DTFT - Transformada de Fourier de Tempo Discreto

Seja uma sequência x[n]. A DTFT de x[n] é definida como:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

Este somatório converge se a sequência x[n] for absolutamente somável, i.e. :

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

A resposta em frequência $H(e^{j\Omega})$ é a DTFT de h[n].

Determine a DTFT da sequência x[n].

$$x[n] = \alpha^n \cdot u[n] \quad , \quad |\alpha| < 1$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \alpha^n \cdot u[n] \cdot e^{-j\Omega n} =$$

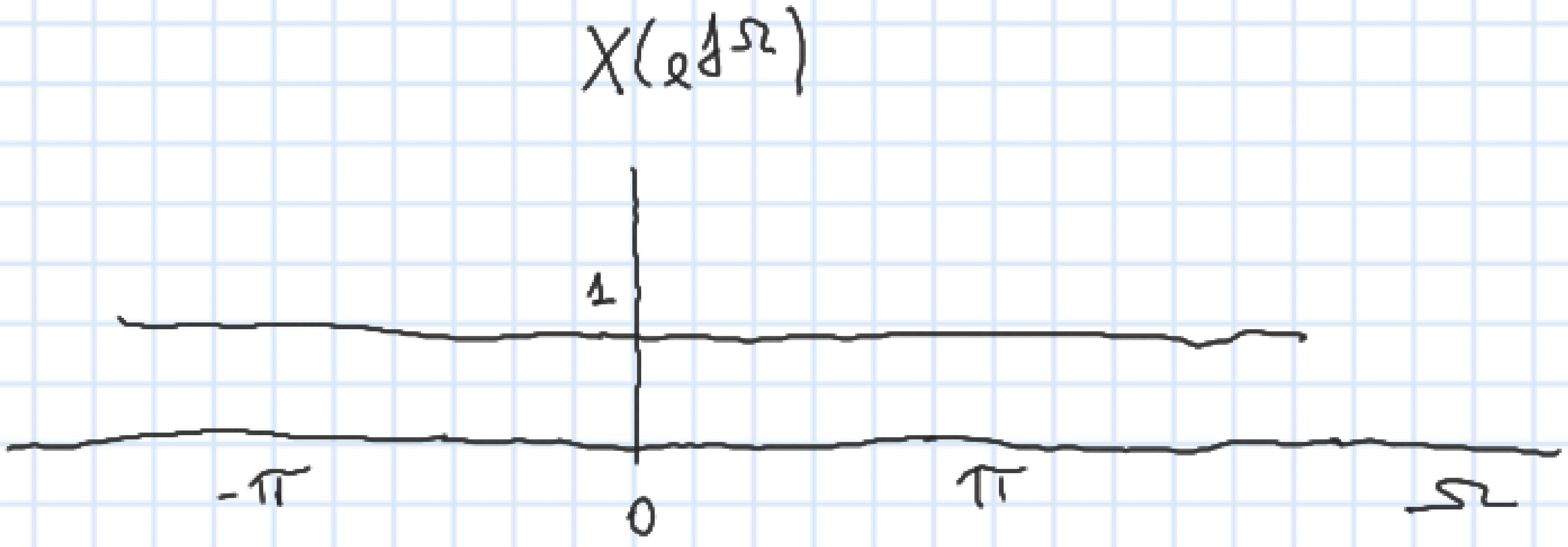
$$= \sum_{n=0}^{\infty} \alpha^n \cdot e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{e^{j\Omega}}\right)^n \qquad \left|\frac{\alpha}{e^{j\Omega}}\right| = \frac{|\alpha|}{|e^{j\Omega}|} = \frac{|\alpha|}{1} < 1$$

$$\boxed{X(e^{j\Omega}) = \frac{1}{1 - \alpha \cdot e^{-j\Omega}}}$$

Calcule a DTFT do impulso unitário.

$$x[n] = \delta[n]$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta[n] \cdot e^{-j\Omega n} = 1$$



