Exercise with Solution - Quantum Teleportation

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In the following pages, I present a detailed derivation of each stage of the quantum teleportation circuit, each labelled $|\psi\rangle_i$, with $i \in \{0,1,2,3,4\}$, using the Bell state $|\beta_{01}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$. To visually follow the quantum teleportation protocol, please see Fig. (1).

Let us denote our Bell state by

$$|\beta_{01}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

- $|\psi_0\rangle$

$$\begin{aligned}
|\psi\rangle_0 =& |\psi\rangle \otimes |\beta_{01}\rangle \\
=& (\alpha|0\rangle + \beta|1\rangle) \otimes \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right) \\
=& \frac{1}{\sqrt{2}}(\alpha|0\rangle(|00\rangle - |11\rangle) + \beta|1\rangle(|00\rangle - |11\rangle)) \\
=& \frac{1}{\sqrt{2}}(\alpha|0\rangle|00\rangle - \alpha|0\rangle|11\rangle + \beta|1\rangle|00\rangle - \beta|1\rangle|11\rangle) \\
=& \frac{1}{\sqrt{2}}(\alpha|00\rangle|0\rangle - \alpha|01\rangle|1\rangle + \beta|10\rangle|0\rangle - \beta|11\rangle|1\rangle)
\end{aligned}$$

So,

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(\alpha|00\rangle|0\rangle - \alpha|01\rangle|1\rangle + \beta|10\rangle|0\rangle - \beta|11\rangle|1\rangle) \tag{1}$$

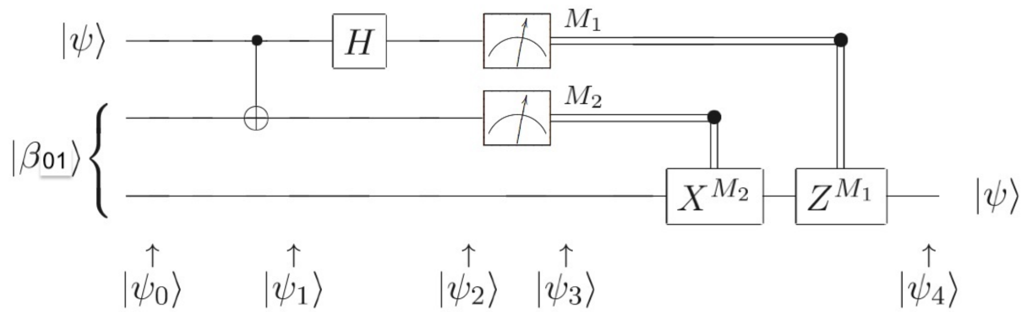

Figure 1.13. Quantum circuit for teleporting a qubit. The two top lines represent Alice's system, while the bottom line is Bob's system. The meters represent measurement, and the double lines coming out of them carry classical bits (recall that single lines denote qubits).

Figure 1: Quantum Teleportation Circuit (figure taken from [1]).

- $|\psi_1\rangle$

$$
\begin{aligned}
|\psi_1\rangle =& \hat{C}_{\mathrm{NOT}} \otimes \hat{\mathbb{I}}\ |\psi_0\rangle \\
=& (\hat{C}_{\mathrm{NOT}} \otimes \hat{\mathbb{I}}) \frac{1}{\sqrt{2}} (\alpha|00\rangle|0\rangle - \alpha|01\rangle|1\rangle + \beta|10\rangle|0\rangle - \beta|11\rangle|1\rangle) \\
=& \frac{1}{\sqrt{2}} [\hat{C}_{\mathrm{NOT}} \otimes \hat{\mathbb{I}}\ (\alpha|00\rangle|0\rangle) - \hat{C}_{\mathrm{NOT}} \otimes \hat{\mathbb{I}}\ (\alpha|01\rangle|1\rangle) + \hat{C}_{\mathrm{NOT}} \otimes \hat{\mathbb{I}}\ (\beta|10\rangle|0\rangle) - \hat{C}_{\mathrm{NOT}} \otimes \hat{\mathbb{I}}\ (\beta|11\rangle|1\rangle)] \\
=& \frac{1}{\sqrt{2}} [\alpha(\hat{C}_{\mathrm{NOT}}\ |00\rangle \otimes \hat{\mathbb{I}}\ |0\rangle) - \alpha(\hat{C}_{\mathrm{NOT}}\ |01\rangle \otimes \hat{\mathbb{I}}|1\rangle) + \beta(\hat{C}_{\mathrm{NOT}}\ |10\rangle \otimes \hat{\mathbb{I}}|0\rangle) - \beta(\hat{C}_{\mathrm{NOT}}\ |11\rangle \otimes \hat{\mathbb{I}}|1\rangle)] \\
=& \frac{1}{\sqrt{2}} (\alpha|00\rangle|0\rangle - \alpha|01\rangle|1\rangle + \beta|11\rangle|0\rangle - \beta|10\rangle|1\rangle) \\
=& \frac{1}{\sqrt{2}} (\alpha|0\rangle|00\rangle - \alpha|0\rangle|11\rangle + \beta|1\rangle|10\rangle - \beta|1\rangle|01\rangle)
\end{aligned}
$$

