Exercises with Solutions - Postulates of QM and The Bell Circuit

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Exercise 1. Let

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$
$$B_c = \{|0\rangle, |1\rangle\}$$
$$B_d = \{|+\rangle, |-\rangle\}$$
$$M_c = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$$
$$M_d = \{|+\rangle\langle +|, |-\rangle\langle -|\}$$

Compute the following.

1. Measure $|\psi\rangle$ using M_c. Assuming measurement outcomes are labeled a_0, a_1 then compute $p(a_0), p(a_1), |\psi\rangle_{a_0}^{\text{pm}}, |\psi\rangle_{a_1}^{\text{pm}}$

   Solution:

   Let us remember that the probability of measuring the outcome a_i, $i \in \{0, 1\}$, is given by

   $$p(a_i) = \langle\psi|\hat{M}_{a_i}^{\dagger}\hat{M}_{a_i}|\psi\rangle,$$

   where $|\psi\rangle$ is an arbitrary qubit, and

   $$\hat{M}_{a_0} = |0\rangle\langle 0| = \hat{M}_{a_0}^{\dagger}$$
   $$\hat{M}_{a_1} = |1\rangle\langle 1| = \hat{M}_{a_1}^{\dagger}$$

   We are now in position to start working on the exercise.

$$
\begin{aligned}
p(a_0) &= \left(\frac{\sqrt{3}}{2}\langle 0| - \frac{i}{2}\langle 1|\right)\left(|0\rangle\langle 0||0\rangle\langle 0|\right)\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right) \\
&= \left(\frac{\sqrt{3}}{2}\langle 0| - \frac{i}{2}\langle 1|\right)\left(|0\rangle\langle 0|\right)\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right) \\
&= \left(\frac{\sqrt{3}}{2}\langle 0|0\rangle\langle 0| - \frac{i}{2}\langle 1|0\rangle\langle 0|\right)\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right) \\
&= \frac{\sqrt{3}}{2}\langle 0|\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right) \\
&= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\langle 0|0\rangle + \frac{\sqrt{3}}{2}\cdot\frac{i}{2}\langle 0|1\rangle \\
&= \frac{3}{4},
\end{aligned}
$$

where we have made use of the orthonormality of the basis.

Now, we have seen that

$$
|\psi\rangle^{\mathrm{pm}}_{a_i} = \frac{\hat{M}_{a_i}|\psi\rangle}{\sqrt{p(a_i)}} = \frac{1}{\sqrt{p(a_i)}}\hat{M}_{a_i}|\psi\rangle, \qquad i \in \{0, 1\},
$$

which, using the previous result, yields

$$
\begin{aligned}
|\psi\rangle^{\mathrm{pm}}_{a_0} &= \frac{1}{\sqrt{\frac{3}{4}}}|0\rangle\langle 0|\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right) \\
&= \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}|0\rangle\langle 0|0\rangle + \frac{\frac{i}{2}}{\frac{\sqrt{3}}{2}}|0\rangle\langle 0|1\rangle \\
&= |0\rangle.
\end{aligned}
$$

Analogously[1], we now work with a_1.

$$
\begin{aligned}
p(a_1) &= \left(\frac{\sqrt{3}}{2}\langle 0| - \frac{i}{2}\langle 1|\right)\Big(|1\rangle\langle 1||1\rangle\langle 1|\Big)\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right) \\
&= \left(\frac{\sqrt{3}}{2}\langle 0| - \frac{i}{2}\langle 1|\right)\Big(|1\rangle\langle 1|\Big)\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right) \\
&= \left(\frac{\sqrt{3}}{2}\langle 0|1\rangle\langle 1| - \frac{i}{2}\langle 1|1\rangle\langle 1|\right)\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right) \\
&= -\frac{i}{2}\langle 1|\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right) \\
&= -\frac{\sqrt{3}}{2}\cdot\frac{i}{2}\langle 1|0\rangle - \frac{i}{2}\cdot\frac{i}{2}\langle 1|1\rangle \\
&= -\frac{i^2}{4} \\
&= \frac{1}{4},
\end{aligned}
$$

and the post-measurement state is given by the equation in red,

$$
\begin{aligned}
|\psi\rangle_{a_1}^{\mathrm{pm}} &= \frac{1}{\sqrt{\frac{1}{4}}}|1\rangle\langle 1|\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right) \\
&= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}|1\rangle\langle 1|0\rangle + \frac{\frac{i}{2}}{\frac{1}{2}}|1\rangle\langle 1|1\rangle \\
&= i|1\rangle,
\end{aligned}
$$

now since we have an undetectable global phase, i, we say that that the post-measurement state is

$$
|\psi\rangle_{a_1}^{\mathrm{pm}} = |1\rangle.
$$

[1]Notice that, since there are only two possible outcomes, once we know $p(a_0)$ we can use the fact that $p(a_0) + p(a_1) = 1$ to compute $p(a_1) = 1 - p(a_0) = 1 - \frac{3}{4} = \frac{1}{4}$. Here, we do the whole calculation to further illustrate the use of the measurement operators.