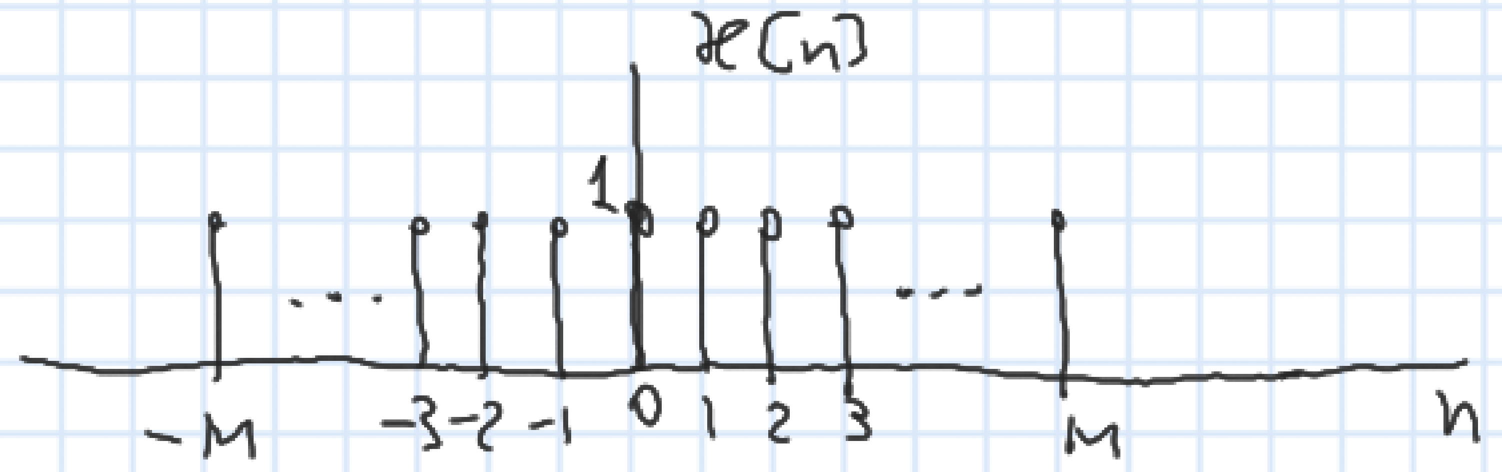
Transformada Inversa de Fourier

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n} \cdot d\Omega$$

$$x[n] \longleftrightarrow X(e^{j\Omega})$$

Calcule a DTFT do pulso retangular simétrico

$$x[n] = \begin{cases} 1, & p/\ |n| \leq M \\ 0, & c.c. \end{cases}$$



$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n} = \sum_{n=-M}^{M} e^{-j\Omega n} =$$

$$= \frac{e^{j\Omega M} - e^{-j\Omega(M+1)}}{1 - e^{-j\Omega}} = \frac{e^{j\Omega M} \cdot \left(1 - e^{-j\Omega(2M+1)}\right)}{1 - e^{-j\Omega}} =$$

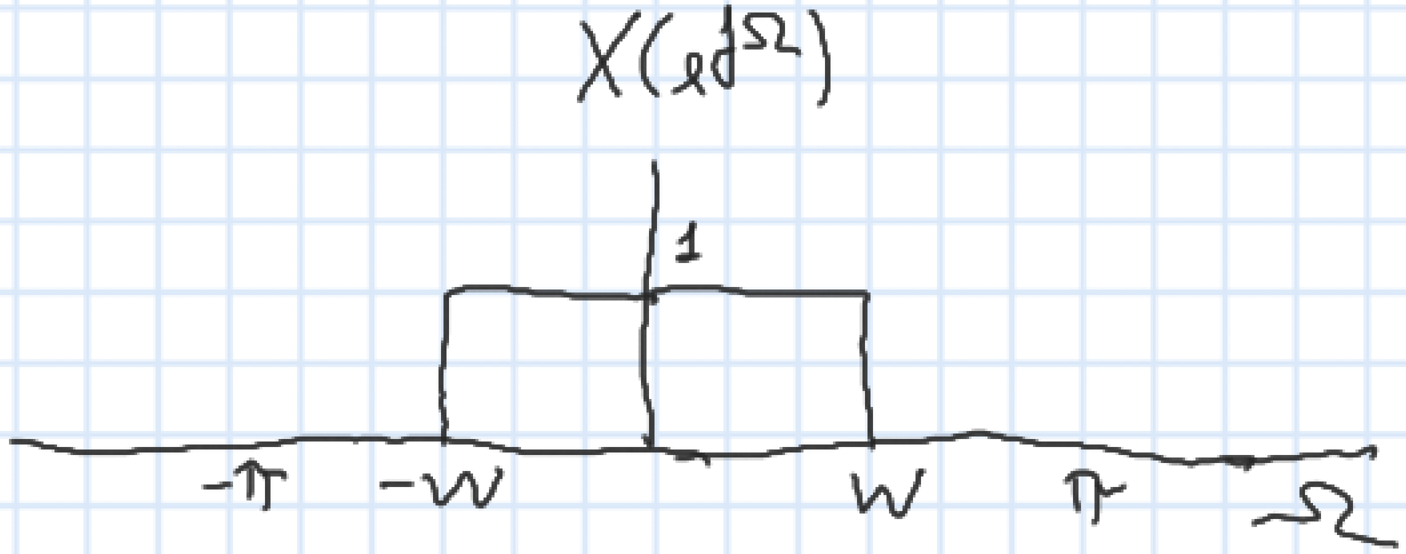
$$\frac{e^{j\Omega M} \cdot e^{-j\frac{\Omega(2M+1)}{2}}}{e^{-j\frac{\Omega}{2}}} \cdot \frac{\left(e^{j\frac{\Omega(2M+1)}{2}} - e^{-j\frac{\Omega(2M+1)}{2}}\right)}{\left(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}\right)} =$$

$$= \frac{e^{j\Omega M} \cdot e^{-j\frac{\Omega(2M+1)}{2}}}{e^{-j\frac{\Omega}{2}}} \cdot \frac{\cancel{2j} \cdot \text{SEN}\left(\frac{\Omega(2M+1)}{2}\right)}{\cancel{2j} \cdot \text{SEN}\left(\frac{\Omega}{2}\right)} = \boxed{\frac{\text{SEN}\left(\frac{\Omega(2M+1)}{2}\right)}{\text{SEN}\left(\frac{\Omega}{2}\right)}}$$

$$j(\cancel{\Omega M} - \cancel{\Omega M} - \cancel{\tfrac{\Omega}{2}} + \cancel{\tfrac{\Omega}{2}})$$

