Toffoli Implementation

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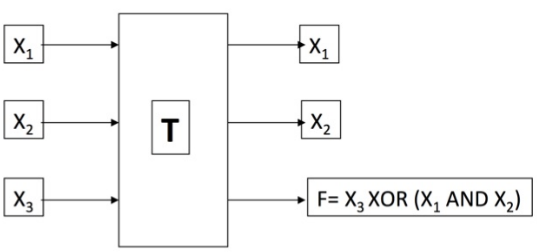
1 Introduction

The goal of this lecture is to implement a quantum circuit that simulates the behaviour of the Toffoli gate. Let us remember the structure and truth table of the Toffoli gate (Fig. (1)).



Input			Output		
X_1	X_2	X_3	X_1	X_2	F
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	**1**	**1**	**1**
1	1	1	**1**	**1**	**0**

Figure 1: The Toffoli Gate and Truth Table.

Remember that:

1. Universal Computation
   $\text{NOT}(X_3) = \text{Toffoli}(1, 1, X_3)$
   $\text{AND}(X_1, X_2) = \text{Toffoli}(X_1, X_2, 0)$
   $\text{NAND}(X_1, X_2) = \text{Toffoli}(X_1, X_2, 1)$

2. Note that the Toffoli gate behaves like a double controlled-not: $110 \rightarrow 111$ and $111 \rightarrow 110$

The quantum circuit $\hat{\mathbf{T}}$ presented in Fig. (2) simulates the Toffoli gate. We shall devote this lecture to understand it and analize it.

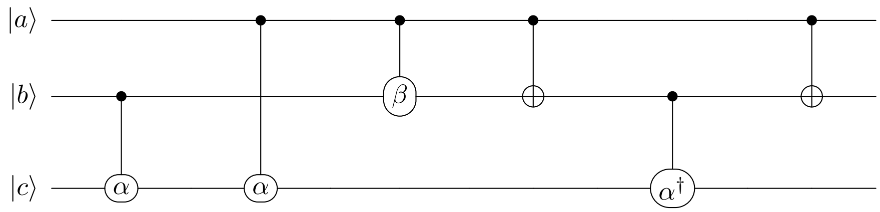


Figure (2): Quantum circuit $\hat{\mathbf{T}}$.

where

$$\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \qquad \alpha^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \qquad \beta^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

In the following sections, we shall compute the controlled gates based on α, $\alpha^{\dagger}$, and β.

2 Rules of engagement

Three important rules:

1. From matrix notation to outer product representation.

$$A = \begin{array}{c} \\ |0\rangle \\ |1\rangle \end{array} \begin{array}{c} \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \Longleftrightarrow \hat{A} = a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$$

2. Tensor product of outer products.

$$|w\rangle\langle x| \otimes |y\rangle\langle z| = |wy\rangle\langle xz|$$

3. General form of a controlled operation.

$$\hat{C}_{\hat{A}} = |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1|\hat{A}$$

3 Quantum Gates I

- α and $\hat{\alpha}$

$$\alpha = \frac{1}{\sqrt{2}} \begin{array}{c c} & \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \end{array} \iff \hat{\alpha} = \frac{1}{\sqrt{2}} \left[1|0\rangle\langle 0| - i|0\rangle\langle 1| - i|1\rangle\langle 0| + 1|1\rangle\langle 1| \right] \implies$$

$$\boxed{\hat{\alpha} = \frac{1}{\sqrt{2}} \left[|0\rangle\langle 0| - i|0\rangle\langle 1| - i|1\rangle\langle 0| + |1\rangle\langle 1| \right]}$$

$$\boxed{\hat{\alpha}|0\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle\langle 0| - i|0\rangle\langle 1| - i|1\rangle\langle 0| + |1\rangle\langle 1| \right] |0\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle - i|1\rangle \right]}$$

$$\boxed{\hat{\alpha}|1\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle\langle 0| - i|0\rangle\langle 1| - i|1\rangle\langle 0| + |1\rangle\langle 1| \right] |1\rangle = \frac{1}{\sqrt{2}} \left[-i|0\rangle + |1\rangle \right]}$$

Note that

$$\alpha \times \alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+i^2 & -i-i \\ -i-i & i^2+1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In other words,

$$\boxed{\alpha \times \alpha = -i\sigma_x}$$

Furthermore, it is straightforward to see that

$$\boxed{\hat{\alpha} \times \hat{\alpha} = -i\hat{\sigma}_x}$$

- $\alpha^\dagger$ and $\hat{\alpha}^\dagger$

$$\alpha^\dagger = \frac{1}{\sqrt{2}} \begin{array}{c c} & \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \end{array} \iff \hat{\alpha}^\dagger = \frac{1}{\sqrt{2}} \left[1|0\rangle\langle 0| + i|0\rangle\langle 1| + i|1\rangle\langle 0| + 1|1\rangle\langle 1| \right] \implies$$

$$\boxed{\hat{\alpha}^\dagger = \frac{1}{\sqrt{2}} \left[|0\rangle\langle 0| + i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1| \right]}$$

$$\boxed{\hat{\alpha}^\dagger|0\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle\langle 0| + i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1| \right] |0\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle + i|1\rangle \right]}$$

$$\boxed{\hat{\alpha}^\dagger|1\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle\langle 0| + i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1| \right] |1\rangle = \frac{1}{\sqrt{2}} \left[i|0\rangle + |1\rangle \right]}$$

- β and $\hat{\beta}$

$$\beta = \begin{array}{c c} & \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \end{array} \iff \hat{\beta} = 1|0\rangle\langle 0| + 0|0\rangle\langle 1| + 0|1\rangle\langle 0| + i|1\rangle\langle 1| \implies$$

$$\boxed{\hat{\beta} = |0\rangle\langle 0| + i|1\rangle\langle 1|}$$

$$\boxed{\hat{\beta}|0\rangle = [|0\rangle\langle 0| + i|1\rangle\langle 1|]\,|0\rangle = |0\rangle}$$

$$\boxed{\hat{\beta}|1\rangle = [|0\rangle\langle 0| + i|1\rangle\langle 1|]\,|1\rangle = i|1\rangle}$$

- $\beta^\dagger$ and $\hat{\beta}^\dagger$

$$\beta^\dagger = \begin{array}{c c} & \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \end{array} \iff \hat{\beta}^\dagger = 1|0\rangle\langle 0| + 0|0\rangle\langle 1| + 0|1\rangle\langle 0| + -i|1\rangle\langle 1| \implies$$

$$\boxed{\hat{\beta}^\dagger = |0\rangle\langle 0| - i|1\rangle\langle 1|}$$

$$\boxed{\hat{\beta}^\dagger|0\rangle = [|0\rangle\langle 0| - i|1\rangle\langle 1|]\,|0\rangle = |0\rangle}$$

$$\boxed{\hat{\beta}^\dagger|1\rangle = [|0\rangle\langle 0| - i|1\rangle\langle 1|]\,|1\rangle = -i|1\rangle}$$

