m of the line containing the points $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x},$$

provided $x_1 \neq x_0$, that is, $\Delta x \neq 0$.

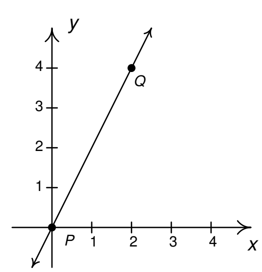
A couple of notes about Equation A.3 are in order. First, don't ask why we use the letter 'm' to represent slope. There are many explanations out there, but apparently no one really knows for sure.[1] Secondly, the stipulation $x_1 \neq x_0$ (or $\Delta x \neq 0$) ensures that we aren't trying to divide by zero. The reader is invited to pause to think about what is happening geometrically when the 'change in x' is 0; the anxious reader can skip along to the next example.

Example A.5.1. Find the slope of the line containing the following pairs of points, if it exists. Plot each pair of points and the line containing them.

1. $P(0,0)$, $Q(2,4)$
2. $P(-1,2)$, $Q(3,4)$
3. $P(-2,3)$, $Q(2,-3)$
4. $P(-3,2)$, $Q(4,2)$
5. $P(2,3)$, $Q(2,-1)$
6. $P(2,3)$, $Q(2.1,-1)$

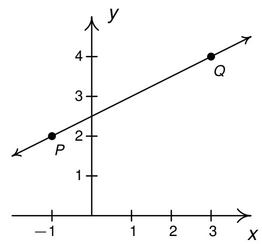
Solution. In each of these examples, we apply the slope formula, Equation A.3.

1. $$m = \frac{4-0}{2-0} = \frac{4}{2} = 2$$

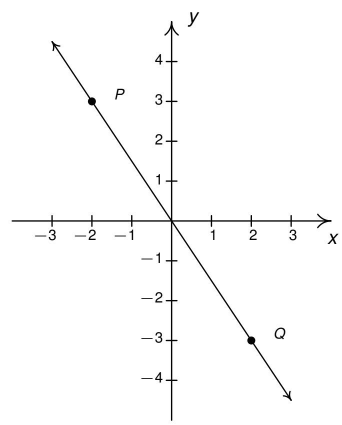


[1]See www.mathforum.org or www.mathworld.wolfram.com for discussions on this topic.

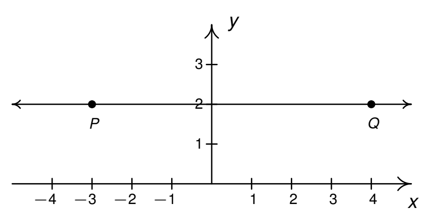
2. $m = \dfrac{4-2}{3-(-1)} = \dfrac{2}{4} = \dfrac{1}{2}$



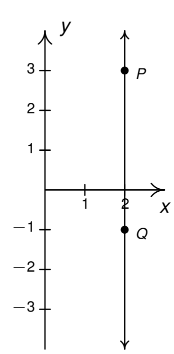
3. $m = \dfrac{-3-3}{2-(-2)} = \dfrac{-6}{4} = -\dfrac{3}{2}$



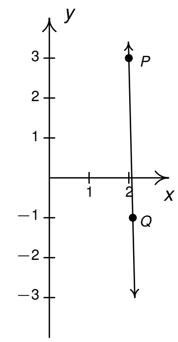
4. $m = \dfrac{2-2}{4-(-3)} = \dfrac{0}{7} = 0$



5. $m = \dfrac{-1-3}{2-2} = \dfrac{-4}{0}$, which is undefined



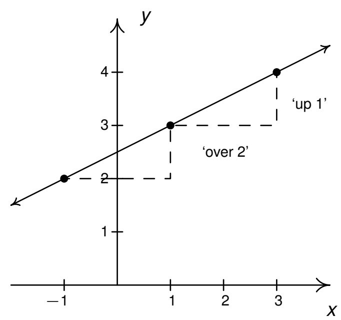
6. $m = \dfrac{-1-3}{2.1-2} = \dfrac{-4}{0.1} = -40$



□

A few comments about Example A.5.1 are in order. First, if the slope is positive then the resulting line is said to be 'increasing', meaning as we move from left to right,[2] the y-values are getting larger.[3] Similarly, if the slope is negative, we say the line is 'decreasing', since as we move from left to right, the y-values are getting smaller. A slope of 0 results in a horizontal line which we say is 'constant', since the y-values here remain unchanged as we move from left to right, and an undefined slope results in a vertical line.[4]

Second, the larger the slope is in absolute value, the steeper the line. You may recall from Intermediate Algebra that slope can be described as the ratio '$\frac{\text{rise}}{\text{run}}$'. For example, if the slope works out to be $\frac{1}{2}$, we can interpret this as a 'rise' of 1 unit upward for every 'run' of 2 units to the right:



In this way, we may view the slope as 'the **rate of change** of y with respect to x'. From the expression

$$m = \frac{\Delta y}{\Delta x}$$

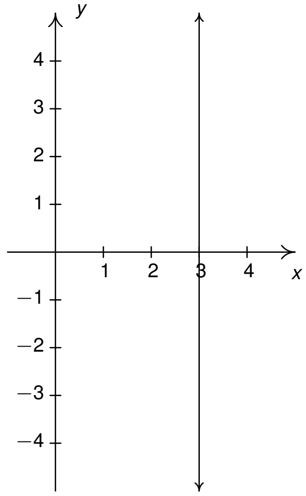
we get $\Delta y = m\Delta x$ so that the y-values change 'm' times as fast as the x-values. We'll have more to say about this concept in Section 1.2 when we explore applications of linear functions; presently, we will keep our attention focused on the analytic geometry of lines. To that end, our next task is to find algebraic equations that describe lines and we start with a discussion of vertical and horizontal lines.

[2]That is, as we increase the x-values . . .

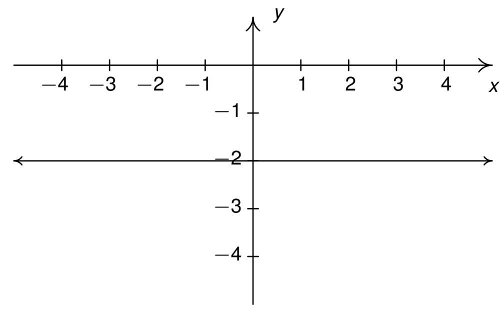
[3]We'll have more to say about this idea in Section 1.2.

[4]Some authors use the unfortunate moniker 'no slope' when a slope is undefined. It's easy to confuse the notions of 'no slope' with 'slope of 0'. For this reason, we will describe slopes of vertical lines as 'undefined'.

Consider the two lines shown below: V (for 'V'ertical Line) and H (for 'H'orizontal Line).



The line V



The line H

All of the points on the line V have an x-coordinate of 3. Conversely, any point with an x-coordinate of 3 lies on the line V. Said differently, the point (x, y) lies on V if and only if $x = 3$. Because of this, we say the equation $x = 3$ *describes* the line V, or, said differently, the *graph* of the equation $x = 3$ is the line V.

In Section 5.5, we'll spend a great deal of time talking about graphing equations. For now, it suffices to know that a graph of an equation is a plot of all of the points which make the equation true. So to graph $x = 3$, we plot all of the points (x, y) which satisfy $x = 3$ and this gives us our vertical line V.

Turning our attention to H, we note that every point on H has a y-coordinate of -2, and vice-versa. Hence the equation $y = -2$ describes the line H, or the graph of the equation $y = -2$ is H. In general:

Equation A.4. Equations of Vertical and Horizontal Lines

- The graph of the equation $x = a$ in the xy-plane is a **vertical line** through $(a, 0)$.
- The graph of the equation $y = b$ in the xy-plane is a **horizontal line** through $(0, b)$.

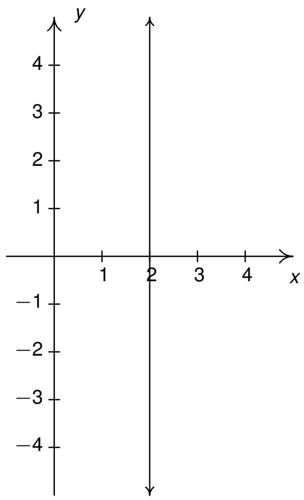
Of course, we may be working on axes which aren't labeled with the 'usual' x's and y's. In this case, we understand Equation A.4 to say 'horizontal axis label $= a$' describes a *vertical* line through $(a, 0)$ and 'vertical axis label $= b$' describes a *horizontal* line through $(0, b)$.

Example A.5.2.

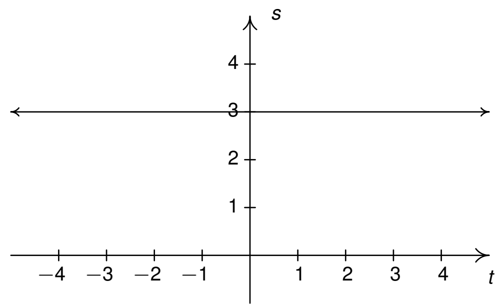
1. Graph the following equations in the xy-plane:

   (a) $y = 3$ (b) $x = -117$

2. Find the equation of each of the given lines.



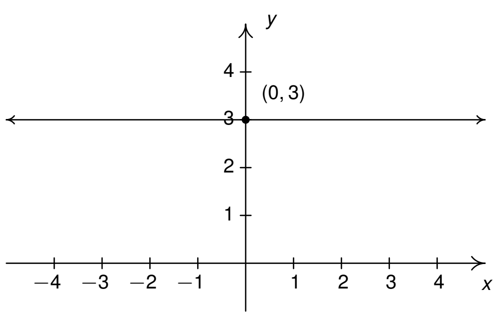
Line L_1



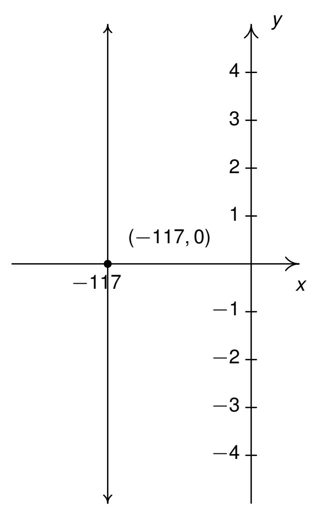
Line L_2

Solution.

1. Since we're in the familiar xy-plane, the graph of $y = 3$ is a horizontal line through $(0, 3)$, shown below on the left and the graph of $x = -117$ is a vertical line through $(-117, 0)$. We scale the x-axis differently than the y-axis to produce the graph below on the right.



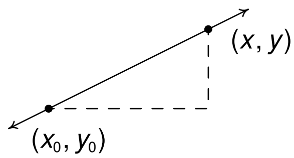
The line $y = 3$



The line $x = -117$

2. Since L_1 is a vertical line through $(2, 0)$, and the horizontal axis is labeled with 'x', the equation of L_1 is $x = 2$. Since L_2 is a horizontal line through $(0, 3)$ and the vertical axis is labeled as 's', the equation of this line is $s = 3$. □

Using the concept of slope, we can develop equations for the other varieties of lines. Suppose a line has a slope of m and contains the point (x_0, y_0). Suppose (x, y) is another point on the line, as indicated below.



Equation A.3 yields

$$\begin{aligned} m &= \frac{y - y_0}{x - x_0} \\ m(x - x_0) &= y - y_0 \\ y - y_0 &= m(x - x_0) \end{aligned}$$

which is known as the **point-slope form** of a line.

Equation A.5. The **point-slope form** of the line with slope m containing the point (x_0, y_0) is the equation

$$y - y_0 = m(x - x_0)$$

A few remarks about Equation A.5 are in order. First, note that if the slope $m = 0$, then the line is horizontal and Equation A.5 reduces to $y - y_0 = 0$ or $y = y_0$, as prescribed by Equation A.4.[5] Second, we may need to change the letters in Equation A.5 from 'x' and 'y' depending on the context, so while Equation A.5 should be committed to memory, it should be understood that 'x' refers to whichever variable is used to label the horizontal axis, and y refers to whichever variable is used to label the vertical axis. Lastly, while Equation A.5 is, by far, the easiest way to *construct* the equation of a line given a point and a slope, more often than not, the equation is solved for y and simplified into the form below.

Equation A.6. The **slope-intercept form** of the line with slope m and y-intercept $(0, b)$ is the equation

$$y = mx + b$$

Equation A.6 is probably[6] a familiar sight from Intermediate Algebra. You may recall from that class that the 'intercept' in 'slope-intercept' comes from the fact that this line 'intercepts' or crosses the y-axis at the point $(0, b)$.[7] If we set the slope, $m = 0$, we obtain $y = b$, the formula for Horizontal Lines first introduced in Equation A.4. Hence, any line which has a defined slope m can be represented in both point-slope and slope-intercept forms. The only exceptions are vertical lines.[8] There is one equation - the aptly named 'general form' - which describes every type of line and it is presented on the next page.

[5] Here we have y_0 as the constant whereas in the Equation we used the letter b. The form y = constant is what matters.
[6] Hopefully?
[7] We can verify this algebraically by setting $x = 0$ in the equation $y = mx + b$ and obtaining $y = b$.
[8] We'll have more to say about this in Section 1.2.

Equation A.7. Every line may be represented by an equation of the form $Ax + By = C$, where A, B and C are real numbers for which A and B aren't both zero. This is called *a* **general form** of the line.

Note the indefinite article 'a' in Equation A.7. The line $y = 5$ is a general form for the horizontal line through $(0, 5)$, but so are $3y = 15$ and $0.5y = 2.5$. The reader is left to ponder the use of the definite article 'the' in Equations A.5 and A.6. Regardless of *which* form the equation of a line takes, note that the variables involved are all raised to the first power.[9] For instance, there are no $\sqrt{x}$ terms, no y^2 terms or any variables appearing in denominators. Let's look at a few examples.

Example A.5.3.

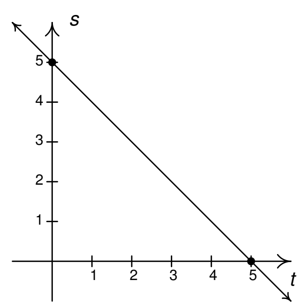
1. Graph the following equations in the xy-plane:

   (a) $y = 3x - 1$

   (b) $2x + 4y = 3$

2. Find the slope-intercept form of the line containing the points $(-1, 3)$ and $(2, 1)$.

3. Find the slope-intercept form of the equation of the line below:



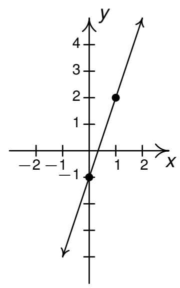
Solution.

1. To graph a line, we need just two points on that line. There are several ways to do this, and we showcase two of them here. For the first equation, we recognize that $y = 3x - 1$ is in slope-intercept form, $y = mx + b$, with $m = 3$ and $b = -1$. This immediately gives us one point on the graph – the y-intercept $(0, -1)$. From here, we use the slope $m = 3 = \frac{3}{1}$ and move one unit to the right and three units up, to obtain a second point on the line, $(1, 2)$. Connecting these points gives us the graph on the left at the top of the next page.

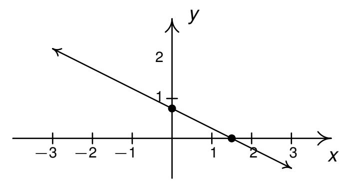
   The second equation, $2x + 4y = 3$, is a general form of a line. To get two points here, we choose 'convenient' values for one of the variables, and solve for the other variable. Choosing $x = 0$, for example, reduces $2x + 4y = 3$ to $4y = 3$, or $y = \frac{3}{4}$. This means the point $\left(0, \frac{3}{4}\right)$ is on the graph. Choosing $y = 0$ gives $2x = 3$, or $x = \frac{3}{2}$. This gives is a second point on the line, $\left(\frac{3}{2}, 0\right)$.[10] Our graph of $2x + 4y = 3$ is on the right at the top of the next page.

[9] Recall, $x = x^1$, $y = y^1$, etc.

[10] You may recall, that this is the x-intercept of the line.



$y = 3x - 1$



$2x + 4y = 3$

2. We'll assume we're using the familiar (x, y) axis labels and begin by finding the slope of the line using Equation A.3: $m = \frac{\Delta y}{\Delta x} = \frac{1-3}{2-(-1)} = -\frac{2}{3}$. Next, we substitute this result, along with one of the given points, into the point-slope equation of the line, Equation A.5. We have two options for the point (x_0, y_0). We'll use $(-1, 3)$ and leave it to the reader to check that using $(2, 1)$ results in the same equation. Substituting into the point-slope form of the line, we get

$$\begin{aligned}
y - y_0 &= m(x - x_0) \\
y - 3 &= -\frac{2}{3}(x - (-1)) \\
y - 3 &= -\frac{2}{3}(x + 1) \\
y - 3 &= -\frac{2}{3}x - \frac{2}{3} \\
y &= -\frac{2}{3}x - \frac{2}{3} + 3 \\
y &= -\frac{2}{3}x + \frac{7}{3}.
\end{aligned}$$

We can check our answer by showing that both $(-1, 3)$ and $(2, 1)$ are on the graph of $y = -\frac{2}{3}x + \frac{7}{3}$ algebraically by showing that the equation holds true when we substitute $x = -1$ and $y = 3$ and when $x = 2$ and $y = 1$.

3. From the graph, we see that the points $(0, 5)$ and $(5, 0)$ are on the line, so we may proceed as we did in the previous problem. Here, however, we use 't' in place of 'x' and 's' in place of 'y' in accordance to the axis labels given. We find the slope $m = \frac{\Delta s}{\Delta t} = \frac{0-5}{5-0} = -1$. As before, we have two points to choose from to substitute into the point-slope formula, and, as before, we'll select one of them, $(0, 5)$ and leave the computations with $(5, 0)$ to the reader.

$$\begin{aligned}
s - s_0 &= m(t - t_0) \\
s - 5 &= (-1)(t - 0) \\
s - 5 &= -t \\
s &= -t + 5.
\end{aligned}$$

As before we can check this line contains both points $(t, s) = (0, 5)$ and $(t, s) = (5, 0)$ algebraically. □

While every point on a line holds value and meaning,[11] we've reminded you of certain points, called 'intercepts,' which hold special enough significance to be singled out. Formally, we define these as follows.

Definition A.12. Given a graph of an equation in the xy-plane:

- A point on a graph which is also on the x-axis is called an ***x*-intercept** of the graph. To determine the x-intercept(s) of a graph, set $y = 0$ in the equation and solve for x.

  NOTE: x-intercepts always have the form: $(x_0, 0)$.

- A point on a graph which is also on the y-axis is called an ***y*-intercept** of the graph. To determine the y-intercept(s) of a graph, set $x = 0$ in the equation and solve for y.

  NOTE: y-intercepts always have the form: $(0, y_0)$.

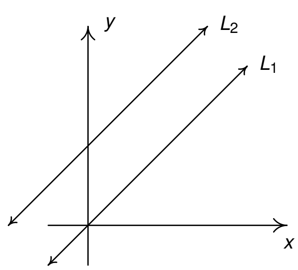
As usual, the labels of the axes in the problem will dictate the labels on the intercepts. If we're working in the vw-plane, for instance, there would be v- and w-intercepts.

The last little bit of analytic geometry we need to review about lines are the concepts of 'parallel' and 'perpendicular' lines. Parallel lines do not intersect,[12] and hence, parallel lines necessarily have the same slope. Perpendicular lines intersect at a right (90°) angle. The relationship between these slopes is somewhat more complicated, and is summarized below.

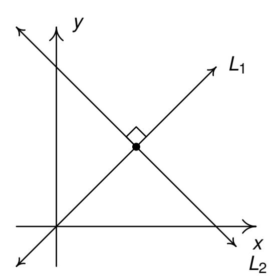
Theorem A.3. Suppose line L_1 has slope m_1 and line L_2 has slope m_2:

- L_1 and L_2 are parallel (written $L_1 \parallel L_2$) if and only if $m_1 = m_2$.
- If $m_1 \neq 0$ and $m_2 \neq 0$ then L_1 and L_2 are perpendicular (written $L_1 \perp L_2$) if and only if $m_1 m_2 = -1$.

  NOTE: $m_1 m_2 = -1$ is equivalent to $m_2 = -\dfrac{1}{m_1}$, so that perpendicular lines have slopes which are 'opposite reciprocals' of one another.



$L_1 \parallel L_2$, $m_1 = m_2$



$L_1 \perp L_2$, $m_1 m_2 = -1$

A few remarks about Theorem A.3 are in order. First off, the theorem assumes that the slopes of the lines exist. The reader is encouraged to think about the case when one (or both) of the slopes don't exist. Along those same lines, the reader is encouraged to think about why the stipulations $m_1 \neq 0$ and $m_2 \neq 0$ appear

[11] Lines missing points - even one - usually belie some algebraic pathology which we'll discuss in more detail in Chapter 3.
[12] Well, at least in Euclidean Geometry . . .

in the statement regarding slopes of perpendicular lines, and what happens in this case as well. (Think geometrically!) In Exercise 41, you'll prove the assertion about the slopes of perpendicular lines. For now, we accept it as true and use it in the following example.

Example A.5.4. For line $y = 2x - 1$ and the point $(3, 4)$, find:

1. the equation of the line parallel to the given line which contains the given point.
2. the equation of the line perpendicular to the given line which contains the given point. Check your answers by graphing them, along with the original line, using a graphing utility.

Solution.

1. Since $y = 2x - 1$ is already in slope-intercept form, we have the slope $m = 2$. To find the line parallel to this line containing $(3, 4)$, we use the point-slope form with $m = 2$ to get:

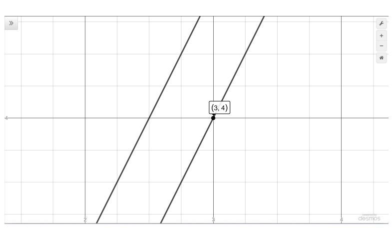
$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 4 &= 2(x - 3) \\ y - 4 &= 2x - 6 \\ y &= 2x - 2 \end{aligned}$$

   Algebraically, we can verify that the slope is indeed 2 and that when $x = 3$ we get $y = 4$. Using a graphing utility with a window centered at the point $(3, 4)$, we graph both $y = 2x - 1$ and $y = 2x - 2$ below on the left and observe that they appear to be parallel.

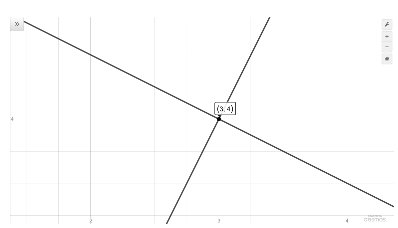
2. To find the line perpendicular to $y = 2x - 1$ containing $(3, 4)$, we use the slope $m = -\frac{1}{2}$ in the point-slope formula:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 4 &= -\frac{1}{2}(x - 3) \\ y - 4 &= -\frac{1}{2}x + \frac{3}{2} \\ y &= -\frac{1}{2}x + \frac{11}{2} \end{aligned}$$

   Algebraically, we check that the slope is $m = -\frac{1}{2}$ and when $x = 3$ we get $y = 4$ as required. When checking using our graphing utility, we centered the viewing window at $(3, 4)$ and had to 'square' it, removing its default aspect ratio, to truly observe the perpendicular nature of the lines.



$y = 2x - 1$ and
$\boldsymbol{y = 2x - 2}$

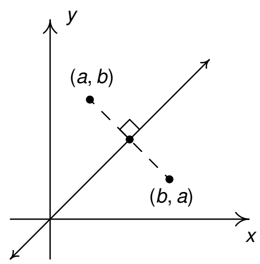


$y = 2x - 1$ and
$\boldsymbol{y = -\frac{1}{2}x + \frac{11}{2}}$

Our last example with lines sets up a fourth kind of symmetry which will be revisited in Section 5.6.

Example A.5.5. Show that the points (a, b) and (b, a) in the xy-plane are symmetric about the line $y = x$.

Solution. If $a = b$ then $(a, b) = (a, a) = (b, a)$ and this point lies on the line $y = x$.[13] To prove the claim for the case when $a \neq b$, we will show that the line $y = x$ is a perpendicular bisector of the line segment with endpoints (a, b) and (b, a), as illustrated below.



To show the 'perpendicular' part, we first note the slope of the line containing (a, b) and (b, a) is

$$m = \frac{a-b}{b-a} = \frac{\cancel{(a-b)}}{-\cancel{(b-a)}} = -1$$

Since the slope of $y = x = 1x + 0$ is $m = 1$, we see that the slopes of these two lines are negative reciprocals. Hence, $y = x$ and the line segment with endpoints (a, b) and (b, a) are perpendicular. For the 'bisector' part, we use Equation A.2 to find the midpoint of the line segment with endpoints (a, b) and (b, a):

$$\begin{aligned} M &= \left(\frac{a+b}{2}, \frac{b+a}{2}\right) \\ &= \left(\frac{a+b}{2}, \frac{a+b}{2}\right) \end{aligned}$$

Since the x and y coordinates of this point are the same, we find that the midpoint lies on the line $y = x$. $\square$

[13] Please ask your instructor if lying on the line counts as being 'symmetric about the line' or not.

A.5.1 Exercises

In Exercises 1 - 10, find both the point-slope form and the slope-intercept form of the line with the given slope which passes through the given point.

1. $m = 3, \quad P(3, -1)$
2. $m = -2, \quad P(-5, 8)$
3. $m = -1, \quad P(-7, -1)$
4. $m = \frac{2}{3}, \quad P(-2, 1)$
5. $m = -\frac{1}{5}, \quad P(10, 4)$
6. $m = \frac{1}{7}, \quad P(-1, 4)$
7. $m = 0, \quad P(3, 117)$
8. $m = -\sqrt{2}, \quad P(0, -3)$
9. $m = -5, \quad P(\sqrt{3}, 2\sqrt{3})$
10. $m = 678, \quad P(-1, -12)$

In Exercises 11 - 20, find the slope-intercept form of the line which passes through the given points.

11. $P(0, 0), \ Q(-3, 5)$
12. $P(-1, -2), \ Q(3, -2)$
13. $P(5, 0), \ Q(0, -8)$
14. $P(3, -5), \ Q(7, 4)$
15. $P(-1, 5), \ Q(7, 5)$
16. $P(4, -8), \ Q(5, -8)$
17. $P\left(\frac{1}{2}, \frac{3}{4}\right), \ Q\left(\frac{5}{2}, -\frac{7}{4}\right)$
18. $P\left(\frac{2}{3}, \frac{7}{2}\right), \ Q\left(-\frac{1}{3}, \frac{3}{2}\right)$
19. $P\left(\sqrt{2}, -\sqrt{2}\right), \ Q\left(-\sqrt{2}, \sqrt{2}\right)$
20. $P\left(-\sqrt{3}, -1\right), \ Q\left(\sqrt{3}, 1\right)$

In Exercises 21 - 26, graph the line. Find the slope, y-intercept and x-intercept, if any exist.

21. $y = 2x - 1$
22. $y = 3 - x$
23. $y = 3$
24. $y = 0$
25. $y = \frac{2}{3}x + \frac{1}{3}$
26. $y = \dfrac{1 - x}{2}$
27. Graph $3v + 2w = 6$ on both the vw- and wv-axes. What characteristics to both graphs share? What's different?
28. Find all of the points on the line $y = 2x + 1$ which are 4 units from the point $(-1, 3)$.

In Exercises 29 - 34, you are given a line and a point which is not on that line. Find the line parallel to the given line which passes through the given point.

29. $y = 3x + 2,\ P(0,0)$

30. $y = -6x + 5,\ P(3,2)$

31. $y = \frac{2}{3}x - 7,\ P(6,0)$

32. $y = \dfrac{4-x}{3},\ P(1,-1)$

33. $y = 6,\ P(3,-2)$

34. $x = 1,\ P(-5,0)$

In Exercises 35 - 40, you are given a line and a point which is not on that line. Find the line perpendicular to the given line which passes through the given point.

35. $y = \frac{1}{3}x + 2,\ P(0,0)$

36. $y = -6x + 5,\ P(3,2)$

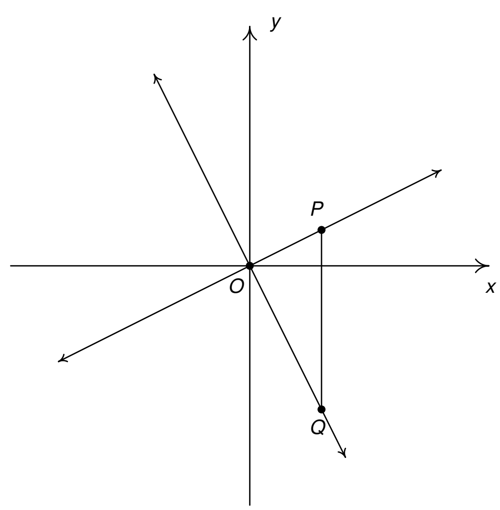
37. $y = \frac{2}{3}x - 7,\ P(6,0)$

38. $y = \dfrac{4-x}{3},\ P(1,-1)$

39. $y = 6,\ P(3,-2)$

40. $x = 1,\ P(-5,0)$

41. We shall now prove that $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ if and only if $m_1 \cdot m_2 = -1$. To make our lives easier we shall assume that $m_1 > 0$ and $m_2 < 0$. We can also "move" the lines so that their point of intersection is the origin without messing things up, so we'll assume $b_1 = b_2 = 0$. (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points $O(0,0)$, $P(1,m_1)$ and $Q(1,m_2)$ gives us the following set up.



The line $y = m_1x$ will be perpendicular to the line $y = m_2x$ if and only if $\triangle OPQ$ is a right triangle. Let d_1 be the distance from O to P, let d_2 be the distance from O to Q and let d_3 be the distance from P to Q. Use the Pythagorean Theorem to show that $\triangle OPQ$ is a right triangle if and only if $m_1 \cdot m_2 = -1$ by showing $d_1^2 + d_2^2 = d_3^2$ if and only if $m_1 \cdot m_2 = -1$.

A.5.2 Answers

1. $y + 1 = 3(x - 3)$
   $y = 3x - 10$

2. $y - 8 = -2(x + 5)$
   $y = -2x - 2$

3. $y + 1 = -(x + 7)$
   $y = -x - 8$

4. $y - 1 = \frac{2}{3}(x + 2)$
   $y = \frac{2}{3}x + \frac{7}{3}$

5. $y - 4 = -\frac{1}{5}(x - 10)$
   $y = -\frac{1}{5}x + 6$

6. $y - 4 = \frac{1}{7}(x + 1)$
   $y = \frac{1}{7}x + \frac{29}{7}$

7. $y - 117 = 0$
   $y = 117$

8. $y + 3 = -\sqrt{2}(x - 0)$
   $y = -\sqrt{2}x - 3$

9. $y - 2\sqrt{3} = -5(x - \sqrt{3})$
   $y = -5x + 7\sqrt{3}$

10. $y + 12 = 678(x + 1)$
    $y = 678x + 666$

11. $y = -\frac{5}{3}x$

12. $y = -2$

13. $y = \frac{8}{5}x - 8$

14. $y = \frac{9}{4}x - \frac{47}{4}$

15. $y = 5$

16. $y = -8$

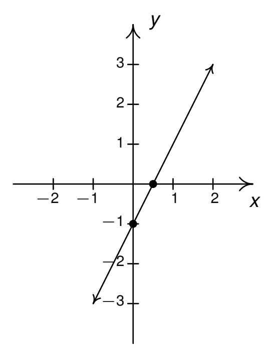
17. $y = -\frac{5}{4}x + \frac{11}{8}$

18. $y = 2x + \frac{13}{6}$

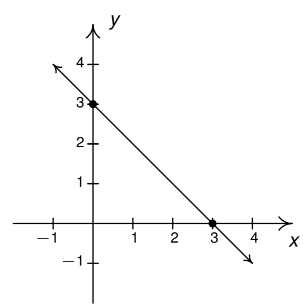
19. $y = -x$

20. $y = \frac{\sqrt{3}}{3}x$

21. $y = 2x - 1$
    slope: $m = 2$
    y-intercept: $(0, -1)$
    x-intercept: $\left(\frac{1}{2}, 0\right)$



22. $y = 3 - x$
    slope: $m = -1$
    y-intercept: $(0, 3)$
    x-intercept: $(3, 0)$

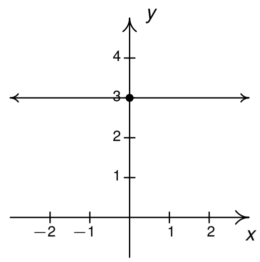


23. $y = 3$

    slope: $m = 0$

    y-intercept: $(0, 3)$

    x-intercept: none

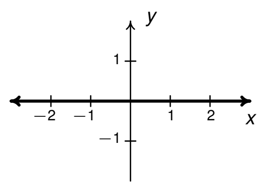


24. $y = 0$

    slope: $m = 0$

    y-intercept: $(0, 0)$

    x-intercept: $\{(x, 0) \mid x \text{ is a real number}\}$

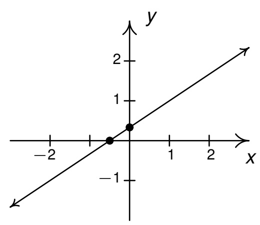


25. $y = \frac{2}{3}x + \frac{1}{3}$

    slope: $m = \frac{2}{3}$

    y-intercept: $\left(0, \frac{1}{3}\right)$

    x-intercept: $\left(-\frac{1}{2}, 0\right)$

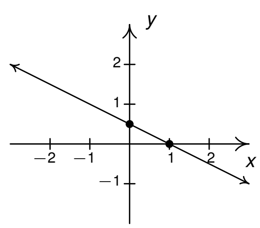


26. $y = \dfrac{1 - x}{2}$

    slope: $m = -\frac{1}{2}$

    y-intercept: $\left(0, \frac{1}{2}\right)$

    x-intercept: $(1, 0)$

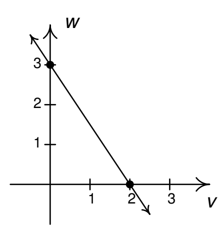


27. $w = -\frac{3}{2}v + 3$

    slope: $m = -\frac{3}{2}$

    w-intercept: $(0, 3)$

    v-intercept: $(2, 0)$

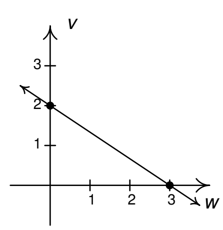


    $v = -\frac{2}{3}w + 2$

    slope: $m = -\frac{2}{3}$

    v-intercept: $(0, 2)$

    w-intercept: $(3, 0)$



28. $(-1, -1)$ and $\left(\frac{11}{5}, \frac{27}{5}\right)$

29. $y = 3x$
30. $y = -6x + 20$
31. $y = \frac{2}{3}x - 4$
32. $y = -\frac{1}{3}x - \frac{2}{3}$
33. $y = -2$
34. $x = -5$
35. $y = -3x$
36. $y = \frac{1}{6}x + \frac{3}{2}$
37. $y = -\frac{3}{2}x + 9$
38. $y = 3x - 4$
39. $x = 3$
40. $y = 0$

A.6 Systems of Two Linear Equations in Two Unknowns

This section of the Appendix combines ideas from Section A.4 and A.5 so that we can start to solve systems of linear equations. Before we get ahead of ourselves, let's review a few definitions.

Definition A.13. A **linear equation in two variables** is an equation of the form $a_1x + a_2y = c$ where a_1, a_2 and c are real numbers and at least one of a_1 and a_2 is nonzero.

For reasons which will become clear when you study Chapter 9, we are using subscripts in Definition A.13 to indicate different, but fixed, real numbers and those subscripts have no mathematical meaning beyond that. For example, $3x - \frac{y}{2} = 0.1$ is a linear equation in two variables with $a_1 = 3$, $a_2 = -\frac{1}{2}$ and $c = 0.1$. We can also consider $x = 5$ to be a linear equation in two variables[1] by identifying $a_1 = 1$, $a_2 = 0$, and $c = 5$.

If a_1 and a_2 are both 0, then depending on c, we get either an equation which is *always* true, called an **identity**, or an equation which is *never* true, called a **contradiction**. (If $c = 0$, then we get $0 = 0$, which is always true. If $c \neq 0$, then we'd have $0 \neq 0$, which is never true.) Even though identities and contradictions have a large role to play throughout Chapter 9, we do not consider them linear equations. The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. The reader should consider why these do not satisfy Definition A.13.

We know from our work is Sections A.5 that the graphs of linear equations are lines. If we couple two or more linear equations together, in effect to find the points of intersection of two or more lines, we obtain a **system of linear equations in two variables**. Our first example explores the basic techniques for solving these systems. Remember - if we are looking for points in the plane, then both the x and y values are important. This is a key distinction between solving one equation and solving a *system* of equations.

Example A.6.1. Solve the following systems of equations. Check your answer algebraically and graphically. (Said another way, make sure both x and y are correct!)

1. $\left\{ \begin{array}{rcl} 2x - y & = & 1 \\ y & = & 3 \end{array} \right.$
2. $\left\{ \begin{array}{rcr} 3x + 4y & = & -2 \\ -3x - y & = & 5 \end{array} \right.$
3. $\left\{ \begin{array}{rcl} \frac{x}{3} - \frac{4y}{5} & = & \frac{7}{5} \\ \frac{2x}{9} + \frac{y}{3} & = & \frac{1}{2} \end{array} \right.$
4. $\left\{ \begin{array}{rcl} 2x - 4y & = & 6 \\ 3x - 6y & = & 9 \end{array} \right.$
5. $\left\{ \begin{array}{rcr} 6x + 3y & = & 9 \\ 4x + 2y & = & 12 \end{array} \right.$
6. $\left\{ \begin{array}{rcr} x - y & = & 0 \\ x + y & = & 2 \\ -2x + y & = & -2 \end{array} \right.$

Solution.

1. Our first system is nearly solved for us. The second equation tells us that $y = 3$. To find the corresponding value of x, we **substitute** this value for y into the the first equation to obtain $2x - 3 = 1$, so that $x = 2$. Our solution to the system is $(2, 3)$. To check this algebraically, we substitute $x = 2$ and $y = 3$ into each equation and see that they are satisfied. We see $2(2) - 3 = 1$, and $3 = 3$, as

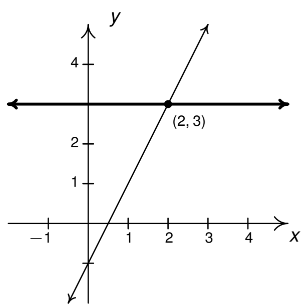
[1] Critics may argue that $x = 5$ is clearly an equation in one variable. It can also be considered an equation in 117 variables with the coefficients of 116 variables set to 0. As with many conventions in Mathematics, the context will clarify the situation.

required. To check our answer graphically, we graph the lines $2x - y = 1$ and $y = 3$ and verify that they intersect at $(2, 3)$.

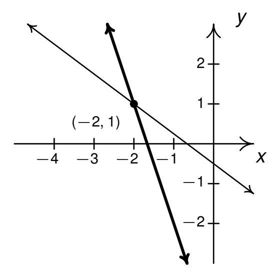
2. To solve the second system, we use the **addition** method to **eliminate** the variable x. We take the two equations as given and 'add equals to equals' to obtain

$$\begin{array}{rrcr} & 3x + 4y & = & -2 \\ + & (-3x - y & = & 5) \\ \hline & 3y & = & 3 \end{array}$$

This gives us $y = 1$. We now substitute $y = 1$ into either of the two equations, say $-3x - y = 5$, to get $-3x - 1 = 5$ so that $x = -2$. Our solution is $(-2, 1)$. Substituting $x = -2$ and $y = 1$ into the first equation gives $3(-2) + 4(1) = -2$, which is true, and, likewise, when we check $(-2, 1)$ in the second equation, we get $-3(-2) - 1 = 5$, which is also true. Geometrically, the lines $3x + 4y = -2$ and $-3x - y = 5$ intersect at $(-2, 1)$.



$2x - y = 1$
$\mathbf{y = 3}$



$3x + 4y = -2$
$\mathbf{-3x - y = 5}$

3. The equations in the third system are more approachable if we clear denominators. We multiply both sides of the first equation by 15 and both sides of the second equation by 18 to obtain the kinder, gentler system

$$\left\{ \begin{array}{rcr} 5x - 12y & = & 21 \\ 4x + 6y & = & 9 \end{array} \right.$$

Adding these two equations directly fails to eliminate either of the variables, but we note that if we multiply the first equation by 4 and the second by -5, we will be in a position to eliminate the x term

$$\begin{array}{rrcr} & 20x - 48y & = & 84 \\ + & (-20x - 30y & = & -45) \\ \hline & -78y & = & 39 \end{array}$$

From this we get $y = -\frac{1}{2}$. We can temporarily avoid too much unpleasantness by choosing to substitute $y = -\frac{1}{2}$ into one of the equivalent equations we found by clearing denominators, say into

$5x - 12y = 21$. We get $5x + 6 = 21$ which gives $x = 3$. Our answer is $\left(3, -\frac{1}{2}\right)$. At this point, we have no choice – in order to check an answer algebraically, we must see if the answer satisfies both of the *original* equations, so we substitute $x = 3$ and $y = -\frac{1}{2}$ into both $\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$ and $\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}$. We leave it to the reader to verify that the solution is correct. Graphing both of the lines involved with considerable care yields an intersection point of $\left(3, -\frac{1}{2}\right)$. (The picture is on the next page.)

4. An eerie calm settles over us as we cautiously approach our fourth system. Do its friendly integer coefficients belie something more sinister? We note that if we multiply both sides of the first equation by 3 and both sides of the second equation by -2, we are ready to eliminate the x

$$\begin{array}{rrcr} & 6x - 12y & = & 18 \\ + & (-6x + 12y & = & -18) \\ \hline & 0 & = & 0 \end{array}$$

We eliminated not only the x, but the y as well and we are left with the identity $0 = 0$. This means that these two different linear equations are, in fact, equivalent. In other words, if an ordered pair (x, y) satisfies the equation $2x - 4y = 6$, it *automatically* satisfies the equation $3x - 6y = 9$.

This system has infinitely many solutions and one way to describe the solution set to this system is to use the roster method[2] and write $\{(x, y) \mid 2x - 4y = 6\}$. While this is correct (and corresponds exactly to what's happening graphically, as we shall see shortly), we take this opportunity to introduce the notion of a **parametric solution to a system**.

Our first step is to solve $2x - 4y = 6$ for one of the variables, say $y = \frac{1}{2}x - \frac{3}{2}$. For each value of x, the formula $y = \frac{1}{2}x - \frac{3}{2}$ determines the corresponding y-value of a solution. Since we have no restriction on x, it is called a **free variable**. We let $x = t$, a so-called 'parameter', and get $y = \frac{1}{2}t - \frac{3}{2}$. Our set of solutions can then be described as $\left\{\left(t, \frac{1}{2}t - \frac{3}{2}\right) \mid -\infty < t < \infty\right\}$.[3]

For specific values of t, we can generate solutions. For example, $t = 0$ gives us the solution $\left(0, -\frac{3}{2}\right)$; $t = 117$ gives us $(117, 57)$, and while we can check that each of these particular solutions satisfy both equations, the question is how do we check our general answer algebraically? Same as always.

