Universidad Nacional Autónoma de México

FACULTAD DE CIENCIAS

TAREA 01

Computación Cuántica I

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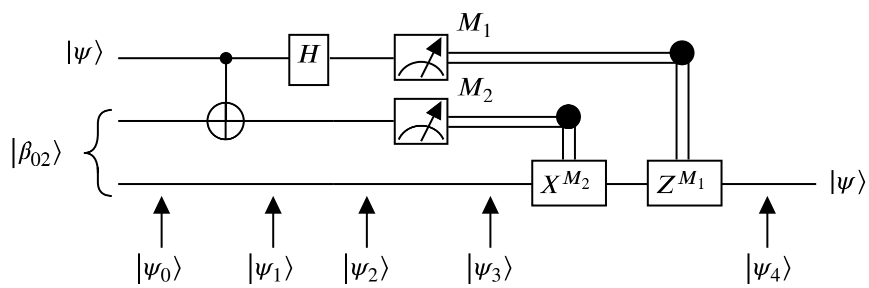
1. Basándose en las lecturas de clase de nuestro curso, calcule el protocolo de teletransportación cuántica usando el estado de Bell:

$$|\beta_{02}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

Y el estado $|\psi\rangle$ a teletransportar definido como:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$



- Proporciona una derivación matemática completa de los siguientes estados cuánticos:

$$|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \text{ y } |\psi_4\rangle$$

- Además, explique completamente la estrategia a seguir para transformar su qubit al estado:

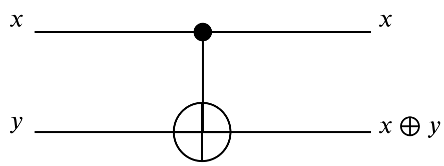
$$|\psi\rangle$$

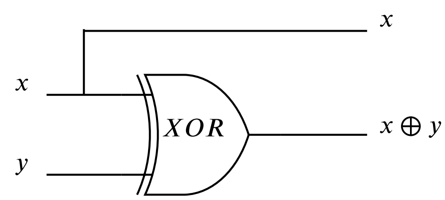
Tomando en cuenta todo lo anterior, procedemos a calcular el estado $|\psi_0\rangle$.

- $|\psi_0\rangle$

$$\begin{aligned}
|\psi_0\rangle &= |\psi\rangle \otimes |\beta_{02}\rangle \\
&= (\alpha |0\rangle + \beta |1\rangle) \otimes \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}(\alpha |0\rangle (|01\rangle + |10\rangle) + \beta |1\rangle (|01\rangle + |10\rangle)) \\
&= \frac{1}{\sqrt{2}}(\alpha |0\rangle |01\rangle + \alpha |0\rangle |10\rangle + \beta |1\rangle |01\rangle + \beta |1\rangle |10\rangle) \\
&= \frac{1}{\sqrt{2}}(\alpha |00\rangle |1\rangle + \alpha |01\rangle |0\rangle + \beta |10\rangle |1\rangle + \beta |11\rangle |0\rangle)
\end{aligned}$$

$$\therefore |\psi_0\rangle = \frac{1}{\sqrt{2}}(\alpha |00\rangle |1\rangle + \alpha |01\rangle |0\rangle + \beta |10\rangle |1\rangle + \beta |11\rangle |0\rangle)$$





Input		*Output*	
x	y	x	$x \oplus y$
$\|0>$	$\|0>$	$\|0>$	$\|0>$
$\|0>$	$\|1>$	$\|0>$	$\|1>$
$\|1>$	$\|0>$	$\|1>$	$\|1>$
$\|1>$	$\|1>$	$\|1>$	$\|0>$

Input		*Output*	
x	y	x	$x \oplus y$
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