2. Measure $|\psi\rangle$ using M_d. Assuming measurement outcomes are labeled a_+, a_- then compute $p(a_+), p(a_-), |\psi\rangle^{\text{pm}}_{a_+}, |\psi\rangle^{\text{pm}}_{a_-}$

   Solution:

   In this exercise, it is useful to express the ket $|\psi\rangle$ in the diagonal basis, $\{|-\rangle, |+\rangle\}$. Since we know that

$$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$
$$|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle,$$

   then we can write

$$\begin{aligned}
|\psi\rangle &= \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle \\
&= \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) + \frac{i}{2}\left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right) \\
&= \left(\frac{\sqrt{3}+i}{2\sqrt{2}}\right)|+\rangle + \left(\frac{\sqrt{3}-i}{2\sqrt{2}}\right)|-\rangle
\end{aligned}$$

   In this case, we will use the operators

$$\hat{M}_{a_+} = |+\rangle\langle+| = \hat{M}^\dagger_{a_+}$$
$$\hat{M}_{a_-} = |-\rangle\langle-| = \hat{M}^\dagger_{a_-}$$

   and use the equation in magenta, with $i \in \{+, -\}$, to calculate the probabilities $p(a_-)$ and $p(a_+)$. Let us start with a_-,

$$
\begin{aligned}
p(a_-) &= \left(\frac{\sqrt{3}-i}{2\sqrt{2}}\langle+| + \frac{\sqrt{3}+i}{2\sqrt{2}}\langle-|\right)\Big(|-\rangle\langle-||-\rangle\langle-|\Big)\left(\frac{\sqrt{3}+i}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}|-\rangle\right) \\
&= \left(\frac{\sqrt{3}-i}{2\sqrt{2}}\langle+| + \frac{\sqrt{3}+i}{2\sqrt{2}}\langle-|\right)\Big(|-\rangle\langle-|\Big)\left(\frac{\sqrt{3}+i}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}|-\rangle\right) \\
&= \left(\frac{\sqrt{3}-i}{2\sqrt{2}}\langle+|-\rangle\langle-| + \frac{\sqrt{3}+i}{2\sqrt{2}}\langle-|-\rangle\langle-|\right)\left(\frac{\sqrt{3}+i}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}|-\rangle\right) \\
&= \frac{\sqrt{3}+i}{2\sqrt{2}}\langle-|\left(\frac{\sqrt{3}+i}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}|-\rangle\right) \\
&= \frac{\sqrt{3}+i}{2\sqrt{2}}\cdot\frac{\sqrt{3}+i}{2\sqrt{2}}\langle-|+\rangle + \frac{\sqrt{3}+i}{2\sqrt{2}}\cdot\frac{\sqrt{3}-i}{2\sqrt{2}}\langle-|-\rangle \\
&= \frac{(\sqrt{3})^2+1}{8} \\
&= \frac{3+1}{8} \\
&= \frac{4}{8} \\
&= \frac{1}{2}
\end{aligned}
$$

Now we use the equation in red, with $i \in \{+,-\}$, to get

$$
\begin{aligned}
|\psi\rangle^{\mathrm{pm}}_{a_-} &= \frac{1}{\sqrt{\frac{1}{2}}}|-\rangle\langle-|\left(\frac{\sqrt{3}+i}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}|-\rangle\right) \\
&= \frac{\frac{\sqrt{3}+i}{2\sqrt{2}}}{\sqrt{\frac{1}{2}}}|-\rangle\langle-|+\rangle + \frac{\frac{\sqrt{3}-i}{2\sqrt{2}}}{\sqrt{\frac{1}{2}}}|-\rangle\langle-|-\rangle \\
&= \frac{\sqrt{3}-i}{2\sqrt{2}}\frac{\sqrt{2}}{1}|-\rangle \\
&= \frac{\sqrt{3}-i}{2}|-\rangle.
\end{aligned}
$$

The constant multiplying the ket $|-\rangle$ is only a global phase, which can be ignored:

$$\begin{aligned}
e^{-i\frac{\pi}{6}} &= \cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6} \\
&= \frac{\sqrt{3}}{2} - i\frac{1}{2} \\
&= \frac{\sqrt{3}}{2} - \frac{i}{2} \\
&= \frac{\sqrt{3}-i}{2}
\end{aligned}$$

