Projective Synchronization

Image Analysis and Computer Vision Project

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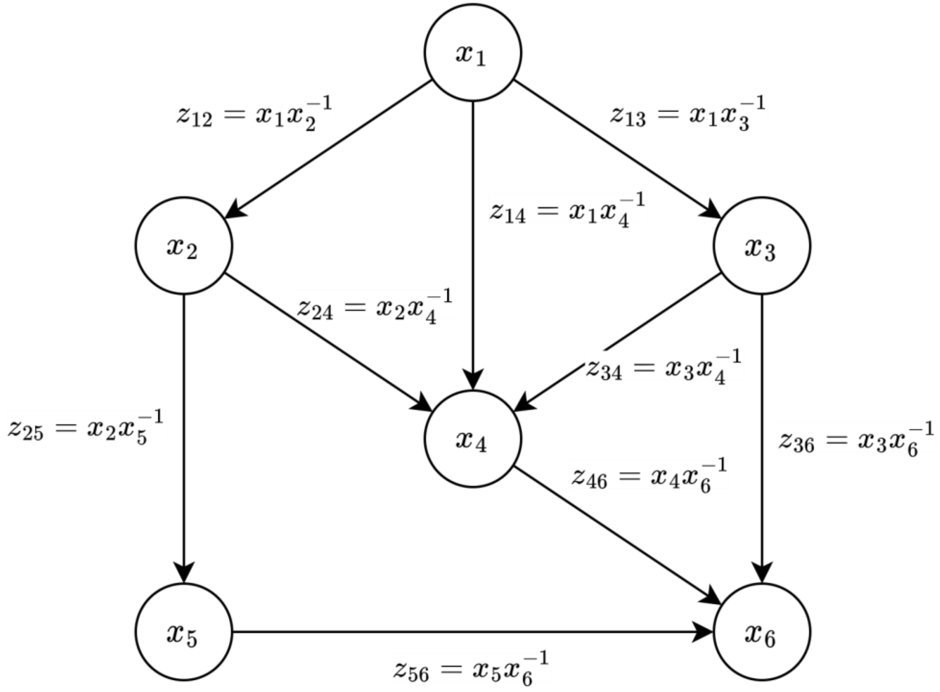
The synchronization problem

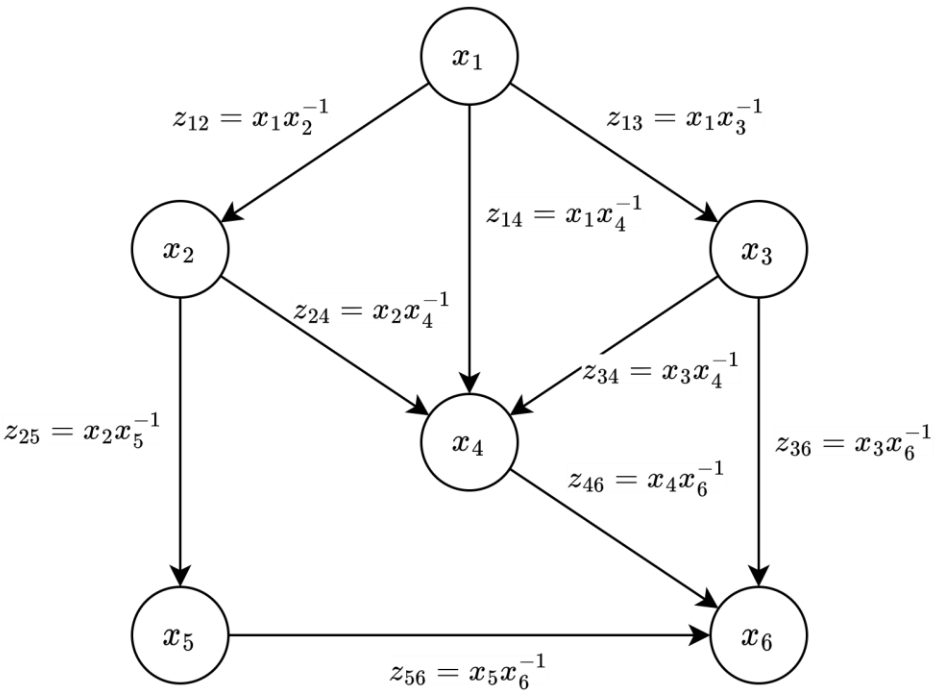


In the synchronization problem, there is a network where:

- The node states x_i are unknown
- A certain relation (measurement) z_{ij} between (some) pair of nodes (x_i, x_j) is know

The goal is to find the unknown states x_i

The synchronization problem

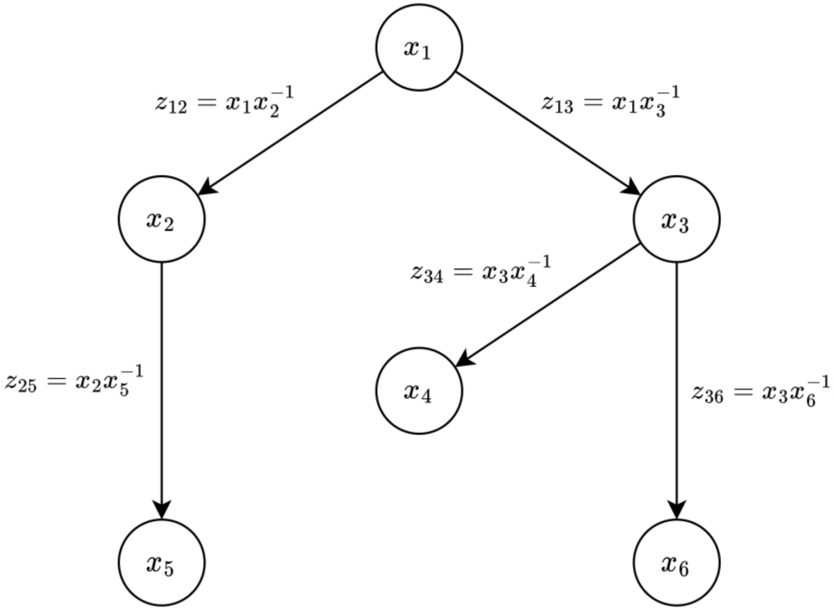


The problem is represented by a graph, where the measurements are labeling of the edges.

