Universidad Nacional Autónoma de México

FACULTAD DE CIENCIAS

EXAMEN 01

Computación Cuántica I

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1. Explique brevemente cual es el objetivo del:

(a) Protocolo de Teletransportacion Cuántica.
Es transmitir el estado cuántico completo de una partícula desde un emisor (Alice) a un receptor (Bob), utilizando un par de partículas entrelazadas y un canal clásico.

(b) Protocolo de Superdense Coding.
Es transmitir dos bits de información clásica utilizando únicamente un qubit entrelazado y un canal clásico.

2. Si

$$|\phi\rangle = i\,|0\rangle \;\text{ y }\; |\psi\rangle = \frac{e^{i\varphi}}{\sqrt{5}}\,|0\rangle + \beta\,|1\rangle$$

(a) Calcule el valor de β, para tener estados normalizados.
¿Existe una solucion única?

$$\langle\psi|\psi\rangle = \Big(\langle 0|\,\frac{e^{-i\varphi}}{\sqrt{5}} + \beta^*\,\langle 1|\Big)\Big(\frac{e^{i\varphi}}{\sqrt{5}}\,|0\rangle + \beta\,|1\rangle\Big) = \frac{1}{5} + |\beta|^2 = 1$$

$$|\beta|^2 = 1 - \frac{1}{5} = \frac{4}{5} \Rightarrow \beta = \frac{2}{\sqrt{5}}e^{i\varphi} \to \frac{2}{\sqrt{5}}$$

(b) Calcule: $\langle\psi|\phi\rangle$

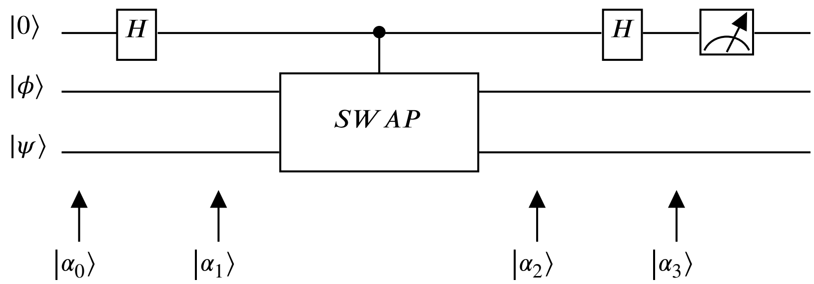
$$\begin{aligned}\langle\psi|\phi\rangle &= \Big(\langle 0|\,\frac{e^{-i\varphi}}{\sqrt{5}} + \langle 1|\,\frac{2}{\sqrt{5}}\Big)i\,|0\rangle \\ &= i\,\frac{e^{-i\varphi}}{\sqrt{5}} \\ &= \frac{e^{-i\varphi+\frac{\pi}{2}i}}{\sqrt{5}}\end{aligned}$$

(c) Calcule: $Control - \hat{Z}(|\psi\rangle \otimes |\phi\rangle)$;
Es decir, $|\psi\rangle$ es el control y $|\phi\rangle$ el objetivo, con:

$$\hat{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\begin{aligned}C - \hat{Z}(|\psi\rangle\,|\phi\rangle) &= C - \hat{Z}\Big[\Big(\frac{e^{i\varphi}}{\sqrt{5}}\,|0\rangle + \frac{2}{\sqrt{5}}\,|1\rangle\Big) \otimes i\,|0\rangle\Big] \\ &= \frac{e^{i\varphi}}{\sqrt{5}}C - \hat{Z}(|0\rangle\,|0\rangle) + \frac{2}{\sqrt{5}}C - \hat{Z}(|1\rangle\,|0\rangle) \\ &= \frac{e^{i\varphi}}{\sqrt{5}}\,|0\rangle\,|0\rangle + \frac{2}{\sqrt{5}}\,|1\rangle\,|0\rangle\end{aligned}$$

3. Para el circuito de la figura siguiente, calcule las probabilidades de los estados $|0\rangle$ y $|1\rangle$ en el primer **qubit**, en términos de $|\langle\phi|\psi\rangle|^2$.



$$|\alpha_0\rangle = |0\rangle|\phi\rangle|\psi\rangle$$

$$\begin{aligned}
|\alpha_1\rangle &= \left(\hat{H}|0\rangle\right)|\phi\rangle|\psi\rangle \\
&= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|\phi\rangle|\psi\rangle \\
&= \frac{1}{\sqrt{2}}|0\rangle|\phi\psi\rangle + \frac{1}{\sqrt{2}}|1\rangle|\phi\psi\rangle
\end{aligned}$$

$$|\alpha_2\rangle = \frac{1}{\sqrt{2}}|0\rangle|\phi\rangle|\psi\rangle + \frac{1}{\sqrt{2}}|1\rangle|\psi\rangle|\phi\rangle$$

$$\begin{aligned}
|\alpha_3\rangle &= \frac{1}{\sqrt{2}}\hat{H}|0\rangle|\phi\rangle|\psi\rangle + \frac{1}{\sqrt{2}}\hat{H}|1\rangle|\psi\rangle|\phi\rangle \\
&= \frac{1}{\sqrt{2}}\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}|\phi\rangle|\psi\rangle + \frac{|0\rangle - |1\rangle}{\sqrt{2}}|\psi\rangle|\phi\rangle\right] \\
&= \frac{1}{2}\left[|0\rangle\left(|\phi\rangle|\psi\rangle + |\psi\rangle|\phi\rangle\right) + |1\rangle\left(|\phi\rangle|\psi\rangle - |\psi\rangle|\phi\rangle\right)\right]
\end{aligned}$$