Therefore,

$$|\psi\rangle^{\mathrm{pm}}_{a_-} = |-\rangle$$

Now let us turn our attention to $p(a_+)$,[2]

$$\begin{aligned}
p(a_+) &= \left(\frac{\sqrt{3}-i}{2\sqrt{2}}\langle+| + \frac{\sqrt{3}+i}{2\sqrt{2}}\langle-|\right)\Big(|+\rangle\langle+||+\rangle\langle+|\Big)\left(\frac{\sqrt{3}+i}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}|-\rangle\right) \\
&= \left(\frac{\sqrt{3}-i}{2\sqrt{2}}\langle+| + \frac{\sqrt{3}+i}{2\sqrt{2}}\langle-|\right)\Big(|+\rangle\langle+|\Big)\left(\frac{\sqrt{3}+i}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}|-\rangle\right) \\
&= \left(\frac{\sqrt{3}-i}{2\sqrt{2}}\langle+|+\rangle\langle+| + \frac{\sqrt{3}+i}{2\sqrt{2}}\langle-|+\rangle\langle+|\right)\left(\frac{\sqrt{3}+i}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}|-\rangle\right) \\
&= \frac{\sqrt{3}-i}{2\sqrt{2}}\langle+|\left(\frac{\sqrt{3}+i}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}|-\rangle\right) \\
&= \frac{\sqrt{3}-i}{2\sqrt{2}}\cdot\frac{\sqrt{3}+i}{2\sqrt{2}}\langle+|+\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}\cdot\frac{\sqrt{3}-i}{2\sqrt{2}}\langle+|-\rangle \\
&= \frac{(\sqrt{3})^2+1}{8} \\
&= \frac{3+1}{8} \\
&= \frac{4}{8} \\
&= \frac{1}{2}
\end{aligned}$$

[2]In the same spirit of the previous footnote, we could omit the subsequent calculation and simply write $p(a_+) = 1 - p(a_1) = 1 - \frac{1}{2} = \frac{1}{2}$ which of course yields the same result that is shown in the main text.

We can now calculate $|\psi\rangle_{a_+}^{\text{pm}}$ as

$$
\begin{aligned}
|\psi\rangle_{a_+}^{\text{pm}} &= \frac{1}{\sqrt{\frac{1}{2}}}|+\rangle\langle+|\left(\frac{\sqrt{3}+i}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}|-\rangle\right) \\
&= \frac{\frac{\sqrt{3}+i}{2\sqrt{2}}}{\sqrt{\frac{1}{2}}}|+\rangle\langle+|+\rangle + \frac{\frac{\sqrt{3}-i}{2\sqrt{2}}}{\sqrt{\frac{1}{2}}}|+\rangle\langle+|-\rangle \\
&= \frac{\sqrt{3}+i}{2\sqrt{2}}\frac{\sqrt{2}}{1}|+\rangle \\
&= \frac{\sqrt{3}+i}{2}|+\rangle.
\end{aligned}
$$

Now the global phase in the ket $|+\rangle$ is

$$
\begin{aligned}
e^{i\frac{\pi}{6}} &= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} \\
&= \frac{\sqrt{3}}{2} + i\frac{1}{2} \\
&= \frac{\sqrt{3}}{2} + \frac{i}{2} \\
&= \frac{\sqrt{3}+i}{2}.
\end{aligned}
$$

Therefore,

$$
|\psi\rangle_{a_+}^{\text{pm}} = |+\rangle.
$$

Exercise 2. Let

$$\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$$
$$\hat{\sigma}_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$
$$\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$
$$\hat{H} = \tfrac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$
$$|\psi\rangle_1 = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle$$
$$|\psi\rangle_2 = \tfrac{1}{2}|0\rangle + i\tfrac{\sqrt{3}}{2}|1\rangle$$
$$|\psi\rangle_3 = \cos\frac{3\pi}{4}|0\rangle + i\sin\frac{3\pi}{4}|1\rangle$$

Compute the following

1. $\hat{H} \otimes \hat{\sigma}_x(|\psi\rangle_1 \otimes |\psi\rangle_2)$

   Solution:

$$\hat{H} \otimes \hat{\sigma}_x(|\psi\rangle_1 \otimes |\psi\rangle_2) = \hat{H}|\psi\rangle_1 \otimes \hat{\sigma}_x|\psi\rangle_2$$

