Continuous Perceptron

I. STANDARD PERCEPTRON

We consider a perceptron with binary or continuous variables, that we shall assume take value in some measure space Ω. We also impose the norm $q_* N$, with of course $q_* = 1$ for binary synapsis. In particular we shall consider $\Omega = [-\infty, +\infty]$, $\Omega = [-1, +1]$ and $\Omega = \{-1, +1\}$.

We will allow for a generic energy function $f\left(\frac{\sum_i \xi_i^\mu W_i}{\sqrt{N}}\right)$.

A. Entropy Calculation

The replicated partition function of the model is given by

$$\overline{Z^n} = \int_\Omega \prod_{a,i} \mathrm{d}W_i^a \prod_a \delta\left(\sum_i (W_i^a)^2 - q_* N\right) \prod_{\mu,a} f\left(\frac{\sum_i \xi_i^\mu W_i^a}{\sqrt{N}}\right) \tag{1}$$

where q_* is an external fixed normalization (generally $q_* = 1$) and where f is a generic energetic term (e.g. $f(x) = \theta(x)$).

After some manipulations

$$\overline{Z^n} = \int \prod_a \frac{\mathrm{d}\hat{q}_{aa}}{2\pi} \prod_{a<b} \frac{\mathrm{d}\hat{q}_{ab}\mathrm{d}q_{ab}}{2\pi} e^{N\phi[\hat{q},q]} \tag{2}$$

where the replicated action is given by

$$\phi[\hat{q}, q] = -\frac{1}{2}\sum_{a,b} \hat{q}_{ab}\, q_{ab} + G_S + \alpha G_E \tag{3}$$

where we identified $q_{aa} = q_*$ and

$$G_S[\hat{q}] = \log \int_\Omega \prod_a \mathrm{d}\, W_a \; e^{\frac{1}{2}\sum_{ab} \hat{q}_{ab} W_a W_b} \tag{4}$$

$$G_E[q] = \log \int \prod_a \frac{\mathrm{d}\,\hat{u}_a\,\mathrm{d}\,u_a}{2\pi} \; e^{-\frac{1}{2}\sum_{ab} q_{ab}\hat{u}_a\hat{u}_b + i\hat{u}_a u_a} \prod_a f(u_a) \tag{5}$$

Saddle point equation read

$$\hat{q}_{ab} = -\alpha \ll \hat{u}_a \hat{u}_b \gg_E \qquad a > b \tag{6}$$

$$q_{ab} = \ll W_a W_b \gg_S \qquad a > b \tag{7}$$

$$q_{aa} \equiv q_* = \ll W_a^2 \gg_S \tag{8}$$

Last equations should be solved implicitly for $\hat{q}_{aa}$.

B. RS ansatz

$$\frac{1}{n}\phi[\hat{q}, q] \sim -\frac{1}{2}(q_*\hat{q}_1 - q_0\hat{q}_0) + \mathcal{G}_s + \alpha\mathcal{G}_E \tag{9}$$

$$\mathcal{G}_S[\hat{q}] = \int \mathcal{D}z_0 \log \int_\Omega \mathrm{d}W\ e^{\frac{1}{2}(\hat{q}_1-\hat{q}_0)W^2+\sqrt{\hat{q}_0}z_0 W} \tag{10}$$

$$\mathcal{G}_E[q] = \int \mathcal{D}z_0 \log \int \mathcal{D}u\ f\left(\sqrt{q_0}z_0 + \sqrt{q_* - q_0}u\right) \tag{11}$$

