

Postulates of Quantum Mechanics and the Bell State Quantum Circuit

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Table of Contents

- 1 Postulates of Quantum Mechanics
 - State space and definition of qubit
 - Quantum Evolution
 - Quantum Measurement
 - Composite Quantum Systems
- 2 Our First Quantum Circuit!



State space and definition of qubit



Postulate 1. State Space

To each isolated physical system we associate a Hilbert space \mathcal{H} , hereinafter known as the **state space** of the system. The physical system is completely described by its **state vector**, which is a unit vector $|\psi\rangle \in \mathcal{H}$. The dimension of \mathcal{H} depends on the specific degrees of freedom of the physical property under consideration.



Postulate 1. State Space

Postulate 1 implies that a linear combination of state vectors is a state vector. This is known as the **superposition principle**. In particular, any vector state $|\psi\rangle$ may be described as a superposition of basis states $\{|e_i\rangle\}$ in \mathcal{H} , i.e. $|\psi\rangle = \sum_i c_i |e_i\rangle$, $c_i \in \mathbb{C}$.



Postulate 1. Definition of qubit

In quantum computing, information is stored, manipulated and measured in the form of qubits.

A qubit may be mathematically represented as a unit vector in a two-dimensional Hilbert space $|\psi\rangle \in \mathcal{H}^2$.

A qubit $|\psi\rangle$ may be written in general form as

$$|\psi\rangle = \alpha|p\rangle + \beta|q\rangle$$

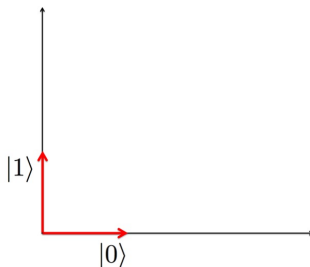
where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$ and $\{|p\rangle, |q\rangle\}$ is an arbitrary basis spanning \mathcal{H}^2 .



Postulate 1. Definition of qubit

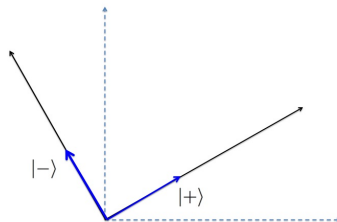
A most important consequence of the vectorial nature of a qubit is the possibility of writing it as a linear combination of elements of any basis.

For example, we may choose $|p\rangle = |0\rangle$ and $|q\rangle = |1\rangle$ to write a qubit. The basis $\{|0\rangle, |1\rangle\}$ is known as **the computational basis**.



Postulate 1. Definition of qubit

We may also choose the **the diagonal basis** $\{|+\rangle, |-\rangle\}$ to write a qubit.



Postulate 1. Definition of qubit

Choosing a concrete vector basis and values for corresponding complex coefficients is known as **preparing a qubit**. For instance,

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

In computer science parlance, preparing a qubit (or a set of qubits) is equivalent to variable initialization in a computer program.



Quantum Evolution



Postulate 2. Evolution by Unitary Operator

The evolution of a closed quantum system with state vector $|\Psi\rangle$ is described by a Unitary operator. The state of a system at time t_2 according to its state at time t_1 is given by

$$|\Psi(t_2)\rangle = \hat{U}|\Psi(t_1)\rangle.$$

Postulate 2 only describes the mathematical properties that an evolution operator must have. The specific evolution operator required to describe the behaviour of a particular quantum system depends on the system itself.



Postulate 2. Evolution by Schrödinger equation

Quantum evolution can also be written in terms of differential equations.

The time evolution of a closed quantum system can be described by the Schrödinger equation:

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{\mathbf{H}} |\psi(t)\rangle$$

where $\hbar = \frac{h}{2\pi}$, h is Planck's constant (about $6.62607004 \times 10^{-34} Js$), and $\hat{\mathbf{H}}$ is a Hermitian operator known as the *Hamiltonian* of the system.



Examples of Unitary Operators (1/2)

Pauli operators

$$\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|; \hat{\sigma}_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|; \hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\text{If } |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \langle 0| = (1, 0), \text{ and } \langle 1| = (0, 1)$$

Then we produce the following matrix representations of Pauli operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



Examples of Unitary Operators (2/2)

The **Hadamard** operator

$$\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

Again, if $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\langle 0| = (1, 0)$, and $\langle 1| = (0, 1)$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



(Already known) Example of Evolution by Unitary Operator

Let $\hat{\sigma}_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ and $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$. Compute $\hat{\sigma}_y|\psi\rangle$.

$$\begin{aligned}
 \hat{\sigma}_y|\psi\rangle &= (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle) \\
 &= -\frac{\sqrt{3}i\langle 1|0\rangle}{2}|0\rangle - \frac{i^2\langle 1|1\rangle}{2}|0\rangle + \frac{\sqrt{3}i\langle 0|0\rangle}{2}|1\rangle + \frac{i^2\langle 0|1\rangle}{2}|1\rangle \\
 &= \frac{1}{2}|0\rangle + \frac{\sqrt{3}i}{2}|1\rangle
 \end{aligned}$$



Example of Evolution by Unitary Operator

In computer science parlance,

- Designing Unitary operators is equivalent to writing a computer program.
- Applying Unitary operators to qubits is equivalent to running a computer program.



