Quantum Parallelism

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Let us now assume that we have an **unknown** function f with the following domain and range:

$$f : \{0, 1\}^n \to \{0, 1\}$$

By unknown I mean that we do not know any mathematical description of f, we only have access to a black box that, given a binary string $\vec{x}$, it (the black box) produces the value $f(\vec{x})$.

Hence, we are able to produce ordered pairs $(\vec{x}, f(\vec{x}))$, one at a time.

Let us suppose that we want to know the exact rules of operation of f.

Using classical computers, we would have to compute $f(\vec{x})$ for all the potential inputs $\vec{x} = 0, 1, \ldots, 2^n - 1$. We could to this job either by

- Serial substitution, which would take a very large amount of time for large n, or
- An exponential number of computers, each one ready to compute $f(\vec{x})$ for a given value of $\vec{x}$.

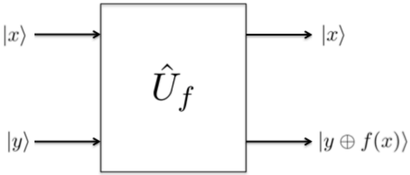
Instead, we could use quantum mechanics in order to build a (macro) quantum gate which would be able to do this job in a single step.

Let us start with the basic case:

$$f : \{0, 1\} \to \{0, 1\}$$

Let us postulate the existence of the following gate $\hat{U}_f$:

$$\hat{U}_f : |x\rangle|y\rangle \to |x\rangle|y \oplus f(x)\rangle$$



$$|x, y\rangle \to |x, y \oplus f(x)\rangle$$

$$|0, 0\rangle \to |0, 0 \oplus f(0)\rangle$$

$$|0, 1\rangle \to |0, 1 \oplus f(0)\rangle$$

$$|1, 0\rangle \to |1, 0 \oplus f(1)\rangle$$

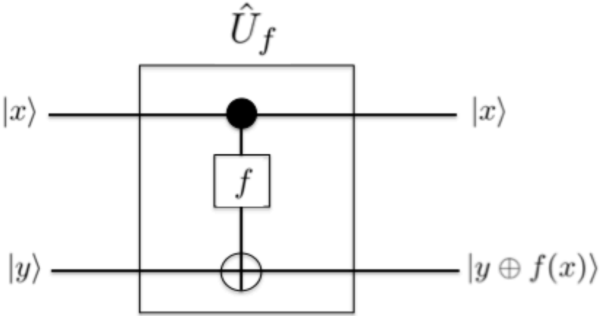
$$|1, 1\rangle \to |1, 1 \oplus f(1)\rangle$$

XOR Operation		
X	Y	X⊕Y
0	0	0
0	1	1
1	0	1
1	1	0

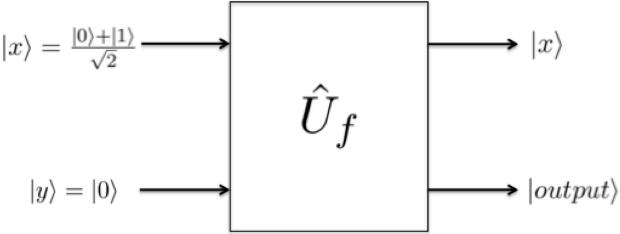
Now, note that:

- $f(0) = 0$ or $f(0) = 1$.
  - Suppose $f(0) = 0$. Then, $0 \oplus f(0) = 0 \oplus 0 = 0 = f(0)$. Consequently, $0 \oplus f(0) = f(0)$ for $f(0) = 0$.
  - Suppose $f(0) = 1$. Then, $0 \oplus f(0) = 0 \oplus 1 = 1 = f(0)$. Consequently, $0 \oplus f(0) = f(0)$ for $f(0) = 1$.
  - **Thus, $0 \oplus f(0) = f(0)$ regardless the value of $f(0)$.**
- $f(1) = 0$ or $f(1) = 1$.
  - Suppose $f(1) = 0$. Then, $0 \oplus f(1) = 0 \oplus 0 = 0 = f(1)$. Consequently, $0 \oplus f(1) = f(1)$ for $f(1) = 0$.
  - Suppose $f(1) = 1$. Then, $0 \oplus f(1) = 0 \oplus 1 = 1 = f(1)$. Consequently, $0 \oplus f(1) = f(1)$ for $f(1) = 1$.
  - **Thus, $0 \oplus f(1) = f(1)$ regardless the value of $f(1)$.**

Operator $\hat{U}_f$ can be created via a variation of the C-NOT gate:



Let us now run $\hat{U}_f$ with $|x\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|y\rangle = |0\rangle$ as input, i.e.