For the binary Percetron $\hat{q}_1 = 0, q_* = 1, \int dW = \sum_{W=\pm 1}$.
Saddle points condition, obtained by derivation of last equations, read

$$\hat{q}_0 = \int \mathcal{D}z_0 \left(\frac{\int \mathcal{D}u\ f'\left(\sqrt{q_0}z_0 + \sqrt{q_* - q_0}u\right)}{\int \mathcal{D}u\ f\left(\sqrt{q_0}z_0 + \sqrt{q_* - q_0}u\right)} \right)^2 \tag{12}$$

$$q_0 = \int \mathcal{D}z_0\ \frac{\int_\Omega \mathrm{d}\,W\ e^{\frac{1}{2}(\hat{q}_1-\hat{q}_0)W^2+\sqrt{\hat{q}_0}z_0 W}\left(W^2 - \frac{1}{\sqrt{\hat{q}_0}}z_0 W\right)}{\int_\Omega \mathrm{d}\,W\ e^{\frac{1}{2}(\hat{q}_1-\hat{q}_0)W^2+\sqrt{\hat{q}_0}z_0 W}} \tag{13}$$

$$q_* = \int \mathcal{D}z_0\ \frac{\int_\Omega \mathrm{d}\,W\ e^{\frac{1}{2}(\hat{q}_1-\hat{q}_0)W^2+\sqrt{\hat{q}_0}z_0 W}\ W^2}{\int_\Omega \mathrm{d}\,W\ e^{\frac{1}{2}(\hat{q}_1-\hat{q}_0)W^2+\sqrt{\hat{q}_0}z_0 W}} \tag{14}$$

1. case $\int_{-\infty}^{+\infty} dW$

$$\mathcal{G}_S[\hat{q}] = \frac{1}{2}\log(2\pi) + \frac{\hat{q}_0}{2(\hat{q}_0 - \hat{q}_1)} - \frac{1}{2}\log(\hat{q}_0 - \hat{q}_1) \tag{15}$$

$$q_0 = \frac{\hat{q}_0}{(\hat{q}_0 - \hat{q}_1)^2} \tag{16}$$

$$q_* = \frac{2\hat{q}_0 - \hat{q}_1}{(\hat{q}_0 - \hat{q}_1)^2} \tag{17}$$

Last equation solved for $\hat{q}_1$ gives

$$\hat{q}_1 = \frac{-1 + 2\hat{q}_0 q_* - \sqrt{1 + 4\hat{q}_0 q_*}}{2q_*} \tag{18}$$

2. case $\int_{-1}^{+1} dW$

$$G_S = -\frac{1}{2}\log a + \int \mathcal{D}z_0\ \log\left[\sqrt{\frac{\pi}{2}}e^{-\frac{b^2}{2a}}\left(\mathrm{erfi}\left(\frac{a-b}{\sqrt{2}\sqrt{a}}\right) + \mathrm{erfi}\left(\frac{a+b}{\sqrt{2}\sqrt{a}}\right)\right)\right] \tag{19}$$

3. case $\sum_{W=\pm 1}$

$$\mathcal{G}_S[\hat{q}] = \frac{1}{2}(\hat{q}_1 - \hat{q}_0) + \int \mathcal{D}z_0 \log 2\cosh\left(\sqrt{\hat{q}_0}z_0\right) \tag{20}$$

$$q_0 = \int \mathcal{D}z_0\ \tanh^2\left(\sqrt{\hat{q}_0}z_0\right) \tag{21}$$

$\hat{q}_1$ in this case is undetermined and can be set to 0

4. case $f(x)=\theta(x)$

$$\mathcal{G}_E[q]=\int\mathcal{D}z_0\log H\left(-\frac{\sqrt{q_0}z_0}{\sqrt{q_*-q_0}}\right) \tag{22}$$

$$\hat{q}_0=\frac{1}{q_*-q_0}\int\mathcal{D}z_0\left(\frac{G\left(-\frac{\sqrt{q_0}z_0}{\sqrt{q_*-q_0}}\right)}{H\left(-\frac{\sqrt{q_0}z_0}{\sqrt{q_*-q_0}}\right)}\right)^2 \tag{23}$$