Measurement of quantum states



Quantum Measurement

In quantum mechanics, measurement is a non-trivial and highly counter-intuitive process because:

a) Measurement outcomes are inherently probabilistic. Regardless of the carefulness in the preparation of a measurement procedure, the possible outcomes of such measurement are produced/generated according to a certain probability distribution.



Quantum Measurement

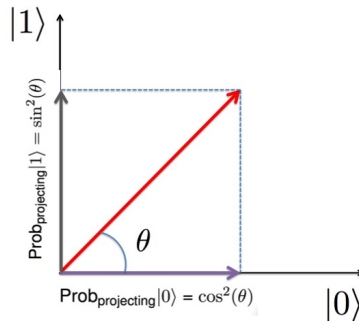
b) Pre- and post-measurement quantum states are different.

Once a measurement has been performed, a quantum system is unavoidably altered due to the interaction with the measurement apparatus. Consequently, for an arbitrary quantum system, pre-measurement and post-measurement quantum states are different in general.



Quantum Measurement

c) Measuring \Leftrightarrow Projecting. Measuring a quantum system is equivalent to projecting (as in analytic geometry) a vector onto a vector basis.



Quantum Measurement

The **KEY** points to remember are:

- 1 Measuring a quantum state is equivalent to **information retrieval**.
- 2 Measuring a quantum system is a **probabilistic process**.
- 3 Measuring a quantum system makes irreversible changes in the information contained in that quantum system, i.e. **in general, pre-measurement and post-measurement states of a quantum system are different**.
- 4 Measuring a quantum system is equivalent to **projecting** its corresponding quantum state onto one of the vectors of the **chosen** measurement basis.



Quantum Measurement

Suppose that we have a quantum system $|\psi\rangle$ living in an n -dimensional Hilbert space \mathcal{H}^n .

The dimension of the Hilbert space is equal to the number of degrees of freedom of the quantum system and, consequently, it is also **equal to the number of different possible outcomes** of a measurement performed on such quantum system.



Quantum Measurement

For example, let us focus on the spin of an electron. Experimental results show that performing a measurement on the spin of an electron always produces **one** of **two** possible outcomes: **spin up** or **spin down**.

So, the spin of an electron is an example of a quantum property that has two degrees of freedom. **Therefore, a quantum state representing the spin of an electron must live in a two-dimensional Hilbert space.**



Quantum Measurement

We may write a quantum state $|\psi\rangle$ that represents the spin of an electron as follows:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $|0\rangle$ represents the quantum state **spin up**, $|1\rangle$ represents the quantum state **spin down**, and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.



Formal description of Quantum Measurement

The mathematical description of quantum measurement requires the following components:

- A quantum state $|\psi\rangle$ and the dimension n of the Hilbert space \mathcal{H} in which $|\psi\rangle$ lives.
- A set of measurement outcomes $\{a_i | i \in \{0, 1, \dots, n-1\}\}$.
- An orthonormal basis $B = \{|i\rangle | i \in \{0, 1, \dots, n-1\}\}$ of \mathcal{H} . The elements of B will be the vectors on which we will project $|\psi\rangle$.
- A set of measurement operators $\{\hat{M}_{a_i} = |i\rangle\langle i|\}$ which will be built using the elements of basis B . Index i labels the different measurement outcomes, which act on the state space of the system being measured.
- Mathematical expressions for computing probability distributions and post-measurement quantum states.



Formal description of Quantum Measurement

Let $|\psi\rangle \in \mathcal{H}^n$ be the state of a quantum system immediately before the measurement. Also, let $\{a_i\}$ be the set of measurement outcomes and $\{\hat{M}_{a_i} = |i\rangle\langle i|\}$ be the set of measurement operators built using basis $B = \{|i\rangle\}$, where $i \in \{0, 1, \dots, n-1\}$.

Then, the probability that outcome a_i occurs is given by

$$p(a_i) = \langle\psi|\hat{M}_{a_i}^\dagger\hat{M}_{a_i}|\psi\rangle$$

where $\hat{M}_{a_i}^\dagger$ is the result of applying the dagger operator \dagger to the projection operator \hat{M}_{a_i} .

