Superdense Coding

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The goal of this lecture is to study the Superdense Coding protocol, a quantum-mechanical procedure that uses a Bell state to send two bits using only one qubit.

1 A review of the Bell State Circuit

The behaviour of superdense coding is closely related to the Bell state circuit $\hat{B} = \hat{C}_{\text{not}}(\hat{H} \otimes \hat{\mathbb{I}})$, which we shall quickly review in this section. The Bell state circuit $\hat{B}$ is presented in Fig. (1) and its behavior on entries $\{|a\rangle, |b\rangle\}$, $a, b \in \{0, 1\}$ is presented in Eqs. (1, 2, 3, and 4)

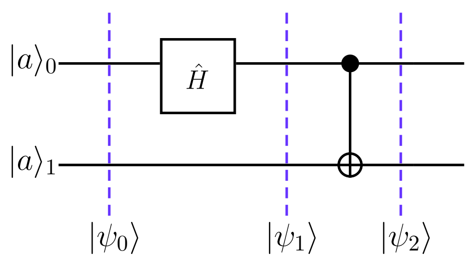

Figure 1: The Bell State Circuit $\hat{B}$

$$\hat{B}|00\rangle = \hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|00\rangle) = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{1}$$

$$\hat{B}|01\rangle = \hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|01\rangle) = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \tag{2}$$

$$\hat{B}|10\rangle = \hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|10\rangle) = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \tag{3}$$

$$\hat{B}|11\rangle = \hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|11\rangle) = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{4}$$

Indeed, the goal of circuit $\hat{B}$ is to produce the four Bell states. Now, **an alternative view on circuit $\hat{B}$ is to think of it as a quantum circuit that encodes classical bits into bipartite quantum states**.

For example, let us focus on the following transformation (Eq. (5))

$$00 \xrightarrow{\text{Qubit preparation}} |0\rangle|0\rangle \xrightarrow{\hat{B}} \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{5}$$

Transformation presented in Eq. (5) can be seen as a process consisting of two parts:

1. Encoding a pair of classical bits 00 into a pair of qubits initialised as $|0\rangle|0\rangle$.
2. Transforming qubits $|0\rangle|0\rangle$ to the Bell state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ via the circuit $\hat{C}_{\text{not}}(\hat{H} \otimes \hat{\mathbb{I}})$.

Following the same rationale, we may use the Bell state circuit to run the following encodings:

$$01 \xrightarrow{\text{Qubit preparation}} |0\rangle|1\rangle \xrightarrow{\hat{B}} \frac{|01\rangle + |10\rangle}{\sqrt{2}} \tag{6}$$

$$10 \xrightarrow{\text{Qubit preparation}} |1\rangle|0\rangle \xrightarrow{\hat{B}} \frac{|00\rangle - |11\rangle}{\sqrt{2}} \tag{7}$$

$$11 \xrightarrow{\text{Qubit preparation}} |1\rangle|1\rangle \xrightarrow{\hat{B}} \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{8}$$

Now, suppose that we are given a Bell state $|\Psi\rangle$ that was produced following one of the transformations presented in Eqs. (5, 6, 7 or 8). **Would it be possible at all to retrieve the classical bits that were encoded in $|\Psi\rangle$?**

The answer is **Yes, indeed!**. We would need to follow two steps:

i) Since any unitary circuit is reversible then the Bell state circuit $\hat{B}$ is reversible. Thus, we need to compute $\hat{B}^{-1}$ $(= \hat{B}^\dagger)$ and feed it with $|\Psi\rangle$ in order to produce the corresponding qubit pair $|a\rangle|b\rangle$, $a, b \in \{0, 1\}$.

ii) Measure $|a\rangle|b\rangle$ using the set of measurement operators $B_c = \{|00\rangle\langle 00|, |01\rangle\langle 01|, |10\rangle\langle 10|, |11\rangle\langle 11|\}$. Note that $|a\rangle|b\rangle$ is always an element of the canonical basis of $\mathcal{H}^4$, i.e. $|a\rangle|b\rangle \in \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ therefore *measuring $|a\rangle|b\rangle$ with B_c will always be a deterministic measurement*.