✪ $|\psi_1\rangle$

$$
\begin{aligned}
|\psi_1\rangle &= \hat{C}_{NOT} \otimes \hat{\mathbf{I}} \, |\psi_0\rangle \\
&= (\hat{C}_{NOT} \otimes \hat{\mathbf{I}}) \frac{1}{\sqrt{2}} (\alpha \, |00\rangle \, |1\rangle + \alpha \, |01\rangle \, |0\rangle + \beta \, |10\rangle \, |1\rangle + \beta \, |11\rangle \, |0\rangle) \\
&= \frac{1}{\sqrt{2}} [\hat{C}_{NOT} \otimes \hat{\mathbf{I}} (\alpha \, |00\rangle \, |1\rangle) + \hat{C}_{NOT} \otimes \hat{\mathbf{I}} (\alpha \, |01\rangle \, |0\rangle) + \hat{C}_{NOT} \otimes \hat{\mathbf{I}} (\beta \, |10\rangle \, |1\rangle) + \hat{C}_{NOT} \otimes \hat{\mathbf{I}} (\beta \, |11\rangle \, |0\rangle)] \\
&= \frac{1}{\sqrt{2}} [\alpha (\hat{C}_{NOT} \, |00\rangle \otimes \hat{\mathbf{I}} \, |1\rangle) + \alpha (\hat{C}_{NOT} \, |01\rangle \otimes \hat{\mathbf{I}} \, |0\rangle) + \beta (\hat{C}_{NOT} \, |10\rangle \otimes \hat{\mathbf{I}} \, |1\rangle) + \beta (\hat{C}_{NOT} \, |11\rangle \otimes \hat{\mathbf{I}} \, |0\rangle)] \\
&= \frac{1}{\sqrt{2}} [\alpha \, |00\rangle \, |1\rangle + \alpha \, |01\rangle \, |0\rangle + \beta \, |11\rangle \, |1\rangle + \beta \, |10\rangle \, |0\rangle] \\
&= \frac{1}{\sqrt{2}} [\alpha \, |0\rangle \, |01\rangle + \alpha \, |0\rangle \, |10\rangle + \beta \, |1\rangle \, |11\rangle + \beta \, |1\rangle \, |00\rangle]
\end{aligned}
$$

$$
\therefore |\psi_1\rangle = \frac{1}{\sqrt{2}} (\alpha \, |0\rangle \, |01\rangle + \alpha \, |0\rangle \, |10\rangle + \beta \, |1\rangle \, |11\rangle + \beta \, |1\rangle \, |00\rangle)
$$

- $\left|\psi_2\right\rangle$

$$
\begin{aligned}
\left|\psi_2\right\rangle &= (\hat{H}\otimes\hat{\mathbf{I}}\otimes\hat{\mathbf{I}})\left|\psi_1\right\rangle \\
&= (\hat{H}\otimes\hat{\mathbf{I}}\otimes\hat{\mathbf{I}})\frac{1}{\sqrt{2}}(\alpha\left|0\right\rangle\left|01\right\rangle+\alpha\left|0\right\rangle\left|10\right\rangle+\beta\left|1\right\rangle\left|11\right\rangle+\beta\left|1\right\rangle\left|00\right\rangle) \\
&= (\hat{H}\otimes\hat{\mathbf{I}}\otimes\hat{\mathbf{I}})\frac{1}{\sqrt{2}}(\alpha\left|0\right\rangle\left|0\right\rangle\left|1\right\rangle+\alpha\left|0\right\rangle\left|1\right\rangle\left|0\right\rangle+\beta\left|1\right\rangle\left|1\right\rangle\left|1\right\rangle+\beta\left|1\right\rangle\left|0\right\rangle\left|0\right\rangle) \\
&= \frac{1}{\sqrt{2}}[\alpha\hat{H}\left|0\right\rangle\otimes\hat{\mathbf{I}}\left|0\right\rangle\otimes\hat{\mathbf{I}}\left|1\right\rangle+\alpha\hat{H}\left|0\right\rangle\otimes\hat{\mathbf{I}}\left|1\right\rangle\otimes\hat{\mathbf{I}}\left|0\right\rangle+\beta\hat{H}\left|1\right\rangle\otimes\hat{\mathbf{I}}\left|1\right\rangle\otimes\hat{\mathbf{I}}\left|1\right\rangle+\beta\hat{H}\left|1\right\rangle\otimes\hat{\mathbf{I}}\left|0\right\rangle\otimes\hat{\mathbf{I}}\left|0\right\rangle] \\
&= \frac{1}{\sqrt{2}}\left[\alpha\hat{H}\left(\frac{\left|0\right\rangle+\left|1\right\rangle}{\sqrt{2}}\right)\left|01\right\rangle+\alpha\hat{H}\left(\frac{\left|0\right\rangle+\left|1\right\rangle}{\sqrt{2}}\right)\left|10\right\rangle+\beta\hat{H}\left(\frac{\left|0\right\rangle-\left|1\right\rangle}{\sqrt{2}}\right)\left|11\right\rangle+\beta\hat{H}\left(\frac{\left|0\right\rangle-\left|1\right\rangle}{\sqrt{2}}\right)\left|00\right\rangle\right] \\
&= \frac{1}{\sqrt{2}^2}[\alpha\left|001\right\rangle+\alpha\left|101\right\rangle+\alpha\left|010\right\rangle+\alpha\left|110\right\rangle+\beta\left|011\right\rangle-\beta\left|111\right\rangle+\beta\left|000\right\rangle-\beta\left|100\right\rangle] \\
&= \frac{1}{2}[\alpha\left|001\right\rangle+\beta\left|000\right\rangle+\alpha\left|101\right\rangle-\beta\left|100\right\rangle+\alpha\left|010\right\rangle+\beta\left|011\right\rangle+\alpha\left|110\right\rangle-\beta\left|111\right\rangle] \\
&= \frac{1}{2}[\left|00\right\rangle(\alpha\left|1\right\rangle+\beta\left|0\right\rangle)+\left|10\right\rangle(\alpha\left|1\right\rangle-\beta\left|0\right\rangle)+\left|01\right\rangle(\alpha\left|0\right\rangle+\beta\left|1\right\rangle)+\left|11\right\rangle(\alpha\left|0\right\rangle-\beta\left|1\right\rangle)]
\end{aligned}
$$

