PhD notes I: Physical quantities and Mueller matrix decomposition

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Physical quantities and Mueller matrix decomposition

1 Physical quantities

[**Chapter 6, pg. 199**] *J. J. Gil and R. Ossikovski, Polarized light and the Mueller matrix approach, Series in Optics and Optoelectronics, CRC Press, Taylor and Francis Group (2016)*

The sources of polarimetric purity (sources of depolarization) are studied from the complementary attributes: diattenuation, polarizance and degree of spherical purity. The **depolarization** of a medium represented by a Mueller matrix M is closely related to the randomness of the covariance matrix H (or the coherency matrix C) associated with M. The analysis of the eigenvalues of H provides information about the number and relative weights of the pure constituents of the medium and leads to the definition of IPPs.

Generic Mueller matrix block form:

$$\hat{M} = m_{00} \begin{bmatrix} 1 & D^T \\ P & m \end{bmatrix} \tag{1}$$

where

$$\boldsymbol{D} \equiv \frac{1}{m_{00}}(m_{01}, m_{02}, m_{03})^T \quad \boldsymbol{P} \equiv \frac{1}{m_{00}}(m_{10}, m_{20}, m_{30})^T \quad m = \frac{1}{m_{00}} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{2}$$

Components of purity D (diattenuation), P (polarizance), and P_S (**degree of spherical purity**), defined between 0 and 1:

$$D = \frac{\sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2}}{m_{00}} \qquad P = \frac{\sqrt{m_{10}^2 + m_{20}^2 + m_{30}^2}}{m_{00}} \qquad P_S = \frac{||m||_2}{\sqrt{3}} = \frac{\sqrt{\sum_{k,l=1}^{3} m_{k,l}^2}}{m_{00}\sqrt{3}} \tag{3}$$

Degree of polarimetric purity P_Δ:

$$P_\Delta = \frac{1}{\sqrt{3}}\sqrt{D^2 + P^2 + 3P_S^2} \tag{4}$$

Diattenuation of M is a measure of the relative difference between the maximum and minimum intensities of the output states with respect to all possible input polarization states. For D=0: media do not exhibit diattenuation i.e. does not exist and input state for which the output state has intensity 0; while D=1 (**analyzer medium**) implies that there exists an input state whose output intensity is 0 (completely orthogonal). When D=1 and P=1 **nondepolarizing polarizer**, when D=1 and P<1 **depolarizing analyzer**. Polarizance of M can be considered a measure of the relative difference between the maximum and minimum intensities of the output states with respect to all possible polarization states incoming in the reverse direction. For P=0, media do not exhibit diattenuation; P=1 **polarizer medium**, there exists an input state for which the output intensity is 0. Regardless of the input state, the output is totally polarized with a Stokes vector proportional to input state.

Diattenuation and **Polarizance** show a dual role depending on the direction of propagation of light: D is both diattenuation of M and polarizance of M^r (reverse), while P is both the polarizance of M and diattenuation of M^r. The **degree of polarizance** measures their joint contribution to polarimetric purity: global measure of the diattenuation and polarizance of a MM (restricted to be from 0 to 1).

$$P_P = \sqrt{\frac{P^2 + D^2}{2}} \tag{5}$$

$P_P = 1$ means **pure polarizer**; $P_P = 0$ when MM does not exhibit any polarizing or diattenuating effect (P=D=0) so MM has the form of a **nonpolarizing MM**.

The **linear diattenuation** D_L and **circular diattenuation** D_C of a MM with $D = (D_1, D_2, D_3)^T$ are defined as

$$D_L = \sqrt{D_1^2 + D_2^2} > 0 \qquad\qquad D_C = D_3 \quad , \quad -1 < D_C < 1 \tag{6}$$

when $D_C = 0$ the diattenuation only affects s_1 and s_2 Stokes components (linear ones); conversely, when $D_L = 0$ only s_3 is affected.

The **linear polarizance** P_L and **circular polarizance** P_C of a MM with $D = (P_1, P_2, P_3)^T$ are defined as

$$P_L = \sqrt{P_1^2 + P_2^2} > 0 \qquad\qquad P_C = P_3 \quad , \quad -1 < P_C < 1 \tag{7}$$

when $P_C = 0$ the polarizance only affects s_1 and s_2 Stokes components (linear ones); conversely, when $P_L = 0$ only s_3 is affected.

Degree of polarimetric purity P_Δ: can be considered the result of two separated contributions: degree of polarizance P_P and the degree of spherical purity P_S

$$P_\Delta = \frac{1}{\sqrt{3}}\sqrt{D^2 + P^2 + 3P_S^2} = \sqrt{\frac{2P_P^2}{3} + P_S^2} \tag{8}$$

due to

$$P_P = \sqrt{\frac{P^2 + D^2}{2}} \qquad\qquad P_S = \frac{||m||_2}{\sqrt{3}} = \frac{\sqrt{\sum_{k,l=1}^{3} m_{k,l}^2}}{m_{00}\sqrt{3}} \tag{9}$$

2 MM algebra for the analysis of polarimetric measurements

[**Depolarization**] *J. J. Gil, Review on Mueller matrix algebra for the analysis of polarimetric measurements, J. Appl. Remote Sens. 8, 081599 (2014).*

The measured MMs contain up to 16 (4x4) **independent** parameters for each **measurement configuration** (spectral profile of the wave probe of the polarimeter, angle of incidence, observation direction, etc.) and for each **spatially resolved element** of the sample (imaging polarimetry).

★ When dealing with MM decompositions, it is useful to distinguish the cases of

(i) **Pure** Mueller matrices (**Mueller-Jones or nondepolarizing)** associated with systems whose depolarization index $P_\Delta = 1$ so that they do not depolarize any totally polarized input beam.

(ii) **Nonpure** Mueller matrices (**depolarizing**) which depolarize some or all totally polarized input states ($P_\Delta < 1$).

2.1 Intro: Parallel decompositions

Parallel decompositions consist of representing a **MM as convex sum of MM**. The **physical meaning** of parallel decompositions is that the incoming light beam is shared among a set of pencils that interact with a number of optical components spatially distributed in the irradiated area without overlapping, in such a manner that the emerging pencils are incoherently recombined into a single output light beam.

$$\vec{s} \rightarrow [M] \rightarrow \vec{s}' \qquad\qquad M = \sum_i p_i M_i \qquad so \quad that \tag{10}$$

$$p_0\vec{s} \rightarrow [M_0] \rightarrow \vec{s_0}' \qquad\qquad p_1\vec{s} \rightarrow [M_1] \rightarrow \vec{s_1}' \qquad (...) \qquad p_i\vec{s} \rightarrow [M_i] \rightarrow \vec{s_i}' \tag{11}$$

Polarimetric subtraction of MM: identify a parallel component M_0 subtractable from the whole system **M** in such a manner that the cross-section p_0 of M_0 and the MM of the remainder system M_X are calculated:

$$p_0\vec{s} \rightarrow [M_0] \rightarrow \vec{s_0}' \qquad\qquad (1-p_0)\vec{s} \rightarrow [M_X] \rightarrow \vec{s_1}' \tag{12}$$

$$M_X = \frac{M - p_0 M_0}{1 - p_0} \tag{13}$$

2.2 Intro: Serial decompositions

Serial decompositions consist of representing a general Mueller matrix as a **product of particular MM**. The **physical meaning** of serial decompositions is that the whole system is considered as a cascade of polarization components so that the incoming light interacts **sequentially** with them. This arrangement of the components constitutes the serial equivalent system:

$$\vec{s} \to [M] \to \vec{s}' \qquad \vec{s} \to M_0 \to M_1 \to M_2 \to M_3 \to \vec{s}' \qquad so \quad that \tag{14}$$

$$M = M_3 M_2 M_1 M_0 \qquad product \quad from \quad right \quad to \quad left \tag{15}$$

2.3 Intro: Differential decompositions

Differential decompositions consist of **identifying an elementary representative of a given integral MM and then separating the mean values of the elementary properties from the depolarizing sources given by their uncertainties and by the anisotropic absorptions**. When the depolarization occurs uniformly and continuously along the optical path, a differential depolarizing MM can be properly defined as $m = ln(M)/L$, where L is the optical path length. Otherwise, an equivalent differential Mueller matrix can mathematically be defined, but without a direct interpretation in terms of the nature of the sample.

$$\vec{s} \to [M] \to \vec{s}' \qquad \vec{s} \to dM \to \vec{s}' \qquad so \quad that \tag{16}$$

$$\frac{dM}{dz} = mM \quad \to \quad \int \frac{dM}{M} = \int m dz \quad \to \quad ln(M) = mL \tag{17}$$

$$m = m_m + m_u = nondepolarizing + depolarizing \tag{18}$$

2.4 Non-depolarizing (pure) systems

A nondepolarizing system can be represented by means of:

(i) Jones Matrix **T**
(ii) Mueller-Jones Matrix $M_J(T)$
(iii) Rank 1 Covariance Matrix $H_J(M_J) = H_J(T)$
(iv) Four-dimensional complex vector **w**: $H_J = trH_J(w \otimes w^\dagger)$

*Global phase factors in **T** and **w** are physically irrelevant in polarimetric intensity measurements: pure systems are always characterized by up to 7 independent real parameters*

For a **pure system** characterized by a Jones matrix

$$T = \begin{bmatrix} t_0 & t_1 \\ t_2 & t_3 \end{bmatrix} \tag{19}$$

The relation between $M_J(T)$ and **T**:

$$M_J = L(T \otimes T^*)L^{-1} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix} \qquad L^{-1} = \frac{1}{2}L^{\dagger} \tag{20}$$

The corresponding covariance matrix expressed in terms of t_i:

$$H_J = \frac{1}{2}(t \otimes t^{\dagger}) \qquad t = (t_0, t_1, t_2, t_3)^T \tag{21}$$

Relation between **w** and **T**:

$$t = (\sqrt{2trH})w \qquad \rightarrow \qquad T = (\sqrt{2trH})W \qquad W = \begin{bmatrix} w_0 & w_1 \\ w_2 & w_3 \end{bmatrix} \tag{22}$$

2.5 Depolarizing (non-pure) systems

A depolarizing MM can be considered as a **parallel combination** (convex sum or an ensemble average) **of pure Mueller matrices** M_{Ji} **with the same transmittance** $(M_{Ji})_{00} = m_{00}$:

$$M = \sum_i^n p_i M_{Ji} = \sum_i^n p_i[L(T_i \otimes T_i^*)L^{-1}] \qquad \sum_i^n p_i = 1 \tag{23}$$

whose covariance matrix **H**:

$$H(M) = \sum_i^n p_i H_i(M_{Ji}) = \frac{1}{4}\sum_{k,l=0}^{3} m_{kl}(\sigma_k \otimes \sigma_l) \tag{24}$$

Elements of **M(H)**:

$$m_{kl} = tr[(\sigma_k \otimes \sigma_l)H] \tag{25}$$

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{26}$$

Unlike the case of a **pure covariance matrix** H_J, $H(M)$ has **more than one nonzero eigenvalue** (rank **H**>1).

