Section 7.8 **Convolution.**

https://en.wikipedia.org/wiki/Convolution

Definition.

(not multiplication)

The convolution of f, g, denoted $f * g$, is the function defined by

$$f * g = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

Theorem. ① $\mathcal{L}\{(f*g)(t)\}(s) = \mathcal{L}\{f(t)\}(s) \cdot \mathcal{L}\{g(t)\}(s)$

② $\mathcal{L}^{-1}\{F(s)G(s)\} = (\mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\})(t)$

Properties of the convolution.

1. $f * g = g * f$
2. $f * (c_1 g_1 + c_2 g_2) = c_1 f * g_1 + c_2 f * g_2$ $\qquad c_1, c_2$ constants
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0 * f = 0$

Example 1. Find the Laplace transform of the function

$$h(t) = \int_0^t (t-\tau)^2 \cos(2\tau)\, d\tau$$

$f(t) = t^2 \qquad g(t) = \cos 2t$

$\mathcal{L}\{t^2\}(s) \cdot \mathcal{L}\{\cos 2t\}(s) = \ldots$

Example 2. Express $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4(s^2+1)}\right\}$ in terms of a convolution integral.

$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{1}{6}t^3 * \sin t = \int_0^t \frac{1}{6}(t-\tau)^3 \cdot \sin\tau\, d\tau$

Example 3. Use convolution to solve the IVPs.

1. $y'' + y = \cos(t); \quad y(0) = 0, y'(0) = 0.$

$s^2Y + Y$

$Y(s^2+1) = \frac{s}{s^2+1}$

$Y = \frac{s}{(s^2+1)^2}$

$y = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)}\right\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$

$y = \sin * \cos = \int_0^t \sin(t-\tau)\cos\tau \, d\tau$

trig for $\sin a \cos b$

$\int_0^t \sin(t-\tau)\cos(\tau)\,d\tau$

$= \int_0^t \frac{\sin\tau + \sin(t-2\tau)}{2}\,d\tau$

$= \frac{1}{2}\int_0^t \sin\tau + \sin(t-2\tau)\,d\tau$

$= -\frac{1}{2}\cos\tau + \frac{\cos(t-2\tau)}{4}\Big|_0^t$

2. $y'' + y' = e^t; \quad y(0) = 0, y'(0) = 1.$

$s^2Y - 1 + sY = \mathcal{L}\{e^t\}$

$Y(s^2+s) = \frac{1}{s-1} + \frac{s-1}{s-1} = \frac{s}{s-1}$

$Y = \frac{s}{(s^2+s)(s-1)} = \frac{1}{(s+1)(s-1)}$

$y = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{-t} * e^t$

$\int_0^t e^{-(t-\tau)} \cdot e^{\tau}\,d\tau = e^{-t}\int_0^t e^{2\tau}\,d\tau$

$= e^{-t}\cdot\left(\frac{1}{2}e^{2\tau}\Big|_0^t\right) = e^{-t}\cdot\left(\frac{1}{2}e^{2t} - \frac{1}{2}\right)$