Lower your Expectations: Tight Probabilistic Reachable Sets for Uncertain Saturating Systems

Carlo Karam[1], Matteo Tacchi-Bénard[1], Mirko Fiacchini[1]

[1] GIPSA-lab, Grenoble INP, Université Grenoble-Alpes, CNRS

Overview and Motivation

System Characteristics

We have a **discrete-time LTI system** $x_{k+1} = Ax_k + Bu_k + w_k$, subject to **zero-mean, i.i.d. and unbounded noise** of **variance** W. We wish to satisfy **state chance constraints** $\Pr\{x_k \in \mathcal{X}\} \geq 1 - \varepsilon$, and **hard input constraints** $u_k \in \mathcal{U}$. **Stochastic Model Predictive Control (SMPC)** is the usual choice for such a task.

Standard Approach

The **input is parameterized** as $u_k = v_k + Ke_k$, and the **state is split** into $x_k = z_k + e_k$. The **nominal dynamics** follow $z_{k+1} = Az_k + Bv_k$, and the **error** evolves as $e_{k+1} = (A + BK)e_k + w_k$. The nominal system is controlled via MPC using **tightened constraints** $\mathcal{X} \ominus \mathcal{R}$, where $\mathcal{R}$ is the **probabilistic reachable set (PRS)** of the error.

Our Contribution

$u = v_k + Ke_k$ will be unbounded due to e_k, and we can **only guarantee chance constraints on the inputs**. We propose to **saturate the dynamics** $\boldsymbol{x_{k+1} = Ax_k + B\varphi(u_k) + w_k}$, where $e_{k+1} = Ae_k + B(\varphi(v_k + Ke_k) - v_k) + w_k$. We construct PRS for this **nonlinear stochastic system** that can be integrated in a SMPC scheme [1].

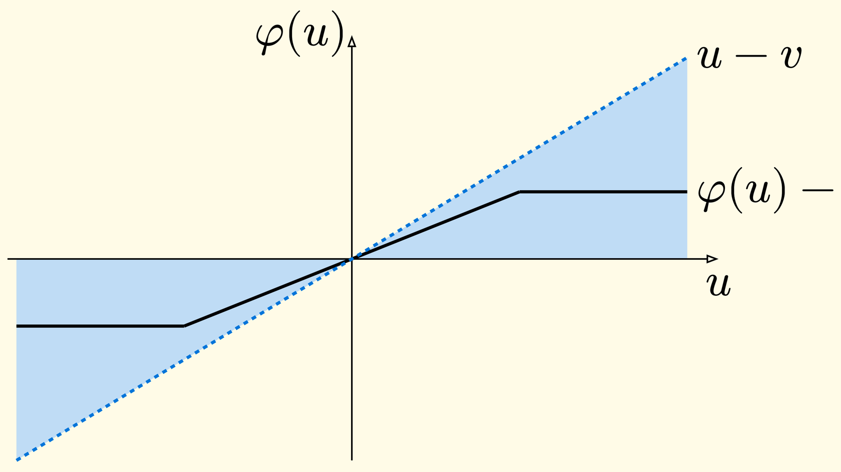
Handling Saturation Nonlinearity

Main idea – Over-approximate the saturation mapping by a convex bound valid for all admissible inputs $v \in \mathcal{U}$, to enable reachability analysis.

Sector Nonlinearity

The saturation function $\varphi(u) - v$ can be enclosed in a so-called sector:

$$\begin{cases} 0 \leq \varphi(K_i e + v_i) - v_i \leq K_i e & \text{if } K_i e \geq 0 \\ K_i e \leq \varphi(K_i e + v_i) - v_i \leq 0 & \text{if } K_i e < 0 \end{cases}$$

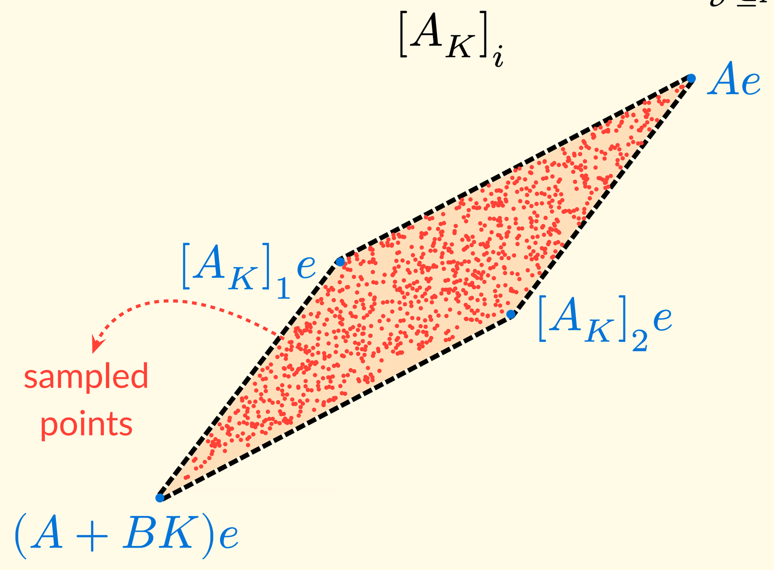


Takeaway – We can bound the nonlinear dynamics by a class of linear functions.