- The graph must be connected
- Nodes are element of a group $(\Sigma,*)$
- Labels z_{ij} must be consistent:
  - $z_{ij} = x_i * x_j^{-1}$
  - if $\{(x_1, x_2), (x_2, x_3), \ldots, (x_l, x_1)\}$ forms a cycle then $z_{12} * z_{23} * \cdots * z_{l1} = 1_\Sigma$ (in the noise free case)

The reference solution: spanning tree approach

A basic approach consist in finding a minimum spanning tree of the measurement graph and use the consistency property to find the unknown labels of the nodes. The solution has an ambiguity: the reference node used as a root of the spanning tree (e.g. the node with the highest out degree).



If there is no noise, the solution is exact:

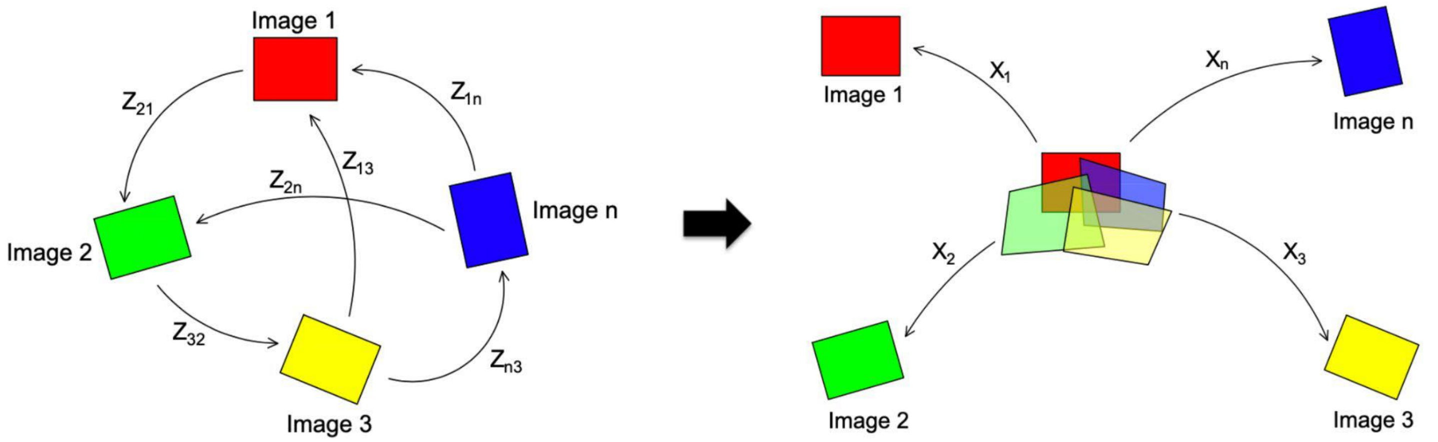
- $x_2 = x_1 * z_{12}^{-1}$
- $x_3 = x_1 * z_{13}^{-1}$
- $x_4 = x_3 * z_{34}^{-1}$
- $x_5 = x_2 * z_{25}^{-1}$
- $x_6 = x_3 * z_{36}^{-1}$

The solution is defined up to x_1, that can be set to any element of Σ (e.g. 1_Σ)

Homography synchronization

Homography synchronization: Σ is the set of 2-dimensional homographies, i.e. 3x3 invertible matrix defined up to a scale.

Homography synchronization can be used to solve the image mosaicing problem.



Homography synchronization – Spectral solution

To remove the scale, the measurements Z_{ij} (pairwise homographies) are divided by the 3-rd root of the determinant, obtaining the $SL(3)$ group.

The problem can be reduced to an eigenvector problem (spectral solution):

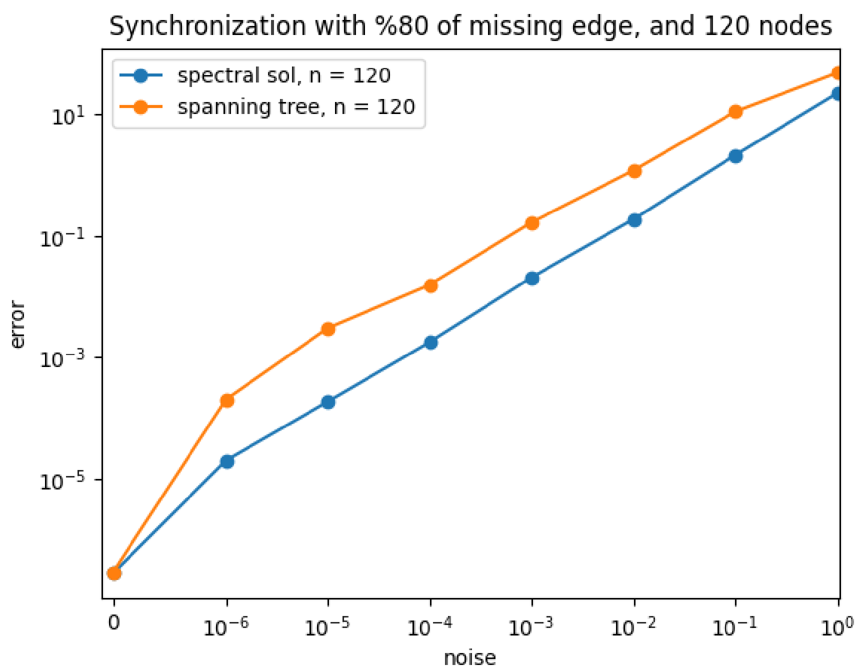
$$X = \begin{bmatrix} X_1 \\ \dots \\ X_n \end{bmatrix} \; Z = \begin{bmatrix} Z_{11} & \cdots & Z_{n1} \\ \vdots & \ddots & \vdots \\ Z_{n1} & \cdots & Z_{nn} \end{bmatrix} = XX^{-b} \text{, where } Z_{ij} = I_d.$$