- $\hat{C}_{\hat{\alpha}}$

$$\begin{aligned}\hat{C}_{\hat{\alpha}} &= |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha} \\ &= |0\rangle\langle 0| \otimes [|0\rangle\langle 0| + |1\rangle\langle 1|] + |1\rangle\langle 1| \otimes \tfrac{1}{\sqrt{2}} [|0\rangle\langle 0| - i|0\rangle\langle 1| - i|1\rangle\langle 0| + |1\rangle\langle 1|] \\ &= |00\rangle\langle 00| + |01\rangle\langle 01| + \tfrac{1}{\sqrt{2}} [|10\rangle\langle 10| - i|10\rangle\langle 11| - i|11\rangle\langle 10| + |11\rangle\langle 11|] \implies\end{aligned}$$

$$\hat{C}_{\hat{\alpha}} = |00\rangle\langle 00| + |01\rangle\langle 01| + \tfrac{1}{\sqrt{2}}|10\rangle\langle 10| - \tfrac{i}{\sqrt{2}}|10\rangle\langle 11| - \tfrac{i}{\sqrt{2}}|11\rangle\langle 10| + \tfrac{1}{\sqrt{2}}|11\rangle\langle 11|$$

$$\boxed{\hat{C}_{\hat{\alpha}} = |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}}$$

$$\boxed{\hat{C}_{\hat{\alpha}} = |00\rangle\langle 00| + |01\rangle\langle 01| + \tfrac{1}{\sqrt{2}}|10\rangle\langle 10| - \tfrac{i}{\sqrt{2}}|10\rangle\langle 11| - \tfrac{i}{\sqrt{2}}|11\rangle\langle 10| + \tfrac{1}{\sqrt{2}}|11\rangle\langle 11|}$$

$$\boxed{\hat{C}_{\hat{\alpha}}|00\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |00\rangle = |0\rangle \otimes \hat{\mathbb{I}}|0\rangle = |00\rangle}$$

$$\boxed{\hat{C}_{\hat{\alpha}}|01\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |01\rangle = |0\rangle \otimes \hat{\mathbb{I}}|1\rangle = |01\rangle}$$

$$\boxed{\hat{C}_{\hat{\alpha}}|10\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |10\rangle = |1\rangle \otimes \hat{\alpha}|0\rangle = |1\rangle \otimes \frac{1}{\sqrt{2}} [|0\rangle - i|1\rangle] = \frac{1}{\sqrt{2}} [|10\rangle - i|11\rangle]}$$

$$\boxed{\hat{C}_{\hat{\alpha}}|11\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |11\rangle = |1\rangle \otimes \hat{\alpha}|1\rangle = |1\rangle \otimes \frac{1}{\sqrt{2}} [-i|0\rangle + |1\rangle] = \frac{1}{\sqrt{2}} [-i|10\rangle + |11\rangle]}$$

- $\hat{C}_{\hat{\alpha}^\dagger}$

$$
\begin{aligned}
\hat{C}_{\hat{\alpha}^\dagger} &= |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}^\dagger \\
&= |0\rangle\langle 0| \otimes [|0\rangle\langle 0| + |1\rangle\langle 1|] + |1\rangle\langle 1| \otimes \tfrac{1}{\sqrt{2}} [|0\rangle\langle 0| + i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|] \implies \\
\hat{C}_{\hat{\alpha}^\dagger} &= |00\rangle\langle 00| + |01\rangle\langle 01| + \tfrac{1}{\sqrt{2}}|10\rangle\langle 10| + \tfrac{i}{\sqrt{2}}|10\rangle\langle 11| + \tfrac{i}{\sqrt{2}}|11\rangle\langle 10| + \tfrac{1}{\sqrt{2}}|11\rangle\langle 11|
\end{aligned}
$$

$$\boxed{\hat{C}_{\hat{\alpha}^\dagger} = |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}^\dagger}$$

$$\boxed{\hat{C}_{\hat{\alpha}^\dagger} = |00\rangle\langle 00| + |01\rangle\langle 01| + \tfrac{1}{\sqrt{2}}|10\rangle\langle 10| + \tfrac{i}{\sqrt{2}}|10\rangle\langle 11| + \tfrac{i}{\sqrt{2}}|11\rangle\langle 10| + \tfrac{1}{\sqrt{2}}|11\rangle\langle 11|}$$

$$\boxed{\hat{C}_{\hat{\alpha}^\dagger}|00\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}^\dagger\right]|00\rangle = |0\rangle \otimes \hat{\mathbb{I}}|0\rangle = |00\rangle}$$

$$\boxed{\hat{C}_{\hat{\alpha}^\dagger}|01\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}^\dagger\right]|01\rangle = |0\rangle \otimes \hat{\mathbb{I}}|1\rangle = |01\rangle}$$

$$\boxed{\hat{C}_{\hat{\alpha}^\dagger}|10\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}^\dagger\right]|10\rangle = |1\rangle \otimes \hat{\alpha}^\dagger|0\rangle = |1\rangle \otimes \frac{1}{\sqrt{2}}[|0\rangle + i|1\rangle] = \frac{1}{\sqrt{2}}[|10\rangle + i|11\rangle]}$$

$$\boxed{\hat{C}_{\hat{\alpha}^\dagger}|11\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}^\dagger\right]|11\rangle = |1\rangle \otimes \hat{\alpha}^\dagger|1\rangle = |1\rangle \otimes \frac{1}{\sqrt{2}}[i|0\rangle + |1\rangle] = \frac{1}{\sqrt{2}}[i|10\rangle + |11\rangle]}$$

- $\hat{C}_{\hat{\beta}}$

$$\begin{aligned}\hat{C}_{\hat{\beta}} &= |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta} \\ &= |0\rangle\langle 0| \otimes [|0\rangle\langle 0| + |1\rangle\langle 1|] + |1\rangle\langle 1| \otimes [|0\rangle\langle 0| + i|1\rangle\langle 1|] \implies \\ \hat{C}_{\hat{\beta}} &= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + i|11\rangle\langle 11|\end{aligned}$$

$$\boxed{\hat{C}_{\hat{\beta}} = |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta}}$$

$$\boxed{\hat{C}_{\hat{\beta}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + i|11\rangle\langle 11|}$$

$$\boxed{\hat{C}_{\hat{\beta}}|00\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta}\right]|00\rangle = |0\rangle \otimes \hat{\mathbb{I}}|0\rangle = |00\rangle}$$

$$\boxed{\hat{C}_{\hat{\beta}}|01\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta}\right]|01\rangle = |0\rangle \otimes \hat{\mathbb{I}}|1\rangle = |01\rangle}$$

$$\boxed{\hat{C}_{\hat{\beta}}|10\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta}\right]|10\rangle = |1\rangle \otimes \hat{\beta}|0\rangle = |10\rangle}$$

$$\boxed{\hat{C}_{\hat{\beta}}|11\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta}\right]|11\rangle = |1\rangle \otimes \hat{\beta}|1\rangle = i|11\rangle}$$

- $\hat{C}_{\hat{\beta}^\dagger}$

$$\begin{aligned}\hat{C}_{\hat{\beta}^\dagger} &= |0\rangle\langle 0| \otimes \hat{\mathbb{1}} + |1\rangle\langle 1| \otimes \hat{\beta}^\dagger \\ &= |0\rangle\langle 0| \otimes [|0\rangle\langle 0| + |1\rangle\langle 1|] + |1\rangle\langle 1| \otimes [|0\rangle\langle 0| - i|1\rangle\langle 1|] \implies \\ \hat{C}_{\hat{\beta}} &= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - i|11\rangle\langle 11|\end{aligned}$$

$$\boxed{\hat{C}_{\hat{\beta}^\dagger} = |0\rangle\langle 0| \otimes \hat{\mathbb{1}} + |1\rangle\langle 1| \otimes \hat{\beta}^\dagger}$$

$$\boxed{\hat{C}_{\hat{\beta}^\dagger} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - i|11\rangle\langle 11|}$$

$$\boxed{\hat{C}_{\hat{\beta}^\dagger}|00\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{1}} + |1\rangle\langle 1| \otimes \hat{\beta}^\dagger\right]|00\rangle = |0\rangle \otimes \hat{\mathbb{1}}|0\rangle = |00\rangle}$$

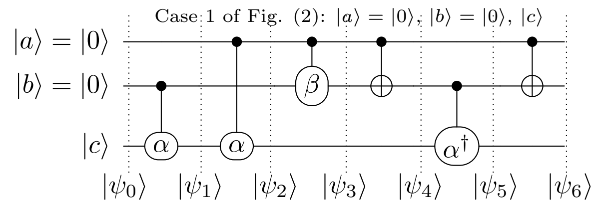
$$\boxed{\hat{C}_{\hat{\beta}^\dagger}|01\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{1}} + |1\rangle\langle 1| \otimes \hat{\beta}^\dagger\right]|01\rangle = |0\rangle \otimes \hat{\mathbb{1}}|1\rangle = |01\rangle}$$

$$\boxed{\hat{C}_{\hat{\beta}^\dagger}|10\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{1}} + |1\rangle\langle 1| \otimes \hat{\beta}^\dagger\right]|10\rangle = |1\rangle \otimes \hat{\beta}^\dagger|0\rangle = |10\rangle}$$

$$\boxed{\hat{C}_{\hat{\beta}^\dagger}|11\rangle = \left[|0\rangle\langle 0| \otimes \hat{\mathbb{1}} + |1\rangle\langle 1| \otimes \hat{\beta}^\dagger\right]|11\rangle = |1\rangle \otimes \hat{\beta}^\dagger|1\rangle = -i|11\rangle}$$