5. case $f(x)=H^\beta\left(-\frac{x}{\sqrt{1-q_*}}\right)$ (likelihood)

$$\mathcal{G}_E[q]=\int\mathcal{D}z_0\log\int\mathcal{D}u\ H^\beta\left(-\frac{\sqrt{q_0}z_0+\sqrt{q_*-q_0}u}{\sqrt{1-q_*}}\right) \tag{24}$$

C. Parisi-Franz with binary perceptron

Now we couple with our system, that we shall denote with $\tilde{W}$, a binary perceptron W. We enforce an overlap $S=\frac{1}{N}\sum_i W_i\,\mathrm{sign}(\tilde{W}_i)$. For binary $\tilde{W}$ this corresponds to a standard overlap.

$$\phi_{FP}(S)=-\frac{1}{2}\hat{Q}\left(1-Q\right)+\hat{s}_0s_0-\hat{s}_1s_1-\gamma S+G_S+\alpha G_E \tag{25}$$

$$G_S=\int\mathrm{D}\,z_0\,\frac{\int_\Omega\mathrm{d}\,\tilde{W}\int\mathcal{D}\eta\ e^{\frac{1}{2}(\hat{q}_1-\hat{q}_0)\tilde{W}^2+\sqrt{\hat{q}_0}z_0\tilde{W}}A_S(\tilde{W},\eta,z_0)}{\int_\Omega\mathrm{d}\,\tilde{W}\ e^{\frac{1}{2}(\hat{q}_1-\hat{q}_0)\tilde{W}^2+\sqrt{\hat{q}_0}z_0\tilde{W}}} \tag{26}$$

$$A_S(\tilde{W},\eta,z_0)=\log 2\cosh\left((\hat{s}_1-\hat{s}_0)\tilde{W}+\gamma\,\mathrm{sign}(\tilde{W})+\sqrt{\frac{\hat{Q}\hat{q}_0-\hat{s}_0^2}{\hat{q}_0}}\eta+\frac{\hat{s}_0}{\sqrt{\hat{q}_0}}z_0\right) \tag{27}$$

$$G_E=\int\mathcal{D}z_0\frac{\int\mathcal{D}\eta\mathcal{D}z_1\ f\left(\sqrt{q_0}z_0+\sqrt{a}z_1+\frac{s_1-s_0}{\sqrt{b}}\eta\right)\log H\left(-\frac{\sqrt{b}\eta+\frac{s_0}{\sqrt{q_0}}z_0}{\sqrt{1-Q}}\right)}{\int\mathcal{D}z_1\ f\left(\sqrt{q_0}z_0+\sqrt{q_*-q_0}z_1\right)} \tag{28}$$

$$a=q_*-q_0-\frac{\left(s_1-s_0\right)^2}{\left(Q-s_0\right)}\left(1-\frac{s_0\left(q_0-s_0\right)}{\left(Qq_0-s_0^2\right)}\right) \tag{29}$$

$$b=\frac{Qq_0-s_0^2}{q_0} \tag{30}$$

Saddle point equations read

$$\hat{Q} = -2\alpha \frac{\partial G_E}{\partial Q} \tag{31}$$

$$Q = 1 - 2\frac{\partial G_S}{\partial \hat{Q}} \tag{32}$$

$$s_0 = -\frac{\partial G_S}{\partial \hat{s}_0} \tag{33}$$

$$\hat{s}_0 = -\alpha \frac{\partial G_E}{\partial s_0} \tag{34}$$

$$s_1 = \frac{\partial G_S}{\partial \hat{s}_1} \tag{35}$$

$$\hat{s}_1 = \alpha \frac{\partial G_E}{\partial s_1} \tag{36}$$

$$S = \frac{\partial G_S}{\partial \gamma} \tag{37}$$

In the case of binary $\tilde{W}$ we can set $\gamma = 0$, $s_1 = S$, and drop last two equations.