In order to compute $\hat{B}^{-1}$, we need to remember the following rules.

- $\hat{U}$ is a Unitary operator $\Leftrightarrow$

$$\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \hat{\mathbb{I}} \tag{9}$$

  We can see that the inverse of $\hat{U}$ is the operator $\hat{U}^\dagger$, i.e. $\hat{U}^\dagger = \hat{U}^{-1}$. So, $\hat{B}^{-1} = \hat{B}^\dagger$, hence we need to compute $\hat{B}^\dagger$.

- Let $\hat{A}$, $\hat{B}$ be two linear operators and $|v\rangle\langle w|$ an outer product $\Rightarrow$

$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger \tag{10}$$

$$(\hat{A} \otimes \hat{B})^\dagger = \hat{A}^\dagger \otimes \hat{B}^\dagger \tag{11}$$

$$(|v\rangle\langle w|)^\dagger = |w\rangle\langle v| \tag{12}$$

So,

$$\left[\hat{C}_{\text{not}}(\hat{H}\otimes\hat{\mathbb{I}})\right]^{\dagger}=(\hat{H}\otimes\hat{\mathbb{I}})^{\dagger}\hat{C}_{\text{not}}^{\dagger}=(\hat{H}^{\dagger}\otimes\hat{\mathbb{I}}^{\dagger})\hat{C}_{\text{not}}^{\dagger}\tag{13}$$

where

$$\hat{H}=\frac{1}{\sqrt{2}}(|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|-|1\rangle\langle 1|)$$

implies

$$\hat{H}^{\dagger}=\frac{1}{\sqrt{2}}(|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|-|1\rangle\langle 1|)\tag{14}$$

and

$$\hat{C}_{\text{not}}=|00\rangle\langle 00|+|01\rangle\langle 01|+|10\rangle\langle 11|+|11\rangle\langle 10|$$

makes

$$\hat{C}_{\text{not}}^{\dagger}=|00\rangle\langle 00|+|01\rangle\langle 01|+|10\rangle\langle 11|+|11\rangle\langle 10|\tag{15}$$

Let us now process $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$:

$$\hat{C}_{\text{not}}^{\dagger}\left[\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right]=\frac{1}{\sqrt{2}}\left[\hat{C}_{\text{not}}^{\dagger}|00\rangle+\hat{C}_{\text{not}}^{\dagger}|11\rangle\right]=\frac{1}{\sqrt{2}}\left[|00\rangle+|10\rangle\right]$$

Furthermore,

$$\hat{H}^{\dagger}\otimes\hat{\mathbb{I}}^{\dagger}\left[\frac{1}{\sqrt{2}}\left[|00\rangle+|10\rangle\right]\right]=\frac{1}{\sqrt{2}}\left[\hat{H}^{\dagger}|0\rangle\otimes\hat{\mathbb{I}}^{\dagger}|0\rangle+\hat{H}^{\dagger}|1\rangle\otimes\hat{\mathbb{I}}^{\dagger}|0\rangle\right]$$

Since $\hat{H}^{\dagger}|0\rangle=\frac{1}{\sqrt{2}}[|0\rangle+|1\rangle]$ and $\hat{H}^{\dagger}|1\rangle=\frac{1}{\sqrt{2}}[|0\rangle-|1\rangle]$ then

$$\hat{H}^{\dagger}|0\rangle\otimes\hat{\mathbb{I}}^{\dagger}|0\rangle=\frac{1}{\sqrt{2}}[|0\rangle+|1\rangle]\otimes|0\rangle=\frac{1}{\sqrt{2}}[|00\rangle+|10\rangle]$$

and

$$\hat{H}^{\dagger}|1\rangle\otimes\hat{\mathbb{I}}^{\dagger}|0\rangle=\frac{1}{\sqrt{2}}[|0\rangle-|1\rangle]\otimes|0\rangle=\frac{1}{\sqrt{2}}[|00\rangle-|10\rangle]$$