4 Mathematical behaviour of the Toffoli quantum circuit

Case 1. $|a\rangle = |0\rangle$, $|b\rangle = |0\rangle$, $|c\rangle$. Let us now run $\hat{\mathbf{T}}$ (Fig. (2)) for $|a\rangle = |0\rangle$, $|b\rangle = |0\rangle$, $|c\rangle$.



$$
\begin{aligned}
|\psi_0\rangle &= |a\rangle \otimes |b\rangle \otimes |c\rangle \\
&= |0\rangle \otimes |0\rangle \otimes |c\rangle \\
&= |00c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_1\rangle &= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}}\right] |\psi_0\rangle \\
&= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}}\right] |00c\rangle \\
&= \hat{\mathbb{I}}|0\rangle \otimes \hat{C}_{\hat{\alpha}}|0c\rangle \\
&= \hat{\mathbb{I}}|0\rangle \otimes \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |0c\rangle\Big) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \Big((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|0c\rangle + (|1\rangle\langle 1| \otimes \hat{\alpha})|0c\rangle\Big) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \Big(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}|c\rangle + \langle 1|0\rangle|1\rangle \otimes \hat{\alpha}|c\rangle\Big) \\
&= |00c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_2\rangle &= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |\psi_1\rangle \\
&= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |0\rangle_{a'}|0\rangle_{b'}|c\rangle_{c'} \\
&= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |0\rangle_{b'}|0\rangle_{a'}|c\rangle_{c'} \\
&= \hat{\mathbb{I}}_{b'}|0\rangle_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}|0\rangle_{a'}|c\rangle_{c'}
\end{aligned}
$$

Remove subscripts in order not to clutter calculations:

$$
\begin{aligned}
&= \hat{\mathbb{I}}|0\rangle \otimes \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |0c\rangle\Big) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \Big((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|0c\rangle + (|1\rangle\langle 1| \otimes \hat{\alpha})|0c\rangle\Big) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \Big(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}|c\rangle + \langle 1|0\rangle|1\rangle \otimes \hat{\alpha}|c\rangle\Big)
\end{aligned}
$$

Back to subscripts:

$$
\begin{aligned}
&= |0\rangle_{b'} \otimes |0\rangle_{a'}|c\rangle_{c'} \\
&= |0\rangle_{a'}|0\rangle_{b'}|c\rangle_{c'} \\
&= |00c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_3\rangle &= \left[\hat{C}_{\hat{\beta}} \otimes \hat{\mathbb{I}}\right] |\psi_2\rangle \\
&= \left[\hat{C}_{\hat{\beta}} \otimes \hat{\mathbb{I}}\right] |00c\rangle \\
&= \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta}\right] |00\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= \Big((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|00\rangle + (|1\rangle\langle 1| \otimes \hat{\beta})|00\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= \Big(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}|0\rangle + \langle 1|0\rangle|1\rangle \otimes \hat{\beta}|0\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= |00c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_4\rangle &= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |\psi_3\rangle \\
&= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |00c\rangle \\
&= \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\sigma}_x\right] |00\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= \Big((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|00\rangle + (|1\rangle\langle 1| \otimes \hat{\sigma}_x)|00\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= \Big(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}|0\rangle + \langle 1|0\rangle|1\rangle \otimes \hat{\sigma}_x|0\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= |00c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_5\rangle &= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}^\dagger}\right] |\psi_4\rangle \\
&= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}^\dagger}\right] |00c\rangle \\
&= \hat{\mathbb{I}}|0\rangle \otimes \hat{C}_{\hat{\alpha}^\dagger}|0c\rangle \\
&= \hat{\mathbb{I}}|0\rangle \otimes \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}^\dagger\right] |0c\rangle\Big) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \Big((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|0c\rangle + (|1\rangle\langle 1| \otimes \hat{\alpha}^\dagger)|0c\rangle\Big) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \Big(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}|c\rangle + \langle 1|0\rangle|1\rangle \otimes \hat{\alpha}^\dagger|c\rangle\Big) \\
&= |00c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_6\rangle &= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |\psi_5\rangle \\
&= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |00c\rangle \\
&= \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\sigma}_x\right] |00\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= \Big((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|00\rangle + (|1\rangle\langle 1| \otimes \hat{\sigma}_x)|00\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= \Big(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}|0\rangle + \langle 1|0\rangle|1\rangle \otimes \hat{\sigma}_x|0\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= |00c\rangle
\end{aligned}
$$

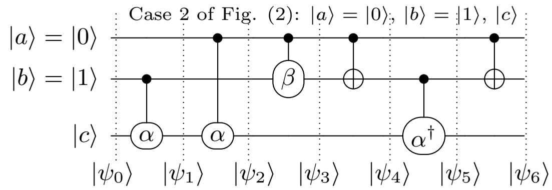
So,

$|\mathbf{00c}\rangle \xrightarrow{\hat{\mathbf{T}}} |\mathbf{00c}\rangle$, i.e.

$|\mathbf{000}\rangle \xrightarrow{\hat{\mathbf{T}}} |\mathbf{000}\rangle$

$|\mathbf{001}\rangle \xrightarrow{\hat{\mathbf{T}}} |\mathbf{001}\rangle$

Case 2. $|a\rangle = |0\rangle,\ |b\rangle = |1\rangle,\ |c\rangle$. Now, let us now run $\hat{\mathbf{T}}$ (Fig. (2)) for $|a\rangle = |0\rangle$, $|b\rangle = |1\rangle$, $|c\rangle$.



$$
\begin{aligned}
|\psi_0\rangle &= |a\rangle \otimes |b\rangle \otimes |c\rangle \\
&= |0\rangle \otimes |1\rangle \otimes |c\rangle \\
&= |01c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_1\rangle &= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}}\right] |\psi_0\rangle \\
&= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}}\right] |01c\rangle \\
&= \hat{\mathbb{I}}|0\rangle \otimes \hat{C}_{\hat{\alpha}}|1c\rangle \\
&= \hat{\mathbb{I}}|0\rangle \otimes \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |1c\rangle\right) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \left((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|1c\rangle + (|1\rangle\langle 1| \otimes \hat{\alpha})|1c\rangle\right) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \left(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}|c\rangle + \langle 1|1\rangle|1\rangle \otimes \hat{\alpha}|c\rangle\right) \\
&= |01\rangle[\hat{\alpha}|c\rangle] \quad \text{(Keep in mind that } \hat{\alpha}|c\rangle \text{ is a qubit.)}
\end{aligned}
$$

$$
\begin{aligned}
|\psi_2\rangle &= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |\psi_1\rangle \\
&= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |0\rangle_{a'}|1\rangle_{b'}[\hat{\alpha}|c\rangle]_{c'} \\
&= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |1\rangle_{b'}|0\rangle_{a'}[\hat{\alpha}|c\rangle]_{c'} \\
&= \hat{\mathbb{I}}_{b'}|1\rangle_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}|0\rangle_{a'}[\hat{\alpha}|c\rangle]_{c'}
\end{aligned}
$$

Remove subscripts in order not to clutter calculations:

$$
\begin{aligned}
&= \hat{\mathbb{I}}|1\rangle \otimes \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |0\rangle[\hat{\alpha}|c\rangle]\right) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \left(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] + \langle 1|0\rangle|1\rangle \otimes \hat{\alpha}[\hat{\alpha}|c\rangle]\right)
\end{aligned}
$$