1. case $\sum_{\tilde{W}=\pm 1}$

$$G_S = \int \mathrm{D}\, z_0 \, \frac{\sum_{\tilde{W}=\pm 1} \int \mathcal{D}\eta \; e^{\sqrt{\hat{q}_0} z_0 \tilde{W}} A_S(\tilde{W}, \eta, z_0)}{2\cosh\left(\sqrt{\hat{q}_0} z_0 \tilde{W}\right)} \tag{38}$$

$$A_S(\tilde{W}, \eta, z_0) = \log 2\cosh\left((\hat{s}_1 - \hat{s}_0)\tilde{W} + \sqrt{\frac{\hat{Q}\hat{q}_0 - \hat{s}_0^2}{\hat{q}_0}}\eta + \frac{\hat{s}_0}{\sqrt{\hat{q}_0}} z_0\right) \tag{39}$$

2. case $\int_{-\infty}^{+\infty} d\tilde{W}$

assuming $\hat{q}_0 > \hat{q}_1$.

$$G_S = \int \mathrm{D}\, z_0 \mathcal{D}\tilde{W} \mathcal{D}\eta \; A_S \tag{40}$$

$$A_S = \log 2\cosh\left(\frac{(\hat{s}_1 - \hat{s}_0)}{\hat{q}_0 - \hat{q}_1}\left(\sqrt{\hat{q}_0 - \hat{q}_1}\tilde{W} + \sqrt{\hat{q}_0} z_0\right) + \gamma \,\mathrm{sign}\left(\sqrt{\hat{q}_0 - \hat{q}_1}\tilde{W} + \sqrt{\hat{q}_0} z_0\right) + \sqrt{\frac{\hat{Q}\hat{q}_0 - \hat{s}_0^2}{\hat{q}_0}}\eta + \frac{\hat{s}_0}{\sqrt{\hat{q}_0}} z_0\right) \tag{41}$$

which can be simplified to

$$G_S = \int \mathcal{D}\tilde{W} \mathcal{D} z_0 \;\log 2\cosh\left[\left(\frac{\hat{s}_1 - \hat{s}_0}{\hat{q}_0 - \hat{q}_1}\sqrt{2\hat{q}_0 - \hat{q}_1} + \frac{\hat{s}_0}{\sqrt{2\hat{q}_0 - \hat{q}_1}}\right)\tilde{W} + \gamma\,\mathrm{sign}(\tilde{W}) + \sqrt{\hat{Q} - \frac{\hat{s}_0^2}{2\hat{q}_0 - \hat{q}_1}} z_0\right] \tag{42}$$

3. case $\int_{-1}^{+1} d\tilde{W}$

assuming $\hat{q}_0 > \hat{q}_1$.

$$G_S = \int \mathrm{D}\, z_0 \frac{\int \mathcal{D}\tilde{W}\mathcal{D}\eta\ I\left(-1 < \frac{\sqrt{\hat{q}_0-\hat{q}_1}\tilde{W}+\sqrt{\hat{q}_0}z_0}{\hat{q}_0-\hat{q}_1} < 1\right)\ A_S}{\int \mathcal{D}\tilde{W}\ I\left(-1 < \frac{\sqrt{\hat{q}_0-\hat{q}_1}\tilde{W}+\sqrt{\hat{q}_0}z_0}{\hat{q}_0-\hat{q}_1} < 1\right)} \tag{43}$$

$$A_S = \log 2\cosh\left(\frac{(\hat{s}_1-\hat{s}_0)}{\hat{q}_0-\hat{q}_1}\left(\sqrt{\hat{q}_0-\hat{q}_1}\tilde{W}+\sqrt{\hat{q}_0}z_0\right)+\gamma\,\mathrm{sign}\left(\sqrt{\hat{q}_0-\hat{q}_1}\tilde{W}+\sqrt{\hat{q}_0}z_0\right)+\sqrt{\frac{\hat{Q}\hat{q}_0-\hat{s}_0^2}{\hat{q}_0}}\eta+\frac{\hat{s}_0}{\sqrt{\hat{q}_0}}z_0\right) \tag{44}$$