Hence

$$\hat{H}^{\dagger}\otimes\hat{\mathbb{I}}^{\dagger}\left[\frac{1}{\sqrt{2}}\left[|00\rangle+|10\rangle\right]\right]=\frac{1}{(\sqrt{2})^2}\left[|00\rangle+|10\rangle+|00\rangle-|10\rangle\right]=\frac{2}{2}|00\rangle=|00\rangle$$

Therefore

$$\left(\hat{H}^{\dagger}\otimes\hat{\mathbb{I}}^{\dagger}\right)\hat{C}_{\text{not}}^{\dagger}\left[\frac{1}{\sqrt{2}}\left[|00\rangle+|11\rangle\right]\right]=|00\rangle\tag{16}$$

So, applying $\hat{B}^\dagger$ to $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ outputs $|00\rangle$. Now, we want to retrieve the classical bits (namely, 00) that were encoded in $|00\rangle$. What can we do?

The answer is to measure $|00\rangle$ using the following measuring machinery:

- Measurement outcomes. The pairs of classical bits encoded in Eqs. (5, 6, 7 or 8)) were $00, 01, 10,$ and 11. So, it is reasonable to expect that measuring the output of $\hat{B}^\dagger\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right)$ will produce one of the following outcomes: $00, 01, 10, 11$. We shall associate each outcome with the following measurement operators:
- Let $B_c = \{|00\rangle, |01\rangle, |10\rangle,\ |11\rangle\}$ be the computational basis of $\mathcal{H}^4$. Based on the elements of B_c, we create the following measurement operators:

$$\{\hat{M}_{00} = |00\rangle\langle 00|, \hat{M}_{01} = |01\rangle\langle 01|, \hat{M}_{10} = |10\rangle\langle 10|, \hat{M}_{11} = |11\rangle\langle 11|\}$$

- Remember that the probability of getting a_i as outcome is given by

$$p(a_i) = \langle\psi|\hat{M}^\dagger_{a_i}\hat{M}_{a_i}|\psi\rangle$$

  Thus

$$p(00) = \langle 00|\hat{M}_{00}|00\rangle \tag{17}$$

$$p(01) = \langle 00|\hat{M}_{01}|00\rangle \tag{18}$$

$$p(10) = \langle 00|\hat{M}_{10}|00\rangle \tag{19}$$

$$p(11) = \langle 00|\hat{M}_{11}|00\rangle \tag{20}$$

Let us now compute the probabilities of Eqs. (17, 18, 19, 20).

- $p(00) = \langle 00|\hat{M}_{00}|00\rangle$

$$\hat{M}_{00}|00\rangle = (|00\rangle\langle 00|)|00\rangle = \langle 00|00\rangle|00\rangle = |00\rangle \text{ , since } B_c \text{ is orthonormal.}$$

  So,

$$\langle 00|\hat{M}_{00}|00\rangle = \langle 00|00\rangle = 1 \text{ , since } B_c \text{ is orthonormal.}$$

  That is,

$$p(00) = 1$$

- $p(01) = \langle 00|\hat{M}_{01}|00\rangle$

$$\hat{M}_{01}|00\rangle = (|01\rangle\langle 01|)|00\rangle = \langle 01|00\rangle|01\rangle = \vec{0} \text{ , since } B_c \text{ is orthonormal.}$$

  So,

$$\langle 00|\hat{M}_{01}|00\rangle = \langle 00|\vec{0} = (|00\rangle, \vec{0}) = 0$$

  That is,

$$p(01) = 0$$

- $p(10) = \langle 00|\hat{M}_{10}|00\rangle$

$$\hat{M}_{10}|00\rangle = (|10\rangle\langle 10|)|00\rangle = \langle 10|00\rangle|10\rangle = \vec{0} \text{ , since } B_c \text{ is orthonormal.}$$