Back to subscripts:

$$
\begin{aligned}
&= |1\rangle_{b'} \otimes |0\rangle_{a'}[\hat{\alpha}|c\rangle]_{c'} \\
&= |0\rangle_{a'}|1\rangle_{b'}[\hat{\alpha}|c\rangle]_{c'} \\
&= |01\rangle[\hat{\alpha}|c\rangle]
\end{aligned}
$$

$$
\begin{aligned}
|\psi_3\rangle &= \left[\hat{C}_{\hat{\beta}} \otimes \hat{\mathbb{I}}\right] |\psi_2\rangle \\
&= \left[\hat{C}_{\hat{\beta}} \otimes \hat{\mathbb{I}}\right] |01\rangle[\hat{\alpha}|c\rangle] \\
&= \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta}\right] |01\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= \left((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|01\rangle + (|1\rangle\langle 1| \otimes \hat{\beta})|01\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= \left(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}|1\rangle + \langle 1|0\rangle|1\rangle \otimes \hat{\beta}|1\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= |01\rangle[\hat{\alpha}|c\rangle]
\end{aligned}
$$

$$
\begin{aligned}
|\psi_4\rangle &= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |\psi_3\rangle \\
&= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |01\rangle[\hat{\alpha}|c\rangle] \\
&= \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\sigma}_x\right] |01\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= \left((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|01\rangle + (|1\rangle\langle 1| \otimes \hat{\sigma}_x)|01\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= \left(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}|1\rangle + \langle 1|0\rangle|1\rangle \otimes \hat{\sigma}_x|1\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= |01\rangle[\hat{\alpha}|c\rangle]
\end{aligned}
$$

$$
\begin{aligned}
|\psi_5\rangle &= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}^\dagger}\right] |\psi_4\rangle \\
&= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}^\dagger}\right] |01\rangle[\hat{\alpha}|c\rangle] \\
&= \hat{\mathbb{I}}|0\rangle \otimes \hat{C}_{\hat{\alpha}^\dagger}|1\rangle[\hat{\alpha}|c\rangle] \\
&= \hat{\mathbb{I}}|0\rangle \otimes \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}^\dagger\right] |1\rangle[\hat{\alpha}|c\rangle]\right) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \left((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|1\rangle[\hat{\alpha}|c\rangle] + (|1\rangle\langle 1| \otimes \hat{\alpha}^\dagger)|1\rangle[\hat{\alpha}|c\rangle]\right) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \left(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] + \langle 1|1\rangle|1\rangle \otimes \hat{\alpha}^\dagger([\hat{\alpha}|c\rangle])\right) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \left(\langle 1|1\rangle|1\rangle \otimes (\hat{\alpha}^\dagger\hat{\alpha})|c\rangle\right) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \left(\langle 1|1\rangle|1\rangle \otimes \hat{\mathbb{I}}|c\rangle\right) \quad (\hat{\alpha} \text{ is unitary } \Rightarrow \hat{\alpha}^\dagger\hat{\alpha} = \hat{\mathbb{I}}). \\
&= |01c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_6\rangle &= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |\psi_5\rangle \\
&= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |01c\rangle \\
&= \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\sigma}_x\right] |01\rangle\right) \otimes \hat{\mathbb{I}}|c\rangle \\
&= \left((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|01\rangle + (|1\rangle\langle 1| \otimes \hat{\sigma}_x)|01\rangle\right) \otimes \hat{\mathbb{I}}|c\rangle \\
&= \left(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}|1\rangle + \langle 1|0\rangle|1\rangle \otimes \hat{\sigma}_x|1\rangle\right) \otimes \hat{\mathbb{I}}|c\rangle \\
&= |01c\rangle
\end{aligned}
$$

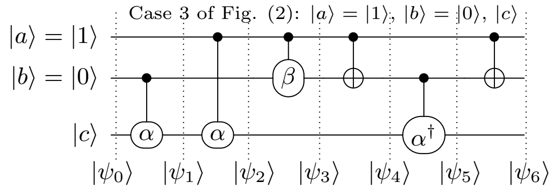
So,

$|\mathbf{01c}\rangle \xrightarrow{\hat{\mathbf{T}}} |\mathbf{01c}\rangle$, i.e.

$|\mathbf{010}\rangle \xrightarrow{\hat{\mathbf{T}}} |\mathbf{010}\rangle$

$|\mathbf{011}\rangle \xrightarrow{\hat{\mathbf{T}}} |\mathbf{011}\rangle$

Case 3. $|a\rangle = |1\rangle$, $|b\rangle = |0\rangle$, $|c\rangle$. Now, let us now compute the outcome of $\hat{\mathbf{T}}$ (Fig. (2)) for $|a\rangle = |1\rangle$, $|b\rangle = |0\rangle$, $|c\rangle$.



$$
\begin{aligned}
|\psi_0\rangle &= |a\rangle \otimes |b\rangle \otimes |c\rangle \\
&= |1\rangle \otimes |0\rangle \otimes |c\rangle \\
&= |10c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_1\rangle &= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}}\right] |\psi_0\rangle \\
&= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}}\right] |10c\rangle \\
&= \hat{\mathbb{I}}|1\rangle \otimes \hat{C}_{\hat{\alpha}}|0c\rangle \\
&= \hat{\mathbb{I}}|1\rangle \otimes \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |0c\rangle\Big) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \Big((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|0c\rangle + (|1\rangle\langle 1| \otimes \hat{\alpha})|0c\rangle\Big) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \Big(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}|c\rangle + \langle 1|0\rangle|1\rangle \otimes \hat{\alpha}|c\rangle\Big) \\
&= |10c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_2\rangle &= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |\psi_1\rangle \\
&= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |1\rangle_{a'}|0\rangle_{b'}|c\rangle_{c'} \\
&= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |0\rangle_{b'}|1\rangle_{a'}|c\rangle_{c'} \\
&= \hat{\mathbb{I}}_{b'}|0\rangle_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}|1\rangle_{a'}|c\rangle_{c'}
\end{aligned}
$$

Remove subscripts in order not to clutter calculations:

$$
\begin{aligned}
&= \hat{\mathbb{I}}|0\rangle \otimes \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |1\rangle|c\rangle\Big) \\
&= \hat{\mathbb{I}}|0\rangle \otimes \Big(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}|c\rangle + \langle 1|1\rangle|1\rangle \otimes \hat{\alpha}|c\rangle\Big)
\end{aligned}
$$

Back to subscripts:

$$
\begin{aligned}
&= |0\rangle_{b'} \otimes |1\rangle_{a'}[\hat{\alpha}|c\rangle]_{c'} \\
&= |1\rangle_{a'}|0\rangle_{b'}[\hat{\alpha}|c\rangle]_{c'} \\
&= |10\rangle[\hat{\alpha}|c\rangle]
\end{aligned}
$$

$$
\begin{aligned}
|\psi_3\rangle &= \left[\hat{C}_{\hat{\beta}} \otimes \hat{\mathbb{I}}\right] |\psi_2\rangle \\
&= \left[\hat{C}_{\hat{\beta}} \otimes \hat{\mathbb{I}}\right] |10\rangle[\hat{\alpha}|c\rangle] \\
&= \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta}\right] |10\rangle\Big) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= \Big((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|10\rangle + (|1\rangle\langle 1| \otimes \hat{\beta})|10\rangle\Big) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= \Big(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}|0\rangle + \langle 1|1\rangle|1\rangle \otimes \hat{\beta}|0\rangle\Big) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= (|1\rangle \otimes |0\rangle) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \text{(Remember: } \hat{\beta}|0\rangle = |0\rangle\text{)}. \\
&= |10\rangle[\hat{\alpha}|c\rangle]
\end{aligned}
$$

$$
\begin{aligned}
|\psi_4\rangle &= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |\psi_3\rangle \\
&= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |10\rangle[\hat{\alpha}|c\rangle] \\
&= \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\sigma}_x\right] |10\rangle\Big) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= \Big((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|10\rangle + (|1\rangle\langle 1| \otimes \hat{\sigma}_x)|10\rangle\Big) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= \Big(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}|0\rangle + \langle 1|1\rangle|1\rangle \otimes \hat{\sigma}_x|0\rangle\Big) \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] \\
&= |11\rangle[\hat{\alpha}|c\rangle]
\end{aligned}
$$

