Cantor Set

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1 Introduction

Georg Cantor's (1845-1918) achievements revolutionized the foundation of mathematics with set theory. Set theory is now considered so fundamental that it seems to border on the obvious but at its introduction it was controversial and revolutionary. The controversial element centered around the problem of whether infinity was a potentiality or could be achieved. Before Cantor it was generally felt that infinity as an actuality did not make sense; one could only speak of a variable increasing without bound as that variable going to infinity. That is to say, it was felt that $n \to \infty$ makes sense but $n = \infty$ does not. Cantor not only found a way to make sense about an actual infinity, as opposed to potential but showed that there are different orders of infinity[1].

Cantor's Theorem is a fundamental result that states that, for any set A, the set of all subsets of A (the power set of A, denoted by $P(A)$) has a strictly greater cardinality than itself. For finite sets, Cantor's theorem can be seen to be true by simple enumeration of the number of subsets.

2 1.6.1

Consider $\tan(\pi x - \frac{\pi}{2})$, this transforms the tangent function to be from (0,1), now there is a bijection from (0,1) to $\mathbb{R}$. Therefore if $\mathbb{R}$ is uncountable then the interval (0,1) must also be uncountable.

3 1.6.2

Consider the proof of Theorem 1.6.1 which states, the open interval (0,1) $= x \subseteq \mathbb{R} : 0 < x < 1$ is uncountable. With this proof, you could start with a contradiction and assume that there does exists a function $f : \mathbb{N} \to (0,1)$ that is 1-1 and onto. For each $m \in \mathbb{N}, f(m)$ is a real number between 0 and 1, and we represent it using the decimal notation $f(m) = .a_{m1}a_{m2}a_{m3}...$ The key assumption about this correspondence is that every real number in (0,1) is assumed to appear somewhere on the list. Now,define a real number $x \in (0,1)$ with the decimal expansion $x = .b_1b_2b_3...$ using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$$

First, there is at least one element where $b_n \neq a_n$ because $x = .b_1b_2b_3...$ for example x differs from the number $f(1)$, at b_1 because it is not equal to a_{11} because of the definition of b_1. Secondly, considering $f(2)$ and $f(n)$, when $x = .b_1b_2b_3...$ then they will always differ in at least one point of the decimal expansion. Lastly, We've assumed $f(n)$ gives us any $\mathbb{R}$ from (0,1) and now we've shown $x \neq f(n)$ for any n, therefore f is not onto.

4 1.6.4

We will show that the set S of sequences of 0's and 1's is uncountable. For purposes of contradiction, let S be countable. Then there's 1-1 correspondence between $\mathbb{N}$ and S such that: $\mathbb{N}$: 1,2,3,4,5,6,...,n.

S:

$$f(1) = (a_{11}, a_{12}, a_{13}, ...)$$
$$f(2) = (a_{21}, a_{22}, a_{23}, ...)$$
$$f(3) = (a_{31}, a_{32}, a_{33}, ...)$$
$$f(4) = (a_{41}, a_{42}, a_{43}, ...)$$
$$f(5) = (a_{51}, a_{52}, a_{53}, ...)$$
$$...$$

If sequence $x = (b_1, b_2, b_3, b_4, b_5, b_6, ...)$

$$b_n = \begin{cases} 0 & \text{if } a_{nn} = 1 \\ 1 & \text{if } a_{nn} = 0 \end{cases}$$

Then $b_1 \neq$ first term of $f(1)$ and $b_n \neq$ nth term of $f(n)$.

$$x \neq f(1)...f(n)$$

Thus there's a contradiction. Thus S is uncountable.

5 1.6.5

(a) Let $A = \{a, b, c\}$, the subsets of A are

$$\emptyset, \{a\}, \{b\}, \{c\}, \{ab\}, \{ac\}, \{bc\}, \text{and } A.$$

Thus the power set of A is

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{ab\}, \{ac\}, \{bc\}, A\}$$

(b) Let A be a finite set with n elements. We will show that $P(A)$ has 2^n elements by induction.

Base: For n=0, A has no element. So the power set only has $\emptyset$.

$$P(A) = \{\emptyset\}$$

The power set of A has $2^0 = 1$ element. For $n = 0$, the result is true.

For $n = 1$, A has one element. So its power set has $\emptyset$ and A.

$$P(A) = \{\emptyset, A\}$$

$P(A)$ contains two elements. Power set of A has $2^1 = 2$ elements. So $n = 1$, result is true.

Induction: Assume the result is true for $n = k$: If A has k elements, then $P(A)$ contains 2^k elements. (want to show that for $n = k + 1$, $P(A)$ has 2^{k+1} elements). Let A have $k + 1$ elements. Remove 1 element, n, from A, then $A \setminus \{n\}$ has k elements. Thus $P(A \setminus \{n\})$ has 2^k elements. It also means there are 2^k subsets of A that do not contain n, and 2^k subsets that do contain n. Thus total number of subsets becomes $2^k + 2^k = 2^{k+1}$. Thus according to Principle of Mathematical Induction, this is true for all values of n. That means if A has n elements, them power set of A, $P(A)$, contains 2^n elements.