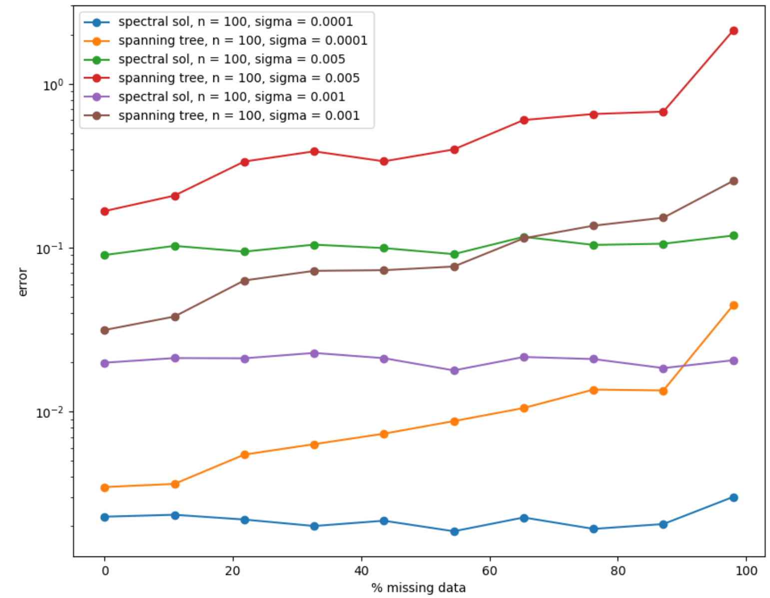
X can be found as the 3 eigenvectors corresponding to the 3 highest eigenvalues of the matrix $(D \otimes I_d)^{-1}\left(Z \circ (A \otimes 1_{3x3})\right)$, where

- D is the degree matrix of the graph, A is the adjacency matrix
- $\circ$ is the Hadamard product and $\otimes$ is the Kronecker product

The solution is the scaled w.r.t. a reference node (e.g. the one with the highest degree).

Homography synchronization





Synthetic datasets generation and error computation

The tests use some synthetic datasets generated in the following way:

- The matrix $X \in \mathbb{R}^{3n\times 3}$ was generated from random transformations $X_i \in \mathbb{R}^{3\times 3}$
- The matrix Z was computed as $Z = XX^{-b}$
- A gaussian noise was added to Z
- Some random transformations were selected to be removed from the graph

Synthetic datasets generation and error computation

The error in the solution U was computed in the following way:

- Each matrix X_i and U_i is vectorized:

$$x_i = [X_i(1,1),\ X_i(1,2), \ldots, X_i(3,3)]^T,\ u_i = [U_i(1,1),\ U_i(1,2), \ldots, U_i(3,3)]^T$$

- The error is the angle between the two vectors:

$$\varepsilon_i = \arccos\left(\frac{x_i^T u_i}{||x_i||\,||u_i||}\right)$$

- The final error is the sum of all ε_i

Projective synchronization

In the projective synchronization problem, Σ is the set of 3-dimensional homographies, i.e. 4x4 invertible matrix defined up to a scale.

Issue w.r.t homography synchronization: Z_{ij} is a 4x4 matrix, so the determinant can be negative, thus dividing by the 4-th root of the determinant can yield a complex matrix, so the problem cannot be formulated in the $SL(4)$ group.