We claim that for any real number t, the pair $\left(t, \frac{1}{2}t - \frac{3}{2}\right)$ satisfies both equations. Substituting $x = t$ and $y = \frac{1}{2}t - \frac{3}{2}$ into $2x - 4y = 6$ gives $2t - 4\left(\frac{1}{2}t - \frac{3}{2}\right) = 6$. Simplifying, we get $2t - 2t + 6 = 6$, which is always true. Similarly, when we make these substitutions in the equation $3x - 6y = 9$, we get $3t - 6\left(\frac{1}{2}t - \frac{3}{2}\right) = 9$ which reduces to $3t - 3t + 9 = 9$, so it checks out, too.

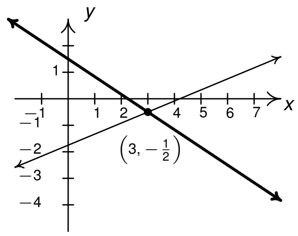
Geometrically, $2x - 4y = 6$ and $3x - 6y = 9$ are the same line, which means that they intersect at every point on their graphs. The reader is encouraged to think about how our parametric solution says exactly that.

---

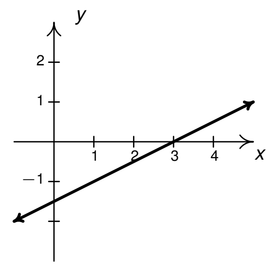
[2]See Section A.1 for a review of this.

[3]Note that we could have just as easily chosen to solve $2x - 4y = 6$ for x to obtain $x = 2y + 3$. Letting y be the parameter t, we have that for any value of t, $x = 2t + 3$, which gives $\{(2t + 3, t) \mid -\infty < t < \infty\}$. There is no one correct way to parameterize the solution set, which is why it is always best to check your answer.

The picture for this system is shown below on the right while the picture for the previous example is shown on the left.



$\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$
$\mathbf{\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}}$



$2x - 4y = 6$
$\mathbf{3x - 6y = 9}$
(Same line.)

5. Multiplying both sides of the first equation by 2 and the both sides of the second equation by -3, we set the stage to eliminate x

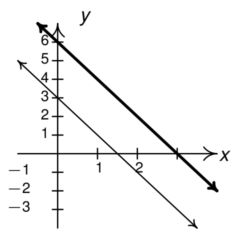
$$\begin{array}{rrcr} & 12x + 6y & = & 18 \\ + & (-12x - 6y & = & -36) \\ \hline & 0 & = & -18 \end{array}$$

As in the previous example, both x and y dropped out of the equation, but we are left with an irrevocable contradiction, $0 = -18$. This tells us that it is impossible to find a pair (x, y) which satisfies both equations; in other words, the system has no solution. Graphically, the lines $6x + 3y = 9$ and $4x + 2y = 12$ are distinct and parallel, so they do not intersect.

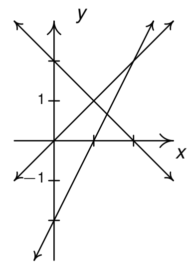
6. We can begin to solve our last system by adding the first two equations

$$\begin{array}{rrcr} & x - y & = & 0 \\ + & (x + y & = & 2) \\ \hline & 2x & = & 2 \end{array}$$

which gives $x = 1$. Substituting this into the first equation gives $1 - y = 0$ so that $y = 1$. We seem to have determined a solution to our system, $(1, 1)$. While this checks in the first two equations, when we substitute $x = 1$ and $y = 1$ into the third equation, we get $-2(1) + (1) = -2$ which simplifies to the contradiction $-1 = -2$. Graphing the lines $x - y = 0$, $x + y = 2$, and $-2x + y = -2$, we see that the first two lines do, in fact, intersect at $(1, 1)$, however, all three lines never intersect at the same point simultaneously, which is what is required if a solution to the system is to be found.



$6x + 3y = 9$
$\mathbf{4x + 2y = 12}$



$y - x = 0$
$y + x = 2$
$-2x + y = -2$

□

A few remarks about Example A.6.1 are in order. Notice that some of the systems of linear equations had solutions while others did not. Those which have solutions are called **consistent**, those with no solution are called **inconsistent**. We also distinguish between the two different types of behavior among consistent systems. Those which admit free variables are called **dependent** and those with no free variables are called **independent**.[4]

Using this new vocabulary, we classify numbers 1, 2 and 3 in Example A.6.1 as consistent independent systems, number 4 is consistent dependent, and numbers 5 and 6 are inconsistent.[5] The system in 6 above is called **overdetermined**, since we have more equations than variables.[6] Not surprisingly, a system with more variables than equations is called **underdetermined**. While the system in number 6 above is overdetermined and inconsistent, there exist overdetermined consistent systems (both dependent and independent) and we leave it to the reader to think about what is happening algebraically and geometrically in these cases. Likewise, there are both consistent and inconsistent underdetermined systems,[7] but a consistent underdetermined system of linear equations is necessarily dependent.[8]

We end this section with a story problem. It is an example of a classic "mixture" problem and should be familiar to most readers. The basic goal here is to create two equations: one which represents

$$\text{stuff} + \text{other stuff} = \text{total stuff}$$

and the other which represents

$$\text{value of stuff} + \text{value of other stuff} = \text{value of total stuff.}$$

[4] In the case of systems of linear equations, regardless of the number of equations or variables, consistent independent systems have exactly one solution. The reader is encouraged to think about why this is the case for linear equations in two variables. Hint: think geometrically.

[5] The adjectives 'dependent' and 'independent' apply only to *consistent* systems – they describe the *type* of solutions. Is there a free variable (dependent) or not (independent)?

[6] If we think if each variable being an unknown quantity, then ostensibly, to recover two unknown quantities, we need two pieces of information - i.e., two equations. Having more than two equations suggests we have more information than necessary to determine the values of the unknowns. While this is not necessarily the case, it does explain the choice of terminology 'overdetermined'.

[7] We need more than two variables to give an example of the latter.

[8] Again, experience with systems with more variables helps to see this here, as does a solid course in Linear Algebra.

Example A.6.2. The Dude-Bros want to create a highly caffeinated, yet still drinkable, fruit punch for their annual "Disturb the Neighbors BBQ and Dance Competition". They plan to add Sasquatch Sweat™ Energy Drink, which has 100 mg. of caffeine per fluid ounce, to Frooty Giggle Delight™, which has only 3 mg. of caffeine per fluid ounce. How much of each component is required to make 5 gallons[9] of a fruit punch that has 80 mg. of caffeine per fluid ounce.

Solution. Let S stand for the number of fluid ounces of Sasquatch Sweat™ Energy Drink and let F be the number of fluid ounces of Frooty Giggle Delight™ that will be added together. The goal is to make 5 gallons and there are 128 fluid ounces per gallon so the first equation is

$$S + F = 640.$$

That equation describes "stuff + other stuff = total stuff" measured in fluid ounces. Now we need to consider the value of the stuff - in this case we need to see how much caffeine is being contributed by each component. Each fluid ounce of Sasquatch Sweat™ contains 100 mg. of caffeine so S fluid ounces would contain $100S$ mg./ of caffeine.

Similarly, the F fluid ounces of Frooty Giggle Delight™ add $3F$ mg. of caffeine to the total mixture. Thus when we go to express "value of stuff + value of other stuff = value of total stuff" we need to figure out how much caffeine is supposed to be in the end product. Well, the goal was 5 gallons of punch that had 80 mg. of caffeine per fluid ounce so the Dude-Bros need to end up with $5 * 128 * 80 = 51200$ mg. of caffeine when they're done. Hence the second is equation is

$$100S + 3F = 51200.$$

By turning the first equation into $F = 640 - S$ and substituting that into the second equation we get

$$100S + 3(640 - S) = 51200$$

which yields $S = \frac{49280}{97} \approx 508.04$ fluid ounces. Back-substituting this value of S into the first equation gives us $F = \frac{12800}{97} \approx 131.96$ fluid ounces.

The reader should take the time to verify that $S = \frac{49280}{97}$ and $F = \frac{12800}{97}$ do indeed satisfy both equations and thus are the solution to the problem. Those are fairly unattractive numbers so we end this example by discussing a way to verify an approximate answer which is *reasonable* without having to fight with fractions. Round S down to 508 and round F up to 132. Clearly $508 + 132 = 640$ so the first equation is still satisfied. Notice that $100 * 508 + 3 * 132 = 51196$ which is really close to 51200. Thus the second equation is nearly satisfied which means the values $S = 508$ and $F = 132$, while not precise, are reasonable.[10] □

[9] Warning: unit conversion ahead!

[10] Just be careful here - sometimes "close enough for the Dude-Bros" is not good enough for your Professor!

A.6.1 Exercises

In Exercises 1 - 8, solve the given system using substitution and/or elimination. Classify each system as consistent independent, consistent dependent, or inconsistent. Check your answers both algebraically and graphically.

1. $\left\{ \begin{array}{rcl} x + 2y & = & 5 \\ x & = & 6 \end{array} \right.$

2. $\left\{ \begin{array}{rcl} 2y - 3x & = & 1 \\ y & = & -3 \end{array} \right.$

3. $\left\{ \begin{array}{rcl} \frac{x+2y}{4} & = & -5 \\ \frac{3x-y}{2} & = & 1 \end{array} \right.$

4. $\left\{ \begin{array}{rcl} \frac{2}{3}x - \frac{1}{5}y & = & 3 \\ \frac{1}{2}x + \frac{3}{4}y & = & 1 \end{array} \right.$

5. $\left\{ \begin{array}{rcl} \frac{1}{2}x - \frac{1}{3}y & = & -1 \\ 2y - 3x & = & 6 \end{array} \right.$

6. $\left\{ \begin{array}{rcl} x + 4y & = & 6 \\ \frac{1}{12}x + \frac{1}{3}y & = & \frac{1}{2} \end{array} \right.$

7. $\left\{ \begin{array}{rcl} 3y - \frac{3}{2}x & = & -\frac{15}{2} \\ \frac{1}{2}x - y & = & \frac{3}{2} \end{array} \right.$

8. $\left\{ \begin{array}{rcl} \frac{5}{6}x + \frac{5}{3}y & = & -\frac{7}{3} \\ -\frac{10}{3}x - \frac{20}{3}y & = & 10 \end{array} \right.$

9. A local buffet charges \$7.50 per person for the basic buffet and \$9.25 for the deluxe buffet (which includes crab legs.) If 27 diners went out to eat and the total bill was \$227.00 before taxes, how many chose the basic buffet and how many chose the deluxe buffet?

10. At The Old Home Fill'er Up and Keep on a-Truckin' Cafe, Mavis mixes two different types of coffee beans to produce a house blend. The first type costs \$3 per pound and the second costs \$8 per pound. How much of each type does Mavis use to make 50 pounds of a blend which costs \$6 per pound?

11. Skippy has a total of \$10,000 to split between two investments. One account offers 3% simple interest, and the other account offers 8% simple interest. For tax reasons, he can only earn \$500 in interest the entire year. How much money should Skippy invest in each account to earn \$500 in interest for the year?

12. A 10% salt solution is to be mixed with pure water to produce 75 gallons of a 3% salt solution. How much of each are needed?

13. This exercise is a follow-up to Example A.6.2. Work with your classmates to explain why mixing 4 gallons of Sasquatch Sweat™ Energy Drink and 1 gallon of Frooty Giggle Delight™ would also produce a mixture that was “close enough for the Dude-Bros”.

A.6.2 Answers

1. Consistent independent
   Solution $\left(6, -\frac{1}{2}\right)$

2. Consistent independent
   Solution $\left(-\frac{7}{3}, -3\right)$

3. Consistent independent
   Solution $\left(-\frac{16}{7}, -\frac{62}{7}\right)$

4. Consistent independent
   Solution $\left(\frac{49}{12}, -\frac{25}{18}\right)$

5. Consistent dependent
   Solution $\left(t, \frac{3}{2}t + 3\right)$
   for all real numbers t

6. Consistent dependent
   Solution $(6 - 4t, t)$
   for all real numbers t

7. Inconsistent
   No solution

8. Inconsistent
   No solution

9. 13 chose the basic buffet and 14 chose the deluxe buffet.

10. Mavis needs 20 pounds of \$3 per pound coffee and 30 pounds of \$8 per pound coffee.

11. Skippy needs to invest \$6000 in the 3% account and \$4000 in the 8% account.

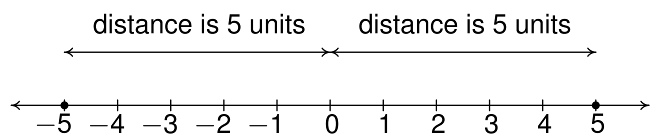
12. 22.5 gallons of the 10% solution and 52.5 gallons of pure water.

A.7 Absolute Value Equations and Inequalities

In this section, we review some of the basic concepts involving the absolute value of a real number x. There are a few different ways to define absolute value and in this section we choose the following definition. (Absolute value will be revisited in much greater depth in Section 1.3 where we present what one can think of as the "precise" definition.)

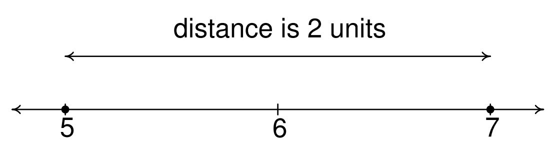
Definition A.14. Absolute Value as Distance: For every real number x, the **absolute value** of x, denoted $|x|$, is the distance between x and 0 on the number line. More generally, if x and c are real numbers, $|x - c|$ is the distance between the numbers x and c on the number line.

For example, $|5| = 5$ and $|-5| = 5$, since each is 5 units from 0 on the number line:



Graphically why $|-5| = 5$ and $|5| = 5$

Computationally, the absolute value 'makes negative numbers positive', though we need to be a little cautious with this description. While $|-7| = 7$, $|5 - 7| \neq 5 + 7$. The absolute value acts as a grouping symbol, so $|5 - 7| = |-2| = 2$, which makes sense since 5 and 7 are two units away from each other on the number line:



Graphically why $|5 - 7| = 2$

We list some of the operational properties of absolute value below.

Theorem A.4. Properties of Absolute Value: Let a and b be real numbers and let n be an integer.[a]

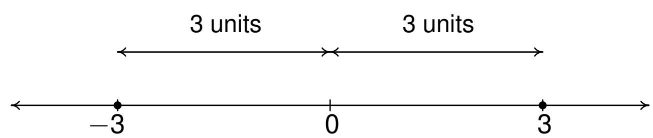
- **Product Rule:** $|ab| = |a||b|$
- **Power Rule:** $|a^n| = |a|^n$ whenever a^n is defined
- **Quotient Rule:** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, provided $b \neq 0$

[a]See page 1320 if you don't remember what an integer is.

The proof of Theorem A.4 is difficult, but not impossible, using the distance definition of absolute value or even the 'it makes negatives positive' notion. It is, however, much easier if one uses the "precise" definition given in Section 1.3 so we will revisit the proof then. For now, let's focus on how to solve basic equations and inequalities involving the absolute value.

A.7.1 Absolute Value Equations

Thinking of absolute value in terms of distance gives us a geometric way to interpret equations. For example, to solve $|x| = 3$, we are looking for all real numbers x whose distance from 0 is 3 units. If we move three units to the right of 0, we end up at $x = 3$. If we move three units to the left, we end up at $x = -3$. Thus the solutions to $|x| = 3$ are $x = \pm 3$.



The solutions to $|x| = 3$ are $x = \pm 3$.

Thinking this way gives us the following.

Theorem A.5. Absolute Value Equations: Suppose x, y and c are real numbers.

- $|x| = 0$ if and only if $x = 0$.
- For $c > 0$, $|x| = c$ if and only if $x = c$ or $x = -c$.
- For $c < 0$, $|x| = c$ has no solution.
- $|x| = |y|$ if and only if $x = y$ or $x = -y$.

  (That is, if two numbers have the same absolute values, they are either the same number or exact opposites of each other.)

Theorem A.5 is our main tool in solving equations involving the absolute value, since it allows us a way to rewrite such equations as compound linear equations.

Strategy for Solving Equations Involving Absolute Value

In order to solve an equation involving the absolute value of a quantity $|X|$:

1. Isolate the absolute value on one side of the equation so it has the form $|X| = c$.
2. Apply Theorem A.5.

The techniques we use to 'isolate the absolute value' are precisely those we used in Section A.4 to isolate the variable when solving linear equations. Time for some practice.

Example A.7.1. Solve each of the following equations.

1. $|3x - 1| = 6$
2. $\dfrac{3 - |y + 5|}{2} = 1$
3. $3|2t + 1| - \sqrt{5} = 0$
4. $4 - |5w + 3| = 5$
5. $\left|3 - x\sqrt[3]{12}\right| = |4x + 1|$
6. $|t - 1| - 3|t + 1| = 0$

Solution.

1. The equation $|3x-1| = 6$ is of already in the form $|X| = c$, so we know that either $3x-1 = 6$ or $3x-1 = -6$. Solving the former gives us at $x = \frac{7}{3}$ and solving the latter yields $x = -\frac{5}{3}$. We may check both of these solutions by substituting them into the original equation and showing that the arithmetic works out.

2. We begin solving $\frac{3-|y+5|}{2} = 1$ by isolating the absolute value to put it in the form $|X| = c$.

$$\begin{array}{rcll} \dfrac{3-|y+5|}{2} & = & 1 & \\ 3-|y+5| & = & 2 & \text{Multiply by 2} \\ -|y+5| & = & -1 & \text{Subtract 3} \\ |y+5| & = & 1 & \text{Divide by } -1 \end{array}$$

At this point, we have $y+5 = 1$ or $y+5 = -1$, so our solutions are $y = -4$ or $y = -6$. We leave it to the reader to check both answers in the original equation.

3. As in the previous example, we first isolate the absolute value. Don't let the $\sqrt{5}$ throw you off - it's just another real number, so we treat it as such:

$$\begin{array}{rcll} 3|2t+1|-\sqrt{5} & = & 0 & \\ 3|2t+1| & = & \sqrt{5} & \text{Add } \sqrt{5} \\ |2t+1| & = & \dfrac{\sqrt{5}}{3} & \text{Divide by 3} \end{array}$$

From here, we have that $2t+1 = \frac{\sqrt{5}}{3}$ or $2t+1 = -\frac{\sqrt{5}}{3}$. The first equation gives $t = \frac{\sqrt{5}-3}{6}$ while the second gives $t = \frac{-\sqrt{5}-3}{6}$ thus we list our answers as $t = \frac{-3\pm\sqrt{5}}{6}$. The reader should enjoy the challenge of substituting both answers into the original equation and following through the arithmetic to see that both answers work.

4. Upon isolating the absolute value in the equation $4-|5w+3| = 5$, we get $|5w+3| = -1$. At this point, we know there cannot be any real solution. By definition, the absolute value is a *distance*, and as such is never negative. We write 'no solution' and carry on.

5. Our next equation already has the absolute value expressions (plural) isolated, so we work from the principle that if $|x| = |y|$, then $x = y$ or $x = -y$. Thus from $\left|3-x\sqrt[3]{12}\right| = |4x+1|$ we get two equations to solve:

$$3-x\sqrt[3]{12} = 4x+1, \qquad \text{and} \qquad 3-x\sqrt[3]{12} = -(4x+1)$$

Notice that the right side of the second equation is $-(4x+1)$ and not simply $-4x+1$. Remember, the expression $4x+1$ represents a single real number so in order to negate it we need to negate the *entire* expression $-(4x+1)$. Moving along, when solving $3-x\sqrt[3]{12} = 4x+1$, we obtain $x = \frac{2}{4+\sqrt[3]{12}}$ and the solution to $3-x\sqrt[3]{12} = -(4x+1)$ is $x = \frac{4}{\sqrt[3]{12}-4}$. As usual, the reader is invited to check these answers by substituting them into the original equation.

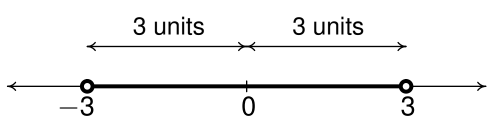
6. We start by isolating one of the absolute value expressions: $|t-1|-3|t+1| = 0$ gives $|t-1| = 3|t+1|$. While this *resembles* the form $|x| = |y|$, the coefficient 3 in $3|t+1|$ prevents it from being an exact match. Not to worry - since 3 is positive, $3 = |3|$ so

$$3|t+1| = |3||t+1| = |3(t+1)| = |3t+3|.$$

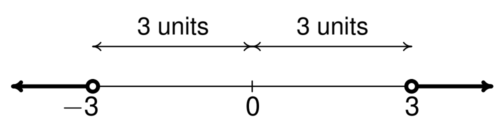
Hence, our equation becomes $|t-1| = |3t+3|$ which results in the two equations: $t-1 = 3t+3$ and $t-1 = -(3t+3)$. The first equation gives $t = -2$ and the second gives $t = -\frac{1}{2}$. The reader is encouraged to check both answers in the original equation. $\square$

A.7.2 Absolute Value Inequalities

We now turn our attention to solving some basic inequalities involving the absolute value. Suppose we wished to solve $|x| < 3$. Geometrically, we are looking for all of the real numbers whose distance from 0 is *less* than 3 units. We get $-3 < x < 3$, or in interval notation, $(-3, 3)$. Suppose we are asked to solve $|x| > 3$ instead. Now we want the distance between x and 0 to be *greater* than 3 units. Moving in the positive direction, this means $x > 3$. In the negative direction, this puts $x < -3$. Our solutions would then satisfy $x < -3$ *or* $x > 3$. In interval notation, we express this as $(-\infty, -3) \cup (3, \infty)$.



The solution to $|x| < 3$ is $(-3, 3)$



The solution to $|x| > 3$ is $(-\infty, -3) \cup (3, \infty)$

Generalizing this notion, we get the following:

Theorem A.6. Inequalities Involving Absolute Value: Let c be a real number.

- If $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.
- If $c \leq 0$, $|x| < c$ has no solution.
- If $c > 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.
- If $c \leq 0$, $|x| > c$ is true for all real numbers.

If the inequality we're faced with involves '$\leq$' or '$\geq$,' we can combine the results of Theorem A.6 with Theorem A.5 as needed.

Strategy for Solving Inequalities Involving Absolute Value

In order to solve an inequality involving the absolute value of a quantity $|X|$:

1. Isolate the absolute value on one side of the inequality.
2. Apply Theorem A.6.

Example A.7.2. Solve the following inequalities.

1. $\left|x - \sqrt[4]{5}\right| > 1$
2. $\dfrac{4 - 2|2x+1|}{4} \geq -\sqrt{3}$
3. $|2x - 1| \leq 3|4 - 8x| - 10$
4. $|2x - 1| \leq 3|4 - 8x| + 10$
5. $2 < |x - 1| \leq 5$
6. $|10x - 5| + |10 - 5x| \leq 0$

Solution.

1. From Theorem A.6, $\left|x - \sqrt[4]{5}\right| > 1$ is equivalent to $x - \sqrt[4]{5} < -1$ or $x - \sqrt[4]{5} > 1$. Solving this compound inequality, we get $x < -1 + \sqrt[4]{5}$ or $x > 1 + \sqrt[4]{5}$. Our answer, in interval notation, is: $\left(-\infty, -1 + \sqrt[4]{5}\right) \cup \left(1 + \sqrt[4]{5}, \infty\right)$. As with linear inequalities, we can only partially check our answer by selecting values of x both inside and outside of the solution intervals to see which values of x satisfy the original inequality and which do not.

2. Our first step in solving $\frac{4-2|2x+1|}{4} \geq -\sqrt{3}$ is to isolate the absolute value.

$$\begin{array}{rcll} \dfrac{4 - 2|2x+1|}{4} & \geq & -\sqrt{3} & \\ 4 - 2|2x+1| & \geq & -4\sqrt{3} & \text{Multiply by 4} \\ -2|2x+1| & \geq & -4 - 4\sqrt{3} & \text{Subtract 4} \\ |2x+1| & \leq & \dfrac{-4 - 4\sqrt{3}}{-2} & \text{Divide by } -2\text{, reverse the inequality} \\ |2x+1| & \leq & 2 + 2\sqrt{3} & \text{Reduce} \end{array}$$

Since we're dealing with '$\leq$' instead of just '$<$,' we can combine Theorems A.6 and A.5 to rewrite this last inequality as:[1] $-(2 + 2\sqrt{3}) \leq 2x + 1 \leq 2 + 2\sqrt{3}$. Subtracting the '1' across both inequalities gives $-3 - 2\sqrt{3} \leq 2x \leq 1 + 2\sqrt{3}$, which reduces to $\frac{-3-2\sqrt{3}}{2} \leq x \leq \frac{1+2\sqrt{3}}{2}$. In interval notation this reads as $\left[\frac{-3-2\sqrt{3}}{2}, \frac{1+2\sqrt{3}}{2}\right]$.

3. There are two absolute values in $|2x - 1| \leq 3|4 - 8x| - 10$, so we cannot directly apply Theorem A.6 here. Notice, however, that $|4 - 8x| = |(-4)(2x - 1)|$. Using this, we get:

$$\begin{array}{rcll} |2x-1| & \leq & 3|4-8x| - 10 & \\ |2x-1| & \leq & 3|(-4)(2x-1)| - 10 & \text{Factor} \\ |2x-1| & \leq & 3|-4||2x-1| - 10 & \text{Product Rule} \\ |2x-1| & \leq & 12|2x-1| - 10 & \\ -11|2x-1| & \leq & -10 & \text{Subtract } 12|2x-1| \\ |2x-1| & \geq & \dfrac{10}{11} & \text{Divide by } -11 \text{ and reduce} \end{array}$$

Now we are allowed to invoke Theorems A.5 and A.6 and write the equivalent compound inequality: $2x - 1 \leq -\frac{10}{11}$ *or* $2x - 1 \geq \frac{10}{11}$. We get $x \leq \frac{1}{22}$ *or* $x \geq \frac{21}{22}$, which when written with interval notation becomes $\left(-\infty, \frac{1}{22}\right] \cup \left[\frac{21}{22}, \infty\right)$.

[1] Note the use of parentheses: $-(2 + 2\sqrt{3})$ as opposed to $-2 + 2\sqrt{3}$.

4. The inequality $|2x - 1| \leq 3|4 - 8x| + 10$ differs from the previous example in exactly one respect: on the right side of the inequality, we have '+10' instead of '-10.' The steps to isolate the absolute value here are identical to those in the previous example, but instead of obtaining $|2x - 1| \geq \frac{10}{11}$ as before, we obtain $|2x - 1| \geq -\frac{10}{11}$. This latter inequality is *always* true. (Absolute value is, by definition, a distance and hence always 0 or greater.) Thus our solution to this inequality is all real numbers.

5. To solve $2 < |x - 1| \leq 5$, we rewrite it as the compound inequality: $2 < |x - 1|$ *and* $|x - 1| \leq 5$. The first inequality, $2 < |x - 1|$, can be re-written as $|x - 1| > 2$ so it is equivalent to $x - 1 < -2$ *or* $x - 1 > 2$. Thus the solution to $2 < |x - 1|$ is $x < -1$ or $x > 3$, which in interval notation is $(-\infty, -1) \cup (3, \infty)$. For $|x - 1| \leq 5$, we combine the results of Theorems A.5 and A.6 to get $-5 \leq x - 1 \leq 5$ so that $-4 \leq x \leq 6$, or $[-4, 6]$.

   Our solution to $2 < |x - 1| \leq 5$ is comprised of values of x which satisfy both parts of the inequality, so we intersect $(-\infty, -1) \cup (3, \infty)$ with $[-4, 6]$ to get our final answer $[-4, -1) \cup (3, 6]$.

6. Our first hope when encountering $|10x - 5| + |10 - 5x| \leq 0$ is that we can somehow combine the two absolute value quantities as we'd done in earlier examples. We leave it to the reader to show, however, that no matter what we try to factor out of the absolute value quantities, what remains inside the absolute values will always be different.

   At this point, we take a step back and look at the equation in a more general way: we are adding two absolute values together and wanting the result to be less than or equal to 0. The absolute value of anything is always 0 or greater, so there are no solutions to: $|10x - 5| + |10 - 5x| < 0$.

   Is it possible that $|10x - 5| + |10 - 5x| = 0$? Only if there is an x where $|10x - 5| = 0$ and $|10 - 5x| = 0$ *at the same time*.[2] The first equation holds only when $x = \frac{1}{2}$, while the second holds only when $x = 2$. Alas, we have no solution.[3] $\square$

The astute reader will have noticed by now that the authors have done nothing in the way of explaining *why* anyone would ever need to know this stuff. Go back and read the New Preface and the introduction to the Appendix. These sections are designed to review skills and concepts that you've already learned. Thus the deeper applications are in the main body of the text as opposed to here in the Appendix.

We close this section with an example of how the properties in Theorem A.4 are used in Calculus. Here, 'ε' is the Greek letter 'epsilon' and it represents a positive real number. Those of you who will be taking Calculus in the future should become *very* familiar with this type of algebraic manipulation.

[2] Do you see why?

[3] Not for lack of trying, however!

$$\begin{array}{rcll}
\left|\dfrac{8-4x}{3}\right| & < & \varepsilon & \\
\dfrac{|8-4x|}{|3|} & < & \varepsilon & \text{Quotient Rule} \\
\dfrac{|-4(x-2)|}{3} & < & \varepsilon & \text{Factor} \\
\dfrac{|-4||x-2|}{3} & < & \varepsilon & \text{Product Rule} \\
\dfrac{4|x-2|}{3} & < & \varepsilon & \\
\dfrac{3}{4}\cdot\dfrac{4|x-2|}{3} & < & \dfrac{3}{4}\cdot\varepsilon & \text{Multiply by } \dfrac{3}{4} \\
|x-2| & < & \dfrac{3}{4}\varepsilon &
\end{array}$$

A.7.3 Exercises

In Exercises 1 - 18, solve the equation.

1. $|x| = 6$
2. $|3t - 1| = 10$
3. $|4 - w| = 7$
4. $4 - |y| = 3$
5. $2|5m + 1| - 3 = 0$
6. $|7x - 1| + 2 = 0$
7. $\dfrac{5 - |x|}{2} = 1$
8. $\dfrac{2}{3}|5 - 2w| - \dfrac{1}{2} = 5$
9. $|3t - \sqrt{2}| + 4 = 6$
10. $\dfrac{|2v + 1| - 3}{4} = \dfrac{1}{2} - |2v + 1|$
11. $|2x + 1| = \dfrac{|2x + 1| - 3}{2}$
12. $\dfrac{|3 - 2y| + 4}{2} = 2 - |3 - 2y|$
13. $|3t - 2| = |2t + 7|$
14. $|3x + 1| = |4x|$
15. $|1 - \sqrt{2}y| = |y + 1|$
16. $|4 - x| - |x + 2| = 0$
17. $|2 - 5z| = 5|z + 1|$
18. $\sqrt{3}|w - 1| = 2|w + 1|$

In Exercises 19 - 30, solve the inequality. Write your answer using interval notation.

19. $|3x - 5| \leq 4$
20. $|7t + 2| > 10$
21. $|2w + 1| - 5 < 0$
22. $|2 - y| - 4 \geq -3$
23. $|3z + 5| + 2 < 1$
24. $2|7 - v| + 4 > 1$
25. $3 - |x + \sqrt{5}| < -3$
26. $|5t| \leq |t| + 3$
27. $|w - 3| < |3 - w|$
28. $2 \leq |4 - y| < 7$
29. $1 < |2w - 9| \leq 3$
30. $3 > 2|\sqrt{3} - x| > 1$
31. With help from your classmates, solve:

    (a) $|5 - |2x - 3|| = 4$

    (b) $|5 - |2x - 3|| < 4$

A.7.4 Answers

1. $x = -6$ or $x = 6$
2. $t = -3$ or $t = \frac{11}{3}$
3. $w = -3$ or $w = 11$
4. $y = -1$ or $y = 1$
5. $m = -\frac{1}{2}$ or $m = \frac{1}{10}$
6. No solution
7. $x = -3$ or $x = 3$
8. $w = -\frac{13}{8}$ or $w = \frac{53}{8}$
9. $t = \frac{\sqrt{2} \pm 2}{3}$
10. $v = -1$ or $v = 0$
11. No solution
12. $y = \frac{3}{2}$
13. $t = -1$ or $t = 9$
14. $x = -\frac{1}{7}$ or $x = 1$
15. $y = 0$ or $y = \frac{2}{\sqrt{2} - 1}$
16. $x = 1$
17. $z = -\frac{3}{10}$
18. $w = \frac{\sqrt{3} \pm 2}{\sqrt{3} \mp 2}$

    See footnote[4]
19. $\left[\frac{1}{3}, 3\right]$
20. $\left(-\infty, -\frac{12}{7}\right) \cup \left(\frac{8}{7}, \infty\right)$
21. $(-3, 2)$
22. $(-\infty, 1] \cup [3, \infty)$
23. No solution
24. $(-\infty, \infty)$
25. $(-\infty, -6 - \sqrt{5}) \cup (6 - \sqrt{5}, \infty)$
26. $\left[-\frac{3}{4}, \frac{3}{4}\right]$
27. No solution
28. $(-3, 2] \cup [6, 11)$
29. $[3, 4) \cup (5, 6]$
30. $\left(\frac{2\sqrt{3} - 3}{2}, \frac{2\sqrt{3} - 1}{2}\right) \cup \left(\frac{2\sqrt{3} + 1}{2}, \frac{2\sqrt{3} + 3}{2}\right)$
31. (a) $x = -3$, or $x = 1$, or $x = 2$, or $x = 6$

    (b) $(-3, 1) \cup (2, 6)$

[4]That is, $w = \frac{\sqrt{3} + 2}{\sqrt{3} - 2}$ or $w = \frac{\sqrt{3} - 2}{\sqrt{3} + 2}$

A.8 Polynomial Arithmetic

In this section, we review the vocabulary and arithmetic of **polynomials**. We start by defining what is meant by the word ‘polynomial’ in general. A more narrow definition of a ‘polynomial function’ will be given in Chapter 2. The general definition suffices for the purposes of this review.

Definition A.15. A **polynomial** is a sum of terms each of which is a real number or a real number multiplied by one or more variables to natural number powers.

Some examples of polynomials are $x^2+x\sqrt{3}+4$, $27x^2y+\frac{7x}{2}$ and 6. Things like $3\sqrt{x}$, $4x-\frac{2}{x+1}$ and $13x^{2/3}y^2$ are **not** polynomials. In the box below we review some of the terminology associated with polynomials.

Definition A.16. Polynomial Vocabulary

- **Constant Terms:** Terms in polynomials without variables are called **constant** terms.
- **Coefficient:** In non-constant terms, the real number factor in the expression is called the **coefficient** of the term.
- **Degree:** The **degree** of a non-constant term is the sum of the exponents on the variables in the term; non-zero constant terms are defined to have degree 0. The degree of a polynomial is the highest degree of the nonzero terms.
- **Like Terms:** Terms in a polynomial are called **like** terms if they have the same variables each with the same corresponding exponents.
- **Simplified:** A polynomial is said to be **simplified** if all arithmetic operations have been completed and there are no longer any like terms.
- **Classification by Number of Terms:** A simplified polynomial is called a
  - **monomial** if it has exactly one nonzero term
  - **binomial** if it has exactly two nonzero terms
  - **trinomial** if it has exactly three nonzero terms

For example, $x^2+x\sqrt{3}+4$ is a trinomial of degree 2. The coefficient of x^2 is 1 and the constant term is 4. The polynomial $27x^2y+\frac{7x}{2}$ is a binomial of degree 3 ($x^2y=x^2y^1$) with constant term 0.

The concept of ‘like’ terms really amounts to finding terms which can be combined using the Distributive Property. For example, in the polynomial $17x^2y-3xy^2+7xy^2$, $-3xy^2$ and $7xy^2$ are like terms, since they have the same variables with the same corresponding exponents. This allows us to combine these two terms as follows:

$$17x^2y-3xy^2+7xy^2 = 17x^2y+(-3)xy^2+7xy^2+17x^2y+(-3+7)xy^2 = 17x^2y+4xy^2$$

Note that even though $17x^2y$ and $4xy^2$ have the same variables, they are not like terms since in the first term we have x^2 and $y = y^1$ but in the second we have $x = x^1$ and $y = y^2$ so the corresponding exponents aren't the same. Hence, $17x^2y + 4xy^2$ is the simplified form of the polynomial.

There are four basic operations we can perform with polynomials: addition, subtraction, multiplication and division. The first three of these follow directly from properties of real number arithmetic and will be discussed together. Division, on the other hand, is a bit more complicated and will be discussed separately.

A.8.1 Polynomial Addition, Subtraction and Multiplication.

Adding and subtracting polynomials comes down to identifying like terms and then adding or subtracting the coefficients of those like terms. Multiplying polynomials comes to us courtesy of the Generalized Distributive Property.

Theorem A.7. Generalized Distributive Property: To multiply a quantity of n terms by a quantity of m terms, multiply each of the n terms of the first quantity by each of the m terms in the second quantity and add the resulting $n \cdot m$ terms together.

In particular, Theorem A.7 says that, before combining like terms, a product of an n-term polynomial and an m-term polynomial will generate $(n \cdot m)$-terms. For example, a binomial times a trinomial will produce six terms some of which may be like terms. Thus the simplified end result may have fewer than six terms but you will start with six terms.

A special case of Theorem A.7 is the famous **F.O.I.L.**, listed here:[1]

Theorem A.8. F.O.I.L: The terms generated from the product of two binomials: $(a + b)(c + d)$ can be verbalized as follows: "Take the sum of

- the product of the **F**irst terms a and c, ac
- the product of the **O**uter terms a and d, ad
- the product of the **I**nner terms b and c, bc
- the product of the **L**ast terms b and d, bd."

That is, $(a + b)(c + d) = ac + ad + bc + bd$.

Theorem A.7 is best proved using the technique known as Mathematical Induction which is covered in Section 10.3. The result is really nothing more than repeated applications of the Distributive Property so it seems reasonable and we'll use it without proof for now. The other major piece of polynomial multiplication is one of the Power Rules of Exponents from page 1340 in Section A.2, namely $a^na^m = a^{n+m}$. The Commutative and Associative Properties of addition and multiplication are also used extensively. We put all of these properties to good use in the next example.

[1] We caved to peer pressure on this one. Apparently all of the cool Precalculus books have FOIL in them even though it's redundant once you know how to distribute multiplication across addition. In general, we don't like mechanical short-cuts that interfere with a student's understanding of the material and FOIL is one of the worst.

Example A.8.1. Perform the indicated operations and simplify.

1. $\left(3x^2 - 2x + 1\right) - (7x - 3)$
2. $4xz^2 - 3z(xz - x + 4)$
3. $(2t + 1)(3t - 7)$
4. $\left(3y - \sqrt[3]{2}\right)\left(9y^2 + 3\sqrt[3]{2}y + \sqrt[3]{4}\right)$
5. $\left(4w - \dfrac{1}{2}\right)^2$
6. $\left[2(x + h) - (x + h)^2\right] - \left(2x - x^2\right)$

Solution.

1. We begin 'distributing the negative' as indicated on page 1334 in Section A.2, then we rearrange and combine like terms:

$$\begin{array}{rcll} \left(3x^2 - 2x + 1\right) - (7x - 3) & = & 3x^2 - 2x + 1 - 7x + 3 & \text{Distribute} \\ & = & 3x^2 - 2x - 7x + 1 + 3 & \text{Rearrange terms} \\ & = & 3x^2 - 9x + 4 & \text{Combine like terms} \end{array}$$

   Our answer is $3x^2 - 9x + 4$.

2. Following in our footsteps from the previous example, we first distribute the $-3z$ through, then rearrange and combine like terms:

$$\begin{array}{rcll} 4xz^2 - 3z(xz - x + 4) & = & 4xz^2 - 3z(xz) + 3z(x) - 3z(4) & \text{Distribute} \\ & = & 4xz^2 - 3xz^2 + 3xz - 12z & \text{Multiply} \\ & = & xz^2 + 3xz - 12z & \text{Combine like terms} \end{array}$$

   We get our final answer: $xz^2 + 3xz - 12z$.

3. At last, we have a chance to use our F.O.I.L. technique:

$$\begin{array}{rcll} (2t + 1)(3t - 7) & = & (2t)(3t) + (2t)(-7) + (1)(3t) + (1)(-7) & \text{F.O.I.L.} \\ & = & 6t^2 - 14t + 3t - 7 & \text{Multiply} \\ & = & 6t^2 - 11t - 7 & \text{Combine like terms} \end{array}$$

   We get $6t^2 - 11t - 7$ as our final answer.

4. We use the Generalized Distributive Property here, multiplying each term in the second quantity first by $3y$, then by $-\sqrt[3]{2}$:

$$\begin{array}{rcl} \left(3y - \sqrt[3]{2}\right)\left(9y^2 + 3\sqrt[3]{2}y + \sqrt[3]{4}\right) & = & 3y\left(9y^2\right) + 3y\left(3\sqrt[3]{2}y\right) + 3y\left(\sqrt[3]{4}\right) \\ & & -\sqrt[3]{2}\left(9y^2\right) - \sqrt[3]{2}\left(3\sqrt[3]{2}y\right) - \sqrt[3]{2}\left(\sqrt[3]{4}\right) \\ & = & 27y^3 + 9y^2\sqrt[3]{2} + 3y\sqrt[3]{4} - 9y^2\sqrt[3]{2} - 3y\sqrt[3]{4} - \sqrt[3]{8} \\ & = & 27y^3 + 9y^2\sqrt[3]{2} - 9y^2\sqrt[3]{2} + 3y\sqrt[3]{4} - 3y\sqrt[3]{4} - 2 \\ & = & 27y^3 - 2 \end{array}$$

   To our surprise and delight, this product reduces to $27y^3 - 2$.

5. Exponents do **not** distribute across powers[2] so we know that $\left(4w - \frac{1}{2}\right)^2 \neq (4w)^2 - \left(\frac{1}{2}\right)^2$. Instead, we proceed as follows:

$$\begin{array}{rcll} \left(4w - \dfrac{1}{2}\right)^2 & = & \left(4w - \dfrac{1}{2}\right)\left(4w - \dfrac{1}{2}\right) & \\ & = & (4w)(4w) + (4w)\left(-\dfrac{1}{2}\right) + \left(-\dfrac{1}{2}\right)(4w) + \left(-\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right) & \text{F.O.I.L.} \\ & = & 16w^2 - 2w - 2w + \dfrac{1}{4} & \text{Multiply} \\ & = & 16w^2 - 4w + \dfrac{1}{4} & \text{Combine like terms} \end{array}$$

Our (correct) final answer is $16w^2 - 4w + \frac{1}{4}$.