$$
\begin{aligned}
|\psi_5\rangle &= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}^\dagger}\right] |\psi_4\rangle \\
&= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}^\dagger}\right] |11\rangle[\hat{\alpha}|c\rangle] \\
&= \hat{\mathbb{I}}|1\rangle \otimes \hat{C}_{\hat{\alpha}^\dagger}|1\rangle[\hat{\alpha}|c\rangle] \\
&= \hat{\mathbb{I}}|1\rangle \otimes \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}^\dagger\right] |1\rangle[\hat{\alpha}|c\rangle]\Big) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \Big((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|1\rangle[\hat{\alpha}|c\rangle] + (|1\rangle\langle 1| \otimes \hat{\alpha}^\dagger)|1\rangle[\hat{\alpha}|c\rangle]\Big) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \Big(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] + \langle 1|1\rangle|1\rangle \otimes \hat{\alpha}^\dagger([\hat{\alpha}|c\rangle])\Big) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \Big(\langle 1|1\rangle|1\rangle \otimes (\hat{\alpha}^\dagger\hat{\alpha})|c\rangle\Big) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \Big(\langle 1|1\rangle|1\rangle \otimes \hat{\mathbb{I}}|c\rangle\Big) \text{ (}\hat{\alpha}\text{ is unitary } \Rightarrow \hat{\alpha}^\dagger\hat{\alpha} = \hat{\mathbb{I}}\text{)}. \\
&= |11c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_6\rangle &= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |\psi_5\rangle \\
&= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |11c\rangle \\
&= \Big(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\sigma}_x\right] |11\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= \Big((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|11\rangle + (|1\rangle\langle 1| \otimes \hat{\sigma}_x)|11\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= \Big(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}|1\rangle + \langle 1|1\rangle|1\rangle \otimes \hat{\sigma}_x|1\rangle\Big) \otimes \hat{\mathbb{I}}|c\rangle \\
&= |10c\rangle
\end{aligned}
$$

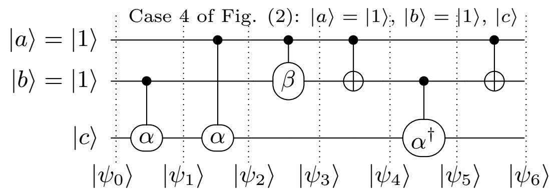
So,

$|\mathbf{10c}\rangle \xrightarrow{\hat{\mathbf{T}}} |\mathbf{10c}\rangle$, i.e.

$|\mathbf{100}\rangle \xrightarrow{\hat{\mathbf{T}}} |\mathbf{100}\rangle$

$|\mathbf{101}\rangle \xrightarrow{\hat{\mathbf{T}}} |\mathbf{101}\rangle$

Case 4. $|a\rangle = |1\rangle,\ |b\rangle = |1\rangle,\ |c\rangle$. Now, let us now compute the outcome of $\hat{\mathbf{T}}$ (Fig. (2)) for the last case, $|a\rangle = |1\rangle,\ |b\rangle = |1\rangle,\ |c\rangle$.



$$
\begin{aligned}
|\psi_0\rangle &= |a\rangle \otimes |b\rangle \otimes |c\rangle \\
&= |1\rangle \otimes |1\rangle \otimes |c\rangle \\
&= |11c\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_1\rangle &= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}}\right] |\psi_0\rangle \\
&= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}}\right] |11c\rangle \\
&= \hat{\mathbb{I}}|1\rangle \otimes \hat{C}_{\hat{\alpha}}|1c\rangle \\
&= \hat{\mathbb{I}}|1\rangle \otimes \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |1c\rangle\right) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \left((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|1c\rangle + (|1\rangle\langle 1| \otimes \hat{\alpha})|1c\rangle\right) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \left(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}|c\rangle + \langle 1|1\rangle|1\rangle \otimes \hat{\alpha}|c\rangle\right) \\
&= |11\rangle[\hat{\alpha}|c\rangle]
\end{aligned}
$$

$$
\begin{aligned}
|\psi_2\rangle &= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |\psi_1\rangle \\
&= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |1\rangle_{a'}|1\rangle_{b'}[\hat{\alpha}|c\rangle]_{c'} \\
&= \left[\hat{\mathbb{I}}_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}\right] |1\rangle_{b'}|1\rangle_{a'}[\hat{\alpha}|c\rangle]_{c'} \\
&= \hat{\mathbb{I}}_{b'}|1\rangle_{b'} \otimes \hat{C}_{\hat{\alpha}\ a',c'}|1\rangle_{a'}[\hat{\alpha}|c\rangle]_{c'}
\end{aligned}
$$

Remove subscripts in order not to clutter calculations:

$$
\begin{aligned}
&= \hat{\mathbb{I}}|1\rangle \otimes \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}\right] |1\rangle[\hat{\alpha}|c\rangle]\right) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \left(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}[\hat{\alpha}|c\rangle] + \langle 1|1\rangle|1\rangle \otimes \hat{\alpha}[\hat{\alpha}|c\rangle]\right) \\
&= \hat{\mathbb{I}}|1\rangle \otimes (|1\rangle \otimes (\hat{\alpha}\hat{\alpha})|c\rangle) \\
&= \hat{\mathbb{I}}|1\rangle \otimes |1\rangle \otimes [-i\hat{\sigma}_x|c\rangle] (\text{Remember: } \hat{\alpha}\hat{\alpha} = -i\hat{\sigma}_x).
\end{aligned}
$$

Back to subscripts:

$$
\begin{aligned}
&= |1\rangle_{b'} \otimes |1\rangle_{a'}[-i\hat{\sigma}_x|c\rangle]_{c'} \\
&= |1\rangle_{a'}|1\rangle_{b'}[-i\hat{\sigma}_x|c\rangle]_{c'} \\
&= |11\rangle[-i\hat{\sigma}_x|c\rangle]
\end{aligned}
$$

$$
\begin{aligned}
|\psi_3\rangle &= \left[\hat{C}_{\hat{\beta}} \otimes \hat{\mathbb{I}}\right] |\psi_2\rangle \\
&= \left[\hat{C}_{\hat{\beta}} \otimes \hat{\mathbb{I}}\right] |11\rangle[-i\hat{\sigma}_x|c\rangle] \\
&= \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta}\right] |11\rangle\right) \otimes \hat{\mathbb{I}}[-i\hat{\sigma}_x|c\rangle] \\
&= \left((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|11\rangle + (|1\rangle\langle 1| \otimes \hat{\beta})|11\rangle\right) \otimes \hat{\mathbb{I}}[-i\hat{\sigma}_x|c\rangle] \\
&= \left(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}|1\rangle + \langle 1|1\rangle|1\rangle \otimes \hat{\beta}|1\rangle\right) \otimes \hat{\mathbb{I}}[-i\hat{\sigma}_x|c\rangle] \\
&= (|1\rangle \otimes i|1\rangle) \otimes \hat{\mathbb{I}}[-i\hat{\sigma}_x|c\rangle] (\text{Remember: } \hat{\beta}|1\rangle = i|1\rangle) \\
&= -i^2|11\rangle[\hat{\sigma}_x|c\rangle] = |11\rangle[\hat{\sigma}_x|c\rangle]
\end{aligned}
$$

$$
\begin{aligned}
|\psi_4\rangle &= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |\psi_3\rangle \\
&= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |11\rangle[\hat{\sigma}_x|c\rangle] \\
&= \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\sigma}_x\right] |11\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\sigma}_x|c\rangle] \\
&= \left((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|11\rangle + (|1\rangle\langle 1| \otimes \hat{\sigma}_x)|11\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\sigma}_x|c\rangle] \\
&= \left(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}|1\rangle + \langle 1|1\rangle|1\rangle \otimes \hat{\sigma}_x|1\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\sigma}_x|c\rangle] \\
&= |10\rangle[\hat{\sigma}_x|c\rangle]
\end{aligned}
$$