Convex Bounds

The sector condition lets us replace the nonlinearity with simpler quadratic constraints. We can leverage that to convexify $f(e, v) = Ae + B(\varphi(Ke + v) - v)$, by bounding it in a polytopic set [2]:

$$f(e, v) \in \mathrm{co}\left\{ \Big(\underbrace{A + \sum_{i \in \mathcal{J}} B_{(i)} K_i}_{[A_K]_i} \Big) e \right\}_{\mathcal{J} \subseteq \mathbb{N}^m}$$



Takeaway – A convex polytope bounding the nonlinear mapping $f(e, v)$ exists for every value of e, for any admissible input v.

Main impact – We can leverage linear systems theory to construct bounds on our nonlinear system, and easily propagate the set evolution.

Computation of Base PRS

Main idea – We transfer the contraction property from the convex dynamics to the nonlinear system to bound the error in ellipsoidal sets.

Contracting Dynamics

We can design gain K, Lyapunov metric P, for a factor $\lambda \in [0, 1)$, such that:

$$[A_K]_i^\top P [A_K]_i \preceq \lambda P.$$

This implies:

$$\mathbb{E}[e_{k+1}^\top P e_{k+1}] \leq \underbrace{\lambda \mathbb{E}[e_k^\top P e_k]}_{\text{contraction}} + \overbrace{\operatorname{tr}(PW)}^{\text{noise effect}}.$$

If the initial state is known $e_0 = 0$, and we have:

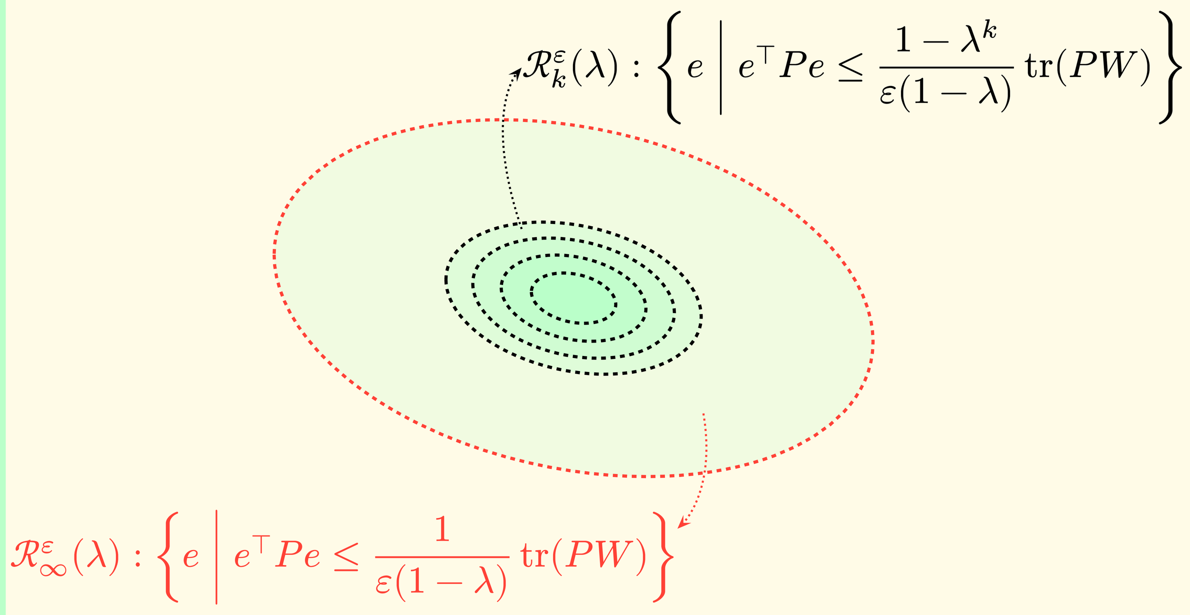
$$\mathbb{E}[e_k^\top P e_k] \leq \frac{1 - \lambda^k}{1 - \lambda} \operatorname{tr}(PW) \quad \xrightarrow{\lim_{k \to \infty}} \quad \mathbb{E}[e_\infty^\top P e_\infty] \leq \frac{1}{1 - \lambda} \operatorname{tr}(PW)$$

Takeaway – The contraction and noise effects equilibrate for $k \to \infty$ for properly designed K, P.

PRS Computation

We compute the PRS using the Markov inequality:

$$\Pr\{\underbrace{e_k^\top P e_k \leq r}_{\text{ellipsoid}}\} \geq 1 - \underbrace{\frac{\mathbb{E}[e_k^\top P e_k]}{r}}_{\text{violation probability } \varepsilon}$$



Takeaway – For a given ε, we can compute ellipsoidal sets that contain the error e with probability $1 - \varepsilon$. We get a sequence of nested PRS that is bounded by a **Probabilistic Ultimate Bound (PUB)**.