• **Cloude's criterion:** The general characterization of **all** the set of physical MM (**both nondepolarizing and depolarizing**) is given by the non-negativity of the eigenvalues of **H**. While the general characterization of a **pure Mueller matrix** M_J is directly achieved from its mathematical link with its corresponding Jones matrix. *Underlying argument: any physical MM is the integral result of the additive combination -coherent, partially coherent of incoherent- of elementary deterministic and well-defined by a Mueller-Jones matrix transformations.*

★ **Only matrices satisfying Cloude's criterion are physically realizable and can properly be called Mueller matrices.**

Mueller, covariance **H**, and coherency **C** matrices corresponding to the same optical system contain identical physical information. The covariance and the coherency matrices are related by a unitary similarity transformation, so that they have the same non-negative eigenvalues λ_0, λ_1, λ_2, λ_3.

Covariance matrix **H**:

$$H(M) = \frac{1}{4}\begin{bmatrix} \begin{array}{c} m_{00}+m_{01} \\ +m_{10}+m_{11} \end{array} & \begin{array}{c} m_{02}+m_{12} \\ +i(m_{03}+m_{13}) \end{array} & \begin{array}{c} m_{20}+m_{21} \\ -i(m_{30}+m_{31}) \end{array} & \begin{array}{c} m_{22}+m_{33} \\ +i(m_{23}-m_{32}) \end{array} \\ \\ \begin{array}{c} m_{02}+m_{12} \\ -i(m_{03}+m_{13}) \end{array} & \begin{array}{c} m_{00}-m_{01} \\ +m_{10}-m_{11} \end{array} & \begin{array}{c} m_{22}-m_{33} \\ -i(m_{23}+m_{32}) \end{array} & \begin{array}{c} m_{20}-m_{21} \\ -i(m_{30}-m_{31}) \end{array} \\ \\ \begin{array}{c} m_{20}+m_{21} \\ +i(m_{30}+m_{31}) \end{array} & \begin{array}{c} m_{22}-m_{33} \\ +i(m_{23}+m_{32}) \end{array} & \begin{array}{c} m_{00}-m_{01} \\ -m_{10}-m_{11} \end{array} & \begin{array}{c} m_{02}-m_{12} \\ +i(m_{03}-m_{13}) \end{array} \\ \\ \begin{array}{c} m_{22}+m_{33} \\ -i(m_{23}-m_{32}) \end{array} & \begin{array}{c} m_{20}-m_{21} \\ +i(m_{30}-m_{31}) \end{array} & \begin{array}{c} m_{02}-m_{12} \\ -i(m_{03}-m_{13}) \end{array} & \begin{array}{c} m_{00}-m_{01} \\ -m_{10}+m_{11} \end{array} \end{bmatrix} \tag{27}$$

Coherency matrix **C**:

$$C(M) = \frac{1}{4}\begin{bmatrix} \begin{array}{c} m_{00}+m_{11} \\ +m_{22}+m_{33} \end{array} & \begin{array}{c} m_{01}+m_{10} \\ -i(m_{23}-m_{32}) \end{array} & \begin{array}{c} m_{02}+m_{20} \\ +i(m_{13}-m_{31}) \end{array} & \begin{array}{c} m_{03}+m_{30} \\ -i(m_{12}-m_{21}) \end{array} \\ \\ \begin{array}{c} m_{01}+m_{10} \\ +i(m_{23}-m_{32}) \end{array} & \begin{array}{c} m_{00}+m_{11} \\ -m_{22}-m_{33} \end{array} & \begin{array}{c} m_{12}+m_{21} \\ +i(m_{03}-m_{30}) \end{array} & \begin{array}{c} m_{13}+m_{31} \\ -i(m_{02}-m_{20}) \end{array} \\ \\ \begin{array}{c} m_{02}+m_{20} \\ -i(m_{13}-m_{31}) \end{array} & \begin{array}{c} m_{12}+m_{21} \\ -i(m_{03}-m_{30}) \end{array} & \begin{array}{c} m_{00}-m_{11} \\ +m_{22}-m_{33} \end{array} & \begin{array}{c} m_{23}+m_{32} \\ +i(m_{01}-m_{10}) \end{array} \\ \\ \begin{array}{c} m_{03}+m_{30} \\ +i(m_{12}-m_{21}) \end{array} & \begin{array}{c} m_{13}+m_{31} \\ +i(m_{02}-m_{20}) \end{array} & \begin{array}{c} m_{23}+m_{32} \\ -i(m_{01}-m_{10}) \end{array} & \begin{array}{c} m_{00}-m_{11} \\ -m_{22}+m_{33} \end{array} \end{bmatrix} \tag{28}$$

2.6 Anisotropy coefficients of a MM

Set of parameters representative of the anisotropic behavior of a linear system: α, β and γ.

Polarization state	Stokes vector
Linear at 0°	s_1 = (1, 1, 0, 0)
Linear at 90°	s_{1-} =(1, -1, 0, 0)
Linear at 45°	s_2 =(1, 0, 1, 0)
Linear at 135°	s_{2-} =(1, 0, -1, 0)
Circular (dextro)	s_3 =(1, 0, 0, 1)
Circular (levo)	s_{3-} =(1, 0, 0, -1)

The following nondepolarizing MM (linear horizontal anisotropy $\hat{M}_L$, linear 45° anisotropy $\hat{M}_{L'}$ and circular anisotropy $\hat{M}_C$) maintain unchanged Stokes except for a positive coefficient ($s_{1\pm}$, $s_{2\pm}$ and $s_{3\pm}$, respectively):

$$s_{1\pm} \to [\hat{M}_L] \to s_{1\pm} \qquad s_{2\pm} \to [\hat{M}_{L'}] \to s_{2\pm} \qquad s_{3\pm} \to [\hat{M}_C] \to s_{3\pm} \tag{29}$$

Each one of these pure Mueller matrices of diattenuating retarders (or retarding diattenuators) can, respectively, be considered as a serial combination of a diattenuator (linear, linear 45 deg, circular) and a retarder (linear, linear 45 deg, circular), which, by virtue of the commutativity of the respective matrix factors, can be placed in either of the two possible relative positions.

- By extending this idea to general depolarizing Mueller matrices:

(i) The quantities $(m_{01}+m_{10})$ and $(m_{23}-m_{32})$ correspond to **linear horizontal anisotropy**.

(ii) The quantities $(m_{02}+m_{20})$ and $(m_{13}-m_{31})$ correspond to **linear 45° anisotropy**.

(iii) The quantities $(m_{03}+m_{30})$ and $(m_{12}-m_{21})$ correspond to **circular anisotropy**.

We define the following **anisotropy coefficients of a MM**:

$$\alpha = \alpha_1 = \sqrt{\frac{1}{\Sigma}[(m_{01}+m_{10})^2 - (m_{23}+m_{32})^2]} \tag{30}$$

$$\beta = \alpha_2 = \sqrt{\frac{1}{\Sigma}[(m_{02}+m_{20})^2 - (m_{13}+m_{31})^2]} \tag{31}$$

$$\gamma = \alpha_3 = \sqrt{\frac{1}{\Sigma}[(m_{03}+m_{30})^2 - (m_{12}+m_{21})^2]} \tag{32}$$

$$\Sigma = 3m_{00}^2 - (m_{11}^2 + m_{22}^2 + m_{33}^2) + 2\Delta \tag{33}$$

$$\Delta = m_{01}m_{10} + m_{02}m_{20} + m_{03}m_{30} - (m_{23}m_{32} + m_{13}m_{31} + m_{12}m_{21}) \tag{34}$$

They can be grouped into the **anisotropy vector a**=$(\alpha, \beta, \gamma)^T$ and so the **degree of anisotropy of MM**:

$$P_a = |a| = \sqrt{\sum_{i=1}^{3} \alpha_i^2} \qquad\qquad P_a \leq P_\Delta \leq 1 \tag{35}$$

The maximum value $P_a = P_\Delta$ is reached **if and only if M is a pure matrix (**$P_\Delta = 1$**).**

2.7 Components of Purity of a MM

Recall the degree of spherical purity P_S, the depolarization index P_Δ and the degree of polarizance P_P (quadratic mean of polarizance P (forward polarizance, polarizing power of the system for light incoming in the forward direction) and diattenuation D (reverse polarizance, polarizing power of the system for light incoming in the reverse direction)):

$$P_S = \frac{||m||_2}{\sqrt{3}} \qquad\qquad P_\Delta = \sqrt{\frac{D^2 + P^2 + 3P_S^2}{3}} \qquad\qquad P_P = \sqrt{\frac{P^2 + D^2}{2}} \tag{36}$$

Note that P(**M**), D(**M**) and P_S(**M**) remain invariable when **M** is pre/post-multiplied by an orthogonal MM (**pure retarder**).

Degree of polarizance	P_P	Restricted to $0 \leq P_P \leq 1$
Max. value	1	Total Pure Polarizer
Min. value	0	Parallel Mixture of Pure Retarders
Particular case	0	$P_\Delta = 1$ and $P_S = 1$: Pure Retarder

Degree of spherical purity	P_S	Restricted to $0 \leq P_S \leq 1$
Max. value	1	Pure Retarder
Min. value	$1/\sqrt{3}$	$P_\Delta = 1$, total polarimetric purity
Zero	0	Sub-matrix elements $m_{kl} = 0$

2.8 Parallel decompositions of MMs

Parallel decomposition → convex sum of particular MMs → Spectral, **Arbitrary** and **Characteristic/Trivial** decompositions.

2.8.1 Spectral (or Cloude) Decomposition of MM : shows that **any linear system can be considered as a parallel combination of up to four pure systems with weights proportional to the eigenvalues of H**. Any physically realizable parallel decomposition must be expressed as a convex linear combination (i.e., the sum of the positive coefficients of the expansion is equal to one), where all addends have the same mean transmittance m_{00}.