$$
\begin{aligned}
|\psi_5\rangle &= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}^\dagger}\right] |\psi_4\rangle \\
&= \left[\hat{\mathbb{I}} \otimes \hat{C}_{\hat{\alpha}^\dagger}\right] |10\rangle[\hat{\sigma}_x|c\rangle] \\
&= \hat{\mathbb{I}}|1\rangle \otimes \hat{C}_{\hat{\alpha}^\dagger}|0\rangle[\hat{\sigma}_x|c\rangle] \\
&= \hat{\mathbb{I}}|1\rangle \otimes \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}^\dagger\right] |0\rangle[\hat{\sigma}_x|c\rangle]\right) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \left((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|0\rangle[\hat{\sigma}_x|c\rangle] + (|1\rangle\langle 1| \otimes \hat{\alpha}^\dagger)|0\rangle[\hat{\sigma}_x|c\rangle]\right) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \left(\langle 0|0\rangle|0\rangle \otimes \hat{\mathbb{I}}[\hat{\sigma}_x|c\rangle] + \langle 1|0\rangle|1\rangle \otimes \hat{\alpha}^\dagger([\hat{\sigma}_x|c\rangle])\right) \\
&= \hat{\mathbb{I}}|1\rangle \otimes \left(|0\rangle \otimes \hat{\mathbb{I}}[\hat{\sigma}_x|c\rangle]\right) \\
&= |10\rangle[\hat{\sigma}_x|c\rangle]
\end{aligned}
$$

$$
\begin{aligned}
|\psi_6\rangle &= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |\psi_5\rangle \\
&= \left[\hat{C}_{\text{not}} \otimes \hat{\mathbb{I}}\right] |10\rangle[\hat{\sigma}_x|c\rangle] \\
&= \left(\left[|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\sigma}_x\right] |10\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\sigma}_x|c\rangle] \\
&= \left((|0\rangle\langle 0| \otimes \hat{\mathbb{I}})|10\rangle + (|1\rangle\langle 1| \otimes \hat{\sigma}_x)|10\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\sigma}_x|c\rangle] \\
&= \left(\langle 0|1\rangle|0\rangle \otimes \hat{\mathbb{I}}|0\rangle + \langle 1|1\rangle|1\rangle \otimes \hat{\sigma}_x|0\rangle\right) \otimes \hat{\mathbb{I}}[\hat{\sigma}_x|c\rangle] \\
&= |11\rangle[\hat{\sigma}_x|c\rangle]
\end{aligned}
$$

So,

$\mathbf{|11c\rangle \xrightarrow{\hat{T}} |11\rangle[\hat{\sigma}_x|c\rangle]}$, i.e.

$\mathbf{|110\rangle \xrightarrow{\hat{T}} |111\rangle}$

$\mathbf{|111\rangle \xrightarrow{\hat{T}} |110\rangle}$

5 Quantum Gates II - Implementation of $\hat{\mathbf{T}}$ in IBM Q

So far, we have mathematically proved that $\hat{\mathbf{T}}$ does simulate the Toffoli gate. Now, let us implement circuit $\hat{\mathbf{T}}$ on IBM's quantum platform.

IBM has a catalog of quantum gates that can be used in both composer and quantum lab platforms. The first step towards this goal is to identify a set of IBM quantum gates that can be used to implement $\hat{\mathbf{T}}$.

- $\hat{C}_{\hat{\alpha}}$

$$\begin{aligned}
\hat{C}_{\hat{\alpha}} &= |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha} \\
&= |0\rangle\langle 0| \otimes [|0\rangle\langle 0| + |1\rangle\langle 1|] + |1\rangle\langle 1| \otimes \tfrac{1}{\sqrt{2}}[|0\rangle\langle 0| - i|0\rangle\langle 1| - i|1\rangle\langle 0| + |1\rangle\langle 1|] \\
&= |00\rangle\langle 00| + |01\rangle\langle 01| + \tfrac{1}{\sqrt{2}}[|10\rangle\langle 10| - i|10\rangle\langle 11| - i|11\rangle\langle 10| + |11\rangle\langle 11|] \implies \\
\hat{C}_{\hat{\alpha}} &= |00\rangle\langle 00| + |01\rangle\langle 01| + \tfrac{1}{\sqrt{2}}|10\rangle\langle 10| - \tfrac{i}{\sqrt{2}}|10\rangle\langle 11| - \tfrac{i}{\sqrt{2}}|11\rangle\langle 10| + \tfrac{1}{\sqrt{2}}|11\rangle\langle 11|
\end{aligned}$$

In matrix form:

$$C_\alpha = \begin{array}{c} \\ |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \begin{array}{c} \begin{array}{cccc} \langle 00| & \langle 01| & \langle 10| & \langle 11| \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \end{array}$$

The CRX(θ) gate from Qiskit Manual is defined as follows:

$$CRX(\theta)q_1q_0 = |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes RX(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ 0 & 0 & -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

Let $\theta = \frac{\pi}{2} \Rightarrow$

$$CRX(\tfrac{\pi}{2})q_1q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\pi}{4} & -i\sin\frac{\pi}{4} \\ 0 & 0 & -i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & 0 & -i\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Therefore

$$\hat{C}_{\hat{\alpha}} = CRX(\tfrac{\pi}{2}).$$

- $\hat{C}_{\hat{\alpha}^\dagger}$

$$
\begin{aligned}
\hat{C}_{\hat{\alpha}^\dagger} &= |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\alpha}^\dagger \\
&= |0\rangle\langle 0| \otimes [|0\rangle\langle 0| + |1\rangle\langle 1|] + |1\rangle\langle 1| \otimes \tfrac{1}{\sqrt{2}}\,[|0\rangle\langle 0| + i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|] \implies \\
\hat{C}_{\hat{\alpha}^\dagger} &= |00\rangle\langle 00| + |01\rangle\langle 01| + \tfrac{1}{\sqrt{2}}|10\rangle\langle 10| + \tfrac{i}{\sqrt{2}}|10\rangle\langle 11| + \tfrac{i}{\sqrt{2}}|11\rangle\langle 10| + \tfrac{1}{\sqrt{2}}|11\rangle\langle 11|
\end{aligned}
$$

In matrix form:

$$
C_{\alpha^\dagger} = \begin{array}{c} \\ |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \begin{array}{c} \begin{array}{cccc} \langle 00| & \langle 01| & \langle 10| & \langle 11| \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \end{array}
$$

We know that our old friend, the CRX(θ) gate from Qiskit Manual, is defined as follows:

$$
CRX(\theta)q_1q_0 = |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes RX(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ 0 & 0 & -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.
$$

Let us now set $\theta = \frac{7}{2}\pi \;\; (= -\frac{\pi}{2}) \Rightarrow$

$$
CRX(\tfrac{7\pi}{2})q_1q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{7\pi}{4} & -i\sin\frac{7\pi}{4} \\ 0 & 0 & -i\sin\frac{7\pi}{4} & \cos\frac{7\pi}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{-\pi}{4} & -i\sin\frac{-\pi}{4} \\ 0 & 0 & -i\sin\frac{-\pi}{4} & \cos\frac{-\pi}{4} \end{pmatrix}
$$

That is,

$$
CRX(\tfrac{7\pi}{2})q_1q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\pi}{4} & -i(-1)\sin\frac{\pi}{4} \\ 0 & 0 & -i(-1)\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.
$$

Therefore,

$$
\hat{C}_{\hat{\alpha}^\dagger} = CRX(\tfrac{7\pi}{2}) = CRX(-\tfrac{\pi}{2})
$$

- $\hat{C}_{\hat{\beta}}$

$$\begin{aligned}\hat{C}_{\hat{\beta}} &= |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta}\\ &= |0\rangle\langle 0| \otimes [|0\rangle\langle 0| + |1\rangle\langle 1|] + |1\rangle\langle 1| \otimes [|0\rangle\langle 0| + i|1\rangle\langle 1|] \implies\\ \hat{C}_{\hat{\beta}} &= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + i|11\rangle\langle 11|\end{aligned}$$

