Section 8.3 **Power Series Solutions to Linear Differential Equations.**

Definition: Suppose $f(x)$ is a function whose Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ at $x = x_0$ has radius of convergence $\rho > 0$. We say that

$f(x)$ is analytic at f(x0) if

its equal to the taylor series.

Facts.

1. Polynomials, $\sin(x), \cos(x)$, and e^x are analytic at every point x_0. $\ln(x)$ is analytic at x_0 whenever $x_0 > 0$.
2. If $f(x)$ and $g(x)$ are analytic at x_0, so then so are $f(x) + g(x)$ and $f(x)g(x)$, and so is $\frac{f(x)}{g(x)}$ as long as $g(x_0) \neq 0$.
3. If f is analytic at x_0 and g is analytic at $f(x_0)$, then $g(f(x))$ is analytic at x_0.

Example 1.

1. $e^{\cos(x)}$ is an. at every point x0
2. $\dfrac{1}{1+x}$ is all except -1=x0
3. $\dfrac{1}{1+x^2}$ is at every x0

We consider a linear homogeneous equation of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \quad (1)$$

which, after dividing by $a_2(x)$ and cancelling any common factors, we can rewrite in **standard form**

$$y'' + p(x)y' + q(x)y = 0. \quad (2)$$

Definition:
A point x_0 is called an ordinary point of the equation

(2) $y'' + p(x)y' + q(x)y = 0$ if p(x) & q(x) are analytic at x0.

Points x_0 that are not ordinary are called singular points.

Theorem: If x_0 is an ordinary point of the equation (2)

$$y'' + p(x)y' + q(x)y = 0$$

radius of convergence

such that the Taylor series for $p(x)$ and $q(x)$ at x_0 is at least $\rho > 0$, then for any choice of initial conditions $y(x_0) = y_0, y'(x_0) = y_1$, the ~~equation (2)~~ IVP has a power series solution at x_0 with radius of convergence at least ρ.

Example 2. Find the first 6 terms in the power series centered at $x_0 = 0$ for the general solution to the equation

$$(1 + x^2)y'' + y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} a_n \cdot n x^{n-1} = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1}$$

$$y'' = \sum_{n=1}^{\infty} a_n n \cdot (n-1) x^{n-2} = \sum_{n=2}^{\infty} a_n (n-1) n x^{n-2}$$

$$(1+x^2) \cdot \sum_{n=2}^{\infty} a_n (n-1) n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

single series

$$\sum_{n=2}^{\infty} a_n (n-1) n x^{n-2} + x^2 \sum_{n=2}^{\infty} a_n (n-1) n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n+2=2}^{\infty} a_{n+2} (n+2)(n+1) x^n + \sum_{n=2}^{\infty} a_n (n-1) n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n + \sum_{n=0}^{\infty} a_n (n-1) n \cdot x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$y = a_0 + a_1 x \dots$$

$$\sum_{n=0}^{\infty} \underbrace{\left(a_{n+2}(n+2)(n+1) + a_n (n-1) n + a_n\right)}_{\text{has to } = 0} x^n = 0$$

$$a_{n+2}(n+2)(n+1) = -a_n - a_n (n-1) n$$
$$= a_n(-1 + (n-1)n)$$

$$a_{n+2} = \frac{a_n(-1 + (n-1)n)}{(n+2)(n+1)}$$ recursive formula

a_0, a_1 are arbitrary constants for diff eq, the C_1, C_2

for $n=0$: $a_2 = a_0 \cdot \frac{-1}{2}$ $\quad a_4 = a_2 \cdot \frac{1}{12}$

for $n=1$: $a_3 = a_1 \cdot \frac{-1}{6}$ $\quad a_5 = a_3 \cdot \frac{5}{20} = \frac{1}{4}$

Example 3. Find the power series centered at $x_0 = 0$ for the general solution to the each equation.

1. $2y'' + xy' + y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} a_n (n-1) n x^{n-2}$$

$$2\sum_{n=2}^{\infty} a_n (n-1) n x^{n-2} + x\sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=2}^{\infty} a_n (n-1) n x^{n-2} + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n+2=2}^{n+2=\infty} a_{n+2}(n+1)(n+2)x^n + \sum_{n=0}^{\infty} a_n n x^n + a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left(2a_{n+2}(n+1)(n+2) + a_n n + a_n\right) x^n = 0$$

$$2a_{n+2}(n+1)(n+2) = -a_n n - a_n$$

$$a_{n+2} = \frac{a_n(-n-1)}{(n+1)(n+2)\cdot 2} = \frac{-a_n}{2(n+2)}$$

$n=0$:

$$a_2 = a_0 \cdot \frac{-1}{2\cdot 2}$$

$$a_3 = a_1 \cdot \frac{-1}{2\cdot 3}$$

$$a_4 = a_2 \cdot \frac{-1}{2\cdot 4} = a_0 \cdot \frac{-1}{2\cdot 2}\cdot\frac{-1}{2\cdot 4}$$

$$a_5 = a_3 \cdot \frac{-1}{2\cdot 5} = a_1 \cdot \frac{-1}{2\cdot 3}\cdot\frac{-1}{2\cdot 5}$$

$$a_6 = a_4 \cdot \frac{-1}{2\cdot 6} = a_0 \cdot \frac{-1}{2\cdot 2}\cdot\frac{-1}{2\cdot 4}\cdot\frac{-1}{2\cdot 6}$$

$$y = a_1 \sum_{n \text{ odd}}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}}\cdot n!!} + a_0 \sum_{n \text{ even}} \frac{(-1)^{\frac{n}{2}}}{2^{\frac{n}{2}}\cdot n!!}$$

separate even & odd

$$a_2 = \frac{-1}{2\cdot 2}a_0 \quad a_4 = \frac{a_0}{2\cdot 2\cdot 2\cdot 4} \quad a_6 = \frac{-a_0}{2\cdot 2\cdot 2\cdot 4\cdot 2\cdot 6}$$

even:

$$a_n = a_0 \cdot (-1)^{\frac{n}{2}} \cdot \frac{1}{2^{\frac{n}{2}}\cdot n!!}$$

odd:

$$a_n = a_1 \cdot (-1)^{\frac{n-1}{2}} \cdot \frac{1}{2^{\frac{n-1}{2}}\cdot n!!}$$

2. $y'' - xy' + 4y = 0$