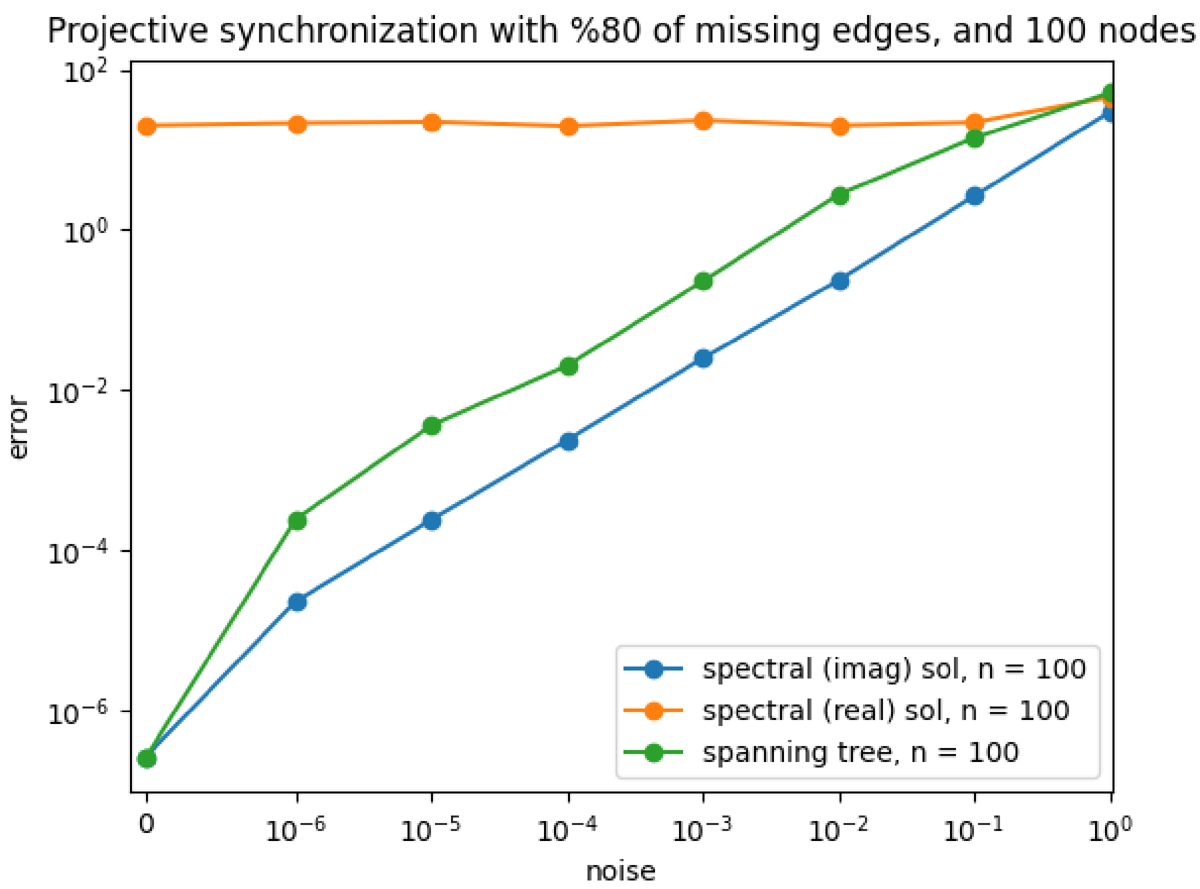
Extending the spectral solution

1) The scale in the labels Z_{ij} can be removed by dividing by the 4-th root of the determinant, which is potentially a complex number, obtaining a matrix Z that is complex
2) A complex solution X can be found by computing the top 4 eigenvectors of the matrix $(D \otimes I_d)^{-1}\big(Z \circ (A \otimes 1_{4x4})\big)$

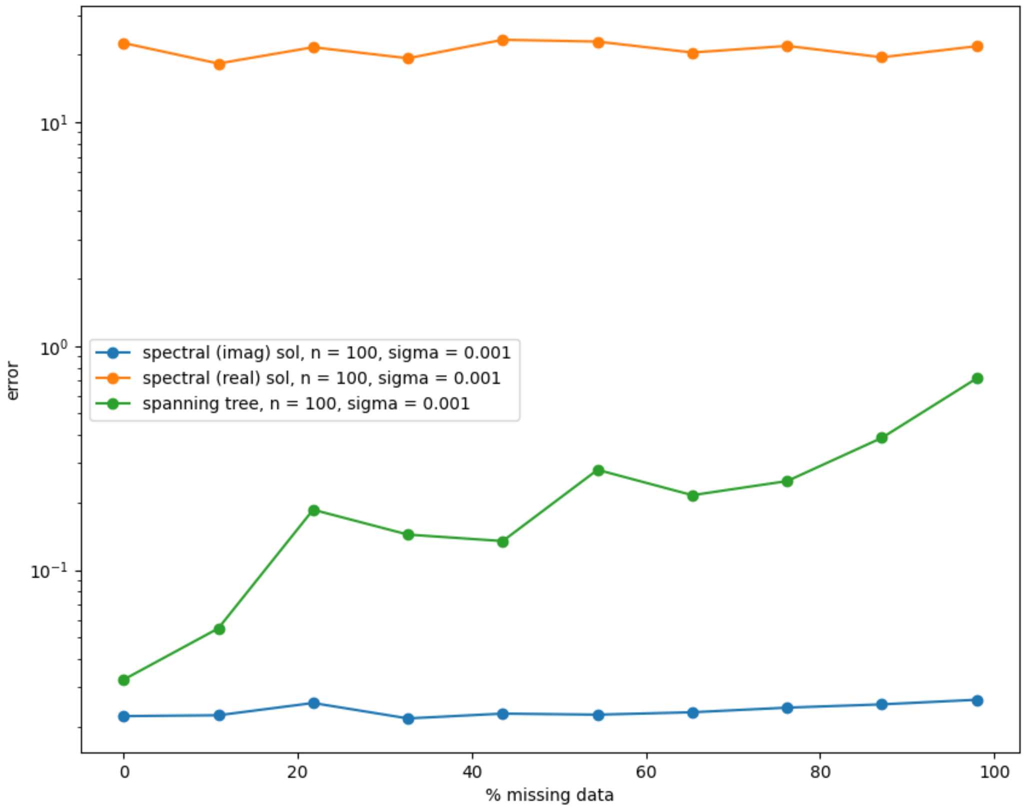
The imaginary part of X have to be removed, there are two possibilities:

- Removing it after computing the eigenvectors, and before scaling by the reference node
- Removing it after scaling the matrices using the reference node