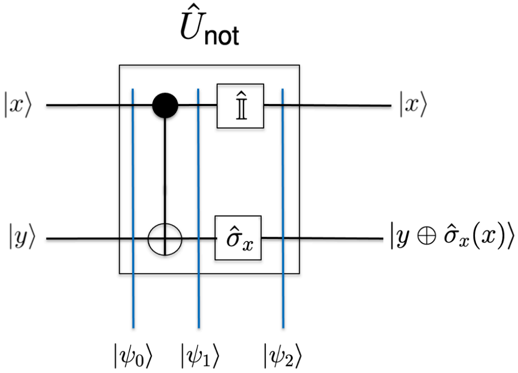
$$
\begin{aligned}
\hat{U}_f|x\rangle|y\rangle &= \hat{U}_f\left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\otimes|0\rangle\right]\\
&= \hat{U}_f\left[\frac{|00\rangle+|10\rangle}{\sqrt{2}}\right]\\
&= \frac{1}{\sqrt{2}}\left[\hat{U}_f|00\rangle+\hat{U}_f|10\rangle\right]\\
&= \frac{1}{\sqrt{2}}\left[|0\rangle|0\oplus f(0)\rangle+|1\rangle|0\oplus f(1)\rangle\right]\\
&= \frac{1}{\sqrt{2}}\left[|0\rangle|f(0)\rangle+|1\rangle|f(1)\rangle\right].
\end{aligned}
$$

This equation comprises both $f(0)$ and $f(1)$ after a *single run of the gate*.

So, gate $\hat{U}_f$ solves the problem of evaluating f, $\forall\ x \in D_f$ in one single step.

Example - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Let us now present a detailed example of quantum parallelism: parallel computation of the logical **NOT** gate.



Example - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Let $|x\rangle = |0\rangle$ and $|y\rangle = |0\rangle$ then

$$|\psi_0\rangle = |0\rangle \otimes |0\rangle$$

$$\begin{aligned}
|\psi_1\rangle &= \hat{C}_{\text{not}}|00\rangle \\
&= (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|)|00\rangle \\
&= |00\rangle
\end{aligned}$$

$$\begin{aligned}
|\psi_2\rangle &= (\hat{\mathbb{I}} \otimes \hat{\sigma}_x)|0\rangle \otimes |0\rangle \\
&= (|0\rangle\langle 0| + |1\rangle\langle 1|)|0\rangle \otimes (|0\rangle\langle 1| + |1\rangle\langle 0|)|0\rangle \\
&= |01\rangle
\end{aligned}$$

So,

$$|\psi_2\rangle = |01\rangle = |0\rangle \otimes \hat{\sigma}_x|0\rangle = |0\rangle|f(0)\rangle$$

Example - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Let $|x\rangle = |1\rangle$ and $|y\rangle = |0\rangle$ then

$$
\begin{aligned}
|\psi_0\rangle &= |1\rangle \otimes |0\rangle \\
\\
|\psi_1\rangle &= \hat{C}_{\text{not}}|10\rangle \\
&= (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|)|10\rangle \\
&= |11\rangle \\
\\
|\psi_2\rangle &= (\hat{\mathbb{I}} \otimes \hat{\sigma}_x)|1\rangle \otimes |1\rangle \\
&= (|0\rangle\langle 0| + |1\rangle\langle 1|)|1\rangle \otimes (|0\rangle\langle 1| + |1\rangle\langle 0|)|1\rangle \\
&= |10\rangle
\end{aligned}
$$