6. Our last example has two levels of grouping symbols. We begin simplifying the quantity inside the brackets, expanding $(x+h)^2$ in the same way we expanded $(4w - \frac{1}{2})^2$ in our previous example:

$$(x+h)^2 = (x+h)(x+h) = (x)(x) + (x)(h) + (h)(x) + (h)(h) = x^2 + 2xh + h^2$$

When we substitute this into our expression, we envelope it in parentheses, as usual, so that we don't forget to distribute the negative.

$$\begin{array}{rcll} \left[2(x+h) - (x+h)^2\right] - \left(2x - x^2\right) & = & \left[2(x+h) - \left(x^2 + 2xh + h^2\right)\right] - \left(2x - x^2\right) & \text{Substitute} \\ & = & \left[2x + 2h - x^2 - 2xh - h^2\right] - \left(2x - x^2\right) & \text{Distribute} \\ & = & 2x + 2h - x^2 - 2xh - h^2 - 2x + x^2 & \text{Distribute} \\ & = & 2x - 2x + 2h - x^2 + x^2 - 2xh - h^2 & \text{Rearrange terms} \\ & = & 2h - 2xh - h^2 & \text{Combine like terms} \end{array}$$

We find no like terms in $2h - 2xh - h^2$ so we are finished. □

We conclude our discussion of polynomial multiplication by showcasing two special products which happen often enough they should be committed to memory.

Theorem A.9. Special Products: Let a and b be real numbers:

- **Perfect Square:** $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$
- **Difference of Two Squares:** $(a-b)(a+b) = a^2 - b^2$

The formulas in Theorem A.9 can be verified by working through the multiplication.[3]

[2]See the remarks following the Properties of Exponents on 1340.
[3]These are both special cases of F.O.I.L.

A.8.2 Polynomial Long Division.

We now turn our attention to polynomial long division. Dividing two polynomials follows the same algorithm, in principle, as dividing two natural numbers so we review that process first. Suppose we wished to divide 2585 by 79. The standard division tableau is given below.

$$
\begin{array}{rr}
 & 32 \\ \cline{2-2}
79 & \multicolumn{1}{|r}{2585} \\
 & -\underline{237}\downarrow \\
 & 215 \\
 & -\underline{158} \\
 & 57
\end{array}
$$

In this case, 79 is called the **divisor**, 2585 is called the **dividend**, 32 is called the **quotient** and 57 is called the **remainder**. We can check our answer by showing:

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

or in this case, $2585 = (79)(32)+57\checkmark$. We hope that the long division tableau evokes warm, fuzzy memories of your formative years as opposed to feelings of hopelessness and frustration. If you experience the latter, keep in mind that the Division Algorithm essentially is a two-step process, iterated over and over again. First, we guess the number of times the divisor goes into the dividend and then we subtract off our guess. We repeat those steps with what's left over until what's left over (the remainder) is less than what we started with (the divisor). That's all there is to it!

The division algorithm for polynomials has the same basic two steps but when we subtract polynomials, we must take care to subtract *like terms* only. As a transition to polynomial division, let's write out our previous division tableau in expanded form.

$$
\begin{array}{rr}
 & 3\cdot 10+2 \\ \cline{2-2}
7\cdot 10+9 & \multicolumn{1}{|r}{2\cdot 10^3 + 5\cdot 10^2 + 8\cdot 10 + 5} \\
 & -\underline{\left(2\cdot 10^3 + 3\cdot 10^2 + 7\cdot 10\right)} \ \downarrow \\
 & 2\cdot 10^2 + 1\cdot 10 + 5 \\
 & -\underline{\left(1\cdot 10^2 + 5\cdot 10 + 8\right)} \\
 & 5\cdot 10 + 7
\end{array}
$$

Written this way, we see that when we line up the digits we are really lining up the coefficients of the corresponding powers of 10 - much like how we'll have to keep the powers of x lined up in the same columns. The big difference between polynomial division and the division of natural numbers is that the value of x is an unknown quantity. So unlike using the known value of 10, when we subtract there can be no regrouping of coefficients as in our previous example. (The subtraction $215 - 158$ requires us to 'regroup' or 'borrow' from the tens digit, then the hundreds digit.) This actually makes polynomial division easier.[4]

[4] In our opinion - you can judge for yourself.

Before we dive into examples, we first state a theorem telling us when we can divide two polynomials, and what to expect when we do so.

Theorem A.10. Polynomial Division: Let d and p be nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, q and r, such that $p = d \cdot q + r$, where either $r = 0$ or the degree of r is strictly less than the degree of d.

Essentially, Theorem A.10 tells us that we can divide polynomials whenever the degree of the divisor is less than or equal to the degree of the dividend. We know we're done with the division when the polynomial left over (the remainder) has a degree strictly less than the divisor. It's time to walk through a few examples to refresh your memory.

Example A.8.2. Perform the indicated division. Check your answer by showing

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

1. $\left(x^3 + 4x^2 - 5x - 14\right) \div (x - 2)$
2. $(2t + 7) \div (3t - 4)$
3. $\left(6y^2 - 1\right) \div (2y + 5)$
4. $\left(w^3\right) \div \left(w^2 - \sqrt{2}\right)$.

Solution.

1. To begin $\left(x^3 + 4x^2 - 5x - 14\right) \div (x - 2)$, we divide the first term in the dividend, namely x^3, by the first term in the divisor, namely x, and get $\frac{x^3}{x} = x^2$. This then becomes the first term in the quotient. We proceed as in regular long division at this point: we multiply the entire divisor, $x - 2$, by this first term in the quotient to get $x^2(x - 2) = x^3 - 2x^2$. We then subtract this result from the dividend.

$$\begin{array}{rrrrr} & x^2 & & & \\ \cline{2-5} x-2 \big) & x^3 & +\,4x^2 & -5x & -14 \\ & -\left(x^3\right. & \left.-\,2x^2\right) & \downarrow & \\ \cline{2-3} & & 6x^2 & -5x & \end{array}$$

Now we 'bring down' the next term of the quotient, namely $-5x$, and repeat the process. We divide $\frac{6x^2}{x} = 6x$, and add this to the quotient polynomial, multiply it by the divisor (which yields $6x(x - 2) = 6x^2 - 12x$) and subtract.

$$\begin{array}{rrrrr} & x^2 & +\,6x & & \\ \cline{2-5} x-2 \big) & x^3 & +\,4x^2 & -\,5x & -14 \\ & -\left(x^3\right. & \left.-\,2x^2\right) & & \downarrow \\ \cline{2-3} & & 6x^2 & -\,5x & \downarrow \\ & & -\left(6x^2\right. & \left.-\,12x\right) & \downarrow \\ \cline{3-4} & & & 7x & -14 \end{array}$$

Finally, we 'bring down' the last term of the dividend, namely -14, and repeat the process. We divide $\frac{7x}{x} = 7$, add this to the quotient, multiply it by the divisor (which yields $7(x-2) = 7x - 14$) and subtract.

$$
\begin{array}{rrrrrr}
 & x^2 & + & 6x & + & 7 \\
\cline{2-6}
x-2 \big) & x^3 & + & 4x^2 & - & 5x \quad - 14 \\
-\left(x^3 - 2x^2\right) & & & & & \\
\cline{2-4}
 & & & 6x^2 & - & 5x \\
 & & & -\left(6x^2 - 12x\right) & & \\
\cline{4-6}
 & & & & & 7x \quad - 14 \\
 & & & & & -\left(7x \quad - 14\right) \\
\cline{6-6}
 & & & & & 0
\end{array}
$$

In this case, we get a quotient of $x^2 + 6x + 7$ with a remainder of 0. To check our answer, we compute

$$(x-2)\left(x^2 + 6x + 7\right) + 0 = x^3 + 6x^2 + 7x - 2x^2 - 12x - 14 = x^3 + 4x^2 - 5x - 14 \checkmark$$

2. To compute $(2t + 7) \div (3t - 4)$, we start as before. We find $\frac{2t}{3t} = \frac{2}{3}$, so that becomes the first (and only) term in the quotient. We multiply the divisor $(3t - 4)$ by $\frac{2}{3}$ and get $2t - \frac{8}{3}$. We subtract this from the divided and get $\frac{29}{3}$.

$$
\begin{array}{rl}
 & \qquad \frac{2}{3} \\
\cline{2-2}
3t-4 \big) & 2t + 7 \\
 & -\left(2t - \frac{8}{3}\right) \\
\cline{2-2}
 & \qquad \frac{29}{3}
\end{array}
$$

Our answer is $\frac{2}{3}$ with a remainder of $\frac{29}{3}$. To check our answer, we compute

$$(3t-4)\left(\frac{2}{3}\right) + \frac{29}{3} = 2t - \frac{8}{3} + \frac{29}{3} = 2t + \frac{21}{3} = 2t + 7 \checkmark$$

3. When we set-up the tableau for $\left(6y^2 - 1\right) \div (2y + 5)$, we must first issue a 'placeholder' for the 'missing' y-term in the dividend, $6y^2 - 1 = 6y^2 + 0y - 1$. We then proceed as before. Since $\frac{6y^2}{2y} = 3y$, $3y$ is the first term in our quotient. We multiply $(2y + 5)$ times $3y$ and subtract it from the dividend.

We bring down the -1, and repeat.

$$\begin{array}{r|rrrr}
\multicolumn{2}{r}{} & 3y & - & \frac{15}{2} \\ \cline{2-5}
2y+5 & 6y^2 + & 0y & - & 1 \\
& -\left(6y^2 + \right. & \left. 15y\right) & & \downarrow \\ \cline{2-3}
& & -15y & - & 1 \\
& & -\left(-15y\right. & - & \left.\frac{75}{2}\right) \\ \cline{3-5}
& & & & \frac{73}{2}
\end{array}$$

Our answer is $3y - \frac{15}{2}$ with a remainder of $\frac{73}{2}$. To check our answer, we compute:

$$(2y+5)\left(3y - \frac{15}{2}\right) + \frac{73}{2} = 6y^2 - 15y + 15y - \frac{75}{2} + \frac{73}{2} = 6y^2 - 1 \checkmark$$

4. For our last example, we need 'placeholders' for both the divisor $w^2 - \sqrt{2} = w^2 + 0w - \sqrt{2}$ and the dividend $w^3 = w^3 + 0w^2 + 0w + 0$. The first term in the quotient is $\frac{w^3}{w^2} = w$, and when we multiply and subtract this from the dividend, we're left with just $0w^2 + w\sqrt{2} + 0 = w\sqrt{2}$.

$$\begin{array}{r|rrrr}
\multicolumn{2}{r}{} & & w & \\ \cline{2-5}
w^2+0w-\sqrt{2} & w^3 & +0w^2 + & 0w & +0 \\
- & \left(w^3\right. & +0w^2 - & \left.w\sqrt{2}\right) & \downarrow \\ \cline{2-4}
& & 0w^2 + & w\sqrt{2} & +0
\end{array}$$

Since the degree of $w\sqrt{2}$ (which is 1) is less than the degree of the divisor (which is 2), we are done.[5] Our answer is w with a remainder of $w\sqrt{2}$. To check, we compute:

$$\left(w^2 - \sqrt{2}\right) w + w\sqrt{2} = w^3 - w\sqrt{2} + w\sqrt{2} = w^3 \checkmark$$

□

[5] Since $\frac{0w^2}{w^2} = 0$, we could proceed, write our quotient as $w + 0$, and move on... but even pedants have limits.

A.8.3 Exercises

In Exercises 1 - 15, perform the indicated operations and simplify.

1. $(4 - 3x) + (3x^2 + 2x + 7)$
2. $t^2 + 4t - 2(3 - t)$
3. $q(200 - 3q) - (5q + 500)$
4. $(3y - 1)(2y + 1)$
5. $\left(3 - \dfrac{x}{2}\right)(2x + 5)$
6. $-(4t + 3)(t^2 - 2)$
7. $2w(w^3 - 5)(w^3 + 5)$
8. $(5a^2 - 3)(25a^4 + 15a^2 + 9)$
9. $(x^2 - 2x + 3)(x^2 + 2x + 3)$
10. $(\sqrt{7} - z)(\sqrt{7} + z)$
11. $(x - \sqrt[3]{5})^3$
12. $(x - \sqrt[3]{5})(x^2 + x\sqrt[3]{5} + \sqrt[3]{25})$
13. $(w - 3)^2 - (w^2 + 9)$
14. $(x+h)^2 - 2(x+h) - (x^2 - 2x)$
15. $(x - [2 + \sqrt{5}])(x - [2 - \sqrt{5}])$

In Exercises 16 - 27, perform the indicated division. Check your answer by showing

dividend = (divisor)(quotient) + remainder

16. $(5x^2 - 3x + 1) \div (x + 1)$
17. $(3y^2 + 6y - 7) \div (y - 3)$
18. $(6w - 3) \div (2w + 5)$
19. $(2x + 1) \div (3x - 4)$
20. $(t^2 - 4) \div (2t + 1)$
21. $(w^3 - 8) \div (5w - 10)$
22. $(2x^2 - x + 1) \div (3x^2 + 1)$
23. $(4y^4 + 3y^2 + 1) \div (2y^2 - y + 1)$
24. $w^4 \div (w^3 - 2)$
25. $(5t^3 - t + 1) \div (t^2 + 4)$
26. $(t^3 - 4) \div (t - \sqrt[3]{4})$
27. $(x^2 - 2x - 1) \div (x - [1 - \sqrt{2}])$

In Exercises 28 - 33 verify the given formula by showing the left hand side of the equation simplifies to the right hand side of the equation.

28. **Perfect Cube:** $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
29. **Difference of Cubes:** $(a - b)(a^2 + ab + b^2) = a^3 - b^3$
30. **Sum of Cubes:** $(a + b)(a^2 - ab + b^2) = a^3 + b^3$
31. **Perfect Quartic:** $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
32. **Difference of Quartics:** $(a - b)(a + b)(a^2 + b^2) = a^4 - b^4$
33. **Sum of Quartics:** $(a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2) = a^4 + b^4$
34. With help from your classmates, determine under what conditions $(a + b)^2 = a^2 + b^2$. What about $(a + b)^3 = a^3 + b^3$? In general, when does $(a + b)^n = a^n + b^n$ for a natural number $n \geq 2$?

A.8.4 Answers

1. $3x^2 - x + 11$
2. $t^2 + 6t - 6$
3. $-3q^2 + 195q - 500$
4. $6y^2 + y - 1$
5. $-x^2 + \frac{7}{2}x + 15$
6. $-4t^3 - 3t^2 + 8t + 6$
7. $2w^7 - 50w$
8. $125a^6 - 27$
9. $x^4 + 2x^2 + 9$
10. $7 - z^2$
11. $x^3 - 3x^2\sqrt[3]{5} + 3x\sqrt[3]{25} - 5$
12. $x^3 - 5$
13. $-6w$
14. $h^2 + 2xh - 2h$
15. $x^2 - 4x - 1$
16. quotient: $5x - 8$, remainder: 9
17. quotient: $3y + 15$, remainder: 38
18. quotient: 3, remainder: 18
19. quotient: $\frac{2}{3}$, remainder: $\frac{11}{3}$
20. quotient: $\frac{t}{2} - \frac{1}{4}$, remainder: $-\frac{15}{4}$
21. quotient: $\frac{w^2}{5} + \frac{2w}{5} + \frac{4}{5}$, remainder: 0
22. quotient: $\frac{2}{3}$, remainder: $-x + \frac{1}{3}$
23. quotient: $2y^2 + y + 1$, remainder: 0
24. quotient: w, remainder: $2w$
25. quotient: $5t$, remainder: $-21t + 1$
26. quotient:[6] $t^2 + t\sqrt[3]{4} + 2\sqrt[3]{2}$, remainder: 0
27. quotient: $x - 1 - \sqrt{2}$, remainder: 0

[6]Note: $\sqrt[3]{16} = 2\sqrt[3]{2}$.

A.9 Basic Factoring Techniques

Now that we have reviewed the basics of polynomial arithmetic it's time to review the basic techniques of factoring polynomial expressions. Our goal is to apply these techniques to help us solve certain specialized classes of non-linear equations. Given that 'factoring' literally means to resolve a product into its factors, it is, in the purest sense, 'undoing' multiplication. If this sounds like division to you then you've been paying attention. Let's start with a numerical example.

Suppose we are asked to factor 16337. We could write $16337 = 16337 \cdot 1$, and while this is technically a factorization of 16337, it's probably not an answer the poser of the question would accept. Usually, when we're asked to factor a natural number, we are being asked to resolve it into to a product of so-called 'prime' numbers.[1] Recall that **prime numbers** are defined as natural numbers whose only (natural number) factors are themselves and 1. They are, in essence, the 'building blocks' of natural numbers as far as multiplication is concerned. Said differently, we can build - via multiplication - any natural number given enough primes.

So how do we find the prime factors of 16337? We start by dividing each of the primes: 2, 3, 5, 7, etc., into 16337 until we get a remainder of 0. Eventually, we find that $16337 \div 17 = 961$ with a remainder of 0, which means $16337 = 17 \cdot 961$. So factoring and division are indeed closely related - factors of a number are precisely the divisors of that number which produce a zero remainder.[2] We continue our efforts to see if 961 can be factored down further, and we find that $961 = 31 \cdot 31$. Hence, 16337 can be 'completely factored' as $17 \cdot 31^2$. (This factorization is called the **prime factorization** of 16337.)

In factoring natural numbers, our building blocks are prime numbers, so to be completely factored means that every number used in the factorization of a given number is prime. One of the challenges when it comes to factoring polynomial expressions is to explain what it means to be 'completely factored'. In this section, our 'building blocks' for factoring polynomials are 'irreducible' polynomials as defined below.

Definition A.17. A polynomial is said to be **irreducible** if it cannot be written as the product of polynomials of lower degree.

While Definition A.17 seems straightforward enough, sometimes a greater level of specificity is required. For example, $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. While $x - \sqrt{3}$ and $x + \sqrt{3}$ are perfectly fine polynomials, factoring which requires irrational numbers is usually saved for a more advanced treatment of factoring.[3] For now, we will restrict ourselves to factoring using rational coefficients. So, while the polynomial $x^2 - 3$ can be factored using irrational numbers, it is called irreducible **over the rationals**, since there are no polynomials with *rational* coefficients of smaller degree which can be used to factor it.[4]

Since polynomials involve terms, the first step in any factoring strategy involves pulling out factors which are common to all of the terms. For example, in the polynomial $18x^2y^3 - 54x^3y^2 - 12xy^2$, each coefficient is a multiple of 6 so we can begin the factorization as $6(3x^2y^3 - 9x^3y^2 - 2xy^2)$. The remaining coefficients: 3, 9 and 2, have no common factors so 6 was the greatest common factor. What about the variables? Each

[1] As mentioned in Section A.2, this is possible, in only one way, thanks to the Fundamental Theorem of Arithmetic.

[2] We'll refer back to this when we get to Section 2.2.

[3] See Section 2.3.

[4] If this isn't immediately obvious, don't worry - in some sense, it shouldn't be. We'll talk more about this later.

term contains an x, so we can factor an x from each term. When we do this, we are effectively dividing each term by x which means the exponent on x in each term is reduced by 1: $6x(3xy^3 - 9x^2y^2 - 2y^2)$. Next, we see that each term has a factor of y in it. In fact, each term has at least *two* factors of y in it, since the lowest exponent on y in each term is 2. This means that we can factor y^2 from each term. Again, factoring out y^2 from each term is tantamount to dividing each term by y^2 so the exponent on y in each term is reduced by *two*: $6xy^2(3xy - 9x^2 - 2)$. Just like we checked our division by multiplication in the previous section, we can check our factoring here by multiplication, too. $6xy^2(3xy - 9x^2 - 2) = (6xy^2)(3xy) - (6xy^2)(9x^2) - (6xy^2)(2) = 18x^2y^3 - 54x^3y^2 - 12xy^2$ ✓. We summarize how to find the Greatest Common Factor (G.C.F.) of a polynomial expression below.

Finding the G.C.F. of a Polynomial Expression

- If the coefficients are integers, find the G.C.F. of the coefficients.

  NOTE 1: If all of the coefficients are negative, consider the negative as part of the G.C.F..

  NOTE 2: If the coefficients involve fractions, get a common denominator, combine numerators, reduce to lowest terms and apply this step to the polynomial in the numerator.

- If a variable is common to all of the terms, the G.C.F. contains that variable to the smallest exponent which appears among the terms.

For example, to factor $-\frac{3}{5}z^3 - 6z^2$, we would first get a common denominator and factor as:

$$-\frac{3}{5}z^3 - 6z^2 = \frac{-3z^3 - 30z^2}{5} = \frac{-3z^2(z+10)}{5} = -\frac{3z^2(z+10)}{5} = -\frac{3}{5}z^2(z+10)$$

We now list some common factoring formulas, each of which can be verified by multiplying out the right side of the equation. While they all should look familiar - this is a review section after all - some should look more familiar than others since they appeared as 'special product' formulas in the previous section.

Common Factoring Formulas

- **Perfect Square Trinomials:** $a^2 + 2ab + b^2 = (a+b)^2$ and $a^2 - 2ab + b^2 = (a-b)^2$

- **Difference of Two Squares:** $a^2 - b^2 = (a-b)(a+b)$

  NOTE: In general, the sum of squares, $a^2 + b^2$ is irreducible over the rationals.

- **Sum of Two Cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

  NOTE: In general, $a^2 - ab + b^2$ is irreducible over the rationals.

- **Difference of Two Cubes:** $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

  NOTE: In general, $a^2 + ab + b^2$ is irreducible over the rationals.

The example on the next page gives us practice with these formulas.

Example A.9.1. Factor the following polynomials completely over the rationals. That is, write each polynomial as a product polynomials of lowest degree which are irreducible over the rationals.

1. $18x^2 - 48x + 32$
2. $64y^2 - 1$
3. $75t^4 + 30t^3 + 3t^2$
4. $w^4z - wz^4$
5. $81 - 16t^4$
6. $x^6 - 64$

Solution.

1. Our first step is to factor out the G.C.F. which in this case is 2. To match what is left with one of the special forms, we rewrite $9x^2 = (3x)^2$ and $16 = 4^2$. Since the 'middle' term is $-24x = -2(4)(3x)$, we see that we have a perfect square trinomial.

$$\begin{array}{rcll} 18x^2 - 48x + 32 & = & 2(9x^2 - 24x + 16) & \text{Factor out G.C.F.} \\ & = & 2((3x)^2 - 2(4)(3x) + (4)^2) & \\ & = & 2(3x-4)^2 & \text{Perfect Square Trinomial: } a = 3x,\ b = 4 \end{array}$$

   Our final answer is $2(3x-4)^2$. To check our work, we multiply out $2(3x-4)^2$ to show that it equals $18x^2 - 48x + 32$.

2. For $64y^2 - 1$, we note that the G.C.F. of the terms is just 1, so there is nothing (of substance) to factor out of both terms. Since $64y^2 - 1$ is the difference of two terms, one of which is a square, we look to the Difference of Squares Formula for inspiration. Seeing $64y^2 = (8y)^2$ and $1 = 1^2$, we get

$$\begin{array}{rcll} 64y^2 - 1 & = & (8y)^2 - 1^2 & \\ & = & (8y-1)(8y+1) & \text{Difference of Squares, } a = 8y,\ b = 1 \end{array}$$

   As before, we can check our answer by multiplying out $(8y-1)(8y+1)$ to show that it equals $64y^2-1$.

3. The G.C.F. of the terms in $75t^4+30t^3+3t^2$ is $3t^2$, so we factor that out first. We identify what remains as a perfect square trinomial:

$$\begin{array}{rcll} 75t^4 + 30t^3 + 3t^2 & = & 3t^2(25t^2 + 10t + 1) & \text{Factor out G.C.F.} \\ & = & 3t^2((5t)^2 + 2(1)(5t) + 1^2) & \\ & = & 3t^2(5t+1)^2 & \text{Perfect Square Trinomial, } a = 5t,\ b = 1 \end{array}$$

   Our final answer is $3t^2(5t+1)^2$, which the reader is invited to check.

4. For $w^4z - wz^4$, we identify the G.C.F. as wz and once we factor it out a difference of cubes is revealed:

$$\begin{array}{rcll} w^4z - wz^4 & = & wz(w^3 - z^3) & \text{Factor out G.C.F.} \\ & = & wz(w-z)(w^2 + wz + z^2) & \text{Difference of Cubes, } a = w,\ b = z \end{array}$$

   Our final answer is $wz(w-z)(w^2+wz+z^2)$. The reader is strongly encouraged to multiply this out to see that it reduces to $w^4z - wz^4$.

5. The G.C.F. of the terms in $81 - 16t^4$ is just 1 so there is nothing of substance to factor out from both terms. With just a difference of two terms, we are limited to fitting this polynomial into either the Difference of Two Squares or Difference of Two Cubes formula. Since the variable here is t^4, and 4 is a multiple of 2, we can think of $t^4 = (t^2)^2$. This means that we can write $16t^4 = (4t^2)^2$ which is a perfect square. (Since 4 is not a multiple of 3, we cannot write t^4 as a perfect cube of a polynomial.) Identifying $81 = 9^2$ and $16t^4 = (4t^2)^2$, we apply the Difference of Squares Formula to get:

$$\begin{array}{rcll} 81 - 16t^4 & = & 9^2 - (4t^2)^2 & \\ & = & (9 - 4t^2)(9 + 4t^2) & \text{Difference of Squares, } a = 9,\ b = 4t^2 \end{array}$$

At this point, we have an opportunity to proceed further. Identifying $9 = 3^2$ and $4t^2 = (2t)^2$, we see that we have another difference of squares in the first quantity, which we can reduce. (The sum of two squares in the second quantity <u>cannot</u> be factored over the rationals.)

$$\begin{array}{rcll} 81 - 16t^4 & = & (9 - 4t^2)(9 + 4t^2) & \\ & = & (3^2 - (2t)^2)(9 + 4t^2) & \\ & = & (3 - 2t)(3 + 2t)(9 + 4t^2) & \text{Difference of Squares, } a = 3,\ b = 2t \end{array}$$

As always, the reader is encouraged to multiply out $(3 - 2t)(3 + 2t)(9 + 4t^2)$ to check the result.

6. With a G.C.F. of 1 and just two terms, $x^6 - 64$ is a candidate for both the Difference of Squares and the Difference of Cubes formulas. Notice that we can identify $x^6 = (x^3)^2$ and $64 = 8^2$ (both perfect squares), but also $x^6 = (x^2)^3$ and $64 = 4^3$ (both perfect cubes). If we follow the Difference of Squares approach, we get:

$$\begin{array}{rcll} x^6 - 64 & = & (x^3)^2 - 8^2 & \\ & = & (x^3 - 8)(x^3 + 8) & \text{Difference of Squares, } a = x^3 \text{ and } b = 8 \end{array}$$

At this point, we have an opportunity to use both the Difference and Sum of Cubes formulas:

$$\begin{array}{rcll} x^6 - 64 & = & (x^3 - 2^3)(x^3 + 2^3) & \\ & = & (x - 2)(x^2 + 2x + 2^2)(x + 2)(x^2 - 2x + 2^2) & \text{Sum / Difference of Cubes, } a = x,\ b = 2 \\ & = & (x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4) & \text{Rearrange factors} \end{array}$$

From this approach, our final answer is $(x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4)$.

Following the Difference of Cubes Formula approach, we get

$$\begin{array}{rcll} x^6 - 64 & = & (x^2)^3 - 4^3 & \\ & = & (x^2 - 4)((x^2)^2 + 4x^2 + 4^2) & \text{Difference of Cubes, } a = x^2,\ b = 4 \\ & = & (x^2 - 4)(x^4 + 4x^2 + 16) & \end{array}$$

At this point, we recognize $x^2 - 4$ as a difference of two squares:

$$\begin{array}{rcll} x^6 - 64 & = & (x^2 - 2^2)(x^4 + 4x^2 + 16) & \\ & = & (x - 2)(x + 2)(x^4 + 4x^2 + 16) & \text{Difference of Squares, } a = x,\ b = 2 \end{array}$$

Unfortunately, the remaining factor $x^4 + 4x^2 + 16$ is not a perfect square trinomial - the middle term would have to be $8x^2$ for this to work - so our final answer using this approach is $(x-2)(x+2)(x^4 + 4x^2 + 16)$. This isn't as factored as our result from the Difference of Squares approach which was $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$. While it is true that $x^4+4x^2+16 = (x^2-2x+4)(x^2+2x+4)$, there is no 'intuitive' way to motivate this factorization at this point.[5] The moral of the story? When given the option between using the Difference of Squares and Difference of Cubes, start with the Difference of Squares. Our final answer to this problem is $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$. The reader is strongly encouraged to show that this reduces down to $x^6 - 64$ after performing all of the multiplication. □

The formulas on page 1425, while useful, can only take us so far. Thus we need to review some additional factoring strategies which should be good friends from back in the day!

Additional Factoring Formulas

- **'un-F.O.I.L.ing':** Given a trinomial $Ax^2 + Bx + C$, try to reverse the F.O.I.L. process.

  That is, find a, b, c and d such that $Ax^2 + Bx + C = (ax+b)(cx+d)$.

  NOTE: This means $ac = A$, $bd = C$ and $B = ad + bc$.

- **Factor by Grouping:** If the expression contains four terms with no common factors among the four terms, try 'factor by grouping':

$$ac + bc + ad + bd = (a+b)c + (a+b)d = (a+b)(c+d)$$

The techniques of 'un-F.O.I.L.ing' and 'factoring by grouping' are difficult to describe in general but should make sense to you with enough practice. Be forewarned - like all 'Rules of Thumb', these strategies work just often enough to be useful, but you can be sure there are exceptions which will defy any advice given here and will require some 'inspiration' to solve.[6] Even though Chapter 2 will give us more powerful factoring methods, we'll find that, in the end, there is no single algorithm for factoring which works for every polynomial. In other words, there will be times when you just have to try something and see what happens.

Example A.9.2. Factor the following polynomials completely over the integers.[7]

1. $x^2 - x - 6$
2. $2t^2 - 11t + 5$
3. $36 - 11y - 12y^2$
4. $18xy^2 - 54xy - 180x$
5. $2t^3 - 10t^2 + 3t - 15$
6. $x^4 + 4x^2 + 16$

[5] Of course, this begs the question, "How do we know $x^2 - 2x + 4$ and $x^2 + 2x + 4$ are irreducible?" (We were told so on page 1425, but no reason was given.) Stay tuned! We'll get back to this in due course.

[6] Jeff will be sure to pepper the Exercises with these.

[7] This means that all of the coefficients in the factors will be integers. In a rare departure from form, Carl decided to avoid fractions in this set of examples. Don't get complacent, though, because fractions will return with a vengeance soon enough.

Solution.

1. The G.C.F. of the terms $x^2 - x - 6$ is 1 and $x^2 - x - 6$ isn't a perfect square trinomial (Think about why not.) so we try to reverse the F.O.I.L. process and look for integers a, b, c and d such that $(ax + b)(cx + d) = x^2 - x - 6$. To get started, we note that $ac = 1$. Since a and c are meant to be integers, that leaves us with either a and c both being 1, or a and c both being -1. We'll go with $a = c = 1$, since we can factor[8] the negatives into our choices for b and d. This yields $(x + b)(x + d) = x^2 - x - 6$. Next, we use the fact that $bd = -6$. The product is negative so we know that one of b or d is positive and the other is negative. Since b and d are integers, one of b or d is ± 1 and the other is ∓ 6 OR one of b or d is ± 2 and the other is ∓ 3. After some guessing and checking,[9] we find that $x^2 - x - 6 = (x + 2)(x - 3)$.

2. As with the previous example, we check the G.C.F. of the terms in $2t^2 - 11t + 5$, determine it to be 1 and see that the polynomial doesn't fit the pattern for a perfect square trinomial. We now try to find integers a, b, c and d such that $(at + b)(ct + d) = 2t^2 - 11t + 5$. Since $ac = 2$, we have that one of a or c is 2, and the other is 1. (Once again, we ignore the negative options.) At this stage, there is nothing really distinguishing a from c so we choose $a = 2$ and $c = 1$. Now we look for b and d so that $(2t + b)(t + d) = 2t^2 - 11t + 5$. We know $bd = 5$ so one of b or d is ± 1 and the other ± 5. Given that bd is positive, b and d must have the same sign. The negative middle term $-11t$ guides us to guess $b = -1$ and $d = -5$ so that we get $(2t - 1)(t - 5) = 2t^2 - 11t + 5$. We verify our answer by multiplying.[10]

3. Once again, we check for a nontrivial G.C.F. and see if $36 - 11y - 12y^2$ fits the pattern of a perfect square. Twice disappointed, we rewrite $36 - 11y - 12y^2 = -12y^2 - 11y + 36$ for notational convenience. We now look for integers a, b, c and d such that $-12y^2 - 11y + 36 = (ay + b)(cy + d)$. Since $ac = -12$, we know that one of a or c is ± 1 and the other ± 12 OR one of them is ± 2 and the other is ± 6 OR one of them is ± 3 while the other is ± 4. Since their product is -12, however, we know one of them is positive, while the other is negative. To make matters worse, the constant term 36 has its fair share of factors, too. Our answers for b and d lie among the pairs ± 1 and ± 36, ± 2 and ± 18, ± 4 and ± 9, or ± 6. Since we know one of a or c will be negative, we can simplify our choices for b and d and just look at the positive possibilities. After some guessing and checking,[11] we find $(-3y + 4)(4y + 9) = -12y^2 - 11y + 36$.

4. Since the G.C.F. of the terms in $18xy^2 - 54xy - 180x$ is $18x$, we begin the problem by factoring it out first: $18xy^2 - 54xy - 180x = 18x(y^2 - 3y - 10)$. We now focus our attention on $y^2 - 3y - 10$. We can take a and c to both be 1 which yields $(y + b)(y + d) = y^2 - 3y - 10$. Our choices for b and d are among the factor pairs of -10: ± 1 and ± 10 or ± 2 and ± 5, where one of b or d is positive and the other is negative. We find $(y - 5)(y + 2) = y^2 - 3y - 10$. Our final answer is $18xy^2 - 54xy - 180x = 18x(y - 5)(y + 2)$.

[8] Pun intended!

[9] The authors have seen some strange gimmicks that allegedly help students with this step. We don't like them so we're sticking with good old-fashioned guessing and checking.

[10] That's the 'checking' part of 'guessing and checking'.

[11] Some of these guesses can be more 'educated' than others. Since the middle term is relatively 'small,' we don't expect the 'extreme' factors of 36 and 12 to appear, for instance.

5. Since $2t^3 - 10t^2 - 3t + 15$ has four terms, we are pretty much resigned to factoring by grouping. The strategy here is to factor out the G.C.F. from two *pairs* of terms, and see if this reveals a common factor. If we group the first two terms, we can factor out a $2t^2$ to get $2t^3 - 10t^2 = 2t^2(t - 5)$. We now try to factor something out of the last two terms that will leave us with a factor of $(t - 5)$. Sure enough, we can factor out a -3 from both: $-3t + 15 = -3(t - 5)$. Hence, we get

$$2t^3 - 10t^2 - 3t + 15 = 2t^2(t - 5) - 3(t - 5) = (2t^2 - 3)(t - 5)$$

Now the question becomes can we factor $2t^2 - 3$ over the integers? This would require integers a, b, c and d such that $(at + b)(ct + d) = 2t^2 - 3$. Since $ab = 2$ and $cd = -3$, we aren't left with many options - in fact, we really have only four choices: $(2t - 1)(t + 3)$, $(2t + 1)(t - 3)$, $(2t - 3)(t + 1)$ and $(2t + 3)(t - 1)$. None of these produces $2t^2 - 3$ - which means it's irreducible over the integers - thus our final answer is $(2t^2 - 3)(t - 5)$.

6. Our last example, $x^4 + 4x^2 + 16$, is our old friend from Example A.9.1. As noted there, it is not a perfect square trinomial, so we could try to reverse the F.O.I.L. process. This is complicated by the fact that our highest degree term is x^4, so we would have to look at factorizations of the form $(x + b)(x^3 + d)$ as well as $(x^2 + b)(x^2 + d)$. We leave it to the reader to show that neither of those work. This is an example of where 'trying something' pays off. Even though we've stated that it is not a perfect square trinomial, it's pretty close. Identifying $x^4 = (x^2)^2$ and $16 = 4^2$, we'd have $(x^2 + 4)^2 = x^4 + 8x^2 + 16$, but instead of $8x^2$ as our middle term, we only have $4x^2$. We could add in the extra $4x^2$ we need, but to keep the balance, we'd have to subtract it off. Doing so produces an unexpected opportunity:

$$\begin{array}{rcll} x^4 + 4x^2 + 16 & = & x^4 + 4x^2 + 16 + (4x^2 - 4x^2) & \text{Adding and subtracting the same term} \\ & = & x^4 + 8x^2 + 16 - 4x^2 & \text{Rearranging terms} \\ & = & (x^2 + 4)^2 - (2x)^2 & \text{Factoring perfect square trinomial} \\ & = & [(x^2 + 4) - 2x][(x^2 + 4) + 2x] & \text{Difference of Squares: } a = (x^2 + 4),\ b = 2x \\ & = & (x^2 - 2x + 4)(x^2 + 2x + 4) & \text{Rearraging terms} \end{array}$$

We leave it to the reader to check that neither $x^2 - 2x + 4$ nor $x^2 + 2x + 4$ factor over the integers, so we are done. □

A.9.1 Solving Equations by Factoring

Many students wonder why they are forced to learn how to factor. Simply put, factoring is our main tool for solving the non-linear equations which arise in many of the applications of Mathematics.[12] We use factoring in conjunction with the Zero Product Property of Real Numbers which was first stated on page 1333 and is given here again for reference.

The Zero Product Property of Real Numbers: If a and b are real numbers with $ab = 0$ then either $a = 0$ or $b = 0$ or both.

[12] Also known as 'story problems' or 'real-world examples'.

Consider the equation $6x^2 + 11x = 10$. To see how the Zero Product Property is used to help us solve this equation, we first set the equation equal to zero and then apply the techniques from Example A.9.2:

$$\begin{array}{rclr}
6x^2 + 11x & = & 10 & \\
6x^2 + 11x - 10 & = & 0 & \text{Subtract 10 from both sides} \\
(2x+5)(3x-2) & = & 0 & \text{Factor} \\
2x + 5 = 0 & \text{or} & 3x - 2 = 0 & \text{Zero Product Property} \\
x = -\frac{5}{2} & \text{or} & x = \frac{2}{3} & a = 2x+5,\, b = 3x-2
\end{array}$$

The reader should check that both of these solutions satisfy the original equation.

It is critical that you see the importance of setting the expression equal to 0 before factoring. Otherwise, we'd get something silly like:

$$\begin{array}{rcll}
6x^2 + 11x & = & 10 & \\
x(6x+11) & = & 10 & \text{Factor}
\end{array}$$

What we **cannot** deduce from this equation is that $x = 10$ or $6x + 11 = 10$ or that $x = 2$ and $6x + 11 = 5$. (It's wrong and you should feel bad if you do it.) It is precisely because 0 plays such a special role in the arithmetic of real numbers (as the Additive Identity) that we can assume a factor is 0 when the product is 0. No other real number has that ability.

We summarize the **correct** equation solving strategy below.

Strategy for Solving Non-linear Equations

1. Put all of the nonzero terms on one side of the equation so that the other side is 0.
2. Factor.
3. Use the Zero Product Property of Real Numbers and set each factor equal to 0.
4. Solve each of the resulting equations.

Let's finish the section with a collection of examples in which we use this strategy.

Example A.9.3. Solve the following equations.

1. $3x^2 = 35 - 16x$
2. $t = \dfrac{1 + 4t^2}{4}$
3. $(y-1)^2 = 2(y-1)$
4. $\dfrac{w^4}{3} = \dfrac{8w^3 - 12}{12} - \dfrac{w^2 - 4}{4}$
5. $z(z(18z+9) - 50) = 25$
6. $x^4 - 8x^2 - 9 = 0$

Solution.

1. We begin by gathering all of the nonzero terms to one side getting 0 on the other and then we proceed to factor and apply the Zero Product Property.

$$\begin{array}{rcll} 3x^2 & = & 35-16x & \\ 3x^2+16x-35 & = & 0 & \text{Add } 16x \text{, subtract } 35 \\ (3x-5)(x+7) & = & 0 & \text{Factor} \\ 3x-5=0 & \text{or} & x+7=0 & \text{Zero Product Property} \\ x=\frac{5}{3} & \text{or} & x=-7 & \end{array}$$

   We check our answers by substituting each of them into the original equation. Plugging in $x=\frac{5}{3}$ yields $\frac{25}{3}$ on both sides while $x=-7$ gives 147 on both sides.

2. To solve $t=\frac{1+4t^2}{4}$, we first clear fractions then move all of the nonzero terms to one side of the equation, factor and apply the Zero Product Property.

$$\begin{array}{rcll} t & = & \dfrac{1+4t^2}{4} & \\ 4t & = & 1+4t^2 & \text{Clear fractions (multiply by 4)} \\ 0 & = & 1+4t^2-4t & \text{Subtract 4} \\ 0 & = & 4t^2-4t+1 & \text{Rearrange terms} \\ 0 & = & (2t-1)^2 & \text{Factor (Perfect Square Trinomial)} \end{array}$$

   At this point, we get $(2t-1)^2=(2t-1)(2t-1)=0$, so, the Zero Product Property gives us $2t-1=0$ in both cases.[13] Our final answer is $t=\frac{1}{2}$, which we invite the reader to check.

3. Following the strategy outlined above, the first step to solving $(y-1)^2=2(y-1)$ is to gather the nonzero terms on one side of the equation with 0 on the other side and factor.

$$\begin{array}{rcll} (y-1)^2 & = & 2(y-1) & \\ (y-1)^2-2(y-1) & = & 0 & \text{Subtract } 2(y-1) \\ (y-1)[(y-1)-2] & = & 0 & \text{Factor out G.C.F.} \\ (y-1)(y-3) & = & 0 & \text{Simplify} \\ y-1=0 & \text{or} & y-3=0 & \\ y=1 & \text{or} & y=3 & \end{array}$$

   Both of these answers are easily checked by substituting them into the original equation.

   An alternative method to solving this equation is to begin by dividing both sides by $(y-1)$ to simplify things outright. As we saw in Example A.4.1, however, whenever we divide by a variable quantity, we make the explicit assumption that this quantity is nonzero. Thus we must stipulate that $y-1\neq 0$.

$$\begin{array}{rcll} \dfrac{(y-1)^2}{(y-1)} & = & \dfrac{2(y-1)}{(y-1)} & \text{Divide by } (y-1) \text{ - this assumes } (y-1)\neq 0 \\ y-1 & = & 2 & \\ y & = & 3 & \end{array}$$

[13] More generally, given a positive power p, the only solution to $x^p=0$ is $x=0$.