$x[n]$	$X(e^{j\Omega})$
$\delta[n]$	1
$\alpha^n \cdot u[n]$, $\lvert\alpha\rvert < 1$	$\dfrac{1}{1-\alpha e^{-j\Omega}}$
$\begin{cases} 1, \text{ p/ } \lvert n\rvert \leq M \\ 0, \text{ c.c.} \end{cases}$	$\dfrac{\text{SEN}\left(\frac{\Omega(2M+1)}{2}\right)}{\text{SEN}\left(\frac{\Omega}{2}\right)}$
$\dfrac{\text{SEN}(W \cdot n)}{\pi \cdot n}$	$\begin{cases} 1, \text{ p/ } \lvert\Omega\rvert < W \\ 0, \text{ c.c.} \end{cases}$

Determine a sequência x[n] que deu origem à DTFT $X(e^{j\Omega}) = \begin{cases} 1, & p/ \ |\Omega| < W \\ 0, & c.c. \end{cases}$



$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) \cdot e^{j\Omega n} \cdot d\Omega$$

$$x[n] = \frac{1}{2\pi} \int_{-W}^{W} e^{j\Omega n} \cdot d\Omega$$

$\int e^{x} dx = e^{x} + C$

$\int e^{u} du = e^{u} + c$

$e^{j\Omega n} = e^{u} \qquad u = j\Omega n \qquad du = jn \cdot d\Omega$

$$x[n] = \frac{1}{2\pi jn} \int_{-W}^{W} e^{j\Omega n} \cdot jn \cdot d\Omega =$$

$$= \frac{1}{2\pi jn} \cdot e^{j\Omega n} \Big|_{-W}^{W} = \frac{1}{2\pi jn} \left(e^{jWn} - e^{-jWn} \right) = \frac{1}{\pi n} \cdot \left(\frac{e^{jWn} - e^{-jWn}}{2j} \right) =$$

$$\boxed{x[n] = \frac{1}{\pi . n} \cdot SEN(W . n)}$$

Determine a DTFT do sinal x[n] $= 3 \cdot 2^{n} \cdot u[-n]$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} 3 \cdot 2^{n} \cdot u[-n] \cdot e^{-j\Omega n} =$$

$$= \sum_{n=-\infty}^{0} 3 \cdot 2^{n} \cdot e^{-j\Omega n} = \sum_{n=0}^{\infty} 3 \cdot 2^{-n} \cdot e^{j\Omega n} = 3 \cdot \sum_{n=0}^{\infty} \left(\frac{e^{j\Omega}}{2}\right)^{n} =$$

$$= \boxed{\frac{3}{1 - \frac{1}{2} \cdot e^{j\Omega}}}$$

Determine a DTFT de x[n] $= \left(\frac{1}{2}\right)^n \cdot u[n-2]$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \cdot u[n-2] \cdot e^{-j\Omega n} =$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n \cdot e^{-j\Omega n} = \sum_{n=2}^{\infty} \left(\frac{\frac{1}{2}}{e^{j\Omega}}\right)^n = \boxed{\frac{\frac{1}{4} \cdot e^{-j\Omega 2}}{1 - \frac{1}{2} \cdot e^{-j\Omega}}}$$

Propriedades da DTFT

Sejam os pares $x[n] \longleftrightarrow X(e^{j\Omega})$ e $y[n] \longleftrightarrow Y(e^{j\Omega})$

- LINEARIDADE

$$\alpha . x[n] + \beta . y[n] \longleftrightarrow \alpha . X(e^{j\Omega}) + \beta . Y(e^{j\Omega})$$

- DESLOCAMENTO NO TEMPO

$$x[n-n_0] \longleftrightarrow e^{-j\Omega . n_0} . X(e^{j\Omega})$$

- MULTIPLICAÇÃO POR EXPONENCIAL COMPLEXA

$$e^{j\Omega_0 . n} . x[n] \longleftrightarrow X(e^{j(\Omega-\Omega_0)})$$

- MODULAÇÃO

$$x[n] . y[n] \longleftrightarrow \frac{1}{2\pi} . X(e^{j\Omega}) \circledast Y(e^{j\Omega})$$

- CONVOLUÇÃO

$$x[n] * y[n] \longleftrightarrow X(e^{j\Omega}) . Y(e^{j\Omega})$$

Determine a DTFT do sinal $x[n] = \left(\frac{1}{2}\right)^n \cdot u[n-2]$

SEJA $a[n] = \left(\frac{1}{2}\right)^n \cdot u[n] \longleftrightarrow A(e^{j\Omega}) = \dfrac{1}{1-\frac{1}{2}e^{-j\Omega}}$

$$a[n-2] = \left(\frac{1}{2}\right)^{n-2} \cdot u[n-2] \longleftrightarrow e^{-j2\Omega} \cdot A(e^{j\Omega})$$

$$x[n] = \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{n-2} \cdot u[n-2] = \frac{1}{4} \cdot a[n-2] \longleftrightarrow \frac{1}{4} \cdot e^{-j2\Omega} \cdot A(e^{j\Omega})$$

$$\boxed{X(e^{j\Omega}) = \frac{\frac{1}{4} e^{-j2\Omega}}{1-\frac{1}{2}e^{-j\Omega}}}$$

