Exercises with Solutions - Complex Numbers

Quantum Computing Course
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Exercise 1. Let $z_1 = 4 - 5i$ and $z_2 = 3 + 6i$. Compute the following:

1. $z_1 z_2$

$$\begin{aligned} z_1 z_2 &= (4-5i)(3+6i) \\ &= (4)(3) + (-5i)(3) + (4)(6i) + (-5i)(6i) \\ &= 12 - 15i + 24i - 30i^2 \\ &= 12 - 15i + 24i - 30(\sqrt{-1})^2 \\ &= 12 - 15i + 24i - 30(-1) \\ &= 12 - 15i + 24i + 30 \\ &= 12 + 30 + (-15 + 24)i \\ &= 42 + 9i \end{aligned}$$

2. $z_1^2 - z_2^3$

$$\begin{aligned} z_1^2 &= (4-5i)^2 \\ &= (4-5i)(4-5i) \\ &= 16 - 20i - 20i + 25i^2 \\ &= 16 - 25 + (-20 - 20)i \\ &= -9 - 40i \end{aligned}$$

$$\begin{aligned} z_2^3 &= (3+6i)^3 \\ &= (3+6i)(3+6i)(3+6i) \\ &= (9 + 18i + 18i + 36i^2)(3+6i) \\ &= (9 - 36 + (18+18)i)(3+6i) \\ &= (-27 + 36i)(3+6i) \\ &= -81 + 108i - 162i + 216i^2 \\ &= -81 - 216 + (108 - 162)i \\ &= -297 - 54i \end{aligned}$$

So,

$$\begin{aligned} z_1^2 - z_2^3 &= (-9 - 40i) - (-297 - 54i) \\ &= -9 - 40i + 297 + 54i \\ &= -9 + 297 + (-40 + 54)i \\ &= 288 + 14i \end{aligned}$$

Exercise 2. Let $z_1 = 1 - 2i$ and $z_2 = 3 + 4i$. Compute the following:

1. $\bar{z_1}$

$$\begin{aligned} \bar{z_1} &= \overline{1-2i} \\ &= \overline{1+(-2)i} \\ &= 1-(-2)i \\ &= 1+(-1)(-1)(2)i \\ &= 1+2i \end{aligned}$$

2. $(z_1 - z_2^2)^*$

$$\begin{aligned} z_2^2 &= (3+4i)(3+4i) \\ &= 9+12i+12i+16i^2 \\ &= 9-16+(12+12)i \\ &= -7+24i \end{aligned}$$

Then

$$\begin{aligned} z_1 - z_2^2 &= 1-2i-(-7+24i) \\ &= 1-2i+7-24i \\ &= 8-26i \end{aligned}$$

So,

$$\begin{aligned} (z_1 - z_2^2)^* &= (8-26i)^* \\ &= (8+(-26)i)^* \\ &= 8-(-26)i \\ &= 8+26i \end{aligned}$$

3. $\|z_2 z_1\|$

$$\begin{aligned} z_2 z_1 &= (1-2i)(3+4i) \\ &= 3-6i+4i-8i^2 \\ &= 3+8+(-6+4)i \\ &= 11-2i \end{aligned}$$

So,

$$\begin{aligned} \|z_2 z_1\| &= \|11-2i\| \\ &= \|11+(-2)i\| \\ &= \sqrt{(11)^2+(-2)^2} \\ &= \sqrt{121+4} \\ &= \sqrt{125} \\ &\approx 11.1803 \end{aligned}$$

4. $\| \, \|z_1\| + \|z_2\| i \, \|^2$

$$\begin{aligned} \|z_1\| &= \|1 - 2i\| \\ &= \|1 + (-2)i\| \\ &= \sqrt{(1)^2 + (-2)^2} \\ &= \sqrt{1+4} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} \|z_2\| &= \|3 + 4i\| \\ &= \sqrt{(3)^2 + (4)^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

So,

$$\begin{aligned} \| \, \|z_1\| + \|z_2\| i \, \|^2 &= \|\sqrt{5} + 5i\|^2 \\ &= \left[\sqrt{(\sqrt{5})^2 + (5)^2}\right]^2 \\ &= 5 + 25 \\ &= 30 \end{aligned}$$

Exercise 3. Let $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{3}$. Compute the following:

1. $e^{i\theta}$

$$\begin{aligned} e^{i\theta} &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \end{aligned}$$

since $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

2. $(e^{i\theta})^4$

Note that $(e^{i\theta})^4 = e^{i\theta}e^{i\theta}e^{i\theta}e^{i\theta}$ and, since $e^{i\zeta}e^{i\nu} = e^{i(\zeta+\nu)}$ then

$$\begin{aligned} (e^{i\theta})^4 &= e^{i\theta}e^{i\theta}e^{i\theta}e^{i\theta} \\ &= e^{i(\theta+\theta+\theta+\theta)} \\ &= e^{i4\theta} \\ &= e^{i\frac{4\pi}{4}} \\ &= e^{i\pi} \\ &= \cos\pi + i\sin\pi \\ &= -1 + 0i \\ &= -1 \end{aligned}$$

since $\cos\pi = -1$ and $\sin\pi = 0$.

3. $\|e^{i\phi}\|$ and $\overline{e^{i\phi}}$

We know that

$$\begin{aligned} e^{i\phi} &= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}i \end{aligned}$$

since $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

So,

$$\begin{aligned} \|e^{i\phi}\| &= \|\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\| \\ &= \sqrt{(\tfrac{1}{2})^2 + (\tfrac{\sqrt{3}}{2})^2} \\ &= \sqrt{\tfrac{1}{4} + \tfrac{3}{4}} \\ &= 1 \text{ (as expected. Why?)} \end{aligned}$$

Also, since $e^{i\phi} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ then

$$\begin{aligned} \overline{e^{i\phi}} &= \overline{\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i} \\ &= \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i \end{aligned}$$