Sean los operadores de medición para el **qubit** 0:

$$\begin{aligned}
\hat{M}_0 &= |0\rangle\langle 0| \\
\hat{M}_1 &= |1\rangle\langle 1|
\end{aligned}$$

Tenemos que la probabilidad de 0 es:

$$\begin{aligned}
\langle\alpha_3|\hat{M}_0^\dagger\hat{M}_0|\alpha_3\rangle &= \langle\alpha_3|0\rangle\langle 0|0\rangle\langle 0|\alpha_3\rangle \\
&= \langle\alpha_3|0\rangle\langle 0|\alpha_3\rangle \\
&= \frac{1}{4}\left[\left(\langle\phi|\langle\psi| + \langle\psi|\langle\phi|\right)\left(|\phi\rangle|\psi\rangle + |\psi\rangle|\phi\rangle\right)\right] \\
&= \frac{1}{4}\Big[\left(\langle\phi|\otimes\langle\psi|\right)\left(|\phi\rangle\otimes|\psi\rangle\right) \\
&+ \left(\langle\phi|\otimes\langle\psi|\right)\left(|\psi\rangle\otimes|\phi\rangle\right) \\
&+ \left(\langle\psi|\otimes\langle\phi|\right)\left(|\phi\rangle\otimes|\psi\rangle\right) \\
&+ \left(\langle\psi|\otimes\langle\phi|\right)\left(|\psi\rangle\otimes|\phi\rangle\right)\Big]
\end{aligned}$$

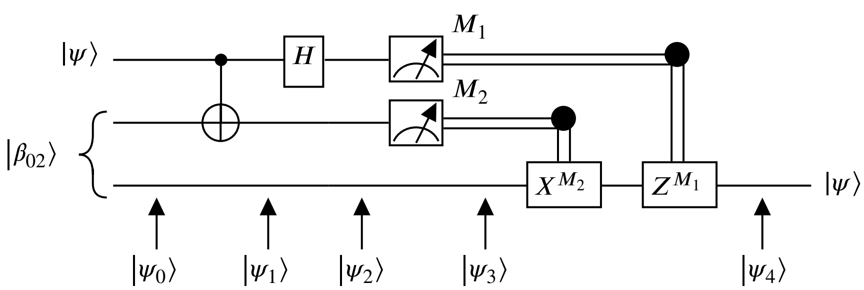
4. Describa, con todo detalle, el protocolo de Teletransportacion Cuántica de un **qubit** dado por:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Utilizando el estado de Bell:

$$|\Psi^{+}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

Recordando que:

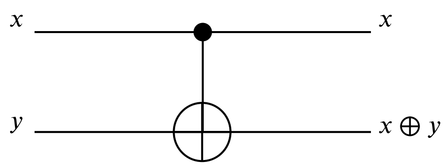


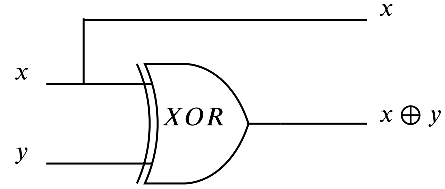
Tomando en cuenta todo lo anterior, procedemos a calcular el estado $|\psi_0\rangle$.

- $|\psi_0\rangle$

$$\begin{aligned}
|\psi_0\rangle &= |\psi\rangle \otimes |\beta_{02}\rangle \\
&= (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}(\alpha|0\rangle(|01\rangle + |10\rangle) + \beta|1\rangle(|01\rangle + |10\rangle)) \\
&= \frac{1}{\sqrt{2}}(\alpha|0\rangle|01\rangle + \alpha|0\rangle|10\rangle + \beta|1\rangle|01\rangle + \beta|1\rangle|10\rangle) \\
&= \frac{1}{\sqrt{2}}(\alpha|00\rangle|1\rangle + \alpha|01\rangle|0\rangle + \beta|10\rangle|1\rangle + \beta|11\rangle|0\rangle)
\end{aligned}$$

$$\therefore |\psi_0\rangle = \frac{1}{\sqrt{2}}(\alpha|00\rangle|1\rangle + \alpha|01\rangle|0\rangle + \beta|10\rangle|1\rangle + \beta|11\rangle|0\rangle)$$





Input		*Output*	
x	y	x	$x \oplus y$
$\|0>$	$\|0>$	$\|0>$	$\|0>$
$\|0>$	$\|1>$	$\|0>$	$\|1>$
$\|1>$	$\|0>$	$\|1>$	$\|1>$
$\|1>$	$\|1>$	$\|1>$	$\|0>$

Input		*Output*	
x	y	x	$x \oplus y$
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