Determine a DTFT de x[n] = $\left(\frac{1}{3}\right)^{n-1} \cdot u[n+1]$

$$a[n] = \left(\frac{1}{3}\right)^{n} \cdot u[n] \longleftrightarrow A(e^{j\Omega}) = \frac{1}{1-\frac{1}{3}e^{-j\Omega}}$$

$$a[n+1] = \left(\frac{1}{3}\right)^{n+1} \cdot u[n+1] \longleftrightarrow e^{j\Omega} \cdot A(e^{j\Omega})$$

$$x[n] = \left(\frac{1}{3}\right)^{-2} \cdot \left(\frac{1}{3}\right)^{n+1} \cdot u[n+1] = 9 \cdot a[n+1] \longleftrightarrow 9 \cdot e^{j\Omega} \cdot A(e^{j\Omega})$$

$$X(e^{j\Omega}) = \frac{9 \cdot e^{j\Omega}}{1-\frac{1}{3}e^{-j\Omega}}$$

Determine a DTFT de x[n] $= \cos\left(\frac{\pi}{2} \cdot n\right) \cdot \left(\frac{1}{2}\right)^n \cdot u[n]$

$$x[n] = \left(\frac{e^{j\frac{\pi}{2} \cdot n} + e^{-j\frac{\pi}{2} n}}{2}\right) \cdot \underbrace{\left(\frac{1}{2}\right)^n \cdot u[n]}_{a(n)}$$

$$x(n) = \frac{1}{2}\left(e^{j\frac{\pi}{2} \cdot n} \cdot a[n] + e^{-j\frac{\pi}{2} n} \cdot a[n]\right) \longrightarrow \frac{1}{2}\left(A\left(e^{j\left(\Omega - \frac{\pi}{2}\right)}\right) + A\left(e^{j\left(\Omega + \frac{\pi}{2}\right)}\right)\right)$$

$$A(e^{j\Omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\Omega}}$$

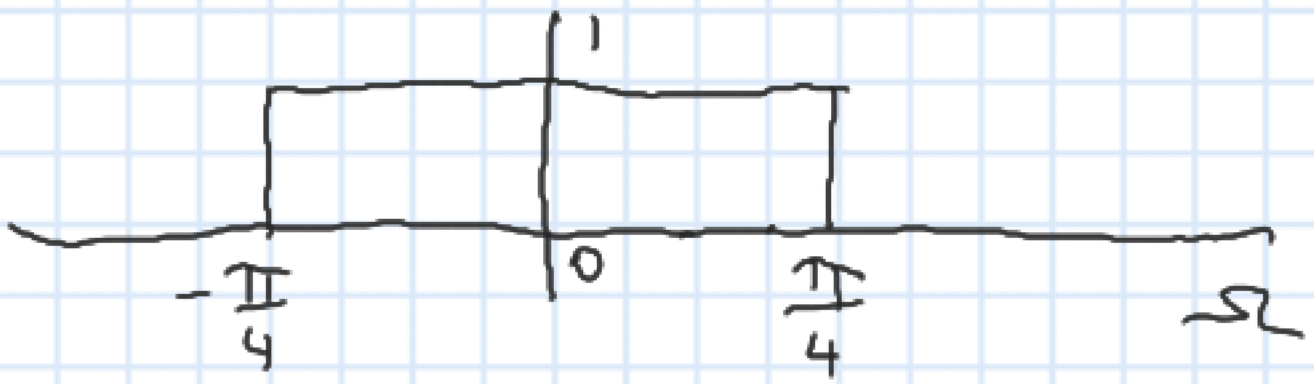
$$\boxed{X(e^{j\Omega}) = \frac{\frac{1}{2}}{1 - \frac{1}{2} e^{-j\left(\Omega - \frac{\pi}{2}\right)}} + \frac{\frac{1}{2}}{1 - \frac{1}{2} e^{-j\left(\Omega + \frac{\pi}{2}\right)}}}$$

Determine $x[n] = \underbrace{\frac{\text{SEN}\left(\frac{\pi}{4} \cdot n\right)}{\pi \cdot n}}_{a[n]} * \underbrace{\frac{\text{SEN}\left(\frac{\pi}{8} \cdot n\right)}{\pi \cdot n}}_{b[n]}$