Finally, the post-measurement quantum state that corresponds to measurement outcome a_i is given by

$$|\psi\rangle_{pm}^{a_i} = \frac{\hat{M}_{a_i}|\psi\rangle}{\sqrt{p(a_i)}}$$



Quantum Measurement - Example 01

Let $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ be a qubit (i.e., a mathematical representation of a two-dimensional quantum system) defined over the **computational** basis $\{|0\rangle, |1\rangle\}$.

Measure $|\psi\rangle$ with the basis $\{|0\rangle, |1\rangle\}$, i.e compute

$$p(a_0), p(a_1), |\psi\rangle_{pm}^{a_0}, |\psi\rangle_{pm}^{a_1}$$

where measurement outcomes are labelled as a_0, a_1 .



Quantum Measurement - Example 01

The first step is to compute measurement operators $\hat{M}_{a_0}, \hat{M}_{a_1}$ and corresponding operators $\hat{M}_{a_0}^\dagger, \hat{M}_{a_1}^\dagger$.

- Since we will use the computational basis $\{|0\rangle, |1\rangle\}$ to measure, then measurement operators \hat{M}_{a_0} and \hat{M}_{a_1} are defined as:

$$\hat{M}_{a_0} = |0\rangle\langle 0| \text{ and } \hat{M}_{a_1} = |1\rangle\langle 1|$$

- Note that $\hat{M}_{a_0}^\dagger = (|0\rangle\langle 0|)^\dagger = |0\rangle\langle 0|$ and $\hat{M}_{a_1}^\dagger = (|1\rangle\langle 1|)^\dagger = |1\rangle\langle 1|$

- Furthermore,

$$\hat{M}_{a_0}^\dagger \hat{M}_{a_0} = (|0\rangle\langle 0|)(|0\rangle\langle 0|) = |0\rangle\langle 0| = \hat{M}_{a_0} \text{ (Exercise 04, Outer Product Section)}$$

$$\hat{M}_{a_1}^\dagger \hat{M}_{a_1} = (|1\rangle\langle 1|)(|1\rangle\langle 1|) = |1\rangle\langle 1| = \hat{M}_{a_1} \text{ (Exercise 04, Outer Product Section)}$$



Quantum Measurement - Example 01

Let us now calculate the probability of outcome a_0

$$\begin{aligned}
 p(a_0) &= \langle \psi | \hat{M}_{a_0}^\dagger \hat{M}_{a_0} | \psi \rangle = \langle \psi | \hat{M}_{a_0} | \psi \rangle \\
 &= \left(\frac{1}{\sqrt{2}} \langle 0| + \frac{1}{\sqrt{2}} \langle 1| \right) \left[(|0\rangle \langle 0|) \left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) \right] \\
 &= \left(\frac{1}{\sqrt{2}} \langle 0| + \frac{1}{\sqrt{2}} \langle 1| \right) \left(\frac{1}{\sqrt{2}} \langle 0|0\rangle |0\rangle + \frac{1}{\sqrt{2}} \langle 0|1\rangle |0\rangle \right) \\
 &= \left(\frac{1}{\sqrt{2}} \langle 0| + \frac{1}{\sqrt{2}} \langle 1| \right) \left(\frac{1}{\sqrt{2}} |0\rangle \right) \\
 &= \left(\frac{1}{\sqrt{2}} \right)^2 \langle 0|0\rangle + \left(\frac{1}{\sqrt{2}} \right)^2 \langle 1|0\rangle \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\text{So, } p(a_0) = \frac{1}{2}$$



Quantum Measurement - Example 01

We now compute the post-measurement quantum state $|\psi\rangle_{pm}^{a_0}$

$$\begin{aligned} |\psi\rangle_{pm}^{a_0} &= \frac{\hat{M}_{a_0} |\psi\rangle}{\sqrt{p(a_0)}} \\ &= \frac{\frac{1}{\sqrt{2}} |0\rangle}{\frac{1}{\sqrt{2}}} \\ &= |0\rangle \end{aligned}$$

$$\text{So, } |\psi\rangle_{pm}^{a_0} = |0\rangle$$



Quantum Measurement - Example 01

As for outcome a_1 :

$$\begin{aligned}
 p(a_1) &= \langle \psi | \hat{M}_{a_1}^\dagger \hat{M}_{a_1} | \psi \rangle = \langle \psi | \hat{M}_{a_1} | \psi \rangle \\
 &= \left(\frac{1}{\sqrt{2}} \langle 0| + \frac{1}{\sqrt{2}} \langle 1| \right) \left[(|1\rangle \langle 1|) \left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) \right] \\
 &= \left(\frac{1}{\sqrt{2}} \langle 0| + \frac{1}{\sqrt{2}} \langle 1| \right) \left(\frac{1}{\sqrt{2}} \langle 1|0\rangle |1\rangle + \frac{1}{\sqrt{2}} \langle 1|1\rangle |1\rangle \right) \\
 &= \left(\frac{1}{\sqrt{2}} \langle 0| + \frac{1}{\sqrt{2}} \langle 1| \right) \left(\frac{1}{\sqrt{2}} |1\rangle \right) \\
 &= \left(\frac{1}{\sqrt{2}} \right)^2 \langle 0|1\rangle + \left(\frac{1}{\sqrt{2}} \right)^2 \langle 1|1\rangle \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\text{So, } p(a_1) = \frac{1}{2}$$