$$
\therefore \left|\psi_2\right\rangle = \frac{1}{2}(\left|00\right\rangle(\alpha\left|1\right\rangle+\beta\left|0\right\rangle)+\left|10\right\rangle(\alpha\left|1\right\rangle-\beta\left|0\right\rangle)+\left|01\right\rangle(\alpha\left|0\right\rangle+\beta\left|1\right\rangle)+\left|11\right\rangle(\alpha\left|0\right\rangle-\beta\left|1\right\rangle))
$$

- $\left|\psi_3\right\rangle$

Notemos como es que $\left|\psi_3\right\rangle$ es el estado cuántico producido después de medir los qubits de el remitente.
En este paso, mediremos los dos qubits de el remitente, $\left|\psi\right\rangle$ y su qubit EPR, simultáneamente.
Recordando los operadores de medida de un qubit:

$$
\hat{P}_{a0}^{\left|\psi\right\rangle} = \left|0\right\rangle\left\langle 0\right| \qquad \hat{P}_{a1}^{\left|\psi\right\rangle} = \left|1\right\rangle\left\langle 1\right| \qquad \hat{P}_{b0}^{\left|\beta_{00}\right\rangle} = \left|0\right\rangle\left\langle 0\right| \qquad \hat{P}_{b1}^{\left|\beta_{00}\right\rangle} = \left|1\right\rangle\left\langle 1\right|
$$

Y también recordando que usamos las etiquetas $\{a_0, a_1\}$ y $\{b_0, b_1\}$ para referirnos a los posibles resultados de medida para $\left|\psi\right\rangle$ y el qubit EPR de el remitente, respectivamente, tendremos que sólo cuatro resultados son posibles; $\{a_0, b_0\}$, $\{a_0, b_1\}$, $\{a_1, b_0\}$ o $\{a_1, b_1\}$.
Es por esto que tenemos los siguientes operadores de medida de dos qubits:

$$
\begin{aligned}
\hat{P}_{\{a_0,b_0\}} &= \hat{P}_{\{a_0\}}^{\left|\psi\right\rangle}\otimes\hat{P}_{\{b_0\}}^{\left|\beta_{00}\right\rangle} = \left|0_{a_0}\right\rangle\left\langle 0_{a_0}\right|\otimes\left|0_{b_0}\right\rangle\left\langle 0_{b_0}\right| = \left|0_{a_0}0_{b_0}\right\rangle\left\langle 0_{a_0}0_{b_0}\right| = \left|00\right\rangle\left\langle 00\right| \\
\hat{P}_{\{a_0,b_1\}} &= \hat{P}_{\{a_0\}}^{\left|\psi\right\rangle}\otimes\hat{P}_{\{b_1\}}^{\left|\beta_{00}\right\rangle} = \left|0_{a_0}\right\rangle\left\langle 0_{a_0}\right|\otimes\left|1_{b_1}\right\rangle\left\langle 1_{b_1}\right| = \left|0_{a_0}1_{b_1}\right\rangle\left\langle 0_{a_0}1_{b_1}\right| = \left|01\right\rangle\left\langle 01\right| \\
\hat{P}_{\{a_1,b_0\}} &= \hat{P}_{\{a_1\}}^{\left|\psi\right\rangle}\otimes\hat{P}_{\{b_0\}}^{\left|\beta_{00}\right\rangle} = \left|1_{a_1}\right\rangle\left\langle 1_{a_1}\right|\otimes\left|0_{b_0}\right\rangle\left\langle 0_{b_0}\right| = \left|1_{a_1}0_{b_0}\right\rangle\left\langle 1_{a_1}0_{b_0}\right| = \left|10\right\rangle\left\langle 10\right| \\
\hat{P}_{\{a_1,b_1\}} &= \hat{P}_{\{a_1\}}^{\left|\psi\right\rangle}\otimes\hat{P}_{\{b_1\}}^{\left|\beta_{00}\right\rangle} = \left|1_{a_1}\right\rangle\left\langle 1_{a_1}\right|\otimes\left|1_{b_1}\right\rangle\left\langle 1_{b_1}\right| = \left|1_{a_1}1_{b_1}\right\rangle\left\langle 1_{a_1}1_{b_1}\right| = \left|11\right\rangle\left\langle 11\right|
\end{aligned}
$$

Teniendo en mente lo anterior, vamos a calcular la distribución de probabilidad y los estados posteriores a la medición para los resultados:
$\{a_0, b_0\}$, $\{a_0, b_1\}$, $\{a_1, b_0\}$, $\{a_1, b_1\}$.