Note that in this approach, we obtain the $y = 3$ solution, but we 'lose' the $y = 1$ solution. How did that happen? Assuming $y - 1 \neq 0$ is equivalent to assuming $y \neq 1$. This is an issue because $y = 1$ is a solution to the original equation and it was 'divided out' too early. The moral of the story? If you decide to divide by a variable expression, double check that you aren't excluding any solutions.[14]

4. Proceeding as before, we clear fractions, gather the nonzero terms on one side of the equation, have 0 on the other and factor.

$$\begin{array}{rcll}
\dfrac{w^4}{3} & = & \dfrac{8w^3 - 12}{12} - \dfrac{w^2 - 4}{4} & \\
12\left(\dfrac{w^4}{3}\right) & = & 12\left(\dfrac{8w^3 - 12}{12} - \dfrac{w^2 - 4}{4}\right) & \text{Multiply by 12} \\
4w^4 & = & (8w^3 - 12) - 3(w^2 - 4) & \text{Distribute} \\
4w^4 & = & 8w^3 - 12 - 3w^2 + 12 & \text{Distribute} \\
0 & = & 8w^3 - 12 - 3w^2 + 12 - 4w^4 & \text{Subtract } 4w^4 \\
0 & = & 8w^3 - 3w^2 - 4w^4 & \text{Gather like terms} \\
0 & = & w^2(8w - 3 - 4w^2) & \text{Factor out G.C.F.}
\end{array}$$

At this point, we apply the Zero Product Property to deduce that $w^2 = 0$ or $8w - 3 - 4w^2 = 0$. From $w^2 = 0$, we get $w = 0$. To solve $8w - 3 - 4w^2 = 0$, we rearrange terms and factor: $-4w^2 + 8w - 3 = (2w - 1)(-2w + 3) = 0$. Applying the Zero Product Property again, we get $2w - 1 = 0$ (which gives $w = \frac{1}{2}$), or $-2w + 3 = 0$ (which gives $w = \frac{3}{2}$). Our final answers are $w = 0$, $w = \frac{1}{2}$ and $w = \frac{3}{2}$. The reader is encouraged to check each of these answers in the original equation. (You need the practice with fractions!)

5. For our next example, we begin by subtracting the 25 from both sides then work out the indicated operations before factoring by grouping.

$$\begin{array}{rcll}
z(z(18z + 9) - 50) & = & 25 & \\
z(z(18z + 9) - 50) - 25 & = & 0 & \text{Subtract 25} \\
z(18z^2 + 9z - 50) - 25 & = & 0 & \text{Distribute} \\
18z^3 + 9z^2 - 50z - 25 & = & 0 & \text{Distribute} \\
9z^2(2z + 1) - 25(2z + 1) & = & 0 & \text{Factor} \\
(9z^2 - 25)(2z + 1) & = & 0 & \text{Factor}
\end{array}$$

At this point, we use the Zero Product Property and get $9z^2 - 25 = 0$ or $2z + 1 = 0$. The latter gives $z = -\frac{1}{2}$ whereas the former factors as $(3z - 5)(3z + 5) = 0$. Applying the Zero Product Property again gives $3z - 5 = 0$ (so $z = \frac{5}{3}$) or $3z + 5 = 0$ (so $z = -\frac{5}{3}$.) Our final answers are $z = -\frac{1}{2}$, $z = \frac{5}{3}$ and $z = -\frac{5}{3}$, each of which is good fun to check.

6. The nonzero terms of the equation $x^4 - 8x^2 - 9 = 0$ are already on one side of the equation so we proceed to factor. This trinomial doesn't fit the pattern of a perfect square so we attempt to reverse

[14]You will see other examples throughout this text where dividing by a variable quantity does more harm than good. Keep this basic one in mind as you move on in your studies - it's a good cautionary tale.

the F.O.I.L.ing process. With an x^4 term, we have two possible forms to try: $(ax^2 + b)(cx^2 + d)$ and $(ax^3 + b)(cx + d)$. We leave it to you to show that $(ax^3 + b)(cx + d)$ does not work and we show that $(ax^2 + b)(cx^2 + d)$ does.

Since the coefficient of x^4 is 1, we take $a = c = 1$. The constant term is -9 so we know b and d have opposite signs and our choices are limited to two options: either b and d come from ± 1 and ± 9 OR one is 3 while the other is -3. After some trial and error, we get $x^4 - 8x^2 - 9 = (x^2 - 9)(x^2 + 1)$. Hence $x^4 - 8x^2 - 9 = 0$ reduces to $(x^2 - 9)(x^2 + 1) = 0$. The Zero Product Property tells us that either $x^2 - 9 = 0$ or $x^2 + 1 = 0$. To solve the former, we factor: $(x - 3)(x + 3) = 0$, so $x - 3 = 0$ (hence, $x = 3$) or $x + 3 = 0$ (hence, $x = -3$). The equation $x^2 + 1 = 0$ has no (real) solution, since for any real number x, x^2 is always 0 or greater. Thus $x^2 + 1$ is always positive. Our final answers are $x = 3$ and $x = -3$. As always, the reader is invited to check both answers in the original equation. □

A.9.2 Exercises

In Exercises 1 - 30, factor completely over the integers. Check your answer by multiplication.

1. $2x - 10x^2$
2. $12t^5 - 8t^3$
3. $16xy^2 - 12x^2y$
4. $5(m+3)^2 - 4(m+3)^3$
5. $(2x-1)(x+3) - 4(2x-1)$
6. $t^2(t-5) + t - 5$
7. $w^2 - 121$
8. $49 - 4t^2$
9. $81t^4 - 16$
10. $9z^2 - 64y^4$
11. $(y+3)^2 - 4y^2$
12. $(x+h)^3 - (x+h)$
13. $y^2 - 24y + 144$
14. $25t^2 + 10t + 1$
15. $12x^3 - 36x^2 + 27x$
16. $m^4 + 10m^2 + 25$
17. $27 - 8x^3$
18. $t^6 + t^3$
19. $x^2 - 5x - 14$
20. $y^2 - 12y + 27$
21. $3t^2 + 16t + 5$
22. $6x^2 - 23x + 20$
23. $35 + 2m - m^2$
24. $7w - 2w^2 - 3$
25. $3m^3 + 9m^2 - 12m$
26. $x^4 + x^2 - 20$
27. $4(t^2-1)^2 + 3(t^2-1) - 10$
28. $x^3 - 5x^2 - 9x + 45$
29. $3t^2 + t - 3 - t^3$
30. [15] $y^4 + 5y^2 + 9$

In Exercises 31 - 45, find all rational number solutions. Check your answers.

31. $(7x+3)(x-5) = 0$
32. $(2t-1)^2(t+4) = 0$
33. $(y^2+4)(3y^2+y-10) = 0$
34. $4t = t^2$
35. $y + 3 = 2y^2$
36. $26x = 8x^2 + 21$
37. $16x^4 = 9x^2$
38. $w(6w+11) = 10$
39. $2w^2 + 5w + 2 = -3(2w+1)$
40. $x^2(x-3) = 16(x-3)$
41. $(2t+1)^3 = (2t+1)$
42. $a^4 + 4 = 6 - a^2$
43. $\dfrac{8t^2}{3} = 2t + 3$
44. $\dfrac{x^3+x}{2} = \dfrac{x^2+1}{3}$
45. $\dfrac{y^4}{3} - y^2 = \dfrac{3}{2}(y^2+3)$
46. With help from your classmates, factor $4x^4 + 8x^2 + 9$.
47. With help from your classmates, find an equation which has 3, $-\frac{1}{2}$, and 117 as solutions.

[15] $y^4 + 5y^2 + 9 = (y^4 + 6y^2 + 9) - y^2$

A.9.3 Answers

1. $2x(1-5x)$
2. $4t^3(3t^2-2)$
3. $4xy(4y-3x)$
4. $-(m+3)^2(4m+7)$
5. $(2x-1)(x-1)$
6. $(t-5)(t^2+1)$
7. $(w-11)(w+11)$
8. $(7-2t)(7+2t)$
9. $(3t-2)(3t+2)(9t^2+4)$
10. $(3z-8y^2)(3z+8y^2)$
11. $-3(y-3)(y+1)$
12. $(x+h)(x+h-1)(x+h+1)$
13. $(y-12)^2$
14. $(5t+1)^2$
15. $3x(2x-3)^2$
16. $(m^2+5)^2$
17. $(3-2x)(9+6x+4x^2)$
18. $t^3(t+1)(t^2-t+1)$
19. $(x-7)(x+2)$
20. $(y-9)(y-3)$
21. $(3t+1)(t+5)$
22. $(2x-5)(3x-4)$
23. $(7-m)(5+m)$
24. $(-2w+1)(w-3)$
25. $3m(m-1)(m+4)$
26. $(x-2)(x+2)(x^2+5)$
27. $(2t-3)(2t+3)(t^2+1)$
28. $(x-3)(x+3)(x-5)$
29. $(t-3)(1-t)(1+t)$
30. $(y^2-y+3)(y^2+y+3)$
31. $x=-\frac{3}{7}$ or $x=5$
32. $t=\frac{1}{2}$ or $t=-4$
33. $y=\frac{5}{3}$ or $y=-2$
34. $t=0$ or $t=4$
35. $y=-1$ or $y=\frac{3}{2}$
36. $x=\frac{3}{2}$ or $x=\frac{7}{4}$
37. $x=0$ or $x=\pm\frac{3}{4}$
38. $w=-\frac{5}{2}$ or $w=\frac{2}{3}$
39. $w=-5$ or $w=-\frac{1}{2}$
40. $x=3$ or $x=\pm 4$
41. $t=-1$, $t=-\frac{1}{2}$, or $t=0$
42. $a=\pm 1$
43. $t=-\frac{3}{4}$ or $t=\frac{3}{2}$
44. $x=\frac{2}{3}$
45. $y=\pm 3$

A.10 Quadratic Equations

In Section A.9.1, we reviewed how to solve basic non-linear equations by factoring. The astute reader should have noticed that all of the equations in that section were carefully constructed so that the polynomials could be factored using the integers. To demonstrate just how contrived the equations had to be, we can solve $2x^2 + 5x - 3 = 0$ by factoring, $(2x - 1)(x + 3) = 0$, from which we obtain $x = \frac{1}{2}$ and $x = -3$. If we change the 5 to a 6 and try to solve $2x^2 + 6x - 3 = 0$, however, we find that this polynomial doesn't factor over the integers and we are stuck. It turns out that there are two real number solutions to this equation, but they are *irrational* numbers, and the goal of this section is to review the techniques which allow us to find these solutions.[1] In this section, we focus our attention on **quadratic** equations.

Definition A.18. An equation is said to be **quadratic** in a variable x if it can be written in the form $ax^2 + bx + c = 0$ where a, b and c are expressions which do not involve x and $a \neq 0$.

Think of quadratic equations as equations that are one degree up from linear equations - instead of the highest power of x being just $x = x^1$, it's x^2. The simplest class of quadratic equations to solve are the ones in which $b = 0$. In that case, we have the following.

Solving Quadratic Equations by Extracting Square Roots

If c is a real number with $c \geq 0$, the solutions to $x^2 = c$ are $x = \pm\sqrt{c}$.

Note: If $c < 0$, $x^2 = c$ has no real number solutions.

There are a couple different ways to see why Extracting Square Roots works, both of which are demonstrated by solving the equation $x^2 = 3$. If we follow the procedure outlined in the previous section, we subtract 3 from both sides to get $x^2 - 3 = 0$ and we now try to factor $x^2 - 3$. As mentioned in the remarks following Definition A.17, we could think of $x^2 - 3 = x^2 - (\sqrt{3})^2$ and apply the Difference of Squares formula to factor $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. We solve $(x - \sqrt{3})(x + \sqrt{3}) = 0$ by using the Zero Product Property as before by setting each factor equal to zero: $x - \sqrt{3} = 0$ and $x + \sqrt{3} - 0$. We get the answers $x = \pm\sqrt{3}$. In general, if $c \geq 0$, then $\sqrt{c}$ is a real number, so $x^2 - c = x^2 - (\sqrt{c})^2 = (x - \sqrt{c})(x + \sqrt{c})$. Replacing the '3' with 'c' in the above discussion gives the general result.

Another way to view this result is to visualize 'taking the square root' of both sides: since $x^2 = c$, $\sqrt{x^2} = \sqrt{c}$. How do we simplify $\sqrt{x^2}$? We have to exercise a bit of caution here. Note that $\sqrt{(5)^2}$ and $\sqrt{(-5)^2}$ both simplify to $\sqrt{25} = 5$. In both cases, $\sqrt{x^2}$ returned a *positive* number, since the negative in -5 was 'squared away' *before* we took the square root. In other words, $\sqrt{x^2}$ is x if x is positive, or, if x is negative, we make x positive - that is, $\sqrt{x^2} = |x|$, the absolute value of x. So from $x^2 = 3$, we 'take the square root' of both sides of the equation to get $\sqrt{x^2} = \sqrt{3}$. This simplifies to $|x| = \sqrt{3}$, which by Theorem A.5 is equivalent to $x = \sqrt{3}$ or $x = -\sqrt{3}$. Replacing the '3' in the previous argument with 'c,' gives the general result.

As you might expect, Extracting Square Roots can be applied to more complicated equations. Consider the equation below. We can solve it by Extracting Square Roots provided we first isolate the quantity that

[1] While our discussion in this section departs from factoring, we'll see in Chapter 2 that the same correspondence between factoring and solving equations holds whether or not the polynomial factors over the integers.

is being squared :

$$
\begin{array}{rcll}
2\left(x+\frac{3}{2}\right)^2-\frac{15}{2} & = & 0 & \\
2\left(x+\frac{3}{2}\right)^2 & = & \frac{15}{2} & \text{Add } \frac{15}{2} \\
\left(x+\frac{3}{2}\right)^2 & = & \frac{15}{4} & \text{Divide by 2} \\
x+\frac{3}{2} & = & \pm\sqrt{\frac{15}{4}} & \text{Extract Square Roots} \\
x+\frac{3}{2} & = & \pm\frac{\sqrt{15}}{2} & \text{Property of Radicals} \\
x & = & -\frac{3}{2}\pm\frac{\sqrt{15}}{2} & \text{Subtract } \frac{3}{2} \\
x & = & -\frac{3\pm\sqrt{15}}{2} & \text{Add fractions}
\end{array}
$$

Let's return to the equation $2x^2+6x-3=0$ from the beginning of the section. We leave it to the reader to expand the left side and show that

$$2\left(x+\frac{3}{2}\right)^2-\frac{15}{2}=2x^2+6x-3.$$

In other words, we can solve $2x^2+6x-3=0$ by *transforming* into an equivalent equation. This process, you may recall, is called 'Completing the Square.' We'll revisit Completing the Square in Section 1.4 in more generality and for a different purpose but for now we revisit the steps needed to complete the square to solve a quadratic equation.

Solving Quadratic Equations: Completing the Square

To solve a quadratic equation $ax^2+bx+c=0$ by Completing the Square:

1. Subtract the constant c from both sides.
2. Divide both sides by a, the coefficient of x^2. (Remember: $a \neq 0$.)
3. Add $\left(\frac{b}{2a}\right)^2$ to both sides of the equation. (That's half the coefficient of x, squared.)
4. Factor the left hand side of the equation as $\left(x+\frac{b}{2a}\right)^2$.
5. Extract Square Roots.
6. Subtract $\frac{b}{2a}$ from both sides.

To refresh our memories, we apply this method to solve $3x^2 - 24x + 5 = 0$:

$$\begin{array}{rcll}
3x^2 - 24x + 5 & = & 0 & \\
3x^2 - 24x & = & -5 & \text{Subtract } c = 55 \\
x^2 - 8x & = & -\dfrac{5}{3} & \text{Divide by } a = 3 \\
x^2 - 8x + 16 & = & -\dfrac{5}{3} + 16 & \text{Add } \left(\frac{b}{2a}\right)^2 = (-4)^2 = 16 \\
(x-4)^2 & = & \dfrac{43}{3} & \text{Factor: Perfect Square Trinomial} \\
x - 4 & = & \pm\sqrt{\dfrac{43}{3}} & \text{Extract Square Roots} \\
x & = & 4 \pm \sqrt{\dfrac{43}{3}} & \text{Add 4}
\end{array}$$

At this point, we use properties of fractions and radicals to 'rationalize' the denominator:[2]

$$\sqrt{\frac{43}{3}} = \sqrt{\frac{43 \cdot 3}{3 \cdot 3}} = \frac{\sqrt{129}}{\sqrt{9}} = \frac{\sqrt{129}}{3}$$

We can now get a common (integer) denominator which yields:

$$x = 4 \pm \sqrt{\frac{43}{3}} = 4 \pm \frac{\sqrt{129}}{3} = \frac{12 \pm \sqrt{129}}{3}$$

The key to Completing the Square is that the procedure always produces a perfect square trinomial. To see why this works *every single time*, we start with $ax^2 + bx + c = 0$ and follow the procedure:

$$\begin{array}{rcll}
ax^2 + bx + c & = & 0 & \\
ax^2 + bx & = & -c & \text{Subtract } c \\
x^2 + \dfrac{bx}{a} & = & -\dfrac{c}{a} & \text{Divide by } a \neq 0 \\
x^2 + \dfrac{bx}{a} + \left(\dfrac{b}{2a}\right)^2 & = & -\dfrac{c}{a} + \left(\dfrac{b}{2a}\right)^2 & \text{Add } \left(\dfrac{b}{2a}\right)^2
\end{array}$$

(Hold onto the line above for a moment.) Here's the heart of the method - we need to show that

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2$$

To show this, we start with the right side of the equation and apply the Perfect Square Formula from Theorem A.9

$$\left(x + \frac{b}{2a}\right)^2 = x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 \checkmark$$

[2]Recall that this means we want to get a denominator with rational (more specifically, integer) numbers.

With just a few more steps we can solve the general equation $ax^2 + bx + c = 0$ so let's pick up the story where we left off. (The line on the previous page we told you to hold on to.)

$$
\begin{array}{rcll}
x^2 + \dfrac{bx}{a} + \left(\dfrac{b}{2a}\right)^2 & = & -\dfrac{c}{a} + \left(\dfrac{b}{2a}\right)^2 & \\
\left(x + \dfrac{b}{2a}\right)^2 & = & -\dfrac{c}{a} + \dfrac{b^2}{4a^2} & \text{Factor: Perfect Square Trinomial} \\
\left(x + \dfrac{b}{2a}\right)^2 & = & -\dfrac{4ac}{4a^2} + \dfrac{b^2}{4a^2} & \text{Get a common denominator} \\
\left(x + \dfrac{b}{2a}\right)^2 & = & \dfrac{b^2 - 4ac}{4a^2} & \text{Add fractions} \\
x + \dfrac{b}{2a} & = & \pm\sqrt{\dfrac{b^2 - 4ac}{4a^2}} & \text{Extract Square Roots} \\
x + \dfrac{b}{2a} & = & \pm\dfrac{\sqrt{b^2 - 4ac}}{2a} & \text{Properties of Radicals} \\
x & = & -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a} & \text{Subtract } \dfrac{b}{2a} \\
x & = & \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} & \text{Add fractions.}
\end{array}
$$

Lo and behold, we have derived the legendary **Quadratic Formula**!

Theorem A.11. Quadratic Formula: The solution(s) to $ax^2 + bx + c = 0$ with $a \neq 0$ is/are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can check our earlier solutions to $2x^2 + 6x - 3 = 0$ and $3x^2 - 24x + 5 = 0$ using the Quadratic Formula. For $2x^2 + 6x - 3 = 0$, we identify $a = 2$, $b = 6$ and $c = -3$. The quadratic formula gives:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(2)(-3)}}{2(2)} = \frac{-6 \pm \sqrt{36 + 24}}{4} = \frac{-6 \pm \sqrt{60}}{4}$$

Using properties of radicals ($\sqrt{60} = 2\sqrt{15}$), this reduces to $\frac{2(-3 \pm \sqrt{15})}{4} = \frac{-3 \pm \sqrt{15}}{2}$. We leave it to the reader to show these two answers are the same as $-\frac{3 \pm \sqrt{15}}{2}$, as required.[3]

For $3x^2 - 24x + 5 = 0$, we identify $a = 3$, $b = -24$ and $c = 5$. Here, we get:

$$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(3)(5)}}{2(3)} = \frac{24 \pm \sqrt{516}}{6}$$

Since $\sqrt{516} = 2\sqrt{129}$, this reduces to $x = \frac{12 \pm \sqrt{129}}{3}$.

[3] Think about what $-(3 \pm \sqrt{15})$ is really telling you.

It is worth noting that the Quadratic Formula applies to all quadratic equations - even ones we could solve using other techniques. For example, to solve $2x^2 + 5x - 3 = 0$ we identify $a = 2$, $b = 5$ and $c = -3$. Plugging those into the Quadratic Formula yields:

$$x = \frac{-5 \pm \sqrt{5^2 - 4(2)(-3)}}{2(2)} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4}$$

At this point, we have $x = \frac{-5+7}{4} = \frac{1}{2}$ and $x = \frac{-5-7}{4} = \frac{-12}{4} = -3$ - the same two answers we obtained factoring. We can also use it to solve $x^2 = 3$, if we wanted to. From $x^2 - 3 = 0$, we have $a = 1$, $b = 0$ and $c = -3$. The Quadratic Formula produces

$$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(3)}}{2(1)} = \frac{\pm\sqrt{12}}{2} = \pm\frac{2\sqrt{3}}{2} = \pm\sqrt{3}$$

As this last example illustrates, while the Quadratic Formula *can* be used to solve every quadratic equation, that doesn't mean it *should* be used. Many times other methods are more efficient. We now provide a more comprehensive approach to solving Quadratic Equations.

Strategies for Solving Quadratic Equations

- If the variable appears in the squared term only, isolate it and Extract Square Roots.
- Otherwise, put the nonzero terms on one side of the equation so that the other side is 0.
  - Try factoring.
  - If the expression doesn't factor easily, use the Quadratic Formula.

The reader is encouraged to pause for a moment to think about why 'Completing the Square' doesn't appear in our list of strategies despite the fact that we've spent the majority of the section so far talking about it.[4] Let's get some practice solving quadratic equations, shall we?

Example A.10.1. Find all real number solutions to the following equations.

1. $3 - (2w - 1)^2 = 0$
2. $5x - x(x - 3) = 7$
3. $(y - 1)^2 = 2 - \dfrac{y + 2}{3}$
4. $5(25 - 21x) = \dfrac{59}{4} - 25x^2$
5. $-4.9t^2 + 10t\sqrt{3} + 2 = 0$
6. $2x^2 = 3x^4 - 6$

Solution.

1. Since $3 - (2w - 1)^2 = 0$ contains a perfect square, we isolate it first then extract square roots:

$$\begin{array}{rcll}
3 - (2w-1)^2 & = & 0 & \\
3 & = & (2w-1)^2 & \text{Add } (2w-1)^2 \\
\pm\sqrt{3} & = & 2w - 1 & \text{Extract Square Roots} \\
1 \pm \sqrt{3} & = & 2w & \text{Add 1} \\
\dfrac{1 \pm \sqrt{3}}{2} & = & w & \text{Divide by 2}
\end{array}$$

[4]Unacceptable answers include "Jeff and Carl are mean" and "It was one of Carl's Pedantic Rants".

We find our two answers $w = \frac{1 \pm \sqrt{3}}{2}$. The reader is encouraged to check both answers by substituting each into the original equation.[5]

2. To solve $5x - x(x-3) = 7$, we perform the indicated operations and set one side equal to 0.

$$\begin{array}{rcll} 5x - x(x-3) & = & 7 & \\ 5x - x^2 + 3x & = & 7 & \text{Distribute} \\ -x^2 + 8x & = & 7 & \text{Gather like terms} \\ -x^2 + 8x - 7 & = & 0 & \text{Subtract 7} \end{array}$$

At this point, we attempt to factor and find $-x^2 + 8x - 7 = (x-1)(-x+7)$. Using the Zero Product Property, we get $x - 1 = 0$ or $-x + 7 = 0$. Our answers are $x = 1$ or $x = 7$, which are easily verified.

3. Even though we have a perfect square in $(y-1)^2 = 2 - \frac{y+2}{3}$, Extracting Square Roots won't help matters since we have a y on the other side of the equation. Our strategy here is to perform the indicated operations (and clear the fraction for good measure) and get 0 on one side of the equation.

$$\begin{array}{rcll} (y-1)^2 & = & 2 - \dfrac{y+2}{3} & \\ y^2 - 2y + 1 & = & 2 - \dfrac{y+2}{3} & \text{Perfect Square Trinomial} \\ 3(y^2 - 2y + 1) & = & 3\left(2 - \dfrac{y+2}{3}\right) & \text{Multiply by 3} \\ 3y^2 - 6y + 3 & = & 6 - 3\left(\dfrac{y+2}{3}\right) & \text{Distribute} \\ 3y^2 - 6y + 3 & = & 6 - (y+2) & \\ 3y^2 - 6y + 3 - 6 + (y+2) & = & 0 & \text{Subtract 6, Add } (y+2) \\ 3y^2 - 5y - 1 & = & 0 & \end{array}$$

A cursory attempt at factoring bears no fruit, so we run this through the Quadratic Formula with $a = 3$, $b = -5$ and $c = -1$.

$$\begin{array}{rcl} y & = & \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} \\ y & = & \dfrac{5 \pm \sqrt{25 + 12}}{6} \\ y & = & \dfrac{5 \pm \sqrt{37}}{6} \end{array}$$

Since 37 is prime, we have no way to reduce $\sqrt{37}$. Thus, our final answers are $y = \frac{5 \pm \sqrt{37}}{6}$. The reader is encouraged to supply the details of the challenging verification of the answers.

[5] It's excellent practice working with radicals and fractions so we really, *really* want you to take the time to do it.

4. We proceed as before; our goal is to gather the nonzero terms on one side of the equation.

$$\begin{array}{rcll}
5(25-21x) & = & \dfrac{59}{4} - 25x^2 & \\
125 - 105x & = & \dfrac{59}{4} - 25x^2 & \text{Distribute} \\
4(125-105x) & = & 4\left(\dfrac{59}{4} - 25x^2\right) & \text{Multiply by 4} \\
500 - 420x & = & 59 - 100x^2 & \text{Distribute} \\
500 - 420x - 59 + 100x^2 & = & 0 & \text{Subtract 59, Add } 100x^2 \\
100x^2 - 420x + 441 & = & 0 & \text{Gather like terms}
\end{array}$$

With highly composite numbers like 100 and 441, factoring seems inefficient at best,[6] so we apply the Quadratic Formula with $a = 100$, $b = -420$ and $c = 441$:

$$\begin{array}{rcl}
x & = & \dfrac{-(-420) \pm \sqrt{(-420)^2 - 4(100)(441)}}{2(100)} \\
& = & \dfrac{420 \pm \sqrt{176000 - 176400}}{200} \\
& = & \dfrac{420 \pm \sqrt{0}}{200} \\
& = & \dfrac{420 \pm 0}{200} \\
& = & \dfrac{420}{200} \\
& = & \dfrac{21}{10}
\end{array}$$

To our surprise and delight we obtain just one answer, $x = \frac{21}{10}$.

5. Our next equation $-4.9t^2 + 10t\sqrt{3} + 2 = 0$, already has 0 on one side of the equation, but with coefficients like -4.9 and $10\sqrt{3}$, factoring with integers is not an option. We could make things a *bit* easier by clearing the decimal (by multiplying through by 10) to get $-49t^2 + 100t\sqrt{3} + 20 = 0$ but we simply cannot rid ourselves of the irrational number $\sqrt{3}$. The Quadratic Formula is our only recourse. With $a = -49$, $b = 100\sqrt{3}$ and $c = 20$ we get:

[6] This is actually the Perfect Square Trinomial $(10x - 21)^2$.

$$
\begin{aligned}
t &= \frac{-100\sqrt{3} \pm \sqrt{(100\sqrt{3})^2 - 4(-49)(20)}}{2(-49)} \\
&= \frac{-100\sqrt{3} \pm \sqrt{30000 + 3920}}{-98} \\
&= \frac{-100\sqrt{3} \pm \sqrt{33920}}{-98} \\
&= \frac{-100\sqrt{3} \pm 8\sqrt{530}}{-98} \\
&= \frac{2(-50\sqrt{3} \pm 4\sqrt{530})}{2(-49)} \\
&= \frac{-50\sqrt{3} \pm 4\sqrt{530}}{-49} && \text{Reduce} \\
&= \frac{-(-50\sqrt{3} \pm 4\sqrt{530})}{49} && \text{Properties of Negatives} \\
&= \frac{50\sqrt{3} \mp 4\sqrt{530}}{49} && \text{Distribute}
\end{aligned}
$$

You'll note that when we 'distributed' the negative in the last step, we changed the '$\pm$' to a '$\mp$.' While this is technically correct, at the end of the day both symbols mean 'plus or minus',[7] so we can write our answers as $t = \frac{50\sqrt{3} \pm 4\sqrt{530}}{49}$. Checking these answers are a true test of arithmetic mettle.

6. At first glance, the equation $2x^2 = 3x^4 - 6$ seems misplaced. The highest power of the variable x here is 4, not 2, so this equation isn't a quadratic equation - at least not in terms of the variable x. It is, however, an example of an equation that is 'Quadratic in Disguise'.[8] We introduce a new variable u to help us see the pattern - specifically we let $u = x^2$. Thus $u^2 = (x^2)^2 = x^4$. So in terms of the variable u, the equation $2x^2 = 3x^4 - 6$ is $2u = 3u^2 - 6$. The latter is a quadratic equation, which we can solve using the usual techniques:

$$
\begin{aligned}
2u &= 3u^2 - 6 \\
0 &= 3u^2 - 2u - 6 && \text{Subtract } 2u
\end{aligned}
$$

After a few attempts at factoring, we resort to the Quadratic Formula with $a = 3$, $b = -2$ and $c = -6$

[7] There are instances where we need both symbols, however. For example, the Sum and Difference of Cubes Formulas (page 1425) can be written as a single formula: $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$. In this case, all of the 'top' symbols are read to give the sum formula; the 'bottom' symbols give the difference formula.

[8] More formally, **quadratic in form.** Carl likes 'Quadratics in Disguise' since it reminds him of the tagline of one of his beloved childhood cartoons and toy lines.

to get the following:

$$\begin{aligned}
u &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} \\
&= \frac{2 \pm \sqrt{4+72}}{6} \\
&= \frac{2 \pm \sqrt{76}}{6} \\
&= \frac{2 \pm \sqrt{4 \cdot 19}}{6} \\
&= \frac{2 \pm 2\sqrt{19}}{6} && \text{Properties of Radicals} \\
&= \frac{2(1 \pm \sqrt{19})}{2(3)} && \text{Factor} \\
&= \frac{1 \pm \sqrt{19}}{3} && \text{Reduce}
\end{aligned}$$

We've solved the equation for u, but what we still need to solve the original equation[9] - which means we need to find the corresponding values of x. Since $u = x^2$, we have two equations:

$$x^2 = \frac{1+\sqrt{19}}{3} \quad \text{or} \quad x^2 = \frac{1-\sqrt{19}}{3}$$

We can solve the first equation by extracting square roots to get $x = \pm\sqrt{\frac{1+\sqrt{19}}{3}}$. The second equation, however, has no real number solutions because $\frac{1-\sqrt{19}}{3}$ is a negative number. For our final answers we can rationalize the denominator[10] to get:

$$x = \pm\sqrt{\frac{1+\sqrt{19}}{3}} = \pm\sqrt{\frac{1+\sqrt{19}}{3} \cdot \frac{3}{3}} = \pm\frac{\sqrt{3+3\sqrt{19}}}{3}$$

As with the previous exercise, the very challenging check is left to the reader. □

Our last example above, the 'Quadratic in Disguise', hints that the Quadratic Formula is applicable to a wider class of equations than those which are strictly quadratic. We give some general guidelines to recognizing these beasts in the wild on the next page.

[9]Or, you've solved the equation for 'you' (u), now you have to solve it for your instructor (x).

[10]We'll say more about this technique in Section A.13.

Identifying Quadratics in Disguise

An equation is a 'Quadratic in Disguise' if it can be written in the form: $ax^{2m} + bx^m + c = 0$.
In other words:

- There are exactly three terms, two with variables and one constant term.
- The exponent on the variable in one term is *exactly twice* the variable on the other term.

To transform a Quadratic in Disguise to a quadratic equation, let $u = x^m$ so $u^2 = (x^m)^2 = x^{2m}$. This transforms the equation into $au^2 + bu + c = 0$.

For example, $3x^6 - 2x^3 + 1 = 0$ is a Quadratic in Disguise, since $6 = 2 \cdot 3$. If we let $u = x^3$, we get $u^2 = (x^3)^2 = x^6$, so the equation becomes $3u^2 - 2u + 1 = 0$. However, $3x^6 - 2x^2 + 1 = 0$ is *not* a Quadratic in Disguise, since $6 \neq 2 \cdot 2$. The substitution $u = x^2$ yields $u^2 = (x^2)^2 = x^4$, not x^6 as required. We'll see more instances of 'Quadratics in Disguise' in later sections.

We close this section with a review of the **discriminant** of a quadratic equation as defined below.

Definition A.19. The Discriminant: Given a quadratic equation $ax^2 + bx + c = 0$, the quantity $b^2 - 4ac$ is called the **discriminant** of the equation.

The discriminant is the radicand of the square root in the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It *discriminates* between the nature and number of solutions we get from a quadratic equation. The results are summarized below.

Theorem A.12. Discriminant Theorem: Given a Quadratic Equation $ax^2 + bx + c = 0$, let $D = b^2 - 4ac$ be the discriminant.

- If $D > 0$, there are two distinct real number solutions to the equation.
- If $D = 0$, there is one repeated real number solution.

  Note: 'Repeated' here comes from the fact that 'both' solutions $\frac{-b \pm 0}{2a}$ reduce to $-\frac{b}{2a}$.
- If $D < 0$, there are no real solutions.

For example, the equation $x^2 + x - 1 = 0$ has two real number solutions since the discriminant works out to be $(1)^2 - 4(1)(-1) = 5 > 0$. This results in a $\pm\sqrt{5}$ in the Quadratic Formula which then generates two different answers. On the other hand, $x^2 + x + 1 = 0$ has no real solutions since here, the discriminant is $(1)^2 - 4(1)(1) = -3 < 0$ which generates a $\pm\sqrt{-3}$ in the Quadratic Formula. The equation $x^2 + 2x + 1 = 0$ has discriminant $(2)^2 - 4(1)(1) = 0$ so in the Quadratic Formula we get a $\pm\sqrt{0} = 0$ thereby generating just one solution. More can be said as well. For example, the discriminant of $6x^2 - x - 40 = 0$ is 961. This is a perfect square, $\sqrt{961} = 31$, which means our solutions are rational numbers. When our solutions are rational numbers, the quadratic actually factors nicely. In our example $6x^2 - x - 40 = (2x + 5)(3x - 8)$. Admittedly, if you've already computed the discriminant, you're most of the way done with the problem and probably wouldn't take the time to experiment with factoring the quadratic at this point – but we'll see another use for this analysis of the discriminant in Example A.12.1.

A.10.1 Exercises

In Exercises 1 - 21, find all real solutions. Check your answers, as directed by your instructor.

1. $3\left(x - \frac{1}{2}\right)^2 = \frac{5}{12}$
2. $4 - (5t + 3)^2 = 3$
3. $3(y^2 - 3)^2 - 2 = 10$
4. $x^2 + x - 1 = 0$
5. $3w^2 = 2 - w$
6. $y(y + 4) = 1$
7. $\frac{z}{2} = 4z^2 - 1$
8. $0.1v^2 + 0.2v = 0.3$
9. $x^2 = x - 1$
10. $3 - t = 2(t + 1)^2$
11. $(x - 3)^2 = x^2 + 9$
12. $(3y - 1)(2y + 1) = 5y$
13. $w^4 + 3w^2 - 1 = 0$
14. $2x^4 + x^2 = 3$
15. $(2 - y)^4 = 3(2 - y)^2 + 1$
16. $3x^4 + 6x^2 = 15x^3$
17. $6p + 2 = p^2 + 3p^3$
18. $10v = 7v^3 - v^5$
19. $y^2 - \sqrt{8}y = \sqrt{18}y - 1$
20. $x^2\sqrt{3} = x\sqrt{6} + \sqrt{12}$
21. $\frac{v^2}{3} = \frac{v\sqrt{3}}{2} + 1$

In Exercises 22 - 27, find all real solutions and use a calculator to approximate your answers, rounded to two decimal places.

22. $5.54^2 + b^2 = 36$
23. $\pi r^2 = 37$
24. $54 = 8r\sqrt{2} + \pi r^2$
25. $-4.9t^2 + 100t = 410$
26. $x^2 = 1.65(3 - x)^2$
27. $(0.5 + 2A)^2 = 0.7(0.1 - A)^2$

In Exercises 28 - 30, use Theorem A.5 along with the techniques in this section to find all real solutions to the following.

28. $|x^2 - 3x| = 2$
29. $|2x - x^2| = |2x - 1|$
30. $|x^2 - x + 3| = |4 - x^2|$
31. Prove that for every nonzero number p, $x^2 + xp + p^2 = 0$ has no real solutions.
32. Solve for t: $-\frac{1}{2}gt^2 + vt + h = 0$. Assume $g > 0$, $v \geq 0$ and $h \geq 0$.

A.10.2 Answers

1. $x = \dfrac{3 \pm \sqrt{5}}{6}$
2. $t = -\dfrac{4}{5}, -\dfrac{2}{5}$
3. $y = \pm 1, \pm\sqrt{5}$
4. $x = \dfrac{-1 \pm \sqrt{5}}{2}$
5. $w = -1, \dfrac{2}{3}$
6. $y = -2 \pm \sqrt{5}$
7. $z = \dfrac{1 \pm \sqrt{65}}{16}$
8. $v = -3, 1$
9. No real solution.
10. $t = \dfrac{-5 \pm \sqrt{33}}{4}$
11. $x = 0$
12. $y = \dfrac{2 \pm \sqrt{10}}{6}$
13. $w = \pm\sqrt{\dfrac{\sqrt{13} - 3}{2}}$
14. $x = \pm 1$
15. $y = \dfrac{4 \pm \sqrt{6 + 2\sqrt{13}}}{2}$
16. $x = 0, \dfrac{5 \pm \sqrt{17}}{2}$
17. $p = -\dfrac{1}{3}, \pm\sqrt{2}$
18. $v = 0, \pm\sqrt{2}, \pm\sqrt{5}$
19. $y = \dfrac{5\sqrt{2} \pm \sqrt{46}}{2}$
20. $x = \dfrac{\sqrt{2} \pm \sqrt{10}}{2}$
21. $v = -\dfrac{\sqrt{3}}{2}, 2\sqrt{3}$
22. $b = \pm\dfrac{\sqrt{13271}}{50} \approx \pm 2.30$
23. $r = \pm\sqrt{\dfrac{37}{\pi}} \approx \pm 3.43$
24. $r = \dfrac{-4\sqrt{2} \pm \sqrt{54\pi + 32}}{\pi}$, $r \approx -6.32, 2.72$
25. $t = \dfrac{500 \pm 10\sqrt{491}}{49}$, $t \approx 5.68, 14.73$
26. $x = \dfrac{99 \pm 6\sqrt{165}}{13}$, $x \approx 1.69, 13.54$
27. $A = \dfrac{-107 \pm 7\sqrt{70}}{330}$, $A \approx -0.50, -0.15$
28. $x = 1, 2, \dfrac{3 \pm \sqrt{17}}{2}$
29. $x = \pm 1, 2 \pm \sqrt{3}$
30. $x = -\dfrac{1}{2}, 1, 7$
31. The discriminant is: $D = p^2 - 4p^2 = -3p^2 < 0$. Since $D < 0$, there are no real solutions.
32. $t = \dfrac{v \pm \sqrt{v^2 + 2gh}}{g}$

A.11 Complex Numbers

The results of Section A.10 tell us that the equation $x^2 + 1 = 0$ has no real number solutions. However, it *would* have solutions if we could make sense of $\sqrt{-1}$. The **Complex Numbers** do just that - they give us a mechanism for working with $\sqrt{-1}$. As such, the set of complex numbers fill in an algebraic gap left by the set of real numbers.

Here's the basic plan. There is no real number x with $x^2 = -1$, since for any real number $x^2 \geq 0$. However, we could formally extract square roots and write $x = \pm\sqrt{-1}$. We build the complex numbers by relabeling the quantity $\sqrt{-1}$ as i, the unfortunately misnamed **imaginary unit**.[1] The number i, while not a real number, is defined so that it plays along well with real numbers and acts very much like any other radical expression. For instance, $3(2i) = 6i$, $7i - 3i = 4i$, $(2 - 7i) + (3 + 4i) = 5 - 3i$, and so forth. The key properties which distinguish i from the real numbers are listed below.

Definition A.20. The imaginary unit i satisfies the two following properties:

1. $i^2 = -1$
2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$

Property 1 in Definition A.20 establishes that i does act as a square root[2] of -1, and property 2 establishes what we mean by the 'principal square root' of a negative real number. In property 2, it is important to remember the restriction on c. For example, it is perfectly acceptable to say $\sqrt{-4} = i\sqrt{4} = i(2) = 2i$. However, $\sqrt{-(-4)} \neq i\sqrt{-4}$, otherwise, we'd get

$$2 = \sqrt{4} = \sqrt{-(-4)} = i\sqrt{-4} = i(2i) = 2i^2 = 2(-1) = -2,$$

which is unacceptable. The moral of this story is that the general properties of radicals do not apply for even roots of negative quantities. With Definition A.20 in place, we can define the set of **complex numbers**.

Definition A.21. A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit. The set of complex numbers is denoted $\mathbb{C}$.

Complex numbers include things you'd normally expect, like $3 + 2i$ and $\frac{2}{5} - i\sqrt{3}$. However, don't forget that a or b could be zero, which means numbers like $3i$ and 6 are also complex numbers. In other words, don't forget that the complex numbers *include* the real numbers,[3] so 0 and $\pi - \sqrt{21}$ are both considered complex numbers. The arithmetic of complex numbers is as you would expect. The only things you need to remember are the two properties in Definition A.20. The next example should help recall how these animals behave.

[1] Some Technical Mathematics textbooks label it 'j'. While it carries the adjective 'imaginary', these numbers have essential real-world implications. For example, every electronic device owes its existence to the study of 'imaginary' numbers.

[2] Note the use of the indefinite article 'a'. Whatever beast is chosen to be i, $-i$ is the other square root of -1.

[3] To use the language of Section A.1.2, $\mathbb{R} \subseteq \mathbb{C}$.

Example A.11.1. Perform the indicated operations.

1. $(1-2i)-(3+4i)$
2. $(1-2i)(3+4i)$
3. $\dfrac{1-2i}{3-4i}$
4. $\sqrt{-3}\sqrt{-12}$
5. $\sqrt{(-3)(-12)}$
6. $(x-[1+2i])(x-[1-2i])$

Solution.

1. As mentioned earlier, we treat expressions involving i as we would any other radical. We distribute and combine like terms:

$$\begin{array}{rcll} (1-2i)-(3+4i) & = & 1-2i-3-4i & \text{Distribute} \\ & = & -2-6i & \text{Gather like terms} \end{array}$$

   Technically, we'd have to rewrite our answer $-2-6i$ as $(-2)+(-6)i$ to be (in the strictest sense) 'in the form $a+bi$'. That being said, even pedants have their limits, so $-2-6i$ is good enough.