Quantum Measurement - Example 01

Finally, we compute the post-measurement quantum state $|\psi\rangle_{pm}^{a_1}$:

$$\begin{aligned} |\psi\rangle_{pm}^{a_1} &= \frac{\hat{M}_{a_1} |\psi\rangle}{\sqrt{p(a_1)}} \\ &= \frac{\frac{1}{\sqrt{2}} |1\rangle}{\frac{1}{\sqrt{2}}} \\ &= |1\rangle \end{aligned}$$

$$\text{So, } |\psi\rangle_{pm}^{a_1} = |1\rangle$$



Quantum Measurement - Exercise 02

Let $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ be the same qubit of Exercise 01.

Now, let us measure $|\psi\rangle$ with the **diagonal** basis $\{|+\rangle, |-\rangle\}$, i.e. compute

$$p(a_+), p(a_-), |\psi\rangle_{pm}^{a_+}, |\psi\rangle_{pm}^{a_-}$$

where measurement outcomes are labelled as a_+, a_- .



Quantum Measurement - Example 01

As in Exercise 01, the first step is to compute measurement operators \hat{M}_{a+} , \hat{M}_{a-} and corresponding operators \hat{M}_{a+}^\dagger , \hat{M}_{a-}^\dagger .

- Since we will use the diagonal basis $\{|+\rangle, |-\rangle\}$ to measure, then measurement operators \hat{M}_{a+} and \hat{M}_{a-} are defined as:

$$\hat{M}_{a+} = |+\rangle\langle+| \text{ and } \hat{M}_{a-} = |-\rangle\langle-|$$

- Note that $\hat{M}_{a+}^\dagger = (|+\rangle\langle+|)^\dagger = |+\rangle\langle+|$ and $\hat{M}_{a-}^\dagger = (|-\rangle\langle-|)^\dagger = |-\rangle\langle-|$

- It can be easily proved that

$$\hat{M}_{a+}^\dagger \hat{M}_{a+} = (|+\rangle\langle+|)(|+\rangle\langle+|) = |+\rangle\langle+| = \hat{M}_{a+}$$

$$\hat{M}_{a-}^\dagger \hat{M}_{a-} = (|-\rangle\langle-|)(|-\rangle\langle-|) = |-\rangle\langle-| = \hat{M}_{a-}$$



Quantum Measurement - Exercise 02

Now, let us calculate the probability of getting outcome a_+ :

$$\begin{aligned}
 p(a_+) &= \langle \psi | \hat{M}_{a_+}^\dagger \hat{M}_{a_+} | \psi \rangle = \langle \psi | \hat{M}_{a_+} | \psi \rangle \\
 &= \left(\frac{1}{\sqrt{2}} \langle 0| + \frac{1}{\sqrt{2}} \langle 1| \right) [(|+\rangle \langle +|) \left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right)] \\
 &= (\langle +|) [(|+\rangle \langle +|) (|+\rangle)] \\
 &= (\langle +|) [(\langle +|+ \rangle) (|+\rangle)] \\
 &= \langle +|+ \rangle \\
 &= 1
 \end{aligned}$$

So, $p(a_+) = 1$



Quantum Measurement - Exercise 02

Corresponding post-measurement quantum state $|\psi\rangle_{pm}^{a+}$ is:

$$\begin{aligned} |\psi\rangle_{pm}^{a+} &= \frac{\hat{M}_{a+} |\psi\rangle}{\sqrt{p(a+)}} \\ &= \frac{|+\rangle}{1} \\ &= |+\rangle \end{aligned}$$



Quantum Measurement - Exercise 02

As for outcome a_- :

$$\begin{aligned}
 p(a_-) &= \langle \psi | \hat{M}_{a_-}^\dagger \hat{M}_{a_-} | \psi \rangle = \langle \psi | \hat{M}_{a_-} | \psi \rangle \\
 &= \left(\frac{1}{\sqrt{2}} \langle 0 | + \frac{1}{\sqrt{2}} \langle 1 | \right) [(| - \rangle \langle - |) \left(\frac{1}{\sqrt{2}} | 0 \rangle + \frac{1}{\sqrt{2}} | 1 \rangle \right)] \\
 &= (\langle + |) [(| - \rangle \langle - |) (| + \rangle)] \\
 &= (\langle + |) [(\langle - | + \rangle) (| + \rangle)] \\
 &= (\langle + |) (\langle 0 | + \rangle) \\
 &= (\langle + |) (| \vec{0} \rangle) \\
 &= 0
 \end{aligned}$$