So,

$$
|\psi_1\rangle = \frac{1}{\sqrt{2}} (\alpha|0\rangle|00\rangle - \alpha|0\rangle|11\rangle + \beta|1\rangle|10\rangle - \beta|1\rangle|01\rangle) \tag{2}
$$

- $|\psi_2\rangle$

$$
\begin{aligned}
|\psi_2\rangle =&(\hat{H}\otimes\hat{\mathbb{I}}\otimes\hat{\mathbb{I}})\ |\psi_1\rangle \\
=&(\hat{H}\otimes\hat{\mathbb{I}}\otimes\hat{\mathbb{I}})\ \frac{1}{\sqrt{2}}(\alpha|0\rangle|00\rangle-\alpha|0\rangle|11\rangle+\beta|1\rangle|10\rangle-\beta|1\rangle|01\rangle) \\
=&(\hat{H}\otimes\hat{\mathbb{I}}\otimes\hat{\mathbb{I}})\ (\frac{1}{\sqrt{2}}[\alpha|0\rangle|0\rangle|0\rangle-\alpha|0\rangle|1\rangle|1\rangle+\beta|1\rangle|1\rangle|0\rangle-\beta|1\rangle|0\rangle|1\rangle]) \\
=&\frac{1}{\sqrt{2}}[\alpha\hat{H}|0\rangle\otimes\hat{\mathbb{I}}|0\rangle\otimes\hat{\mathbb{I}}|0\rangle-\alpha\hat{H}|0\rangle\otimes\hat{\mathbb{I}}|1\rangle\otimes\hat{\mathbb{I}}|1\rangle+\beta\hat{H}|1\rangle\otimes\hat{\mathbb{I}}|1\rangle\otimes\hat{\mathbb{I}}|0\rangle-\beta\hat{H}|1\rangle\otimes\hat{\mathbb{I}}|0\rangle\otimes\hat{\mathbb{I}}|1\rangle] \\
=&\frac{1}{\sqrt{2}}\left[\alpha\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|00\rangle-\alpha\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|11\rangle+\beta\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)|10\rangle-\beta\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)|01\rangle\right] \\
=&\frac{1}{\sqrt{2}^2}[\alpha|000\rangle+\alpha|100\rangle-\alpha|011\rangle-\alpha|111\rangle+\beta|010\rangle-\beta|110\rangle-\beta|001\rangle+\beta|101\rangle] \\
=&\frac{1}{2}[|00\rangle(\alpha|0\rangle-\beta|1\rangle)+|01\rangle(-\alpha|1\rangle+\beta|0\rangle)+|10\rangle(\alpha|0\rangle+\beta|1\rangle)+|11\rangle(-\alpha|1\rangle-\beta|0\rangle)] \\
=&\frac{1}{2}[|00\rangle(\alpha|0\rangle-\beta|1\rangle)+|01\rangle(-\alpha|1\rangle+\beta|0\rangle)+|10\rangle(\alpha|0\rangle+\beta|1\rangle)+|11\rangle(\alpha|1\rangle+\beta|0\rangle)]
\end{aligned}
$$

since $-\alpha|1\rangle-\beta|0\rangle=e^{i\pi}(\alpha|0\rangle+\beta|1\rangle)$, i.e. $-\alpha|1\rangle-\beta|0\rangle\equiv\alpha|1\rangle+\beta|0\rangle$.

So,

$$
|\psi_2\rangle=\frac{1}{2}[|00\rangle(\alpha|0\rangle-\beta|1\rangle)+|01\rangle(-\alpha|1\rangle+\beta|0\rangle)+|10\rangle(\alpha|0\rangle+\beta|1\rangle)+|11\rangle(\alpha|1\rangle+\beta|0\rangle)] \tag{3}
$$

where I use the notation $|ij\rangle_A$, $i,j\in\{0,1\}$ (as in $|00\rangle_A$) to stress the fact that those two qubits are (and have always been) under the domain of Alice, i.e. Alice can manipulate them and measure them at will.