I) $p(a_0, b_0) = \langle \psi_2 | \hat{P}_{\{a_0,b_0\}} |\psi_2\rangle$

$$\begin{aligned}
\hat{P}_{\{a_0,b_0\}} |\psi_2\rangle &= |00\rangle\langle 00| \left[\frac{1}{2}(|00\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|1\rangle - \beta|0\rangle) + |01\rangle(\alpha|0\rangle + \beta|1\rangle) + |11\rangle(\alpha|0\rangle - \beta|1\rangle))\right] \\
&= \frac{1}{2}\Big[(\langle 00|00\rangle|00\rangle(\alpha|1\rangle + \beta|0\rangle) + \langle 00|10\rangle|00\rangle(\alpha|1\rangle - \beta|0\rangle) \\
&+ \langle 00|01\rangle|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \langle 00|11\rangle|00\rangle(\alpha|0\rangle - \beta|1\rangle))\Big] \\
&= \frac{1}{2}|00\rangle(\alpha|1\rangle + \beta|0\rangle)
\end{aligned}$$

Por ende:

$$\hat{P}_{\{a_0,b_0\}} |\psi_2\rangle = \frac{1}{2}|00\rangle(\alpha|1\rangle + \beta|0\rangle)$$

Ahora:

$$\begin{aligned}
\langle \psi_2 | \hat{P}_{\{a_0,b_0\}} |\psi_2\rangle &= \frac{1}{2}[\langle 00|(\alpha^*\langle 1| + \beta^*\langle 0|) + \langle 10|(\alpha^*\langle 1| - \beta^*\langle 0|) + \langle 01|(\alpha^*\langle 0| + \beta^*\langle 1|) + \langle 11|(\alpha^*\langle 0| - \beta^*\langle 1|)] \\
&[\frac{1}{2}|00\rangle(\alpha|1\rangle + \beta|0\rangle)] \\
&= \frac{1}{4}\Big[\langle 00|00\rangle(\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|1\rangle + \beta|0\rangle) + \langle 10|00\rangle(\alpha^*\langle 1| - \beta^*\langle 0|)(\alpha|1\rangle + \beta|0\rangle) \\
&+ \langle 01|00\rangle(\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|1\rangle + \beta|0\rangle) + \langle 11|00\rangle(\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|1\rangle + \beta|0\rangle)\Big] \\
&= \frac{1}{4}\Big[(\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|1\rangle + \beta|0\rangle)\Big] \\
&= \frac{1}{4}\Big[\alpha^*\alpha\langle 1|1\rangle + \alpha^*\beta\langle 1|0\rangle + \beta^*\alpha\langle 0|1\rangle + \beta^*\beta\langle 0|0\rangle\Big] \\
&= \frac{1}{4}\Big[||\alpha||^2 + ||\beta||^2\Big] \\
&= \frac{1}{4}
\end{aligned}$$

$$p(a_0, b_0) = \langle \psi_2 | \hat{P}_{\{a_0,b_0\}} |\psi_2\rangle = \frac{1}{4}$$

El correspondiente estado post-medición $|\psi\rangle^{pm}_{\{a_0,b_0\}}$ esta dado por:

$$|\psi\rangle^{pm}_{\{a_0,b_0\}} = \frac{\hat{P}_{\{a_0,b_0\}} |\psi_2\rangle}{\sqrt{\langle \psi_2 | \hat{P}_{\{a_0,b_0\}} |\psi_2\rangle}} = \frac{\frac{1}{2}|00\rangle(\alpha|1\rangle + \beta|0\rangle)}{\sqrt{1/4}} = |00\rangle(\alpha|1\rangle + \beta|0\rangle)$$

Por lo tanto:

$$p(a_0, b_0) = \langle \psi_2 | \hat{P}_{\{a_0,b_0\}} |\psi_2\rangle = \frac{1}{4}$$

$$|\psi\rangle^{pm}_{\{a_0,b_0\}} = |00\rangle(\alpha|1\rangle + \beta|0\rangle)$$

