# CSci 243 Homework 5

Due: Wed Oct 12, end of day **My name**

1. (12 points) For each of these partial sequences of integers, determine the next term of the sequence, and then provide a general formula or rule to generates terms of the sequence. For full credit, give a closed form ($a_n = ..$) instead of a recursive formula. Explain your derivations.

   $+1$ $+3$ $+5$ $+7$ $+9$ $+11$

   (a) $3, 4, 7, 12, 19, 28, 39, 52, 67, 84, 103, \ldots$ 124 $\qquad a_n = 3 + (n-1)^2$

   $3$ $3+1$ $3+4$ $3+9$ $3+16$ $3+25$ $3+36$ $3+49$ $3+64$ $3+81$ $3+100$ $3+121$

   (b) $7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, \ldots$ 62 $\qquad a_n = 2 + 5n$

   $+5$ $+5$ $+5$

   (c) $1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, \ldots$ $\qquad a_n =$

   1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

   (d) $3, 9, 81, 6561, 43046721, \ldots$ $3^{32}$ $\qquad a_n = 3^{2^{(n-1)}}$

   $3^1$ $3^2$ $3^4$ $3^8$ $3^{16}$

2. (4 points) Compute each of these double sums.

   (a) $\sum_{i=1}^{2} \sum_{j=2}^{4} (i + 5j)$ $= 99$

   (b) $\sum_{i=0}^{2} \sum_{j=0}^{3} (2i + 3j)$ $= 78$

   (c) $\sum_{i=1}^{3} \sum_{j=0}^{2} i$ $= 18$

   (d) $\sum_{i=0}^{2} \sum_{j=1}^{3} ij^2$ $= 42$

3. (6 points) Compute each of these sums and show all your work.

   (a) $\sum_{i=0}^{n} 5^{i+1} - 5^i$

   (b) $\sum_{i=0}^{2n} (-2)^i$

4. (8 points) Find the closed form expression as a function of $n$ of the following double summation:

$$\sum_{j=1}^{n} \sum_{k=j}^{n} \frac{1}{k}$$

5. (5 points) Prove by induction that $\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$.

(a) $\sum_{i=1}^{2}\sum_{j=2}^{4}(i+5j)$

$= \sum_{i=1}^{2}\left[\sum_{j=2}^{4}(i+5j)\right]$

$= \sum_{i=1}^{2}\left[(i+10)+(i+15)+(i+20)\right]$ or $\sum_{i=1}^{2}\left[3i+45\right]$

$= \left[(1+10)+(1+15)+(1+20)\right]+\left[(2+10)+(2+15)+(2+20)\right]$ or $\left[(3+45)+(6+45)\right]$

$= 48+51$

$= 99$

(b) $\sum_{i=0}^{2}\sum_{j=0}^{3}(2i+3j)$

$= \sum_{i=0}^{2}\left[\sum_{j=0}^{3}(2i+3j)\right]$

$= \sum_{i=0}^{2}\left[(2i+0)+(2i+3)+(2i+6)+(2i+9)\right]$

$= \sum_{i=0}^{2}\left[8i+18\right]$

$= \left[(18)+(8+18)+(16+18)\right] =$

$= 18+18+18+24$

$= 78$

(c) $\sum_{i=1}^{3}\sum_{j=0}^{2} i$

$= \sum_{i=1}^{3}\left[\sum_{j=0}^{2} i\right]$

$= \sum_{i=1}^{3}\left[i+i+i\right]$

$= \sum_{i=1}^{3}\left[3i\right]$

$= 3+6+9$

$= 18$

(d) $\sum_{i=0}^{2}\sum_{j=1}^{3} ij^2$

$= \sum_{i=0}^{2}\left[\sum_{j=1}^{3} ij^2\right]$

$= \sum_{i=0}^{2}\left[(i)+(4i)+(9i)\right]$

$= \sum_{i=0}^{2}\left[14i\right]$

$= 0+14+28$

$= 42$

Notes on sums from video - Kimberly Brehm 2.4.3. Intro to Summation & 2.4.4 Summation Properties and Formulas

## Sigma Notation

to express sum of the terms of the sequence $a_n = \{ a_m, a_{m+1}, a_{m+2}, \dots a_n \}$

$$\sum_{i=m}^{n} a_i$$

i is the iterator variable
i starts as m, increment by 1 up to n
↳ take all these values and add them together