In matrix form:

$$C_\beta = \begin{array}{c c} & \begin{array}{cccc} \langle 00| & \langle 01| & \langle 10| & \langle 11| \end{array} \\ \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \end{array}$$

In Qiskit parlance (CPhase(λ) gate on Qiskit Manual):

$$CPhase(\lambda) = |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes P(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\lambda} \end{pmatrix}.$$

Since Euler's formula states that

$$e^{i\lambda} = \cos\lambda + i\sin\lambda,$$

let us set $\lambda = \frac{\pi}{2}$, which implies that

$$e^{i\frac{\pi}{2}} = \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2} = 0 + i(1) = i$$

So,

$$CPhase(\lambda)q_1q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\lambda} \end{pmatrix}.$$

Then

$$CPhase(\tfrac{\pi}{2})q_1q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\frac{\pi}{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

Therefore

$$\hat{C}_{\hat{\beta}} = CPhase(\tfrac{\pi}{2}).$$

- $\hat{C}_{\hat{\beta}^\dagger}$

$$\begin{aligned}\hat{C}_{\hat{\beta}^\dagger} &= |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\beta}^\dagger \\ &= |0\rangle\langle 0| \otimes [|0\rangle\langle 0| + |1\rangle\langle 1|] + |1\rangle\langle 1| \otimes [|0\rangle\langle 0| - i|1\rangle\langle 1|] \implies \\ \hat{C}_{\hat{\beta}^\dagger} &= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - i|11\rangle\langle 11|\end{aligned}$$

In matrix form:

$$C_\beta = \begin{array}{c} \\ |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \begin{array}{c} \begin{array}{cccc} \langle 00| & \langle 01| & \langle 10| & \langle 11| \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \end{array}$$

In Qiskit parlance (CPhase(λ) gate on Qiskit Manual):

$$CPhase(\lambda) = |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes P(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\lambda} \end{pmatrix}.$$

Since Euler's formula states that

$$e^{i\lambda} = \cos\lambda + i\sin\lambda,$$

let us set $\lambda = \frac{3\pi}{2}$, which implies that

$$e^{i\frac{3\pi}{2}} = \cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2} = 0 + i(-1) = -i$$

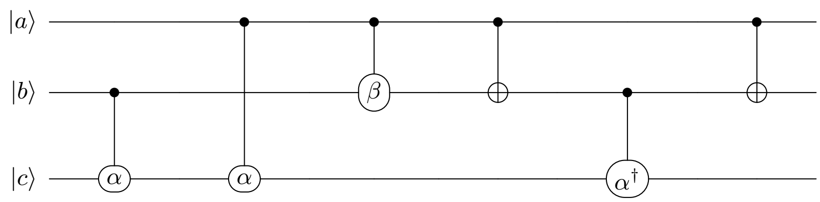
Then,

$$CPhase(\tfrac{3\pi}{2})q_1q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\frac{3\pi}{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

Therefore

$$\hat{C}_{\hat{\beta}^\dagger} = CPhase(\tfrac{3\pi}{2}).$$

So, our quantum circuit $\hat{\mathbf{T}}$ (Fig. (2));



Copy of Fig. (2). Quantum circuit $\hat{\mathbf{T}}$.

turns into the following circuit

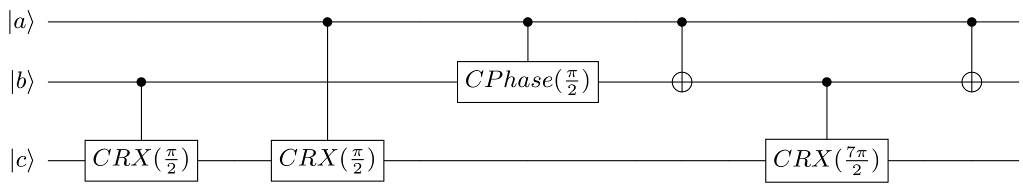


Fig. (3). Quantum circuit $\hat{\mathbf{T}}$ with Qiskit matrices.

The last step of this journey consists of building our circuit on IBM Quantum Composer. Fig. (4) presents circuit $\hat{\mathbf{T}}$ on IBM Quantum Composer. A key point here is to remember that on IBM Quantum Composer, qubits are presented in inverse significance order with respect to the standard qubit order in quantum computing textbooks. In other words, $|a\rangle \otimes |b\rangle \otimes |c\rangle$ on circuit $\hat{\mathbf{T}}$ is `q[2]q[1]q[0]` on IBM Quantum Composer.

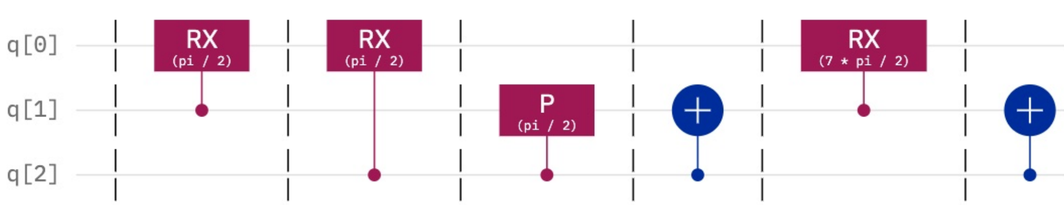


Figure (4): Circuit $\hat{\mathbf{T}}$ on IBM Quantum Composer.

In the following pages, we present a full implementation of circuit $\hat{\mathbf{T}}$ on IBM Quantum Composer with initial values $|000\rangle$, $|111\rangle$, and $|110\rangle$. For each input, run the QASM code (provided as companion files to this document) and test it using the `Inspect` tool.

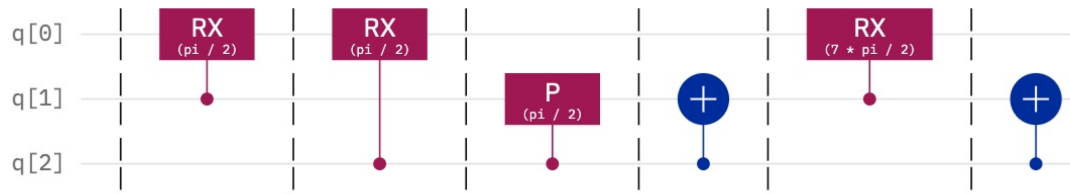


Figure (5): (repeated) Circuit $\hat{\mathbf{T}}$ on IBM Quantum Composer.

Circuit from Fig. (5) has been produced using the following QASM code:

```
// Basic Toffoli Circuit on IBM Quantum Composer
// No qubit initialization
// No measurement devices
// Just for comparison purposes
OPENQASM 2.0;
include "qelib1.inc";

// Three quantum channels
qreg q[3];

// 1. The barrier command is used to prevent
//    IBM's traspiler to change the structure
//    of our circuit.
// 2. Apply CRX gate with argument pi/2
//    The control qubit is the second least significant qubit
//    The target qubit is the least significant qubit
barrier q[0], q[1], q[2];
crx(pi / 2) q[1], q[0];

// Apply CRX gate with argument pi/2
// The control qubit is the most significant qubit
// The target qubit is the least significant qubit
barrier q[0], q[1], q[2];
crx(pi / 2) q[2], q[0];

// Apply CPhase gate with argument pi/2
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cp(pi / 2) q[2], q[1];

// Apply Cnot gate
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cx q[2], q[1];

// Apply CRX gate with argument 7*pi/2
// The control qubit is the second least significant qubit
// The target qubit is the least significant qubit
barrier q[0], q[1], q[2];
crx(7 * pi / 2) q[1], q[0];

// Apply Cnot gate
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cx q[2], q[1];
```

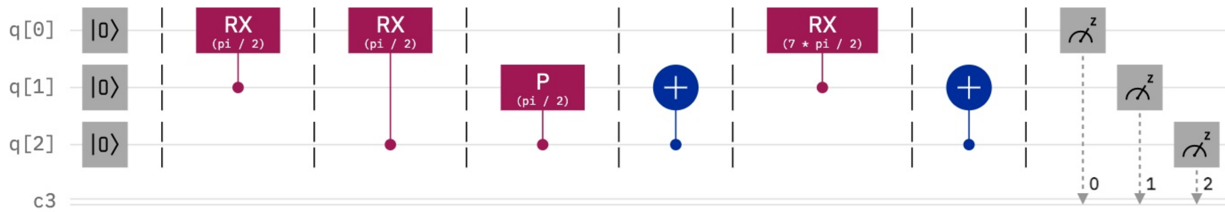


Figure (6): Full $\hat{\mathbf{T}}$ Circuit on IBM Quantum Composer initialized in $|000\rangle$.

