CSci 243 Homework 5

Due: Wed Oct 12, end of day Carlo Mehegan

1. (12 points) For each of these partial sequences of integers, determine the next term of the sequence, and then provide a general formula or rule to generates terms of the sequence. For full credit, give a closed form ($a_n = ..$) instead of a recursive formula. Explain your derivations.

   (a) $3, 4, 7, 12, 19, 28, 39, 52, 67, 84, 103, \ldots 124$. $a_n = 3 + (n-1)^2$

   (b) $7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, \ldots 62$. $a_n = 2 + 5n$

   (c) $1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, \ldots$ Not enough info.

   (d) $3, 9, 81, 6561, 43046721, \ldots 3^{32}$. $a_n = 3^{2^{n-1}}$

2. (4 points) Compute each of these double sums.

   (a) $\sum_{i=1}^{2} \sum_{j=2}^{4} (i + 5j) = 99$

   (b) $\sum_{i=0}^{2} \sum_{j=0}^{3} (2i + 3j) = 78$

   (c) $\sum_{i=1}^{3} \sum_{j=0}^{2} i = 18$

   (d) $\sum_{i=0}^{2} \sum_{j=1}^{3} ij^2 = 42$

3. (6 points) Compute each of these sums and show all your work.

   (a) $\sum_{i=0}^{n} 5^{i+1} - 5^i$
   $= \sum_{i=0}^{n} 5^{i+1} - \sum_{i=0}^{n} 5^i$
   $= \sum_{i=0}^{n} 5^i * 5 - \sum_{i=0}^{n} 5^i$
   $= (5 * \sum_{i=0}^{n} 5^i) - \sum_{i=0}^{n} 5^i$
   $= 4 * \sum_{i=0}^{n} 5^i$
   $= 4(\frac{5^{n+1}-1}{5-1})$
   $= 4(\frac{5^{n+1}-1}{4})$
   $= 5^{n+1} - 1$

   (b) $\sum_{i=0}^{2n} (-2)^i$
   $= \frac{(-2)^{2n+1}-1}{-2-1}$
   $= \frac{(-2)^{2n+1}-1}{-3}$

4. (8 points) Find the closed form expression as a function of n of the following double summation:

$$\sum_{j=1}^{n} \sum_{k=j}^{n} \frac{1}{k}$$

$\sum_{k=j}^{n} \frac{1}{k} < lnn + 1$
$\Rightarrow \sum_{j=1}^{n} (\sum_{k=j}^{n} \frac{1}{k}) < \sum_{j=1}^{n} (lnn + 1)$
$\Rightarrow \sum_{j=1}^{n} (\sum_{k=j}^{n} \frac{1}{k}) < \sum_{j=1}^{n} (lnn) + \sum_{j=1}^{n} 1$
$\Rightarrow \sum_{j=1}^{n} (\sum_{k=j}^{n} \frac{1}{k}) < \sum_{j=1}^{n} (lnn) + n$

5. (5 points) Prove by induction that $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

   Base case: $n = 1$

   $1^2 = \frac{1}{6}(1)(1+1)(2+1)$

   $1 = \frac{1}{6}(2)(3)$

   $1 = 1$

   Inductive hypothesis: for all k, $\sum_{i=1}^{k} i^2 = \frac{1}{6}k(k+1)(2k+1)$.

   Inductive step:

   want to prove $\sum_{i=1}^{k+1} i^2 = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$

   $\sum_{i=1}^{k+1} i^2 = (k+1)^2 + \sum_{i=1}^{k} i^2 = (k+1)^2 + \frac{1}{6}k(k+1)(2k+1)$

   now show $(k+1)^2 + \frac{1}{6}k(k+1)(2k+1) = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$

   $\Rightarrow k^2 + 2k + 1 + \frac{1}{6}k(2k^2 + 3k + 1) = \frac{1}{6}(k^2 + 3k + 2)(2k + 3)$

   $\Rightarrow \frac{1}{6}(6k^2 + 12k + 6) + \frac{1}{6}(2k^3 + 3k^2 + k) = \frac{1}{6}(2k^3 + 9k^2 + 13k + 6)$

   $\Rightarrow \frac{1}{6}(2k^3 + 9k^2 + 13k + 6) = \frac{1}{6}(2k^3 + 9k^2 + 13k + 6)$