II) $p(a_0, b_1) = \langle\psi_2|\,\hat{P}_{\{a_0,b_1\}}\,|\psi_2\rangle$

$$\begin{aligned}
\hat{P}_{\{a_0,b_1\}}\,|\psi_2\rangle &= |01\rangle\langle 01|\left[\frac{1}{2}(|00\rangle(\alpha|1\rangle+\beta|0\rangle)+|10\rangle(\alpha|1\rangle-\beta|0\rangle)+|01\rangle(\alpha|0\rangle+\beta|1\rangle)+|11\rangle(\alpha|0\rangle-\beta|1\rangle))\right] \\
&= \frac{1}{2}\Big[(\langle 01|00\rangle|01\rangle(\alpha|1\rangle+\beta|0\rangle)+\langle 01|10\rangle|01\rangle(\alpha|1\rangle-\beta|0\rangle) \\
&+\langle 01|01\rangle|01\rangle(\alpha|0\rangle+\beta|1\rangle)+\langle 01|11\rangle|01\rangle(\alpha|0\rangle-\beta|1\rangle))\Big] \\
&= \frac{1}{2}|01\rangle(\alpha|0\rangle+\beta|1\rangle)
\end{aligned}$$

Por ende:

$$\hat{P}_{\{a_0,b_1\}}\,|\psi_2\rangle = \frac{1}{2}|01\rangle(\alpha|0\rangle+\beta|1\rangle)$$

Ahora:

$$\begin{aligned}
\langle\psi_2|\,\hat{P}_{\{a_0,b_1\}}\,|\psi_2\rangle &= \frac{1}{2}[\langle 00|(\alpha^*\langle 1|+\beta^*\langle 0|)+\langle 10|(\alpha^*\langle 1|-\beta^*\langle 0|)+\langle 01|(\alpha^*\langle 0|+\beta^*\langle 1|)+\langle 11|(\alpha^*\langle 0|-\beta^*\langle 1|)] \\
&[\frac{1}{2}|01\rangle(\alpha|0\rangle+\beta|1\rangle)] \\
&= \frac{1}{4}\Big[\langle 00|01\rangle(\alpha^*\langle 1|+\beta^*\langle 0|)(\alpha|0\rangle+\beta|1\rangle)+\langle 10|01\rangle(\alpha^*\langle 1|-\beta^*\langle 0|)(\alpha|0\rangle+\beta|1\rangle) \\
&+\langle 01|01\rangle(\alpha^*\langle 0|+\beta^*\langle 1|)(\alpha|0\rangle+\beta|1\rangle)+\langle 11|01\rangle(\alpha^*\langle 0|-\beta^*\langle 1|)(\alpha|0\rangle+\beta|1\rangle)\Big] \\
&= \frac{1}{4}\Big[(\alpha^*\langle 0|+\beta^*\langle 1|)(\alpha|0\rangle+\beta|1\rangle)\Big] \\
&= \frac{1}{4}\Big[\alpha^*\alpha\langle 0|0\rangle+\alpha^*\beta\langle 0|1\rangle+\beta^*\alpha\langle 1|0\rangle+\beta^*\beta\langle 1|1\rangle\Big] \\
&= \frac{1}{4}\Big[||\alpha||^2+||\beta||^2\Big] \\
&= \frac{1}{4}
\end{aligned}$$

$$p(a_0, b_1) = \langle\psi_2|\,\hat{P}_{\{a_0,b_1\}}\,|\psi_2\rangle = \frac{1}{4}$$

El correspondiente estado post-medición $|\psi\rangle^{pm}_{\{a_0,b_1\}}$ esta dado por:

$$|\psi\rangle^{pm}_{\{a_0,b_1\}} = \frac{\hat{P}_{\{a_0,b_1\}}\,|\psi_2\rangle}{\sqrt{\langle\psi_2|\,\hat{P}_{\{a_0,b_1\}}\,|\psi_2\rangle}} = \frac{\frac{1}{2}|01\rangle(\alpha|0\rangle+\beta|1\rangle)}{\sqrt{1/4}} = |01\rangle(\alpha|0\rangle+\beta|1\rangle)$$

Por lo tanto:

$$p(a_0, b_1) = \langle\psi_2|\,\hat{P}_{\{a_0,b_1\}}\,|\psi_2\rangle = \frac{1}{4}$$