Extending the spectral solution

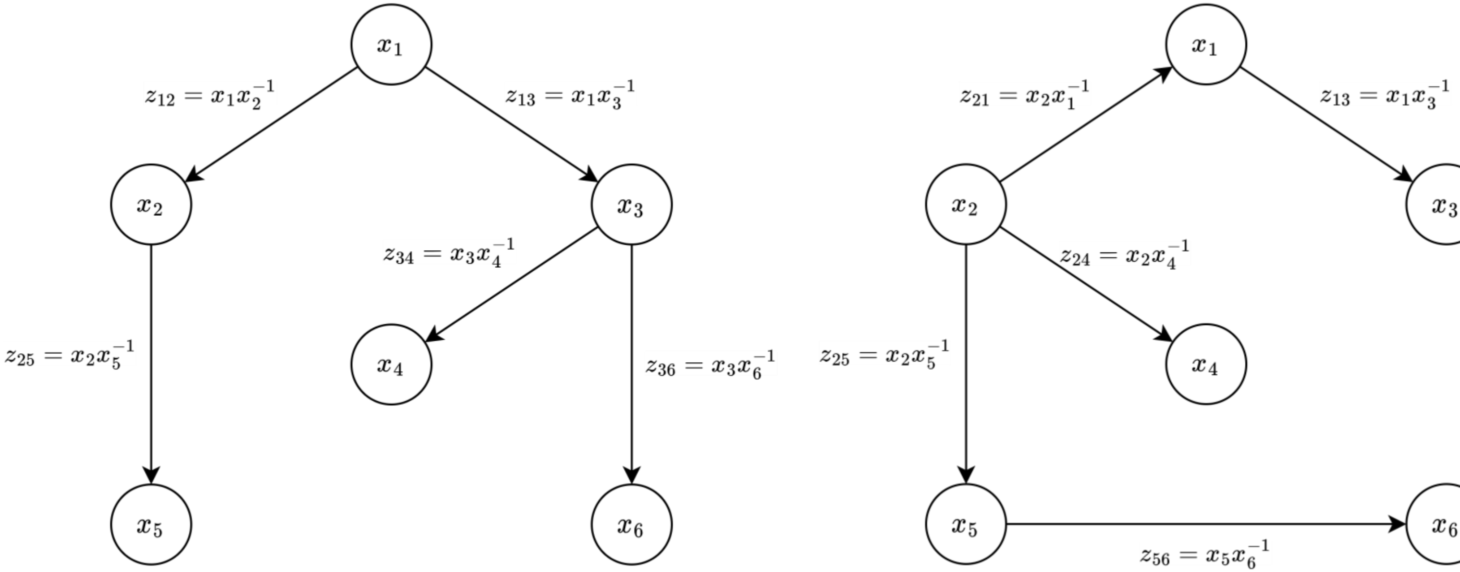


Extending the spectral solution



Multi-Source Propagation

In the spanning tree approach, the root of the tree can be chosen in different ways. Multi-Source Propagation exploit this fact by aggregating different solutions from different spanning trees.



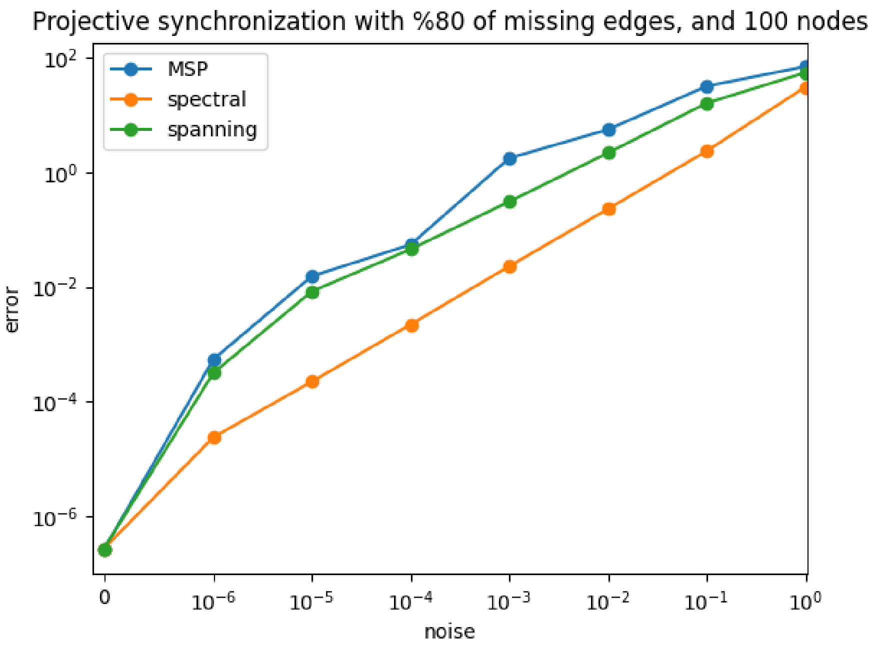
Multi-Source Propagation

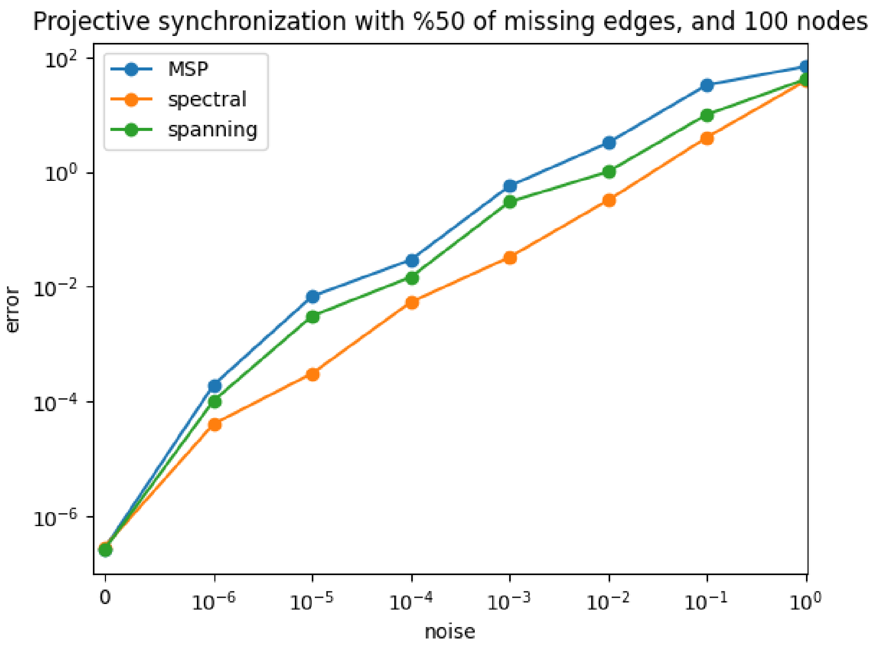
1. Compute the spanning tree solution for 10 different roots: the 10 nodes $n_1, \dots, n_{10}$ with the highest degree, obtaining

$$X^{(i)} = \begin{bmatrix} X_1^{(i)} \\ \dots \\ X_n^{(i)} \end{bmatrix}, \text{ where } X_{n_i}^{(i)} = I_4, \text{ for } i = 1, \dots 10$$