So, $p(a_-) = 0$



Quantum Measurement - Exercise 02

Finally, we compute the post-measurement quantum state $|\psi\rangle_{pm}^{a-}$:

$$\begin{aligned} |\psi\rangle_{pm}^{a-} &= \frac{\hat{M}_{a-} |\psi\rangle}{\sqrt{p(a-)}} \\ &= \frac{|-\rangle}{0} \text{ which, of course, is undefined.} \end{aligned}$$

Why did we get $\frac{|-\rangle}{0}$? The answer is:

Remember that $p(a-) = 0$.

We cannot compute a quantum state that will never exist!



Summary of Quantum Measurement Exercises

- $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ is the quantum state we want to measure.
- Computational basis $\{|0\rangle, |1\rangle\}$
- Diagonal basis $\{|+\rangle, |-\rangle\}$
- Remember that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Measurement basis	Outcomes	Outcome Probabilities and Post-measurement quantum states
$\{ 0\rangle, 1\rangle\}$	a_0, a_1	$p(a_0) = 0.5 \quad \psi\rangle_{pm}^{a_0} = 0\rangle$ $p(a_1) = 0.5 \quad \psi\rangle_{pm}^{a_1} = 1\rangle$
$\{ +\rangle, -\rangle\}$	a_+, a_-	$p(a_+) = 1 \quad \psi\rangle_{pm}^{a_+} = +\rangle$ $p(a_-) = 0 \quad \nexists \psi\rangle_{pm}^{a_-}$



More results (1/3)

Similarly, it is possible to prove the following

- $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ is the quantum state we want to measure.
- Computational basis $\{|0\rangle, |1\rangle\}$
- Diagonal basis $\{|+\rangle, |-\rangle\}$
- Remember that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Measurement basis	Outcomes	Outcome Probabilities and Post-measurement quantum states
$\{ 0\rangle, 1\rangle\}$	a_0, a_1	$p(a_0) = 0.5 \quad \psi\rangle_{pm}^{a_0} = 0\rangle$ $p(a_1) = 0.5 \quad \psi\rangle_{pm}^{a_1} = - 1\rangle = 1\rangle$ get rid of global phase
$\{ +\rangle, -\rangle\}$	a_+, a_-	$p(a_+) = 0 \quad \nexists \psi\rangle_{pm}^{a_+}$ $p(a_-) = 1 \quad \psi\rangle_{pm}^{a_-} = -\rangle$



More results (2/3)

- $|\psi\rangle = |0\rangle$ is the quantum state we want to measure.
- Computational basis $\{|0\rangle, |1\rangle\}$
- Diagonal basis $\{|+\rangle, |-\rangle\}$
- Remember that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Measurement basis	Outcomes	Outcome Probabilities and Post-measurement quantum states
$\{ 0\rangle, 1\rangle\}$	a_0, a_1	$p(a_0) = 1$ $ \psi\rangle_{pm}^{a_0} = 0\rangle$ $p(a_1) = 0$ $\nexists \psi\rangle_{pm}^{a_1}$
$\{ +\rangle, -\rangle\}$	a_+, a_-	$p(a_+) = 0.5$ $ \psi\rangle_{pm}^{a_+} = +\rangle$ $p(a_-) = 0.5$ $ \psi\rangle_{pm}^{a_-} = -\rangle$



More results (3/3)

- $|\psi\rangle = |1\rangle$ is the quantum state we want to measure.
- Computational basis $\{|0\rangle, |1\rangle\}$
- Diagonal basis $\{|+\rangle, |-\rangle\}$
- Remember that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Measurement basis	Outcomes	Outcome Probabilities and Post-measurement quantum states
$\{ 0\rangle, 1\rangle\}$	a_0, a_1	$p(a_0) = 0 \quad \nexists \psi\rangle_{pm}^{a_0}$ $p(a_1) = 1 \quad \psi\rangle_{pm}^{a_1} = 1\rangle$
$\{ +\rangle, -\rangle\}$	a_+, a_-	$p(a_+) = 0.5 \quad \psi\rangle_{pm}^{a_+} = +\rangle$ $p(a_-) = 0.5 \quad \psi\rangle_{pm}^{a_-} = -\rangle$



Composite Quantum Systems



Postulate 4. Composite quantum systems

Quantum registers

Let $|\psi\rangle \in \mathcal{H}^2$ be a qubit. We know that the most general state of a qubit written in terms of the computational basis is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha, \beta \in \mathbb{C}$ and $||\alpha||^2 + ||\beta||^2 = 1$.