So,

$$|\psi_2\rangle = |10\rangle = |1\rangle \otimes \hat{\sigma}_x|1\rangle = |1\rangle|f(1)\rangle$$

Example - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Let $|x\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|y\rangle = |0\rangle$ then

$$
\begin{aligned}
|\psi_0\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle \\
|\psi_0\rangle &= \frac{|00\rangle + |10\rangle}{\sqrt{2}} \\
|\psi_1\rangle &= \hat{C}_{\text{not}} \frac{|00\rangle + |10\rangle}{\sqrt{2}} \\
&= (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|) \left(\frac{|00\rangle + |10\rangle}{\sqrt{2}} \right) \\
&= \frac{|00\rangle + |11\rangle}{\sqrt{2}}
\end{aligned}
$$

Example - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

$$
\begin{aligned}
|\psi_2\rangle &= (\hat{\mathbb{I}} \otimes \hat{\sigma}_x) \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{\sqrt{2}} \left(\hat{\mathbb{I}}|0\rangle \otimes \hat{\sigma}_x|0\rangle + \hat{\mathbb{I}}|1\rangle \otimes \hat{\sigma}_x|1\rangle \right) \\
&= \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \\
&= \frac{|01\rangle + |10\rangle}{\sqrt{2}}
\end{aligned}
$$

So,

$$
|\psi_2\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{|0\rangle|\hat{\sigma}_x(0)\rangle + |1\rangle|\hat{\sigma}_x(1)\rangle}{\sqrt{2}} = \frac{|0\rangle|f(0)\rangle + |1\rangle|f(1)\rangle}{\sqrt{2}}
$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Let us now simulate $\hat{U}_{\text{not}}$ on Matlab.

We start by remembering the Kronecker product which is a matrix representation of the tensor product.

Suppose we have operators $\hat{A}$ and $\hat{B}$ that have matrix representations given by $A = (a_{ij}), B = (b_{ij})$, two matrices of order n, respectively (for simplicity, we take square matrices as the general definition of the Kronecker product of two non-square matrices can be straightforwardly deduced). So,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

The Kronecker product of matrices A and B, **i.e., a matrix representation of the tensor product $\hat{A} \otimes \hat{B}$**, is given by

$$
\begin{aligned}
A \otimes B &= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix} \\
&= \begin{pmatrix} a_{11} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} & \cdots & a_{1n} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \\ \vdots & & \vdots \\ a_{n1} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} & \cdots & a_{nn} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \end{pmatrix}
\end{aligned}
$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

For example, let

$$\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

be the Hadamard operator.

Write

$$\hat{H} \otimes \hat{H}$$

in matrix notation.

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

$$
\begin{aligned}
H \otimes H &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}
\end{aligned}
$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Let us now introduce the matrix form of operators and kets involved in the computation of U_{not}.

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad C_{\text{not}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad ;$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Also,

$$\mathbb{I}\otimes\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\otimes\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Moreover,

$$U_{\text{not}} = (\mathbb{I} \otimes \sigma_x) C_{\text{not}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Furthermore,

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \\ 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \\ 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Finally,

$$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \\ 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \\ 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

In summary,

$$U_{\text{not}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$|0\rangle \otimes |0\rangle = |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|0\rangle \otimes |1\rangle = |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|1\rangle \otimes |0\rangle = |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|1\rangle \otimes |1\rangle = |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Now, we know the following:

1. For $|x\rangle = |0\rangle$ and $|y\rangle = |0\rangle$

$$\hat{U}_{\text{not}}|x\rangle|y\rangle = \hat{U}_{\text{not}}|0\rangle|0\rangle = |0\rangle|f(0)\rangle = |0\rangle|1\rangle \text{ i.e.,}$$
$$\hat{U}_{\text{not}}|00\rangle = |01\rangle$$

2. For $|x\rangle = |1\rangle$ and $|y\rangle = |0\rangle$

$$\hat{U}_{\text{not}}|x\rangle|y\rangle = \hat{U}_{\text{not}}|1\rangle|0\rangle = |1\rangle|f(1)\rangle = |1\rangle|0\rangle \text{ i.e.,}$$
$$\hat{U}_{\text{not}}|10\rangle = |10\rangle$$

How do these two results look like in matrix notation?

