CSci 243 Homework 9

Due: Wednesday, November 16, end of day

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1. (6 points) Give a recursive definition for each of the following sequences $\{a_n\}$ for $n = 1, 2, 3, \ldots$.

   (a) $a_n = 4n - 2$
   Basis step: $F(1) = 4 - 2 = 2$
   Recursive step: $F(n) = F(n-1) + 4$

   (b) $a_n = 1 + (-1)^n$
   Basis step: $F(1) = 1 + (-1)^1 = 0$, $F(2) = 1 + (-1)^2 = 2$
   Recursive step: $F(n) = F(n-2)$

   (c) $a_n = (\frac{1}{2})^n$
   Basis step: $F(1) = \frac{1}{2}$
   Recursive step: $F(n) = F(n-1) * \frac{1}{2}$

2. For string $w = a_1 a_2 \cdots a_n$, the reversal of the string is defined as $w^R = a_n \cdots a_2 a_1$.

   (a) (4 points) What is ε^R? What is $(10110)^R$?
   $\varepsilon^R = \varepsilon$, $(10110)^R = 01101$

   (b) (4 points) Give a recursive definition of the reversal of a string.
   Basis step: $F(a_1 a_2) = (a_1 a_2)^R = a_2 a_1$
   Recursive step: $F(a_1 \ldots a_n) = a_n F(a_1 \ldots a_{n-1})$

   (c) (6 points) Use structural induction to prove that $(w_1 w_2)^R = w_2^R w_1^R$.
   Basis step: $w_1 = \emptyset, w_2 = \emptyset$. The equality holds because $\emptyset^R = \emptyset$.
   I.H.: Assume for two strings w_1 and w_2 the equality holds.
   I.S.: Create two new strings $v_1 = w_1 w_2$, $v_2 = w_2 w_1$, and show $(v_1 v_2)^R = v_2^R v_1^R$.
   $(v_1 v_2)^R = (w_1 w_2 w_2 w_1)^R$
   The pattern $w_1 w_2 w_2 w_1$ is symmetrical, and is the same after being reversed. The contents of the w_1 and w_2 strings also must be reversed, leaving $(w_1^R w_2^R w_2^R w_1^R)$
   Going to the other side of the equality:
   $v_2^R v_1^R = (w_2 w_1)^R (w_1 w_2)^R$
   The contents of the w_1 and w_2 strings also must be reversed, leaving $(w_1^R w_2^R)(w_2^R w_1^R)$. Remove parentheses and now both sides are equal.
   $(v_1 v_2)^R = v_2^R v_1^R \Rightarrow w_1^R w_2^R w_2^R w_1^R = w_1^R w_2^R w_2^R w_1^R$

3. (10 points) A palindrome is a string that reads the same forward and backward, i.e., $w = w^R$. Give a recursive algorithm in pseudocode that checks whether a given string w is a palindrome. What is the time complexity of your algorithm?

```
method isPalindrome (w)
    if w.length == 1 or w.length == 0
        return true
    else if w.length == 2
        return w[0] == w[1]
```

```
    else if w.length > 2
        outer_matching = w[0] == w[w.length-1]
        sub_string = w with the first and last characters removed
        return outer_matching and isPalindrome(sub_string)
```

Time complexity: linear time; each recursive step covers 2 values; $\frac{n}{2}+1$ steps.

4. (10 points) Give a recursive algorithm in pseudocode that finds the maximum number among n integers. What is the time complexity of your algorithm?

```
method findMax ([a_1, ..., a_n])
    if n == 2
        return max(a_1, a_2)
    else if n > 2
        return max(a_n, findMax([a_1, ... a_n-1]))
```

Time complexity: linear time; there are n steps, one for each value in the list.

5. An employee joined a company in 2015 with a starting salary of \$50,000. Every year this employee receives a raise of \$1000 plus 4% of the salary of the previous year.

   - (3 points) Set up a recurrence relation for the salary of this employee n years after 2015.
     $a_0 = 50,000$
     $a_n = a_{n-1} * 1.04 + 1000$
   - (4 points) Find an explicit formula for the salary of this employee n years after 2015.
     $a_n = 50,000 * 1.04^n + 1000 * 1.04^{n-1} + 1000 * 1.04^{n-2} + \ldots + 1000 * 1.04^2 + 1000 * 1.04^1 + 1000$
     $= 50,000 * 1.04^n + \frac{1000 * 1.04(1-1.04^{n-1})}{1-1.04} + 1000$
     $= 50,000 * 1.04^n + \frac{1040(1-1.04^{n-1})}{-0.04} + 1000$
   - (3 points) What will the salary of this employee be in 2030?
     $a_{15} = 50,000 * 1.04^{15} + \frac{1040(1-1.04^{14})}{-0.04} + 1000$
     $= 110,071.375$