1 Diffie-Hellman key agreement

1.1 Finding a generator

a. (1) the element tested from 1 to 10; (2) the order of each such element; and (3) what (sub)group that element generates.

Element	Order mod 2027	Generated Group
1	1	Trivial subgroup
2	2026	Full group Z*_2027
3	1013	Large prime-order subgroup
4	1013	Large prime-order subgroup
5	2026	Full group Z*_2027
6	2026	Full group Z*_2027
7	2026	Full group Z*_2027
8	2026	Full group Z*_2027
9	1013	Large prime-order subgroup
10	1013	Large prime-order subgroup
6	2026	Full group Z*_2027
7	2026	Full group Z*_2027
8	2026	Full group Z*_2027
9	1013	Large prime-order subgroup
10	1013	Large prime-order subgroup

b.

- *full group g=2 → order 2026*

  2^{2026} mod 2027 ≡ 1, for all proper divisors d|2026, $2^d \neq 1$ mod 2027,

  2^1 mod 2027 ≠ 1, 2^2 mod 2027 ≠ 1, 2^{1013} mod 2027 ≠ 1, 2^{2026} mod 2027 = 1

  Conclusion: g=2 passes all checks

- *large prime-order subgroup g=3 → order 1013*

  3^{1013} mod 2027 ≡ 1, 3^1 mod 2027 ≠ 1

  $g^q \equiv 1$ mod p, and $g \neq 1$ mod p

  Conclusion: g=3

1.2 Performing key exchange

a.

$a = 8wy = 8 \times 62 = 496$, $b = 9xz = 9 \times 28 = 252$, $p = 2027$, $g = 2$

b.

$A = g^a \bmod p = 2^{496} \bmod 2027 = 1450$, $B = g^b \bmod p = 2^{252} \bmod 2027 = 872$

c.

$s = B^a \bmod p = 872^{496} \bmod 2027 = 902$, $s = A^b \bmod p = 1450^{252} \bmod 2027 = 902$

They are agreeing on session key: 902

1.3 Sabotaging the protocol

a.

$p = 2027$, $g = 2$, Alice's private key: $a = 496$, Malicious Bot's private key: $b^1 = 1013$,

Both derive the same session key: $s = A^{b1} \bmod p = 1450^{1013}$, $\bmod 2027 = 1$

b.

Alpha (α)	Alice's Message A	Shared Secret with Bob	Shared Secret with Bot
496	1450	902	1
497	873	68	2026
498	1746	513	1
499	1465	1396	2026
500	903	1112	1
501	1806	758	2026
502	1585	174	1
503	1143	1730	2026
504	259	472	1
505	518	103	2026

c.

The session keys are predictable and guessable, because one party uses a fixed exponent corresponding to a low-order subgroup. This breaks the security of Diffie-Hellman, as an attacker could easily guess or brute force the key.

2 The RSA cryptosystem

2.1 Textbook RSA encryption

a.

$e = 28 \times 6 + 401 = 168 + 401 = 569$

b.

$p = 2027$, $q = 2593$, $n = 2027 \times 2593 = 5256011$, $\phi(n) = (2026) \times (2592) = 5251392$,

$e = 569$, my public key is: $(n = 5256011, e = 569)$

c.

$d = e^{-1} \bmod \phi(n) = 569^{-1} \bmod 5251392 = 230729$,

my full RSA private key is: $n = 5256011$, $d = 230729$

d.

q	r1	r2	s1	s2	t1	t2
9229	569	91	0	1	1	-9229
6	91	23	1	-6	-9229	55375
3	23	22	-6	19	55375	-175354
1	22	1	19	-25	-175354	230729
22	1	0	-25	569	230729	-5251392

$\gcd(5251392, 569) = 1$,
Modular inverse $d = 569^{-1} \bmod 5251392 = 230729$,
$569 \cdot 230729 - 5251392 \cdot 25 = 1$, $d = 230729$

e.