$$|\psi\rangle^{pm}_{\{a_0,b_1\}} = |01\rangle(\alpha|0\rangle+\beta|1\rangle)$$

III) $p(a_1, b_0) = \langle\psi_2|\, \hat{P}_{\{a_1,b_0\}} \,|\psi_2\rangle$

$$\begin{aligned}
\hat{P}_{\{a_1,b_0\}} |\psi_2\rangle &= |10\rangle\langle 10| \left[\frac{1}{2}(|00\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|1\rangle - \beta|0\rangle) + |01\rangle(\alpha|0\rangle + \beta|1\rangle) + |11\rangle(\alpha|0\rangle - \beta|1\rangle))\right] \\
&= \frac{1}{2}\Big[(\langle 10|00\rangle|10\rangle(\alpha|1\rangle + \beta|0\rangle) + \langle 10|10\rangle|10\rangle(\alpha|1\rangle - \beta|0\rangle) \\
&+ \langle 10|01\rangle|10\rangle(\alpha|0\rangle + \beta|1\rangle) + \langle 10|11\rangle|10\rangle(\alpha|0\rangle - \beta|1\rangle))\Big] \\
&= \frac{1}{2}|10\rangle(\alpha|1\rangle - \beta|0\rangle)
\end{aligned}$$

Por ende:

$$\hat{P}_{\{a_1,b_0\}} |\psi_2\rangle = \frac{1}{2}|10\rangle(\alpha|1\rangle - \beta|0\rangle)$$

Ahora:

$$\begin{aligned}
\langle\psi_2|\, \hat{P}_{\{a_1,b_0\}} |\psi_2\rangle &= \frac{1}{2}[\langle 00|(\alpha^*\langle 1| + \beta^*\langle 0|) + \langle 10|(\alpha^*\langle 1| - \beta^*\langle 0|) + \langle 01|(\alpha^*\langle 0| + \beta^*\langle 1|) + \langle 11|(\alpha^*\langle 0| - \beta^*\langle 1|)] \\
&[\frac{1}{2}|10\rangle(\alpha|1\rangle - \beta|0\rangle)] \\
&= \frac{1}{4}\Big[\langle 00|10\rangle(\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|1\rangle - \beta|0\rangle) + \langle 10|10\rangle(\alpha^*\langle 1| - \beta^*\langle 0|)(\alpha|1\rangle - \beta|0\rangle) \\
&+ \langle 01|10\rangle(\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|1\rangle - \beta|0\rangle) + \langle 11|10\rangle(\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|1\rangle - \beta|0\rangle)\Big] \\
&= \frac{1}{4}\Big[(\alpha^*\langle 1| - \beta^*\langle 0|)(\alpha|1\rangle - \beta|0\rangle)\Big] \\
&= \frac{1}{4}\Big[\alpha^*\alpha\langle 1|1\rangle - \alpha^*\beta\langle 1|0\rangle - \beta^*\alpha\langle 0|1\rangle + \beta^*\beta\langle 0|0\rangle\Big] \\
&= \frac{1}{4}\Big[||\alpha||^2 + ||\beta||^2\Big] \\
&= \frac{1}{4}
\end{aligned}$$

$$p(a_1, b_0) = \langle\psi_2|\, \hat{P}_{\{a_1,b_0\}} |\psi_2\rangle = \frac{1}{4}$$

El correspondiente estado post-medición $|\psi\rangle^{pm}_{\{a_1,b_0\}}$ esta dado por:

$$|\psi\rangle^{pm}_{\{a_1,b_0\}} = \frac{\hat{P}_{\{a_1,b_0\}} |\psi_2\rangle}{\sqrt{\langle\psi_2|\, \hat{P}_{\{a_1,b_0\}} |\psi_2\rangle}} = \frac{\frac{1}{2}|10\rangle(\alpha|1\rangle - \beta|0\rangle)}{\sqrt{1/4}} = |10\rangle(\alpha|1\rangle - \beta|0\rangle)$$

Por lo tanto:

$$p(a_1, b_0) = \langle\psi_2|\, \hat{P}_{\{a_1,b_0\}} |\psi_2\rangle = \frac{1}{4}$$

$$|\psi\rangle^{pm}_{\{a_1,b_0\}} = |10\rangle(\alpha|1\rangle - \beta|0\rangle)$$

IV) $p(a_1, b_1) = \langle \psi_2 | \hat{P}_{\{a_1,b_1\}} | \psi_2 \rangle$

$$\begin{aligned}
\hat{P}_{\{a_1,b_1\}} |\psi_2\rangle &= |11\rangle \langle 11| \left[\frac{1}{2}(|00\rangle (\alpha |1\rangle + \beta |0\rangle) + |10\rangle (\alpha |1\rangle - \beta |0\rangle) + |01\rangle (\alpha |0\rangle + \beta |1\rangle) + |11\rangle (\alpha |0\rangle - \beta |1\rangle))\right] \\
&= \frac{1}{2}\Big[(\langle 11|00\rangle |11\rangle (\alpha |1\rangle + \beta |0\rangle) + \langle 11|10\rangle |11\rangle (\alpha |1\rangle - \beta |0\rangle) \\
&+ \langle 11|01\rangle |11\rangle (\alpha |0\rangle + \beta |1\rangle) + \langle 11|11\rangle |11\rangle (\alpha |0\rangle - \beta |1\rangle))\Big] \\
&= \frac{1}{2} |11\rangle (\alpha |0\rangle - \beta |1\rangle)
\end{aligned}$$