  So,

$$\langle 00|\hat{M}_{10}|00\rangle = \langle 00|\vec{0} = (|00\rangle, \vec{0}) = 0$$

  That is,

$$p(10) = 0$$

- $p(11) = \langle 00|\hat{M}_{11}|00\rangle$

$$\hat{M}_{11}|00\rangle = (|11\rangle\langle 11|)|00\rangle = \langle 11|00\rangle|11\rangle = \vec{0} \text{ , since } B_c \text{ is orthonormal.}$$

  So,

$$\langle 00|\hat{M}_{11}|00\rangle = \langle 00|\vec{0} = (|00\rangle, \vec{0}) = 0$$

  That is,

$$p(11) = 0$$

In summary,

$$p(00) = 1, p(01) = 0, p(10) = 0, p(11) = 0$$

That is, the only possible outcome of measuring $|00\rangle$ is the pair of classical bits 00. The other three pairs of classical bits, namely $01, 10, 11$ can never be the outcome of measuring $|00\rangle$ with the measurement operators $\{\hat{M}_{00} = |00\rangle\langle 00|, \hat{M}_{01} = |01\rangle\langle 01|, \hat{M}_{10} = |10\rangle\langle 10|, \hat{M}_{11} = |11\rangle\langle 11|\}$.

So, measuring $|00\rangle$ with $\{\hat{M}_{00} = |00\rangle\langle 00|, \hat{M}_{01} = |01\rangle\langle 01|, \hat{M}_{10} = |10\rangle\langle 10|, \hat{M}_{11} = |11\rangle\langle 11|\}$ is the right strategy to extract the pair of classical bits 00 that we encoded in Eq. (5).

The other three calculations, those corresponding to encoding $01, 10,$ and 11 into $|01\rangle$, $|1|0\rangle$, and $|11\rangle$ are left as an exercise to the reader.

2 Superdense coding

The goal of the superdense coding protocol is to encode and transmit two classical bits using only one qubit. To do so, we use Bell states as a quantum communication resource. Alice also has a quantum channel (say, an optical fiber) to send her qubit to Bob when the time comes to do so.

The protocol begins by producing a Bell state $|\Psi\rangle$ that is to be shared by Alice and Bob, that is, Alice is in possesion of one of the qubits that constitute $|\Psi\rangle$ while Bob owns the other qubit.

Now, as it is usually the case with Alice and Bob, they are in different locations and Alice is told by her boss that she has to send two classical bits to Bob. Of course, Alice can only use her qubit to send Bob that pair of classical bits.

Let us suppose that the Bell state that Alice and Bob share is given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] = \frac{1}{\sqrt{2}}\left[|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right]$$

We use the subindices A and B to emphasize the fact that one qubit belongs to Alice (A) and the other belongs to Bob (B).

Now, since Alice knows quantum computing, she knows the Bell state circuit $\hat{B}$ as well as the fact that $\hat{B}$ can be utilized to encode classical bits. She also knows that following the procedure presented in the previous section, $\hat{B}^\dagger$ can be used to retrieve the classical bits a, b that were encoded on each Bell state.

So, she wonders: *is it possible to apply local operations to my qubit to transform the Bell state $|\Psi\rangle$ to the Bell state that corresponds to the classical bits as shown in Eqs. (5, 6, 7 or 8)?*

The answer is **Yes, indeed!**. To show how to do that, let us remember the following operators.

$$\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0| \tag{21}$$

$$\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1| \tag{22}$$

$$\hat{\sigma}_y = i\left[|1\rangle\langle 0| - |0\rangle\langle 1|\right] \tag{23}$$

From Eq. (23), we find that

$$i\hat{\sigma}_y = |0\rangle\langle 1| - |1\rangle\langle 0| \tag{24}$$

The procedure that Alice will follow is

i) For sending classical bits 00, Alice does not do anything to her qubit, i.e.