We split the integral in two parts

$$I\left(0 < \frac{\sqrt{\hat{q}_0-\hat{q}_1}\tilde{W}+\sqrt{\hat{q}_0}z_0}{\hat{q}_0-\hat{q}_1} < 1\right)\ A_S^+ \tag{45}$$

$$\begin{aligned}
A_S^+ &= \log 2\cosh\left(\left(\frac{\hat{s}_0}{\sqrt{\hat{q}_0}}+\frac{(\hat{s}_1-\hat{s}_0)}{\hat{q}_0-\hat{q}_1}\sqrt{\hat{q}_0}\right)z_0+\gamma+\sqrt{\frac{\hat{Q}\hat{q}_0-\hat{s}_0^2}{\hat{q}_0}}\eta+\frac{(\hat{s}_1-\hat{s}_0)}{\sqrt{\hat{q}_0-\hat{q}_1}}\tilde{W}\right) &(46)\\
&= \log 2\cosh\left(\left(\frac{\hat{s}_0}{\sqrt{\hat{q}_0}}+\frac{(\hat{s}_1-\hat{s}_0)}{\hat{q}_0-\hat{q}_1}\sqrt{\hat{q}_0}\right)z_0+\gamma+\sqrt{\frac{\hat{Q}\hat{q}_0-\hat{s}_0^2}{\hat{q}_0}+\frac{(\hat{s}_1-\hat{s}_0)^2}{\hat{q}_0-\hat{q}_1}}W'\right) &(47)
\end{aligned}$$

$$\begin{aligned}
I\left(0 < \frac{\sqrt{\hat{q}_0-\hat{q}_1}\tilde{W}+\sqrt{\hat{q}_0}z_0}{\hat{q}_0-\hat{q}_1} < 1\right) &= I\left(0 < \frac{\sqrt{\hat{q}_0}}{\hat{q}_0-\hat{q}_1}z_0+\frac{1}{\sqrt{\hat{q}_0-\hat{q}_1}}\left(\frac{\sqrt{b}}{\sqrt{a+b}}W'-\frac{\sqrt{a}}{\sqrt{a+b}}\eta'\right) < 1\right) &(48)\\
&= I\left(0 < d-\frac{1}{\sqrt{\hat{q}_0-\hat{q}_1}}\frac{\sqrt{a}}{\sqrt{a+b}}\eta' < 1\right) &(49)\\
&= I\left(-1+d < \frac{1}{\sqrt{\hat{q}_0-\hat{q}_1}}\frac{\sqrt{a}}{\sqrt{a+b}}\eta' < d\right) &(50)\\
&= I\left(\frac{-1+d}{\frac{1}{\sqrt{\hat{q}_0-\hat{q}_1}}\frac{\sqrt{a}}{\sqrt{a+b}}} < \eta' < \frac{d}{\frac{1}{\sqrt{\hat{q}_0-\hat{q}_1}}\frac{\sqrt{a}}{\sqrt{a+b}}}\right) &(51)
\end{aligned}$$

where

$$\begin{aligned}
a &= \frac{\hat{Q}\hat{q}_0-\hat{s}_0^2}{\hat{q}_0} &(52)\\
b &= \frac{(\hat{s}_1-\hat{s}_0)^2}{\hat{q}_0-\hat{q}_1} &(53)\\
d &= \frac{\sqrt{\hat{q}_0}}{\hat{q}_0-\hat{q}_1}z_0+\frac{1}{\sqrt{\hat{q}_0-\hat{q}_1}}\frac{\sqrt{b}}{\sqrt{a+b}}W' &(54)
\end{aligned}$$

....
assuming $\hat{q}_1 > \hat{q}_0$. We change the sign of z_0.