Moreover, we know that measuring $|\psi\rangle$ will produce one of two mutually exclusive outcomes that we could labeled as 0 and 1.



Postulate 4. Composite quantum systems

Quantum registers

Now, what about a **pair** of qubits?

What is the most general state of a pair of qubits?

- Let $|\psi\rangle_1 \in \mathcal{H}_1$ where $|\psi\rangle_1 = a|0\rangle + b|1\rangle$ and $|\psi\rangle_2 \in \mathcal{H}_2$ where $|\psi\rangle_2 = c|0\rangle + d|1\rangle$.
- Note that , if we measure **both** qubits **at once**, we will get **one** out of **four** possible outcomes: 0 and 0, 0 and 1, 1 and 0, 1 and 1.



Postulate 4. Composite quantum systems

Quantum registers

So, we may think inductively and propose an ansatz for mathematical representation of two qubits:

$$|\phi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$$

where $|\phi\rangle$ is the result of 'mixing' $|\psi\rangle_1$ and $|\psi\rangle_2$,
 $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$, and $\sum_i ||\alpha_i||^2 = 1$.



Postulate 4. Composite quantum systems

Quantum registers

In fact, the expression

$$|\phi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$$

is an example of a well established method employed to describe *multipartite* quantum systems: the tensor product.



Postulate 4. Composite quantum systems

We now focus on the mathematical description of a composite quantum system, i.e. **a system made up of several different physical systems**.

The state space of a composite quantum system is the tensor product of the component system state spaces.

If we have n quantum systems expressed as *state vectors*, labeled $|\psi\rangle_1, |\psi\rangle_2, \dots, |\psi\rangle_n$ then the joint state of the total system is given by $|\psi\rangle_T = |\psi\rangle_1 \otimes |\psi\rangle_2 \otimes \dots \otimes |\psi\rangle_n$.

Please note this crucial property: $\bigotimes_{i=1}^n \mathcal{H}_i^2 = \mathcal{H}^{2^n}$.



Postulate 4. Composite quantum systems

Let $\hat{A} : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ and $\hat{B} : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ be linear operators. Then, the following equalities hold $\forall |a\rangle_1, |a\rangle_2 \in \mathcal{H}_1, |b\rangle_1, |b\rangle_2 \in \mathcal{H}_2, \alpha \in \mathbb{C}$:

$$1) \alpha(|a\rangle_1 \otimes |b\rangle_1) = (\alpha|a\rangle_1) \otimes |b\rangle_1 = |a\rangle_1 \otimes (\alpha|b\rangle_1)$$

$$2) (|a\rangle_1 + |a\rangle_2) \otimes |b\rangle_1 = |a\rangle_1 \otimes |b\rangle_1 + |a\rangle_2 \otimes |b\rangle_1$$

$$3) |a\rangle_1 \otimes (|b\rangle_1 + |b\rangle_2) = |a\rangle_1 \otimes |b\rangle_1 + |a\rangle_1 \otimes |b\rangle_2$$

$$4) \hat{A} \otimes \hat{B}(|a\rangle_1 \otimes |b\rangle_1) = \hat{A}|a\rangle_1 \otimes \hat{B}|b\rangle_1$$

$$5) \text{ Let } |a\rangle_i \in \mathcal{H}_1, |b\rangle_i \in \mathcal{H}_2 \text{ and } \alpha_i \in \mathbb{C} \Rightarrow \hat{A} \otimes \hat{B}(\sum_i \alpha_i |a\rangle_i \otimes |b\rangle_i) = \sum_i \alpha_i \hat{A}|a\rangle_i \otimes \hat{B}|b\rangle_i$$



Postulate 4. Composite quantum systems

- We may write $|a\rangle \otimes |b\rangle$ as $|ab\rangle$ or $|a, b\rangle$.
- Moreover, the tensor product of $|a\rangle$ with itself n times may be written as $|a\rangle \otimes |a\rangle \otimes \dots \otimes |a\rangle = |a\rangle^{\otimes n}$.



Postulate 4. Composite quantum systems

Example 1. Let $|\psi\rangle_1 \in \mathcal{H}_1$, $|\psi\rangle_2 \in \mathcal{H}_2$ where $|\psi\rangle_1 = a|0\rangle + b|1\rangle$ and $|\psi\rangle_2 = c|0\rangle + d|1\rangle$. Then

$$\begin{aligned}
 |\psi\rangle_1 \otimes |\psi\rangle_2 &= (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \\
 &= (a|0\rangle + b|1\rangle) \otimes c|0\rangle + (a|0\rangle + b|1\rangle) \otimes d|1\rangle \\
 &= ac|0\rangle|0\rangle + bc|1\rangle|0\rangle + ad|0\rangle|1\rangle + bd|1\rangle|1\rangle \\
 &= ac|00\rangle + bc|10\rangle + ad|01\rangle + bd|11\rangle \\
 &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \\
 &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle
 \end{aligned}$$