- $|\psi_1\rangle$

$$
\begin{aligned}
|\psi_1\rangle &= \hat{C}_{NOT} \otimes \hat{\mathbf{I}}\,|\psi_0\rangle \\
&= (\hat{C}_{NOT} \otimes \hat{\mathbf{I}})\frac{1}{\sqrt{2}}(\alpha\,|00\rangle\,|1\rangle + \alpha\,|01\rangle\,|0\rangle + \beta\,|10\rangle\,|1\rangle + \beta\,|11\rangle\,|0\rangle) \\
&= \frac{1}{\sqrt{2}}[\hat{C}_{NOT} \otimes \hat{\mathbf{I}}(\alpha\,|00\rangle\,|1\rangle) + \hat{C}_{NOT} \otimes \hat{\mathbf{I}}(\alpha\,|01\rangle\,|0\rangle) + \hat{C}_{NOT} \otimes \hat{\mathbf{I}}(\beta\,|10\rangle\,|1\rangle) + \hat{C}_{NOT} \otimes \hat{\mathbf{I}}(\beta\,|11\rangle\,|0\rangle)] \\
&= \frac{1}{\sqrt{2}}[\alpha(\hat{C}_{NOT}\,|00\rangle \otimes \hat{\mathbf{I}}\,|1\rangle) + \alpha(\hat{C}_{NOT}\,|01\rangle \otimes \hat{\mathbf{I}}\,|0\rangle) + \beta(\hat{C}_{NOT}\,|10\rangle \otimes \hat{\mathbf{I}}\,|1\rangle) + \beta(\hat{C}_{NOT}\,|11\rangle \otimes \hat{\mathbf{I}}\,|0\rangle)] \\
&= \frac{1}{\sqrt{2}}[\alpha\,|00\rangle\,|1\rangle + \alpha\,|01\rangle\,|0\rangle + \beta\,|11\rangle\,|1\rangle + \beta\,|10\rangle\,|0\rangle] \\
&= \frac{1}{\sqrt{2}}[\alpha\,|0\rangle\,|01\rangle + \alpha\,|0\rangle\,|10\rangle + \beta\,|1\rangle\,|11\rangle + \beta\,|1\rangle\,|00\rangle]
\end{aligned}
$$

$$
\therefore |\psi_1\rangle = \frac{1}{\sqrt{2}}(\alpha\,|0\rangle\,|01\rangle + \alpha\,|0\rangle\,|10\rangle + \beta\,|1\rangle\,|11\rangle + \beta\,|1\rangle\,|00\rangle)
$$

- $|\psi_2\rangle$

$$
\begin{aligned}
|\psi_2\rangle &= (\hat{H}\otimes\hat{\mathbf{I}}\otimes\hat{\mathbf{I}})\,|\psi_1\rangle \\
&= (\hat{H}\otimes\hat{\mathbf{I}}\otimes\hat{\mathbf{I}})\frac{1}{\sqrt{2}}(\alpha\,|0\rangle\,|01\rangle+\alpha\,|0\rangle\,|10\rangle+\beta\,|1\rangle\,|11\rangle+\beta\,|1\rangle\,|00\rangle) \\
&= (\hat{H}\otimes\hat{\mathbf{I}}\otimes\hat{\mathbf{I}})\frac{1}{\sqrt{2}}(\alpha\,|0\rangle\,|0\rangle\,|1\rangle+\alpha\,|0\rangle\,|1\rangle\,|0\rangle+\beta\,|1\rangle\,|1\rangle\,|1\rangle+\beta\,|1\rangle\,|0\rangle\,|0\rangle) \\
&= \frac{1}{\sqrt{2}}[\alpha\hat{H}\,|0\rangle\otimes\hat{\mathbf{I}}\,|0\rangle\otimes\hat{\mathbf{I}}\,|1\rangle+\alpha\hat{H}\,|0\rangle\otimes\hat{\mathbf{I}}\,|1\rangle\otimes\hat{\mathbf{I}}\,|0\rangle+\beta\hat{H}\,|1\rangle\otimes\hat{\mathbf{I}}\,|1\rangle\otimes\hat{\mathbf{I}}\,|1\rangle+\beta\hat{H}\,|1\rangle\otimes\hat{\mathbf{I}}\,|0\rangle\otimes\hat{\mathbf{I}}\,|0\rangle] \\
&= \frac{1}{\sqrt{2}}\left[\alpha\hat{H}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|01\rangle+\alpha\hat{H}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|10\rangle+\beta\hat{H}\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)|11\rangle+\beta\hat{H}\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)|00\rangle\right] \\
&= \frac{1}{\sqrt{2}^2}[\alpha\,|001\rangle+\alpha\,|101\rangle+\alpha\,|010\rangle+\alpha\,|110\rangle+\beta\,|011\rangle-\beta\,|111\rangle+\beta\,|000\rangle-\beta\,|100\rangle] \\
&= \frac{1}{2}[\alpha\,|001\rangle+\beta\,|000\rangle+\alpha\,|101\rangle-\beta\,|100\rangle+\alpha\,|010\rangle+\beta\,|011\rangle+\alpha\,|110\rangle-\beta\,|111\rangle] \\
&= \frac{1}{2}[|00\rangle\,(\alpha\,|1\rangle+\beta\,|0\rangle)+|10\rangle\,(\alpha\,|1\rangle-\beta\,|0\rangle)+|01\rangle\,(\alpha\,|0\rangle+\beta\,|1\rangle)+|11\rangle\,(\alpha\,|0\rangle-\beta\,|1\rangle)]
\end{aligned}
$$

$$
\therefore |\psi_2\rangle = \frac{1}{2}(|00\rangle\,(\alpha\,|1\rangle+\beta\,|0\rangle)+|10\rangle\,(\alpha\,|1\rangle-\beta\,|0\rangle)+|01\rangle\,(\alpha\,|0\rangle+\beta\,|1\rangle)+|11\rangle\,(\alpha\,|0\rangle-\beta\,|1\rangle))
$$