Covariance matrix **H** can be expressed as the convex linear combination of 4 rank 1 covariance matrices (pure systems):

$$H = \sum_{i=0}^{3} \frac{\lambda_i}{trH} H_i \qquad \leftrightarrow \qquad M = \sum_{i=0}^{3} \frac{\lambda_i}{m_{00}} M_{Ji} \tag{37}$$

$$H_i = (trH)(u_i \otimes u_i^\dagger) \qquad\qquad trH = m_{00} \tag{38}$$

2.8.2 Arbitrary Decomposition of MM : **H** into a convex linear combination of pure covariance matrices (rank = 1). The minimum number of pure matrices of the equivalent system is equal to rank**H**.

$$H = \sum_{i=0}^{3} p_i H_i \qquad \leftrightarrow \qquad M = \sum_{i=0}^{3} p_i M_{Ji} \tag{39}$$

$$\sum_{i=0}^{3} p_i = 1 \qquad\qquad trH = trH_i = m_{00} \qquad\qquad (M_{Ji})_{00} = m_{00} \tag{40}$$

$$H_i = H_i^\dagger = [(trH)(w_i \otimes w_i^\dagger)] \qquad\qquad |w_i| = 1 \tag{41}$$

$$p_i = \frac{tr\left(diag(\lambda_0, \lambda_1, \lambda_2, \lambda_3)[w_i \otimes w_i^\dagger]\right)}{tr(H)} \qquad\qquad p_i > 0 \tag{42}$$

2.8.3 Arbitrary Decomposition of Stokes : given a partially polarized state of polarization Stokes vector **s**

$$s = s_0 \begin{bmatrix} 1 \\ P\vec{u} \end{bmatrix} \qquad\qquad \vec{u} = (u_1, u_2, u_3)^T = (cos2\chi cos2\varphi, cos2\chi sin2\varphi, sin2\chi)^T \tag{43}$$

being s_0 the intensity (energy flux of light wave), P the degree or polarization and $\vec{u}$ unit vector that summarizes information about azimuth φ and ellipticity $tan(\chi)$:

$$0 \leq \varphi < \pi \qquad\qquad -\pi/4 \leq \chi \leq \pi/4 \tag{44}$$

For $1 < P$, **s** can always be considered as an incoherent composition of **two** totally (pure) polarized states:

$$s = s_0 \begin{bmatrix} 1 \\ P\vec{u} \end{bmatrix} = p_0 s_0 + (1-p_0)s_1 \qquad ; \qquad s_0 = s_0 \begin{bmatrix} 1 \\ \vec{v} \end{bmatrix} \qquad s_1 = s_0 \begin{bmatrix} 1 \\ \vec{w} \end{bmatrix} \tag{45}$$

$$p_0 = \frac{1-P^2}{2(1-P\vec{u}^T\vec{v})} \quad , \quad \vec{w} = \frac{P\vec{u} - p_0\vec{v}}{1-p_0} \quad ; \quad |\vec{u}| = |\vec{v}| = |\vec{w}| = 1 \tag{46}$$

2.8.4 Characteristic / Trivial Decomposition of MM : some of the **pure components** can be grouped into **depolarizing** components. The characteristic decomposition of the 4x4 covariance matrix **H** associated with **M** can be formulated as a **convex sum of covariance matrices, all with the same trace**:

$$H = \frac{\lambda_0 - \lambda_1}{trH} H_0 + 2\frac{\lambda_1 - \lambda_2}{trH} H_1 + 3\frac{\lambda_2 - \lambda_3}{trH} H_2 + 4\frac{\lambda_3}{trH} H_3 \tag{47}$$

$$H_0 = trH[U \cdot diag(1,0,0,0) \cdot U^\dagger] = (trH)(\vec{u_0} \otimes \vec{u_0}^\dagger) \qquad Pure \quad component \tag{48}$$

$$H_1 = \frac{1}{2} trH[U \cdot diag(1,1,0,0) \cdot U^\dagger] \qquad Non-pure \quad component \tag{49}$$

$$H_2 = \frac{1}{3} trH[U \cdot diag(1,1,1,0) \cdot U^\dagger] \qquad Non-pure \quad component \tag{50}$$

$$H_3 = \frac{1}{4} trH[U \cdot diag(1,1,1,1) \cdot U^\dagger] \qquad Non-pure \quad component \tag{51}$$

Non-pure components with rank$\mathbf{H}_i = i+1$ and degenerate nonzero i+1 eigenvalues.

★ The characteristic decomposition of a covariance matrix **H** is formulated as a **convex sum** of:

- a pure component $\mathbf{H}_0$ determined by $\vec{u_0}$
- a nonpure component containing equiprobable mixture of two pure components (namely, the first two spectral components)
- a nonpure component containing an equiprobable mixture of three pure components (namely, the first three spectral components)
- an ideal depolarizer (a nonpure component contain- ing an equiprobable mixture of the four spectral components)

$$M = P_1 M_{J0}(H_{J0}) + (P_2 - P_1)M_1(H_1) + (P_3 - P_2)M_2(H_2) + (1-P_3)M_3(H_3) \tag{52}$$

When $P_3 = 1$, the ideal depolarizer (4th component) vanishes. So that 3rd and 2nd components can be considered, respectively, a 3D and 2D ideal depolarizers embedded into the 4D system. $P_2 = 1$ entails a maximum of 2 components and $P_1 = 1$ entails $P_\Delta = 1$ so that the system is a pure component.

2.9 Serial Decompositions of Pure MM

Recall MM of pure retarders ($\mathbf{M}_R$) and homogeneous/pure (physical eigenvectors are mutually orthogonal) diattenuators ($\mathbf{M}_D$) *both MM are normal*:

$$M_R = m_{00}\begin{bmatrix} 1 & 0^T \\ \vec{0} & m_R \end{bmatrix} \qquad 0^T = (0,0,0)^T \qquad m_R^T = m_R^{-1} \quad det(m_R) = det(M_R) = 1 \tag{53}$$

$$M_D = m_{00}\begin{bmatrix} 1 & \vec{D}^T \\ \vec{D} & m_D \end{bmatrix} \qquad m_D = \sqrt{1-D^2}I + \frac{1}{D^2}[1-\sqrt{1-D^2}]\vec{D}\times\vec{D}^T \tag{54}$$

so that M_R and m_R are both orthogonal MM, m_D is symmetric and M_D is a symmetric pure MM. Since $(M_R)_{00} = 1$ (**$\mathbf{M}_R$ represents a transparent retarder)**.

$\bullet_1$ In general, an **orthogonal MM** $M_R(\varphi,\chi,\Delta)$ represents a **transparent elliptic retarder** characterized by:

a) the pair of angles (φ,χ) that determine $\rightarrow$ azimuths $(\varphi,\varphi+\pi/2)$ and ellipticities $(tan\chi, -tan\chi)$ of orthogonal eigenstates.

b) the retardance Δ introduced between both eigenstates.

c) the rotation axis is given by the eigenvector with value 1.

$\bullet_2$ In general, a **symmetric MM** $M_D(\psi,\nu,k_1,k_2)$ represents a **diattenuator** characterized by:

a) the pair of angles (ψ,ν) that determine $\rightarrow$ azimuths $(\psi,\psi+\pi/2)$ and ellipticities $(tan\nu, -tan\nu)$ of orthogonal eigenstates.

b) the respective intensity transmittances (k_1,k_2) for eigenstates.

c) the diattenuation vector $\mathbf{D}^T = D(cos\nu cos\psi, cos\nu sin\psi, sin\nu)^T$ determines the angles (ψ,ν) as well as the diattenuation polarizance D of M_D.

d) the expressions of k_1 and k_2 in terms of D and the mean transmittance m_{00} are

$$k_1 = m_{00}(1+D) \qquad\qquad k_2 = m_{00}(1-D) \tag{55}$$

2.9.1 Polar decomposition of Pure Systems

$$\vec{s} \to [M_J] \to \vec{s}' \qquad \vec{s} \to M_D \to M_R \to \vec{s}' \qquad so \quad that \tag{56}$$

A) Equivalent diattenuator:

$$Eigenstate \quad (\psi, \nu) \to [M_D] \to k_1 \quad Transmittance \tag{57}$$

$$Eigenstate \quad (\psi + \pi/2, -\nu) \to [M_D] \to k_2 \quad Transmittance \tag{58}$$

B) Equivalent retarder:

$$Eigenstate \quad (\varphi, \chi) \to [M_R] \to \Delta \quad Retardance \tag{59}$$

$$Eigenstate \quad (\varphi + \pi/2, -\chi) \to [M_R] \to \Delta \quad Retardance \tag{60}$$

Being:

$$M_J(T) = M_D M_R = M_R M_D^{'} \qquad M_D^{'} = M_R^{-1} M_D M_R \tag{61}$$

where $\mathbf{M}_D$ and $\mathbf{M}_D^{'}$ are the **symmetric pure** MM associated with **normal pure diattenuators** $\mathbf{T}_D$ and $\mathbf{T}_D^{'}$, and $\mathbf{M}_R$ is the **orthogonal** MM associated with a **pure retarder** $\mathbf{T}_R$. So that, $\mathbf{M}_J$ depends up to 7 parameters: ψ, ν, k_1, k_2, φ, χ and $\Delta \to$ the 4 singular values are given by

$$k_1 = m_{00}(1 + D) \qquad k_2 = m_{00}(1 - D) \qquad \sqrt{k_1 k_2} \qquad \sqrt{k_1 k_2} \tag{62}$$

2.9.2 General Decomposition of Pure Systems : recall the MM of a **horizontal linear diattenuator** whose principal transmittance axes with ($k_1 \geq k_2$ transmittance) are aligned along the lab axes XY:

$$M_{D,L0} = m_{00} \begin{bmatrix} 1 & cos\theta & 0 & 0 \\ cos\theta & 1 & 0 & 0 \\ 0 & 0 & sin\theta & 0 \\ 0 & 0 & 0 & sin\theta \end{bmatrix} \qquad cos\theta = D = P \qquad (0 \leq \theta \leq \pi/2) \tag{63}$$

Additionally, the MM of a **normal diattenuator (in general, elliptic)** M_D can be represented by means of a **serial combination** composed by an aligned **linear diattenuator** $\mathrm{M}_{D,L0}$ sandwiched by 2 identical **linear retarders** $\mathrm{M}_{R,L}$ and $\mathrm{M}_{R,L}^{\dagger}$ whose fast axis are crossed: **serial combination of 2 retarders is equivalent to another retarder**.