- $|\psi_3\rangle$

  Now, $|\psi_3\rangle$ is the quantum state produced after measuring Alice's qubits.

  In this step, we shall measure Alice's two qubits, $|\psi\rangle$ and her EPR qubit, *simultaneously*. Let us start by defining the following one-qubit measurement operators:

$$\hat{P}_{a_0}^{|\psi\rangle} = |0\rangle\langle 0| \tag{4a}$$

$$\hat{P}_{a_1}^{|\psi\rangle} = |1\rangle\langle 1| \tag{4b}$$

$$\hat{P}_{b_0}^{|\beta_{00}\rangle} = |0\rangle\langle 0| \tag{4c}$$

$$\hat{P}_{b_1}^{|\beta_{00}\rangle} = |1\rangle\langle 1| \tag{4d}$$

  Operators from Eqs. (4a,4b) would be used to measure $|\psi\rangle$, while operators from Eqs. (4c, 4d) would be employed to measure Alice's qubit from the EPR pair she shares with Bob.

  Notes:
  1. We use the labels $\{a_0, a_1\}$ and $\{b_0, b_1\}$ to refer to the *possible* measurement outcomes for $|\psi\rangle$ and Alice's EPR qubit, respectively.
  2. Hence, only four outcomes are possible: $\{a_0, b_0\}$, $\{a_0, b_1\}$, $\{a_1, b_0\}$ or $\{a_1, b_1\}$.

  Based on Eqs. (4a-4d) we now define the following two-qubit measurement operators:

$$\hat{P}_{\{a_0,b_0\}} = \hat{P}_{a_0}^{|\psi\rangle} \otimes \hat{P}_{b_0}^{|\beta_{00}\rangle} = |0_{a_0}\rangle\langle 0_{a_0}| \otimes |0_{b_0}\rangle\langle 0_{b_0}| = |0_{a_0}0_{b_0}\rangle\langle 0_{a_0}0_{b_0}| = |00\rangle\langle 00| \tag{5a}$$

$$\hat{P}_{\{a_0,b_1\}} = \hat{P}_{a_0}^{|\psi\rangle} \otimes \hat{P}_{b_1}^{|\beta_{00}\rangle} = |0_{a_0}\rangle\langle 0_{a_0}| \otimes |1_{b_1}\rangle\langle 1_{b_1}| = |0_{a_0}1_{b_1}\rangle\langle 0_{a_0}1_{b_1}| = |01\rangle\langle 01| \tag{5b}$$

$$\hat{P}_{\{a_1,b_0\}} = \hat{P}_{a_1}^{|\psi\rangle} \otimes \hat{P}_{b_0}^{|\beta_{00}\rangle} = |1_{a_1}\rangle\langle 1_{a_1}| \otimes |0_{b_0}\rangle\langle 0_{b_0}| = |1_{a_1}0_{b_0}\rangle\langle 1_{a_1}0_{b_0}| = |10\rangle\langle 10| \tag{5c}$$

$$\hat{P}_{\{a_1,b_1\}} = \hat{P}_{a_1}^{|\psi\rangle} \otimes \hat{P}_{b_1}^{|\beta_{00}\rangle} = |1_{a_1}\rangle\langle 1_{a_1}| \otimes |1_{b_1}\rangle\langle 1_{b_1}| = |1_{a_1}1_{b_1}\rangle\langle 1_{a_1}1_{b_1}| = |11\rangle\langle 11| \tag{5d}$$

  Let us now calculate the probability distribution and post-measurement states for outcomes $\{a_0, b_0\}$, $\{a_0, b_1\}$, $\{a_1, b_0\}$, $\{a_1, b_1\}$.

1. $p(a_0, b_0) = \langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle$

$$\begin{aligned}
\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle =&|00\rangle\langle 00|\left[\frac{1}{2}[|00\rangle(\alpha|0\rangle - \beta|1\rangle) + |01\rangle(-\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle + \beta|1\rangle) + |11\rangle(\alpha|1\rangle + \beta|0\rangle)]\right] \\
=&\frac{1}{2}\big[\,\langle 00|00\rangle\,|00\rangle(\alpha|0\rangle - \beta|1\rangle) + \langle 00|01\rangle\,|00\rangle(-\alpha|1\rangle + \beta|0\rangle) \\
&+ \langle 00|10\rangle\,|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \langle 00|11\rangle\,|00\rangle(\alpha|1\rangle + \beta|0\rangle)\big] \\
=&\frac{1}{2}|00\rangle(\alpha|0\rangle - \beta|1\rangle)
\end{aligned}$$