## Basic Properties

- $\sum_{k=1}^{n} c\, a_k = c\left(\sum_{k=1}^{n} a_k\right)$ can take out constants
- $\sum_{k=1}^{n} c = \sum_{k=1}^{n} cn$ just a constant = constant times n
- $\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$ can split at + or −
- $\sum_{k=1}^{n} a_k = \sum_{k=1}^{j} a_k + \sum_{k=j+1}^{n} a_k$ splitting sum in two by limits
- $\sum_{k=1}^{n} a_k = \sum_{k=1+m}^{n+m} a_{k-m}$ for $r \in W$ add to limits, but subtract from iterator

## Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

note i=1

Summation Formula for geometric progression/sequence

$$\sum_{k=0}^{n} ar^k = \frac{ar^{n+1} - a}{r-1}$$

note k=0

also

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

and

$$\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

I dont think I need these

ex. $\sum_{k=0}^{7} \left(\overset{a}{2}\cdot \overset{r}{3}^k + \overset{a}{5}\cdot \overset{r}{2}^k\right)$

$= \sum_{k=0}^{7} 2\cdot 3^k + \sum_{k=0}^{7} 5\cdot 2^k$

$= \frac{2(3)^8 - 2}{2} + \frac{5(2)^8 - 5}{1}$

$= \frac{13122 - 2}{2} + 1280 - 5$

$= 6560 + 1275$

$= 7835$

(a) $\sum_{i=0}^{n} 5^{i+1} - 5^i$

$$= \sum_{i=0}^{n} 5^{i+1} - \sum_{i=0}^{n} 5^i \qquad = 4\left(\frac{5^{n+1}-1}{5-1}\right)$$

$$\sum_{i=0}^{n} 5^i \cdot 5 - \sum_{i=0}^{n} 5^i \qquad = \cancel{4}\left(\frac{5^{n+1}-1}{\cancel{4}}\right)$$

$$\left(5 \sum_{i=0}^{n} 5^i\right) - \sum_{i=0}^{n} 5^i$$

$$= 4 \sum_{i=0}^{n} 5^i$$

(b) $\sum_{i=0}^{2n}(-2)^i$

$$= \sum_{i=0}^{2n} (-2)^i$$

$$= \frac{(-2)^{2n+1}-1}{-2-1}$$

$$= \frac{(-2)^{2n+1}-1}{-3}$$

4. (8 points) Find the closed form expression as a function of $n$ of the following double summation:

$$\sum_{j=1}^{n}\sum_{k=j}^{n}\frac{1}{k}$$

$$= \sum_{j=1}^{n}\left(\sum_{k=j}^{n}\frac{1}{k}\right)$$

$$\sum_{k=j}^{n}\frac{1}{k} < \ln n + 1$$

$$\sum_{j=1}^{n}\left(\sum_{k=j}^{n}\frac{1}{k}\right) < \sum_{j=1}^{n}(\ln n + 1)$$

$$< \sum_{j=1}^{n}(\ln n) + \sum_{j=1}^{n} 1$$

$$< \sum_{j=1}^{n}(\ln n) + n$$

$$<$$

5. (5 points) Prove by induction that $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

base case $n=1$

$1^2 = \frac{1}{6}(1)(1+1)(2(1)+1)$

$1 = \frac{1}{6}(2)(3)$

$1 = 1$ ✓

induction hypothesis

for all $k$, $\sum_{i=1}^{k} i^2 = \frac{1}{6}k(k+1)(2k+1)$

induction step

$\sum_{i=1}^{k+1} i^2 = \frac{1}{6}(k+1)(k+1+1)(2(k+1)+1)$

$(k+1)^2 + \sum_{i=1}^{k} i^2$

$(k+1)^2 + \frac{1}{6}k(k+1)(2k+1) = \frac{1}{6}(k+1)(k+2)(2k+3)$

$k^2+2k+1 + \frac{1}{6}k(2k^2+3k+1) = \frac{1}{6}(k^2+3k+2)(2k+3)$

$k^2+2k+1 + \frac{1}{6}(2k^3+3k^2+k) = \frac{1}{6}(2k^3+9k^2+13k+6)$

$\frac{1}{6}(6k^2+12k+6)$

$\frac{1}{6}(2k^3+9k^2+13k+6) = \frac{1}{6}(2k^3+9k^2+13k+6)$ ✓ done