$$M_D = M_{R,L} M_{D,L0} M_{R,L}^T \tag{64}$$

So that, any **pure MM** can be written as an **aligned linear diattenuator pre- and post-multiplied by respective retarders (one is linear while the other is elliptic)**:

$$M_J = M_{R2} M_{D,L0} M_{R1} \qquad M_{R1} = M_{R,L}^T M_R \qquad M_{R2} = M_{R,L} \tag{65}$$

2.9.3 Symmetric Decomposition of Pure Systems

$$\vec{s} \to [M_J] \to \vec{s}' \qquad \vec{s} \to M_{R1} \to M_{R,D} \to M_{R2} \to \vec{s}' \qquad so \quad that \tag{66}$$

Equivalent linear diattenuating retarder:

$$Eigenstate \quad (\varphi', \chi') \to [M_{R,D}] \to k_1 \quad \Delta' \quad Transmittance \quad Retardance \tag{67}$$

$$Eigenstate \quad (\varphi' + \pi/2, -\chi') \to [M_R] \to k_2 \quad \Delta' \quad Transmittance \quad Retardance \tag{68}$$

Recall **polar decomposition** as $M_J = M_D M_R$ so that it has 2 forms depending on the order of the diattenuator and the retarder. It is worth to find symmetric structure's equivalent system:

$$M_J = M_{R,L2} M_{R,D,L} M_{R,L1} \tag{69}$$

where $M_{R,L}$ are **linear retarders** and $M_{R,D,L}$ is given by a **serial combination** of a **linear diattenuator and a linear retarder** ($M_{D,L0}$ and $M_{R,L0}$) whose eigenaxes are aligned with XY lab:

$$M_{D,L0} = m_{00} \begin{bmatrix} 1 & cos\theta & 0 & 0 \\ cos\theta & 1 & 0 & 0 \\ 0 & 0 & sin\theta & 0 \\ 0 & 0 & 0 & sin\theta \end{bmatrix} \qquad M_{R,L0} = m_{00} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cos\Delta & sin\Delta \\ 0 & 0 & -sin\Delta & cos\Delta \end{bmatrix} \tag{70}$$

so that

$$M_{R,D,L} = M_{R,L0} M_{D,L0} = m_{00} \begin{bmatrix} 1 & cos\theta & 0 & 0 \\ cos\theta & 1 & 0 & 0 \\ 0 & 0 & sin\theta cos\Delta & sin\theta sin\Delta \\ 0 & 0 & -sin\theta sin\Delta & sin\theta cos\Delta \end{bmatrix} \tag{71}$$

2.10 Serial Decompositions of Non-pure (depolarizing) MM

2.10.1 Generalized Polar decomposition (Lu-Chipman) : MM decomposed into a **serial combination** of **diattenuator**, **retarder** and **depolarizer with polarizance**.

$$M \equiv m_{00} \begin{bmatrix} 1 & D^T \\ P & m \end{bmatrix} = m_{00} \hat{M}_{\Delta P} M_R \hat{M}_D \tag{72}$$

$$\hat{M}_{\Delta P} = \begin{bmatrix} 1 & \vec{0}^T \\ P_\Delta & m_\Delta \end{bmatrix} \qquad M_R = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & m_R \end{bmatrix} \qquad \hat{M}_D = \begin{bmatrix} 1 & \vec{D}^T \\ \vec{D} & m_D(D) \end{bmatrix} \tag{73}$$

where **normalized** $\hat{M}_D$ is a **pure normal diattenuator**; M_R is a **pure retarder** and the **normalized** $\hat{M}_{\Delta P}$ is a **depolarizer with nonzero polarizance and zero diattenuation**. This decomposition has an **asymmetric** formulation, so that the order of the depolarizer and the pure component can be interchanged: in **reverse** decomposition, the depolarizer exhibits **nonzero diattenuation and zero polarizance**.

2.10.2 Symmetric Decomposition : any Mueller matrix $\mathbf{M}$ corresponds to one of the following categories

(i) **Type-I**: when $\mathbf{GM}^T\mathbf{GM}$ is diagonalizable and $\mathbf{M}$ can be written in the normal form $\mathbf{M}{=}\mathbf{M}_{J2}\mathbf{M}_{\Delta I}\mathbf{M}_{J1}$ where the **depolarizer** $\mathbf{M}_{\Delta I}$ is a diagonal matrix.

(ii) **Type-II**: when $\mathbf{GM}^T\mathbf{GM}$ is not diagonalizable and $\mathbf{M}$ can be written in the normal form $\mathbf{M}{=}\mathbf{M}_{J2}\mathbf{M}_{\Delta II}\mathbf{M}_{J1}$ where the **depolarizer** $\mathbf{M}_{\Delta II}$ is a non-diagonal matrix with all non-diagonal elements equal to zero except $(\mathbf{M}_{\Delta II})_{01}$.

Depending on whether the matrix $\mathbf{GM}^T\mathbf{GM}$ is diagonalizable or not (Type-I / Type-II), the depolarizer $\mathbf{M}_\Delta$ of the normal form $\mathbf{M}{=}\mathbf{M}_{J2}\mathbf{M}_\Delta\mathbf{M}_{J1}$ can be written:

Type-I:

$$M_{\Delta d} = diag(d_0, d_1, d_2, d_3) \qquad 0 \leq |d_3| \leq d_2 \leq d_1 \leq d_0 \qquad 0 < d_0 \tag{74}$$

being (d_0, d_1, d_2, d_3) the square root of the non-negative eigenvalues $(\rho_0, \rho_1, \rho_2, \rho_3)$ of $\mathbf{GM}^T\mathbf{GM}$.

Type-II:

$$M_{\Delta nd} = \begin{bmatrix} 2a_0 & -a_0 & 0 & 0 \\ a_0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{bmatrix} \qquad 0 \leq a_2 \leq a_0 \tag{75}$$

being (a_0, a_1, a_2, a_3) the square root of the non-negative eigenvalues $(\rho_0, \rho_1, \rho_2, \rho_3)$ of $\mathbf{GM}^T\mathbf{GM}$.

The values of the elements of the equivalent depolarizer M_Δ are restricted by:

- Up to 4 conditions derived from Cloude's:

$$d_0 \geq -d_1 - d_2 - d_3 \qquad d_0 \geq -d_1 + d_2 + d_3 \qquad d_0 \geq d_1 - d_2 + d_3 \qquad d_0 \geq d_1 + d_2 - d_3 \tag{76}$$

$$0 \leq a_2 \leq a_0 \tag{77}$$

- Passivity conditions (forward and reverse):

$$d_0 \leq 1 \qquad a_0 \leq 1/3 \tag{78}$$

which are particular forms of the general transmittance conditions for a passive MM:

$$m_{00}(1 + D) \leq 1 \qquad m_{00}(1 + P) \leq 1 \tag{79}$$

When **M** is a pure MM (necessarily Type-I) with P=D, the previous par of equations is reduced to a single one. For a Type-I MM, $d_0 \leq 1$ is necessary and sufficient; for a Type-II, $a_0 \leq 1/3$ is necessary but not sufficient.

By considering the polar decomposition of the pure components of the symmetric decomposition:

$$M = M_{J2} M_{\Delta} M_{J1} = M_{D2} M_{R2} M_{\Delta} M_{R1} M_{D1} \tag{80}$$

how to obtain the MM of the components? define **N=GM**^{T}**GM** and **N'=MGM**^{T}**G**:

(i) **N**≠0, **N'**≠0: both are different from 0 matrix. Both M_{D1} and M_{D2} are invertible ($D_1 < 1$, $D_2 < 1$) and so their eigenvectors $\mathbf{D}_1$ and $\mathbf{D}_2$ are calculated from **M:**

$$M^T G M G (1, D_1^T)^T = d_0^2 (1, D_1^T)^T \qquad (1, D_2^T)^T = MG(1, D_1^T)^T / [MG(1, D_1^T)^T]_0 \tag{81}$$

$$MGM^T G (1, D_2^T)^T = d_0^2 (1, D_2^T)^T \qquad (1, D_1^T)^T = M^T G(1, D_2^T)^T / [M^T G(1, D_2^T)^T]_0 \tag{82}$$

Normal (or homogeneous) diattenuators M_{D1} and M_{D2} have the form:

$$M_D = m_{00} \begin{bmatrix} 1 & \vec{D}^T \\ \vec{D} & m_D \end{bmatrix} \tag{83}$$

with maximum mean transmittances

$$m_{00}(M_{D1}) = \frac{1}{1 + D_1} \qquad\qquad m_{00}(M_{D2}) = \frac{1}{1 + D_2} \tag{84}$$

Their inverse matrices are given by:

$$M_{Di}^{-1} = \frac{1}{1 - D_i} \begin{bmatrix} 1 & -\vec{D_i}^T \\ -\vec{D_i} & m_{D_i} \end{bmatrix} = \frac{1 + D_i}{1 - D_i} G M_{Di} G \tag{85}$$

$$m_D = \sqrt{1 - D_i^2} I + \frac{1}{D_i^2} [1 - \sqrt{1 - D_i^2}] \vec{D_i} \times \vec{D_i}^T \qquad (i = 1, 2) \tag{86}$$

so that

$$M_{R2} M_{\Delta} M_{R1} = M_{D2}^{-1} M M_{D2}^{-1} \tag{87}$$

can be calculated from the SVD of $M_{D2}^{-1} M M_{D2}^{-1}$ (Type-I).

(ii) **N**≠0, **N'**=0: corresponds to a depolarizing analyzer:

$$M_{DA} = s_1^T s_2 \quad s_1^T G s_1 > 0 \quad s_2^T G s_2 = 0 \quad (\textit{partially and totally polarized}) \tag{88}$$

$$M_{DA} = M_{D2} M_{ID} M_{D1} \quad Type - I \tag{89}$$

where M_{D2} is a normal partial diattenuator, M_{D1} a normal total diattenuator and $M_{ID} = diag(d_0, 0, 0, 0)$ ideal depolarizer.