That is,

$$\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle = \frac{1}{2}|00\rangle(\alpha|0\rangle - \beta|1\rangle) \tag{6}$$

Now,

$$\begin{aligned}
\langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle =&\frac{1}{2}\big[\langle 00|(\alpha^*\langle 0| - \beta^*\langle 1|) + \langle 01|(-\alpha^*\langle 1| + \beta^*\langle 0|) + \langle 10|(\alpha^*\langle 0| + \beta^*\langle 1|) + \langle 11|(\alpha^*\langle 1| + \beta^*\langle 0|)\big] \\
&\big[\frac{1}{2}|00\rangle(\alpha|0\rangle - \beta|1\rangle)\big] \\
=&\frac{1}{4}\big[\,\langle 00|00\rangle\,(\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|0\rangle - \beta|1\rangle) + \langle 01|00\rangle\,(-\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|0\rangle - \beta|1\rangle) \\
&+ \langle 10|00\rangle\,(\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|0\rangle - \beta|1\rangle) + \langle 11|00\rangle\,(\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|0\rangle - \beta|1\rangle)\big] \\
=&\frac{1}{4}\big[(\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|0\rangle - \beta|1\rangle)\big] \\
=&\frac{1}{4}\big[\alpha^*\alpha\,\langle 0|0\rangle - \alpha^*\beta\,\langle 0|1\rangle - \beta^*\alpha\,\langle 1|0\rangle + \beta^*\beta\,\langle 1|1\rangle\,\big] \\
=&\frac{1}{4}\big[||\alpha||^2 + ||\beta||^2\big] \\
=&\frac{1}{4}
\end{aligned}$$

That is,

$$p(a_0, b_0) = \langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle = \frac{1}{4} \tag{7}$$

The corresponding post-measurement state $|\psi\rangle^{\text{pm}}_{\{a_0,b_0\}}$ is given by

$$|\psi\rangle^{\mathbf{pm}}_{\{a_0,b_0\}} = \frac{\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle}{\sqrt{\langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle}} = \frac{\frac{1}{2}|00\rangle(\alpha|0\rangle - \beta|1\rangle)}{\sqrt{1/4}} = |00\rangle(\alpha|0\rangle - \beta|1\rangle)$$

Therefore,

$$p(a_0, b_0) = \langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle = \frac{1}{4} \tag{8}$$

$$|\psi\rangle^{\text{pm}}_{\{a_0,b_0\}} = |00\rangle_A(\alpha|0\rangle_B - \beta|1\rangle_B) \tag{9}$$

where subindex A is used to explicitly state that qubits $|00\rangle_A$ are on Alice's hands, while subindex B is used to explicitly state that qubit $\alpha|0\rangle_B - \beta|1\rangle_B$ is on Bob's hands.

2. $p(a_0, b_1) = \langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle$

$$
\begin{aligned}
\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle =& |01\rangle\langle 01|\left[\frac{1}{2}[|00\rangle(\alpha|0\rangle - \beta|1\rangle) + |01\rangle(-\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle + \beta|1\rangle) + |11\rangle(\alpha|1\rangle + \beta|0\rangle)]\right] \\
=& \frac{1}{2}\big[\,\langle 01|00\rangle\,|01\rangle(\alpha|0\rangle - \beta|1\rangle) + \langle 01|01\rangle\,|01\rangle(-\alpha|1\rangle + \beta|0\rangle) \\
&+ \langle 01|10\rangle\,|01\rangle(\alpha|0\rangle + \beta|1\rangle) + \langle 01|11\rangle\,|01\rangle(\alpha|1\rangle + \beta|0\rangle)\big] \\
=& \frac{1}{2}|01\rangle(-\alpha|1\rangle + \beta|0\rangle)
\end{aligned}
$$

That is,

$$
\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle = \frac{1}{2}|01\rangle(-\alpha|1\rangle + \beta|0\rangle) \tag{10}
$$