Based on our previous (and precious) calculations, we can tell the following:

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

So,

1. $$U_{\text{not}}|00\rangle = |01\rangle \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

2. $$U_{\text{not}}|10\rangle = |10\rangle \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Now, let us see the parallel power of U_{not}.

Let $|x\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} = \frac{|0\rangle}{\sqrt{2}} + \frac{|1\rangle}{\sqrt{2}}$ and $|y\rangle = |0\rangle$. So, the total initial state is given by

$$|x\rangle \otimes |y\rangle = |xy\rangle = \frac{|00\rangle}{\sqrt{2}} + \frac{|10\rangle}{\sqrt{2}}$$

In matrix notation, $|xy\rangle$ has the following form:

$$|xy\rangle = \frac{|00\rangle}{\sqrt{2}} + \frac{|10\rangle}{\sqrt{2}} = \frac{|00\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

We also know that

$$\hat{U}_{\text{not}}\left(\frac{|00\rangle + |10\rangle}{\sqrt{2}}\right) = \frac{|0\rangle|f(0)\rangle + |1\rangle|f(1)\rangle}{\sqrt{2}} = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

being the matrix representation of $\frac{|01\rangle+|10\rangle}{\sqrt{2}}$ given by

$$\frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}$$

Matlab Simulation - Quantum Parallelism and the NOT logical gate ($\hat{\sigma}_x$)

Finally, the computation of $\hat{U}_{\text{not}}\left(\frac{|00\rangle+|10\rangle}{\sqrt{2}}\right)$ in matrix notation is given by

$$U_{\text{not}}\left(\frac{|00\rangle + |10\rangle}{\sqrt{2}}\right) = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \Rightarrow$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Let us see how this works in Matlab!

Generalized Quantum Parallelism

Generalized Quantum Parallelism

$$f : \{0,1\}^n \to \{0,1\}$$

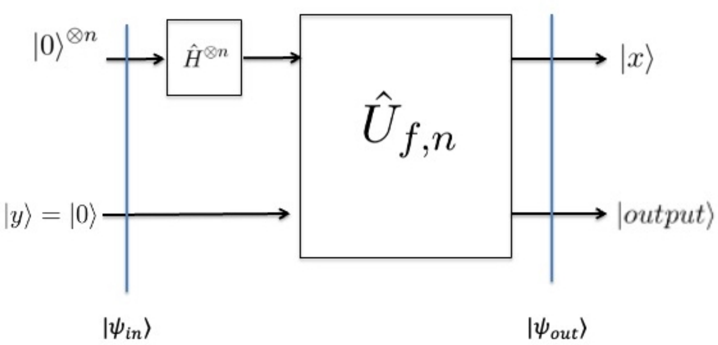
Let us now focus on

$$f : \{0,1\}^n \to \{0,1\}$$

We define $\hat{U}_{f,n}$, a generalization of $\hat{U}_f$, as follows:

$$\hat{U}_{f,n}|\vec{x}\rangle|y\rangle \to |\vec{x}\rangle|y \oplus f(\vec{x})\rangle$$

where $\vec{x} \in \{0,1\}^n$ is an n-bit string and $f(\vec{x}) \in \{0,1\}$



$$|\psi_{in}\rangle = \hat{H}^{\otimes n} \otimes \hat{\mathbb{I}}\left(|0\rangle^{\otimes n} \otimes |0\rangle\right) = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |\vec{x}\rangle \otimes |0\rangle$$

$$
\begin{aligned}
|\psi_{out}\rangle &= \hat{U}_{f,n}|\psi_{in}\rangle \\
&= \hat{U}_{f,n}\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|\vec{x}\rangle \otimes |0\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}\hat{U}_{f,n}|\vec{x}\rangle \otimes |0\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|\vec{x}\rangle \otimes |0 \oplus f(\vec{x})\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|\vec{x}\rangle \otimes |f(\vec{x})\rangle
\end{aligned}
$$

Therefore, f has been evaluated $\forall\ x \in D_f$ in a single step!