No-Cloning Theorem and Quantum Teleportation

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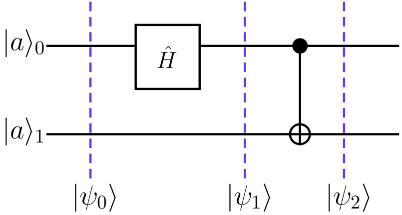
Bell States

Introduction to Quantum Circuits

We now introduce a quantum circuit to compute Bell states

Bell State Circuit (1/4)



Let

$$|a\rangle_0 = |0\rangle \text{ and } |a\rangle_1 = |0\rangle$$

Also, remember that

$$\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

$$\hat{C}_{\text{not}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

Bell State Circuit (2/4)

So,

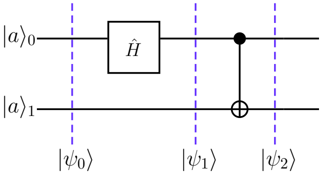
$$
\begin{aligned}
|\psi\rangle_0 &= |0\rangle \otimes |0\rangle = |00\rangle \\
|\psi\rangle_1 &= (\hat{H} \otimes \hat{\mathbb{I}})(|0\rangle \otimes |0\rangle) = \hat{H}|0\rangle \otimes \hat{\mathbb{I}}|0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \\
&= \tfrac{1}{\sqrt{2}}|0\rangle|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle|0\rangle \\
|\psi\rangle_2 &= \hat{C}_{\text{not}}(\tfrac{1}{\sqrt{2}}|0\rangle|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle|0\rangle) \\
&= \tfrac{1}{\sqrt{2}}|0\rangle|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle|1\rangle = \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle
\end{aligned}
$$

So,

$$
|\psi\rangle_2 = \frac{|00\rangle + |11\rangle}{\sqrt{2}}
$$

Bell State Circuit (3/4)

Exercise



Compute $|\psi\rangle_2$ for

$|a\rangle_0 = |0\rangle$ and $|a\rangle_1 = |1\rangle$
$|a\rangle_0 = |1\rangle$ and $|a\rangle_1 = |0\rangle$
$|a\rangle_0 = |1\rangle$ and $|a\rangle_1 = |1\rangle$

Bell State Circuit (4/4)

Answers

$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|00\rangle) = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|01\rangle) = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|10\rangle) = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|11\rangle) = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

These states are known as the **Bell states**

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Quantum Entanglement (1/2)

Bell states are examples of entangled states. Bell states are key features of a quantum information transmission protocol known as quantum teleportation.

Quantum entanglement is a unique type of correlation shared between components of a quantum system.

Quantum entanglement and the principle of superposition are two of the main features behind the power of quantum computation and quantum information theory.

Quantum Entanglement (2/2)

Entangled quantum systems are sometimes best used collectively, that is, sometimes an optimal use of entangled quantum systems for information storage and retrieval includes manipulating and measuring those systems as a whole, rather than on an individual basis.

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The No-Cloning Theorem

The No-Cloning Theorem

The No-Cloning Theorem.
It is not possible to create an identical copy of an unknown quantum state.

Proof.
Let us suppose that a Unitary operator $\hat{U}$ exists such that, for two independent quantum states $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ (where α, β are unknown complex numbers) and $|e\rangle$ we have:

$$\hat{U}|\psi\rangle \otimes |e\rangle = |\psi\rangle \otimes |\psi\rangle$$

The No-Cloning Theorem

On the one hand, $\hat{U}$ must fulfill the following property:

$$
\begin{aligned}
\hat{U}|\psi\rangle \otimes |e\rangle &= \hat{U}\Big(\alpha|0\rangle + \beta|1\rangle\Big) \otimes |e\rangle \\
&= \Big(\alpha|0\rangle + \beta|1\rangle\Big) \otimes \Big(\alpha|0\rangle + \beta|1\rangle\Big) \\
&= \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle \\
&= \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle
\end{aligned}
$$

The No-Cloning Theorem

On the other hand, linearity implies the following:

$$\begin{aligned}
\hat{U}|\psi\rangle \otimes |e\rangle &= \hat{U}\Big(\alpha|0\rangle + \beta|1\rangle\Big) \otimes |e\rangle \\
&= \hat{U}\Big(\alpha|0\rangle|e\rangle + \beta|1\rangle|e\rangle\Big) \\
&= \alpha\hat{U}|0\rangle|e\rangle + \beta\hat{U}|1\rangle|e\rangle \\
&= \alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle \\
&= \alpha|00\rangle + \beta|11\rangle
\end{aligned}$$