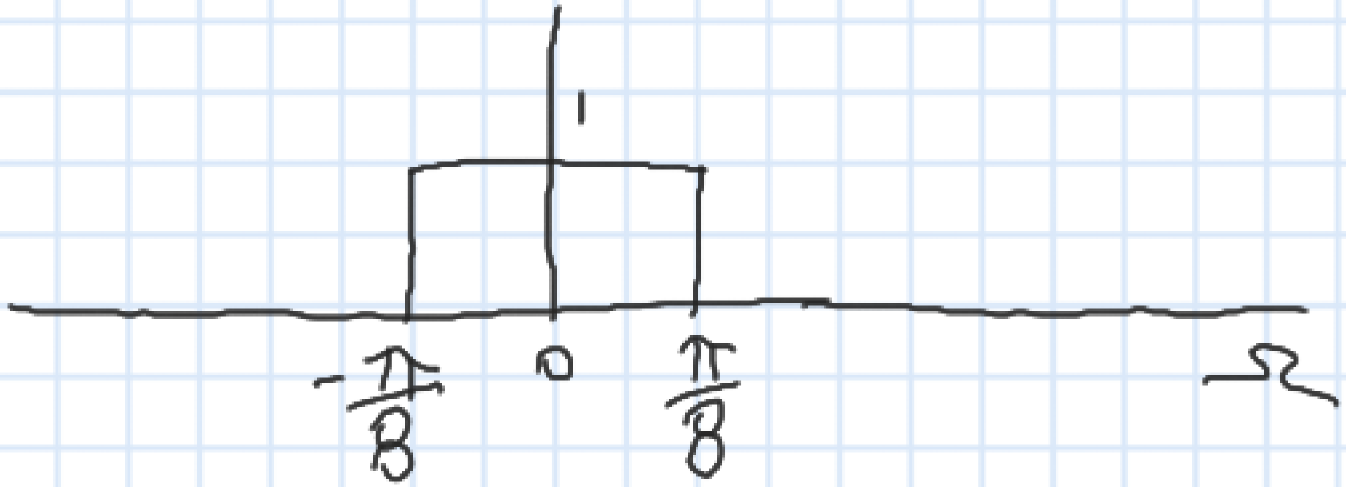
$$x[n] = a[n] * b[n]$$

$$X(e^{j\Omega}) = A(e^{j\Omega}) \cdot B(e^{j\Omega})$$

$A(e^{j\Omega})$



$B(e^{j\Omega})$



$$A(e^{j\Omega}) \cdot B(e^{j\Omega}) = B(e^{j\Omega})$$

$$X(e^{j\Omega}) = B(e^{j\Omega})$$

$$x[n] = b[n] = \boxed{\frac{\text{SEN}\left(\frac{\pi}{8} \cdot n\right)}{\pi \cdot n}}$$

$$x[n] = e^{j\frac{\pi}{2}\cdot n} \cdot \left(\frac{1}{3}\right)^{n-1} \cdot u[n-1]$$

$$a[n] = \left(\frac{1}{3}\right)^{n} \cdot u[n]$$

$$x[n] = e^{j\frac{\pi}{2}\cdot n} \cdot a[n-1]$$

$$b[n] = a[n-1]$$

$$x[n] = e^{j\frac{\pi}{2}\cdot n} \cdot b[n]$$

$$B(e^{j\Omega}) = e^{-j\Omega} \cdot A(e^{j\Omega})$$

$$B(e^{j\Omega}) = \frac{e^{-j\Omega}}{1 - \frac{1}{3}e^{-j\Omega}}$$

$$X(e^{j\Omega}) = B(e^{j(\Omega - \frac{\pi}{2})})$$

$$X(e^{j\Omega}) = \boxed{\frac{e^{-j(\Omega - \frac{\pi}{2})}}{1 - \frac{1}{3}e^{-j(\Omega - \frac{\pi}{2})}}}$$

$$x[n] = e^{j\frac{\pi}{2}(n-1)} \cdot \left(\frac{1}{3}\right)^{n-1} \cdot u[n-1]$$

$$a[n] = e^{j\frac{\pi}{2}\cdot n} \cdot \underbrace{\left(\frac{1}{3}\right)^{n} \cdot u[n]}_{b[n]} = e^{j\frac{\pi}{2}n} \cdot b[n] \longleftrightarrow B\left(e^{j(\Omega - \frac{\pi}{2})}\right)$$

$$x[n] = a[n-1]$$

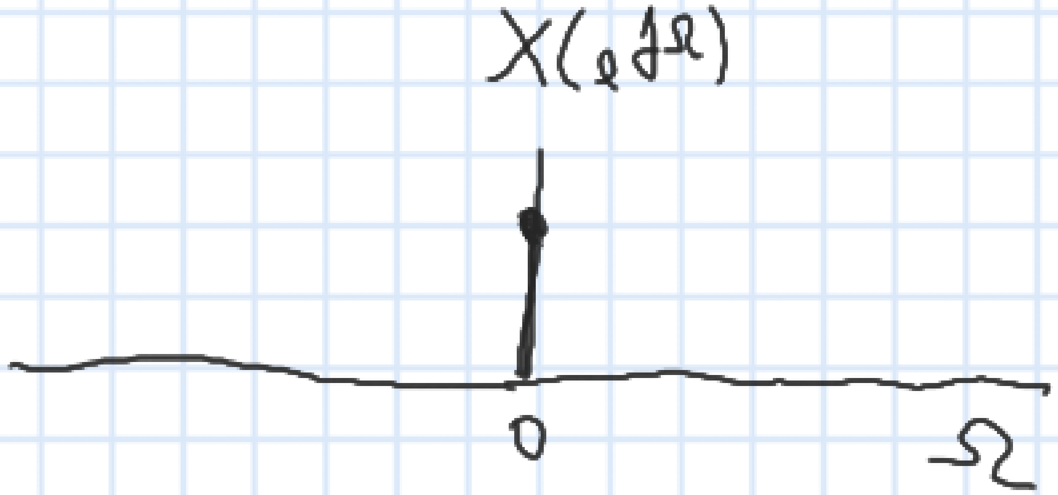
$$X(e^{j\Omega}) = e^{-j\Omega} \cdot A(e^{j\Omega})$$

$$B(e^{j\Omega}) = \frac{1}{1 - \frac{1}{3} e^{-j\Omega}}$$

$$A(e^{j\Omega}) = \frac{1}{1 - \frac{1}{3} e^{-j(\Omega - \frac{\pi}{2})}}$$

$$X(e^{j\Omega}) = \boxed{\frac{e^{-j\Omega}}{1 - \frac{1}{3} e^{-j(\Omega - \frac{\pi}{2})}}}$$