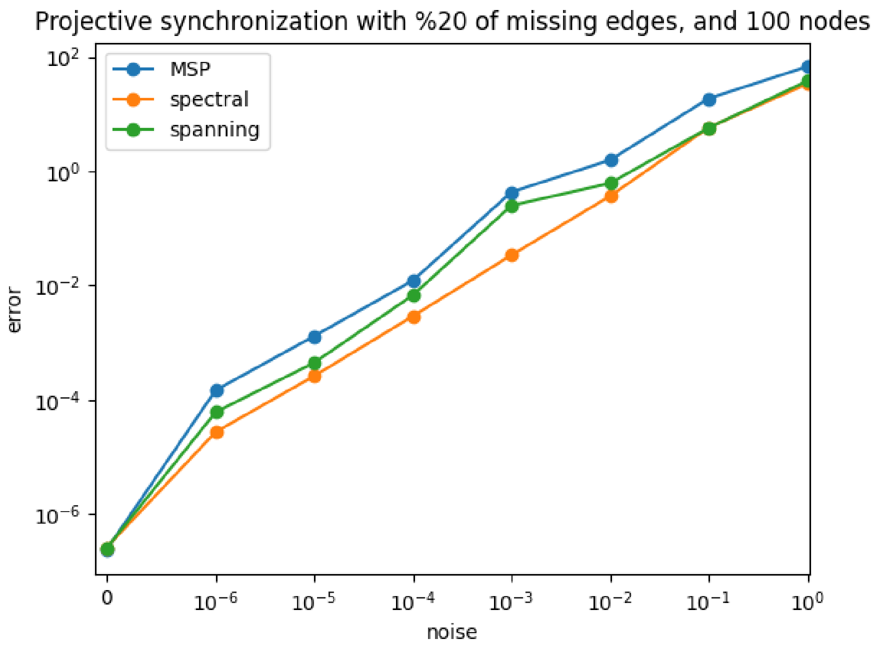
2. Align each solution to the same reference node (the node n_m with the highest degree): ${X'}_j^{(i)} = X_j^{(i)} {X_{n_m}^{(i)}}^{-1}$
3. Normalize each matrix using the norm of the vectorized form of the matrix and the sum of the elements.
4. Aggregate the 10 results by computing the average for each node

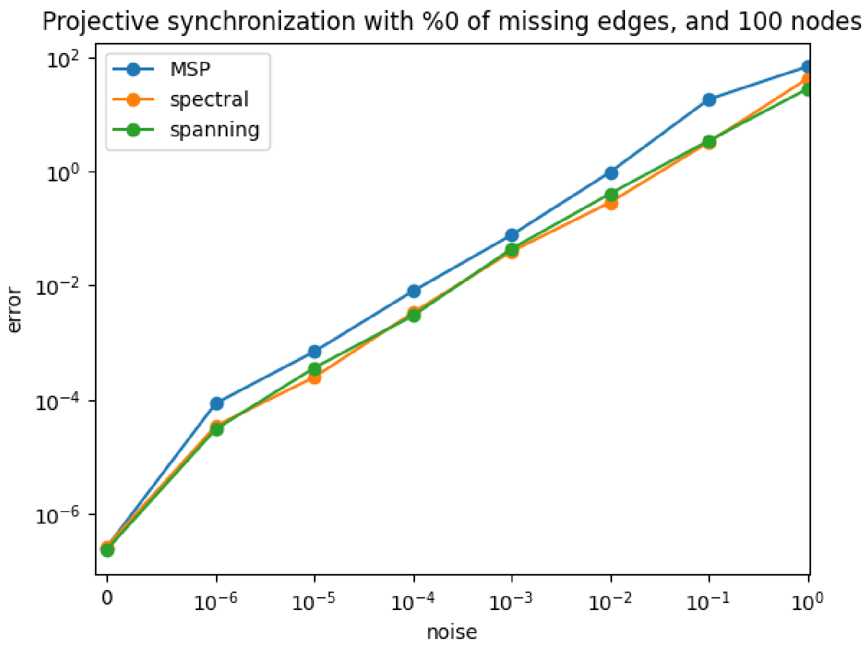
Multi-Source Propagation



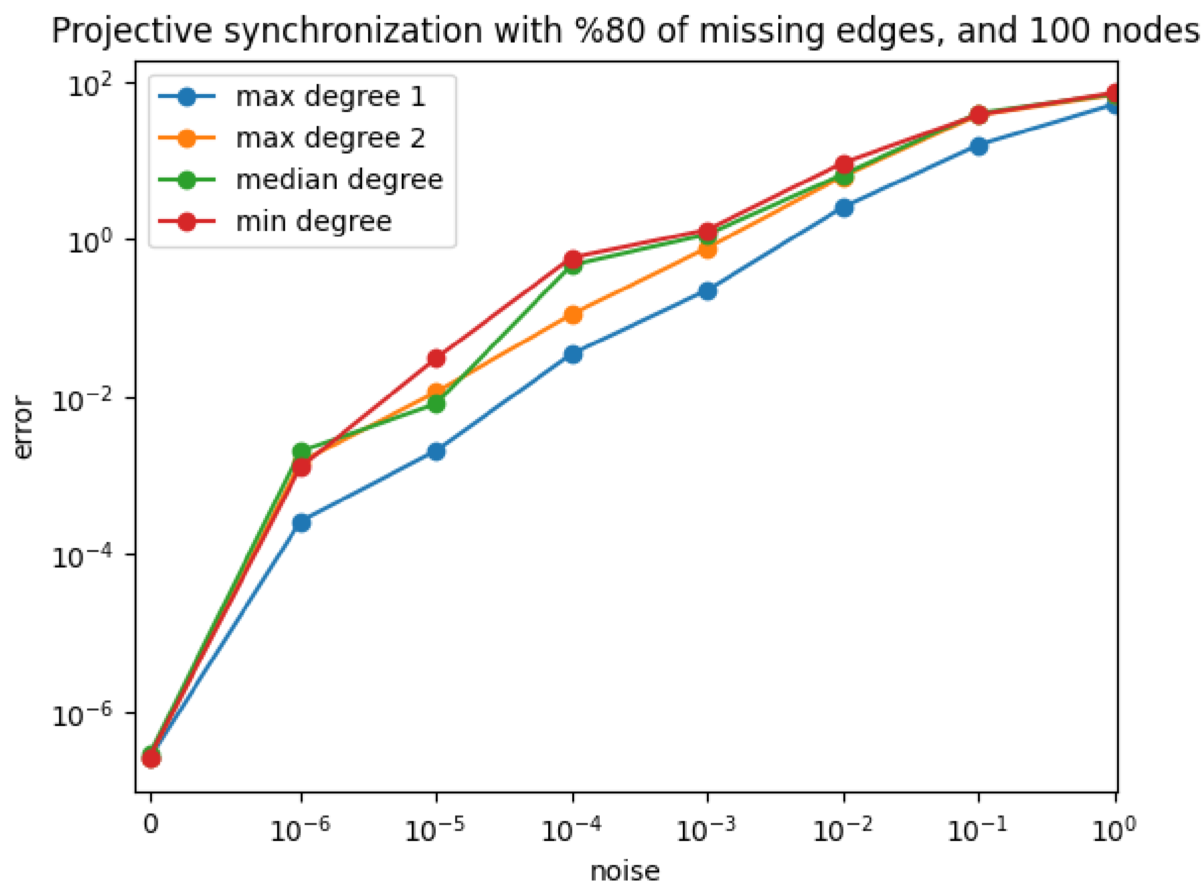


Multi-Source Propagation





Multi-Source Propagation



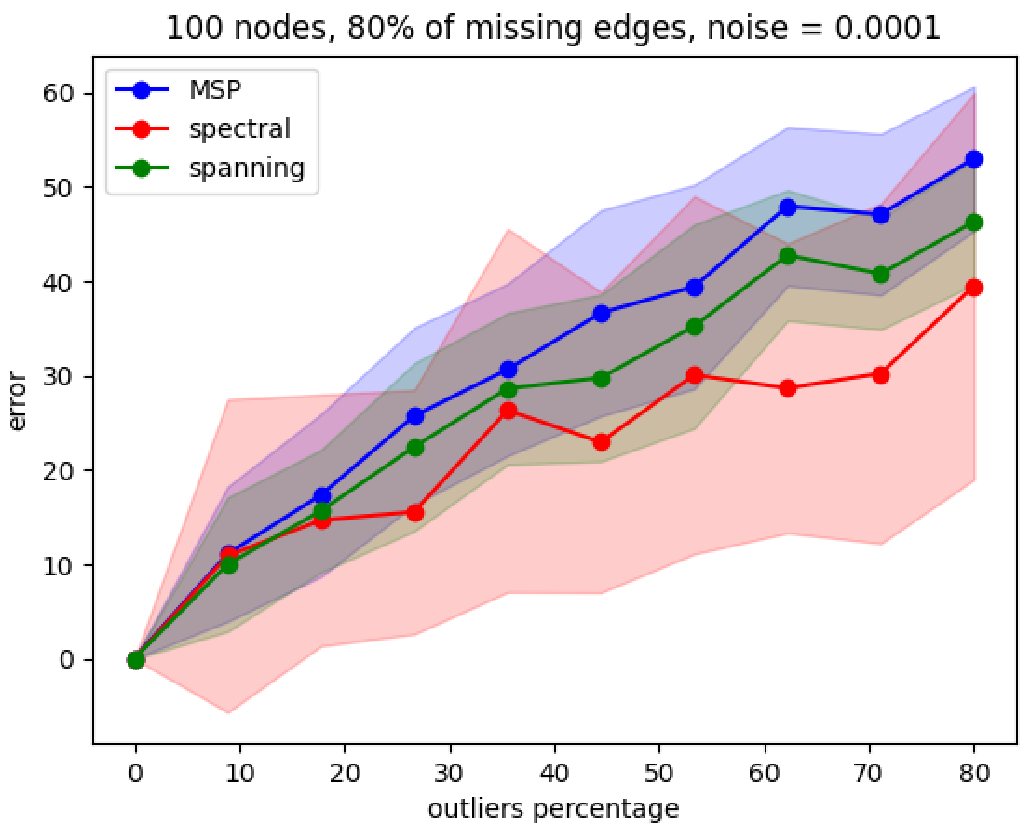
Tests with outliers

Given a certain percentage of outliers α some synthetic datasets have been generated in the following way:

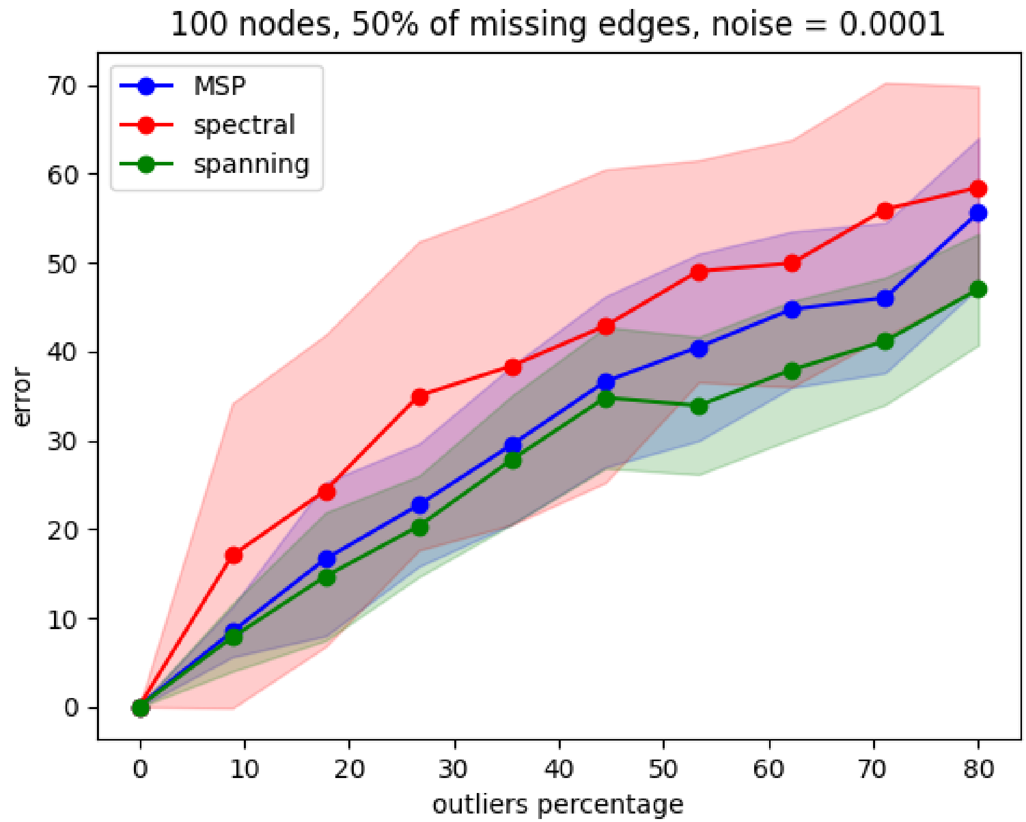
- An initial block vector $X \in \mathbb{R}^{4n\times 4}$ was generated from random transformations $X_i \in \mathbb{R}^{4\times 4}$, from which the matrix $Z = XX^{-b}$ was computed
- A gaussian noise was added to Z
- Some random edges were deleted from the graph
- A percentage α of transformations (i,j) were selected from the remaining transformations and replaced with totally random transformations R_{ij} :

$$Z_{ij} = R_{ij} \, , \, Z_{ji} = R_{ij}^{-1}$$

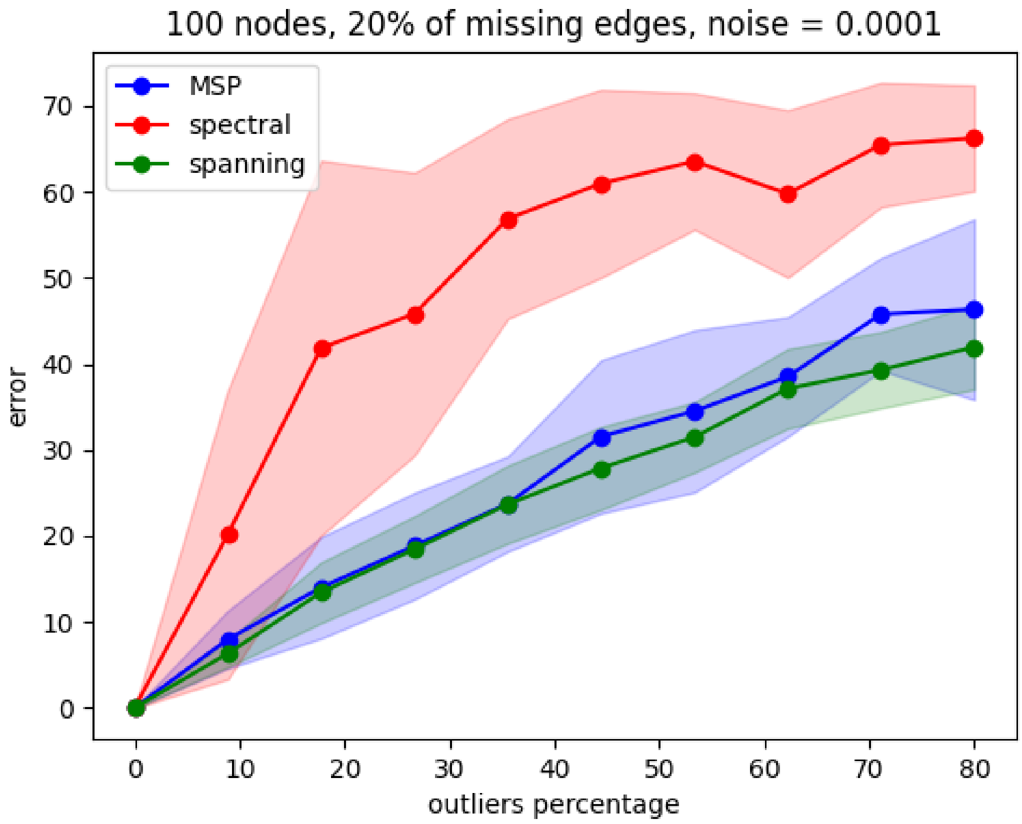
Tests with outliers



Tests with outliers



Tests with outliers



Conclusions

- The extension of the spectral solution using complex matrices permit to obtain similar results w.r.t. the homography synchronization problem.

- The Multi Source Propagation does not improve the spanning tree solution because of numerical errors due to many matrix multiplications

- The extension of spectral solution to the projective synchronization gives the best results when there are no outliers, especially when there are many missing edges (the spanning tree is deeper). Further analysis is required to understand this behaviour