Now,

$$
\begin{aligned}
\langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle =& \frac{1}{2}\big[\langle 00|(\alpha^*\langle 0| - \beta^*\langle 1|) + \langle 01|(-\alpha^*\langle 1| + \beta^*\langle 0|) + \langle 10|(\alpha^*\langle 0| + \beta^*\langle 1|) + \langle 11|(\alpha^*\langle 1| + \beta^*\langle 0|)\big] \\
&\big[\frac{1}{2}|01\rangle(-\alpha|1\rangle + \beta|0\rangle)\big] \\
=& \frac{1}{4}\big[\,\langle 00|01\rangle\,(\alpha^*\langle 0| - \beta^*\langle 1|)(-\alpha|1\rangle + \beta|0\rangle) + \langle 01|01\rangle\,(-\alpha^*\langle 1| + \beta^*\langle 0|)(-\alpha|1\rangle + \beta|0\rangle) \\
&+ \langle 10|01\rangle\,(\alpha^*\langle 0| + \beta^*\langle 1|)(-\alpha|1\rangle + \beta|0\rangle) + \langle 11|01\rangle\,(\alpha^*\langle 1| + \beta^*\langle 0|)(-\alpha|1\rangle + \beta|0\rangle)\big] \\
=& \frac{1}{4}\big[(-\alpha^*\langle 1| + \beta^*\langle 0|)(-\alpha|1\rangle + \beta|0\rangle)\big] \\
=& \frac{1}{4}\big[(-1)(-1)\alpha^*\alpha\,\langle 1|1\rangle - \alpha^*\beta\,\langle 1|0\rangle - \beta^*\alpha\,\langle 0|1\rangle + \beta^*\beta\,\langle 0|0\rangle\,\big] \\
=& \frac{1}{4}\big[||\alpha||^2 + ||\beta||^2\big] \\
=& \frac{1}{4}
\end{aligned}
$$

That is,

$$
p(a_0, b_1) = \langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle = \frac{1}{4} \tag{11}
$$

The corresponding post-measurement state $|\psi\rangle^{\mathrm{pm}}_{\{a_0,b_1\}}$ is given by

$$
|\psi\rangle^{\mathbf{pm}}_{\{a_0,b_1\}} = \frac{\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle}{\sqrt{\langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle}} = \frac{\frac{1}{2}|01\rangle(-\alpha|1\rangle + \beta|0\rangle)}{\sqrt{1/4}} = |01\rangle(-\alpha|1\rangle + \beta|0\rangle)
$$

Therefore,

$$
p(a_0, b_1) = \langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle = \frac{1}{4} \tag{12}
$$

$$
|\psi\rangle^{\mathrm{pm}}_{\{a_0,b_1\}} = |01\rangle_A(-\alpha|1\rangle_B + \beta|0\rangle_B) \tag{13}
$$

where subindex A is used to explicitly state that qubits $|01\rangle_A$ are on Alice's hands, while subindex B is used to explicitly state that qubit $-\alpha|1\rangle_B + \beta|0\rangle_B$ is on Bob's hands.

3. $p(a_1, b_0) = \langle \psi_2 | \hat{P}_{\{a_1,b_0\}} | \psi_2 \rangle$

$$\begin{aligned}
\hat{P}_{\{a_1,b_0\}} |\psi_2\rangle =& |10\rangle\langle 10| \left[\frac{1}{2} [|00\rangle(\alpha|0\rangle - \beta|1\rangle) + |01\rangle(-\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle + \beta|1\rangle) + |11\rangle(\alpha|1\rangle + \beta|0\rangle)] \right] \\
=& \frac{1}{2} \big[\langle 10|00\rangle \, |10\rangle(\alpha|0\rangle - \beta|1\rangle) + \langle 10|01\rangle \, |10\rangle(-\alpha|1\rangle + \beta|0\rangle) \\
& + \langle 10|10\rangle \, |10\rangle(\alpha|0\rangle + \beta|1\rangle) + \langle 10|11\rangle \, |10\rangle(\alpha|1\rangle + \beta|0\rangle) \big] \\
=& \frac{1}{2} |10\rangle(\alpha|0\rangle + \beta|1\rangle)
\end{aligned}$$

That is,

$$\hat{P}_{\{a_1,b_0\}} |\psi_2\rangle = \frac{1}{2} |10\rangle(\alpha|0\rangle + \beta|1\rangle) \tag{14}$$