Postulate 4. Composite quantum systems

Example 2. Let $|\psi\rangle_1 \in \mathcal{H}_1$, $|\psi\rangle_2 \in \mathcal{H}_2$, $|\psi\rangle_3 \in \mathcal{H}_3$ where

$$|\psi\rangle_1 = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$|\psi\rangle_2 = \frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|1\rangle$$

$$|\psi\rangle_3 = \cos\frac{3\pi}{4}|0\rangle + i\sin\frac{3\pi}{4}|1\rangle$$

Compute

$$|\psi\rangle_1 \otimes |\psi\rangle_2$$

$$|\psi\rangle_1 \otimes |\psi\rangle_3$$

$$|\psi\rangle_1 \otimes |\psi\rangle_2 \otimes |\psi\rangle_3$$



Postulate 4. Composite quantum systems

Example 3.

We know that

$$\hat{A} \otimes \hat{B}(|a\rangle_1 \otimes |b\rangle_1) = \hat{A}|a\rangle_1 \otimes \hat{B}|b\rangle_1$$

Since

$$\hat{H}^{\otimes 2}|0\rangle^{\otimes 2} = \hat{H}^{\otimes 2}|00\rangle = \hat{H} \otimes \hat{H}(|0\rangle \otimes |0\rangle)$$

verify that

$$\hat{H} \otimes \hat{H}(|0\rangle \otimes |0\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$



Postulate 4. Composite quantum systems

Example 4. We know that $\hat{A} \otimes \hat{B}(|a\rangle_1 \otimes |b\rangle_1) = \hat{A}|a\rangle_1 \otimes \hat{B}|b\rangle_1$.
Let

$$\begin{aligned}\hat{\sigma}_x &= |0\rangle\langle 1| + |1\rangle\langle 0| \\ \hat{\sigma}_y &= -i|0\rangle\langle 1| + i|1\rangle\langle 0| \\ \hat{\sigma}_z &= |0\rangle\langle 0| - |1\rangle\langle 1| \\ \hat{H} &= \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \\ |\psi\rangle_1 &= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \\ |\psi\rangle_2 &= \frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|1\rangle \\ |\psi\rangle_3 &= \cos\frac{3\pi}{4}|0\rangle + i\sin\frac{3\pi}{4}|1\rangle\end{aligned}$$

Compute as many combinations as you may wish, for instance

$$\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{H}(|\psi\rangle_1 \otimes |\psi\rangle_2 \otimes |\psi\rangle_3)$$



Postulate 4. Composite quantum systems

Example 5. Let $\hat{H}^{\otimes 2}$ be the tensor product of the Hadamard operator with itself. Prove that

$$\begin{aligned}\hat{H}^{\otimes 2} = & \frac{1}{2}(|00\rangle\langle 00| + |01\rangle\langle 00| + |10\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 01| \\ & - |01\rangle\langle 01| + |10\rangle\langle 01| - |11\rangle\langle 01| + |00\rangle\langle 10| + |01\rangle\langle 10| \\ & - |10\rangle\langle 10| - |11\rangle\langle 10| + |00\rangle\langle 11| - |01\rangle\langle 11| - |10\rangle\langle 11| \\ & + |11\rangle\langle 11|)\end{aligned}$$

Key mathematical rule:

$$|a\rangle\langle b| \otimes |c\rangle\langle d| = |ac\rangle\langle bd|$$

For instance,

$$|0\rangle\langle 1| \otimes |0\rangle\langle 0| = |0\rangle|0\rangle\langle 1| \langle 0| = |00\rangle\langle 10|$$



Postulate 4. Composite quantum systems

Example 6. Let $\hat{H}^{\otimes 2}$ be the tensor product of the Hadamard operator with itself and let $|\psi\rangle = |0\rangle \otimes |0\rangle = |00\rangle$. Prove that

$$\begin{aligned}
 \hat{H}^{\otimes 2} |00\rangle &= \frac{1}{2}(|00\rangle\langle 00| + |01\rangle\langle 00| + |10\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 01| \\
 &\quad - |01\rangle\langle 01| + |10\rangle\langle 01| - |11\rangle\langle 01| + |00\rangle\langle 10| + |01\rangle\langle 10| \\
 &\quad - |10\rangle\langle 10| - |11\rangle\langle 10| + |00\rangle\langle 11| - |01\rangle\langle 11| - |10\rangle\langle 11| \\
 &\quad + |11\rangle\langle 11|)|00\rangle \\
 &= \frac{1}{2}(\langle 00|00\rangle|00\rangle + \langle 00|00\rangle|01\rangle + \langle 00|00\rangle|10\rangle + \langle 00|00\rangle|11\rangle) \\
 &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)
 \end{aligned}$$

That is, the notion of inner product on orthonormal bases easily extends to \mathcal{H}^{2^n} .