- $|\psi_3\rangle$
  Notemos como es que $|\psi_3\rangle$ es el estado cuántico producido después de medir los qubits de el remitente.
  En este paso, mediremos los dos qubits de el remitente, $|\psi\rangle$ y su qubit EPR, simultáneamente.
  Recordando los operadores de medida de un qubit:

$$
\hat{P}_{a0}^{|\psi\rangle} = |0\rangle\langle 0| \qquad \hat{P}_{a1}^{|\psi\rangle} = |1\rangle\langle 1| \qquad \hat{P}_{b0}^{|\beta_{00}\rangle} = |0\rangle\langle 0| \qquad \hat{P}_{b1}^{|\beta_{00}\rangle} = |1\rangle\langle 1|
$$

Y también recordando que usamos las etiquetas $\{a_0, a_1\}$ y $\{b_0, b_1\}$ para referirnos a los posibles resultados de medida para $|\psi\rangle$ y el qubit EPR de el remitente, respectivamente, tendremos que sólo cuatro resultados son posibles; $\{a_0, b_0\}$, $\{a_0, b_1\}$, $\{a_1, b_0\}$ o $\{a_1, b_1\}$.
Es por esto que tenemos los siguientes operadores de medida de dos qubits:

$$
\begin{aligned}
\hat{P}_{\{a_0,b_0\}} &= \hat{P}_{\{a_0\}}^{|\psi\rangle}\otimes\hat{P}_{\{b_0\}}^{|\beta_{00}\rangle} = \left|0_{a_0}\right\rangle\left\langle 0_{a_0}\right|\otimes\left|0_{b_0}\right\rangle\left\langle 0_{b_0}\right| = \left|0_{a_0}0_{b_0}\right\rangle\left\langle 0_{a_0}0_{b_0}\right| = |00\rangle\langle 00| \\
\hat{P}_{\{a_0,b_1\}} &= \hat{P}_{\{a_0\}}^{|\psi\rangle}\otimes\hat{P}_{\{b_1\}}^{|\beta_{00}\rangle} = \left|0_{a_0}\right\rangle\left\langle 0_{a_0}\right|\otimes\left|1_{b_1}\right\rangle\left\langle 1_{b_1}\right| = \left|0_{a_0}1_{b_1}\right\rangle\left\langle 0_{a_0}1_{b_1}\right| = |01\rangle\langle 01| \\
\hat{P}_{\{a_1,b_0\}} &= \hat{P}_{\{a_1\}}^{|\psi\rangle}\otimes\hat{P}_{\{b_0\}}^{|\beta_{00}\rangle} = \left|1_{a_1}\right\rangle\left\langle 1_{a_1}\right|\otimes\left|0_{b_0}\right\rangle\left\langle 0_{b_0}\right| = \left|1_{a_1}0_{b_0}\right\rangle\left\langle 1_{a_1}0_{b_0}\right| = |10\rangle\langle 10| \\
\hat{P}_{\{a_1,b_1\}} &= \hat{P}_{\{a_1\}}^{|\psi\rangle}\otimes\hat{P}_{\{b_1\}}^{|\beta_{00}\rangle} = \left|1_{a_1}\right\rangle\left\langle 1_{a_1}\right|\otimes\left|1_{b_1}\right\rangle\left\langle 1_{b_1}\right| = \left|1_{a_1}1_{b_1}\right\rangle\left\langle 1_{a_1}1_{b_1}\right| = |11\rangle\langle 11|
\end{aligned}
$$

Teniendo en mente lo anterior, vamos a calcular la distribución de probabilidad y los estados posteriores a la medición para los resultados:
$\{a_0, b_0\}$, $\{a_0, b_1\}$, $\{a_1, b_0\}$, $\{a_1, b_1\}$.

I) $p(a_0, b_0) = \langle\psi_2|\, \hat{P}_{\{a_0,b_0\}} \,|\psi_2\rangle$

$$
\begin{aligned}
\hat{P}_{\{a_0,b_0\}} |\psi_2\rangle &= |00\rangle\langle 00| \left[\frac{1}{2}(|00\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|1\rangle - \beta|0\rangle) + |01\rangle(\alpha|0\rangle + \beta|1\rangle) + |11\rangle(\alpha|0\rangle - \beta|1\rangle))\right] \\
&= \frac{1}{2}\Big[(\langle 00|00\rangle\,|00\rangle(\alpha|1\rangle + \beta|0\rangle) + \langle 00|10\rangle\,|00\rangle(\alpha|1\rangle - \beta|0\rangle) \\
&+ \langle 00|01\rangle\,|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \langle 00|11\rangle\,|00\rangle(\alpha|0\rangle - \beta|1\rangle))\Big] \\
&= \frac{1}{2}|00\rangle(\alpha|1\rangle + \beta|0\rangle)
\end{aligned}
$$

Por ende:

$$\hat{P}_{\{a_0,b_0\}} |\psi_2\rangle = \frac{1}{2}|00\rangle(\alpha|1\rangle + \beta|0\rangle)$$