Now,

$$\begin{aligned}
\langle \psi_2 | \hat{P}_{\{a_1,b_0\}} | \psi_2 \rangle =& \frac{1}{2} \big[\langle 00|(\alpha^* \langle 0| - \beta^* \langle 1|) + \langle 01|(-\alpha^* \langle 1| + \beta^* \langle 0|) + \langle 10|(\alpha^* \langle 0| + \beta^* \langle 1|) + \langle 11|(\alpha^* \langle 1| + \beta^* \langle 0|) \big] \\
& \big[\frac{1}{2} |10\rangle(\alpha|0\rangle + \beta|1\rangle) \big] \\
=& \frac{1}{4} \big[\langle 00|10\rangle \, (\alpha^* \langle 0| - \beta^* \langle 1|)(\alpha|0\rangle + \beta|1\rangle) + \langle 01|10\rangle \, (-\alpha^* \langle 1| + \beta^* \langle 0|)(\alpha|0\rangle + \beta|1\rangle) \\
& + \langle 10|10\rangle \, (\alpha^* \langle 0| + \beta^* \langle 1|)(\alpha|0\rangle + \beta|1\rangle) + \langle 11|10\rangle \, (\alpha^* \langle 1| + \beta^* \langle 0|)(\alpha|0\rangle + \beta|1\rangle) \big] \\
=& \frac{1}{4} \big[(\alpha^* \langle 0| + \beta^* \langle 1|)(\alpha|0\rangle + \beta|1\rangle) \big] \\
=& \frac{1}{4} \big[\alpha^* \alpha \, \langle 0|0\rangle + \alpha^* \beta \, \langle 0|1\rangle + \beta^* \alpha \, \langle 1|0\rangle + \beta^* \beta \, \langle 1|1\rangle \, \big] \\
=& \frac{1}{4} \big[||\alpha||^2 + ||\beta||^2 \big] \\
=& \frac{1}{4}
\end{aligned}$$

That is,

$$p(a_1, b_0) = \langle \psi_2 | \hat{P}_{\{a_1,b_0\}} | \psi_2 \rangle = \frac{1}{4} \tag{15}$$

The corresponding post-measurement state $|\psi\rangle^{\text{pm}}_{\{a_1,b_0\}}$ is given by

$$|\psi\rangle^{\mathbf{pm}}_{\{a_1,b_0\}} = \frac{\hat{P}_{\{a_1,b_0\}} |\psi_2\rangle}{\sqrt{\langle \psi_2 | \hat{P}_{\{a_1,b_0\}} | \psi_2 \rangle}} = \frac{\frac{1}{2} |10\rangle(\alpha|0\rangle + \beta|1\rangle)}{\sqrt{1/4}} = |10\rangle(\alpha|0\rangle + \beta|1\rangle)$$

Therefore,

$$p(a_1, b_0) = \langle \psi_2 | \hat{P}_{\{a_1,b_0\}} | \psi_2 \rangle = \frac{1}{4} \tag{16}$$

$$|\psi\rangle^{\text{pm}}_{\{a_1,b_0\}} = |10\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) \tag{17}$$

where subindex A is used to explicitly state that qubits $|10\rangle_A$ are on Alice's hands, while subindex B is used to explicitly state that qubit $\alpha|0\rangle_B + \beta|1\rangle_B$ is on Bob's hands.

4. $p(a_1, b_1) = \langle \psi_2 | \hat{P}_{\{a_1,b_1\}} | \psi_2 \rangle$

$$
\begin{aligned}
\hat{P}_{\{a_1,b_1\}} |\psi_2\rangle =& |11\rangle\langle 11| \left[\frac{1}{2} [|00\rangle(\alpha|0\rangle - \beta|1\rangle) + |01\rangle(-\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle + \beta|1\rangle) + |11\rangle(\alpha|1\rangle + \beta|0\rangle)] \right] \\
=& \frac{1}{2} \big[\langle 11|00\rangle \, |11\rangle(\alpha|0\rangle - \beta|1\rangle) + \langle 11|01\rangle \, |11\rangle(-\alpha|1\rangle + \beta|0\rangle) \\
& + \langle 11|10\rangle \, |11\rangle(\alpha|0\rangle + \beta|1\rangle) + \langle 11|11\rangle \, |11\rangle(\alpha|1\rangle + \beta|0\rangle) \big] \\
=& \frac{1}{2} |11\rangle(\alpha|1\rangle + \beta|0\rangle)
\end{aligned}
$$

That is,

$$
\hat{P}_{\{a_1,b_1\}} |\psi_2\rangle = \frac{1}{2} |11\rangle(\alpha|1\rangle + \beta|0\rangle) \tag{18}
$$

