CSci 243 Homework 2

Due: Wednesday, Sept 21
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1. Using logical identities and laws, show the logic equivalence of

   (a) (5 points) $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$
   $p \leftrightarrow q$
   $\equiv (p \rightarrow q) \wedge (q \rightarrow p)$
   $\equiv (\neg p \vee q) \wedge (\neg q \vee p)$
   $\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge \neg q) \vee (q \wedge p)$
   $\equiv (\neg p \wedge \neg q) \vee F \vee F \vee (q \wedge p)$
   $\equiv (\neg p \wedge \neg q) \vee (q \wedge p)$
   $\equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

   (b) (5 points) $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$
   $\neg p \leftrightarrow \neg q$
   $\equiv (\neg p \rightarrow \neg q) \wedge (\neg q \rightarrow \neg p)$
   $\equiv (q \rightarrow p) \wedge (p \rightarrow q)$
   $\equiv p \leftrightarrow q$

2. (10 points) Prove that if $3n+2$ is even integer then n is even, using

   (a) a proof by contraposition.
   Contraposition: Assume $\neg q$, show $\neg q \rightarrow \neg p$
   $\neg p$ = $3n+2$ is odd
   $\neg q$ = n is odd
   Adding 2 to an odd/even number does not change whether it is odd/even, so $3n+2$ shares the same odd/even nature as $3n$.
   We know 3 is odd, and we assume n is odd.
   An odd number multiplied by an odd number is always an odd number.
   Therefore, $3n$ is odd, and $3n+2$ is odd as well.
   $\neg q \rightarrow \neg p \equiv p \rightarrow q$

   (b) a proof by contradiction.
   Contradiction: Assume p and $\neg q$ are true
   p = $3n+2$ is even
   $\neg q$ = n is odd
   Adding 2 to an odd/even number does not change whether it is odd/even, so $3n+2$ shares the same odd/even nature as $3n$.
   We know 3 is odd, and we assume n is even.
   An odd number multiplied by an even number is always an even number.
   Therefore, $3n$ is even, and $3n+2$ is even as well.
   $(\neg q \wedge q)$ contradiction

   (Hint: see a similar set of proofs in the book).

3. (5 points) Disprove the statement that "for any integer $x > 1$, there are positive integers y, z, such that $x^2 = y^2 + z^2$", by finding a counterexample.

   Counterexample: $x = 3$

   Starting with positive integers $y = 0, z = 0$ and showing every combination, the smallest $y^2 + z^2$ values are:

   $y = 0, z = 0, y^2 + z^2 = 0$

   $y = 0, z = 1, y^2 + z^2 = 1$

   $y = 1, z = 1, y^2 + z^2 = 2$

   $y = 0, z = 2, y^2 + z^2 = 4$

   $y = 1, z = 2, y^2 + z^2 = 5$

   $y^2 + z^2$ has already passed 3, showing that there are no positive integers y, z such that $x^2 = y^2 + z^2$.

4. (5 points) Prove that there are no positive perfect cubes (i.e., x^3) less than 1000 that are the sum of the cubes of two positive integers (i.e., $\exists x, y, z > 0(x^3 < 1000 \wedge x^3 = y^3 + z^3)$).

   There are 9 positive perfect cubes less than 1000.

   $1, 8, 27, 64, 125, 216, 343, 512, 729$

   We want to find if any pairs of these cubes can be summed to equal one of the others.

   The two cubes used to sum to a cube n will always have a value less than n.

   First try using cubes $(n-2) + (n-3)$ to sum to n.

$(n-2)$	$(n-3)$	$(n-2)+(n-3)$	n
343	216	559	729
216	125	341	512
125	64	189	343
64	27	91	216
27	8	35	125
8	1	9	64

   From this table, we can see that $(n-2) + (n-3)$ is always less than n. This means that any combination of cubes smaller than $n-1$ cannot be large enough to equal n. So, any sum of two cubes that is equal to (or greater than) n will involve $n-1$ as one of the numbers used.

   If $n = (n-1) +$ some number m, then $n - (n-1) = m$.

n	$(n-1)$	$n-(n-1)$
729	512	217
512	343	169
343	216	127
216	125	91
125	64	61
64	27	37
27	8	19
8	1	7

   From this table, we see every value m that, when added to $n-1$, sums to n.

None of these numbers m are equal to any cubes of positive integers.

Therefore, there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

5. (5 points) Prove that $3 \cdot 10^{100} + 11$ is not a perfect square **OR** $3 \cdot 10^{100} + 12$ is not a perfect square. Is your proof constructive or nonconstructive?

   Assume $3 \cdot 10^{100} + 12$ is a perfect square.

   If $3 \cdot 10^{100} + 12$ is a perfect square, then $\sqrt{3 \cdot 10^{100} + 12}$ is an integer.

   $\sqrt{3 \cdot 10^{100} + 12}$ can be simplified:

   $= \sqrt{3(10^{100} + 4)}$

   $= \sqrt{3}\sqrt{10^{100} + 4}$

   $\sqrt{3}$ is an irrational number.

   The product of an irrational number and any other number is an irrational number.

   Therefore, $\sqrt{3}\sqrt{10^{100} + 4}$ is an irrational number.

   Contradiction: If $3 \cdot 10^{100} + 12$ is a perfect square, then $\sqrt{3 \cdot 10^{100} + 12}$ is an integer, but integers cannot be irrational numbers.

   Therefore, $3 \cdot 10^{100} + 12$ is not a perfect square.

   This proof is nonconstructive because it uses contradiction.

6. (5 points) Given three numbers, prove that at least one pair of them has nonnegative product.

   Given three numbers x, y, z

   If all three numbers are positive:

      $x, y, z > 0$

      The product of two positive numbers is positive

      $xy, xz, yz > 0$

   If one of the numbers is negative and two are positive:

      $x, y > 0, z < 0$

      The product of two positive numbers is positive

      $xy > 0$

   If two of the numbers are negative and one is positive:

      $x > 0, y, z < 0$

      The product of two negative numbers is positive

      $yz > 0$

   If all three numbers are negative:

      $x, y, z < 0$

      The product of two negative numbers is positive

      $xy, xz, yz > 0$

   Therefore, given three numbers, the product of at least one pair of them will be nonnegative.