2. Using the Distributive Property (a.k.a. F.O.I.L.), we get

$$\begin{array}{rcll} (1-2i)(3+4i) & = & (1)(3)+(1)(4i)-(2i)(3)-(2i)(4i) & \text{F.O.I.L.} \\ & = & 3+4i-6i-8i^2 & \\ & = & 3-2i-8(-1) & i^2=-1 \\ & = & 3-2i+8 & \\ & = & 11-2i & \end{array}$$

3. How in the world are we supposed to simplify $\frac{1-2i}{3-4i}$? Well, we deal with the denominator $3-4i$ as we would any other denominator containing two terms, one of which is a square root.[4] We multiply both numerator and denominator by $3+4i$, the (complex) conjugate of $3-4i$. Doing so produces

$$\begin{array}{rcll} \dfrac{1-2i}{3-4i} & = & \dfrac{(1-2i)(3+4i)}{(3-4i)(3+4i)} & \text{Equivalent Fractions} \\ & = & \dfrac{3+4i-6i-8i^2}{9-16i^2} & \text{F.O.I.L.} \\ & = & \dfrac{3-2i-8(-1)}{9-16(-1)} & i^2=-1 \\ & = & \dfrac{11-2i}{25} & \\ & = & \dfrac{11}{25}-\dfrac{2}{25}i & \end{array}$$

4. We use property 2 of Definition A.20 first, then apply the rules of radicals applicable to real numbers to get $\sqrt{-3}\sqrt{-12}=\left(i\sqrt{3}\right)\left(i\sqrt{12}\right)=i^2\sqrt{3\cdot 12}=-\sqrt{36}=-6$.

[4]See subsection A.13.1 for a more thorough treatment of this type of maneuver.

5. We adhere to the order of operations here and perform the multiplication before the radical to get $\sqrt{(-3)(-12)} = \sqrt{36} = 6$.

6. We brute force multiply using the distributive property and find that

$$\begin{aligned}
(x-[1+2i])(x-[1-2i]) &= x^2 - x[1-2i] - x[1+2i] + [1-2i][1+2i] && \text{F.O.I.L.} \\
&= x^2 - x + 2ix - x - 2ix + 1 - 2i + 2i - 4i^2 && \text{Distribute} \\
&= x^2 - 2x + 1 - 4(-1) && \text{Gather like terms} \\
&= x^2 - 2x + 5 && i^2 = -1
\end{aligned}$$

This type of factoring will be revisited in Section 2.4. □

In the previous example, we used the 'conjugate' idea from Section A.13 to divide two complex numbers. More generally, the **complex conjugate** of a complex number $a+bi$ is the number $a-bi$. The notation commonly used for complex conjugation is a 'bar': $\overline{a+bi} = a-bi$. For example, $\overline{3+2i} = 3-2i$ and $\overline{3-2i} = 3+2i$. To find $\overline{6}$, we note that $\overline{6} = \overline{6+0i} = 6-0i = 6$, so $\overline{6} = 6$. Similarly, $\overline{4i} = -4i$, since $\overline{4i} = \overline{0+4i} = 0-4i = -4i$. Note that $\overline{3+\sqrt{5}} = 3+\sqrt{5}$, not $3-\sqrt{5}$, since $\overline{3+\sqrt{5}} = \overline{3+\sqrt{5}+0i} = 3+\sqrt{5}-0i = 3+\sqrt{5}$. Here, the conjugation specified by the 'bar' notation involves reversing the sign before $i = \sqrt{-1}$, not before $\sqrt{5}$. The properties of the conjugate are summarized in the following theorem.

Theorem A.13. Properties of the Complex Conjugate: Let z and w be complex numbers.

- $\overline{\overline{z}} = z$
- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- $\overline{z^n} = (\overline{z})^n$, for any natural number n
- z is a real number if and only if $\overline{z} = z$.

Theorem A.13 says in part that complex conjugation works well with addition, multiplication and powers. The proofs of these properties can best be achieved by writing out $z = a+bi$ and $w = c+di$ for real numbers a, b, c and d. Next, we compute the left and right sides of each equation and verify that they are the same.

The proof of the first property is a very quick exercise.[5] To prove the second property, we compare $\overline{z+w}$ with $\overline{z}+\overline{w}$. We have $\overline{z}+\overline{w} = \overline{a+bi} + \overline{c+di} = a-bi+c-di$. To find $\overline{z+w}$, we first compute

$$z+w = (a+bi)+(c+di) = (a+c)+(b+d)i$$

so

$$\overline{z+w} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i = a+c-bi-di = a-bi+c-di = \overline{z}+\overline{w}$$

As such, we have established $\overline{z+w} = \overline{z}+\overline{w}$. The proof for multiplication works similarly. The proof that the conjugate works well with powers can be viewed as a repeated application of the product rule, and is best

[5] Trust us on this.

proved using a technique called Mathematical Induction.[6] The last property is a characterization of real numbers. If z is real, then $z = a + 0i$, so $\overline{z} = a - 0i = a = z$. On the other hand, if $z = \overline{z}$, then $a + bi = a - bi$ which means $b = -b$ so $b = 0$. Hence, $z = a + 0i = a$ and is real.

We now return to the business of solving quadratic equations. Consider $x^2 - 2x + 5 = 0$. The discriminant $b^2 - 4ac = -16$ is negative, so we know by Theorem A.12 there are no *real* solutions, since the Quadratic Formula would involve the term $\sqrt{-16}$. Complex numbers, however, are built just for such situations, so we can go ahead and apply the Quadratic Formula to get:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Example A.11.2. Find the complex solutions to the following equations.[7]

1. $\dfrac{2x}{x+1} = x + 3$
2. $2t^4 = 9t^2 + 5$
3. $z^3 + 1 = 0$

Solution.

1. Clearing fractions yields a quadratic equation so we then proceed as in Section A.10.

$$\begin{array}{rcll}
\dfrac{2x}{x+1} & = & x+3 & \\
2x & = & (x+3)(x+1) & \text{Multiply by } (x+1) \text{ to clear denominators} \\
2x & = & x^2 + x + 3x + 3 & \text{F.O.I.L.} \\
2x & = & x^2 + 4x + 3 & \text{Gather like terms} \\
0 & = & x^2 + 2x + 3 & \text{Subtract } 2x
\end{array}$$

From here, we apply the Quadratic Formula

$$\begin{array}{rcll}
x & = & \dfrac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} & \text{Quadratic Formula} \\
 & = & \dfrac{-2 \pm \sqrt{-8}}{2} & \text{Simplify} \\
 & = & \dfrac{-2 \pm i\sqrt{8}}{2} & \text{Definition of } i \\
 & = & \dfrac{-2 \pm i2\sqrt{2}}{2} & \text{Product Rule for Radicals} \\
 & = & \dfrac{\cancel{2}(-1 \pm i\sqrt{2})}{\cancel{2}} & \text{Factor and reduce} \\
 & = & -1 \pm i\sqrt{2} &
\end{array}$$

We get two answers: $x = -1 + i\sqrt{2}$ and its conjugate $x = -1 - i\sqrt{2}$. Checking both of these answers reviews all of the salient points about complex number arithmetic and is therefore strongly encouraged.

[6]See Section 10.3.

[7]Remember, all real numbers are complex numbers, so 'complex solutions' means both real and non-real answers.

2. Since we have three terms, and the exponent on one term ('4' on t^4) is exactly twice the exponent on the other ('2' on t^2), we have a Quadratic in Disguise. We proceed accordingly.

$$\begin{array}{rcll} 2t^4 & = & 9t^2+5 & \\ 2t^4-9t^2-5 & = & 0 & \text{Subtract } 9t^2 \text{ and } 5 \\ (2t^2+1)(t^2-5) & = & 0 & \text{Factor} \\ 2t^2+1=0 & \text{or} & t^2=5 & \text{Zero Product Property} \end{array}$$

From $2t^2+1=0$ we get $2t^2=-1$, or $t^2=-\frac{1}{2}$. We extract square roots as follows:

$$t=\pm\sqrt{-\frac{1}{2}}=\pm i\sqrt{\frac{1}{2}}=\pm i\frac{\sqrt{1}}{\sqrt{2}}=\pm i\frac{1}{\sqrt{2}}=\pm\frac{i\sqrt{2}}{2},$$

where we have rationalized the denominator per convention. From $t^2=5$, we get $t=\pm\sqrt{5}$. In total, we have four complex solutions - two real: $t=\pm\sqrt{5}$ and two non-real: $t=\pm\frac{i\sqrt{2}}{2}$.

3. To find the *real* solutions to $z^3+1=0$, we can subtract the 1 from both sides and extract cube roots: $z^3=-1$, so $z=\sqrt[3]{-1}=-1$. It turns out there are two more non-real complex number solutions to this equation. To get at these, we factor:

$$\begin{array}{rcll} z^3+1 & = & 0 & \\ (z+1)(z^2-z+1) & = & 0 & \text{Factor (Sum of Two Cubes)} \\ z+1=0 & \text{or} & z^2-z+1=0 & \end{array}$$

From $z+1=0$, we get our real solution $z=-1$. From $z^2-z+1=0$, we apply the Quadratic Formula to get:

$$z=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}=\frac{1\pm\sqrt{-3}}{2}=\frac{1\pm i\sqrt{3}}{2}$$

Thus we get *three* solutions to $z^3+1=0$ - one real: $z=-1$ and two non-real: $z=\frac{1\pm i\sqrt{3}}{2}$. As always, the reader is encouraged to test their algebraic mettle and check these solutions. □

It is no coincidence that the non-real solutions to the equations in Example A.11.2 appear in complex conjugate pairs. Any time we use the Quadratic Formula to solve an equation with real coefficients, the answers will form a complex conjugate pair owing to the $\pm$ in the Quadratic Formula. This leads us to a generalization of Theorem A.12 which we state below.

Theorem A.14. Discriminant Theorem: Given a Quadratic Equation $ax^2+bx+c=0$, where a, b and c are real numbers, let $D=b^2-4ac$ be the discriminant.

- If $D>0$, there are two distinct real number solutions to the equation.
- If $D=0$, there is one (repeated) real number solution.

  Note: 'Repeated' here comes from the fact that 'both' solutions $\frac{-b\pm 0}{2a}$ reduce to $-\frac{b}{2a}$.
- If $D<0$, there are two non-real solutions which form a complex conjugate pair.

We will have much more to say about complex solutions to equations in Section 2.4 and we will revisit Theorem A.14 then.

A.11.1 Exercises

In Exercises 1 - 10, use the given complex numbers z and w to find and simplify the following.

- $z + w$
- zw
- z^2
- $\dfrac{1}{z}$
- $\dfrac{z}{w}$
- $\dfrac{w}{z}$
- $\overline{z}$
- $z\overline{z}$
- $(\overline{z})^2$

1. $z = 2 + 3i$, $w = 4i$
2. $z = 1 + i$, $w = -i$
3. $z = i$, $w = -1 + 2i$
4. $z = 4i$, $w = 2 - 2i$
5. $z = 3 - 5i$, $w = 2 + 7i$
6. $z = -5 + i$, $w = 4 + 2i$
7. $z = \sqrt{2} - i\sqrt{2}$, $w = \sqrt{2} + i\sqrt{2}$
8. $z = 1 - i\sqrt{3}$, $w = -1 - i\sqrt{3}$
9. $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$, $w = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$
10. $z = -\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$, $w = -\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i$

In Exercises 11 - 18, simplify the quantity.

11. $\sqrt{-49}$
12. $\sqrt{-9}$
13. $\sqrt{-25}\sqrt{-4}$
14. $\sqrt{(-25)(-4)}$
15. $\sqrt{-9}\sqrt{-16}$
16. $\sqrt{(-9)(-16)}$
17. $\sqrt{-(-9)}$
18. $-\sqrt{(-9)}$

We know that $i^2 = -1$ which means $i^3 = i^2 \cdot i = (-1) \cdot i = -i$ and $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$. In Exercises 19 - 26, use this information to simplify the given power of i.

19. i^5
20. i^6
21. i^7
22. i^8
23. i^{15}
24. i^{26}
25. i^{117}
26. i^{304}

In Exercises 27 - 35, find all complex solutions.

27. $3x^2 + 6 = 4x$
28. $15t^2 + 2t + 5 = 3t(t^2 + 1)$
29. $3y^2 + 4 = y^4$
30. $\dfrac{2}{1 - w} = w$
31. $\dfrac{y}{3} - \dfrac{3}{y} = y$
32. $\dfrac{x^3}{2x - 1} = \dfrac{x}{3}$
33. $x = \dfrac{2}{\sqrt{5} - x}$
34. $\dfrac{5y^4 + 1}{y^2 - 1} = 3y^2$
35. $z^4 = 16$
36. Multiply and simplify: $\left(x - [3 - i\sqrt{23}]\right)\left(x - [3 + i\sqrt{23}]\right)$

A.11.2 Answers

1. For $z = 2 + 3i$ and $w = 4i$

- $z + w = 2 + 7i$
- $zw = -12 + 8i$
- $z^2 = -5 + 12i$
- $\frac{1}{z} = \frac{2}{13} - \frac{3}{13}i$
- $\frac{z}{w} = \frac{3}{4} - \frac{1}{2}i$
- $\frac{w}{z} = \frac{12}{13} + \frac{8}{13}i$
- $\overline{z} = 2 - 3i$
- $z\overline{z} = 13$
- $(\overline{z})^2 = -5 - 12i$

2. For $z = 1 + i$ and $w = -i$

- $z + w = 1$
- $zw = 1 - i$
- $z^2 = 2i$
- $\frac{1}{z} = \frac{1}{2} - \frac{1}{2}i$
- $\frac{z}{w} = -1 + i$
- $\frac{w}{z} = -\frac{1}{2} - \frac{1}{2}i$
- $\overline{z} = 1 - i$
- $z\overline{z} = 2$
- $(\overline{z})^2 = -2i$

3. For $z = i$ and $w = -1 + 2i$

- $z + w = -1 + 3i$
- $zw = -2 - i$
- $z^2 = -1$
- $\frac{1}{z} = -i$
- $\frac{z}{w} = \frac{2}{5} - \frac{1}{5}i$
- $\frac{w}{z} = 2 + i$
- $\overline{z} = -i$
- $z\overline{z} = 1$
- $(\overline{z})^2 = -1$

4. For $z = 4i$ and $w = 2 - 2i$

- $z + w = 2 + 2i$
- $zw = 8 + 8i$
- $z^2 = -16$
- $\frac{1}{z} = -\frac{1}{4}i$
- $\frac{z}{w} = -1 + i$
- $\frac{w}{z} = -\frac{1}{2} - \frac{1}{2}i$
- $\overline{z} = -4i$
- $z\overline{z} = 16$
- $(\overline{z})^2 = -16$

5. For $z = 3 - 5i$ and $w = 2 + 7i$

- $z + w = 5 + 2i$
- $zw = 41 + 11i$
- $z^2 = -16 - 30i$
- $\frac{1}{z} = \frac{3}{34} + \frac{5}{34}i$
- $\frac{z}{w} = -\frac{29}{53} - \frac{31}{53}i$
- $\frac{w}{z} = -\frac{29}{34} + \frac{31}{34}i$
- $\overline{z} = 3 + 5i$
- $z\overline{z} = 34$
- $(\overline{z})^2 = -16 + 30i$

6. For $z = -5 + i$ and $w = 4 + 2i$

- $z + w = -1 + 3i$
- $zw = -22 - 6i$
- $z^2 = 24 - 10i$
- $\frac{1}{z} = -\frac{5}{26} - \frac{1}{26}\,i$
- $\frac{z}{w} = -\frac{9}{10} + \frac{7}{10}\,i$
- $\frac{w}{z} = -\frac{9}{13} - \frac{7}{13}\,i$
- $\overline{z} = -5 - i$
- $z\overline{z} = 26$
- $(\overline{z})^2 = 24 + 10i$

7. For $z = \sqrt{2} - i\sqrt{2}$ and $w = \sqrt{2} + i\sqrt{2}$

- $z + w = 2\sqrt{2}$
- $zw = 4$
- $z^2 = -4i$
- $\frac{1}{z} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}\,i$
- $\frac{z}{w} = -i$
- $\frac{w}{z} = i$
- $\overline{z} = \sqrt{2} + i\sqrt{2}$
- $z\overline{z} = 4$
- $(\overline{z})^2 = 4i$

8. For $z = 1 - i\sqrt{3}$ and $w = -1 - i\sqrt{3}$

- $z + w = -2i\sqrt{3}$
- $zw = -4$
- $z^2 = -2 - 2i\sqrt{3}$
- $\frac{1}{z} = \frac{1}{4} + \frac{\sqrt{3}}{4}\,i$
- $\frac{z}{w} = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$
- $\frac{w}{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
- $\overline{z} = 1 + i\sqrt{3}$
- $z\overline{z} = 4$
- $(\overline{z})^2 = -2 + 2i\sqrt{3}$

9. For $z = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$ and $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$

- $z + w = i\sqrt{3}$
- $zw = -1$
- $z^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$
- $\frac{1}{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
- $\frac{z}{w} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
- $\frac{w}{z} = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$
- $\overline{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
- $z\overline{z} = 1$
- $(\overline{z})^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$

10. For $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$ and $w = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$

- $z + w = -\sqrt{2}$
- $zw = 1$
- $z^2 = -i$
- $\frac{1}{z} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$
- $\frac{z}{w} = -i$
- $\frac{w}{z} = i$
- $\overline{z} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$
- $z\overline{z} = 1$
- $(\overline{z})^2 = i$

11. $7i$

12. $3i$

13. -10

14. 10

15. -12

16. 12

17. 3

18. $-3i$

19. $i^5 = i^4 \cdot i = 1 \cdot i = i$

20. $i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1$

21. $i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

22. $i^8 = i^4 \cdot i^4 = \left(i^4\right)^2 = (1)^2 = 1$

23. $i^{15} = \left(i^4\right)^3 \cdot i^3 = 1 \cdot (-i) = -i$

24. $i^{26} = \left(i^4\right)^6 \cdot i^2 = 1 \cdot (-1) = -1$

25. $i^{117} = \left(i^4\right)^{29} \cdot i = 1 \cdot i = i$

26. $i^{304} = \left(i^4\right)^{76} = 1^{76} = 1$

27. $x = \dfrac{2 \pm i\sqrt{14}}{3}$

28. $t = 5, \pm \dfrac{i\sqrt{3}}{3}$

29. $y = \pm 2, \pm i$

30. $w = \dfrac{1 \pm i\sqrt{7}}{2}$

31. $y = \pm \dfrac{3i\sqrt{2}}{2}$

32. $x = 0, \dfrac{1 \pm i\sqrt{2}}{3}$

33. $x = \dfrac{\sqrt{5} \pm i\sqrt{3}}{2}$

34. $y = \pm i, \pm \dfrac{i\sqrt{2}}{2}$

35. $z = \pm 2, \pm 2i$

36. $x^2 - 6x + 32$

A.12 Rational Expressions and Equations

We now turn our attention to rational expressions - that is, algebraic fractions - and equations which contain them. The reader is encouraged to keep in mind the properties of fractions listed on page 1335 because we will need them along the way. Before we launch into reviewing the basic arithmetic operations of rational expressions, we take a moment to review how to simplify them properly. As with numeric fractions, we 'cancel common *factors*,' not common *terms*. That is, in order to simplify rational expressions, we first *factor* the numerator and denominator. For example:

$$\frac{x^4+5x^3}{x^3-25x} \neq \frac{x^4+5\cancel{x^3}}{\cancel{x^3}-25x}$$

but, rather

$$\begin{aligned}
\frac{x^4+5x^3}{x^3-25x} &= \frac{x^3(x+5)}{x(x^2-25)} && \text{Factor G.C.F.} \\
&= \frac{x^3(x+5)}{x(x-5)(x+5)} && \text{Difference of Squares} \\
&= \frac{\cancelto{x^2}{x^3}\cancel{(x+5)}}{\cancel{x}(x-5)\cancel{(x+5)}} && \text{Cancel common factors} \\
&= \frac{x^2}{x-5}
\end{aligned}$$

This equivalence holds provided the factors being canceled aren't 0. Since a factor of x and a factor of $x+5$ were canceled, $x \neq 0$ and $x+5 \neq 0$, so $x \neq -5$. We usually stipulate this as:

$$\frac{x^4+5x^3}{x^3-25x} = \frac{x^2}{x-5}, \qquad \text{provided } x \neq 0,\ x \neq -5$$

While we're talking about common mistakes, please notice that

$$\frac{5}{x^2+9} \neq \frac{5}{x^2} + \frac{5}{9}$$

Just like their numeric counterparts, you don't add algebraic fractions by *adding denominators* of fractions with *common numerators* - it's the other way around:[1]

$$\frac{x^2+9}{5} = \frac{x^2}{5} + \frac{9}{5}$$

It's time to review the basic arithmetic operations with rational expressions.

[1]One of the most common errors students make on college Mathematics placement tests is that they forget how to add algebraic fractions correctly. This places many students into remedial classes even though they are probably ready for college-level Math. We urge you to really study this section with great care so that you don't fall into that trap.

Example A.12.1. Perform the indicated operations and simplify.

1. $\dfrac{2x^2-5x-3}{x^4-4} \div \dfrac{x^2-2x-3}{x^5+2x^3}$

2. $\dfrac{5}{w^2-9} - \dfrac{w+2}{w^2-9}$

3. $\dfrac{3}{y^2-8y+16} + \dfrac{y+1}{16y-y^3}$

4. $\dfrac{\dfrac{2}{4-(x+h)} - \dfrac{2}{4-x}}{h}$

5. $2t^{-3} - (3t)^{-2}$

6. $10x(x-3)^{-1} + 5x^2(-1)(x-3)^{-2}$

Solution.

1. As with numeric fractions, we divide rational expressions by 'inverting and multiplying'. Before we get too carried away however, we factor to see what, if any, factors cancel.

$$\begin{aligned}
\frac{2x^2-5x-3}{x^4-4} \div \frac{x^2-2x-3}{x^5+2x^3} &= \frac{2x^2-5x-3}{x^4-4} \cdot \frac{x^5+2x^3}{x^2-2x-3} && \text{Invert and multiply} \\
&= \frac{(2x^2-5x-3)(x^5+2x^3)}{(x^4-4)(x^2-2x-3)} && \text{Multiply fractions} \\
&= \frac{(2x+1)(x-3)x^3(x^2+2)}{(x^2-2)(x^2+2)(x-3)(x+1)} && \text{Factor} \\
&= \frac{(2x+1)\cancel{(x-3)}x^3\cancel{(x^2+2)}}{(x^2-2)\cancel{(x^2+2)}\cancel{(x-3)}(x+1)} && \text{Cancel common factors} \\
&= \frac{x^3(2x+1)}{(x+1)(x^2-2)} && \text{Provided } x \neq 3
\end{aligned}$$

The '$x \neq 3$' is mentioned since a factor of $(x-3)$ was canceled as we reduced the expression. We also canceled a factor of (x^2+2). Why is there no stipulation as a result of canceling this factor? Because $x^2+2 \neq 0$ for all real x. (See Section A.11 for details.) At this point, we *could* go ahead and multiply out the numerator and denominator to get

$$\frac{x^3(2x+1)}{(x+1)(x^2-2)} = \frac{2x^4+x^3}{x^3+x^2-2x-2}$$

but for most of the applications where this kind of algebra is needed (solving equations, for instance), it is best to leave things factored. Your instructor will let you know whether to leave your answer in factored form or not.[2]

[2]Speaking of factoring, do you remember why x^2-2 can't be factored over the integers?

2. As with numeric fractions we need common denominators in order to subtract. This is already the case here so we proceed by subtracting the numerators.

$$\begin{aligned} \frac{5}{w^2-9} - \frac{w+2}{w^2-9} &= \frac{5-(w+2)}{w^2-9} && \text{Subtract fractions} \\ &= \frac{5-w-2}{w^2-9} && \text{Distribute} \\ &= \frac{3-w}{w^2-9} && \text{Combine like terms} \end{aligned}$$

At this point, we need to see if we can reduce this expression so we proceed to factor. It first appears as if we have no common factors among the numerator and denominator until we recall the property of 'factoring negatives' from Page 1334: $3 - w = -(w - 3)$. This yields:

$$\begin{aligned} \frac{3-w}{w^2-9} &= \frac{-(w-3)}{(w-3)(w+3)} && \text{Factor} \\ &= \frac{-\cancel{(w-3)}}{\cancel{(w-3)}(w+3)} && \text{Cancel common factors} \\ &= \frac{-1}{w+3} && \text{Provided } w \neq 3 \end{aligned}$$

The stipulation $w \neq 3$ comes from the cancellation of the $(w - 3)$ factor.

3. In this next example, we are asked to add two rational expressions with *different* denominators. As with numeric fractions, we must first find a *common denominator*. To do so, we start by factoring each of the denominators.

$$\begin{aligned} \frac{3}{y^2-8y+16} + \frac{y+1}{16y-y^3} &= \frac{3}{(y-4)^2} + \frac{y+1}{y(16-y^2)} && \text{Factor} \\ &= \frac{3}{(y-4)^2} + \frac{y+1}{y(4-y)(4+y)} && \text{Factor some more} \end{aligned}$$

To find the common denominator, we examine the factors in the first denominator and note that we need a factor of $(y - 4)^2$. We now look at the second denominator to see what other factors we need. We need a factor of y and $(4 + y) = (y + 4)$. What about $(4 - y)$? As mentioned in the last example, we can factor this as: $(4 - y) = -(y - 4)$. Using properties of negatives, we 'migrate' this negative out to the front of the fraction, turning the addition into subtraction. We find the (least) common denominator to be $(y - 4)^2 y(y + 4)$. We can now proceed to multiply the numerator and denominator of each fraction by whatever factors are missing from their respective denominators to produce equivalent expressions with common denominators.

$$\begin{aligned}
\frac{3}{(y-4)^2}+\frac{y+1}{y(4-y)(4+y)} &= \frac{3}{(y-4)^2}+\frac{y+1}{y(-(y-4))(y+4)} \\
&= \frac{3}{(y-4)^2}-\frac{y+1}{y(y-4)(y+4)} \\
&= \frac{3}{(y-4)^2}\cdot\frac{y(y+4)}{y(y+4)}-\frac{y+1}{y(y-4)(y+4)}\cdot\frac{(y-4)}{(y-4)} && \text{Equivalent Fractions} \\
&= \frac{3y(y+4)}{(y-4)^2y(y+4)}-\frac{(y+1)(y-4)}{y(y-4)^2(y+4)} && \text{Multiply Fractions}
\end{aligned}$$

At this stage, we can subtract numerators and simplify. We'll keep the denominator factored (in case we can reduce down later), but in the numerator, since there are no common factors, we proceed to perform the indicated multiplication and combine like terms.

$$\begin{aligned}
\frac{3y(y+4)}{(y-4)^2y(y+4)}-\frac{(y+1)(y-4)}{y(y-4)^2(y+4)} &= \frac{3y(y+4)-(y+1)(y-4)}{(y-4)^2y(y+4)} && \text{Subtract numerators} \\
&= \frac{3y^2+12y-(y^2-3y-4)}{(y-4)^2y(y+4)} && \text{Distribute} \\
&= \frac{3y^2+12y-y^2+3y+4}{(y-4)^2y(y+4)} && \text{Distribute} \\
&= \frac{2y^2+15y+4}{y(y+4)(y-4)^2} && \text{Gather like terms}
\end{aligned}$$

We would like to factor the numerator and cancel factors it has in common with the denominator. After a few attempts, it appears as if the numerator doesn't factor, at least over the integers. As a check, we compute the discriminant of $2y^2+15y+4$ and get $15^2-4(2)(4)=193$. This isn't a perfect square so we know that the quadratic equation $2y^2+15y+4=0$ has irrational solutions. This means $2y^2+15y+4$ can't factor over the integers[3] so we are done.

4. In this example, we have a compound fraction, and we proceed to simplify it as we did its numeric counterparts in Example A.2.1. Specifically, we start by multiplying the numerator and denominator of the 'big' fraction by the least common denominator of the 'little' fractions inside of it - in this case we need to use $(4-(x+h))(4-x)$ - to remove the compound nature of the 'big' fraction. Once we

[3]See the remarks following Theorem A.12.

have a more normal looking fraction, we can proceed as we have in the previous examples.

$$\begin{aligned}
\frac{\frac{2}{4-(x+h)}-\frac{2}{4-x}}{h} &= \frac{\left(\frac{2}{4-(x+h)}-\frac{2}{4-x}\right)}{h}\cdot\frac{(4-(x+h))(4-x)}{(4-(x+h))(4-x)} && \text{Equivalent fractions}\\
&= \frac{\left(\frac{2}{4-(x+h)}-\frac{2}{4-x}\right)\cdot(4-(x+h))(4-x)}{h(4-(x+h))(4-x)} && \text{Multiply}\\
&= \frac{\frac{2(4-(x+h))(4-x)}{4-(x+h)}-\frac{2(4-(x+h))(4-x)}{4-x}}{h(4-(x+h))(4-x)} && \text{Distribute}\\
&= \frac{\frac{2\cancel{(4-(x+h))}(4-x)}{\cancel{(4-(x+h))}}-\frac{2(4-(x+h))\cancel{(4-x)}}{\cancel{(4-x)}}}{h(4-(x+h))(4-x)} && \text{Reduce}\\
&= \frac{2(4-x)-2(4-(x+h))}{h(4-(x+h))(4-x)}
\end{aligned}$$

Now we can clean up and factor the numerator to see if anything cancels. (This why we kept the denominator factored.)

$$\begin{aligned}
\frac{2(4-x)-2(4-(x+h))}{h(4-(x+h))(4-x)} &= \frac{2[(4-x)-(4-(x+h))]}{h(4-(x+h))(4-x)} && \text{Factor out G.C.F.}\\
&= \frac{2[4-x-4+(x+h)]}{h(4-(x+h))(4-x)} && \text{Distribute}\\
&= \frac{2[4-4-x+x+h]}{h(4-(x+h))(4-x)} && \text{Rearrange terms}\\
&= \frac{2h}{h(4-(x+h))(4-x)} && \text{Gather like terms}\\
&= \frac{2\cancel{h}}{\cancel{h}(4-(x+h))(4-x)} && \text{Reduce}\\
&= \frac{2}{(4-(x+h))(4-x)} && \text{Provided } h\neq 0
\end{aligned}$$

Your instructor will let you know if you are to expand the denominator or not.[4]

5. At first glance, it doesn't seem as if there is anything that can be done with $2t^{-3}-(3t)^{-2}$ because the exponents on the variables are different. However, since the exponents are negative, these are actually rational expressions. In the first term, the -3 exponent applies to the t *only* but in the second

[4]We'll keep it factored because in Calculus it needs to be factored.

term, the exponent -2 applies to *both* the 3 and the t, as indicated by the parentheses. One way to proceed is as follows:

$$\begin{aligned} 2t^{-3} - (3t)^{-2} &= \frac{2}{t^3} - \frac{1}{(3t)^2} \\ &= \frac{2}{t^3} - \frac{1}{9t^2} \end{aligned}$$

We see that we are being asked to subtract two rational expressions with different denominators, so we need to find a common denominator. The first fraction contributes a t^3 to the denominator, while the second contributes a factor of 9. Thus our common denominator is $9t^3$, so we are missing a factor of '9' in the first denominator and a factor of 't' in the second.

$$\begin{aligned} \frac{2}{t^3} - \frac{1}{9t^2} &= \frac{2}{t^3} \cdot \frac{9}{9} - \frac{1}{9t^2} \cdot \frac{t}{t} && \text{Equivalent Fractions} \\ &= \frac{18}{9t^3} - \frac{t}{9t^3} && \text{Multiply} \\ &= \frac{18 - t}{9t^3} && \text{Subtract} \end{aligned}$$

We find no common factors among the numerator and denominator so we are done.

A second way to approach this problem is by factoring. We can extend the concept of the 'Polynomial G.C.F.' to these types of expressions and we can follow the same guidelines as set forth on page 1425 to factor out the G.C.F. of these two terms. The key ideas to remember are that we take out each factor with the *smallest* exponent and that factoring is the same as dividing. We first note that $2t^{-3} - (3t)^{-2} = 2t^{-3} - 3^{-2}t^{-2}$ and we see that the smallest power on t is -3. Thus we want to factor out t^{-3} from both terms. It's clear that this will leave 2 in the first term, but what about the second term? Since factoring is the same as dividing, we would be dividing the second term by t^{-3} which thanks to the properties of exponents is the same as *multiplying* by $\frac{1}{t^{-3}} = t^3$. The same holds for 3^{-2}. Even though there are no factors of 3 in the first term, we can factor out 3^{-2} by multiplying it by $\frac{1}{3^{-2}} = 3^2 = 9$. We put these ideas together below.

$$\begin{aligned} 2t^{-3} - (3t)^{-2} &= 2t^{-3} - 3^{-2}t^{-2} && \text{Properties of Exponents} \\ &= 3^{-2}t^{-3}(2(3)^2 - t^1) && \text{Factor} \\ &= \frac{1}{3^2}\frac{1}{t^3}(18 - t) && \text{Rewrite} \\ &= \frac{18 - t}{9t^3} && \text{Multiply} \end{aligned}$$

While both ways are valid, one may be more of a natural fit than the other depending on the circumstances and temperament of the student.

6. As with the previous example, we show two different yet equivalent ways to approach simplifying $10x(x-3)^{-1} + 5x^2(-1)(x-3)^{-2}$. First up is what we'll call the 'common denominator approach' where we rewrite the negative exponents as fractions and proceed from there.

- *Common Denominator Approach*:

$$
\begin{aligned}
10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2} &= \frac{10x}{x-3}+\frac{5x^2(-1)}{(x-3)^2} \\
&= \frac{10x}{x-3}\cdot\frac{x-3}{x-3}-\frac{5x^2}{(x-3)^2} && \text{Equivalent Fractions} \\
&= \frac{10x(x-3)}{(x-3)^2}-\frac{5x^2}{(x-3)^2} && \text{Multiply} \\
&= \frac{10x(x-3)-5x^2}{(x-3)^2} && \text{Subtract} \\
&= \frac{5x(2(x-3)-x)}{(x-3)^2} && \text{Factor out G.C.F.} \\
&= \frac{5x(2x-6-x)}{(x-3)^2} && \text{Distribute} \\
&= \frac{5x(x-6)}{(x-3)^2} && \text{Combine like terms}
\end{aligned}
$$

  Both the numerator and the denominator are completely factored with no common factors so we are done.

- *'Factoring Approach'*: In this case, the G.C.F. is $5x(x-3)^{-2}$. Factoring this out of both terms gives:

$$
\begin{aligned}
10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2} &= 5x(x-3)^{-2}(2(x-3)^1-x) && \text{Factor} \\
&= \frac{5x}{(x-3)^2}(2x-6-x) && \text{Rewrite, distribute} \\
&= \frac{5x(x-6)}{(x-3)^2} && \text{Multiply}
\end{aligned}
$$

  As expected, we got the same reduced fraction as before. □

Next, we review the solving of equations which involve rational expressions. As with equations involving numeric fractions, our first step in solving equations with algebraic fractions is to clear denominators. In doing so, we run the risk of introducing what are known as **extraneous** solutions - 'answers' which don't satisfy the original equation. As we illustrate the techniques used to solve these basic equations, see if you can find the step which creates the problem for us.

Example A.12.2. Solve the following equations.

1. $1 + \dfrac{1}{x} = x$
2. $\dfrac{t^3 - 2t + 1}{t - 1} = \dfrac{1}{2}t - 1$
3. $\dfrac{3}{1 - w\sqrt{2}} - \dfrac{1}{2w + 5} = 0$
4. $3(x^2 + 4)^{-1} + 3x(-1)(x^2 + 4)^{-2}(2x) = 0$
5. Solve $x = \dfrac{2y + 1}{y - 3}$ for y.
6. Solve $\dfrac{1}{f} = \dfrac{1}{S_1} + \dfrac{1}{S_2}$ for S_1.

Solution.

1. Our first step is to clear the fractions by multiplying both sides of the equation by x. In doing so, we are implicitly assuming $x \neq 0$; otherwise, we would have no guarantee that the resulting equation is equivalent to our original equation.[5]

$$\begin{array}{rcll}
1 + \dfrac{1}{x} & = & x & \\
\left(1 + \dfrac{1}{x}\right) x & = & (x)x & \text{Provided } x \neq 0 \\
1(x) + \dfrac{1}{x}(x) & = & x^2 & \text{Distribute} \\
x + \dfrac{x}{x} & = & x^2 & \text{Multiply} \\
x + 1 & = & x^2 & \\
0 & = & x^2 - x - 1 & \text{Subtract } x\text{, subtract } 1 \\
x & = & \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} & \text{Quadratic Formula} \\
x & = & \dfrac{1 \pm \sqrt{5}}{2} & \text{Simplify}
\end{array}$$

We obtain two answers, $x = \frac{1 \pm \sqrt{5}}{2}$. Neither of these are 0 thus neither contradicts our assumption that $x \neq 0$. The reader is invited to check both of these solutions.[6]

[5]See page 1369.

[6]The check relies on being able to 'rationalize' the denominator - a skill we haven't reviewed yet. (Come back after you've read Section A.13.1 if you want to!) Additionally, the positive solution to this equation is the famous Golden Ratio.

2. To solve the equation, we clear denominators. Here, we need to assume $t-1 \neq 0$, or $t \neq 1$.

$$\begin{array}{rcll}
\dfrac{t^3-2t+1}{t-1} & = & \dfrac{1}{2}t-1 & \\
\left(\dfrac{t^3-2t+1}{t-1}\right)\cdot 2(t-1) & = & \left(\dfrac{1}{2}t-1\right)\cdot 2(t-1) & \text{Provided } t \neq 1 \\
\dfrac{(t^3-2t+1)(2\cancel{(t-1)})}{\cancel{(t-1)}} & = & \dfrac{1}{\cancel{2}}t(\cancel{2}(t-1))-1(2(t-1)) & \text{Multiply, distribute} \\
2(t^3-2t+1) & = & t^2-t-2t+2 & \text{Distribute} \\
2t^3-4t+2 & = & t^2-3t+2 & \text{Distribute, combine like terms} \\
2t^3-t^2-t & = & 0 & \text{Subtract } t^2\text{, add } 3t\text{, subtract } 2 \\
t(2t^2-t-1) & = & 0 & \text{Factor} \\
t=0 & \text{or} & 2t^2-t-1=0 & \text{Zero Product Property} \\
t=0 & \text{or} & (2t+1)(t-1)=0 & \text{Factor} \\
t=0 & \text{or} & 2t+1=0 \quad \text{or} \quad t-1=0 & \\
t & = & 0,\ -\dfrac{1}{2} \text{ or } 1 &
\end{array}$$

We assumed that $t \neq 1$ in order to clear denominators. Sure enough, the candidate $t = 1$ doesn't check in the original equation since it causes division by 0. In this case, we call $t = 1$ an *extraneous* solution. Note that $t = 1$ *does* work in every equation *after* we clear denominators. In general, multiplying by variable expressions can produce these 'extra' solutions, which is why checking our answers is always encouraged.[7] The other two candidates, $t = 0$ and $t = -\frac{1}{2}$, are solutions.

3. As before, we begin by clearing denominators. Here, we assume $1 - w\sqrt{2} \neq 0$ (so $w \neq \frac{1}{\sqrt{2}}$) and $2w + 5 \neq 0$ (so $w \neq -\frac{5}{2}$).

$$\begin{array}{rcll}
\dfrac{3}{1-w\sqrt{2}}-\dfrac{1}{2w+5} & = & 0 & \\
\left(\dfrac{3}{1-w\sqrt{2}}-\dfrac{1}{2w+5}\right)(1-w\sqrt{2})(2w+5) & = & 0(1-w\sqrt{2})(2w+5) & w \neq \dfrac{1}{\sqrt{2}},-\dfrac{5}{2} \\
\dfrac{3\cancel{(1-w\sqrt{2})}(2w+5)}{\cancel{(1-w\sqrt{2})}}-\dfrac{1(1-w\sqrt{2})\cancel{(2w+5)}}{\cancel{(2w+5)}} & = & 0 & \text{Distribute} \\
3(2w+5)-(1-w\sqrt{2}) & = & 0 &
\end{array}$$

The result is a *linear* equation in w so we gather the terms with w on one side of the equation and

[7] Contrast this with what happened in Example A.9.3 when we divided by a variable and 'lost' a solution.

put everything else on the other. We factor out w and divide by its coefficient.

$$
\begin{array}{rcll}
3(2w+5)-(1-w\sqrt{2}) & = & 0 & \\
6w+15-1+w\sqrt{2} & = & 0 & \text{Distribute} \\
6w+w\sqrt{2} & = & -14 & \text{Subtract } 14 \\
(6+\sqrt{2})w & = & -14 & \text{Factor} \\
w & = & -\dfrac{14}{6+\sqrt{2}} & \text{Divide by } 6+\sqrt{2}
\end{array}
$$

This solution is different than our excluded values, $\frac{1}{\sqrt{2}}$ and $-\frac{5}{2}$, so we keep $w=-\frac{14}{6+\sqrt{2}}$ as our final answer. The reader is invited to check this in the original equation.

4. To solve our next equation, we have two approaches to choose from: we could rewrite the quantities with negative exponents as fractions and clear denominators, or we can factor. We showcase each technique below.

   - *Clearing Denominators Approach*: We rewrite the negative exponents as fractions and clear denominators. In this case, we multiply both sides of the equation by $(x^2+4)^2$, which is never 0. (Think about that for a moment.) As a result, we need not exclude any x values from our solution set.

$$
\begin{array}{rcll}
3(x^2+4)^{-1}+3x(-1)(x^2+4)^{-2}(2x) & = & 0 & \\
\dfrac{3}{x^2+4}+\dfrac{3x(-1)(2x)}{(x^2+4)^2} & = & 0 & \text{Rewrite} \\
\left(\dfrac{3}{x^2+4}-\dfrac{6x^2}{(x^2+4)^2}\right)(x^2+4)^2 & = & 0(x^2+4)^2 & \text{Multiply} \\
\dfrac{3\cancelto{(x^2+4)}{(x^2+4)^2}}{\cancel{(x^2+4)}}-\dfrac{6x^2\cancel{(x^2+4)^2}}{\cancel{(x^2+4)^2}} & = & 0 & \text{Distribute} \\
3(x^2+4)-6x^2 & = & 0 & \\
3x^2+12-6x^2 & = & 0 & \text{Distribute} \\
-3x^2 & = & -12 & \text{Combine like terms, subtract } 12 \\
x^2 & = & 4 & \text{Divide by } -3 \\
x & = & \pm\sqrt{4}=\pm 2 & \text{Extract square roots}
\end{array}
$$

     We leave it to the reader to show that both $x=-2$ and $x=2$ satisfy the original equation.

   - *Factoring Approach*: Since the equation is already set equal to 0, we're ready to factor. Following the guidelines presented in Example A.12.1, we factor out $3(x^2+4)^{-2}$ from both terms and look to see if more factoring can be done:

$$\begin{array}{rclr}
3(x^2+4)^{-1}+3x(-1)(x^2+4)^{-2}(2x) & = & 0 & \\
3(x^2+4)^{-2}((x^2+4)^1+x(-1)(2x)) & = & 0 & \text{Factor} \\
3(x^2+4)^{-2}(x^2+4-2x^2) & = & 0 & \\
3(x^2+4)^{-2}(4-x^2) & = & 0 & \text{Gather like terms} \\
3(x^2+4)^{-2}=0 & \text{or} & 4-x^2=0 & \text{Zero Product Property} \\
\dfrac{3}{x^2+4}=0 & \text{or} & 4=x^2 &
\end{array}$$

The first equation yields no solutions (Think about this for a moment.) while the second gives us $x=\pm\sqrt{4}=\pm 2$ as before.