Ahora:

$$
\begin{aligned}
\langle\psi_2|\, \hat{P}_{\{a_0,b_0\}} \,|\psi_2\rangle &= \frac{1}{2}[\langle 00|(\alpha^*\langle 1| + \beta^*\langle 0|) + \langle 10|(\alpha^*\langle 1| - \beta^*\langle 0|) + \langle 01|(\alpha^*\langle 0| + \beta^*\langle 1|) + \langle 11|(\alpha^*\langle 0| - \beta^*\langle 1|)] \\
&\quad [\frac{1}{2}|00\rangle(\alpha|1\rangle + \beta|0\rangle)] \\
&= \frac{1}{4}\Big[\langle 00|00\rangle(\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|1\rangle + \beta|0\rangle) + \langle 10|00\rangle(\alpha^*\langle 1| - \beta^*\langle 0|)(\alpha|1\rangle + \beta|0\rangle) \\
&+ \langle 01|00\rangle(\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|1\rangle + \beta|0\rangle) + \langle 11|00\rangle(\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|1\rangle + \beta|0\rangle)\Big] \\
&= \frac{1}{4}\Big[(\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|1\rangle + \beta|0\rangle)\Big] \\
&= \frac{1}{4}\Big[\alpha^*\alpha\langle 1|1\rangle + \alpha^*\beta\langle 1|0\rangle + \beta^*\alpha\langle 0|1\rangle + \beta^*\beta\langle 0|0\rangle\Big] \\
&= \frac{1}{4}\Big[||\alpha||^2 + ||\beta||^2\Big] \\
&= \frac{1}{4}
\end{aligned}
$$

$$p(a_0, b_0) = \langle\psi_2|\, \hat{P}_{\{a_0,b_0\}} \,|\psi_2\rangle = \frac{1}{4}$$

El correspondiente estado post-medición $|\psi\rangle^{pm}_{\{a_0,b_0\}}$ esta dado por:

$$|\psi\rangle^{pm}_{\{a_0,b_0\}} = \frac{\hat{P}_{\{a_0,b_0\}} |\psi_2\rangle}{\sqrt{\langle\psi_2|\, \hat{P}_{\{a_0,b_0\}} \,|\psi_2\rangle}} = \frac{\frac{1}{2}|00\rangle(\alpha|1\rangle + \beta|0\rangle)}{\sqrt{1/4}} = |00\rangle(\alpha|1\rangle + \beta|0\rangle)$$

Por lo tanto:

$$p(a_0, b_0) = \langle\psi_2|\, \hat{P}_{\{a_0,b_0\}} \,|\psi_2\rangle = \frac{1}{4}$$

$$|\psi\rangle^{pm}_{\{a_0,b_0\}} = |00\rangle(\alpha|1\rangle + \beta|0\rangle)$$

II) $p(a_0, b_1) = \langle\psi_2| \hat{P}_{\{a_0,b_1\}} |\psi_2\rangle$

$$\begin{aligned}
\hat{P}_{\{a_0,b_1\}} |\psi_2\rangle &= |01\rangle\langle 01| \left[\frac{1}{2}(|00\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|1\rangle - \beta|0\rangle) + |01\rangle(\alpha|0\rangle + \beta|1\rangle) + |11\rangle(\alpha|0\rangle - \beta|1\rangle))\right] \\
&= \frac{1}{2}\Big[(\langle 01|00\rangle|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \langle 01|10\rangle|01\rangle(\alpha|1\rangle - \beta|0\rangle) \\
&+ \langle 01|01\rangle|01\rangle(\alpha|0\rangle + \beta|1\rangle) + \langle 01|11\rangle|01\rangle(\alpha|0\rangle - \beta|1\rangle))\Big] \\
&= \frac{1}{2}|01\rangle(\alpha|0\rangle + \beta|1\rangle)
\end{aligned}$$

Por ende:

$$\hat{P}_{\{a_0,b_1\}} |\psi_2\rangle = \frac{1}{2}|01\rangle(\alpha|0\rangle + \beta|1\rangle)$$

Ahora:

$$\begin{aligned}
\langle\psi_2| \hat{P}_{\{a_0,b_1\}} |\psi_2\rangle &= \frac{1}{2}[\langle 00|(\alpha^*\langle 1| + \beta^*\langle 0|) + \langle 10|(\alpha^*\langle 1| - \beta^*\langle 0|) + \langle 01|(\alpha^*\langle 0| + \beta^*\langle 1|) + \langle 11|(\alpha^*\langle 0| - \beta^*\langle 1|)] \\
&\quad [\frac{1}{2}|01\rangle(\alpha|0\rangle + \beta|1\rangle)] \\
&= \frac{1}{4}\Big[\langle 00|01\rangle(\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|0\rangle + \beta|1\rangle) + \langle 10|01\rangle(\alpha^*\langle 1| - \beta^*\langle 0|)(\alpha|0\rangle + \beta|1\rangle) \\
&+ \langle 01|01\rangle(\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|0\rangle + \beta|1\rangle) + \langle 11|01\rangle(\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|0\rangle + \beta|1\rangle)\Big] \\
&= \frac{1}{4}\Big[(\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|0\rangle + \beta|1\rangle)\Big] \\
&= \frac{1}{4}\Big[\alpha^*\alpha\langle 0|0\rangle + \alpha^*\beta\langle 0|1\rangle + \beta^*\alpha\langle 1|0\rangle + \beta^*\beta\langle 1|1\rangle\Big] \\
&= \frac{1}{4}\Big[||\alpha||^2 + ||\beta||^2\Big] \\
&= \frac{1}{4}
\end{aligned}$$

$$p(a_0, b_1) = \langle\psi_2| \hat{P}_{\{a_0,b_1\}} |\psi_2\rangle = \frac{1}{4}$$