Por ende:

$$\hat{P}_{\{a_1,b_1\}} |\psi_2\rangle = \frac{1}{2} |11\rangle (\alpha |0\rangle - \beta |1\rangle)$$

Ahora:

$$\begin{aligned}
\langle \psi_2 | \hat{P}_{\{a_1,b_1\}} |\psi_2\rangle &= \frac{1}{2}[\langle 00| (\alpha^* \langle 1| + \beta^* \langle 0|) + \langle 10| (\alpha^* \langle 1| - \beta^* \langle 0|) + \langle 01| (\alpha^* \langle 0| + \beta^* \langle 1|) + \langle 11| (\alpha^* \langle 0| - \beta^* \langle 1|)] \\
&[\frac{1}{2} |11\rangle (\alpha |0\rangle - \beta |1\rangle)] \\
&= \frac{1}{4}\Big[\langle 00|11\rangle (\alpha^* \langle 1| + \beta^* \langle 0|)(\alpha |0\rangle - \beta |1\rangle) + \langle 10|11\rangle (\alpha^* \langle 1| - \beta^* \langle 0|)(\alpha |0\rangle - \beta |1\rangle) \\
&+ \langle 01|11\rangle (\alpha^* \langle 0| + \beta^* \langle 1|)(\alpha |0\rangle - \beta |1\rangle) + \langle 11|11\rangle (\alpha^* \langle 0| - \beta^* \langle 1|)(\alpha |0\rangle - \beta |1\rangle)\Big] \\
&= \frac{1}{4}\Big[(\alpha^* \langle 0| - \beta^* \langle 1|)(\alpha |0\rangle - \beta |1\rangle)\Big] \\
&= \frac{1}{4}\Big[\alpha^* \alpha \langle 0|0\rangle - \alpha^* \beta \langle 0|1\rangle - \beta^* \alpha \langle 1|0\rangle + \beta^* \beta \langle 1|1\rangle \Big] \\
&= \frac{1}{4}\Big[||\alpha||^2 + ||\beta||^2\Big] \\
&= \frac{1}{4}
\end{aligned}$$

$$p(a_1, b_1) = \langle \psi_2 | \hat{P}_{\{a_1,b_1\}} |\psi_2\rangle = \frac{1}{4}$$

El correspondiente estado post-medición $|\psi\rangle^{pm}_{\{a_1,b_1\}}$ esta dado por:

$$|\psi\rangle^{pm}_{\{a_1,b_1\}} = \frac{\hat{P}_{\{a_1,b_1\}} |\psi_2\rangle}{\sqrt{\langle \psi_2 | \hat{P}_{\{a_1,b_1\}} |\psi_2\rangle}} = \frac{\frac{1}{2} |11\rangle (\alpha |0\rangle - \beta |1\rangle)}{\sqrt{1/4}} = |11\rangle (\alpha |0\rangle - \beta |1\rangle)$$

Por lo tanto:

$$p(a_1, b_1) = \langle \psi_2 | \hat{P}_{\{a_1,b_1\}} |\psi_2\rangle = \frac{1}{4}$$

$$|\psi\rangle^{pm}_{\{a_1,b_1\}} = |11\rangle (\alpha |0\rangle - \beta |1\rangle)$$

- $|\psi_4\rangle$

Con todo el procedimiento realizado anteriormente tenemos cuatro posibles casos:

Caso I) Salida $\{a_0, b_0\}$

La probabilidad de obtener $\{a_0, b_0\}$ es:

$$p(a_0, b_0) = \langle\psi_2| \hat{P}_{\{a_0,b_0\}} |\psi_2\rangle = \frac{1}{4}$$

Además, el estado cuántico post-medición para el resultado $\{a_0, b_0\}$ es:

$$|\psi\rangle^{pm}_{\{a_0,b_0\}} = |00\rangle (\alpha |1\rangle + \beta |0\rangle)$$

En este caso, el remitente sabe que sus qubits están en el estado $|00\rangle$, además, sabe que el qubit de el destinatario está en el estado $\alpha |1\rangle + \beta |0\rangle$, ahora, recordando que:

$$\begin{aligned}
\hat{\sigma}_x(\alpha |1\rangle + \beta |0\rangle) &= (|0\rangle \langle 1| + |1\rangle \langle 0|)(\alpha |1\rangle + \beta |0\rangle) \\
&= \alpha \langle 1|1\rangle |0\rangle + \beta \langle 1|0\rangle |0\rangle + \alpha \langle 0|1\rangle |1\rangle + \beta \langle 0|0\rangle |1\rangle \\
&= \alpha |0\rangle + \beta |1\rangle
\end{aligned}$$