Main impact – The nonlinearity and stochasticity are embedded in the ellipsoidal sets. Tightening the constraints by $\mathcal{R}_k^\varepsilon(\lambda)$ ensures both state and input constraints.

PRS Tightening and Refinement

Main idea – We account for the system's frequent operation in the linearity zone (no saturation) to obtain tighter bounds on $\mathbb{E}[e_k^\top P e_k]$.

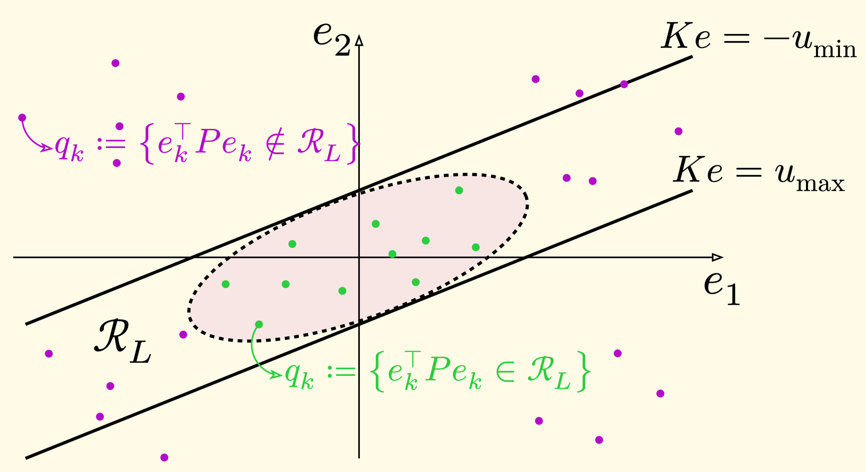
Region of Linearity

A strong contraction $\lambda_L \leq \lambda$ holds without saturation:

$$(A + BK)^\top P (A + BK) \preceq \lambda_L P,$$

This behavior occurs in the *region of linearity*, which we model as an ellipsoid [3] with scaling r_L given by:

$$r_L = \min_{1 \leq i \leq m} \frac{(u_{\max} - v_i)^2}{K P^{-1} K^\top}$$



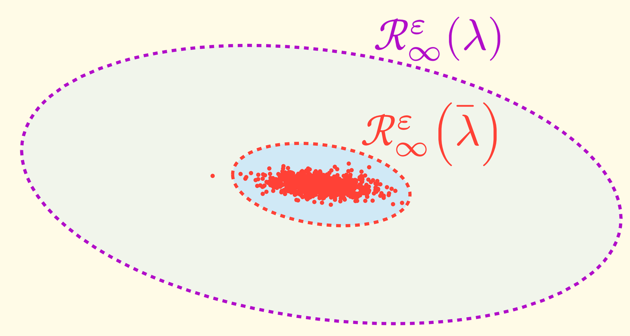
Takeaway – Stronger contraction occurs in the region $\mathcal{R}_L := \{e \mid e_k^\top P e_k \leq r_L\}$, an ellipsoidal area where the system frequently operates.

New Expectation Bound and PRS

We can compute a new **tighter** bound on $\mathbb{E}[e_k^\top P e_k]$ by averaging the contributions of the distinct contractions:

$$\begin{aligned} \mathbb{E}[e_{k+1}^\top P e_{k+1}] &\leq \lambda_L \mathbb{E}[e_k^\top P e_k] \cdot \Pr\{e_k^\top P e_k \leq r_L\} \\ &\quad + \lambda \mathbb{E}[e_k^\top P e_k] \cdot \Pr\{e_k^\top P e_k \geq r_L\} + \operatorname{tr}(PW) \\ &\leq \bar{\lambda} \mathbb{E}[e_k^\top P e_k] + \operatorname{tr}(PW) \end{aligned}$$

With this bound, we obtain **much tighter sets**:



Takeaway – Under certain conditions, we can compute tighter PRS and PUB that reduce conservatism.

Main Impact – The new sets capture the error dynamics more accurately, yielding larger tightened constraints.

Conclusion and Outlook

We presented a constructive analytical method to design probabilistic reachable sets and ultimate bounds in the presence of hard input constraints on a linear system subject to potentially unbounded additive disturbances. By adopting a saturated dynamics formulation, we derive contraction bounds allowing the construction of tight PRS and PUB.

Our current work is focused on implementing this approach in a **full SMPC scheme** with closed-loop guarantees, and extending the framework to non-convex sets using the **Moment Sum of Squares** hierarchy to further reduce the conservatism induced by the convexity assumption on the sets.

References

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[3] S. Tarbouriech, G. Garcia, J. M. Gomes Da Silva, and I. Queinnec, *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer, 2011.