El correspondiente estado post-medición $|\psi\rangle^{pm}_{\{a_0,b_1\}}$ esta dado por:

$$|\psi\rangle^{pm}_{\{a_0,b_1\}} = \frac{\hat{P}_{\{a_0,b_1\}} |\psi_2\rangle}{\sqrt{\langle\psi_2| \hat{P}_{\{a_0,b_1\}} |\psi_2\rangle}} = \frac{\frac{1}{2}|01\rangle(\alpha|0\rangle + \beta|1\rangle)}{\sqrt{1/4}} = |01\rangle(\alpha|0\rangle + \beta|1\rangle)$$

Por lo tanto:

$$p(a_0, b_1) = \langle\psi_2| \hat{P}_{\{a_0,b_1\}} |\psi_2\rangle = \frac{1}{4}$$

$$|\psi\rangle^{pm}_{\{a_0,b_1\}} = |01\rangle(\alpha|0\rangle + \beta|1\rangle)$$

III) $p(a_1, b_0) = \langle\psi_2|\, \hat{P}_{\{a_1,b_0\}}\, |\psi_2\rangle$

$$\begin{aligned}
\hat{P}_{\{a_1,b_0\}}\, |\psi_2\rangle &= |10\rangle\langle 10| \left[\frac{1}{2}(|00\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|1\rangle - \beta|0\rangle) + |01\rangle(\alpha|0\rangle + \beta|1\rangle) + |11\rangle(\alpha|0\rangle - \beta|1\rangle))\right] \\
&= \frac{1}{2}\Big[(\langle 10|00\rangle|10\rangle(\alpha|1\rangle + \beta|0\rangle) + \langle 10|10\rangle|10\rangle(\alpha|1\rangle - \beta|0\rangle) \\
&+ \langle 10|01\rangle|10\rangle(\alpha|0\rangle + \beta|1\rangle) + \langle 10|11\rangle|10\rangle(\alpha|0\rangle - \beta|1\rangle))\Big] \\
&= \frac{1}{2}|10\rangle(\alpha|1\rangle - \beta|0\rangle)
\end{aligned}$$

Por ende:

$$\hat{P}_{\{a_1,b_0\}}\, |\psi_2\rangle = \frac{1}{2}|10\rangle(\alpha|1\rangle - \beta|0\rangle)$$

Ahora:

$$\begin{aligned}
\langle\psi_2|\, \hat{P}_{\{a_1,b_0\}}\, |\psi_2\rangle &= \frac{1}{2}[\langle 00|(\alpha^*\langle 1| + \beta^*\langle 0|) + \langle 10|(\alpha^*\langle 1| - \beta^*\langle 0|) + \langle 01|(\alpha^*\langle 0| + \beta^*\langle 1|) + \langle 11|(\alpha^*\langle 0| - \beta^*\langle 1|)] \\
&[\frac{1}{2}|10\rangle(\alpha|1\rangle - \beta|0\rangle)] \\
&= \frac{1}{4}\Big[\langle 00|10\rangle(\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|1\rangle - \beta|0\rangle) + \langle 10|10\rangle(\alpha^*\langle 1| - \beta^*\langle 0|)(\alpha|1\rangle - \beta|0\rangle) \\
&+ \langle 01|10\rangle(\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|1\rangle - \beta|0\rangle) + \langle 11|10\rangle(\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|1\rangle - \beta|0\rangle)\Big] \\
&= \frac{1}{4}\Big[(\alpha^*\langle 1| - \beta^*\langle 0|)(\alpha|1\rangle - \beta|0\rangle)\Big] \\
&= \frac{1}{4}\Big[\alpha^*\alpha\langle 1|1\rangle - \alpha^*\beta\langle 1|0\rangle - \beta^*\alpha\langle 0|1\rangle + \beta^*\beta\langle 0|0\rangle\Big] \\
&= \frac{1}{4}\Big[||\alpha||^2 + ||\beta||^2\Big] \\
&= \frac{1}{4}
\end{aligned}$$

$$p(a_1, b_0) = \langle\psi_2|\, \hat{P}_{\{a_1,b_0\}}\, |\psi_2\rangle = \frac{1}{4}$$

El correspondiente estado post-medición $|\psi\rangle^{pm}_{\{a_1,b_0\}}$ esta dado por:

$$|\psi\rangle^{pm}_{\{a_1,b_0\}} = \frac{\hat{P}_{\{a_1,b_0\}}\, |\psi_2\rangle}{\sqrt{\langle\psi_2|\, \hat{P}_{\{a_1,b_0\}}\, |\psi_2\rangle}} = \frac{\frac{1}{2}|10\rangle(\alpha|1\rangle - \beta|0\rangle)}{\sqrt{1/4}} = |10\rangle(\alpha|1\rangle - \beta|0\rangle)$$

Por lo tanto:

$$p(a_1, b_0) = \langle\psi_2|\, \hat{P}_{\{a_1,b_0\}}\, |\psi_2\rangle = \frac{1}{4}$$

$$|\psi\rangle^{pm}_{\{a_1,b_0\}} = |10\rangle(\alpha|1\rangle - \beta|0\rangle)$$