Determine x[n] a partir de $X(e^{j\Omega}) = \delta(\Omega)$



$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n} \cdot d\Omega =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega) e^{j\Omega n} \cdot d\Omega = \frac{1}{2\pi}$$

$$\int_{-\infty}^{\infty} \delta(t) \cdot x(t)\, dt = x(0)$$

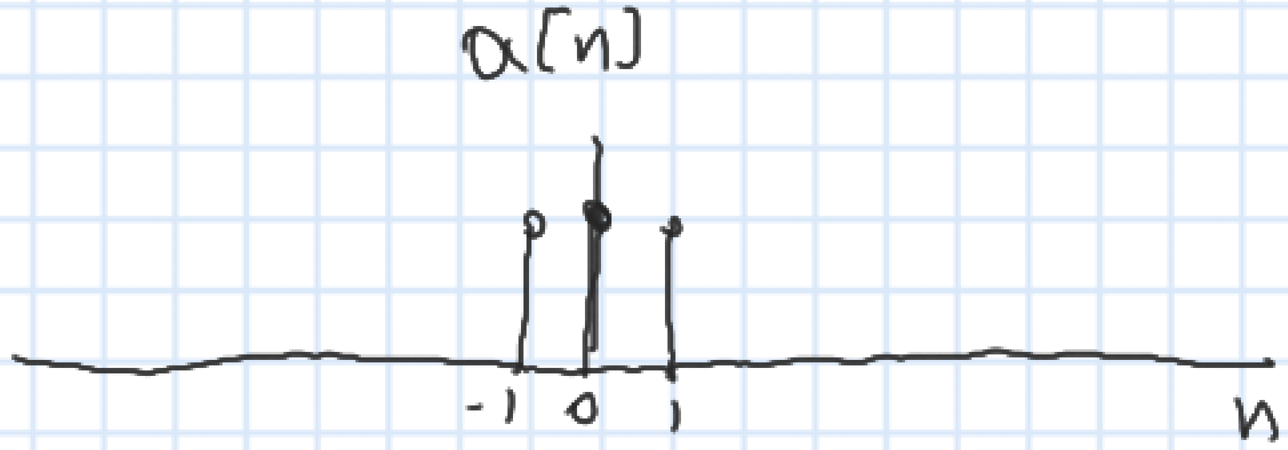
$$\int_{-\infty}^{\infty} \delta(t-t_0) \cdot x(t)\, dt = x(t_0)$$

Determine x[n] a partir de $X(e^{j\Omega}) = \cos(4\Omega)\cdot \underbrace{\frac{\text{SEN}\left(\frac{3}{2}\Omega\right)}{\text{SEN}\left(\frac{\Omega}{2}\right)}}_{A(e^{j\Omega})}$

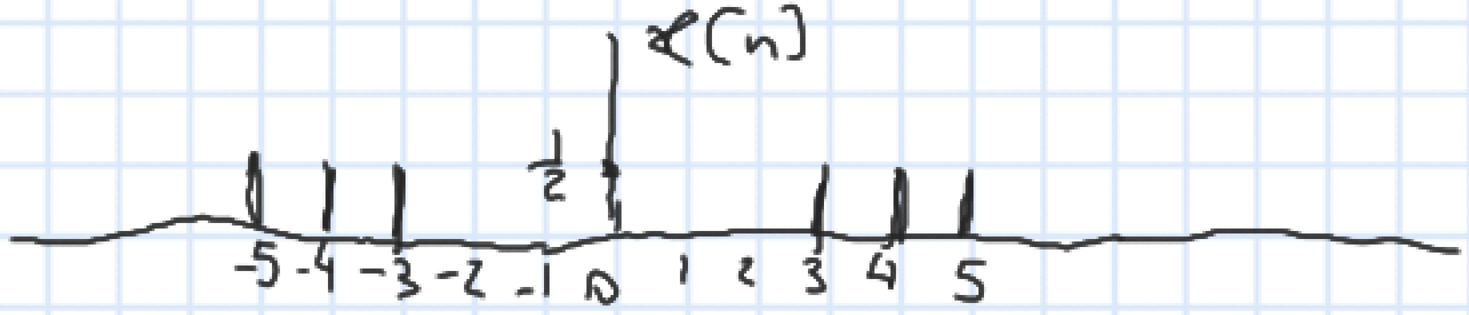
$$X(e^{j\Omega}) = \left(\frac{e^{j4\Omega} + e^{-j4\Omega}}{2}\right)\cdot A(e^{j\Omega})$$

$$X(e^{j\Omega}) = \frac{1}{2}\left(e^{j4\Omega}\cdot A(e^{j\Omega}) + e^{-j4\Omega}\cdot A(e^{j\Omega})\right) \frown x[n] = \frac{1}{2}\left(a[n+4] + a[n-4]\right)$$

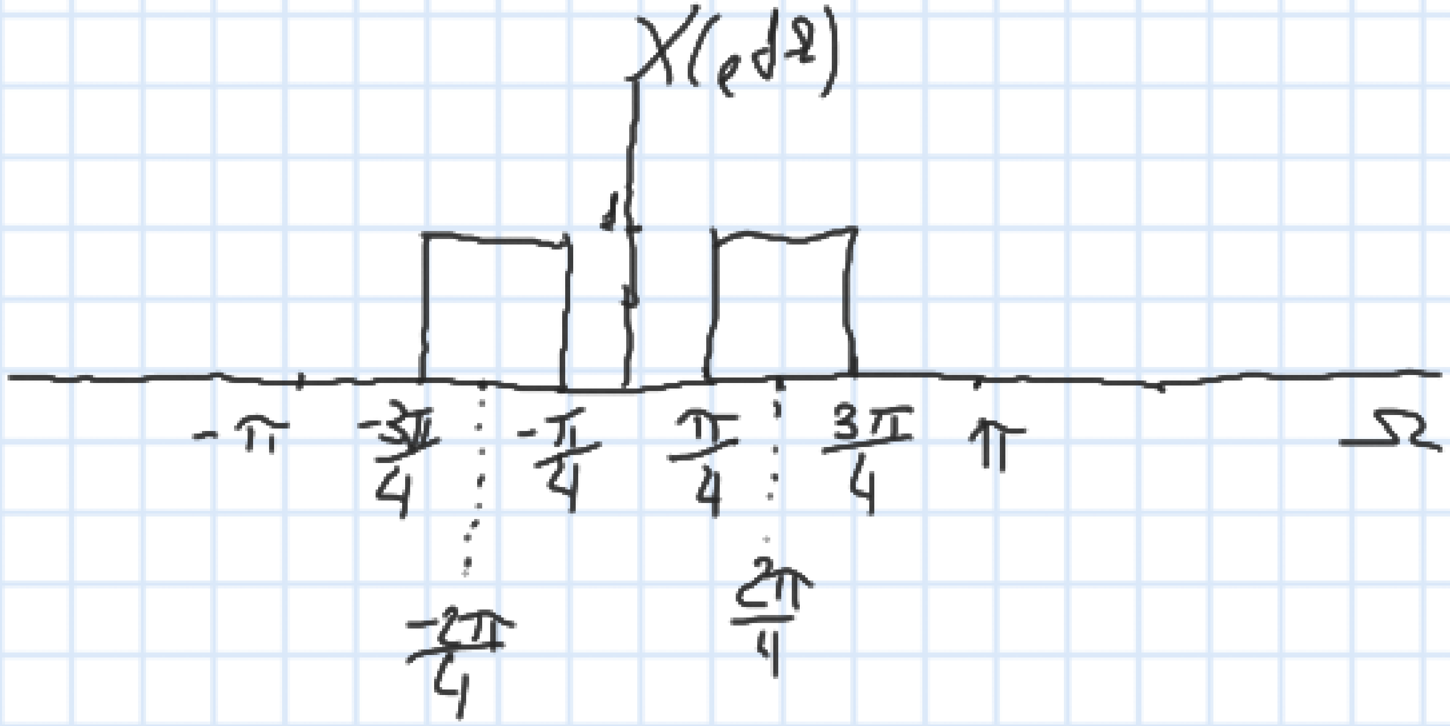
$$A(e^{j\Omega}) = \frac{\text{SEN}\left(\frac{3}{2}\Omega\right)}{\text{SEN}\left(\frac{\Omega}{2}\right)} \frown a[n] = \begin{cases} 1, & \text{p/ } |n| \leq 1 \\ 0, & \text{c.c.} \end{cases}$$