$$|\Psi\rangle_f = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

ii) For sending classical bits 01, Alice applies the $\hat{\sigma}_x$ operator (Eq. (21)) to her qubit:

$$|\Psi\rangle_f = \hat{\sigma}_x \otimes \hat{\mathbb{1}}\left[\frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]\right] = \frac{1}{\sqrt{2}}\left[|10\rangle + |01\rangle\right] = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

iii) For sending classical bits 10, Alice applies the $\hat{\sigma}_z$ operator (Eq. (22)) to her qubit:

$$|\Psi\rangle_f = \hat{\sigma}_z \otimes \hat{\mathbb{I}} \left[\frac{1}{\sqrt{2}} [|00\rangle + |11\rangle]\right] = \frac{1}{\sqrt{2}} [|00\rangle - |11\rangle] = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

iv) For sending classical bits 11, Alice applies the $i\hat{\sigma}_y$ operator (Eq. (24)) to her qubit:

$$|\Psi\rangle_f = i\hat{\sigma}_y \otimes \hat{\mathbb{I}} \left[\frac{1}{\sqrt{2}} [|00\rangle + |11\rangle]\right] = \frac{1}{\sqrt{2}} [-|10\rangle + |01\rangle] = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Once Alice has applied the operator that corresponds to the pair of classical bits she must send to Bob, **she transmits her qubit to Bob**. Bob, who is now in possession of both qubits, applies $\hat{B}^\dagger$ to $|\Psi\rangle_f$ and then measures the output using the canonical projectors $B_c = \{|00\rangle\langle 00|, |01\rangle\langle 01|, |10\rangle\langle 10|, |11\rangle\langle 11|\}$. The final outcome is a pair of classical bits from the set $\{00, 01, 10, 11\}$.

A visualization of Superdense Coding is presented in Fig. (2) and a summary of the relationship between the Bell Circuit $\hat{B}$ and the operators applied by Alice is given in Table 1.

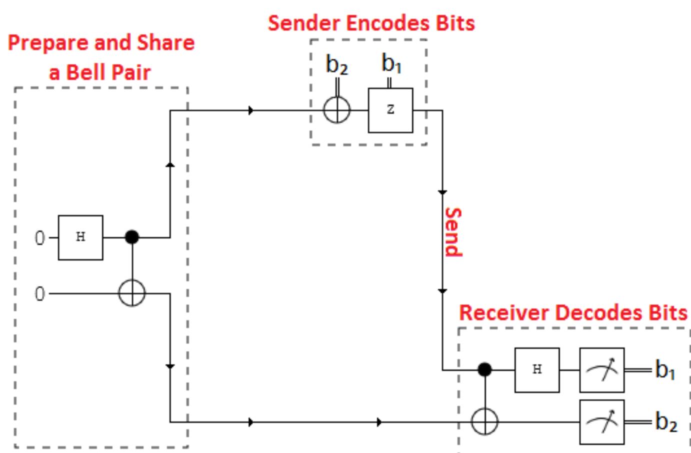

Figure 2: Superdense coding. Image taken from https://en.wikipedia.org/wiki/Superdense_coding

Summary of $\hat{B}$ and Superdense coding			
Classical bits to encode by $\hat{B}$	Bell State produced by $\hat{B}$	Classical bits to be sent by Alice	Bell state received by Bob
00	$\frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$	00	$\frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$
01	$\frac{\|01\rangle+\|10\rangle}{\sqrt{2}}$	01	$\frac{\|01\rangle+\|10\rangle}{\sqrt{2}}$
10	$\frac{\|00\rangle-\|11\rangle}{\sqrt{2}}$	10	$\frac{\|00\rangle-\|11\rangle}{\sqrt{2}}$
11	$\frac{\|01\rangle-\|10\rangle}{\sqrt{2}}$	10	$\frac{\|01\rangle-\|10\rangle}{\sqrt{2}}$

Table 1