Now,

$$
\begin{aligned}
\langle \psi_2 | \hat{P}_{\{a_1,b_1\}} | \psi_2 \rangle =& \frac{1}{2} \big[\langle 00|(\alpha^*\langle 0| - \beta^*\langle 1|) + \langle 01|(-\alpha^*\langle 1| + \beta^*\langle 0|) + \langle 10|(\alpha^*\langle 0| + \beta^*\langle 1|) + \langle 11|(\alpha^*\langle 1| + \beta^*\langle 0|) \big] \\
& \big[\frac{1}{2} |11\rangle(\alpha|1\rangle + \beta|0\rangle) \big] \\
=& \frac{1}{4} \big[\langle 00|11\rangle \, (\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|1\rangle + \beta|0\rangle) + \langle 01|11\rangle \, (-\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|1\rangle + \beta|0\rangle) \\
& + \langle 10|11\rangle \, (\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|1\rangle + \beta|0\rangle) + \langle 11|11\rangle \, (\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|1\rangle + \beta|0\rangle) \big] \\
=& \frac{1}{4} \big[(\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|1\rangle + \beta|0\rangle) \big] \\
=& \frac{1}{4} \big[\alpha^*\alpha \, \langle 1|1\rangle + \alpha^*\beta \, \langle 1|0\rangle + \beta^*\alpha \, \langle 0|1\rangle + \beta^*\beta \, \langle 0|0\rangle \, \big] \\
=& \frac{1}{4} \big[||\alpha||^2 + ||\beta||^2 \big] \\
=& \frac{1}{4}
\end{aligned}
$$

That is,

$$
p(a_1, b_1) = \langle \psi_2 | \hat{P}_{\{a_1,b_1\}} | \psi_2 \rangle = \frac{1}{4} \tag{19}
$$

The corresponding post-measurement state $|\psi\rangle^{\text{pm}}_{\{a_1,b_1\}}$ is given by

$$
|\psi\rangle^{\mathbf{pm}}_{\{a_1,b_1\}} = \frac{\hat{P}_{\{a_1,b_1\}} |\psi_2\rangle}{\sqrt{\langle \psi_2 | \hat{P}_{\{a_1,b_1\}} | \psi_2 \rangle}} = \frac{\frac{1}{2}|11\rangle(\alpha|1\rangle + \beta|0\rangle)}{\sqrt{1/4}} = |11\rangle(\alpha|1\rangle + \beta|0\rangle)
$$

Therefore,

$$
p(a_1, b_1) = \langle \psi_2 | \hat{P}_{\{a_1,b_1\}} | \psi_2 \rangle = \frac{1}{4} \tag{20}
$$

$$
|\psi\rangle^{\text{pm}}_{\{a_1,b_1\}} = |11\rangle_A(\alpha|1\rangle_B + \beta|0\rangle_B) \tag{21}
$$

where subindex A is used to explicitly state that qubits $|11\rangle_A$ are on Alice's hands, while subindex B is used to explicitly state that qubit $\alpha|1\rangle_B + \beta|0\rangle_B$ is on Bob's hands.

- $|\psi_4\rangle$

In summary, we have four cases described by Eqs.(8,9,12,13,16,17,20,21):

Case 1. Outcome $\{a_0, b_0\}$.

The probability of getting outcome $\{a_0, b_0\}$ is

$$p(a_0, b_0) = \langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle = \frac{1}{4}$$

Furthermore, post-measurement quantum state for outcome $\{a_0, b_0\}$ is given by

$$|\psi\rangle^{\mathrm{pm}}_{\{a_0,b_0\}} = |00\rangle_A(\alpha|0\rangle_B - \beta|1\rangle_B)$$

In this case, Alice *knows* that her qubits are in state $|00\rangle_A$. Moreover, she *knows* that Bob's qubit is in the state $\alpha|0\rangle_B - \beta|1\rangle_B$. Now, since

$$\begin{aligned}\hat{\sigma}_z(\alpha|0\rangle_B - \beta|1\rangle_B) =&(|0\rangle\langle 0| - |1\rangle\langle 1|)(\alpha|0\rangle_B - \beta|1\rangle_B)\\ =&\alpha\,\langle 0|0\rangle\,|0\rangle - \beta\,\langle 0|1\rangle\,|0\rangle - \alpha\,\langle 1|0\rangle\,|1\rangle + \beta\,\langle 1|1\rangle\,|1\rangle\\ =&\alpha|0\rangle + \beta|1\rangle\end{aligned}$$

Then Alice calls Bob *via a classical channel* (a telephone line, for instance) to tell him that

$$|\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_z)_B[|00\rangle_A(\alpha|0\rangle_B - \beta|1\rangle_B)] \tag{22}$$

That is, Bob only needs to apply $\hat{\sigma}_z$ operator to his qubit in order to have it ready!

Case 2. Outcome $\{a_0, b_1\}$.