The No-Cloning Theorem

So, we have that

$$\hat{U}|\psi\rangle \otimes |e\rangle = \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle$$

as well as

$$\hat{U}|\psi\rangle \otimes |e\rangle = \alpha|00\rangle + \beta|11\rangle$$

Thus,

$$\alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle = \alpha|00\rangle + \beta|11\rangle$$

The No-Cloning Theorem

$$\alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle = \alpha|00\rangle + \beta|11\rangle$$

is true only for:

$$\left[\alpha = 0 \text{ and } \beta = 1\right] \Leftrightarrow \left[\hat{U}|1\rangle|e\rangle = |1\rangle|1\rangle\right] \tag{1}$$

$$\left[\alpha = 1 \text{ and } \beta = 0\right] \Leftrightarrow \left[\hat{U}|0\rangle|e\rangle = |0\rangle|0\rangle\right] \tag{2}$$

$$\left[\alpha = 0 \text{ and } \beta = 0\right] \tag{3}$$

Eqs. (1,2) correspond to the case of **copying classical information**. Eq. (3) is a most uninteresting case. *Q.E.D.*

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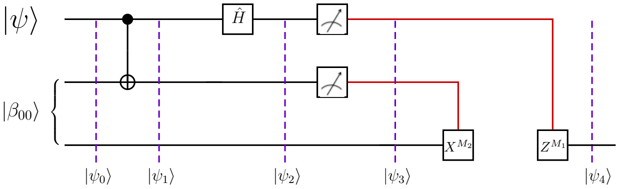
The Quantum Teleportation Protocol

Quantum Teleportation Circuit (1/3)



$$
\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{2}}(\alpha|00\rangle|0\rangle + \alpha|01\rangle|1\rangle + \beta|10\rangle|0\rangle + \beta|11\rangle|1\rangle) \\
|\psi_1\rangle &= \frac{1}{\sqrt{2}}(\alpha|0\rangle|00\rangle + \alpha|0\rangle|11\rangle + \beta|1\rangle|10\rangle + \beta|1\rangle|01\rangle) \\
|\psi_2\rangle &= \tfrac{1}{2}[|00\rangle_A(\alpha|0\rangle + \beta|1\rangle) + |01\rangle_A(\alpha|1\rangle + \beta|0\rangle) + \\
&\quad |10\rangle_A(\alpha|0\rangle - \beta|1\rangle) + |11\rangle_A(\alpha|1\rangle - \beta|0\rangle)]
\end{aligned}
$$

Quantum Teleportation Circuit (2/3)

$|\psi_3\rangle$

$p(a_0, b_0) = \langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle = \frac{1}{4}$ and $|\psi\rangle^{\text{pm}}_{\{a_0,b_0\}} = |00\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B)$

$p(a_0, b_1) = \langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle = \frac{1}{4}$ and $|\psi\rangle^{\text{pm}}_{\{a_0,b_1\}} = |01\rangle_A(\alpha|1\rangle_B + \beta|0\rangle_B)$

$p(a_1, b_0) = \langle\psi_2|\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle = \frac{1}{4}$ and $|\psi\rangle^{\text{pm}}_{\{a_1,b_0\}} = |10\rangle_A(\alpha|0\rangle_B - \beta|1\rangle_B)$

$p(a_1, b_1) = \langle\psi_2|\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle = \frac{1}{4}$ and $|\psi\rangle^{\text{pm}}_{\{a_1,b_1\}} = |11\rangle_A(\alpha|1\rangle_B - \beta|0\rangle_B)$

Quantum Teleportation Circuit (3/3)

$|\psi_4\rangle$

If outcome $\{a_0, b_0\} \Rightarrow |\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_B[|00\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B)]$

If outcome $\{a_0, b_1\} \Rightarrow |\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_x)_B[|01\rangle_A(\alpha|1\rangle_B + \beta|0\rangle_B)]$

If outcome $\{a_1, b_0\} \Rightarrow |\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_z)_B[|10\rangle_A(\alpha|0\rangle_B - \beta|1\rangle_B)]$

If outcome $\{a_1, b_1\} \Rightarrow |\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_z\hat{\sigma}_x)_B[|11\rangle_A(\alpha|1\rangle_B - \beta|0\rangle_B)]$

where $\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$