Step	Bit	Square	Multiply
1	1	1	1024
2	0	1048576	—
3	0	1430675	—
4	0	3615939	—
5	1	4207791	4104975
6	1	1180526	5232105
7	1	3847648	3239313
8	0	1047470	—
9	0	1104650	—
10	1	340707	1987242

Public key $(n = 5256011, e = 569)$, Plaintext $m = 1024$,
$c = 1024^{569} \bmod 5256011 = 1987242$

f.

Ciphertext: $c = 1987242$, Private exponent: $d = 230729$,
Primes: $p = 2027$, $q = 2593$, Modulus: $n = pq = 5256011$,

$d^p = d \bmod (p - 1) = 230729 \bmod 2026 = 1791$,
$d^p = d \bmod (q - 1) = 230729 \bmod 2592 = 41$

$M_p = c^{1791} \bmod 2027 = 1024$,
$M_q = c^{41} \bmod 2593 = 1024$

$q^{-1} \bmod p = 727$,
$p^{-1} \bmod q = 1663$

$M = (q \cdot q^{-1} \bmod p \cdot Mp + p \cdot p^{-1} \bmod q \cdot Mq) \bmod n$
$M = (2593 \cdot 727 \cdot 1024 + 2027 \cdot 1663 \cdot 1024) \bmod 5256011 + 1024$

Decryption result: 1024

2.2 Distinguishing attacks against Textbook RSA and Random-Padded RSA

a.

Public key: ($n = 9005063$, $e = 17$), $C^* = 3155223$, $M1 = 111$, $M2 = 222$

1. Encrypt both M1 and M2 using the public key:
   - $C1 = 111^{17} \bmod 9005063 = 6921938$
   - $C2 = 222^{17} \bmod 9005063 = 3155223$
2. Compare:
   - $C^* = C2$

The plaintext is M2 = 222, encrypting the same message always gets the same ciphertext.

Property exploited: Deterministic encryption is lacking semantic security.

b.

We could try: $C1 = M1^e \bmod n$, $C2 = M2^e \bmod n$,

and compare directly to $C*$, because the scheme is deterministic.

But with random padding, every message is encoded as: $Z = 1000 \cdot R + M$

and since R is chosen randomly for every encryption,

the value of Z changes every time, even for the same M.

So, we cannot predict any specific ciphertext to a known message like before too many possible Z values exist:

- For each M, there are 9004 possible Z values
- And thus 9004 different ciphertexts for each M

3 Digital signatures and authentication

a.

1. The server chooses a crafted challenge $x = h(M^*)$ $x = h(M^*)$, where M* is the message the server wants to forge as if it came from Alice.
2. The server sends this x to Alice in the login protocol.
3. Alice, unaware of the server's intent, signs x and returns:
   $s = Sign_{sk}(x) = Sign_{sk}(h(M^*))$
4. The server now possesses a valid signature s on h(M*), which is precisely what an email signature would look like for message M*.
   The server succeeded forged Alice's signature on an arbitrary email message M*.

b.

1. Modify the challenge x sent by the server so that Alice signs only authentication specific data, e.g.: x^1 = Hash("LOGIN" || x || ServerID || Timestamp)
2. Slice signs $s = Sign_{sk}(x^1)$
3. The login protocol now uses structured data that:
   a. Includes metadata like "LOGIN"
   b. Is tied to the session via timestamp and server identity
   c. Cannot be reused as a hash of any valid email message

