Section 9.3 **Review of Matrices**

1. An $m \times n$ (this is often called the **size** or **dimension** of the matrix) matrix is a table with m rows and n columns.

2. Two matrices are said to be **equal** if they are the same size and each corresponding entry is equal.

3. Special Matrices:

   - A **square** matrix is any matrix whose size (or dimension) is $n \times n$ (i.e. it has the same number of rows as columns.) In a square matrix the diagonal that starts in the upper left and ends in the lower right is often called the **main diagonal.**

   - The **zero** matrix is a matrix all of whose entries are zeroes. The **identity** matrix is a square $n \times n$ matrix, denoted I_n, whose main diagonal consists of all 1's and all the other elements are zero:

- A **diagonal** matrix is a square $n \times n$ matrix of the following form

$$\begin{bmatrix} \lambda_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \lambda_n \end{bmatrix}$$

- A **Column vector** is an $n \times 1$ matrix and a **row vector** is $1 \times n$ matrix.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix}$$

- **Transpose of a Matrix**: If A is an $m \times n$ matrix, then A^T is the $n \times m$ matrix obtained by interchanging rows and columns of A.

4. **Matrix Arithmetic**

- The **sum** or difference of two matrices of the same size is a new matrix of identical size whose entries are the sum or difference of the corresponding entries from the original two matrices. Note that we can't add or subtract entries with different sizes.
- **Scalar multiplication** by a constant gives a new matrix whose entries have all been multiplied by that constant.
- If A, B, and C are matrices of the *same* size and α is a scalar, then
  (a) $A + B = B + A$ (Commutative Law)
  (b) $(A + B) + C = A + (B + C)$ (Associative Law)
  (c) $\alpha(A + B) = \alpha A + \alpha B$ (Distributive Law)
  (d) $(A + B)^T = A^T + B^T$
  (e) $(\alpha A)^T = \alpha A^T$

Example 1. Compute $2A - 3B$ where $A = \begin{bmatrix} 0 & 2 \\ 5 & -1 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -2 \\ 0 & 0 \end{bmatrix}$

- **Matrix multiplication**: If y is a row vector of size $1 \times n$ and x is a column vector of size $n \times 1$ (see above), then the **matrix product** of y and x is defined by

$$yx = \begin{bmatrix} y_1 & y_2 & y_3 & \cdots & y_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_nx_n$$

- If A is an $m \times p$ matrix and matrix B is $p \times n$, then the product AB is an $m \times n$ matrix, and its element in the ith row and jth column is the product of the ith row of A and the jth column of B.
- RULE for multiplying matrices:
  If A is $i \times j$ and B is $k \times \ell$, then AB is well-defined if and only if

  __________, and the dimensions of AB are __________.

5. **Example 1.**

   (a) Let $A = \begin{bmatrix} 1 & 2 & -3 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \end{bmatrix}$. Find BA^T.

(b) Given

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$$

Compute AB and BA when it is possible.

(c) Compute $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 2 & -3 \\ 4 & 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) Compute Ax if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ -1 & -2 & -3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

6. FACTS and LAWS FOR MATRIX MULTIPLICATION: If the size requirements are met for matrices A, B and C, then

   - $AB \neq BA$ (NOT always Commutative) (Since the multiplication of matrices is NOT commutative, you MUST multiply left to right.)

   - $C(A+B) = CA + CB$ (always Distributive)
     $(A+B)C = AC + BC$
   - $(AB)C = A(BC)$ (always Associative)
   - $AB = 0$ does not imply that $A = 0$ or $B = 0$.

   - $AB = AC$ does not imply that $B = C$.

   - $I_n A = A I_n = A$ for any square matrix A of size n.

7. A system of linear equations can be written as a matrix equation $Ax = B$.

8. **Example 2.** Express the following system of linear equations in matrix form:
$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & - & 7x_3 & = & 6 \\ -x_1 & - & 3x_2 & + & 11x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & 1 \end{array}$$

Determinants and Inverses

9. Determinant of a matrix is a function that takes a square matrix and converts it into a number.

10. Determinant of 2×2 matrices.

    - The determinant of a 2×2 matrix is defined by

11. **Matrix Inverse**. Let A be an $n \times n$ matrix. We say that A is __________________

    if there exists another $n \times n$ matrix B such that ____________________________.

    In this case we call B the ________________ of A and write $B =$ ___________.

    A matrix that is not invertible is called ________________.

12. A^{-1} in the case $n = 2$: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

    $A^{-1} =$

13. **Theorem:** A is singular if and only if ______________________.

14. **Solving Systems of Equations with Inverses.**

    Let $Ax = B$ be a linear system of n equations in n unknowns and A^{-1} exists, then
    $x = A^{-1}B$ is the *unique* solution of the system.

Example 3: Solve the system of equations $2x_1 + 3x_2 = 1$, $-x_1 + x_2 = 2$ using matrices.