Es entonces que el remitente llama al destinatario para decirle que

$$|\psi_4\rangle = \hat{\mathbf{I}} \otimes \hat{\mathbf{I}} \otimes \hat{\sigma}_x[|00\rangle (\alpha |1\rangle + \beta |0\rangle)]$$

Caso II) Salida $\{a_0, b_1\}$

La probabilidad de obtener $\{a_0, b_1\}$ es:

$$p(a_0, b_1) = \langle\psi_2| \hat{P}_{\{a_0,b_1\}} |\psi_2\rangle = \frac{1}{4}$$

Además, el estado cuántico post-medición para el resultado $\{a_0, b_1\}$ es:

$$|\psi\rangle^{pm}_{\{a_0,b_1\}} = |01\rangle (\alpha |0\rangle + \beta |1\rangle)$$

En este caso, el remitente sabe que sus qubits están en el estado $|01\rangle$, además, sabe que el qubit de el destinatario está en el estado $\alpha |0\rangle + \beta |1\rangle$ el cual era el qubit que el remitente esperaba enviar, por lo tanto, llama a el destinatario a través de un canal clásico (una línea telefónica, por ejemplo) para decirle que su qubit está listo, es decir:

$$|\psi_4\rangle = \hat{\mathbf{I}} \otimes \hat{\mathbf{I}} \otimes \hat{\mathbf{I}}[|01\rangle (\alpha |0\rangle + \beta |1\rangle)]$$

Caso III) Salida $\{a_1, b_0\}$
La probabilidad de obtener $\{a_1, b_0\}$ es:

$$p(a_1, b_0) = \langle\psi_2|\, \hat{P}_{\{a_1,b_0\}} \,|\psi_2\rangle = \frac{1}{4}$$

Además, el estado cuántico post-medición para el resultado $\{a_1, b_0\}$ es:

$$|\psi\rangle^{pm}_{\{a_1,b_0\}} = |10\rangle\,(\alpha\,|1\rangle - \beta\,|0\rangle)$$

En este caso, el remitente sabe que sus qubits están en el estado $|10\rangle$, además, sabe que el qubit de el destinatario está en el estado $\alpha\,|1\rangle - \beta\,|0\rangle$, ahora, recordando que:

$$\hat{\sigma}_x(\hat{\sigma}_z(\alpha\,|1\rangle - \beta\,|0\rangle)) = \alpha\,|0\rangle + \beta\,|1\rangle$$

Es entonces que el remitente llama al destinatario para decirle que

$$|\psi_4\rangle = \hat{\mathbf{I}} \otimes \hat{\mathbf{I}} \otimes (\hat{\sigma}_x\hat{\sigma}_z)[|10\rangle\,(\alpha\,|1\rangle - \beta\,|0\rangle)]$$

Caso IV) Salida $\{a_1, b_1\}$
La probabilidad de obtener $\{a_1, b_1\}$ es:

$$p(a_1, b_1) = \langle\psi_2|\, \hat{P}_{\{a_1,b_1\}} \,|\psi_2\rangle = \frac{1}{4}$$

Además, el estado cuántico post-medición para el resultado $\{a_1, b_1\}$ es:

$$|\psi\rangle^{pm}_{\{a_1,b_1\}} = |11\rangle\,(\alpha\,|0\rangle - \beta\,|1\rangle)$$

En este caso, el remitente sabe que sus qubits están en el estado $|11\rangle$, además, sabe que el qubit de el destinatario está en el estado $\alpha\,|0\rangle - \beta\,|1\rangle$, ahora, recordando que:

$$\begin{aligned}\hat{\sigma}_z(\alpha\,|0\rangle - \beta\,|1\rangle) &= (|0\rangle\langle 0| - |1\rangle\langle 1|)(\alpha\,|0\rangle - \beta\,|1\rangle)\\ &= \alpha\,\langle 0|0\rangle\,|0\rangle - \beta\,\langle 0|1\rangle\,|0\rangle - \alpha\,\langle 1|0\rangle\,|1\rangle + \beta\,\langle 1|1\rangle\,|1\rangle\\ &= \alpha\,|0\rangle + \beta\,|1\rangle\end{aligned}$$

Es entonces que el remitente llama al destinatario para decirle que

$$|\psi_4\rangle = \hat{\mathbf{I}} \otimes \hat{\mathbf{I}} \otimes \hat{\sigma}_z[|11\rangle\,(\alpha\,|0\rangle - \beta\,|1\rangle)]$$