CSci 243 Homework 3

Due: Wednesday, Wed 28 end of day

Carlo Mehegan

1. (6 points) Prove by induction that $2^n < n!$, $\forall n \geq 4$, where ! denotes the factorial of the number.

   Base case: $n = 4$

   $2^4 < 4!$

   $\Rightarrow 16 < 24$

   Inductive hypothesis: assume for any k, $2^k < k!$

   Inductive step: want to prove $2^{k+1} < (k+1)!$

   $2^{k+1} < (k+1)!$

   $\Rightarrow 2^k 2^1 < (k+1)k!$

   $\Rightarrow 2^k 2^1 < (k+1)k!$

   By definition we know that $(k+1) > 4$ because $k \geq 4$. This means that the left side of the inequality is being multiplied by 2 and the right side is being multiplied by a number greater than 4, which is of course greater than 2.

   By the inductive hypothesis, $2^k < k!$. Since in $2^k 2^1 < (k+1)k!$ the left side is being multiplied by 2, and the right side is being multiplied by at least 5, and $2 < 5$, the inequality does not change. Therefore, $2^{k+1} < (k+1)!$ if $2^k < k!$.

   So, $2^n < n!$, $\forall n \geq 4$.

2. (6 points) Prove by induction that $2^n + 1$ is divisible by 3 for all odd integers n.

   Base case: $n = 1$

   $2^1 + 1 = 3$

   Inductive hypothesis: assume for any odd number k, $2^k + 1$ is divisible by 3.

   definition of "divisible by 3": there exists some number a, such that $2^k + 1 = 3a$.

   Inductive step: want to prove there exists some number b, such that $2^{k+2} + 1 = 3b$.

   $2^{k+2} + 1$

   $\Rightarrow 2^k 2^2 + 1$

   $\Rightarrow 4 \cdot 2^k + 1$

   $\Rightarrow 4 \cdot (3a - 1) + 1$

   $\Rightarrow (12a - 4) + 1$

   $\Rightarrow 12a - 3$

   Let $4a - 1 = b$

   $\Rightarrow 3(4a - 1)$

   $\Rightarrow 3b$

   Therefore, $2^{k+2} + 1$ is divisible by 3 if $2^k + 1$ is divisible by 3.

   So, $2^n + 1$ is divisible by 3 for all odd integers n.

3. (6 points) Here is an inductive proof that all jelly beens in the world have the same color!
Basis ($n = 1$): We pick one jelly bean and clearly it has the same color as itself.
Inductive hypothesis: Assume that any group of $n = k$ jelly beans have the same color.
Inductive step: Take $k+1$ jelly beans and order them in a line. Consider group A of the first $1,\ldots,k$ jelly beans, and group B of the last $2,\ldots,k+1$ jelly beans. By the inductive hypothesis, all jelly beans in group A have the same color, and all jelly beans in group B have the same color. Because the two groups share the jelly beans $2,\ldots,k$, the colors of the two groups must be the same. Thus any $k+1$ jelly beans have the same color.

   You don't need to have eaten jelly beans to know that the above proposition is false. So what is wrong with the proof?

   This proof is wrong because the inductive step does not work for 2 jellybeans, or $n = 2$.

   Following the inductive step with $k = 1, k+1 = 2$:

   group A has the first $1,\ldots,k$ beans; works, same color by inductive hypothesis (or by base case, since only 1 bean).

   group B has the last $2,\ldots,k+1$ beans; works, same color by inductive hypothesis (or by base case, since only 1 bean).

   group A and B share beans $2,...,k$; does not work, because group A and B do not share any beans.

   Therefore, the inductive proof does not work.

4. (7 points) Determine whether these statements are true or false (for the empty set, see p.118 of the book). Write a very brief eplanation (less than one line suffices).

   (a) $\emptyset \in \{\emptyset\}$
   True; the element $\emptyset$ is an element of the set $\{\emptyset\}$.

   (b) $\{\emptyset\} \in \{\{\emptyset\}\}$
   True; the element $\{\emptyset\}$ is an element of the set $\{\{\emptyset\}\}$.

   (c) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
   True; All elements of $\{\emptyset\}$ (one element, $\emptyset$) are within the set $\{\emptyset, \{\emptyset\}\}$ (two elements, $\emptyset$ and $\{\emptyset\}$).

   (d) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
   True; All elements of $\{\{\emptyset\}\}$ (one element, $\{\emptyset\}$) are within the set $\{\emptyset, \{\emptyset\}\}$ (two elements, $\emptyset$ and $\{\emptyset\}$).

   (e) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$
   False; All elements of $\{\{\emptyset\}\}$ (one element, $\{\emptyset\}$) are within the set $\{\{\emptyset\}, \{\emptyset\}\}$ (one repeated element, $\{\emptyset\}$), but because the sets are equivalent, $\{\{\emptyset\}\}$ is not a proper subset of $\{\{\emptyset\}, \{\emptyset\}\}$. The sets are equivalent because repeated elements should only be listed once.

(f) $\{x\} \subseteq \{x\}$
True; All elements of $\{x\}$ (one element, x) are within the set $\{x\}$ (one element, x). These two sets are equivalent.

(g) $\{x\} \in \{x\}$
False; the element $\{x\}$ is not an element of the set $\{x\}$.

5. (6 points) Is each of these sets the power set of a set, where a and b are distinct elements? If yes, give the original set.

   (a) $\emptyset$
   Not a power set of any set; number of elements not equal to a power of 2.

   (b) $\{\emptyset, \{a\}\}$
   Power set of the set $\{a\}$.

   (c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$
   Not a power set of any set; number of elements not equal to a power of 2.

   (d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
   Power set of the set $\{a, b\}$.

6. (4 points) Show that if $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$.

   If every element of A is within C, and every element of B is within D, then every element of $A \times B$ will also be in $C \times D$.

   Proof by implication of $A \subseteq C \wedge B \subseteq D \rightarrow A \times B \subseteq C \times D$

   Assume that $A \subseteq C \wedge B \subseteq D$, prove that $A \times B \subseteq C \times D$

   Let a be an element such that $a \in A$.

   Because $A \subseteq C$, $a \in C$ as well.

   Let b be an element such that $b \in B$.

   Because $B \subseteq D$, $b \in D$ as well.

   Let x be an element (a, b) such that $x \in A \times B$.

   Because $a \in C$ and $b \in D$, $x = (a, b) \in C \times D$.

   Therefore, $A \times B \subseteq C \times D$.

   Therefore, if $A \subseteq C \wedge B \subseteq D$, then $A \times B \subseteq C \times D$.