Postulate 4. Composite quantum systems

Example 7. Introducing \hat{C}_{not} .

Let us introduce \hat{C}_{not} , a two-qubit gate that behaves as follows:

$$\hat{C}_{\text{not}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

Then

$$\hat{C}_{\text{not}}|00\rangle = |00\rangle$$

$$\hat{C}_{\text{not}}|01\rangle = |01\rangle$$

$$\hat{C}_{\text{not}}|10\rangle = |11\rangle$$

$$\hat{C}_{\text{not}}|11\rangle = |10\rangle$$

The operator \hat{C}_{not} acts as a **Controlled Not**: \hat{C}_{not} flips the second qubit iff the first qubit is $|1\rangle$, otherwise \hat{C}_{not} behaves like a buffer.



Postulate 4. Composite quantum systems

The Kronecker product is a matrix representation of the tensor product and it is defined as follows.

Let $A = (a_{ij})$, $B = (b_{ij})$ be two matrices of order $m \times n$ and $p \times q$ respectively. Then $A \otimes B$ is given by

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{pmatrix}.$$

$A \otimes B$ is of order $mp \times nq$.



Postulate 4. Composite quantum systems

Example 8. Let

$$\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

be the Hadamard operator.

Write

$$\hat{H} \otimes \hat{H}$$

in matrix notation.



Postulate 4. Composite quantum systems

Example 8.

$$\begin{aligned}
 H \otimes H &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} 1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & 1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ 1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}
 \end{aligned}$$



Table of Contents

- 1 Postulates of Quantum Mechanics
 - State space and definition of qubit
 - Quantum Evolution
 - Quantum Measurement
 - Composite Quantum Systems
- 2 Our First Quantum Circuit!

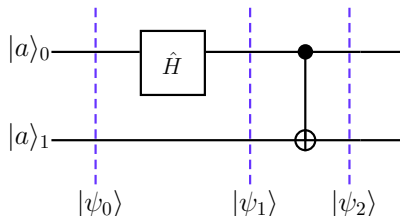


Introduction to Quantum Circuits

We now introduce a quantum circuit to compute **Bell states**.



Bell State Circuit (1/5)



Let

$$|a\rangle_0 = |0\rangle \text{ and } |a\rangle_1 = |0\rangle$$

Also, remember that

$$\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

$$\hat{C}_{\text{not}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$



Bell State Circuit (2/5)

So,

$$|\psi\rangle_0 = |0\rangle \otimes |0\rangle = |00\rangle$$

$$|\psi\rangle_1 = (\hat{H} \otimes \hat{I})(|0\rangle \otimes |0\rangle) = \hat{H}|0\rangle \otimes \hat{I}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|0\rangle$$

$$|\psi\rangle_2 = \hat{C}_{\text{not}}(\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|0\rangle)$$

$$= \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

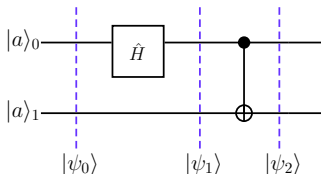
So,

$$|\psi\rangle_2 = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



Bell State Circuit (3/5)

Exercise



Compute $|\psi\rangle_2$ for

$$|a\rangle_0 = |0\rangle \text{ and } |a\rangle_1 = |1\rangle$$

$$|a\rangle_0 = |1\rangle \text{ and } |a\rangle_1 = |0\rangle$$

$$|a\rangle_0 = |1\rangle \text{ and } |a\rangle_1 = |1\rangle$$



Bell State Circuit (4/5)

Answers

$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{I})|00\rangle) = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{I})|01\rangle) = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{I})|10\rangle) = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{I})|11\rangle) = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

These states are known as the **Bell states**

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$



Bell State Circuit (5/5)

An interesting property: try writing the Bell state $|\Phi^+\rangle$ as the tensor product of two arbitrary qubits, i.e. find the values of α , β , γ , and δ such that

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

Try the same procedure with the other three Bell states

$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$



Quantum Entanglement (1/2)

Bell states are examples of entangled states. Bell states are key features of a quantum information transmission protocol known as quantum teleportation.

Quantum entanglement is a unique type of correlation shared between components of a quantum system.

Quantum entanglement and the principle of superposition are two of the main features behind the power of quantum computation and quantum information theory.



Quantum Entanglement (2/2)

Entangled quantum systems are sometimes best used collectively, that is, sometimes an optimal use of entangled quantum systems for information storage and retrieval includes manipulating and measuring those systems as a whole, rather than on an individual basis.