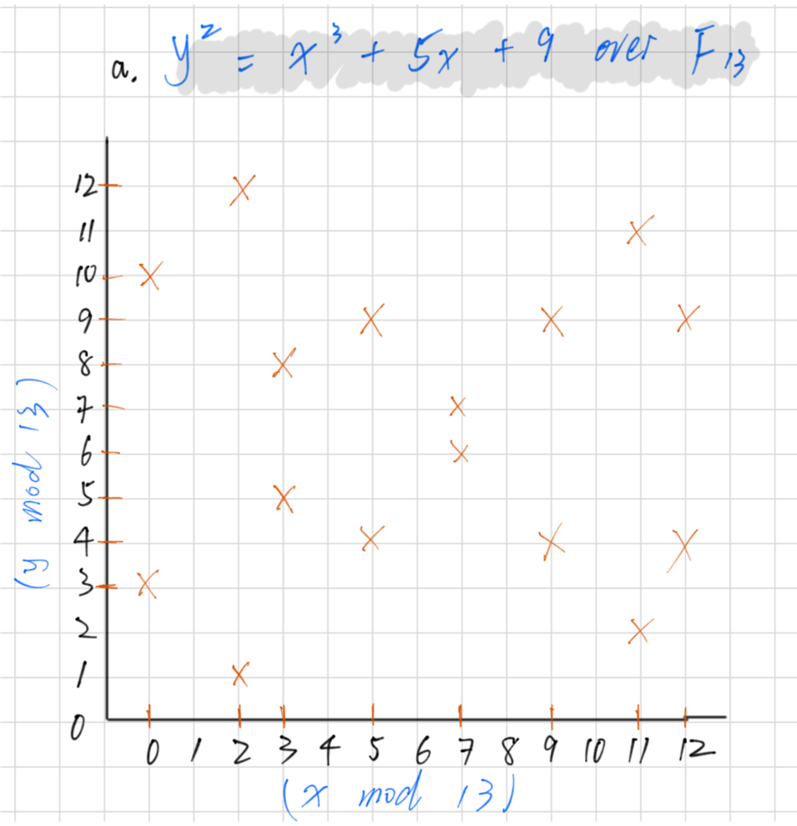
c.

Never reuse the same cryptographic key across different protocols or application. A signature in one protocol might be exploited to forge signatures in another like universal forgery attack.

4 Elliptic-curve cryptography

4.1 Elliptic-curve arithmetic

a.



b.

16 affine points on the curve, and 1 point at infinity,

so the order of my curve is: 17

c.

G = (7,7) & H = (12, 9): on the curve

d.

$\lambda = \frac{3x12+a^1}{2y1}$ mod p,

$x_3 = \lambda^2 - 2_{x1}$ mod p,

$y_3 = \lambda(x_1 - x_3) - y_1$ mod p

the result is: 2G = G + G = 2, 12

e.

I draw a line connecting the points G = (7,7) and H = (12,9) on the plot of the curve over F13. This line intersects the curve at a third point R. Then, I reflect R across the x-axis (mod 13), which gives the result G+H. Since the plot uses a finite field, this reflection is equivalent to negating the y-coordinate modulo 13.

f.

$\lambda = \frac{y2-y1}{x2-x1}$ mod 13 = $\frac{9-7}{12-7} = \frac{2}{5}$ mod 13 = $2 \cdot 5^{-1}$ mod 13

5^{-1} mod 13 = 8, since $5 \cdot 8 = 40 \equiv 1$ mod 13

$\lambda = 2 \cdot 8 = 16 \equiv 3$ mod 13

$x_3 = \lambda^2 - x_1 - x_2$ mod 13 = $3^2 - 7 - 12 = 9 - 19 = -10$ mod 13 = 3

$y_3 = \lambda(x_1 - x_3) - y_1 = 3(7 - 3) - 7 = 3 \cdot 4 - 7 = 12 - 7 = 5$

Result is: G + H = 3, 5

4.2 Elliptic-curve Diffie-Hellman key agreement

a.

The order of point G = (7, 7) under the elliptic curve $y^2 = x^3 + 5x + 9$ mod 13 found: Order of G = 17, G generates the entire group of the curve, since the curve itself has order 17.

b.

Alice's private key: a = 5, generate point: G = (7, 7)

The point: A = a · G = 5 · (7, 7) = (3, 8)

c.

Bob's private key: b = 9, generate point: G = (7, 7)

The public point: B = b · G = 9 · (7, 7) = (9, 9)

d.

From Alice: S = a · B = 5 · (9, 9) = (12, 9)

From Bob: S = b · A = 9 · (3, 8) = (12, 9)

Both Alice and Bob independently derive the same shared secret point (12, 9), it means that the ECDH protocol works correctly with the curve and keys.