5. We are asked to solve this equation for y so we begin by clearing fractions with the stipulation that $y-3\neq 0$ or $y\neq 3$. We are left with a linear equation in the variable y. To solve this, we gather the terms containing y on one side of the equation and everything else on the other. Next, we factor out the y and divide by its coefficient, which in this case turns out to be $x-2$. In order to divide by $x-2$, we stipulate $x-2\neq 0$ or, said differently, $x\neq 2$.

$$\begin{array}{rclr}
x & = & \dfrac{2y+1}{y-3} & \\
x(y-3) & = & \left(\dfrac{2y+1}{y-3}\right)(y-3) & \text{Provided } y\neq 3 \\
xy-3x & = & \dfrac{(2y+1)\cancel{(y-3)}}{\cancel{(y-3)}} & \text{Distribute, multiply} \\
xy-3x & = & 2y+1 & \\
xy-2y & = & 3x+1 & \text{Add } 3x\text{, subtract } 2y \\
y(x-2) & = & 3x+1 & \text{Factor} \\
y & = & \dfrac{3x+1}{x-2} & \text{Divide provided } x\neq 2
\end{array}$$

We highly encourage the reader to check the answer algebraically to see where the restrictions on x and y come into play.[8]

6. Our last example comes from physics and the world of photography.[9] We take a moment here to note that while superscripts in Mathematics indicate exponents (powers), subscripts are used primarily to distinguish one or more variables. In this case, S_1 and S_2 are two *different* variables (much like x and y) and we treat them as such. Our first step is to clear denominators by multiplying both sides by fS_1S_2 - provided each is nonzero. We end up with an equation which is linear in S_1 so we proceed

[8] It involves simplifying a compound fraction!

[9] See this article on focal length.

as in the previous example.

$$\begin{array}{rcll}
\dfrac{1}{f} & = & \dfrac{1}{S_1} + \dfrac{1}{S_2} & \\
\left(\dfrac{1}{f}\right)(fS_1S_2) & = & \left(\dfrac{1}{S_1} + \dfrac{1}{S_2}\right)(fS_1S_2) & \text{Provided } f \neq 0,\ S_1 \neq 0,\ S_2 \neq 0 \\
\dfrac{fS_1S_2}{f} & = & \dfrac{fS_1S_2}{S_1} + \dfrac{fS_1S_2}{S_2} & \text{Multiply, distribute} \\
\dfrac{\cancel{f}S_1S_2}{\cancel{f}} & = & \dfrac{f\cancel{S_1}S_2}{\cancel{S_1}} + \dfrac{fS_1\cancel{S_2}}{\cancel{S_2}} & \text{Cancel} \\
S_1S_2 & = & fS_2 + fS_1 & \\
S_1S_2 - fS_1 & = & fS_2 & \text{Subtract } fS_1 \\
S_1(S_2 - f) & = & fS_2 & \text{Factor} \\
S_1 & = & \dfrac{fS_2}{S_2 - f} & \text{Divide provided } S_2 \neq f
\end{array}$$

As always, the reader is highly encouraged to check the answer.[10] □

[10]. . . and see what the restriction $S_2 \neq f$ means in terms of focusing a camera!

A.12.1 Exercises

In Exercises 1 - 18, perform the indicated operations and simplify.

1. $\dfrac{x^2-9}{x^2} \cdot \dfrac{3x}{x^2-x-6}$

2. $\dfrac{t^2-2t}{t^2+1} \div (3t^2-2t-8)$

3. $\dfrac{4y-y^2}{2y+1} \div \dfrac{y^2-16}{2y^2-5y-3}$

4. $\dfrac{x}{3x-1} - \dfrac{1-x}{3x-1}$

5. $\dfrac{2}{w-1} - \dfrac{w^2+1}{w-1}$

6. $\dfrac{2-y}{3y} - \dfrac{1-y}{3y} + \dfrac{y^2-1}{3y}$

7. $b + \dfrac{1}{b-3} - 2$

8. $\dfrac{2x}{x-4} - \dfrac{1}{2x+1}$

9. $\dfrac{m^2}{m^2-4} + \dfrac{1}{2-m}$

10. $\dfrac{\frac{2}{x}-2}{x-1}$

11. $\dfrac{\frac{3}{2-h}-\frac{3}{2}}{h}$

12. $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$

13. $3w^{-1} - (3w)^{-1}$

14. $-2y^{-1} + 2(3-y)^{-2}$

15. $3(x-2)^{-1} - 3x(x-2)^{-2}$

16. $\dfrac{t^{-1}+t^{-2}}{t^{-3}}$

17. $\dfrac{2(3+h)^{-2}-2(3)^{-2}}{h}$

18. $\dfrac{(7-x-h)^{-1}-(7-x)^{-1}}{h}$

In Exercises 19 - 27, find all real solutions. Be sure to check for extraneous solutions.

19. $\dfrac{x}{5x+4} = 3$

20. $\dfrac{3y-1}{y^2+1} = 1$

21. $\dfrac{1}{w+3} + \dfrac{1}{w-3} = \dfrac{w^2-3}{w^2-9}$

22. $\dfrac{2x+17}{x+1} = x+5$

23. $\dfrac{t^2-2t+1}{t^3+t^2-2t} = 1$

24. $\dfrac{-y^3+4y}{y^2-9} = 4y$

25. $w + \sqrt{3} = \dfrac{3w-w^3}{w-\sqrt{3}}$

26. $\dfrac{2}{x\sqrt{2}-1} - 1 = \dfrac{3}{x\sqrt{2}+1}$

27. $\dfrac{x^2}{(1+x\sqrt{3})^2} = 3$

In Exercises 28 - 30, use Theorem A.5 along with the techniques in this section to find all real solutions.

28. $\left|\dfrac{3n}{n-1}\right| = 3$

29. $\left|\dfrac{2x}{x^2-1}\right| = 2$

30. $\left|\dfrac{2t}{4-t^2}\right| = \left|\dfrac{2}{t-2}\right|$

In Exercises 31 - 33, find all real solutions and use a calculator to approximate your answers, rounded to two decimal places.

31. $2.41 = \dfrac{0.08}{4\pi R^2}$

32. $\dfrac{x^2}{(2.31-x)^2} = 0.04$

33. $1 - \dfrac{6.75 \times 10^{16}}{c^2} = \dfrac{1}{4}$

In Exercises 34 - 39, solve the given equation for the indicated variable.

34. Solve for y: $\dfrac{1-2y}{y+3} = x$

35. Solve for y: $x = 3 - \dfrac{2}{1-y}$

36.[11] Solve for T_2: $\dfrac{V_1}{T_1} = \dfrac{V_2}{T_2}$

37. Solve for t_0: $\dfrac{t_0}{1-t_0t_1} = 2$

38. Solve for x: $\dfrac{1}{x-v_r} + \dfrac{1}{x+v_r} = 5$

39. Solve for R: $P = \dfrac{25R}{(R+4)^2}$

[11] Recall: subscripts on variables have no intrinsic mathematical meaning; they're just used to distinguish one variable from another. In other words, treat quantities like 'V_1' and 'V_2' as two different variables as you would 'x' and 'y.'

A.12.2 Answers

1. $\dfrac{3(x+3)}{x(x+2)},\ x \neq 3$
2. $\dfrac{t}{(3t+4)(t^2+1)},\ t \neq 2$
3. $-\dfrac{y(y-3)}{y+4},\ y \neq -\dfrac{1}{2}, 3, 4$
4. $\dfrac{2x-1}{3x-1}$
5. $-w-1,\ w \neq 1$
6. $\dfrac{y}{3},\ y \neq 0$
7. $\dfrac{b^2-5b+7}{b-3}$
8. $\dfrac{4x^2+x+4}{(x-4)(2x+1)}$
9. $\dfrac{m+1}{m+2},\ m \neq 2$
10. $-\dfrac{2}{x},\ x \neq 1$
11. $\dfrac{3}{4-2h},\ h \neq 0$
12. $-\dfrac{1}{x(x+h)},\ h \neq 0$
13. $\dfrac{8}{3w}$
14. $-\dfrac{2(y^2-7y+9)}{y(y-3)^2}$
15. $-\dfrac{6}{(x-2)^2}$
16. $t^2+t,\ t \neq 0$
17. $-\dfrac{2(h+6)}{9(h+3)^2},\ h \neq 0$
18. $\dfrac{1}{(7-x)(7-x-h)},\ h \neq 0$
19. $x = -\dfrac{6}{7}$
20. $y = 1, 2$
21. $w = -1$
22. $x = -6, 2$
23. No solution.
24. $y = 0, \pm 2\sqrt{2}$
25. $w = -\sqrt{3}, -1$
26. $x = -\dfrac{3\sqrt{2}}{2}, \sqrt{2}$
27. $x = -\dfrac{\sqrt{3}}{2}, -\dfrac{\sqrt{3}}{4}$
28. $n = \dfrac{1}{2}$
29. $x = \dfrac{1 \pm \sqrt{5}}{2}, \dfrac{-1 \pm \sqrt{5}}{2}$
30. $t = -1$
31. $R = \pm\sqrt{\dfrac{0.08}{9.64\pi}} \approx \pm 0.05$
32. $x = -\dfrac{231}{400} \approx -0.58,\ x = \dfrac{77}{200} \approx 0.38$
33. $c = \pm\sqrt{\dfrac{4 \cdot 6.75 \times 10^{16}}{3}} = \pm 3.00 \times 10^8$ (You actually didn't *need* a calculator for this!)
34. $y = \dfrac{1-3x}{x+2},\ y \neq -3,\ x \neq -2$
35. $y = \dfrac{x-1}{x-3},\ y \neq 1,\ x \neq 3$
36. $T_2 = \dfrac{V_2 T_1}{V_1},\ T_1 \neq 0,\ T_2 \neq 0,\ V_1 \neq 0$
37. $t_0 = \dfrac{2}{2t_1+1},\ t_1 \neq -\dfrac{1}{2}$
38. $x = \dfrac{1 \pm \sqrt{25v_r^2+1}}{5},\ x \neq \pm v_r.$
39. $R = \dfrac{-(8P-25) \pm \sqrt{(8P-25)^2 - 64P^2}}{2P} = \dfrac{(25-8P) \pm 5\sqrt{25-16P}}{2P},\ P \neq 0,\ R \neq -4$

A.13 Radical Equations

In this section we review simplifying expressions and solving equations involving radicals. In addition to the product, quotient and power rules stated in Theorem A.1 in Section A.2, we present the following result which states that n^{th} roots and n^{th} powers more or less 'undo' each other.[1]

Theorem A.15. Simplifying n^{th} powers of n^{th} roots and n^{th} roots of n^{th} powers: Suppose n is a natural number, a is a real number and $\sqrt[n]{a}$ is a real number. Then

- $(\sqrt[n]{a})^n = a$
- if n is odd, $\sqrt[n]{a^n} = a$; if n is even, $\sqrt[n]{a^n} = |a|$.

Since $\sqrt[n]{a}$ is *defined* so that $(\sqrt[n]{a})^n = a$, the first claim in the theorem is just a re-wording of Definition A.8. The second part of the theorem breaks down along odd/even exponent lines due to how exponents affect negatives. To see this, consider the specific cases of $\sqrt[3]{(-2)^3}$ and $\sqrt[4]{(-2)^4}$.

In the first case, $\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$, so we have an instance of when $\sqrt[n]{a^n} = a$. The reason that the cube root 'undoes' the third power in $\sqrt[3]{(-2)^3} = -2$ is because the negative is preserved when raised to the third (odd) power. In $\sqrt[4]{(-2)^4}$, the negative 'goes away' when raised to the fourth (even) power: $\sqrt[4]{(-2)^4} = \sqrt[4]{16}$. According to Definition A.8, the fourth root is defined to give only *non-negative* numbers, so $\sqrt[4]{16} = 2$. Here we have a case where $\sqrt[4]{(-2)^4} = 2 = |-2|$, not -2.

In general, we need the absolute values to simplify $\sqrt[n]{a^n}$ only when n is even because a negative to an even power is always positive. In particular, $\sqrt{x^2} = |x|$, not just 'x' (unless we *know* $x \geq 0$.)[2] We practice these formulas in the following example.

Example A.13.1. Perform the indicated operations and simplify.

1. $\sqrt{x^2+1}$
2. $\sqrt{t^2-10t+25}$
3. $\sqrt[3]{48x^{14}}$
4. $\sqrt[4]{\dfrac{\pi r^4}{L^8}}$
5. $2x\sqrt[3]{x^2-4} + 2\left(\dfrac{1}{2(\sqrt[3]{x^2-4})^2}\right)(2x)$
6. $\sqrt{(\sqrt{18y}-\sqrt{8y})^2 + (\sqrt{20}-\sqrt{80})^2}$

Solution.

1. We told you back on page 1346 that roots do not 'distribute' across addition and since x^2+1 cannot be factored over the real numbers, $\sqrt{x^2+1}$ cannot be simplified. It may seem silly to start with this example but it is extremely important that you understand what maneuvers are legal and which ones are not.[3]

[1] See Sections 4.1.2 and 5.6 for a more precise understanding of what we mean here.

[2] This discussion should sound familiar - see the discussion following Definition A.9 and the discussion following 'Extracting the Square Root' on page 1437.

[3] You really do need to understand this otherwise horrible evil will plague your future studies in Math. If you say something totally wrong like $\sqrt{x^2+1} = x+1$ then you may never pass Calculus. PLEASE be careful!

2. Again we note that $\sqrt{t^2 - 10t + 25} \neq \sqrt{t^2} - \sqrt{10t} + \sqrt{25}$, since radicals do *not* distribute across addition and subtraction.[4] In this case, however, we can factor the radicand and simplify as

$$\sqrt{t^2 - 10t + 25} = \sqrt{(t-5)^2} = |t-5|$$

Without knowing more about the value of t, we have no idea if $t-5$ is positive or negative so $|t-5|$ is our final answer.[5]

3. To simplify $\sqrt[3]{48x^{14}}$, we need to look for perfect cubes in the radicand. For the cofficient, we have $48 = 8 \cdot 6 = 2^3 \cdot 6$. To find the largest perfect cube factor in x^{14}, we divide 14 (the exponent on x) by 3 (since we are looking for a perfect *cube*). We get 4 with a remainder of 2. This means $14 = 4 \cdot 3 + 2$, so $x^{14} = x^{4 \cdot 3 + 2} = x^{4 \cdot 3} x^2 = (x^4)^3 x^2$. Putting this altogether gives:

$$\begin{array}{rcll} \sqrt[3]{48x^{14}} & = & \sqrt[3]{2^3 \cdot 6 \cdot (x^4)^3 x^2} & \text{Factor out perfect cubes} \\ & = & \sqrt[3]{2^3}\sqrt[3]{(x^4)^3}\sqrt[3]{6x^2} & \text{Rearrange factors, Product Rule of Radicals} \\ & = & 2x^4\sqrt[3]{6x^2} & \end{array}$$

4. In this example, we are looking for perfect fourth powers in the radicand. In the numerator r^4 is clearly a perfect fourth power. For the denominator, we take the power on the L, namely 12, and divide by 4 to get 3. This means $L^8 = L^{2 \cdot 4} = (L^2)^4$. We get

$$\begin{array}{rcll} \sqrt[4]{\dfrac{\pi r^4}{L^{12}}} & = & \dfrac{\sqrt[4]{\pi r^4}}{\sqrt[4]{L^{12}}} & \text{Quotient Rule of Radicals} \\ & = & \dfrac{\sqrt[4]{\pi}\sqrt[4]{r^4}}{\sqrt[4]{(L^2)^4}} & \text{Product Rule of Radicals} \\ & = & \dfrac{\sqrt[4]{\pi}|r|}{|L^2|} & \text{Simplify} \end{array}$$

Without more information about r, we cannot simplify $|r|$ any further. However, we can simplify $|L^2|$. Regardless of the choice of L, $L^2 \geq 0$. Actually, $L^2 > 0$ because L is in the denominator which means $L \neq 0$. Hence, $|L^2| = L^2$. Our answer simplifies to:

$$\frac{\sqrt[4]{\pi}|r|}{|L^2|} = \frac{|r|\sqrt[4]{\pi}}{L^2}$$

5. After a quick cancellation (two of the 2's in the second term) we need to obtain a common denominator. Since we can view the first term as having a denominator of 1, the common denominator is precisely the denominator of the second term, namely $(\sqrt[3]{x^2 - 4})^2$. With common denominators,

[4]Let $t = 1$ and see what happens to $\sqrt{t^2 - 10t + 25}$ versus $\sqrt{t^2} - \sqrt{10t} + \sqrt{25}$.

[5]In general, $|t-5| \neq |t| - |5|$ and $|t-5| \neq t+5$ so watch what you're doing!

we proceed to add the two fractions. Our last step is to factor the numerator to see if there are any cancellation opportunities with the denominator.

$$\begin{array}{rcll}
2x\sqrt[3]{x^2-4}+2\left(\dfrac{1}{2(\sqrt[3]{x^2-4})^2}\right)(2x) & = & 2x\sqrt[3]{x^2-4}+\cancel{2}\left(\dfrac{1}{\cancel{2}(\sqrt[3]{x^2-4})^2}\right)(2x) & \text{Reduce} \\
& = & 2x\sqrt[3]{x^2-4}+\dfrac{2x}{(\sqrt[3]{x^2-4})^2} & \text{Mutiply} \\
& = & (2x\sqrt[3]{x^2-4})\cdot\dfrac{(\sqrt[3]{x^2-4})^2}{(\sqrt[3]{x^2-4})^2}+\dfrac{2x}{(\sqrt[3]{x^2-4})^2} & \text{Equivalent fractions} \\
& = & \dfrac{2x(\sqrt[3]{x^2-4})^3}{(\sqrt[3]{x^2-4})^2}+\dfrac{2x}{(\sqrt[3]{x^2-4})^2} & \text{Multiply} \\
& = & \dfrac{2x(x^2-4)}{(\sqrt[3]{x^2-4})^2}+\dfrac{2x}{(\sqrt[3]{x^2-4})^2} & \text{Simplify} \\
& = & \dfrac{2x(x^2-4)+2x}{(\sqrt[3]{x^2-4})^2} & \text{Add} \\
& = & \dfrac{2x(x^2-4+1)}{(\sqrt[3]{x^2-4})^2} & \text{Factor} \\
& = & \dfrac{2x(x^2-3)}{(\sqrt[3]{x^2-4})^2} &
\end{array}$$

We cannot reduce this any further because x^2-3 is irreducible over the rational numbers.

6. We begin by working inside each set of parentheses, using the product rule for radicals and combining like terms.

$$\begin{array}{rcl}
\sqrt{(\sqrt{18y}-\sqrt{8y})^2+(\sqrt{20}-\sqrt{80})^2} & = & \sqrt{(\sqrt{9\cdot 2y}-\sqrt{4\cdot 2y})^2+(\sqrt{4\cdot 5}-\sqrt{16\cdot 5})^2} \\
& = & \sqrt{(\sqrt{9}\sqrt{2y}-\sqrt{4}\sqrt{2y})^2+(\sqrt{4}\sqrt{5}-\sqrt{16}\sqrt{5})^2} \\
& = & \sqrt{(3\sqrt{2y}-2\sqrt{2y})^2+(2\sqrt{5}-4\sqrt{5})^2} \\
& = & \sqrt{(\sqrt{2y})^2+(-2\sqrt{5})^2} \\
& = & \sqrt{2y+(-2)^2(\sqrt{5})^2} \\
& = & \sqrt{2y+4\cdot 5} \\
& = & \sqrt{2y+20}
\end{array}$$

To see if this simplifies any further, we factor the radicand: $\sqrt{2y+20}=\sqrt{2(y+10)}$. Finding no perfect square factors, we are done. $\square$

Theorem A.15 allows us to generalize the process of 'Extracting Square Roots' to 'Extracting n$^{\text{th}}$ Roots' which in turn allows us to solve equations[6] of the form $X^n = c$.

Extracting n$^{\text{th}}$ roots:

- If c is a real number and n is odd then the real number solution to $X^n = c$ is $X = \sqrt[n]{c}$.
- If $c \geq 0$ and n is even then the real number solutions to $X^n = c$ are $X = \pm\sqrt[n]{c}$.

  Note: If $c < 0$ and n is even then $X^n = c$ has no real number solutions.

Essentially, we solve $X^n = c$ by 'taking the n$^{\text{th}}$ root' of both sides: $\sqrt[n]{X^n} = \sqrt[n]{c}$. Simplifying the left side gives us just X if n is odd or $|X|$ if n is even. In the first case, $X = \sqrt[n]{c}$, and in the second, $X = \pm\sqrt[n]{c}$. Putting this together with the other part of Theorem A.15, namely $(\sqrt[n]{a})^n = a$, gives us a strategy for solving equations which involve n$^{\text{th}}$ powers and n^{th} roots.

Strategies for Solving Power and Radical Equations

- If the equation involves an n$^{\text{th}}$ power and the variable appears in only one term, isolate the term with the n$^{\text{th}}$ power and extract n$^{\text{th}}$ roots.
- If the equation involves an n$^{\text{th}}$ root and the variable appears in that n$^{\text{th}}$ root, isolate the n$^{\text{th}}$ root and raise both sides of the equation to the n$^{\text{th}}$ power.

  Note: When raising both sides of an equation to an *even* power, be sure to check for extraneous solutions.

The note about 'extraneous solutions' can be demonstrated by the basic equation: $\sqrt{x} = -2$. This equation has no solution since, by definition, $\sqrt{x} \geq 0$ for all real numbers x. However, if we square both sides of this equation, we get $(\sqrt{x})^2 = (-2)^2$ or $x = 4$. However, $x = 4$ doesn't check in the original equation, since $\sqrt{4} = 2$, not -2. Once again, the root[7] of all of our problems lies in the fact that a *negative* number to an *even* power results in a *positive* number. In other words, raising both sides of an equation to an even power does *not* produce an equivalent equation, but rather, an equation which may possess *more* solutions than the original. Hence the cautionary remark above about extraneous solutions.

Example A.13.2. Solve the following equations.

1. $(5x+3)^4 = 16$
2. $1 - \dfrac{(5-2w)^3}{7} = 9$
3. $t + \sqrt{2t+3} = 6$
4. $\sqrt{2} - 3\sqrt[3]{2y+1} = 0$
5. $\sqrt{4x-1} + 2\sqrt{1-2x} = 1$
6. $\sqrt[4]{n^2+2} + n = 0$

For the remaining problems, assume that all of the variables represent positive real numbers.[8]

[6]Well, not entirely. The equation $x^7 = 1$ has seven answers: $x = 1$ and six complex number solutions which we'll find using techniques in Section 14.3.

[7]Pun intended!

[8]That is, you needn't worry that you're multiplying or dividing by 0 or that you're forgetting absolute value symbols.

7. Solve for r: $V = \dfrac{4\pi}{3}(R^3 - r^3)$.

8. Solve for M_1: $\dfrac{r_1}{r_2} = \sqrt{\dfrac{M_2}{M_1}}$

9. Solve for v: $m = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$. Again, assume that no arithmetic rules are violated.

Solution.

1. In our first equation, the quantity containing x is already isolated, so we extract fourth roots. The exponent is even, so when the roots are extracted we need both the positive and negative roots.

$$\begin{array}{rcll} (5x+3)^4 & = & 16 & \\ 5x+3 & = & \pm\sqrt[4]{16} & \text{Extract fourth roots} \\ 5x+3 & = & \pm 2 & \\ 5x+3=2 & \text{or} & 5x+3=-2 & \\ x = -\dfrac{1}{5} & \text{or} & x=-1 & \end{array}$$

We leave it to the reader to verify that both of these solutions satisfy the original equation.

2. In this example, we first need to isolate the quantity containing the variable w. Here, third (cube) roots are required and since the exponent (index) is odd, we do not need the $\pm$:

$$\begin{array}{rcll} 1 - \dfrac{(5-2w)^3}{7} & = & 9 & \\ -\dfrac{(5-2w)^3}{7} & = & 8 & \text{Subtract 1} \\ (5-2w)^3 & = & -56 & \text{Multiply by } -7 \\ 5-2w & = & \sqrt[3]{-56} & \text{Extract cube root} \\ 5-2w & = & \sqrt[3]{(-8)(7)} & \\ 5-2w & = & \sqrt[3]{-8}\sqrt[3]{7} & \text{Product Rule} \\ 5-2w & = & -2\sqrt[3]{7} & \\ -2w & = & -5-2\sqrt[3]{7} & \text{Subtract 5} \\ w & = & \dfrac{-5-2\sqrt[3]{7}}{-2} & \text{Divide by } -2 \\ w & = & \dfrac{5+2\sqrt[3]{7}}{2} & \text{Properties of Negatives} \end{array}$$

The reader should check the answer because it provides a hearty review of arithmetic.

3. To solve $t + \sqrt{2t+3} = 6$, we first isolate the square root, then proceed to square both sides of the equation. In doing so, we run the risk of introducing extraneous solutions so checking our answers

here is a necessity.

$$\begin{array}{rcl r}
t+\sqrt{2t+3} & = & 6 & \\
\sqrt{2t+3} & = & 6-t & \text{Subtract } t \\
(\sqrt{2t+3})^2 & = & (6-t)^2 & \text{Square both sides} \\
2t+3 & = & 36-12t+t^2 & \text{F.O.I.L. / Perfect Square Trinomial} \\
0 & = & t^2-14t+33 & \text{Subtract } 2t \text{ and } 3 \\
0 & = & (t-3)(t-11) & \text{Factor}
\end{array}$$

From the Zero Product Property, we know either $t-3=0$ (which gives $t=3$) or $t-11=0$ (which gives $t=11$). When checking our answers, we find $t=3$ satisfies the original equation, but $t=11$ does not.[9] So our final answer is $t=3$ only.

4. In our next example, we locate the variable (in this case y) beneath a cube root, so we first isolate that root and cube both sides.

$$\begin{array}{rcl r}
\sqrt{2}-3\sqrt[3]{2y+1} & = & 0 & \\
-3\sqrt[3]{2y+1} & = & -\sqrt{2} & \text{Subtract } \sqrt{2} \\
\sqrt[3]{2y+1} & = & \dfrac{-\sqrt{2}}{-3} & \text{Divide by } -3 \\
\sqrt[3]{2y+1} & = & \dfrac{\sqrt{2}}{3} & \text{Properties of Negatives} \\
(\sqrt[3]{2y+1})^3 & = & \left(\dfrac{\sqrt{2}}{3}\right)^3 & \text{Cube both sides} \\
2y+1 & = & \dfrac{(\sqrt{2})^3}{3^3} & \\
2y+1 & = & \dfrac{2\sqrt{2}}{27} & \\
2y & = & \dfrac{2\sqrt{2}}{27}-1 & \text{Subtract } 1 \\
2y & = & \dfrac{2\sqrt{2}}{27}-\dfrac{27}{27} & \text{Common denominators} \\
2y & = & \dfrac{2\sqrt{2}-27}{27} & \text{Subtract fractions} \\
y & = & \dfrac{2\sqrt{2}-27}{54} & \text{Divide by 2 } \left(\text{multiply by } \frac{1}{2}\right)
\end{array}$$

Since we raised both sides to an *odd* power, we don't need to worry about extraneous solutions but we encourage the reader to check the solution just for the fun of it.

[9] It is worth noting that when $t=11$ is substituted into the original equation, we get $11+\sqrt{25}=6$. If the $+\sqrt{25}$ were $-\sqrt{25}$, the solution would check. Once again, when squaring both sides of an equation, we lose track of $\pm$, which is what lets extraneous solutions in the door.

5. In the equation $\sqrt{4x-1}+2\sqrt{1-2x}=1$, we have not one but two square roots. We begin by isolating one of the square roots and squaring both sides.

$$\begin{array}{rcll}
\sqrt{4x-1}+2\sqrt{1-2x} & = & 1 & \\
\sqrt{4x-1} & = & 1-2\sqrt{1-2x} & \text{Subtract } 2\sqrt{1-2x} \text{ from both sides} \\
(\sqrt{4x-1})^2 & = & (1-2\sqrt{1-2x})^2 & \text{Square both sides} \\
4x-1 & = & 1-4\sqrt{1-2x}+(2\sqrt{1-2x})^2 & \text{F.O.I.L. / Perfect Square Trinomial} \\
4x-1 & = & 1-4\sqrt{1-2x}+4(1-2x) & \\
4x-1 & = & 1-4\sqrt{1-2x}+4-8x & \text{Distribute} \\
4x-1 & = & 5-8x-4\sqrt{1-2x} & \text{Gather like terms}
\end{array}$$

At this point, we have just one square root so we proceed to isolate it and square both sides a second time.[10]

$$\begin{array}{rcll}
4x-1 & = & 5-8x-4\sqrt{1-2x} & \\
12x-6 & = & -4\sqrt{1-2x} & \text{Subtract 5, add } 8x \\
(12x-6)^2 & = & (-4\sqrt{1-2x})^2 & \text{Square both sides} \\
144x^2-144x+36 & = & 16(1-2x) & \\
144x^2-144x+36 & = & 16-32x & \\
144x^2-112x+20 & = & 0 & \text{Subtract 16, add } 32x \\
4(36x^2-28x+5) & = & 0 & \text{Factor} \\
4(2x-1)(18x-5) & = & 0 & \text{Factor some more}
\end{array}$$

From the Zero Product Property, we know either $2x-1=0$ or $18x-5=0$. The former gives $x=\frac{1}{2}$ while the latter gives us $x=\frac{5}{18}$. Since we squared both sides of the equation (twice!), we need to check for extraneous solutions. We find $x=\frac{5}{18}$ to be extraneous, so our only solution is $x=\frac{1}{2}$.

6. As usual, our first step in solving $\sqrt[4]{n^2+2}+n=0$ is to isolate the radical. We then proceed to raise both sides to the fourth power to eliminate the fourth root:

$$\begin{array}{rcll}
\sqrt[4]{n^2+2}+n & = & 0 & \\
\sqrt[4]{n^2+2} & = & -n & \text{Subtract } n \\
(\sqrt[4]{n^2+2})^4 & = & (-n)^4 & \text{Raise both sides to the } 4^{\text{th}} \text{ power} \\
n^2+2 & = & n^4 & \text{Properties of Negatives} \\
0 & = & n^4-n^2-2 & \text{Subtract } n^2 \text{ and } 2 \\
0 & = & (n^2-2)(n^2+1) & \text{Factor - this is a `Quadratic in Disguise'}
\end{array}$$

At this point, the Zero Product Property gives either $n^2-2=0$ or $n^2+1=0$. From $n^2-2=0$, we get $n^2=2$, so $n=\pm\sqrt{2}$. From $n^2+1=0$, we get $n^2=-1$, which gives no real solutions.[11] Since we raised both sides to an even (the fourth) power, we need to check for extraneous solutions. We find that $n=-\sqrt{2}$ works but $n=\sqrt{2}$ is extraneous.

[10] To avoid complications with fractions, we'll forego dividing by the coefficient of $\sqrt{1-2x}$, namely -4. This is perfectly fine so long as we don't forget to square it when we square both sides of the equation.

[11] Why is that again?

7. In this problem, we are asked to solve for r. While there are a lot of letters in this equation[12], r appears in only one term: r^3. Our strategy is to isolate r^3 then extract the cube root.

$$\begin{array}{rcll}
V & = & \frac{4\pi}{3}(R^3 - r^3) & \\
3V & = & 4\pi(R^3 - r^3) & \text{Multiply by 3 to clear fractions} \\
3V & = & 4\pi R^3 - 4\pi r^3 & \text{Distribute} \\
3V - 4\pi R^3 & = & -4\pi r^3 & \text{Subtract } 4\pi R^3 \\
\frac{3V - 4\pi R^3}{-4\pi} & = & r^3 & \text{Divide by } -4\pi \\
\frac{4\pi R^3 - 3V}{4\pi} & = & r^3 & \text{Properties of Negatives} \\
\sqrt[3]{\frac{4\pi R^3 - 3V}{4\pi}} & = & r & \text{Extract the cube root}
\end{array}$$

The check is, as always, left to the reader and highly encouraged.

8. The equation we are asked to solve in this example is from the world of Chemistry and is none other than Graham's Law of Effusion. As was mentioned in Example A.12.2, subscripts in Mathematics are used to distinguish between variables and have no arithmetic significance. In this example, r_1, r_2, M_1 and M_2 are as different as x, y, z and 117. Since we are asked to solve for M_1, we locate M_1 and see it is in the denominator of a fraction which is inside of a square root. We eliminate the square root by squaring both sides and proceed from there.

$$\begin{array}{rcll}
\frac{r_1}{r_2} & = & \sqrt{\frac{M_2}{M_1}} & \\
\left(\frac{r_1}{r_2}\right)^2 & = & \left(\sqrt{\frac{M_2}{M_1}}\right)^2 & \text{Square both sides} \\
\frac{r_1^2}{r_2^2} & = & \frac{M_2}{M_1} & \\
r_1^2 M_1 & = & M_2 r_2^2 & \text{Multiply by } r_2^2 M_1 \text{ to clear fractions, assume } r_2, M_1 \neq 0 \\
M_1 & = & \frac{M_2 r_2^2}{r_1^2} & \text{Divide by } r_1^2, \text{ assume } r_1 \neq 0
\end{array}$$

As the reader may expect, checking the answer amounts to a good exercise in simplifying rational and radical expressions. The fact that we are assuming all of the variables represent positive real numbers comes in to play, as well.

9. Our last equation to solve comes from Einstein's Special Theory of Relativity and relates the mass of an object to its velocity as it moves.[13] We are asked to solve for v which is located in just one term,

[12] including a Greek letter, no less!

[13] See this article on the Lorentz Factor.

namely v^2, which happens to lie in a fraction underneath a square root which is itself a denominator. We have quite a lot of work ahead of us!

$$\begin{array}{rclr}
m & = & \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}} & \\
m\sqrt{1-\dfrac{v^2}{c^2}} & = & m_0 & \text{Multiply by } \sqrt{1-\dfrac{v^2}{c^2}} \text{ to clear fractions} \\
\left(m\sqrt{1-\dfrac{v^2}{c^2}}\right)^2 & = & m_0^2 & \text{Square both sides} \\
m^2\left(1-\dfrac{v^2}{c^2}\right) & = & m_0^2 & \text{Properties of Exponents} \\
m^2-\dfrac{m^2v^2}{c^2} & = & m_0^2 & \text{Distribute} \\
-\dfrac{m^2v^2}{c^2} & = & m_0^2-m^2 & \text{Subtract } m^2 \\
m^2v^2 & = & -c^2(m_0^2-m^2) & \text{Multiply by } -c^2 \ (c^2 \neq 0) \\
m^2v^2 & = & -c^2m_0^2+c^2m^2 & \text{Distribute} \\
v^2 & = & \dfrac{c^2m^2-c^2m_0^2}{m^2} & \text{Rearrange terms, divide by } m^2 \ (m^2 \neq 0) \\
v & = & \sqrt{\dfrac{c^2m^2-c^2m_0^2}{m^2}} & \text{Extract Square Roots, } v > 0 \text{ so no } \pm \\
v & = & \dfrac{\sqrt{c^2(m^2-m_0^2)}}{\sqrt{m^2}} & \text{Properties of Radicals, factor} \\
v & = & \dfrac{|c|\sqrt{m^2-m_0^2}}{|m|} & \\
v & = & \dfrac{c\sqrt{m^2-m_0^2}}{m} & c > 0 \text{ and } m > 0 \text{ so } |c| = c \text{ and } |m| = m
\end{array}$$

Checking the answer algebraically would earn the reader great honor and respect on the Algebra battlefield so it is highly recommended.

A.13.1 Rationalizing Denominators and Numerators

In Section A.10, there were a few instances where we needed to 'rationalize' a denominator - that is, take a fraction with radical in the denominator and re-write it as an equivalent fraction without a radical in the denominator. There are various reasons for wanting to do this,[14] but the most pressing reason is

[14]Before the advent of the handheld calculator, rationalizing denominators made it easier to get decimal approximations to fractions containing radicals. However, some (admittedly more abstract) applications remain today – one of which we'll explore in Section A.11; one you'll see in Calculus.

that rationalizing denominators - and numerators as well - gives us an opportunity for more practice with fractions and radicals. To refresh your memory, we rationalize a denominator and a numerator below:

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \frac{7\sqrt[3]{4}}{3} = \frac{7\sqrt[3]{4}\sqrt[3]{2}}{3\sqrt[3]{2}} = \frac{7\sqrt[3]{8}}{3\sqrt[3]{2}} = \frac{7 \cdot 2}{3\sqrt[3]{2}} = \frac{14}{3\sqrt[3]{2}}$$

In general, if the fraction contains either a single term numerator or denominator with an undesirable n^{th} root, we multiply the numerator and denominator by whatever is required to obtain a perfect n^{th} power in the radicand that we want to eliminate. If the fraction contains two terms the situation is somewhat more complicated. To see why, consider the fraction $\frac{3}{4-\sqrt{5}}$. Suppose we wanted to rid the denominator of the $\sqrt{5}$ term. We could try as above and multiply numerator and denominator by $\sqrt{5}$ but that just yields:

$$\frac{3}{4-\sqrt{5}} = \frac{3\sqrt{5}}{(4-\sqrt{5})\sqrt{5}} = \frac{3\sqrt{5}}{4\sqrt{5}-\sqrt{5}\sqrt{5}} = \frac{3\sqrt{5}}{4\sqrt{5}-5}$$

We haven't removed $\sqrt{5}$ from the denominator - we've just shuffled it over to the other term in the denominator. As you may recall, the strategy here is to multiply both the numerator and the denominator by what's called the **conjugate**.

Definition A.22. Congugate of a Square Root Expression: If a, b and c are real numbers with $c > 0$ then the quantities $(a + b\sqrt{c})$ and $(a - b\sqrt{c})$ are **conjugates** of one another.[a] Conjugates multiply according to the Difference of Squares Formula:

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - (b\sqrt{c})^2 = a^2 - b^2c$$

[a]As are $(b\sqrt{c} - a)$ and $(b\sqrt{c} + a)$ because $(b\sqrt{c} - a)(b\sqrt{c} + a) = b^2c - a^2$.

That is, to get the conjugate of a two-term expression involving a square root, you change the '$-$' to a '+,' or vice-versa. For example, the conjugate of $4 - \sqrt{5}$ is $4 + \sqrt{5}$, and when we multiply these two factors together, we get $(4 - \sqrt{5})(4 + \sqrt{5}) = 4^2 - (\sqrt{5})^2 = 16 - 5 = 11$. Hence, to eliminate the $\sqrt{5}$ from the denominator of our original fraction, we multiply both the numerator and the denominator by the *conjugate* of $4 - \sqrt{5}$ to get:

$$\frac{3}{4-\sqrt{5}} = \frac{3(4+\sqrt{5})}{(4-\sqrt{5})(4+\sqrt{5})} = \frac{3(4+\sqrt{5})}{4^2-(\sqrt{5})^2} = \frac{3(4+\sqrt{5})}{16-5} = \frac{12+3\sqrt{5}}{11}$$

What if we had $\sqrt[3]{5}$ instead of $\sqrt{5}$? We could try multiplying $4 - \sqrt[3]{5}$ by $4 + \sqrt[3]{5}$ to get

$$(4 - \sqrt[3]{5})(4 + \sqrt[3]{5}) = 4^2 - (\sqrt[3]{5})^2 = 16 - \sqrt[3]{25},$$

which leaves us with a cube root. What we need to undo the cube root is a perfect cube, which means we look to the Difference of Cubes Formula for inspiration: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. If we take $a = 4$ and $b = \sqrt[3]{5}$, we multiply

$$(4 - \sqrt[3]{5})(4^2 + 4\sqrt[3]{5} + (\sqrt[3]{5})^2) = 4^3 + 4^2\sqrt[3]{5} + 4\sqrt[3]{5} - 4^2\sqrt[3]{5} - 4(\sqrt[3]{5})^2 - (\sqrt[3]{5})^3 = 64 - 5 = 59$$

So if we were charged with rationalizing the denominator of $\frac{3}{4-\sqrt[3]{5}}$, we'd have:

$$\frac{3}{4-\sqrt[3]{5}} = \frac{3(4^2+4\sqrt[3]{5}+(\sqrt[3]{5})^2)}{(4-\sqrt[3]{5})(4^2+4\sqrt[3]{5}+(\sqrt[3]{5})^2)} = \frac{48+12\sqrt[3]{5}+3\sqrt[3]{25}}{59}$$

This sort of thing extends to n$^{\text{th}}$ roots since $(a-b)$ is a factor of $a^n - b^n$ for all natural numbers n, but in practice, we'll stick with square roots with just a few cube roots thrown in for a challenge.[15]

Example A.13.3. Rationalize the indicated numerator or denominator:

1. Rationalize the denominator: $\dfrac{3}{\sqrt[5]{24x^2}}$
2. Rationalize the numerator: $\dfrac{\sqrt{9+h}-3}{h}$

Solution.

1. We are asked to rationalize the denominator, which in this case contains a fifth root. That means we need to work to create fifth powers of each of the factors of the radicand. To do so, we first factor the radicand: $24x^2 = 8 \cdot 3 \cdot x^2 = 2^3 \cdot 3 \cdot x^2$. To obtain fifth powers, we need to multiply by $2^2 \cdot 3^4 \cdot x^3$ inside the radical.

$$\begin{aligned}
\frac{3}{\sqrt[5]{24x^2}} &= \frac{3}{\sqrt[5]{2^3 \cdot 3 \cdot x^2}} & \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^3 \cdot 3 \cdot x^2}\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}} & \text{Equivalent Fractions} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^3 \cdot 3 \cdot x^2 \cdot 2^2 \cdot 3^4 \cdot x^3}} & \text{Product Rule} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^5 \cdot 3^5 \cdot x^5}} & \text{Property of Exponents} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^5}\sqrt[5]{3^5}\sqrt[5]{x^5}} & \text{Product Rule} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{2 \cdot 3 \cdot x} & \text{Product Rule} \\
&= \frac{\cancel{3}\sqrt[5]{4 \cdot 81 \cdot x^3}}{2 \cdot \cancel{3} \cdot x} & \text{Reduce} \\
&= \frac{\sqrt[5]{324x^3}}{2x} & \text{Simplify}
\end{aligned}$$

2. Here, we are asked to rationalize the *numerator*. Since it is a two term numerator involving a square root, we multiply both numerator and denominator by the conjugate of $\sqrt{9+h}-3$, namely $\sqrt{9+h}+3$.

[15]To see what to do about fourth roots, use long division to find $(a^4 - b^4) \div (a-b)$, and apply this to $4 - \sqrt[4]{5}$.

After simplifying, we find an opportunity to reduce the fraction:

$$\begin{aligned}
\frac{\sqrt{9+h}-3}{h} &= \frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)} && \text{Equivalent Fractions} \\
&= \frac{(\sqrt{9+h})^2-3^2}{h(\sqrt{9+h}+3)} && \text{Difference of Squares} \\
&= \frac{(9+h)-9}{h(\sqrt{9+h}+3)} && \text{Simplify} \\
&= \frac{h}{h(\sqrt{9+h}+3)} && \text{Simplify} \\
&= \frac{\cancelto{1}{h}}{\cancel{h}(\sqrt{9+h}+3)} && \text{Reduce} \\
&= \frac{1}{\sqrt{9+h}+3}
\end{aligned}$$

We close this section with an awesome example from Calculus.

Example A.13.4. Simplify the compound fraction $\dfrac{\dfrac{1}{\sqrt{2(x+h)+1}}-\dfrac{1}{\sqrt{2x+1}}}{h}$ then rationalize the numerator of the result.

Solution. We start by multiplying the top and bottom of the 'big' fraction by $\sqrt{2x+2h+1}\sqrt{2x+1}$.

$$\begin{aligned}
\frac{\dfrac{1}{\sqrt{2(x+h)+1}}-\dfrac{1}{\sqrt{2x+1}}}{h} &= \frac{\dfrac{1}{\sqrt{2x+2h+1}}-\dfrac{1}{\sqrt{2x+1}}}{h} \\
&= \frac{\left(\dfrac{1}{\sqrt{2x+2h+1}}-\dfrac{1}{\sqrt{2x+1}}\right)\sqrt{2x+2h+1}\sqrt{2x+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}} \\
&= \frac{\dfrac{\cancel{\sqrt{2x+2h+1}}\sqrt{2x+1}}{\cancel{\sqrt{2x+2h+1}}}-\dfrac{\sqrt{2x+2h+1}\cancel{\sqrt{2x+1}}}{\cancel{\sqrt{2x+1}}}}{h\sqrt{2x+2h+1}\sqrt{2x+1}} \\
&= \frac{\sqrt{2x+1}-\sqrt{2x+2h+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}}
\end{aligned}$$

Next, we multiply the numerator and denominator by the conjugate of $\sqrt{2x+1}-\sqrt{2x+2h+1}$, namely

$\sqrt{2x+1}+\sqrt{2x+2h+1}$, simplify and reduce:

$$\begin{aligned}
\frac{\sqrt{2x+1}-\sqrt{2x+2h+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}} &= \frac{(\sqrt{2x+1}-\sqrt{2x+2h+1})(\sqrt{2x+1}+\sqrt{2x+2h+1})}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})} \\
&= \frac{(\sqrt{2x+1})^2-(\sqrt{2x+2h+1})^2}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})} \\
&= \frac{(2x+1)-(2x+2h+1)}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})} \\
&= \frac{2x+1-2x-2h-1}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})} \\
&= \frac{-2\cancel{h}}{\cancel{h}\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})} \\
&= \frac{-2}{\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})}
\end{aligned}$$

While the denominator is quite a bit more complicated than what we started with, we have done what was asked of us. In the interest of full disclosure, the reason we did all of this was to cancel the original '*h*' from the denominator. That's an awful lot of effort to get rid of just one little *h*, but you'll see the significance of this in Calculus. □

A.13.2 Exercises

In Exercises 1 - 13, perform the indicated operations and simplify.

1. $\sqrt{9x^2}$
2. $\sqrt[3]{8t^3}$
3. $\sqrt{50y^6}$
4. $\sqrt{4t^2+4t+1}$
5. $\sqrt{w^2-16w+64}$
6. $\sqrt{(\sqrt{12x}-\sqrt{3x})^2+1}$
7. $\sqrt{\dfrac{c^2-v^2}{c^2}}$
8. $\sqrt[3]{\dfrac{24\pi r^5}{L^3}}$
9. $\sqrt[4]{\dfrac{32\pi\varepsilon^8}{\rho^{12}}}$
10. $\sqrt{x}-\dfrac{x+1}{\sqrt{x}}$
11. $3\sqrt{1-t^2}+3t\left(\dfrac{1}{2\sqrt{1-t^2}}\right)(-2t)$
12. $2\sqrt[3]{1-z}+2z\left(\dfrac{1}{3\left(\sqrt[3]{1-z}\right)^2}\right)(-1)$
13. $\dfrac{3}{\sqrt[3]{2x-1}}+(3x)\left(-\dfrac{1}{3\left(\sqrt[3]{2x-1}\right)^4}\right)(2)$

In Exercises 14 - 25, find all real solutions.

14. $(2x+1)^3+8=0$
15. $\dfrac{(1-2y)^4}{3}=27$
16. $\dfrac{1}{1+2t^3}=4$
17. $\sqrt{3x+1}=4$
18. $5-\sqrt[3]{t^2+1}=1$
19. $x+1=\sqrt{3x+7}$
20. $y+\sqrt{3y+10}=-2$
21. $3t+\sqrt{6-9t}=2$
22. $2x-1=\sqrt{x+3}$
23. $w=\sqrt[4]{12-w^2}$
24. $\sqrt{x-2}+\sqrt{x-5}=3$
25. $\sqrt{2x+1}=3+\sqrt{4-x}$

In Exercises 26 - 29, solve each equation for the indicated variable. Assume all quantities represent positive real numbers.

26. Solve for h: $I=\dfrac{bh^3}{12}$.
27. Solve for a: $I_0=\dfrac{5\sqrt{3}a^4}{16}$
28. Solve for g: $T=2\pi\sqrt{\dfrac{L}{g}}$
29. Solve for v: $L=L_0\sqrt{1-\dfrac{v^2}{c^2}}$.

In Exercises 30 - 35, rationalize the numerator or denominator, and simplify.

30. $\dfrac{4}{3-\sqrt{2}}$
31. $\dfrac{7}{\sqrt[3]{12x^7}}$
32. $\dfrac{\sqrt{x}-\sqrt{c}}{x-c}$
33. $\dfrac{\sqrt{2x+2h+1}-\sqrt{2x+1}}{h}$
34. $\dfrac{\sqrt[3]{x+1}-2}{x-7}$
35. $\dfrac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$

A.13.3 Answers

1. $3|x|$
2. $2t$
3. $5|y^3|\sqrt{2}$
4. $|2t+1|$
5. $|w-8|$
6. $\sqrt{3x+1}$
7. $\dfrac{\sqrt{c^2-v^2}}{|c|}$
8. $\dfrac{2r\sqrt[3]{3\pi r^2}}{L}$
9. $\dfrac{2\varepsilon^2\sqrt[4]{2\pi}}{|\rho^3|}$
10. $-\dfrac{1}{\sqrt{x}}$
11. $\dfrac{3-6t^2}{\sqrt{1-t^2}}$
12. $\dfrac{6-8z}{3(\sqrt[3]{1-z})^2}$
13. $\dfrac{4x-3}{(2x-1)\sqrt[3]{2x-1}}$
14. $x=-\dfrac{3}{2}$
15. $y=-1,2$
16. $t=-\dfrac{\sqrt[3]{3}}{2}$
17. $x=5$
18. $t=\pm 3\sqrt{7}$
19. $x=3$
20. $y=-3$
21. $t=-\dfrac{1}{3},\dfrac{2}{3}$
22. $x=\dfrac{5+\sqrt{57}}{8}$
23. $w=\sqrt{3}$
24. $x=6$
25. $x=4$
26. $h=\sqrt[3]{\dfrac{12I}{b}}$
27. $a=\dfrac{2\sqrt[4]{I_0}}{\sqrt[4]{5\sqrt{3}}}$
28. $g=\dfrac{4\pi^2L}{T^2}$
29. $v=\dfrac{c\sqrt{L_0^2-L^2}}{L_0}$
30. $\dfrac{12+4\sqrt{2}}{7}$
31. $\dfrac{7\sqrt[3]{18x^2}}{6x^3}$
32. $\dfrac{1}{\sqrt{x}+\sqrt{c}}$
33. $\dfrac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}}$
34. $\dfrac{1}{(\sqrt[3]{x+1})^2+2\sqrt[3]{x+1}+4}$
35. $\dfrac{1}{(\sqrt[3]{x+h})^2+\sqrt[3]{x+h}\sqrt[3]{x}+(\sqrt[3]{x})^2}$

A.14 Variation

In many instances in the sciences, equations are encountered as a result of fundamental natural laws which are typically a result of assuming certain basic relationships between variables. These basic relationships are summarized in the definition below.

Definition A.23. Suppose x, y and z are variable quantities. We say

- y **varies directly with** (or is **directly proportional to**) x if there is a constant k such that

$$y = kx$$

- y **varies inversely with** (or is **inversely proportional to**) x if there is a constant k such that

$$y = \frac{k}{x}$$

- z **varies jointly with** (or is **jointly proportional to**) x and y if there is a constant k such that

$$z = kxy$$

The constant k in the above definitions is called the **constant of proportionality**.

Example A.14.1. Translate the following into mathematical equations using Definition A.23.

1. Hooke's Law: The force F exerted on a spring is directly proportional the extension x of the spring.

2. Boyle's Law: At a constant temperature, the pressure P of an ideal gas is inversely proportional to its volume V. (We explore this one more deeply in Example 3.3.5.

3. The volume V of a right circular cone varies jointly with the height h of the cone and the square of the radius r of the base.

4. Ohm's Law: The current I through a conductor between two points is directly proportional to the voltage V between the points and inversely proportional to the resistance R between the points.

5. Newton's Law of Universal Gravitation: Suppose two objects, one of mass m and one of mass M, are positioned so that the distance between their centers of mass is r. The gravitational force F exerted on the two objects varies directly with the product of the two masses and inversely with the square of the distance between their centers of mass.

Solution.

1. Applying the definition of direct variation, we get $F = kx$ for some constant k.

2. Since P and V are inversely proportional, we write $P = \frac{k}{V}$.

3. There is a bit of ambiguity here. It's clear that the volume and the height of the cone are represented by the quantities V and h, respectively, but does r represent the radius of the base or the square of the radius of the base? It is the former. Usually, if an algebraic operation is specified (like squaring), it is meant to be expressed in the formula. We apply Definition A.23 to get $V = khr^2$.

4. Even though the problem doesn't use the phrase 'varies jointly', it is implied by the fact that the current I is related to two different quantities. Since I varies directly with V but inversely with R, we write $I = \frac{kV}{R}$.

5. We write the product of the masses mM and the square of the distance as r^2. We have that F varies directly with mM and inversely with r^2, so $F = \frac{kmM}{r^2}$. □

A note about units is in order. The formulas given in Example A.14.1 above all have quantities from the "real world" and we would disappoint our friends who teach Science if we didn't remind you to pay attention to units when working with these equations. The natural question that arises is "What units does k have?" The answer is "whatever works" and by that we mean the units on k will be whatever it takes to make the equation have the same units on both sides.

For example, in Hooke's Law we have that $F = kx$. If F is in newtons and x is in meters then k must be in $\frac{\text{newton}}{\text{meter}}$. This can lead to some odd sounding units, such as the units on the constant R in the Ideal Gas Law $PV = nRT$ (see Exercise 11) or no units at all (see Exercise 9a). Unit conversions can mess things up as well - see Exercise 9b for a sample of that kind of nonsense!

We end this section with an example that first requires us to find the value of k and then use it to solve another problem.

Example A.14.2. Suppose it takes 11 pounds of force to hold a spring 2 inches beyond its natural length. What force is required to hold it 7 inches beyond natural length?

Solution. Using Hooke's Law with $F = 11$ pounds and $x = 2$ inches we solve $11 = k * 2$ for k and find $k = 5.5\frac{\text{pound}}{\text{inch}}$. Setting $x = 7$ in Hooke's Law with $k = 5.5$ yields $F = 5.5 * 7 = 38.5$ pounds of force. (Check the units to convince yourself that this worked!)

A.14.1 Exercises

In Exercises 1 - 6, translate the following into mathematical equations.

1. At a constant pressure, the temperature T of an ideal gas is directly proportional to its volume V. (This is Charles's Law)

2. The frequency of a wave f is inversely proportional to the wavelength of the wave λ.

3. The density d of a material is directly proportional to the mass of the object m and inversely proportional to its volume V.

4. The square of the orbital period of a planet P is directly proportional to the cube of the semi-major axis of its orbit a. (This is Kepler's Third Law of Planetary Motion)

5. The drag of an object traveling through a fluid D varies jointly with the density of the fluid ρ and the square of the velocity of the object ν.

6. Suppose two electric point charges, one with charge q and one with charge Q, are positioned r units apart. The electrostatic force F exerted on the charges varies directly with the product of the two charges and inversely with the square of the distance between the charges. (This is Coulomb's Law)

7. According to this webpage, the frequency f of a vibrating string is given by $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\mu}}$ where T is the tension, μ is the linear mass[1] of the string and L is the length of the vibrating part of the string. Express this relationship using the language of variation.

8. According to the Centers for Disease Control and Prevention www.cdc.gov, a person's Body Mass Index B is directly proportional to his weight W in pounds and inversely proportional to the square of his height h in inches.

   (a) Express this relationship as a mathematical equation.

   (b) If a person who was 5 feet, 10 inches tall weighed 235 pounds had a Body Mass Index of 33.7, what is the value of the constant of proportionality?

   (c) Rewrite the mathematical equation found in part 8a to include the value of the constant found in part 8b and then find your Body Mass Index.

9. This exercise refers back to the volume of a right circular cone formula found in Example A.14.1.

   (a) First assume that V, h and r are all measured using the same unit of length. Work with your classmates to show that in this case, the k needed for the volume formula $V = khr^2$ has no units on it.

[1] Also known as the linear density. It is simply a measure of mass per unit length.

(b) Now assume that V is measured in milliliters, h is measured in meters and r is measured in yards. Work with your classmates to find the units on k so that the volume formula $V = khr^2$ makes sense.

10. We know that the circumference of a circle varies directly with its radius with 2π as the constant of proportionality. (That is, we know $C = 2\pi r$.) With the help of your classmates, compile a list of other basic geometric relationships which can be seen as variations.

11. Research the Ideal Gas Law $PV = nRT$ to see what sorts of units are used for the constant R. What other formulations of this law did you find in your research?

A.14.2 Answers

1. $T = kV$

2.[2] $f = \dfrac{k}{\lambda}$

3. $d = \dfrac{km}{V}$

4. $P^2 = ka^3$

5.[3] $D = k\rho\nu^2$

6.[4] $F = \dfrac{kqQ}{r^2}$

7. Rewriting $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\mu}}$ as $f = \dfrac{\frac{1}{2}\sqrt{T}}{L\sqrt{\mu}}$ we see that the frequency f varies directly with the square root of the tension and varies inversely with the length and the square root of the linear mass.

8. (a) $B = \dfrac{kW}{h^2}$ (b)[5] $k = 702.68$ (c) $B = \dfrac{702.68W}{h^2}$

---

[2]The character λ is the lower case Greek letter 'lambda.'

[3]The characters ρ and ν are the lower case Greek letters 'rho' and 'nu,' respectively.

[4]Note the similarity to this formula and Newton's Law of Universal Gravitation as discussed in Example 5.

[5]The CDC uses 703.

Appendix B

Geometry Review

The authors really wanted the Trigonometry portion of Precalculus, Episode IV to start with the definitions of the circular functions so one purpose of this Geometry Review Appendix is to find a home for the material that is prerequisite to those definitions. Another reason for this Appendix is to further support a "co-requisite" approach to teaching a Precalculus[1] class. As is the case with the Algebra Review Appendix, this chapter is not designed for students who have never seen this material before. In fact, our treatment of Geometry is even more brief than that of Algebra because we assume a student who is taking a stand alone college-level Trigonometry class is already proficient in College Algebra, and those learning the Trigonometry portion of a full Precalculus class have ostensibly survived the College Algebra portion. Thus we review only some very basic concepts covered in a typical high school Geometry course. Where appropriate, we have referenced specific sections of the main body of the Precalculus text in an effort to assist faculty who would like to assign the Appendix as "just in time" review reading to their students. This Appendix contains two sections which are briefly described below:

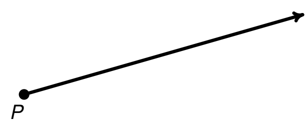
Section B.1 (Angles in Degrees) is a brief review of some of the terminology and concepts from a typical high school Geometry course. Radian measure is deferred until Chapter 11.

Section B.2 (Basic Right Triangle Trigonometry) defines the trigonometric functions in the context of a right triangle using angles measured in degrees. Basic applications are discussed and a proof of the Pythagorean Theorem is given but trigonometric identities are deferred until Chapter 11.

[1] Remember how we define "Precalculus" - to us, Precalculus = College Algebra + College Trigonometry without formal limits. In order to fully support a "co-requisite" approach to a class that has Trigonometry in it, we felt it necessary to provide some material to assist students who have gaps in their Geometry background. The careful reader will note that all of this material was in the main body of our third edition so it can be included nearly seemlessly into a regular Trigonometry class.

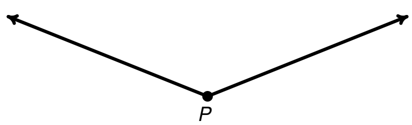
B.1 Angles in Degrees

This section serves as a review of the concept of 'angle' and the use of the degree system to measure angles. Recall that a **ray** is usually described as a 'half-line' and can be thought of as a line segment in which one of the two endpoints is pushed off infinitely distant from the other, as pictured below. The point from which the ray originates is called the **initial point** of the ray.

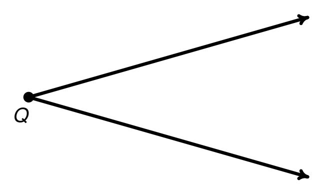


A ray with initial point P.

When two rays share a common initial point they form an **angle** and the common initial point is called the **vertex** of the angle. Two examples of what are commonly thought of as angles are

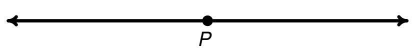


An angle with vertex P.

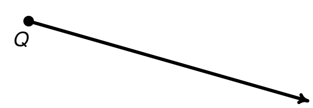


An angle with vertex Q.

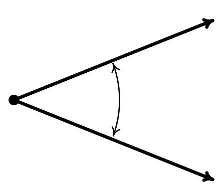
However, the two figures below also depict angles - albeit these are, in some sense, extreme cases. In the first case, the two rays are directly opposite each other forming what is known as a **straight angle**; in the second, the rays are identical so the 'angle' is indistinguishable from the ray itself.

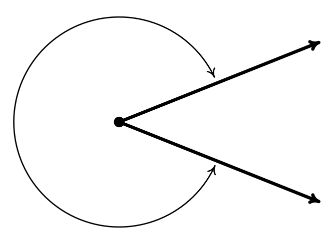


A straight angle.



The **measure of an angle** is a number which indicates the amount of rotation that separates the rays of the angle. There is one immediate problem with this, as pictured below.

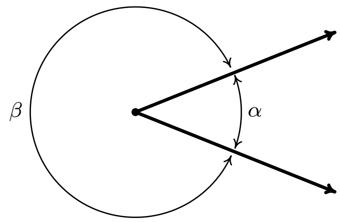




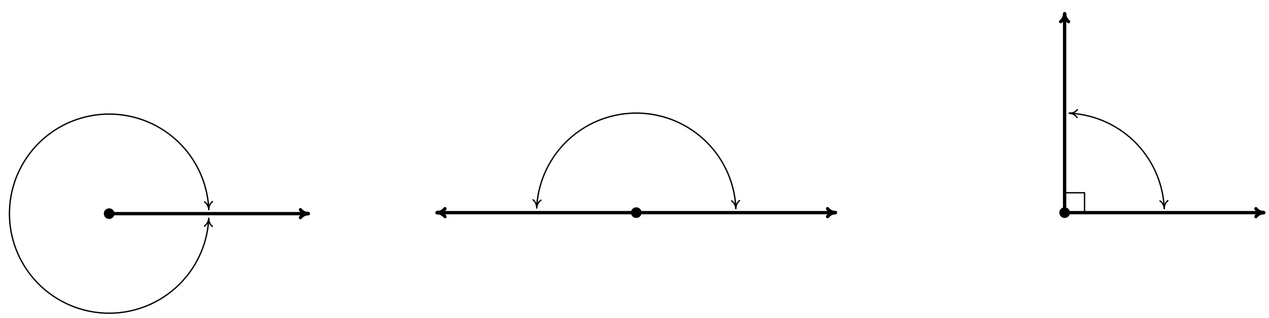
Which amount of rotation are we attempting to quantify? What we have just discovered is that we have at least two angles described by this diagram.[1] Clearly these two angles have different measures because one appears to represent a larger rotation than the other, so we must label them differently. In this book,

[1] The phrase 'at least' will be justified in short order.

we use lower case Greek letters such as α (alpha), β (beta), γ (gamma) and θ (theta) to label angles. So, for instance, we have



One system to measure angles is **degree measure**. Quantities measured in degrees are denoted by the symbol '$^\circ$.' One complete revolution as shown below is 360°, and parts of a revolution are measured proportionately.[2] Thus half of a revolution (a straight angle) measures $\frac{1}{2}(360^\circ) = 180^\circ$, a quarter of a revolution (a **right angle**) measures $\frac{1}{4}(360^\circ) = 90^\circ$ and so on.

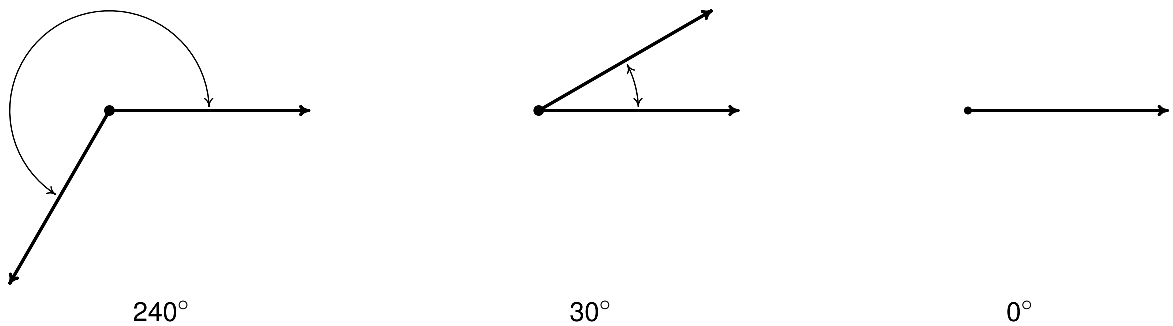


One revolution $\leftrightarrow$ 360° $\qquad$ 180° $\qquad$ 90°

Note that in the above figure, we have used the small square '$\square$' to denote a right angle, as is commonplace in Geometry. Recall that if an angle measures strictly between 0° and 90° it is called an **acute angle** and if it measures strictly between 90° and 180° it is called an **obtuse angle**. It is important to note that, theoretically, we can know the measure of any angle as long as we know the proportion it represents of entire revolution.[3] For instance, the measure of an angle which represents a rotation of $\frac{2}{3}$ of a revolution would measure $\frac{2}{3}(360^\circ) = 240^\circ$, the measure of an angle which constitutes only $\frac{1}{12}$ of a revolution measures $\frac{1}{12}(360^\circ) = 30^\circ$ and an angle which indicates no rotation at all is measured as 0°.

[2] The choice of '360' is most often attributed to the Babylonians.

[3] This is how a protractor is graded.



Using our definition of degree measure, we have that 1° represents the measure of an angle which constitutes $\frac{1}{360}$ of a revolution. Even though it may be hard to draw, it is nonetheless not difficult to imagine an angle with measure smaller than 1°. There are two ways to subdivide degrees. The first, and most familiar, is **decimal degrees**. For example, an angle with a measure of 30.5° would represent a rotation halfway between 30° and 31°, or equivalently, $\frac{30.5}{360} = \frac{61}{720}$ of a full rotation. This can be taken to the limit using Calculus so that measures like $\sqrt{2}^\circ$ make sense.[4] The second way to divide degrees is the **Degree - Minute - Second** (**DMS**) system. In this system, one degree is divided equally into sixty minutes, and in turn, each minute is divided equally into sixty seconds.[5] In symbols, we write $1^\circ = 60'$ and $1' = 60''$, from which it follows that $1^\circ = 3600''$. To convert a measure of 42.125° to the DMS system, we start by noting that $42.125^\circ = 42^\circ + 0.125^\circ$. Converting the partial amount of degrees to minutes, we find $0.125^\circ \left(\frac{60'}{1^\circ}\right) = 7.5' = 7' + 0.5'$. Converting the partial amount of minutes to seconds gives $0.5' \left(\frac{60''}{1'}\right) = 30''$. Putting it all together yields

$$\begin{aligned} 42.125^\circ &= 42^\circ + 0.125^\circ \\ &= 42^\circ + 7.5' \\ &= 42^\circ + 7' + 0.5' \\ &= 42^\circ + 7' + 30'' \\ &= 42^\circ 7' 30'' \end{aligned}$$

On the other hand, to convert $117^\circ 15' 45''$ to decimal degrees, we first compute $15' \left(\frac{1^\circ}{60'}\right) = \frac{1}{4}^\circ$ and $45'' \left(\frac{1^\circ}{3600''}\right) = \frac{1}{80}^\circ$. Then we find

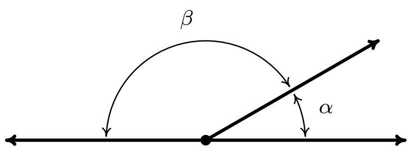
$$\begin{aligned} 117^\circ 15' 45'' &= 117^\circ + 15' + 45'' \\ &= 117^\circ + \frac{1}{4}^\circ + \frac{1}{80}^\circ \\ &= \frac{9381}{80}^\circ \\ &= 117.2625^\circ \end{aligned}$$

Recall that two acute angles are called **complementary angles** if their measures add to 90°. Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles**

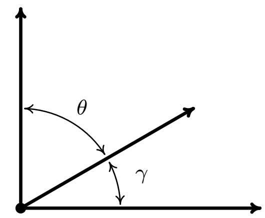
[4] Awesome math pun aside, this is the same idea behind defining irrational exponents in Section 4.2.

[5] Does this kind of system seem familiar?

if their measures add to 180°. In the diagram below, the angles α and β are supplementary angles while the pair γ and θ are complementary angles.



Supplementary Angles



Complementary Angles

In practice, the distinction between the angle itself and its measure is blurred so that the sentence 'α is an angle measuring 42°' is often abbreviated as '$\alpha = 42°$.' It is now time for an example.

Example B.1.1. Let $\alpha = 111.371°$ and $\beta = 37°28'17''$.

1. Convert α to the DMS system. Round your answer to the nearest second.
2. Convert β to decimal degrees. Round your answer to the nearest thousandth of a degree.
3. Sketch α and β.
4. Find a supplementary angle for α.
5. Find a complementary angle for β.

Solution.

1. To convert α to the DMS system, we start with $111.371° = 111° + 0.371°$. Next we convert $0.371° \left(\frac{60'}{1°}\right) = 22.26'$. Writing $22.26' = 22' + 0.26'$, we convert $0.26' \left(\frac{60''}{1'}\right) = 15.6''$. Hence,

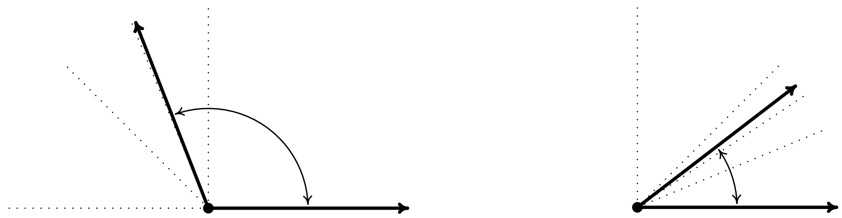
$$\begin{aligned} 111.371° &= 111° + 0.371° \\ &= 111° + 22.26' \\ &= 111° + 22' + 0.26' \\ &= 111° + 22' + 15.6'' \\ &= 111°22'15.6'' \end{aligned}$$

   Rounding to seconds, we obtain $\alpha \approx 111°22'16''$.

2. To convert β to decimal degrees, we convert $28' \left(\frac{1°}{60'}\right) = \frac{7}{15}^{\circ}$ and $17'' \left(\frac{1°}{3600''}\right) = \frac{17}{3600}^{\circ}$. Putting it all together, we have

$$\begin{aligned} 37°28'17'' &= 37° + 28' + 17'' \\ &= 37° + \frac{7}{15}^{\circ} + \frac{17}{3600}^{\circ} \\ &= \frac{134897}{3600}^{\circ} \\ &\approx 37.471° \end{aligned}$$

3. To sketch α, we first note that $90^\circ < \alpha < 180^\circ$. Dividing this range in half, we get $90^\circ < \alpha < 135^\circ$, and once more, we have $90^\circ < \alpha < 112.5^\circ$. This gives us a pretty good estimate for α, as shown below.[6] Proceeding similarly for β, we find $0^\circ < \beta < 90^\circ$, then $0^\circ < \beta < 45^\circ$, $22.5^\circ < \beta < 45^\circ$, and lastly, $33.75^\circ < \beta < 45^\circ$.



Angle α Angle β

4. To find a supplementary angle for α, we seek an angle θ so that $\alpha + \theta = 180^\circ$. We get $\theta = 180^\circ - \alpha = 180^\circ - 111.371^\circ = 68.629^\circ$.

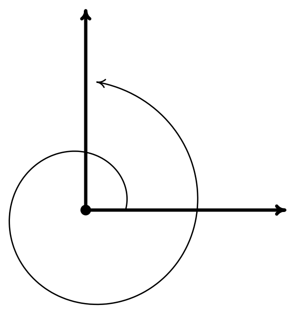
5. To find a complementary angle for β, we seek an angle γ so that $\beta + \gamma = 90^\circ$. We get $\gamma = 90^\circ - \beta = 90^\circ - 37^\circ 28' 17''$. While we could reach for the calculator to obtain an approximate answer, we choose instead to do a bit of sexagesimal[7] arithmetic. We first rewrite $90^\circ = 90^\circ 0' 0'' = 89^\circ 60' 0'' = 89^\circ 59' 60''$. In essence, we are 'borrowing' $1^\circ = 60'$ from the degree place, and then borrowing $1' = 60''$ from the minutes place.[8] This yields, $\gamma = 90^\circ - 37^\circ 28' 17'' = 89^\circ 59' 60'' - 37^\circ 28' 17'' = 52^\circ 31' 43''$. □

Up to this point, we have discussed only angles which measure between 0° and 360°, inclusive. Ultimately, we want to use the arsenal of Algebra which we have stockpiled in Chapters 1 through 10 to not only solve geometric problems involving angles, but also to extend their applicability to other real-world phenomena. A first step in this direction is to extend our notion of 'angle' from merely measuring an extent of rotation to quantities which indicate an amount of rotation along with a **direction**. To that end, we introduce the concept of an **oriented angle**. As its name suggests, in an oriented angle, the direction of the rotation is important. We imagine the angle being swept out starting from an **initial side** and ending at a **terminal side**, as shown below. When the rotation is counter-clockwise[9] from initial side to terminal side, we say that the angle is **positive**; when the rotation is clockwise, we say that the angle is **negative**.

A positive angle, 45° A negative angle, -45°

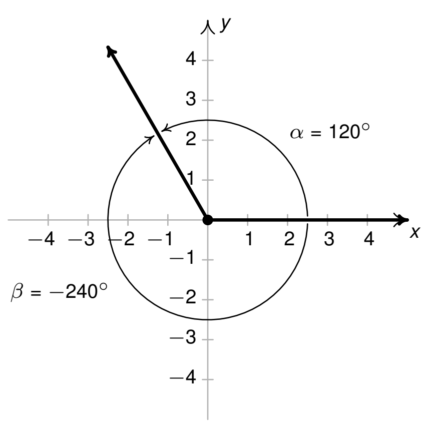
[6]If this process seems hauntingly familiar, it should. Compare this method to the Bisection Method introduced in Section 2.3.3.
[7]Like 'latus rectum,' this is also a real math term.
[8]This is the exact same kind of 'borrowing' you used to do in Elementary School when trying to find 300 − 125. Back then, you were working in a base ten system; here, it is base sixty.
[9]'widdershins'

At this point, we also extend our allowable rotations to include angles which encompass more than one revolution. For example, to sketch an angle with measure 450° we start with an initial side, rotate counter-clockwise one complete revolution (to take care of the 'first' 360°) then continue with an additional 90° counter-clockwise rotation, as seen below.



450°

To further connect angles with the Algebra which has come before, we shall often overlay an angle diagram on the coordinate plane. An angle is said to be in **standard position** if its vertex is the origin and its initial side coincides with the positive horizontal (usually labeled as the x-) axis. Angles in standard position are classified according to where their terminal side lies. For instance, an angle in standard position whose terminal side lies in Quadrant I is called a 'Quadrant I angle'. If the terminal side of an angle lies on one of the coordinate axes, it is called a **quadrantal angle**. Two angles in standard position are called **coterminal** if they share the same terminal side.[10] In the figure below, $\alpha = 120^\circ$ and $\beta = -240^\circ$ are two coterminal Quadrant II angles drawn in standard position. Note that $\alpha = \beta + 360^\circ$, or equivalently, $\beta = \alpha - 360^\circ$. We leave it as an exercise to the reader to verify that coterminal angles always differ by a multiple of 360°.[11] More precisely, if α and β are coterminal angles, then $\beta = \alpha + 360^\circ \cdot k$ where k is an integer.[12]



Two coterminal angles, $\alpha = 120^\circ$ and $\beta = -240^\circ$, in standard position.

[10]Note that by being in standard position they automatically share the same initial side which is the positive x-axis.
[11]It is worth noting that all of the pathologies of Analytic Trigonometry result from this innocuous fact.
[12]Recall that this means $k = 0, \pm 1, \pm 2, \ldots$.

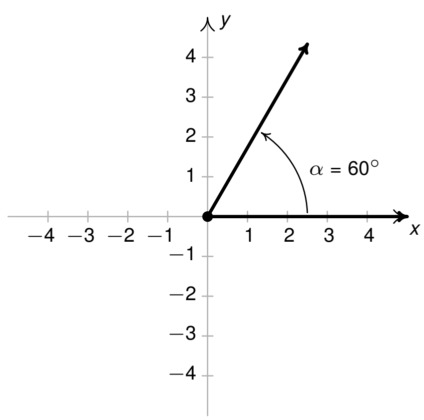
Example B.1.2. Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. $\alpha = 60^\circ$
2. $\beta = -225^\circ$
3. $\gamma = 540^\circ$
4. $\phi = -750^\circ$

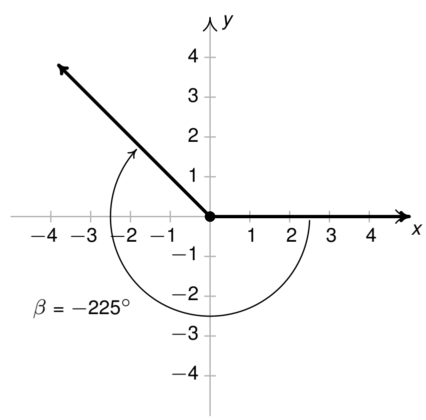
Solution.

1. To graph $\alpha = 60^\circ$, we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of a revolution. We see that α is a Quadrant I angle. To find angles which are coterminal, we look for angles θ of the form $\theta = \alpha + 360^\circ \cdot k$, for some integer k. When $k = 1$, we get $\theta = 60^\circ + 360^\circ = 420^\circ$. Substituting $k = -1$ gives $\theta = 60^\circ - 360^\circ = -300^\circ$. Finally, if we let $k = 2$, we get $\theta = 60^\circ + 720^\circ = 780^\circ$.

2. Since $\beta = -225^\circ$ is negative, we start at the positive x-axis and rotate *clockwise* $\frac{225^\circ}{360^\circ} = \frac{5}{8}$ of a revolution. We see that β is a Quadrant II angle. To find coterminal angles, we proceed as before and compute $\theta = -225^\circ + 360^\circ \cdot k$ for integer values of k. We find 135°, -585° and 495° are all coterminal with -225°.



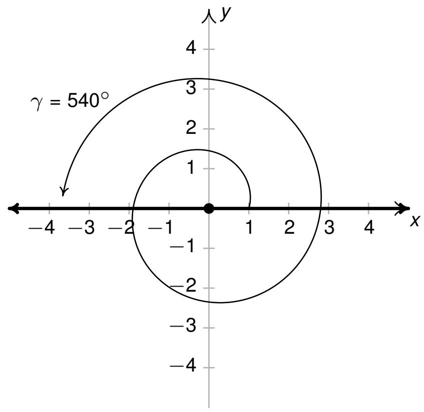
$\alpha = 60^\circ$ in standard position.



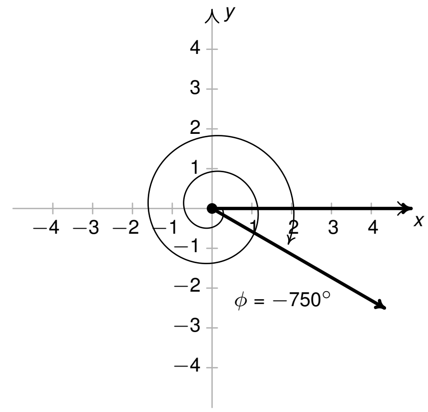
$\beta = -225^\circ$ in standard position.

3. Since $\gamma = 540^\circ$ is positive, we rotate counter-clockwise from the positive x-axis. One full revolution accounts for 360°, with 180°, or $\frac{1}{2}$ of a revolution remaining. Since the terminal side of γ lies on the negative x-axis, γ is a quadrantal angle. All angles coterminal with γ are of the form $\theta = 540^\circ + 360^\circ \cdot k$, where k is an integer. Working through the arithmetic, we find three such angles: 180°, -180° and 900°.

4. The Greek letter ϕ is pronounced 'fee' or 'fie' and since ϕ is negative, we begin our rotation clockwise from the positive x-axis. Two full revolutions account for 720°, with just 30° or $\frac{1}{12}$ of a revolution to go. We find that ϕ is a Quadrant IV angle. To find coterminal angles, we compute $\theta = -750^\circ + 360^\circ \cdot k$ for a few integers k and obtain -390°, -30° and 330°.



$\gamma = 540°$ in standard position.



$\phi = -750°$ in standard position.

□

Note that since there are infinitely many integers, any given angle has infinitely many coterminal angles, and the reader is encouraged to plot the few sets of coterminal angles found in Example B.1.2 to see this. As we'll see in Section B.2 and throughout Chapter 13, degree measure is very popular for many applications involving geometry and modeling physical forces. In Section 11.1, we'll introduce a different method of measuring angles, **radian measure**, which is tied directly to arc length and is useful in other applications involving circular motion and periodic phenomenon.

B.1.1 Exercises

In Exercises 1 - 4, convert the angles into the DMS system. Round each of your answers to the nearest second.

1. 63.75°
2. 200.325°
3. -317.06°
4. 179.999°

In Exercises 5 - 8, convert the angles into decimal degrees. Round each of your answers to three decimal places.

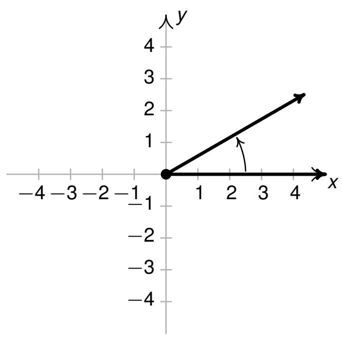
5. $125^\circ 50'$
6. $-32^\circ 10' 12''$
7. $502^\circ 35'$
8. $237^\circ 58' 43''$

In Exercises 9 - 20, graph the oriented angle in standard position. Classify each angle according to where its terminal side lies and then give two coterminal angles, one of which is positive and the other negative.

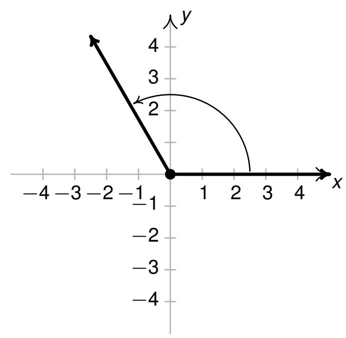
9. 30°
10. 120°
11. 225°
12. 330°
13. -30°
14. -135°
15. -240°
16. -270°
17. 405°
18. 840°
19. -510°
20. -900°
21. With help from your classmates, explain why if (x, y) is a point on the terminal side of an angle α in standard position, then so is $(r\,x, r\,y)$ for any number $r > 0$. What happens if $r < 0$?