IV) $p(a_1, b_1) = \langle\psi_2|\, \hat{P}_{\{a_1,b_1\}}\, |\psi_2\rangle$

$$\begin{aligned}
\hat{P}_{\{a_1,b_1\}} |\psi_2\rangle &= |11\rangle\langle 11| \left[\frac{1}{2}(|00\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|1\rangle - \beta|0\rangle) + |01\rangle(\alpha|0\rangle + \beta|1\rangle) + |11\rangle(\alpha|0\rangle - \beta|1\rangle))\right] \\
&= \frac{1}{2}\Big[(\langle 11|00\rangle|11\rangle(\alpha|1\rangle + \beta|0\rangle) + \langle 11|10\rangle|11\rangle(\alpha|1\rangle - \beta|0\rangle) \\
&+ \langle 11|01\rangle|11\rangle(\alpha|0\rangle + \beta|1\rangle) + \langle 11|11\rangle|11\rangle(\alpha|0\rangle - \beta|1\rangle))\Big] \\
&= \frac{1}{2}|11\rangle(\alpha|0\rangle - \beta|1\rangle)
\end{aligned}$$

Por ende:

$$\hat{P}_{\{a_1,b_1\}} |\psi_2\rangle = \frac{1}{2}|11\rangle(\alpha|0\rangle - \beta|1\rangle)$$

Ahora:

$$\begin{aligned}
\langle\psi_2|\, \hat{P}_{\{a_1,b_1\}}\, |\psi_2\rangle &= \frac{1}{2}[\langle 00|(\alpha^*\langle 1| + \beta^*\langle 0|) + \langle 10|(\alpha^*\langle 1| - \beta^*\langle 0|) + \langle 01|(\alpha^*\langle 0| + \beta^*\langle 1|) + \langle 11|(\alpha^*\langle 0| - \beta^*\langle 1|)] \\
&[\frac{1}{2}|11\rangle(\alpha|0\rangle - \beta|1\rangle)] \\
&= \frac{1}{4}\Big[\langle 00|11\rangle(\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|0\rangle - \beta|1\rangle) + \langle 10|11\rangle(\alpha^*\langle 1| - \beta^*\langle 0|)(\alpha|0\rangle - \beta|1\rangle) \\
&+ \langle 01|11\rangle(\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|0\rangle - \beta|1\rangle) + \langle 11|11\rangle(\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|0\rangle - \beta|1\rangle)\Big] \\
&= \frac{1}{4}\Big[(\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|0\rangle - \beta|1\rangle)\Big] \\
&= \frac{1}{4}\Big[\alpha^*\alpha\langle 0|0\rangle - \alpha^*\beta\langle 0|1\rangle - \beta^*\alpha\langle 1|0\rangle + \beta^*\beta\langle 1|1\rangle\Big] \\
&= \frac{1}{4}\Big[||\alpha||^2 + ||\beta||^2\Big] \\
&= \frac{1}{4}
\end{aligned}$$

$$p(a_1, b_1) = \langle\psi_2|\, \hat{P}_{\{a_1,b_1\}}\, |\psi_2\rangle = \frac{1}{4}$$

El correspondiente estado post-medición $|\psi\rangle^{pm}_{\{a_1,b_1\}}$ esta dado por:

$$|\psi\rangle^{pm}_{\{a_1,b_1\}} = \frac{\hat{P}_{\{a_1,b_1\}}|\psi_2\rangle}{\sqrt{\langle\psi_2|\, \hat{P}_{\{a_1,b_1\}}\, |\psi_2\rangle}} = \frac{\frac{1}{2}|11\rangle(\alpha|0\rangle - \beta|1\rangle)}{\sqrt{1/4}} = |11\rangle(\alpha|0\rangle - \beta|1\rangle)$$

Por lo tanto:

$$p(a_1, b_1) = \langle\psi_2|\, \hat{P}_{\{a_1,b_1\}}\, |\psi_2\rangle = \frac{1}{4}$$

$$|\psi\rangle^{pm}_{\{a_1,b_1\}} = |11\rangle(\alpha|0\rangle - \beta|1\rangle)$$

- $|\psi_4\rangle$

Con todo el procedimiento realizado anteriormente tenemos cuatro posibles casos:

Caso I) Salida $\{a_0, b_0\}$

La probabilidad de obtener $\{a_0, b_0\}$ es:

$$p(a_0, b_0) = \langle\psi_2|\,\hat{P}_{\{a_0,b_0\}}\,|\psi_2\rangle = \frac{1}{4}$$

Además, el estado cuántico post-medición para el resultado $\{a_0, b_0\}$ es:

$$|\psi\rangle^{pm}_{\{a_0,b_0\}} = |00\rangle\,(\alpha\,|1\rangle + \beta\,|0\rangle)$$

En este caso, el remitente sabe que sus qubits están en el estado $|00\rangle$, además, sabe que el qubit de el destinatario está en el estado $\alpha\,|1\rangle + \beta\,|0\rangle$, ahora, recordando que:

$$\begin{aligned}
\hat{\sigma}_x(\alpha\,|1\rangle + \beta\,|0\rangle) &= (|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha\,|1\rangle + \beta\,|0\rangle)\\
&= \alpha\,\langle 1|1\rangle\,|0\rangle + \beta\,\langle 1|0\rangle\,|0\rangle + \alpha\,\langle 0|1\rangle\,|1\rangle + \beta\,\langle 0|0\rangle\,|1\rangle\\
&= \alpha\,|0\rangle + \beta\,|1\rangle
\end{aligned}$$