(iii) **N**=0, **N'**$\neq$0: corresponds to a depolarizing polarizer

$$M_{DP} = s_1^T s_2 \quad s_1^T G s_1 = 0 \quad s_2^T G s_2 > 0 \quad (totally \quad and \quad partially \quad polarized) \tag{90}$$

$$M_{DP} = M_{D2} M_{ID} M_{D1} \quad Type - I \tag{91}$$

where M_{D2} is a normal total diattenuator, M_{D1} a normal partial diattenuator and $M_{ID} = diag(d_0, 0, 0, 0)$ ideal depolarizer.

(iv) **N**=0, **N'**=0: corresponds to a diattenuator (typically, non-normal)

$$M = s_1^T s_2 \quad s_1^T G s_1 = 0 \quad s_2^T G s_2 = 0 \quad (totally \quad polarized) \tag{92}$$

2.11 Arrow Form of a MM

Consider SVD (being l_1, l_2, l_3 the singular values -nonnegative) of the submatrix **m** of **M**:

$$m = m_{RO} m_l m_{RI} \qquad ; \qquad m_{Ri}^{-1} = m_{Ri}^T \quad (i = I, O) \quad m_l = diag(l_1, l_2, l_3) \tag{93}$$

so that pure transparent retarders are built

$$M_{Ri} = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & m_{Ri} \end{bmatrix} \qquad ; \qquad i = I, O \tag{94}$$

and arrow matrix ($1 \geq l_1 \geq l_2 \geq l_3 \geq 0$)

$$M_A(M) = M_{RO}^T M M_{RI}^T = m_{00} \begin{bmatrix} 1 & D_A^T \\ P_A & m_A \end{bmatrix} = m_{00} \begin{bmatrix} 1 & D^T m_{RI}^T \\ m_{RO}^T P & diag(l_1, l_2, l_3) \end{bmatrix} \tag{95}$$

$$M_A(M) = m_{00} \begin{bmatrix} 1 & D^T m_{RI}^T \\ m_{RO}^T P & m_{RO}^T m m_{RI}^T \end{bmatrix} \tag{96}$$

Since $M_A(M)$ is invariant-equivalent to **M** and always contains six zero elements, its main attribute is that it synthesizes all physical information concerning the physical invariants of M in a particularly simple and condensed manner. Invariant quantities besides the azimuths and ellipticities of $D_A(M)$ and $P_A(M)$ vectors: mean transmittance m_{00}, $\mathbf{D}_A$, $\mathbf{P}_A$ and $\mathbf{L}= (l_1, l_2, l_3)^T$.

The degree of spherical purity

$$P_S(M) = P_S(M_A) = \sqrt{(l_1^2 + l_2^2 + l_3^2)/3} \tag{97}$$

Thus, the arrow decomposition of **M** is

$$M = M_{RO} M_A(M) M_{RI} \tag{98}$$

so

$$P_\Delta^2(M) = P_\Delta^2(M_A) = \frac{P^2 + D^2 + L^2}{3} \tag{99}$$

Due to its simple and compact form, the arrow form of a MM seems to be an adequate approach for simplifying the analysis of the transmitting, diattenuating, polarizing, and depolarizing properties of a sample whose MM has been experimentally measured. Physically invariant under arbitrary changes of generalized reference bases for the input and output polarization states quantities (m_{00}, D, P, P_S, P_Δ and IPPs) can be easily obtained from $M_A(M)$.

3 Indices of polarimetric purity

[**IPPs**] *J. J. Gil, Invariant quantities of a Mueller matrix under rotation and retarder transformations, Journal of the Optical Society of America A, vol. 33, 1 (2016)*

Physical quantities involved in a given Mueller matrix **M** (diattenuation, polarizance, depolarization index, IPPs,...) can be identified and properly defined by m_{kl} (**M** elements). Recall the generic Mueller matrix block form:

$$\hat{M} = M/m_{00} = \begin{bmatrix} 1 & D^T \\ P & m \end{bmatrix} \tag{100}$$

$$\boldsymbol{D} \equiv \frac{1}{m_{00}}(m_{01}, m_{02}, m_{03})^T \quad \boldsymbol{P} \equiv \frac{1}{m_{00}}(m_{10}, m_{20}, m_{30})^T \quad m = \frac{1}{m_{00}} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{101}$$

and the mean intensity coefficient (transmittance or reflectance for unpolarized input states) is given by m_{00}.

The degree of polarimetric purity (or, alternatively the polarization entropy H) is given by the **Depolarization index** P_Δ:

$$H = -\sum_{i=0}^{3}(\hat{\lambda}_i log_4 \hat{\lambda}_i) \qquad \hat{\lambda}_i = \frac{\lambda_i}{trH} \quad ; \quad P_\Delta = \frac{1}{3}\sqrt{D^2 + P^2 + 3P_S^2} \qquad 0 \le P_\Delta \le 1 \tag{102}$$

where

$$P_S = \frac{\sqrt{\sum_{k,l=1}^{3} m_{k,l}^2}}{\sqrt{3}} \qquad 0 \le P_S \le 1 \tag{103}$$

so that P_S provides a measure of the contribution to P_Δ that is not directly related to the diattenuation–polarizance properties.

	P_S	P_Δ	Comments
Maximum value	1	1	Pure retarder M_R
Minimum value	$1/\sqrt{3}$	1	Total polarimetric purity of the system
Zero	0	$\sqrt{D^2+P^2}/3$	When m_{ij} of sub-matrix are 0

MM type	P_Δ	Comments
Pure MM	1	System don't depolarize any totally polarized input
Nonpure/depolarizing MM	<1	System depolarize any totally polarized input

Both polarizance P and diattenuation D of a Mueller matrix have a **dual nature** depending on the direction of propagation of light (forward or reverse); D is both the diattenuation of **M** and the polarizance of the **reverse** Mueller matrix M^r (same interaction as **M** but interchanging input and output directions):

$$M^r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot M^T \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{104}$$

It is useful to consider the **degree of polarizance** P_p which provides a measure of the joint contribution of P and D to polarimetric purity:

$$P_P = \sqrt{\frac{P^2 + D^2}{2}} \qquad 0 \leq P_p \leq 1 \tag{105}$$

MM type	P_p	Comments
Pure polarizer	1	System polarizes any totally input
Nonpolarizing	0	Zero diattenuation D and zero polarizance P

The **linear diattenuation** D_L (non-negative parameter) and **circular diattenuation** D_C of **M**: degree of linear polarization and degree of circular polarization of Stokes $s_D = (1, D^T)^T = (1, D_1, D_2, D_3)^T$ are defined as:

$$D_L = \sqrt{D_1^2 + D_2^2} > 0 \qquad D_C = D_3 \quad , \quad -1 < D_C < 1 \tag{106}$$

when $D_C = 0$ the diattenuation only affects s_1 and s_2 Stokes components (linear ones); conversely, when $D_L = 0$ only s_3 is affected.

The **linear polarizance** P_L (non-negative parameter) and **circular polarizance** P_C of **M**: degree of linear polarization and degree of circular polarization of Stokes $s_P = M s_u = M(1, 0^T)^T$, for a polarizance vector $P = (P_1, P_2, P_3)^T$, are defined as:

$$P_L = \sqrt{P_1^2 + P_2^2} > 0 \qquad P_C = P_3 \quad , \quad -1 < P_C < 1 \tag{107}$$

when $P_C = 0$ the polarizance only affects s_1 and s_2 s_P Stokes components (linear ones); conversely, when $P_L = 0$ only s_3 is affected.

Consider the **Singular Value Decomposition** of 3x3 **M** sub-matrix

$$m = m_{RO} m_A m_{RI} \tag{108}$$

with m_{RI} and m_{RO} being proper orthogonal matrices (detm_{RI}=detm_{RO}=+1) and m_A being the diagonal matrix whose entries are defined as follows from the singular values of m

$$m_A = diag(a_1, a_2, \epsilon \cdot a_3) \qquad a_1 \geq a_2 \geq a_3 \tag{109}$$

$$\epsilon = det(M)/|det(M)| \tag{110}$$

3.1 Arrow decomposition of M

$$M = M_{RO} M_A M_{RI} = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & m_{RO} \end{bmatrix} M_A \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & m_{RI} \end{bmatrix} \tag{111}$$

$$M_A = m_{00} \begin{bmatrix} 1 & D_A^T \\ P_A & m_A \end{bmatrix} = M_{RO}^T M_A M_{RI}^T = m_{00} \begin{bmatrix} 1 & D^T m_{RI}^T \\ m_{RO}^T P & m_{RO}^T m m_{RI}^T \end{bmatrix} \tag{112}$$

Note that

$$D_A = D \qquad P_A = P \qquad P_S(M) = P_S(M_A) = \sqrt{(a_1^2 + a_2^2 + a_3^2)/3} \tag{113}$$

Recall that **M** has a biunivocal relation with its associated covariance matrix **H**, which can be submitted to the **Spectral** or **Cloude's decomposition**.

3.2 Spectral / Cloude's decomposition

$$H = (trH) \sum_{i=1}^{4} \hat{\lambda}_i H_{Ji} \tag{114}$$

$$H_{Ji} = u_i \otimes u_i^\dagger \qquad \hat{\lambda}_i = \lambda_i/(trH) = \lambda_i/m_{00} \tag{115}$$

being λ_i the ordered eigenvalues ($\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$) and u_i the corresponding eigenvectors of **H**. The subscript J stresses out that they have only **one non zero value** and are associated with **nondepolarizing or pure** Mueller matrices M_{Ji}. The **spectral decomposition of M**:

$$M = m_{00} \sum_{i=1}^{4} \hat{\lambda}_i \hat{M}_{Ji} \qquad M_{Ji} \quad pure \quad matrices \tag{116}$$

$$H(M)=\frac{1}{4}\begin{bmatrix} \begin{matrix}m_{00}+m_{01}\\+m_{10}+m_{11}\end{matrix} & \begin{matrix}m_{02}+m_{12}\\+i(m_{03}+m_{13})\end{matrix} & \begin{matrix}m_{20}+m_{21}\\-i(m_{30}+m_{31})\end{matrix} & \begin{matrix}m_{22}+m_{33}\\+i(m_{23}-m_{32})\end{matrix} \\ \\ \begin{matrix}m_{02}+m_{12}\\-i(m_{03}+m_{13})\end{matrix} & \begin{matrix}m_{00}-m_{01}\\+m_{10}-m_{11}\end{matrix} & \begin{matrix}m_{22}-m_{33}\\-i(m_{23}+m_{32})\end{matrix} & \begin{matrix}m_{20}-m_{21}\\-i(m_{30}-m_{31})\end{matrix} \\ \\ \begin{matrix}m_{20}+m_{21}\\+i(m_{30}+m_{31})\end{matrix} & \begin{matrix}m_{22}-m_{33}\\+i(m_{23}+m_{32})\end{matrix} & \begin{matrix}m_{00}-m_{01}\\-m_{10}-m_{11}\end{matrix} & \begin{matrix}m_{02}-m_{12}\\+i(m_{03}-m_{13})\end{matrix} \\ \\ \begin{matrix}m_{22}+m_{33}\\-i(m_{23}-m_{32})\end{matrix} & \begin{matrix}m_{20}-m_{21}\\+i(m_{30}-m_{31})\end{matrix} & \begin{matrix}m_{02}-m_{12}\\-i(m_{03}-m_{13})\end{matrix} & \begin{matrix}m_{00}-m_{01}\\-m_{10}+m_{11}\end{matrix} \end{bmatrix} \tag{117}$$

Any **pure MM** satisfies:

$$tr(M^TM)=4m_{00} \qquad \leftrightarrow \qquad \lambda_2=\lambda_3=\lambda_4=0 \qquad \leftrightarrow \qquad P_\Delta=1 \tag{118}$$

Any **ideal depolarizer MM** satisfies:

$$M_{\Delta 0}=diag(m_{00},0,0,0) \qquad \leftrightarrow \qquad \lambda_1=\lambda_2=\lambda_3=\lambda_4=m_{00}/4 \qquad \leftrightarrow \qquad P_\Delta=0 \tag{119}$$

General **depolarizing MM** correspond to $0<P_\Delta<1$ where $\hat{\lambda}_i$ (relative weights in spectral decomposition) take arbitrary values. So that depolarization properties are intrinsically related to the structure of normalized eigenvalues $\hat{\lambda}_i$ of **H**.

A complete description of the polarimetric purity (lack of polarimetric randomness) of a medium (matrix H) requires considering 3 non-dimensional and independent parameters derived from $\hat{\lambda}_i$: IPPs. They contain all the information about the quality of the polarimetric purity in terms of relative measures of the weights of the canonical parallel components of H. Note that P, D and P_S give specific knowledge of the sources of purity but are not sufficient to derive $\hat{\lambda}_i$. **IPPs are invariant: they remain unchanged under variations of the laboratory reference coordinate system.** We define ($\lambda_1\geq\lambda_2\geq\lambda_3\geq\lambda_4\geq 0$) IPPs and depolarization index:

$$P_1=\hat{\lambda}_1-\hat{\lambda}_2 \qquad P_2=\hat{\lambda}_1+\hat{\lambda}_2-2\hat{\lambda}_3 \qquad P_3=\hat{\lambda}_1+\hat{\lambda}_2+\hat{\lambda}_3-3\hat{\lambda}_4 \tag{120}$$

$$P_\Delta=\frac{1}{\sqrt{3}}\sqrt{2P_1^2+\frac{2}{3}P_2^2+\frac{1}{3}P_3^2} \qquad 0\leq P_\Delta\leq 1 \tag{121}$$

IPPs are restricted to $0 \leq P_1 \leq P_2 \leq P_3 \leq 1$:

	Condition	Consequence	Description
If	$P_1 = 1$	$P_2 = P_3 = P_\Delta = 1$	**Pure system (non-depolarizing)**
If	$P_3 = 0$	$P_1 = P_2 = P_\Delta = 0$	**Ideal depolarizer (equiprobable mixture)**

$\bigstar_1$ IPPs provide complete information about polarimetric purity in terms of the weights of the spectral components of **M** (its parallel structure) and constitute a representation complementary to D, P and P_S because IPPs are insensitive to the nature of the medium with respect to diattenuation, polarizance and retardance. The nature of the IPPs lies in the statement that any depolarizer response can be synthesized as an incoherent sum of four pure non-depolarizing components whose relative statistical weight is performed by IPPs. In this way, we can discriminate different types of depolarizers by looking at the statistical weight of each pure component. **The sole knowledge of IPPs does not imply necessarily the knowledge of P, D and P_S; nevertheless, the set of five quantities (IPPs, P and D) is sufficient to calculate P_Δ and P_S.**

$\bigstar_2$ The polarimetric purity of a light beam with fixed direction of propagation [2D polarized light, represented by a 2D coherency matrix] is given by an **only** index of purity, namely the **degree of polarization**.

$\bigstar_3$ For polarization states with fluctuating direction of propagation [3D polarized light, represented by a 3D coherency matrix], a **pair** of indices of purity are required to characterize completely the polarimetric purity: the **degree of directionallity** P_2 and the **degree of polarization** P_1 of the average 2D light beam.

$\bigstar_4$ Therefore, the indices P_1, P_2 and P_3 of a MM (which always has an associated 4D coherency matrix) arise as a natural extension of the concept of **indices of purity** defined for polarized light.

3.3 Characteristic (trivial) decomposition

Particularly, IPPs can be physically interpreted from the so-called **characteristic (or trivial) MM decomposition**:

$$M = m_{00}\left[P_1\hat{M}_J(\hat{H}_J) + (P_2 - P_1)\hat{M}_2(\hat{H}_2) + (P_3 - P_2)\hat{M}_3(\hat{H}_3) + (1 - P_3)\hat{M}_{\Delta 0}(\hat{H_{\Delta 0}})\right] \quad (122)$$

- $M_J = m_{00}\hat{M}_J$ is the **characteristic pure component** whose associated covariance matrix $H_J = m_{00}(u_1 \otimes u_1^\dagger)$ is defined from the eigenvector u_1 with the **largest** eigenvalue of the covariance matrix **H** associated with **M**. **Relative weight of M_J is given by P_1.**

• $M_2 = m_{00}\hat{M}_2$ represents a **2D depolarizer** whose associated covariance matrix is defined $H_2 = \frac{1}{2}m_{00}\sum_{i=1}^{2}(u_i \otimes u_i^\dagger)$; it has 2 (nonzero) eigenvalues and 2 zero eigenvalues, in such a manner that M_2 is constituted by an **equiprobable mixture of two pure components** (namely, the first two spectral components). **Relative weight of M_2 is given by $(P_2 - P_1)$.**

• $M_3 = m_{00}\hat{M}_3$ represents a **3D depolarizer** whose associated covariance matrix is defined $H_3 = \frac{1}{3}m_{00}\sum_{i=1}^{3}(u_i \otimes u_i^\dagger)$; it has 3 (nonzero) eigenvalues and 1 zero eigenvalues, in such a manner that M_3 is constituted by an **equiprobable mixture of three pure components** (namely, the first three spectral components). **Relative weight of M_3 is given by $(P_3 - P_2)$.**

• $M_{\Delta 0} = m_{00} diag(1, 0, 0, 0)$ represents an **ideal depolarizer** (4D depolarizer) whose associated covariance matrix is defined $H_{\Delta 0} = \frac{1}{4}m_{00}I$ (being **I** the identity matrix); it has 4 (nonzero) eigenvalues in such a manner that $M_{\Delta 0}$ is constituted by an **equiprobable mixture of four pure components** (namely, the four spectral components). **Relative weight of $M_{\Delta 0}$ is given by $(1 - P_3)$.**

4 Lu-Chipman decomposition

[Lu-Chipman] *S.-Y. Lu y R. A. Chipman, Interpretation of Mueller matrices based on polar decomposition, Journal of the Optical Society of America A 13, 1106–1113 (1996).*

Polarimetric characteristics of a given sample are encoded in its Mueller matrix in a complex way. This information can be synthesized in a product of three pure MMs, which are function of well-defined polarimetric magnitudes for an easier physical interpretation, by employing the so-called Lu-Chipman decomposition:

$$\hat{M} \equiv m_{00} M_\Delta M_R M_D \tag{123}$$

where m_{00} is related with the irradiance of the sample and those pure matrices are defined as depolarizers (M_Δ), pure retarders (M_R) and pure diattenuators (M_D).

Diattenuator matrix is defined as:

$$\hat{D} = m_{00} \begin{bmatrix} 1 & D^T \\ D & m_D \end{bmatrix} \tag{124}$$

where

$$\boxed{m_D = aI_3 + bDD^T} \qquad a = \sqrt{1-D^2} \qquad b = \frac{1-a}{D^2} \tag{125}$$

$$\hat{M} M_D^{-1} = M' = M_\Delta M_R \tag{126}$$

(From now on, m_{00} is removed); So we compute $M' = \hat{M} M_D^{-1}$ to remove the diattenuator:

$$M' = \hat{M} M_D^{-1} = \begin{bmatrix} 1 & D^T \\ P & m \end{bmatrix} M_D^{-1} = \begin{bmatrix} 1 & \vec{0}^T \\ \frac{P-mD}{a^2} & m' \end{bmatrix} \tag{127}$$

On the other side of equation:

$$M' = M_\Delta M_R = \begin{bmatrix} 1 & \vec{0}^T \\ P_\Delta & m_\Delta \end{bmatrix} \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & m_R \end{bmatrix} = \begin{bmatrix} 1 & \vec{0}^T \\ P_\Delta & m_\Delta m_R \end{bmatrix} \tag{128}$$

So we establish the following relationships:

$$P_\Delta = \frac{P_m D}{1-D^2} \qquad\qquad m' = m_\Delta m_R \tag{129}$$

A pure retarder does not change the polarization degree nor the intensity (s_0) of the incident light but the state of polarization: it rotates the incoming state. Also note that m_Δ is defined as symmetric matrix. Due to that,

$$m_R^{-1} = m_R^T \tag{130}$$

$$m'(m')^T = m_\Delta m_R (m_\Delta m_R)^T = m_\Delta^2 \tag{131}$$

Being $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$ and $\sqrt{\lambda_3}$ the m_Δ eigenvalues, we define the following parameters:

$$k_1 = \sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3} \qquad k_2 = \sqrt{\lambda_1\lambda_2} + \sqrt{\lambda_2\lambda_3} + \sqrt{\lambda_3\lambda_1} \qquad k_3 = \sqrt{\lambda_1\lambda_2\lambda_3} \tag{132}$$

Then, m_Δ submatrix (sign defined by $|m^{'}|$):

$$\boxed{m_\Delta = \pm(m^{'}(m^{'})^T + k_2 I_3)^{-1}(k_1 m^{'}(m^{'})^T + k_3 I_3)} \tag{133}$$

Finally, $m_R = m_\Delta^{-1} m^{'}$ submatrix:

$$\boxed{m_R = \pm(k_1 m^{'}(m^{'})^T + k_3 I_3)^{-1}(m^{'}(m^{'})^T + k_2 m^{'})} \tag{134}$$

5 Inverse decomposition

[Inverse] *R. Ossikovski, A. De Martino, y S. Guyot, Forward and reverse product decompositions of depolarizing Mueller matrices, Optics Letters 32, 689-691 (2007).*

Lu-Chipman analog MM decomposition but in reverse order.

$$\hat{M} = M_D M_R M_\Delta \tag{135}$$

** A MM can have Lu-Chipman decomposition but not an Inverse one $->$ "J. Morio and F. Goudail, Influence of the order of diattenuator, retarder, and polarizer in polar decomposition of Mueller matrices, Optics Letters 29, 2234-2236 (2004)."*

To resolve such problem, if we define the depolarized and diattenuator the following way, we secure there's always an inverse decomposition:

$$M_D = \begin{bmatrix} 1 & P^T \\ P & m_P \end{bmatrix} \qquad M_\Delta = \begin{bmatrix} 1 & D_\Delta \\ \vec{0}^T & m_\Delta \end{bmatrix} \tag{136}$$

where P is shown in (2) and m_P is analogous to m_D in Lu-Chipman:

$$\boxed{m_P = aI_3 + bPP^T} \qquad a = \sqrt{1-P^2} \qquad b = \frac{1-a}{P^2} \tag{137}$$

Note that we have a Depolarizer M_Δ without Polarizance P but with Diattenuation D_Δ :

$$D_\Delta = \frac{D - mP}{1 - P^2} \tag{138}$$

$$\boxed{m_\Delta = \pm((m')^T m' + k_2 I_3)^{-1}(k_1 (m')^T m' + k_3 I_3)} \tag{139}$$

$$\boxed{m_R = m' m_\Delta^{-1}} \tag{140}$$

6 Polar decompositions: Forward and Reverse families

[Polar decompositions] *R. Ossikovski, A. De Martino, y S. Guyot, Forward and reverse product decompositions of depolarizing Mueller matrices, Optics Letters 32, 689-691 (2007).*

One of the specific features of the three-factor polar decomposition is the existence of six possible products of M_D, M_R, and M_Δ due to the non-commutativity of matrix multiplication. These six products can be grouped into two families, the members within each family being equivalent to one another under the following orthogonal transformations generated by the retardance matrix M_R:

$$M_\Delta^{'} = M_R^T M_\Delta M_R \qquad M_\Delta^{''} = M_R M_\Delta M_R^T \tag{141}$$

$$M_D^{'} = M_R M_D M_R^T \qquad M_D^{''} = M_R^T M_D M_R \tag{142}$$

Indeed, Lu-Chipman decomposition generates Forward family and Inverse the Reverse one (we consider them as their canonical form):

$$\odot\, Forward: \qquad M = M_\Delta M_R M_D \qquad M = M_\Delta M_D^{'} M_R \qquad M = M_R M_\Delta^{'} M_D \tag{143}$$

$$\odot\, Reverse: \qquad M = M_D M_R M_\Delta \qquad M = M_R M_D^{''} M_\Delta \qquad M = M_D M_\Delta^{''} M_R \tag{144}$$

As can be seen from the corresponding normal forms, the Forward family physically corresponds to the depolarizer's being at the end of the optical assembly, while the Reverse family corresponds to the depolarizer's being.

Polar decompositions fix $D > 0$ and can happen $m^{'}(m)^T$ being non-diagonalizable.

7 Arrow decomposition

[**Arrow**] *J. J. Gil, Transmittance constraints in serial decompositions of depolarizing Mueller matrices: the arrow form of a Mueller matrix, Journal of the Optical Society of America A 30, 701-707 (2013).*

The aim is to remove any retardance component of the original MM in order to obtain an equivalent form which contains more synthesized information: $M_A(M)$. For such aim, we compute the Singular Value Decomposition (SVD, it's not unique) of the m submatrix:

$$m = m_{RO} m_A m_{RI}^T \qquad m_{Ri}^{-1} = m_{Ri}^T \quad (i = I, O) \quad , \quad m_A = diag(a_1, a_2, a_3) \tag{145}$$

where a_1, a_2, a_3 ($a_1 \geq a_2 \geq a_3 \geq 0$) are the singular values of m. We define the Mueller orthogonal M_{Ri} (pure retarder):

$$M_{Ri} = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & m_{Ri} \end{bmatrix} \qquad i = I, O \tag{146}$$

and eventually we define the Arrow matrix $M_A(M)$:

$$M_A(M) \equiv M_{RO}^T M M_{RI}^T = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & m_{RO}^T \end{bmatrix} \begin{bmatrix} 1 & D^T \\ P & m \end{bmatrix} \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & m_{RI}^T \end{bmatrix} = \begin{bmatrix} 1 & (m_{RI} D)^T \\ m_{RO}^T P & m_{Ri} \end{bmatrix} \tag{147}$$

$$\boxed{M_A(M) = \begin{bmatrix} 1 & D_A^T \\ P_A & m_A \end{bmatrix}} \tag{148}$$

We observe that $M_A(M)$ is invariant-equivalent to M. Since $M_A(M)$ always contains ten nonzero elements, the main attribute of $M_A(M)$ is that it synthesizes all physical information concerning the physical invariants of M ($m_{00}, D, P, P_1, P_2, P_3$) in a particularly simple and condensed manner.

The mean transmittance m_{00} together with the three vectors $D_A, P_A, L = (a_1, a_2, a_3)$ determine all the said parameters besides the respective azimuth and ellipticity of the transformed diattenuation vector $D_A(M)$ and of the transformed polarizance vector $P_A(M)$. Vectors D_A and P_A are obtained through respective proper rotations of D and P so that their modules remain unchanged $D_A = D$, $P_A = P$.

Given a Mueller matrix M, its degree of polarimetric purity P_Δ can be expressed as a quadratic composition of three different contributions, namely the diattenuation D, the polarizance P, and the degree of spherical purity P_S:

$$P_S \equiv \frac{||m||_2}{\sqrt{3}} \qquad 0 \leq P_S \leq 1 \tag{149}$$

$$P_{\Delta}^2 = \frac{1}{3}P^2 + \frac{1}{3}D^2 + P_S^2 \tag{150}$$

For $M_A(M)$ we have

$$P_S(M_A) = \frac{|L|}{\sqrt{3}} \tag{151}$$

$$P_{\Delta}^2(M) = P_{\Delta}^2(M_A) = \frac{1}{3}(P^2 + D^2 + L^2) \tag{152}$$

Thus, due to its simple and compact form, the arrow form of a Muller matrix seems to be an adequate approach for simplifying the analysis of the transmitting, diattenuating, polarizing, and depolarizing properties of a sample whose Mueller matrix has been experimentally measured. In particular, such physical quantities of M as $m_{00}, D, P, P_S, P_{\Delta}$ can be easily obtained from $M_A(M)$. The indices of polarimetric purity P_1, P_2, P_3 of M, which (like $m_{00}, D, P, P_S, P_{\Delta}$) are invariant under arbitrary changes of the generalized reference bases for the representation of the input and output polarization states can also be calculated from $M_A(M)$.

8 Symmetric decomposition

[**Symmetric**] *R. Ossikovski, Analysis of depolarizing Mueller matrices through a symmetric decomposition, Journal of the Optical Society of America A 26, 1109-1118 (2009).*

$$M = M_{D2} M_{R2} M_{\Delta d} M_{R1}^T M_{D1} \tag{153}$$

being M_{D1}, M_{D2} and M_{R1}, M_{R2} pairs of diattenuators and retarders, respectively, and $M_{\Delta d} = diag(d_0, d_1, d_2, d_3)$ the diagonal depolarizer (this factorization is one of the 4 possible by placing the depolarizer in the middle: this decomposition lets us to describe such depolarizer without any diatenuation / retardance presence; however, the other 3 are equivalent to it).

$$M_{Di} = T_{ui} \begin{bmatrix} 1 & D_i^T \\ D_i & m_{D_i} \end{bmatrix} \qquad T_{ui} = \frac{1}{\sqrt{1 - D_i^2}} \quad , \quad m_D = aI_3 + bDD^T \tag{154}$$

$$M_{D_i}^{-1} = G M_{D_i} G \qquad G = diag(1, -1, -1, -1) = Minkowski \tag{155}$$

$$M M_{D1}^{-1} = (MG)(M_{D1}G) = M_{D2}(M_{R2} M_{\Delta d} M_{R1}^T) = M_{D2} M' \tag{156}$$

Due to the fact that we're dealing with a diagonal depolarizer and two pure retarders, M' has null polarizance and diattenuation:

$$M' = M_{R2} M_{\Delta d} M_{R1}^T = \begin{bmatrix} d_0 & \vec{0}^T \\ \vec{0} & m' \end{bmatrix} \quad , \quad d_0 = (M')_{00} \tag{157}$$

By comparing the 1st column of $(MG)(M_{D1}G)$ with the 1st column of $M_{D2}M'$ we obtain:

$$(MG)T_{u1}S_1 = d_0 T_{u2} S_2 \quad , \quad S_1 = (1, D_1)^T \quad , \quad S_2 = (1, D_2)^T \tag{158}$$

Doing the same computation but starting with $M^T M_{D2}^{-1}$ we obtain:

$$(M^T G) T_{u2} S_2 = d_0 T_{u1} S_1 \tag{159}$$

By isolating S_1 and S_2:

$$(M^T G M G) S_1 = d_0^2 S_1 \quad , \quad (M G M^T) S_2 = d_0^2 S_2 \tag{160}$$

We immediately determine D_1 and D_2 so we build M_{D1} and M_{D2} and compute the following $M' = M_{D2}^{-1} M M_{D1}^{-1}$.

Finally, to isolate the retarders we do a SVD of $m^{'}$. Due to the ambiguity that this procedure generates, the 'minimum retardance principle' is proposed: chooses M_{R1} with the minimum retardance value (equivalent to $Tr(m_{R1})$ maxima).

$$m^{'} = m_{R2} m_{\Delta d} m_{R1}^{T} \tag{161}$$

where $m_{\Delta d} = diag(d_1, d_2, d_3)$ and $m_{Ri}^{T} = m_{Ri}^{-1}$, $i = (1, 2)$.

$$M_{R1} = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & m_{R1} \end{bmatrix} \quad , \quad M_{R2} = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & m_{R2} \end{bmatrix} \quad , \quad M_{\Delta d} = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & m_{\Delta d} \end{bmatrix} \tag{162}$$

Symmetric decomposition depends on the existence and how the solutions of the previous mentioned equations $(M^T G M G) S_1 = d_0^2 S_1$ and $(M G M^T) S_2 = d_0^2 S_2$ are.

9 Relationship between Lu-Chipman and Symmetric decomposition

In both decomposition, once we remove the system diattenuation, we define a $M^{'}$ submatrix, $m^{'}$. Then $m^{'}$ contains information about retardance and depolarization. So it is legit to assume that $m^{'}$ represents the same physical system and, therefore, its expression in Lu-Chipman's and Symmetric decomposition must coincide.

$$m^{'}_{LC} = m^{'}_{Sym} \quad \leftrightarrow \quad m_\Delta m_R = m_{R2} m_{\Delta d} m_{R1}^T \quad \leftrightarrow \quad m_\Delta = m_{R2} m_{\Delta d} m_{R1}^T m_R^T \tag{163}$$

By m_Δ and $m_{\Delta d}$ symmetry:

$$m_\Delta^2 = m_\Delta (m_\Delta)^T = (m_{R2} m_{\Delta d} m_{R1}^T m_R^T)(m_{R2} m_{\Delta d} m_{R1}^T m_R^T)^T = \tag{164}$$

$$= (m_{R2} m_{\Delta d}^2 m_{R2}^T) m_{R1}^T m_{R1} m_R^T m_R = m_{R2} m_{\Delta d}^2 m_{R2}^T = \tag{165}$$

$$= m_{R2} m_{\Delta d} m_{R2}^T m_{R2} m_{\Delta d} m_{R2}^T = (m_{R2} m_{\Delta d} m_{R2}^T)^2 \tag{166}$$

$$\boxed{m_\Delta = m_{R2} m_{\Delta d} m_{R2}^T} \tag{167}$$

In Lu-Chipman decomposition, is not necessary m_Δ to be diagonal but this relationship gives us the explicit diagonalization by means of Symmetric decomposition obtained retarders. This is coherent due to the fact that both depolarizators must characterize the same physical system depolarization. In addition, Lu-Chipman depolarizer is the rotation of its analogous in Symmetric decomposition. The second relationship:

$$\boxed{m_R = m_{R2} m_{R1}^T} \tag{168}$$

defines that the Lu-Chipman retarder matrix is equivalent to the product of both retarders (input and output) in Symmetric decomposition. So that, if we know the Symmetric decomposition of a given Mueller matrix we can compute the Lu-Chipman one. This also permits to secure if the obtained retarders by SVD coincide with the physical solution: if we can recover them by using the last m_R relationship.

9.1 Relationship between Inverse and Symmetric decomposition

$$m^{'}_{In} = m^{'}_{Sym} \quad \leftrightarrow \quad m^{'}_{In} = m_{R,In} m_{\Delta,In} \tag{169}$$

$$\boxed{m_{\Delta,In} = m_{R1} m_{\Delta d} m_{R1}^T} \qquad \boxed{m_{R,In} = m_{R2} m_{R1}^T} \qquad \boxed{m_{R,In} = m_R} \tag{170}$$

9.2 Relationship between Lu-Chipman and Inverse decomposition

$$m_{\Delta,In} = m_{R1} m_{R2}^T m_\Delta m_{R2} m_{R1}^T = (m_{R2} m_{R1}^T)^T m_\Delta (m_{R2} m_{R1}^T) \tag{171}$$

$$\boxed{m_{\Delta,In} = m_R^T m_\Delta m_R} \tag{172}$$

10 Canonical forms of depolarizing MM

[**Canonical**] *R. Ossikovski, Canonical forms of depolarizing Mueller matrices, Journal of the Optical Society of America A 27, 123-130 (2010).*

It is shown that any depolarizing MM can be reduced, through a product decomposition, to one of a total of two canonical depolarizer forms, a diagonal and a non-diagonal one: depolarizing MM can be divided into Stokes diagonalizable and Stokes non-diagonalizable ones.

MM	Product of:	Canonical forms	Decomposed
Non-depolarizing	Diattenuator & Retarder	Well-known	Euclidian space
Depolarizing	Diattenuator, Retarder & Depolarizer	5 depolarizer*	Minkowski space

* Depolarizer MM: Normal, type-II, G-polar, polar (forward) and reverse. But **all depolarizer MM can be reduced to only 2 canonical forms: diagonal and non-diagonal.**

Both canonical (diagonal / non-diagonal) forms depend only on the eigenvalues of a special matrix, the matrix **N**, constructed out of the original Mueller matrix M.

Depolarizing MM belong to Minkowski space: we define the Minkowski metric **G**=diag(1,-1,-1,-1). Being **A** a matrix:

the G-adjoint matrix of **A**

$$A^{+} = GA^{T}G \tag{173}$$

the G-self-adjoint (characteristic property of the general depolarizer)

$$H^{+} = H \qquad \to \qquad GH^{T}G = H \tag{174}$$

the G-unitary (characteristic property for a (invertible) non-depolarizing MM)

$$U^{+} = U^{-1} \qquad \to \qquad U^{T}GU = G \tag{175}$$

Matrix **A** can be factored in accordance to G-polar decomposition:

$$A = UH \qquad\qquad det(U) = +1 \tag{176}$$

if **A** is a **MM**, **U** allows for a physical interpretation:

$$M^{T}_{nondepolarizing}GM_{nondepolarizing} = \sqrt{det(M_{nondepolarizing})}G \tag{177}$$

by comparison: **U** can be identified as an invertible $\mathbf{M}_{nondepolarizing}$ normalized to det($\mathbf{M}_{nondepolarizing}$)=+1.

$\bullet_1$ We can also interpret the G-polar decomposition **M=UH** as the factorization of **M** into a non-depolarizing **U** and a depolarizing factor **H** (general depolarizer): the general depolarizer **H** can be factored into **U'** (non-depolarizing MM) and **H'** (reduced depolarizer, takes 2 forms depending on **N** is diagonalizable or not)

$$H = U'^{-1} H' U' \tag{178}$$

$$H_1' = \begin{bmatrix} \sqrt{\rho_0} & 0 & 0 & 0 \\ 0 & \sqrt{\rho_1} & 0 & 0 \\ 0 & 0 & \sqrt{\rho_2} & 0 \\ 0 & 0 & 0 & \epsilon\sqrt{\rho_3} \end{bmatrix} \qquad H_2' = \begin{bmatrix} \sqrt{\rho_0}+\alpha & -\alpha & 0 & 0 \\ \alpha & \sqrt{\rho_0}-\alpha & 0 & 0 \\ 0 & 0 & \sqrt{\rho_2} & 0 \\ 0 & 0 & 0 & \sqrt{\rho_2} \end{bmatrix} \tag{179}$$

$$\rho_i \quad , \quad i = 0,1,2,3 \quad Real \quad eigenvalues \quad of \quad N \tag{180}$$

$$\rho_0 \geq \rho_i \geq 0 \quad , \quad i = 1,2,3 \tag{181}$$

$$\epsilon = \pm 1 \quad (sign \quad of \quad det(M)) \tag{182}$$

$$\alpha > 0 \tag{183}$$

auxiliary matrix (can be diagonalizable or not):

$$N = M^{+}M = GM^{T}GM \tag{184}$$

* In case of H_2', the eigenvalues ρ_i are doubly degenerate, i.e., $\rho_0 = \rho_1$ and $\rho_2 = \rho_3$ since **N** is not diagonalizable.

$\bullet_2$ By factorizing a depolarizing **M** according to the normal form product decomposition:

$$M = L_1 K L_2 \qquad L_1, L_2 \quad invertible \quad non-depolarizing \quad obeying \quad U^{T}GU = G \tag{185}$$

depolarizing matrix factor **K** can be reduced either to

$$K_1 = H_1' \qquad or \qquad K_2 = \begin{bmatrix} \sqrt{n_0} & (\rho_0 - n_0)/\sqrt{n_0} & 0 & 0 \\ 0 & \rho_0/\sqrt{n_0} & 0 & 0 \\ 0 & 0 & \sqrt{\rho_2} & 0 \\ 0 & 0 & 0 & \sqrt{\rho_2} \end{bmatrix} \tag{186}$$

$$\rho_i \quad , \quad i = 0,1,2,3 \quad Real \quad eigenvalues \quad of \quad N \tag{187}$$

$$n_0 > \rho_0 \geq \rho_i \geq 0 \quad , \quad i = 1,2,3 \tag{188}$$

* In case of K_2, the eigenvalues ρ_i are doubly degenerate, i.e., $\rho_0 = \rho_1$ and $\rho_2 = \rho_3$ since **N** is not diagonalizable.

The diagonal depolarizer forms of two different algebraic approaches coincide ($\bullet_1$ and $\bullet_2$):

$$M_{\Delta d} \equiv K_1 \equiv H_1' \tag{189}$$

Non-diagonal forms are apparently different ($K_2 \neq H_2'$) but can be reduced to a common canonical non-diagonal form that depends only on ρ_i eigenvalues of $\mathbf{N}$:

$$M_{\Delta nondepolarizing} = M_D(a_2) K_2' M_D^{-1}(a_2) = \begin{bmatrix} 2\sqrt{\rho_0} & -\sqrt{\rho_0} & 0 & 0 \\ \sqrt{\rho_0} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\rho_2} & 0 \\ 0 & 0 & 0 & \sqrt{\rho_2} \end{bmatrix} \tag{190}$$

★ Any depolarizer MM can be reduced to two canonical forms: diagonal and non-diagonal

$$M_{\Delta d} = \begin{bmatrix} \sqrt{\rho_0} & 0 & 0 & 0 \\ 0 & \sqrt{\rho_1} & 0 & 0 \\ 0 & 0 & \sqrt{\rho_2} & 0 \\ 0 & 0 & 0 & \epsilon\sqrt{\rho_3} \end{bmatrix} \quad M_{\Delta nd} = \begin{bmatrix} 2\sqrt{\rho_0} & -\sqrt{\rho_0} & 0 & 0 \\ \sqrt{\rho_0} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\rho_2} & 0 \\ 0 & 0 & 0 & \sqrt{\rho_2} \end{bmatrix} \tag{191}$$