$$G_S = \int \mathrm{D}\, z_0 \frac{\int \tilde{\mathcal{D}}\tilde{W}\mathcal{D}\eta\ I\left(-1 < \frac{\sqrt{\hat{q}_1-\hat{q}_0}\tilde{W}+\sqrt{\hat{q}_0}z_0}{\hat{q}_1-\hat{q}_0} < 1\right)\ A_S}{\int \tilde{\mathcal{D}}\tilde{W}\ I\left(-1 < \frac{\sqrt{\hat{q}_1-\hat{q}_0}\tilde{W}+\sqrt{\hat{q}_0}z_0}{\hat{q}_1-\hat{q}_0} < 1\right)} \tag{55}$$

where $\tilde{\mathcal{D}}\tilde{W} = \exp(\frac{x^2}{2})/\sqrt{2\pi}\,\mathrm{d}\,x$

$$A_S = \log 2\cosh\left(\frac{(\hat{s}_1-\hat{s}_0)}{\hat{q}_1-\hat{q}_0}\left(\sqrt{\hat{q}_1-\hat{q}_0}\tilde{W}+\sqrt{\hat{q}_0}z_0\right)+\gamma\,\mathrm{sign}\left(\sqrt{\hat{q}_1-\hat{q}_0}\tilde{W}+\sqrt{\hat{q}_0}z_0\right)+\sqrt{\frac{\hat{Q}\hat{q}_0-\hat{s}_0^2}{\hat{q}_0}}\eta-\frac{\hat{s}_0}{\sqrt{\hat{q}_0}}z_0\right) \tag{56}$$

We split the integral in two parts and the make a change of variables $\tilde{W},\eta\to\tilde{W}',\eta'$:

$$I\left(0<\frac{\sqrt{\hat{q}_1-\hat{q}_0}\tilde{W}+\sqrt{\hat{q}_0}z_0}{\hat{q}_1-\hat{q}_0}<1\right)A_S^+ \tag{57}$$

$$A_S^+ = \log 2\cosh\left(\left(-\frac{\hat{s}_0}{\sqrt{\hat{q}_0}}+\frac{(\hat{s}_1-\hat{s}_0)}{\hat{q}_1-\hat{q}_0}\sqrt{\hat{q}_0}\right)z_0+\gamma+\sqrt{\frac{\hat{Q}\hat{q}_0-\hat{s}_0^2}{\hat{q}_0}}\eta+\frac{(\hat{s}_1-\hat{s}_0)}{\sqrt{\hat{q}_1-\hat{q}_0}}\tilde{W}\right) \tag{58}$$

$$= \log 2\cosh\left(\left(-\frac{\hat{s}_0}{\sqrt{\hat{q}_0}}+\frac{(\hat{s}_1-\hat{s}_0)}{\hat{q}_1-\hat{q}_0}\sqrt{\hat{q}_0}\right)z_0+\gamma+\sqrt{\frac{\hat{Q}\hat{q}_0-\hat{s}_0^2}{\hat{q}_0}+\frac{(\hat{s}_1-\hat{s}_0)^2}{\hat{q}_1-\hat{q}_0}}W'\right) \tag{59}$$

....

4. *case* $f(x)=\theta(x)$

$$G_E = \int \mathrm{D}\,z_0 \frac{\int \mathrm{D}\,\eta\; H\left(-\frac{\sqrt{q_0}z_0+\sqrt{\frac{(s_1-s_0)^2}{b}}\eta}{\sqrt{a}}\right)\log H\left(-\frac{\sqrt{b}\eta+\frac{s_0}{\sqrt{q_0}}z_0}{\sqrt{1-Q}}\right)}{H\left(-\frac{\sqrt{q_0}z_0}{\sqrt{q_*-q_0}}\right)} \tag{60}$$

$$a = q_*-q_0-\frac{(s_1-s_0)^2}{(Q-s_0)}\left(1-\frac{s_0\,(q_0-s_0)}{(Qq_0-s_0^2)}\right) \tag{61}$$

$$b = \frac{Qq_0-s_0^2}{q_0} \tag{62}$$