6 1.6.7

Consider Cantor's Theorem, given any set A, there does not exist a function $f : A \to P(A)$ is that is onto. Construct a set $B = \{a \in A : a \notin f(a)\}$, then a one-to-one function could be constructed as:

$$(1) = \begin{cases} a \to \{a, b\} \\ b \to \{a, b\} \\ c \to \{a, b\} \end{cases}$$

$$(2) = \begin{cases} a \to \{c\} \\ b \to \emptyset \\ c \to \{a, c\} \end{cases}$$

These are one-to-one functions that are not onto. And from the part (b) of 1.6.6, ($C = \{1, 2, 3, 4\}$) we have:

$$C \to P(C) = \begin{cases} 1 \to \{1, 2\} \\ 2 \to \{2, 3, 4\} \\ 3 \to \{1, 2, 4\} \\ 4 \to \{1, 2, 3\} \end{cases}$$

Notice that $\{3, 4\}$ is not in the range of the function.

7 1.6.8

(a) Since $B = \{a \in A : a \notin f(a)\}$, suppose $a' \in B$. Then $f(a') = B$. Thus $a' \in A$ mapped by f to a set contains a'. Thus $a' \notin B$. Thus it is a contradiction. Hence, $a' \in B$ leads to the contradiction.

(b)Suppose $a' \notin B$, and $f(a) = B$. $a' \in A$ that mapped by f to a set without a'. Thus by the definition of B, $a' \in B$, and there is a contradiction again. Therefore, $a' \notin B$ is equally unacceptable.

8 1.6.9

We will now show that the power set of $\mathbb{N}$ and the real numbers have the same cardinality. Consider $(0, 1)$, as we have previously shown that $(0, 1)$ has the same cardinality as $\mathbb{R}$. We will show that $P(\mathbb{N})$ and $(0, 1)$ have the same cardinality by giving two injective functions, one from $P(\mathbb{N}) \to (0, 1)$ and one from $(0, 1) \to P(\mathbb{N})$. First we will show a one to one relationship $f : (0, 1) \to P(\mathbb{N})$

If $x \in (0, 1)$ then $x = 0.a_1a_2a_3\ldots$ where $a_n \in 0, 1, \ldots, 9$ and not equal to zero for all $n \in \mathbb{N}$. Define $f : (0, 1) \to P(\mathbb{N})$ by

$$f(x) = \{a_1, 10a_2, 10^2a_3, \ldots\}$$

Now f is one to one since $x_1 \neq x_2$ implies $x_1(= 0.a_1a_2a_3\ldots)$ differs from $x_2(= 0.b_1b_2b_3\ldots)$ at least at one place (say at i^{th} place). Then $10^{i-1}a_i \neq 10^{i-1}b_i$ so $f(x_1) \neq f(x_2)$. Hence f is one to one.

Existence of one to one function $g : P(\mathbb{N}) \in (0, 1)$. Let $x = a_1, a_2, a_3, \ldots \in P(\mathbb{N})$ where $a_n \in \mathbb{N}$. Define $g(x) = 0.b_1b_2b_3\ldots$ where $b_n = 0$ for $g(x_2)$. Hence $g(x_1) \neq g(x_2)$, by, for example, our work from 1.6.2 and 1.6.4.

9 Conclusion

From our work, we have shown Cantor's theorem to be true. An implication of Cantor's Theorem is that there can be no "largest" set. There is an interesting conclusion that follows, there is no such things as a set of all sets. The prevailing opinion in the nineteenth century was that 'completed' infinities could not be studied rigorously; only 'potential' infinity made sense—for example, the process of repeatedly adding one, starting at 1, would never finish and was therefore infinite, but most mathematicians viewed the completed set of positive integers (or any other infinite set) as a dubious concept at best. An infinite set can be placed in one to one correspondence with a proper subset of itself; most mathematicians saw this as a paradox, and 'solved' the problem by declaring that 'infinite sets' simply make no sense.

A few mathematicians went against the grain; Dedekind realized that the 'paradoxical' correspondence between a set and one of its proper subsets could be taken as the definition of an infinite set. Cantor took this notion much further, showing that infinite sets come in an infinite number of sizes. Cantor showed that the rational numbers are countable, that the real numbers are not countable and the theorem accredited to him, as stated in the introduction. Cantor also showed that the set of algebraic numbers is countable. This means that the transcendental numbers (that is, the non-algebraic numbers, like π and e) form an uncountable set—so in fact almost all real numbers are transcendental.

Cantor was constantly ridiculed and had his work refused, which lead to mental instability. Later he died in a mental institution only to have his work, years later, recognized and accepted by other mathematicians[2].

10 Bibliography

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