Es entonces que el remitente llama al destinatario para decirle que

$$|\psi_4\rangle = \hat{\mathbf{I}} \otimes \hat{\mathbf{I}} \otimes \hat{\sigma}_x[|00\rangle\,(\alpha\,|1\rangle + \beta\,|0\rangle)]$$

Caso II) Salida $\{a_0, b_1\}$

La probabilidad de obtener $\{a_0, b_1\}$ es:

$$p(a_0, b_1) = \langle\psi_2|\,\hat{P}_{\{a_0,b_1\}}\,|\psi_2\rangle = \frac{1}{4}$$

Además, el estado cuántico post-medición para el resultado $\{a_0, b_1\}$ es:

$$|\psi\rangle^{pm}_{\{a_0,b_1\}} = |01\rangle\,(\alpha\,|0\rangle + \beta\,|1\rangle)$$

En este caso, el remitente sabe que sus qubits están en el estado $|01\rangle$, además, sabe que el qubit de el destinatario está en el estado $\alpha\,|0\rangle + \beta\,|1\rangle$ el cual era el qubit que el remitente esperaba enviar, por lo tanto, llama a el destinatario a través de un canal clásico (una línea telefónica, por ejemplo) para decirle que su qubit está listo, es decir:

$$|\psi_4\rangle = \hat{\mathbf{I}} \otimes \hat{\mathbf{I}} \otimes \hat{\mathbf{I}}[|01\rangle\,(\alpha\,|0\rangle + \beta\,|1\rangle)]$$

Caso III) Salida $\{a_1, b_0\}$
La probabilidad de obtener $\{a_1, b_0\}$ es:

$$p(a_1, b_0) = \langle\psi_2| \hat{P}_{\{a_1,b_0\}} |\psi_2\rangle = \frac{1}{4}$$

Además, el estado cuántico post-medición para el resultado $\{a_1, b_0\}$ es:

$$|\psi\rangle^{pm}_{\{a_1,b_0\}} = |10\rangle (\alpha |1\rangle - \beta |0\rangle)$$

En este caso, el remitente sabe que sus qubits están en el estado $|10\rangle$, además, sabe que el qubit de el destinatario está en el estado $\alpha |1\rangle - \beta |0\rangle$, ahora, recordando que:

$$\hat{\sigma}_x(\hat{\sigma}_z(\alpha |1\rangle - \beta |0\rangle)) = \alpha |0\rangle + \beta |1\rangle$$

Es entonces que el remitente llama al destinatario para decirle que

$$|\psi_4\rangle = \hat{\mathbf{I}} \otimes \hat{\mathbf{I}} \otimes (\hat{\sigma}_x\hat{\sigma}_z)[|10\rangle (\alpha |1\rangle - \beta |0\rangle)]$$

Caso IV) Salida $\{a_1, b_1\}$
La probabilidad de obtener $\{a_1, b_1\}$ es:

$$p(a_1, b_1) = \langle\psi_2| \hat{P}_{\{a_1,b_1\}} |\psi_2\rangle = \frac{1}{4}$$

Además, el estado cuántico post-medición para el resultado $\{a_1, b_1\}$ es:

$$|\psi\rangle^{pm}_{\{a_1,b_1\}} = |11\rangle (\alpha |0\rangle - \beta |1\rangle)$$

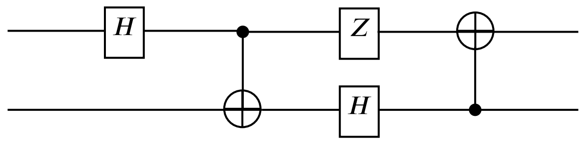
En este caso, el remitente sabe que sus qubits están en el estado $|11\rangle$, además, sabe que el qubit de el destinatario está en el estado $\alpha |0\rangle - \beta |1\rangle$, ahora, recordando que:

$$\begin{aligned}
\hat{\sigma}_z(\alpha |0\rangle - \beta |1\rangle) &= (|0\rangle \langle 0| - |1\rangle \langle 1|)(\alpha |0\rangle - \beta |1\rangle) \\
&= \alpha \langle 0|0\rangle |0\rangle - \beta \langle 0|1\rangle |0\rangle - \alpha \langle 1|0\rangle |1\rangle + \beta \langle 1|1\rangle |1\rangle \\
&= \alpha |0\rangle + \beta |1\rangle
\end{aligned}$$

Es entonces que el remitente llama al destinatario para decirle que

$$|\psi_4\rangle = \hat{\mathbf{I}} \otimes \hat{\mathbf{I}} \otimes \hat{\sigma}_z[|11\rangle (\alpha |0\rangle - \beta |1\rangle)]$$

5. Sea U el circuito dado por el siguiente diagrama:



Calcule y escriba un circuito cuántico que realice la operación U^{-1}.
Recordando que:

$$\hat{U} = \hat{A}\hat{B}\hat{C}\hat{D} \rightarrow \hat{U}^{\dagger} = (\hat{A}\hat{B}\hat{C}\hat{D})^{\dagger} = \hat{D}^{\dagger}\hat{C}^{\dagger}\hat{B}^{\dagger}\hat{A}^{\dagger}$$

Entonces:

$$\hat{U}^{\dagger} = \hat{U} = \hat{U}^{-1} \rightarrow \hat{U}^{-1} = \hat{D}^{-1}\hat{C}^{-1}\hat{B}^{-1}\hat{A}^{-1}$$

Es por esto que $\hat{U}^{-1}$ será: