Departamento de Matemática para a Ciência e Tecnologia

Universidade do Minho

Cálculo

Resolução da Ficha 1-A — Outubro de 2008

1. Calcule :

   (a) $\arcsin(-\frac{\sqrt{2}}{2})$

   Seja $y = \arcsin(-\frac{\sqrt{2}}{2})$.Então, verifica-se as seguintes condições:

$$\begin{cases} y = \arcsin(-\frac{\sqrt{2}}{2}) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ \sin y = -\frac{\sqrt{2}}{2}, \end{cases}$$

   Portanto

$$y = -\frac{\pi}{4}.$$

   (b) $\cot\left(\arcsin(-\frac{4}{5})\right)$

   1^o método

   Seja $x = \arcsin(-\frac{4}{5}) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Então, tem-se que $\cos x > 0$, i.e., $\cos\left(\arcsin(-\frac{4}{5})\right) > 0$.

$$\cot\left(\arcsin(-\frac{4}{5})\right) = \frac{\cos\left(\arcsin(-\frac{4}{5})\right)}{\sin\left(\arcsin(-\frac{4}{5})\right)} = \frac{\sqrt{1-\sin^2\left(\arcsin(-\frac{4}{5})\right)}}{-\frac{4}{5}}$$
$$= \frac{\sqrt{1-(-\frac{4}{5})^2}}{-\frac{4}{5}} = \frac{\sqrt{\frac{9}{25}}}{-\frac{4}{5}} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$$

   2^o método

   Seja $x = \arcsin(-\frac{4}{5})$. Então

$$\sin x = -\frac{4}{5} \wedge x \in 4^oQ.$$

   Da relação

$$1 + \cot^2(x) = \frac{1}{\sin^2(x)}$$

   deduz-se

$$1 + \cot^2(x) = \frac{1}{\sin^2(x)} \Leftrightarrow 1 + \cot^2(x) = \frac{25}{16}$$
$$\Leftrightarrow \cot^2(x) = \frac{9}{16} \Leftrightarrow \cot(x) = \pm\frac{3}{4} \wedge x \in 4^oQ.$$

   Logo

$$\cot(x) = -\frac{3}{4}.$$

(c) Seja $x = \arcsin\left(\frac{1}{2}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ e $z = \arccos(\frac{3}{5}) \in [0, \pi]$. Portanto, $\cos\left(\arcsin\left(\frac{1}{2}\right)\right) > 0$ e $\sin\left(\arccos\left(\frac{3}{5}\right)\right) > 0$. Aplicando a fórmula $\cos(\alpha + \beta)$, obtém-se

$$
\begin{aligned}
&\cos\left[\arcsin(\frac{1}{2}) - \arccos(\frac{3}{5})\right] \\
&= \left\{ \begin{array}{c} \cos\left(\arcsin\left(\frac{1}{2}\right)\right)\cos\left(\arccos\left(\frac{3}{5}\right)\right) \\ +\sin\left(\arcsin\left(\frac{1}{2}\right)\right)\sin\left(\arccos\left(\frac{3}{5}\right)\right) \end{array} \right\} \\
&= \frac{3}{5}\cos\left(\arcsin\left(\frac{1}{2}\right)\right) + \frac{1}{2}\sin\left(\arccos\left(\frac{3}{5}\right)\right) \\
&= \frac{3}{5}\cos\left(\arcsin\left(\frac{1}{2}\right)\right) + \frac{1}{2}\sin\left(\arccos\left(\frac{3}{5}\right)\right) \\
&= \frac{3}{5}\sqrt{1 - \sin^2\left(\arcsin\left(\frac{1}{2}\right)\right)} + \frac{1}{2}\sqrt{1 - \cos^2\left(\arccos\left(\frac{3}{5}\right)\right)} \\
&= \frac{3}{5}\sqrt{1 - \sin^2\left(\arcsin\left(\frac{1}{2}\right)\right)} + \frac{1}{2}\sqrt{1 - \cos^2\left(\arccos\left(\frac{3}{5}\right)\right)} \\
&= \frac{3}{5}\sqrt{1 - \left(\frac{1}{2}\right)^2} + \frac{1}{2}\sqrt{1 - \left(\frac{3}{5}\right)^2} \\
&= \frac{3}{5}\sqrt{\frac{3}{4}} + \frac{1}{2}\sqrt{\frac{16}{25}} = \frac{3\sqrt{3} + 4}{10}
\end{aligned}
$$

2. Determine o número real R tal que:

(a) $R = \arcsin\left(\sin\frac{\pi}{2}\right) + 4\arcsin(-\frac{1}{2}) + 2\arccos(-\frac{\sqrt{2}}{2})$
Seja

$$
\begin{aligned}
&x = \sin\frac{\pi}{2} \in [-1, 1], \\
&y = \arcsin(-\frac{1}{2}) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Leftrightarrow \sin y = -\frac{1}{2} \Leftrightarrow y = -\frac{\pi}{6}, \\
&z = \arccos(-\frac{\sqrt{2}}{2}) \in [0, \pi] \Leftrightarrow \cos z = -\frac{\sqrt{2}}{2} \Leftrightarrow z = \frac{3\pi}{4}.
\end{aligned}
$$

Desta forma, podemos deduzir que:

$$
\begin{aligned}
R &= \arcsin\left(\sin\frac{\pi}{2}\right) + 4\arcsin(-\frac{1}{2}) + 2\arccos(-\frac{\sqrt{2}}{2}) \\
&= \frac{\pi}{2} - 4\frac{\pi}{6} + 2\frac{3\pi}{4} = \frac{\pi}{2} - \frac{2\pi}{3} + \frac{3\pi}{2} = \frac{3\pi - 4\pi + 9\pi}{6} = \frac{4\pi}{3}
\end{aligned}
$$

(b) $R = \cos^2\left(\frac{1}{2}\arccos\frac{1}{3}\right) - \sin^2\left(\frac{1}{2}\arccos\frac{1}{3}\right)$

$$
\begin{aligned}
R &= \cos^2\left(\frac{1}{2}\arccos\frac{1}{3}\right) - \sin^2\left(\frac{1}{2}\arccos\frac{1}{3}\right) \\
&= \cos\left[2\left(\frac{1}{2}\arccos\frac{1}{3}\right)\right] = \cos\left(\arccos\frac{1}{3}\right) = \frac{1}{3}
\end{aligned}
$$

(c) $R = \tan^2(\arcsin\frac{3}{5}) - \cot^2\left(\arccos\frac{4}{5}\right)$
Seja

$$
x = \arcsin\frac{3}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ e} y = \arccos\frac{4}{5} \in [0, \pi].
$$

Então

$$
\cos(\arcsin\frac{3}{5}) > 0 \text{ e } \sin\left(\arccos\frac{4}{5}\right) > 0,
$$

e

$$
\begin{aligned}
R &= \tan^2(\arcsin\frac{3}{5}) - \cot^2\left(\arccos\frac{4}{5}\right) \\
&= \left[\frac{\sin(\arcsin\frac{3}{5})}{\cos(\arcsin\frac{3}{5})}\right]^2 - \left[\frac{\cos\left(\arccos\frac{4}{5}\right)}{\sin\left(\arccos\frac{4}{5}\right)}\right]^2 \\
&= \left[\frac{\frac{3}{5}}{\cos(\arcsin\frac{3}{5})}\right]^2 - \left[\frac{\frac{4}{5}}{\sin\left(\arccos\frac{4}{5}\right)}\right]^2 \\
&= \left[\frac{\frac{3}{5}}{\sqrt{1-\sin^2(\arcsin\frac{3}{5})}}\right]^2 - \left[\frac{\frac{4}{5}}{\sqrt{1-\cos^2\left(\arccos\frac{4}{5}\right)}}\right]^2 \\
&= \left[\frac{\frac{3}{5}}{\sqrt{1-(\frac{3}{5})^2}}\right]^2 - \left[\frac{\frac{4}{5}}{\sqrt{1-\left(\frac{4}{5}\right)^2}}\right]^2 \\
&= \frac{\frac{9}{25}}{1-(\frac{3}{5})^2} - \frac{\frac{16}{25}}{1-\left(\frac{4}{5}\right)^2} = \frac{9}{16} - \frac{16}{9} = \frac{81-256}{16} = -\frac{175}{144}
\end{aligned}
$$

(d) Considere a função real de variável real definida por

$$p(x) = \frac{\pi}{3} - 2\arccos(x+1).$$

i. Calcule: $p(-1) - p(-\frac{3}{2})$.

$$
\begin{aligned}
p(-1) - p(-\frac{3}{2}) &= \frac{\pi}{3} - 2\arccos(0) - \frac{\pi}{3} + 2\arccos(-\frac{3}{2}+1) \\
&= \frac{\pi}{3} - 2\frac{\pi}{2} - \frac{\pi}{3} + 2\arccos(-\frac{1}{2}) \\
&= \frac{\pi}{3} - 2\frac{\pi}{2} - \frac{\pi}{3} + 2\frac{\pi}{3} = -\pi + 2\times\frac{2\pi}{3} = \frac{\pi}{3}
\end{aligned}
$$

ii. Determine o domínio e o contradomínio da função dada p.

$$
\begin{aligned}
D_p &= \{x \in \mathbb{R} : x+1 \in [-1,1]\} \\
&= \{x \in \mathbb{R} : -1 \leq x+1 \leq 1\} \\
&= \{x \in \mathbb{R} : -2 \leq x \leq 0\} = [-2, 0]
\end{aligned}
$$

Sabe-se que $\arccos(x+1) \in D'_{arccos} = [0, \pi]$. Então, deduz-se que:

$$
\begin{gathered}
0 \leq \arccos(x+1) \leq \pi \\
-2\pi \leq -2\arccos(x+1) \leq 0 \\
\frac{\pi}{3} - 2\pi \leq \underbrace{\frac{\pi}{3} - 2\arccos(x+1)}_{p(x)} \leq \frac{\pi}{3}
\end{gathered}
$$

Logo

$$D'_p = \left[-\frac{5\pi}{3}, \frac{\pi}{3}\right]$$

iii. Calcule caso existam, os zeros de p.

$$
\begin{aligned}
p(x) = 0 \Leftrightarrow \frac{\pi}{3} - 2\arccos(x+1) = 0 \Leftrightarrow \arccos(x+1) = \frac{\pi}{6} \\
\Leftrightarrow x+1 = \cos\frac{\pi}{6} \Leftrightarrow x = \cos\frac{\pi}{6} - 1 = \frac{\sqrt{3}-2}{2} \in D_p
\end{aligned}
$$

iv. Caracterize a função inversa de p.

$$y = \frac{\pi}{3} - 2\arccos(x+1) \Leftrightarrow \frac{\pi}{6} - \frac{y}{2} = \arccos(x+1)$$
$$\Leftrightarrow \cos\left(\frac{\pi}{6} - \frac{y}{2}\right) = x+1 \Leftrightarrow x = \cos\left(\frac{3\pi - y}{6}\right) - 1$$

$$\begin{array}{rccc} p^{-1}: & \left[-\frac{5\pi}{3}, \frac{\pi}{3}\right] & \to & [-2,0] \\ & y & \mapsto & \cos\left(\frac{3\pi-y}{6}\right) - 1 \end{array}$$

v. Resolva a seguinte inequação: $p(x) \leq -\frac{\pi}{3}$.

$$p(x) \leq -\frac{\pi}{3} \Leftrightarrow -2\arccos(x+1) \leq -\frac{2\pi}{3}$$
$$\Leftrightarrow \arccos(x+1) \geq \frac{\pi}{3} \Leftrightarrow \arccos(x+1) \geq \arccos(\frac{1}{2})$$

Como $\arccos y$ é uma função decrescente, então

$$x+1 \leq \frac{1}{2} \Leftrightarrow x \leq -\frac{1}{2}$$

e

$$S = \left]-\infty, -\frac{1}{2}\right] \cap D_p = \left]-\infty, -\frac{1}{2}\right] \cap [0,2] = \left[-2, -\frac{1}{2}\right]$$

(e) Determine a expressão das derivadas das funções seguintes:

i. $g(t) = 3t.\arcsin\left(\sqrt{t^2-1}\right)$

$$g'(t) = 3.\arcsin\left(\sqrt{t^2-1}\right) + 3t.\frac{\left(\sqrt{t^2-1}\right)'}{1-\left(\sqrt{t^2-1}\right)^2}$$
$$= 3.\arcsin\left(\sqrt{t^2-1}\right) + 3t.\frac{\frac{1}{2}\frac{2t}{\sqrt{t^2-1}}}{2-t^2}$$
$$= 3\arcsin\left(\sqrt{t^2-1}\right) + \frac{3t^2}{\sqrt{2-t^2}\sqrt{t^2-1}}$$

ii. $f(y) = \frac{1}{\cos y} - \arctan(\frac{y}{2})$

$$f'(y) = \left[\frac{1}{\cos y}\right]' - \left[\arctan(\frac{y}{2})\right]' = \frac{\sin y}{\cos^2 y} - \frac{1/2}{1+\left(\frac{y}{2}\right)^2}$$
$$= \left[\frac{1}{\cos y}\right]' - \left[\arctan(\frac{y}{2})\right]' = \frac{\sin y}{\cos^2 y} - \frac{2}{4+y^2}$$