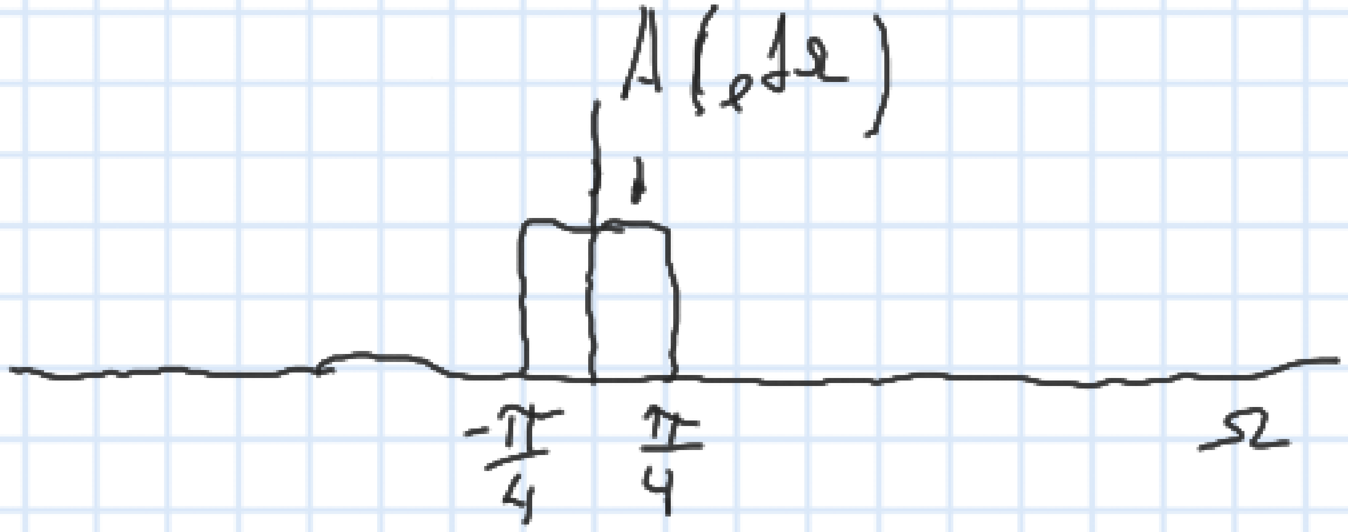


$$x[n] = \begin{cases} \frac{1}{2}, & \text{p/ } |n+4| \leq 1 \text{ e } |n-4| \leq 1 \\ 0, & \text{c.c.} \end{cases}$$



Determine x[n] a partir de $X(e^{j\Omega}) = \begin{cases} 1, & p/ \quad \frac{\pi}{4} < |\Omega| < \frac{3\pi}{4} \\ 0, & c.c. \end{cases}$





$$X(e^{j\Omega}) = A(e^{j(\Omega+\frac{\pi}{2})}) + A(e^{j(\Omega-\frac{\pi}{2})}) \quad \longrightarrow \quad x[n] = e^{-j\frac{\pi}{2}n} \cdot a[n] + e^{j\frac{\pi}{2}n} \cdot a[n]$$

$$A(e^{j\Omega}) = \begin{cases} 1, & p/ \ |\Omega| < \frac{\pi}{4} \\ 0, & c.c. \end{cases}$$

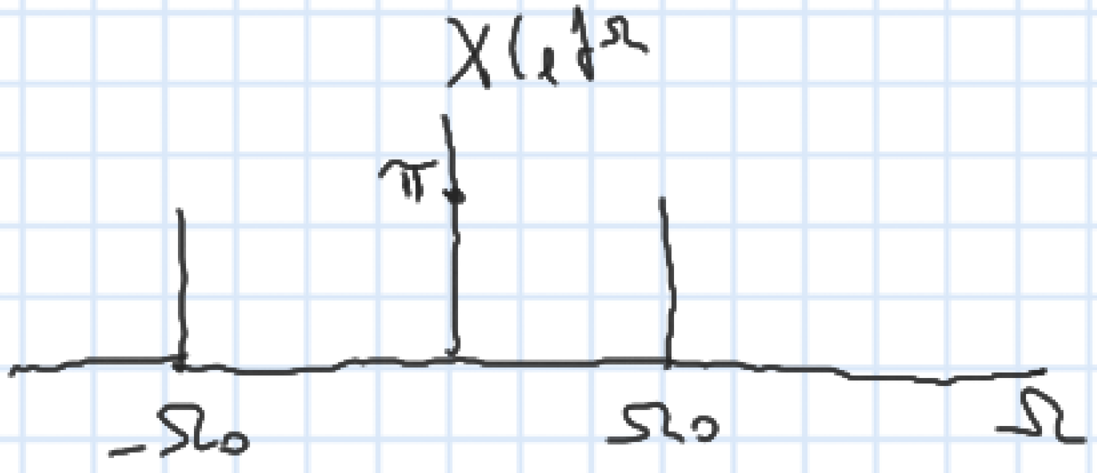
$$x(n) = \left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right) \cdot a[n]$$

$$x[n] = 2 \cdot \cos\left(\frac{\pi}{2}n\right) a[n]$$

$$a[n] = \frac{\operatorname{SEN}\left(\frac{\pi}{4}n\right)}{\pi \cdot n}$$

$$\boxed{x[n] = 2 \cdot \cos\left(\frac{\pi}{2} \cdot n\right) \cdot \frac{\operatorname{SEN}\left(\frac{\pi}{4}n\right)}{\pi \cdot n}}$$

Determine a DTFT de $x[n] = \cos(\Omega_0 n)$



$$x[n] = \frac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right)$$

$$a[n] = \frac{1}{2\pi} \quad \text{—} \quad A(e^{j\Omega}) = \delta(\Omega)$$

$$x[n] = \pi \cdot e^{j\Omega_0 n} \cdot a[n] + \pi \cdot e^{-j\Omega_0 n} \cdot a[n]$$

$$X(e^{j\Omega}) = \pi \cdot A(e^{j(\Omega-\Omega_0)}) + \pi \cdot A(e^{j(\Omega+\Omega_0)})$$

$$X(e^{j\Omega}) = \pi \cdot \delta(\Omega-\Omega_0) + \pi \cdot \delta(\Omega+\Omega_0)$$