Fig. (6) presents the full Quantum circuit $\hat{\mathbf{T}}$ on IBM Quantum Composer, including qubit initialization in $|000\rangle$ and measurement operators. This circuit is produced by the following QASM code:

```
// Full Toffoli Circuit on IBM Quantum Composer
// Qubit initialization in |000>
// With measurement operators
OPENQASM 2.0;
include "qelib1.inc";

// Three quantum channels
qreg q[3];
// Three classical channels
creg c[3];

// Initialize qubits on state |000>
reset q[0];
reset q[1];
reset q[2];
// barrier q[0], q[1], q[2];
// x q[0];
// x q[1];
// x q[2];


// 1. The barrier command is used to prevent
//    IBM's traspiler to change the structure
//    of our circuit.
// 2. Apply CRX gate with argument pi/2
//    The control qubit is the second
//    least significant qubit
//    The target qubit is the
//    least significant qubit
barrier q[0], q[1], q[2];
crx(pi / 2) q[1], q[0];
```

```
// Apply CRX gate with argument pi/2
// The control qubit is the most significant qubit
// The target qubit is the least significant qubit
barrier q[0], q[1], q[2];
crx(pi / 2) q[2], q[0];

// Apply CPhase gate with argument pi/2
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cp(pi / 2) q[2], q[1];

// Apply Cnot gate
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cx q[2], q[1];


// Apply CRX gate with argument 7*pi/2
// The control qubit is the second least significant qubit
// The target qubit is the least significant qubit
barrier q[0], q[1], q[2];
crx(7 * pi / 2) q[1], q[0];

// Apply Cnot gate
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cx q[2], q[1];

// Measure all three qubits simultaneously
barrier q[0], q[1], q[2];
measure q[0] -> c[0];
measure q[1] -> c[1];
measure q[2] -> c[2];
```

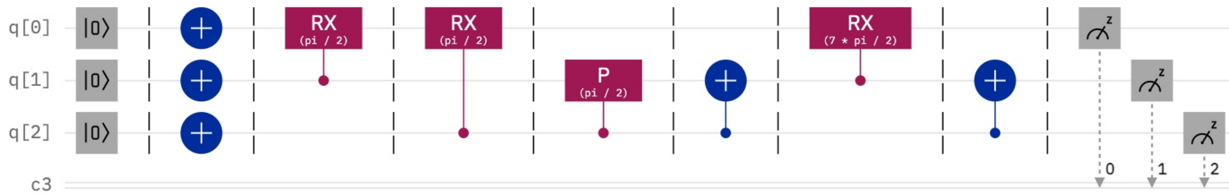


Figure (7): Full $\hat{\mathbf{T}}$ Circuit on IBM Quantum Composer initialized in $|111\rangle$.

Fig. (7) presents the full Quantum circuit $\hat{\mathbf{T}}$ on IBM Quantum Composer, including qubit initialization in $|111\rangle$ and measurement operators. This circuit is produced by the following QASM code:

```
// Full Toffoli Circuit on IBM Quantum Composer
// Qubit initialization in |111>
// With measurement operators
OPENQASM 2.0;
include "qelib1.inc";

// Three quantum channels
qreg q[3];
// Three classical channels
creg c[3];

// Initialize qubits on state |111>
reset q[0];
reset q[1];
reset q[2];
// barrier q[0], q[1], q[2];
x q[0];
x q[1];
x q[2];


// 1. The barrier command is used to prevent
//    IBM's traspiler to change the structure
//    of our circuit.
// 2. Apply CRX gate with argument pi/2
//    The control qubit is the second
//    least significant qubit
//    The target qubit is the
//    least significant qubit
barrier q[0], q[1], q[2];
crx(pi / 2) q[1], q[0];
```

```
// Apply CRX gate with argument pi/2
// The control qubit is the most significant qubit
// The target qubit is the least significant qubit
barrier q[0], q[1], q[2];
crx(pi / 2) q[2], q[0];

// Apply CPhase gate with argument pi/2
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cp(pi / 2) q[2], q[1];

// Apply Cnot gate
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cx q[2], q[1];

// Apply CRX gate with argument 7*pi/2
// The control qubit is the second least significant qubit
// The target qubit is the least significant qubit
barrier q[0], q[1], q[2];
crx(7 * pi / 2) q[1], q[0];

// Apply Cnot gate
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cx q[2], q[1];

// Measure all three qubits simultaneously
barrier q[0], q[1], q[2];
measure q[0] -> c[0];
measure q[1] -> c[1];
measure q[2] -> c[2];
```

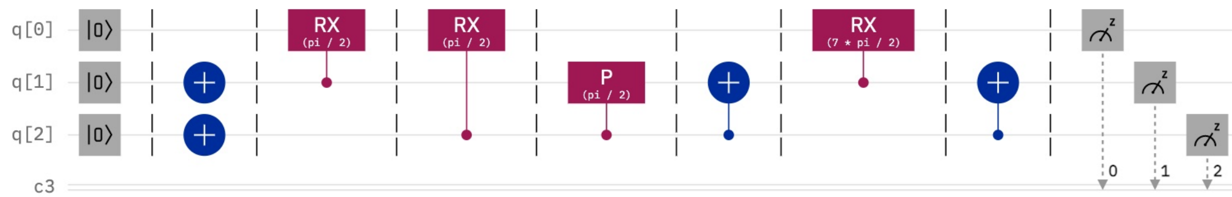


Figure (8): Full $\hat{\mathbf{T}}$ Circuit on IBM Quantum Composer initialized in $|110\rangle$.

Fig. (8) presents the full Quantum circuit $\hat{\mathbf{T}}$ on IBM Quantum Composer, including qubit initialization in $|110\rangle$ and measurement operators. This circuit is produced by the following QASM code:

```
// Full Toffoli Circuit on IBM Quantum Composer
// Qubit initialization in |110>
// With measurement operators
OPENQASM 2.0;
include "qelib1.inc";

// Three quantum channels
qreg q[3];
// Three classical channels
creg c[3];

// Initialize qubits on state |110>
reset q[0];
reset q[1];
reset q[2];
// barrier q[0], q[1], q[2];
// x q[0];
x q[1];
x q[2];


// 1. The barrier command is used to prevent
//    IBM's traspiler to change the structure
//    of our circuit.
// 2. Apply CRX gate with argument pi/2
//    The control qubit is the second
//    least significant qubit
//    The target qubit is the
//    least significant qubit
barrier q[0], q[1], q[2];
crx(pi / 2) q[1], q[0];
```

```
// Apply CRX gate with argument pi/2
// The control qubit is the most significant qubit
// The target qubit is the least significant qubit
barrier q[0], q[1], q[2];
crx(pi / 2) q[2], q[0];

// Apply CPhase gate with argument pi/2
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cp(pi / 2) q[2], q[1];

// Apply Cnot gate
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cx q[2], q[1];


// Apply CRX gate with argument 7*pi/2
// The control qubit is the second least significant qubit
// The target qubit is the least significant qubit
barrier q[0], q[1], q[2];
crx(7 * pi / 2) q[1], q[0];

// Apply Cnot gate
// The control qubit is the most significant qubit
// The target qubit is the second least significant qubit
barrier q[0], q[1], q[2];
cx q[2], q[1];

// Measure all three qubits simultaneously
barrier q[0], q[1], q[2];
measure q[0] -> c[0];
measure q[1] -> c[1];
measure q[2] -> c[2];
```