The probability of getting outcome $\{a_0, b_1\}$ is

$$p(a_0, b_1) = \langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle = \frac{1}{4}$$

Furthermore, post-measurement quantum state for outcome $\{a_0, b_1\}$ is given by

$$|\psi\rangle^{\mathrm{pm}}_{\{a_0,b_1\}} = |01\rangle_A(-\alpha|1\rangle_B + \beta|0\rangle_B)$$

In this case, Alice *knows* that her qubits are in state $|01\rangle_A$. Moreover, she *knows* that Bob's qubit is in the state $-\alpha|1\rangle_B + \beta|0\rangle_B$. Now, since

$$\begin{aligned}\hat{\sigma}_x(\hat{\sigma}_z(-\alpha|1\rangle_B + \beta|0\rangle_B)) =&\hat{\sigma}_x(|0\rangle\langle 0| - |1\rangle\langle 1|)(-\alpha|1\rangle_B + \beta|0\rangle_B)\\ =&\hat{\sigma}_x(-\alpha\,\langle 0|1\rangle\,|0\rangle + \beta\,\langle 0|0\rangle\,|0\rangle + \alpha\,\langle 1|1\rangle\,|1\rangle - \beta\,\langle 1|0\rangle\,|1\rangle)\\ =&\hat{\sigma}_x(\alpha|1\rangle + \beta|0\rangle)\\ =&(|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|1\rangle + \beta|0\rangle)\\ =&\alpha\,\langle 1|1\rangle\,|0\rangle + \beta\,\langle 1|0\rangle\,|0\rangle + \alpha\,\langle 0|1\rangle\,|1\rangle + \beta\,\langle 0|0\rangle\,|1\rangle\\ =&\alpha|0\rangle + \beta|1\rangle\end{aligned}$$

Then Alice calls Bob *via a classical channel* (a telephone line, for instance) to tell him that

$$|\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_x\hat{\sigma}_z)_B[|01\rangle_A(-\alpha|1\rangle_B + \beta|0\rangle_B)] \tag{23}$$

That is, Bob only needs to apply $\hat{\sigma}_x\hat{\sigma}_z$ operators to his qubit in order to have it ready!

Case 3. Outcome $\{a_1, b_0\}$.

The probability of getting outcome $\{a_1, b_0\}$ is

$$p(a_1, b_0) = \langle\psi_2|\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle = \frac{1}{4}$$

Furthermore, post-measurement quantum state for outcome $\{a_1, b_0\}$ is given by

$$|\psi\rangle^{\text{pm}}_{\{a_1,b_0\}} = |10\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B)$$

In this case, Alice *knows* that her qubits are in state $|10\rangle_A$. Moreover, she *knows* that Bob's qubit is in the state $\alpha|0\rangle_B + \beta|1\rangle_B$, that is, the qubit Alice was expected to send! Therefore, she calls Bob *via a classical channel* (a telephone line, for instance) to tell him 'your qubit is ready!', i.e.

$$|\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_B[|10\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B)] \tag{24}$$

Case 4. Outcome $\{a_1, b_1\}$.

The probability of getting outcome $\{a_1, b_1\}$ is

$$p(a_1, b_1) = \langle\psi_2|\hat{P}_{\{a_1,b_1\}}|\psi_2\rangle = \frac{1}{4}$$

Furthermore, post-measurement quantum state for outcome $\{a_1, b_1\}$ is given by

$$|\psi\rangle^{\text{pm}}_{\{a_1,b_1\}} = |11\rangle_A(\alpha|1\rangle_B + \beta|0\rangle_B)$$

In this case, Alice *knows* that her qubits are in state $|11\rangle_A$. Moreover, she *knows* that Bob's qubit is in the state $\alpha|1\rangle_B + \beta|0\rangle_B$. Now, since

$$\begin{aligned}\hat{\sigma}_x(\alpha|1\rangle_B + \beta|0\rangle_B) =& (|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|1\rangle_B + \beta|0\rangle_B)\\ =&\alpha\langle 1|1\rangle|0\rangle + \beta\langle 1|0\rangle|0\rangle + \alpha\langle 0|1\rangle|1\rangle + \beta\langle 0|0\rangle|1\rangle\\ =&\alpha|0\rangle + \beta|1\rangle\end{aligned}$$

Then Alice calls Bob *via a classical channel* (a telephone line, for instance) to tell him that

$$|\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_x)_B[|11\rangle_A(\alpha|1\rangle_B + \beta|0\rangle_B)] \tag{25}$$

That is, Bob only needs to apply $\hat{\sigma}_x$ operator to his qubit in order to have it ready!

References

[1] M.A. Nielsen and I.L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press (2000)