B.1.2 Answers

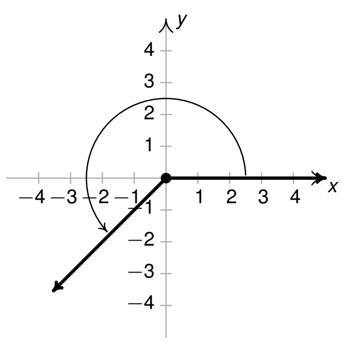
1. $63^\circ 45'$
2. $200^\circ 19' 30''$
3. $-317^\circ 3' 36''$
4. $179^\circ 59' 56''$
5. 125.833°
6. -32.17°
7. 502.583°
8. 237.979°
9. 30° is a Quadrant I angle coterminal with 390° and -330°



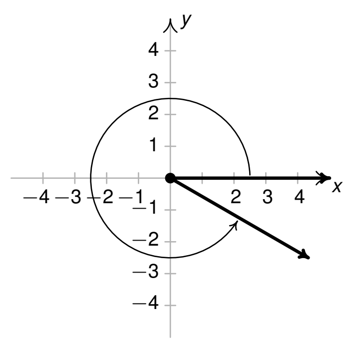
10. 120° is a Quadrant II angle coterminal with 480° and -240°



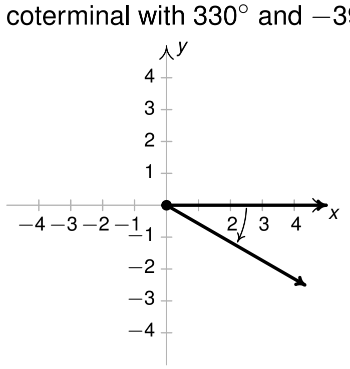
11. 225° is a Quadrant III angle coterminal with 585° and -135°



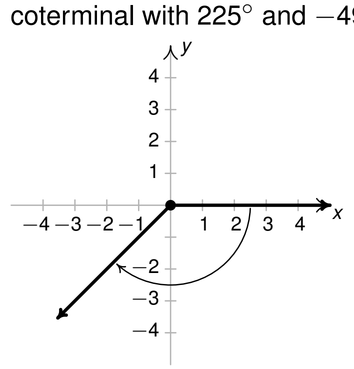
12. 330° is a Quadrant IV angle coterminal with 690° and -30°



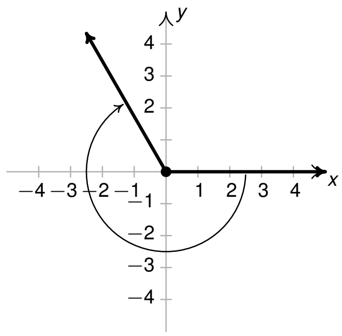
13. -30° is a Quadrant IV angle coterminal with 330° and -390°



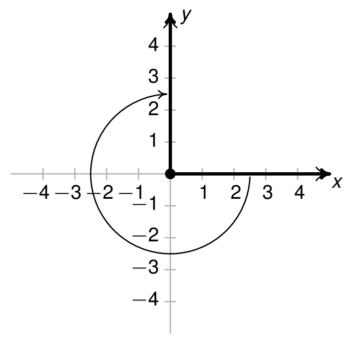
14. -135° is a Quadrant III angle coterminal with 225° and -495°



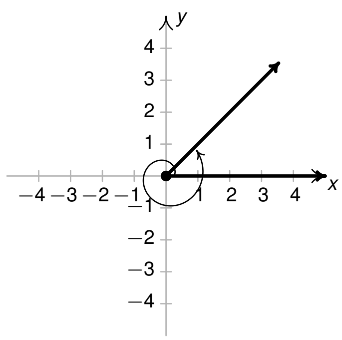
15. -240° is a Quadrant II angle coterminal with 120° and -600°



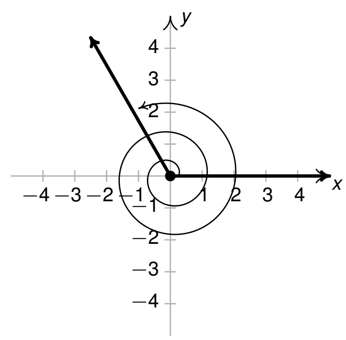
16. -270° is a quadrantal angle coterminal with 90° and -630°



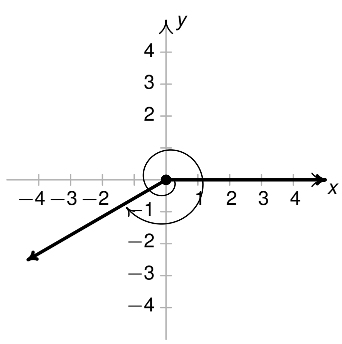
17. 405° is a Quadrant I angle coterminal with 45° and -315°



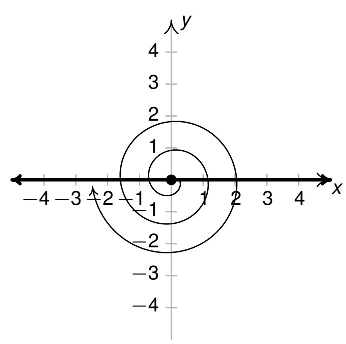
18. 840° is a Quadrant II angle coterminal with 120° and -240°



19. -510° is a Quadrant III angle coterminal with -150° and 210°

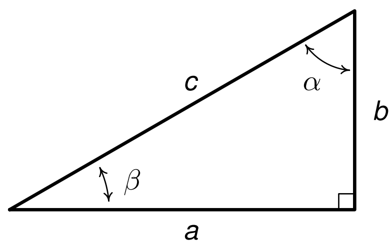


20. -900° is a quadrantal angle coterminal with -180° and 180°



B.2 Right Triangle Trigonometry

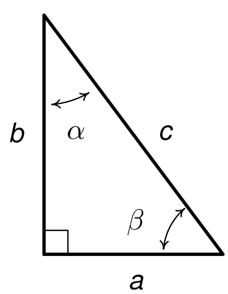
The word 'trigonometry' literally means 'measuring triangles,' so naturally most students' first introduction to trigonometry focuses on triangles. This section focuses on **right triangles**, triangles in which one angle measures 90°. Consider the right triangle below, where, as usual, the small square '□' denotes the right angle, the labels '*a*,' '*b*,' and '*c*' denote the lengths of the sides of the triangle, and α and β represent the (measure of) the non-right angles. As you may recall, the side opposite the right angle is called the **hypotenuse** of the right triangle. Also note that since the sum of the measures of all angles in a triangle must add to 180°, we have $\alpha + \beta + 90^\circ = 180^\circ$, or $\alpha + \beta = 90^\circ$. Said differently, the non-right angles in a right triangle are *complements*.

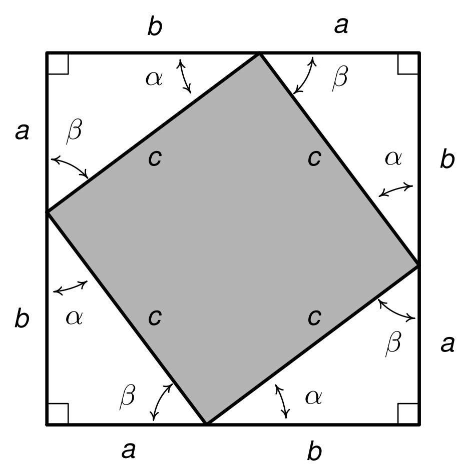


We now state and prove the most famous result about right triangles: **The Pythagorean Theorem**.

Theorem B.1. (**The Pythagorean Theorem**) The square of the length of the hypotenuse of a right triangle is equal to the sums of the squares of the other two sides. More specifically, if *c* is the length of the hypotenuse of a right triangle and *a* and *b* are the lengths of the other two sides, then $a^2 + b^2 = c^2$.

There are several proofs of the Pythagorean Theorem,[1] but the one we choose to reproduce here showcases a nice interplay between algebra and geometry. Consider taking four copies of the right triangle below on the left and arranging them as seen below on the right.





It should be clear that we have produced a large square with a side length of $(a + b)$. What is also true, but may not be obvious, is that the shaded quadrilateral is also a square. We can readily see the shaded

[1] Including one by Mentor, Ohio native President James Garfield.

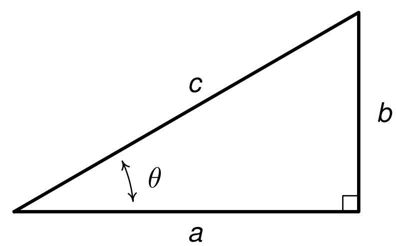
quadrilateral has equal sides of length c. Moreover, since $\alpha + \beta = 90^\circ$, we get the interior angles of the shaded quadrilateral are each 90°. Hence, the shaded quadrilateral is indeed a square.

We finish the proof by computing the area of the of the large square in two ways. First, we square the length of its side: $(a+b)^2$. Next, we add up the areas of the four triangles, each having area $\frac{1}{2}ab$ along with the area of the shaded square, c^2. Equating these to expressions gives: $(a+b)^2 = 4\left(\frac{1}{2}ab\right) + c^2$. Since $(a+b)^2 = a^2 + 2ab + b^2$ and $4\left(\frac{1}{2}ab\right) = 2ab$, we have $a^2 + 2ab + b^2 = 2ab + c^2$ or $a^2 + b^2 = c^2$, as required.

It should be noted that the converse of the Pythagorean Theorem is also true. That is if a, b, and c are the lengths of sides of a triangle and $a^2 + b^2 = c^2$, then c the triangle is a right triangle.[2]

A list of integers (a, b, c) which satisfy the relationship $a^2 + b^2 = c^2$ is called a **Pythagorean Triple**. Some of the more common triples are: $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, and $(8, 15, 17)$. We leave it to the reader to verify these integers satisfy the equation $a^2 + b^2 = c^2$ and suggest committing these triples to memory.

Next, we set about defining characteristic ratios associated with acute angles. Given any acute angle θ, we can imagine θ being an interior angle of a right triangle as seen below.



Focusing on the arrangement of the sides of the triangle with respect to the angle θ, we make the following definitions: the side with length a is called the side of the triangle which is **adjacent** to θ and the side with length b is called the side of the triangle **opposite** θ. As usual, the side labeled 'c' (the side opposite the right angle) is the hypotenuse. Using this diagram, we define three important **trigonometric ratios** of θ.

Definition B.1. Suppose θ is an acute angle residing in a right triangle as depicted above.

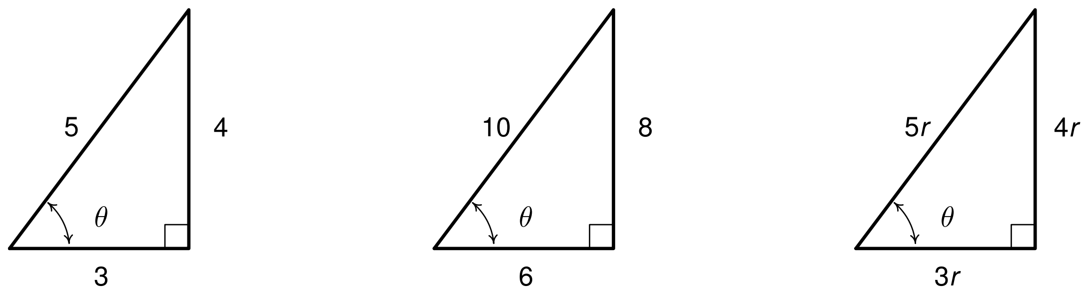
- The **sine** of θ, denoted $\sin(\theta)$ is defined by the ratio: $\sin(\theta) = \dfrac{b}{c}$, or $\dfrac{\text{'length of opposite'}}{\text{'length of hypotenuse'}}$.
- The **cosine** of θ, denoted $\cos(\theta)$ is defined by the ratio: $\cos(\theta) = \dfrac{a}{c}$, or $\dfrac{\text{'length of adjacent'}}{\text{'length of hypotenuse'}}$.
- The **tangent** of θ, denoted $\tan(\theta)$ is defined by the ratio: $\tan(\theta) = \dfrac{b}{a}$, or $\dfrac{\text{'length of opposite'}}{\text{'length of adjacent'}}$.

For example, consider the angle θ indicated in the triangle below on the left. Using Definition B.1, we get $\sin(\theta) = \frac{4}{5}$, $\cos(\theta) = \frac{3}{5}$, and $\tan(\theta) = \frac{4}{3}$. One may well wonder if these trigonometric ratios we've found for θ change if the triangle containing θ changes. For example, if we scale all the sides of the triangle below on the left by a factor of 2, we produce the **similar triangle** below in the middle.[3] Using this triangle to

[2] We will prove this in Section 13.2 by generalizing the Pythagorean Theorem to a formula that works for *all* triangles.

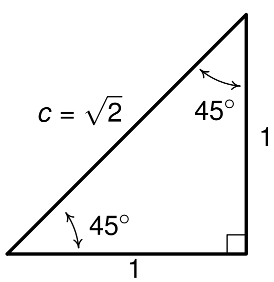
[3] That is, a triangle with the same 'shape' - that is, the same angles.

compute our ratios for θ, we find $\sin(\theta) = \frac{8}{10} = \frac{4}{5}$, $\cos(\theta) = \frac{6}{10} = \frac{3}{5}$, and $\tan(\theta) = \frac{8}{6} = \frac{4}{3}$. Note that the scaling factor, here 2, is common to all sides of the triangle, and, hence, cancels from the numerator and denominator when simplifying each of the ratios.



In general, thanks to the Angle Angle Similarity Postulate, any two *right* triangles which contain our angle θ are similar which means there is a positive constant r so that the sides of the triangle are $3r$, $4r$, and $5r$ as seen above on the right. Hence, regardless of the right triangle in which we choose to imagine θ, $\sin(\theta) = \frac{4r}{5r} = \frac{4}{5}$, $\cos(\theta) = \frac{3r}{5r} = \frac{3}{5}$, and $\tan(\theta) = \frac{4r}{3r} = \frac{4}{3}$. Generalizing this same argument to any acute angle θ assures us that the ratios as described in Definition B.1 are independent of the triangle we use.

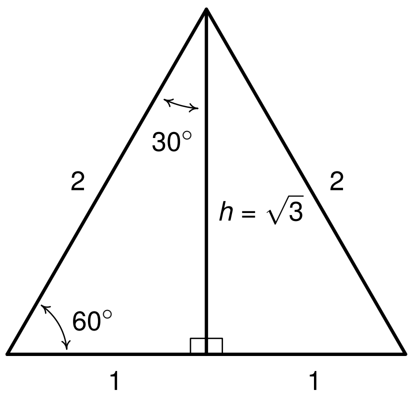
Our next objective is to determine the values of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for some of the more commonly used angles. We begin with 45°. In a right triangle, if one of the non-right angles measures 45°, then the other measures 45° as well. It follows that the two legs of the triangle must be congruent. Since we may choose any right triangle containing a 45° angle for our computations, we choose the length of one (hence both) of the legs to be 1. The Pythagorean Theorem gives the hypotenuse is: $c^2 = 1^2 + 1^2 = 2$, so $c = \sqrt{2}$. (We take only the positive square root here since c represents the length of the hypotenuse here, so, necessarily $c > 0$.) From this, we obtain the values below, and suggest committing them to memory.



- $\sin(45^\circ) = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
- $\cos(45^\circ) = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
- $\tan(45^\circ) = \dfrac{1}{1} = 1$

Note that we have 'rationalized' here to avoid the irrational number $\sqrt{2}$ appearing in the denominator. This is a common convention in trigonometry, and we will adhere to it unless extremely inconvenient.

Next, we investigate 60° and 30° angles. Consider the equilateral triangle below each of whose sides measures 2 units. Each of its interior angles is necessarily 60°, so if we drop an altitude, we produce two $30^\circ - 60^\circ - 90^\circ$ triangles each having a base measuring 1 unit and a hypotenuse of 2 units. Using the Pythagorean Theorem, we can find the height, h of these triangles: $1^2 + h^2 = 2^2$ so $h^2 = 3$ or $h = \sqrt{3}$. Using these, we can find the values of the trigonometric ratios for both 60° and 30°. Again, we recommend committing these values to memory.



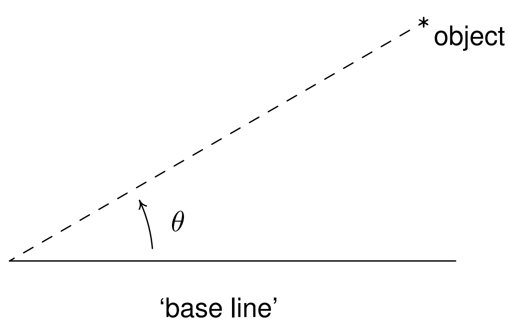
- $\sin(60^\circ) = \dfrac{\sqrt{3}}{2}$
- $\cos(60^\circ) = \dfrac{1}{2}$
- $\tan(60^\circ) = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

- $\sin(30^\circ) = \dfrac{1}{2}$
- $\cos(30^\circ) = \dfrac{\sqrt{3}}{2}$
- $\tan(30^\circ) = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

Since 30° and 60° are complements, the side *adjacent* to the 60° angle is the side *opposite* the 30° and the side *opposite* the 60° angle is the side *adjacent* to the 30° . This sort of 'swapping' is true of all complementary angles and will be generalized in Section 12.2, Theorem 12.6.

Note that the values of the trigonometric ratios we have derived for 30°, 45°, and 60° angles are the *exact* values of these ratios. For these angles, we can conveniently express the exact values of their sines, cosines, and tangents resorting, at worst, to using square roots. The reader may well wonder if, for instance, we can express the exact value of, say, $\sin(42^\circ)$ in terms of radicals. The answer in this case is 'yes' (see here), but, in general, we will not take the time to pursue such representations.[4] Hence, if a problem requests an 'exact' answer involving $\sin(42^\circ)$, we will leave it written as '$\sin(42^\circ)$' and use a calculator to produce a suitable approximation as the situation warrants.

Our first example requires the concept of an 'angle of inclination.' The angle of inclination (or angle of elevation) of an object refers to the angle whose initial side is some kind of base-line (say, the ground), and whose terminal side is the line-of-sight to an object above the base-line. Schematically:



The angle of inclination from the base line to the object is θ

Example B.2.1.

1. The angle of inclination from a point on the ground 30 feet away to the top of Lakeland's Armington Clocktower[5] is 60°. Find the height of the Clocktower to the nearest foot.

[4] We will do a little of this in Section 12.2.

[5] Named in honor of Raymond Q. Armington, Lakeland's Clocktower has been a part of campus since 1972.

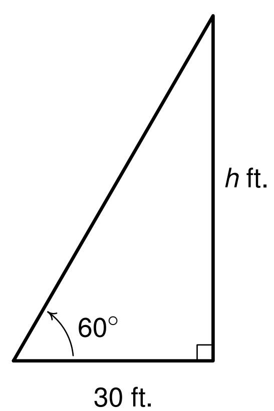
2. The Americans with Disabilities Act (ADA) stipulates the incline on an accessibility ramp be 5°. If a ramp is to be built so that it replaces stairs that measure 21 inches tall, how long does the ramp need to be? Round your answer to the nearest inch.

3. In order to determine the height of a California Redwood tree, two sightings from the ground, one 200 feet directly behind the other, are made. If the angles of inclination were 45° and 30°, respectively, how tall is the tree to the nearest foot?

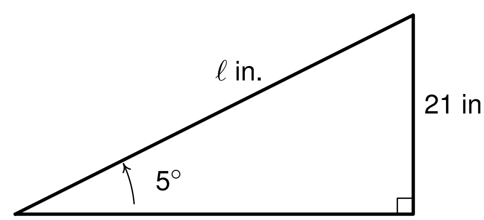
Solution.

1. We can represent the problem situation using a right triangle as shown below on the left. If we let h denote the height of the tower, then we have $\tan(60^\circ) = \frac{h}{30}$. From this we get an exact answer of $h = 30\tan(60^\circ) = 30\sqrt{3}$ feet. Using a calculator, we get the approximation 51.96 which, when rounded to the nearest foot, gives us our answer of 52 feet.

2. We diagram the situation below on the left using ℓ to represent the unknown length of the ramp. We have $\sin(5^\circ) = \frac{21}{\ell}$ so that $\ell = \frac{21}{\sin(5^\circ)} \approx 240.95$ inches. Hence, the ramp is 241 inches long.

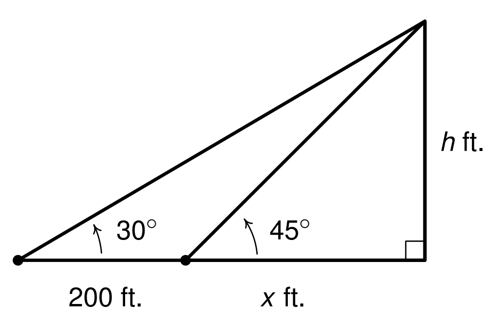


Finding the height of the Clocktower



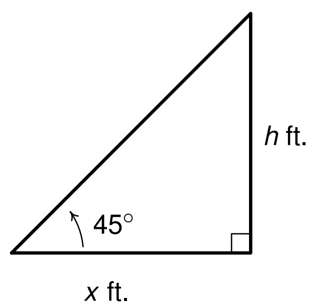
Finding the length of an accessibility ramp.

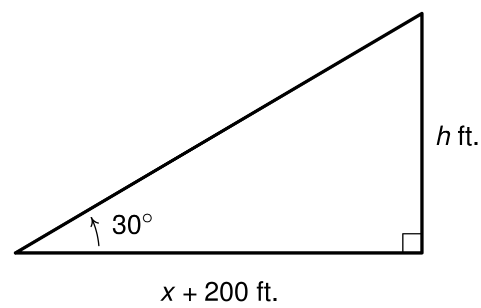
3. Sketching the problem situation below, we find ourselves with two unknowns: the height h of the tree and the distance x from the base of the tree to the first observation point.



Finding the height of a California Redwood

Luckily, we have two right triangles to help us find each unknown, as shown below. From the triangle below on the left, we get $\tan(45^\circ) = \frac{h}{x}$. From the triangle below on the right, we see $\tan(30^\circ) = \frac{h}{x+200}$.





Since $\tan(45^\circ) = 1$, the first equation gives $\frac{h}{x} = 1$, or $x = h$. Substituting this into the second equation gives $\frac{h}{h+200} = \tan(30^\circ) = \frac{\sqrt{3}}{3}$. Clearing fractions, we get $3h = (h + 200)\sqrt{3}$. The result is a linear equation for h, so we expand the right hand side and gather all the terms involving h to one side.

$$\begin{array}{rcl} 3h & = & (h+200)\sqrt{3} \\ 3h & = & h\sqrt{3} + 200\sqrt{3} \\ 3h - h\sqrt{3} & = & 200\sqrt{3} \\ (3-\sqrt{3})h & = & 200\sqrt{3} \\ h & = & \dfrac{200\sqrt{3}}{3-\sqrt{3}} \approx 273.20 \end{array}$$

Hence, the tree is approximately 273 feet tall. □

There are three more trigonometric ratios which are commonly used and they are defined in the same manner the ratios in Definition B.1 are defined. They are listed below.

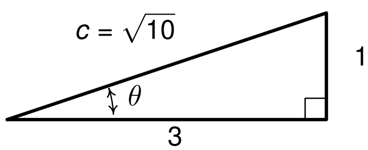
Definition B.2. Suppose θ is an acute angle residing in a right triangle as depicted on page 1508.

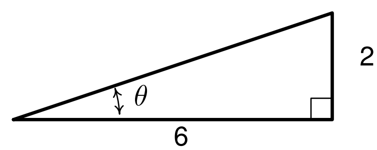
- The **cosecant** of θ, denoted $\csc(\theta)$ is defined by the ratio: $\csc(\theta) = \frac{c}{b}$, or $\frac{\text{‘length of hypotenuse’}}{\text{‘length of opposite’}}$.
- The **secant** of θ, denoted $\sec(\theta)$ is defined by the ratio: $\sec(\theta) = \frac{c}{a}$, or $\frac{\text{‘length of hypotenuse’}}{\text{‘length of adjacent’}}$.
- The **cotangent** of θ, denoted $\cot(\theta)$ is defined by the ratio: $\cot(\theta) = \frac{a}{b}$, or $\frac{\text{‘length of adjacent’}}{\text{‘length of opposite’}}$.

We practice these definitions in the following example.

Example B.2.2. Suppose θ is an acute angle with $\cot(\theta) = 3$. Find the values of the remaining five trigonometric ratios: $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$, $\csc(\theta)$, and $\sec(\theta)$.

Solution. We are given $\cot(\theta) = 3$. So, to proceed, we construct a right triangle in which the length of the side adjacent to θ and the length of the side opposite of θ has a ratio of $3 = \frac{3}{1}$. Note there are infinitely many such right triangles - we have produced two below for reference. We will focus our attention on the triangle below on the left and encourage the reader to work through the details using the triangle below on the right to verify the choice of triangle doesn't matter.





From the diagram, we see immediately $\tan(\theta) = \frac{1}{3}$, but in order to determine the remaining four trigonometric ratios, we need to first find the value of the hypotenuse. The Pythagorean Theorem gives $1^2 + 3^2 = c^2$ so $c^2 = 10$ or $c = \sqrt{10}$. Rationalizing denominators, we find $\sin(\theta) = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$, $\cos(\theta) = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$, $\csc(\theta) = \frac{\sqrt{10}}{1} = \sqrt{10}$ and $\sec(\theta) = \frac{\sqrt{10}}{3}$. $\square$

While we learned all about the trigonometric ratios of θ in Example B.2.2, the identity of θ remains unknown. Since $\sin(\theta) = \frac{\sqrt{10}}{10} \approx 0.316$ is decidedly less than $\sin(30^\circ) = \frac{1}{2} = 0.5$, it stands to reason that $\theta < 30^\circ$. It turns out the calculator can provide for us a decimal approximation of θ by way of the '$\sin^{-1}(x)$' function. Here, the '-1' exponent denotes an inverse function (see Section 5.6) does **not** mean reciprocal.[6] That is, $\sin^{-1}(x)$ (read 'sine-inverse of x') gives an angle whose sine is x. Hence, we may write $\theta = \sin^{-1}\left(\frac{\sqrt{10}}{10}\right) \approx 18.43^\circ$. The functions $\cos^{-1}(x)$ and $\tan^{-1}(x)$ work similarly. Indeed,

$$\theta = \sin^{-1}\left(\frac{\sqrt{10}}{10}\right) = \cos^{-1}\left(\frac{3\sqrt{10}}{10}\right) = \tan^{-1}\left(\frac{1}{3}\right),$$

and the reader is encouraged to use a calculator to verify these statements.

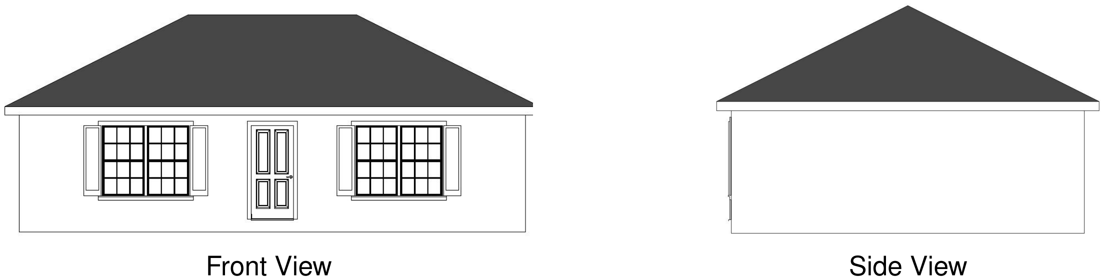
Please note there is **much** more to these inverse functions than the 'angle finder' description use here.[7] That being said, we finish this section showcasing a use for the $\tan^{-1}(x)$ function below.

Example B.2.3. [8] The roof on the house below has a '6/12 pitch'. This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of inclination from the bottom of the roof to the top of the roof. Round your answer to the nearest hundredth of a degree.

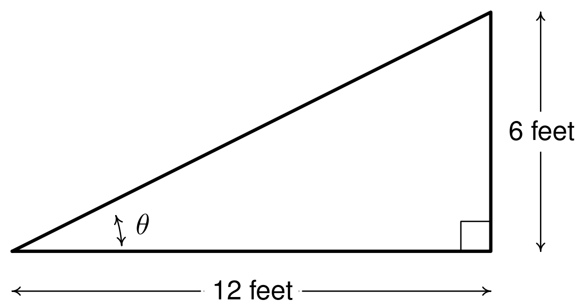
[6]That is, $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$. That being said, $(\sin(x))^{-1} = \frac{1}{\sin(x)} = \csc(x)$.

[7]See Section 12.3 for all of the pedantic details.

[8]The authors would like to thank Dan Stitz for this problem and associated graphics.



Solution. If we divide the side view of the house down the middle, we find that the roof line forms the hypotenuse of a right triangle with legs of length 6 feet and 12 feet as depicted below.

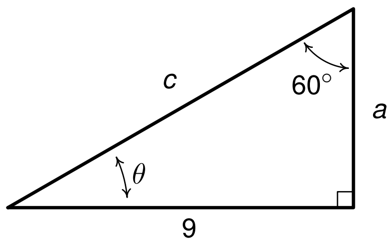


The angle of inclination, θ, satisfies $\tan(\theta) = \frac{6}{12} = \frac{1}{2}$. Hence, $\theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.56^{\circ}$. $\square$

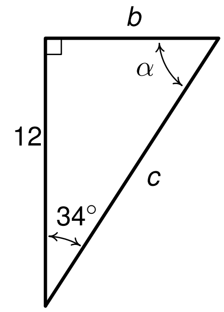
B.2.1 Exercises

In Exercises 1 - 4, find the requested quantities.

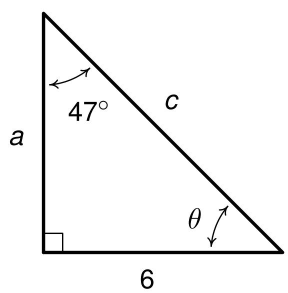
1. Find θ, *a*, and *c*.



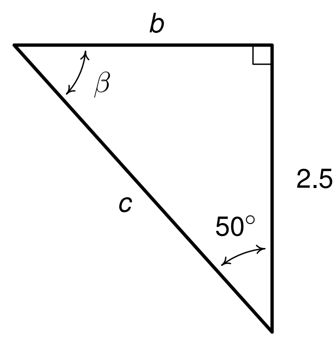
2. Find α, *b*, and *c*.



3. Find θ, *a*, and *c*.



4. Find β, *b*, and *c*.



In Exercises 5 - 10, answer the following questions assuming θ is an angle in a right triangle.

5. If $\theta = 30^\circ$ and the side opposite θ has length 4, how long is the side adjacent to θ?
6. If $\theta = 15^\circ$ and the hypotenuse has length 10, how long is the side opposite θ?
7. If $\theta = 87^\circ$ and the side adjacent to θ has length 2, how long is the side opposite θ?
8. If $\theta = 38.2^\circ$ and the side opposite θ has lengh 14, how long is the hypoteneuse?
9. If $\theta = 2.05^\circ$ and the hypotenuse has length 3.98, how long is the side adjacent to θ?
10. If $\theta = 42^\circ$ and the side adjacent to θ has length 31, how long is the side opposite θ?

In Exercises 11 - 13, find the two acute angles in the right triangle whose sides have the given lengths. Express your answers using degree measure rounded to two decimal places.

11. 3, 4 and 5
12. 5, 12 and 13
13. 336, 527 and 625

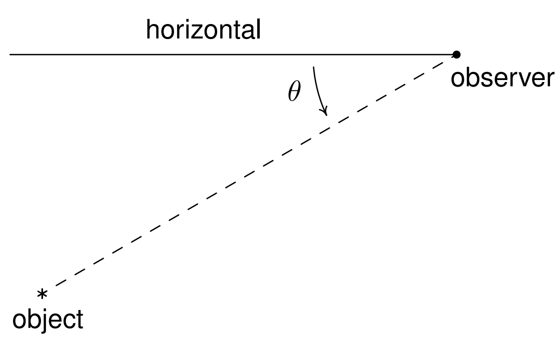
In Exercises 14 - 28, θ is an acute angle. Use the given trigonometric ratio to find the exact values of the remaining trigonometric ratios of θ. Find a decimal approximation to θ, rounded to two decimal places.

14. $\sin(\theta) = \dfrac{3}{5}$
15. $\tan(\theta) = \dfrac{12}{5}$
16. $\csc(\theta) = \dfrac{25}{24}$
17. $\sec(\theta) = 7$
18. $\csc(\theta) = \dfrac{10\sqrt{91}}{91}$
19. $\cot(\theta) = 23$
20. $\tan(\theta) = 2$
21. $\sec(\theta) = 4$
22. $\cot(\theta) = \sqrt{5}$
23. $\cos(\theta) = \dfrac{1}{3}$
24. $\cot(\theta) = 2$
25. $\csc(\theta) = 5$
26. $\tan(\theta) = \sqrt{10}$
27. $\sec(\theta) = 2\sqrt{5}$
28. $\cos(\theta) = 0.4$

29. A tree standing vertically on level ground casts a 120 foot long shadow. The angle of elevation from the end of the shadow to the top of the tree is 21.4°. Find the height of the tree to the nearest foot. With the help of your classmates, research the term *umbra versa* and see what it has to do with the shadow in this problem.

30. The broadcast tower for radio station WSAZ (Home of "Algebra in the Morning with Carl and Jeff") has two enormous flashing red lights on it: one at the very top and one a few feet below the top. From a point 5000 feet away from the base of the tower on level ground the angle of elevation to the top light is 7.970° and to the second light is 7.125°. Find the distance between the lights to the nearest foot.

31. On page 1510 we defined the angle of inclination (also known as the angle of elevation) and in this exercise we introduce a related angle - the angle of depression (also known as the angle of declination). The angle of depression of an object refers to the angle whose initial side is a horizontal line above the object and whose terminal side is the line-of-sight to the object below the horizontal. This is represented schematically below.



The angle of depression from the horizontal to the object is θ

(a) Show that if the horizontal is above and parallel to level ground then the angle of depression (from observer to object) and the angle of inclination (from object to observer) will be congruent because they are alternate interior angles.

(b) From a firetower 200 feet above level ground in the Sasquatch National Forest, a ranger spots a fire off in the distance. The angle of depression to the fire is 2.5°. How far away from the base of the tower is the fire?

(c) The ranger in part 31b sees a Sasquatch running directly from the fire towards the firetower. The ranger takes two sightings. At the first sighting, the angle of depression from the tower to the Sasquatch is 6°. The second sighting, taken just 10 seconds later, gives the the angle of depression as 6.5°. How far did the Saquatch travel in those 10 seconds? Round your answer to the nearest foot. How fast is it running in miles per hour? Round your answer to the nearest mile per hour. If the Sasquatch keeps up this pace, how long will it take for the Sasquatch to reach the firetower from his location at the second sighting? Round your answer to the nearest minute.

32. When I stand 30 feet away from a tree at home, the angle of elevation to the top of the tree is 50° and the angle of depression to the base of the tree is 10°. What is the height of the tree? Round your answer to the nearest foot.

33. From the observation deck of the lighthouse at Sasquatch Point 50 feet above the surface of Lake Ippizuti, a lifeguard spots a boat out on the lake sailing directly toward the lighthouse. The first sighting had an angle of depression of 8.2° and the second sighting had an angle of depression of 25.9°. How far had the boat traveled between the sightings?

34. A guy wire 1000 feet long is attached to the top of a tower. When pulled taut it makes a 43° angle with the ground. How tall is the tower? How far away from the base of the tower does the wire hit the ground?

35. A guy wire 1000 feet long is attached to the top of a tower. When pulled taut it touches level ground 360 feet from the base of the tower. What angle does the wire make with the ground? Express your answer using degree measure rounded to one decimal place.

36. At Cliffs of Insanity Point, The Great Sasquatch Canyon is 7117 feet deep. From that point, a fire is seen at a location known to be 10 miles away from the base of the sheer canyon wall. What angle of depression is made by the line of sight from the canyon edge to the fire? Express your answer using degree measure rounded to one decimal place.

37. Shelving is being built at the Utility Muffin Research Library which is to be 14 inches deep. An 18-inch rod will be attached to the wall and the underside of the shelf at its edge away from the wall, forming a right triangle under the shelf to support it. What angle, to the nearest degree, will the rod make with the wall?

38. A parasailor is being pulled by a boat on Lake Ippizuti. The cable is 300 feet long and the parasailor is 100 feet above the surface of the water. What is the angle of elevation from the boat to the parasailor? Express your answer using degree measure rounded to one decimal place.

39. A tag-and-release program to study the Sasquatch population of the eponymous Sasquatch National Park is begun. From a 200 foot tall tower, a ranger spots a Sasquatch lumbering through the wilderness directly towards the tower. Let θ denote the angle of depression from the top of the tower to a point on the ground. If the range of the rifle with a tranquilizer dart is 300 feet, find the smallest value of θ for which the corresponding point on the ground is in range of the rifle. Round your answer to the nearest hundreth of a degree.

40. The rule of thumb for safe ladder use states that the length of the ladder should be at least four times as long as the distance from the base of the ladder to the wall. Assuming the ladder is resting against a wall which is 'plumb' (that is, makes a 90° angle with the ground), determine the acute angle the ladder makes with the ground, rounded to the nearest tenth of a degree.

As you may have already noticed in working through the exercises, since the six trigonometric ratios are all defined in terms of the three sides of a right triangle, there are several relationships between them. In Exercises 41 - 49, use the diagram on page 1508 along with Definitions B.1 and B.2 to show the following relationships hold for all acute angles.[9]

41. $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$

42. $\csc(\theta) = \dfrac{1}{\sin(\theta)}$

43. $\sec(\theta) = \dfrac{1}{\cos(\theta)}$

For Exercises 44 - 46, it may be helpful to recall that $90^\circ - \theta$ is the measure of the 'other' acute angle in the right triangle besides θ.

44. $\cos(\theta) = \sin\left(90^\circ - \theta\right)$

45. $\csc(\theta) = \sec\left(90^\circ - \theta\right)$

46. $\cot(\theta) = \tan\left(90^\circ - \theta\right)$

For Exercises 47 - 49, it may be helpful to remember that $a^2 + b^2 = c^2$:

47. $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

48. $1 + (\tan(\theta))^2 = (\sec(\theta))^2$

49. $1 + (\cot(\theta))^2 = (\csc(\theta))^2$

[9] These are called trigonometric *identities* and will be studied in greater detail in Section 12.1.

B.2.2 Answers

1. $\theta = 30^\circ$, $a = 3\sqrt{3}$, $c = \sqrt{108} = 6\sqrt{3}$

2. $\alpha = 56^\circ$, $b = 12\tan(34^\circ) = 8.094$, $c = 12\sec(34^\circ) = \dfrac{12}{\cos(34^\circ)} \approx 14.475$

3. $\theta = 43^\circ$, $a = 6\cot(47^\circ) = \dfrac{6}{\tan(47^\circ)} \approx 5.595$, $c = 6\csc(47^\circ) = \dfrac{6}{\sin(47^\circ)} \approx 8.204$

4. $\beta = 40^\circ$, $b = 2.5\tan(50^\circ) \approx 2.979$, $c = 2.5\sec(50^\circ) = \dfrac{2.5}{\cos(50^\circ)} \approx 3.889$

5. The side adjacent to θ has length $4\sqrt{3} \approx 6.928$

6. The side opposite θ has length $10\sin(15^\circ) \approx 2.588$

7. The side opposite θ is $2\tan(87^\circ) \approx 38.162$

8. The hypoteneuse has length $14\csc(38.2^\circ) = \dfrac{14}{\sin(38.2^\circ)} \approx 22.639$

9. The side adjacent to θ has length $3.98\cos(2.05^\circ) \approx 3.977$

10. The side opposite θ has length $31\tan(42^\circ) \approx 27.912$

11. 36.87° and 53.13°

12. 22.62° and 67.38°

13. 32.52° and 57.48°

14. $\sin(\theta) = \frac{3}{5}, \cos(\theta) = \frac{4}{5}, \tan(\theta) = \frac{3}{4}, \csc(\theta) = \frac{5}{3}, \sec(\theta) = \frac{5}{4}, \cot(\theta) = \frac{4}{3}$, $\theta \approx 36.87^\circ$

15. $\sin(\theta) = \frac{12}{13}, \cos(\theta) = \frac{5}{13}, \tan(\theta) = \frac{12}{5}, \csc(\theta) = \frac{13}{12}, \sec(\theta) = \frac{13}{5}, \cot(\theta) = \frac{5}{12}$, $\theta \approx 67.38^\circ$

16. $\sin(\theta) = \frac{24}{25}, \cos(\theta) = \frac{7}{25}, \tan(\theta) = \frac{24}{7}, \csc(\theta) = \frac{25}{24}, \sec(\theta) = \frac{25}{7}, \cot(\theta) = \frac{7}{24}$, $\theta \approx 73.74^\circ$

17. $\sin(\theta) = \frac{4\sqrt{3}}{7}, \cos(\theta) = \frac{1}{7}, \tan(\theta) = 4\sqrt{3}, \csc(\theta) = \frac{7\sqrt{3}}{12}, \sec(\theta) = 7, \cot(\theta) = \frac{\sqrt{3}}{12}$, $\theta \approx 81.79^\circ$

18. $\sin(\theta) = \frac{\sqrt{91}}{10}, \cos(\theta) = \frac{3}{10}, \tan(\theta) = \frac{\sqrt{91}}{3}, \csc(\theta) = \frac{10\sqrt{91}}{91}, \sec(\theta) = \frac{10}{3}, \cot(\theta) = \frac{3\sqrt{91}}{91}$, $\theta \approx 72.54^\circ$

19. $\sin(\theta) = \frac{\sqrt{530}}{530}, \cos(\theta) = \frac{23\sqrt{530}}{530}, \tan(\theta) = \frac{1}{23}, \csc(\theta) = \sqrt{530}, \sec(\theta) = \frac{\sqrt{530}}{23}, \cot(\theta) = 23$, $\theta \approx 2.49^\circ$

20. $\sin(\theta) = \frac{2\sqrt{5}}{5}, \cos(\theta) = \frac{\sqrt{5}}{5}, \tan(\theta) = 2, \csc(\theta) = \frac{\sqrt{5}}{2}, \sec(\theta) = \sqrt{5}, \cot(\theta) = \frac{1}{2}$, $\theta \approx 63.43^\circ$

21. $\sin(\theta) = \frac{\sqrt{15}}{4}, \cos(\theta) = \frac{1}{4}, \tan(\theta) = \sqrt{15}, \csc(\theta) = \frac{4\sqrt{15}}{15}, \sec(\theta) = 4, \cot(\theta) = \frac{\sqrt{15}}{15}$, $\theta \approx 75.52^\circ$

22. $\sin(\theta) = \frac{\sqrt{6}}{6}, \cos(\theta) = \frac{\sqrt{30}}{6}, \tan(\theta) = \frac{\sqrt{5}}{5}, \csc(\theta) = \sqrt{6}, \sec(\theta) = \frac{\sqrt{30}}{5}, \cot(\theta) = \sqrt{5}$, $\theta \approx 24.09^\circ$

23. $\sin(\theta) = \frac{2\sqrt{2}}{3}, \cos(\theta) = \frac{1}{3}, \tan(\theta) = 2\sqrt{2}, \csc(\theta) = \frac{3\sqrt{2}}{4}, \sec(\theta) = 3, \cot(\theta) = \frac{\sqrt{2}}{4}$, $\theta \approx 70.53^\circ$

24. $\sin(\theta) = \frac{\sqrt{5}}{5}, \cos(\theta) = \frac{2\sqrt{5}}{5}, \tan(\theta) = \frac{1}{2}, \csc(\theta) = \sqrt{5}, \sec(\theta) = \frac{\sqrt{5}}{2}, \cot(\theta) = 2$, $\theta \approx 26.57^\circ$

25. $\sin(\theta) = \frac{1}{5}$, $\cos(\theta) = \frac{2\sqrt{6}}{5}$, $\tan(\theta) = \frac{\sqrt{6}}{12}$, $\csc(\theta) = 5$, $\sec(\theta) = \frac{5\sqrt{6}}{12}$, $\cot(\theta) = 2\sqrt{6}$, $\theta \approx 11.54^\circ$

26. $\sin(\theta) = \frac{\sqrt{110}}{11}$, $\cos(\theta) = \frac{\sqrt{11}}{11}$, $\tan(\theta) = \sqrt{10}$, $\csc(\theta) = \frac{\sqrt{110}}{10}$, $\sec(\theta) = \sqrt{11}$, $\cot(\theta) = \frac{\sqrt{10}}{10}$, $\theta \approx 72.45^\circ$

27. $\sin(\theta) = \frac{\sqrt{95}}{10}$, $\cos(\theta) = \frac{\sqrt{5}}{10}$, $\tan(\theta) = \sqrt{19}$, $\csc(\theta) = \frac{2\sqrt{95}}{19}$, $\sec(\theta) = 2\sqrt{5}$, $\cot(\theta) = \frac{\sqrt{19}}{19}$, $\theta \approx 77.08^\circ$

28. $\sin(\theta) = \frac{\sqrt{21}}{5}$, $\cos(\theta) = \frac{2}{5}$, $\tan(\theta) = \frac{\sqrt{21}}{2}$, $\csc(\theta) = \frac{5\sqrt{21}}{21}$, $\sec(\theta) = \frac{5}{2}$, $\cot(\theta) = \frac{2\sqrt{21}}{21}$, $\theta \approx 66.42^\circ$

29. The tree is about 47 feet tall.

30. The lights are about 75 feet apart.

31. (b) The fire is about 4581 feet from the base of the tower.
    (c) The Sasquatch ran $200\cot(6^\circ) - 200\cot(6.5^\circ) \approx 147$ feet in those 10 seconds. This translates to ≈ 10 miles per hour. At the scene of the second sighting, the Sasquatch was ≈ 1755 feet from the tower, which means, if it keeps up this pace, it will reach the tower in about 2 minutes.

32. The tree is about 41 feet tall.

33. The boat has traveled about 244 feet.

34. The tower is about 682 feet tall. The guy wire hits the ground about 731 feet away from the base of the tower.

35. 68.9°

36. 7.7°

37. 51°

38. 19.5°

39. 41.81°

40. 75.5°.