Let us apply each operator on the corresponding ket separately,

$$\begin{aligned}
\hat{H}|\psi\rangle_1 &= \frac{1}{\sqrt{2}}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\big)\big(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\big) \\
&= \frac{1}{2}\big(|0\rangle\langle 0|0\rangle + |0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle - |1\rangle\langle 1|0\rangle\big) \\
&- \frac{1}{2}\big(|0\rangle\langle 0|1\rangle + |0\rangle\langle 1|1\rangle + |1\rangle\langle 0|1\rangle - |1\rangle\langle 1|1\rangle\big) \\
&= \frac{1}{2}\big(|0\rangle + |1\rangle\big) - \frac{1}{2}\big(|0\rangle - |1\rangle\big) \\
&= \frac{1}{2}\big(|0\rangle - |0\rangle\big) + \frac{1}{2}\big(|1\rangle + |1\rangle\big) \\
&= |1\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{\sigma}_x|\psi\rangle_2 &= \big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\big(\frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|1\rangle\big) \\
&= \frac{1}{2}\big(|0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle\big) + i\frac{\sqrt{3}}{2}\big(|0\rangle\langle 1|1\rangle + |1\rangle\langle 0|1\rangle\big) \\
&= \frac{1}{2}|1\rangle + i\frac{\sqrt{3}}{2}|0\rangle
\end{aligned}$$

We gather these results to write

$$\begin{aligned}\hat{H} \otimes \hat{\sigma}_x(|\psi\rangle_1 \otimes |\psi\rangle_2) &= \hat{H}|\psi\rangle_1 \otimes \hat{\sigma}_x|\psi\rangle_2 \\ &= |1\rangle \otimes \big(\frac{1}{2}|1\rangle + i\frac{\sqrt{3}}{2}|0\rangle\big) \\ &= \frac{1}{2}|11\rangle + i\frac{\sqrt{3}}{2}|10\rangle\end{aligned}$$

2. $\hat{\sigma}_y \otimes \hat{\sigma}_x(|\psi\rangle_3 \otimes |\psi\rangle_1)$

   Solution:

$$\hat{\sigma}_y \otimes \hat{\sigma}_x(|\psi\rangle_3 \otimes |\psi\rangle_1) = \hat{\sigma}_y|\psi\rangle_3 \otimes \hat{\sigma}_x|\psi\rangle_1$$

We will proceed as we did before, computing the action of each operator on the corresponding ket, and then we will put the results together.

$$\begin{aligned}\hat{\sigma}_y|\psi\rangle_3 &= \big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\big(\cos\frac{3\pi}{4}|0\rangle + i\sin\frac{3\pi}{4}|1\rangle\big) \\ &= \cos\frac{3\pi}{4}\big(-i|0\rangle\langle 1|0\rangle + i|1\rangle\langle 0|0\rangle\big) + i\sin\frac{3\pi}{4}\big(-i|0\rangle\langle 1|1\rangle + i|1\rangle\langle 0|1\rangle\big) \\ &= i\cos\frac{3\pi}{4}|1\rangle - i^2\sin\frac{3\pi}{4}|0\rangle \\ &= i\cos\frac{3\pi}{4}|1\rangle + \sin\frac{3\pi}{4}|0\rangle\end{aligned}$$

$$\begin{aligned}\hat{\sigma}_x|\psi\rangle_1 &= \big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\big(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\big) \\ &= \frac{1}{\sqrt{2}}\big(|0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle\big) - \frac{1}{\sqrt{2}}\big(|0\rangle\langle 1|1\rangle + |1\rangle\langle 0|1\rangle\big) \\ &= \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle\end{aligned}$$

From the previous calculations, it follows that

$$\begin{aligned}
\hat{\sigma}_y \otimes \hat{\sigma}_x (|\psi\rangle_3 \otimes |\psi\rangle_1) &= \hat{\sigma}_y |\psi\rangle_3 \otimes \hat{\sigma}_x |\psi\rangle_1 \\
&= \big(i\cos\frac{3\pi}{4}|1\rangle + \sin\frac{3\pi}{4}|0\rangle\big) \otimes \big(\frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle\big) \\
&= i\frac{1}{\sqrt{2}} \cdot \cos\frac{3\pi}{4}|1\rangle \otimes |1\rangle - i\frac{1}{\sqrt{2}} \cdot \cos\frac{3\pi}{4}|1\rangle \otimes |0\rangle \\
&+ \frac{1}{\sqrt{2}} \cdot \sin\frac{3\pi}{4}|0\rangle \otimes |1\rangle - \frac{1}{\sqrt{2}} \cdot \sin\frac{3\pi}{4}|0\rangle \otimes |0\rangle \\
&= \frac{i}{\sqrt{2}} \cdot \cos\frac{3\pi}{4}|11\rangle - \frac{i}{\sqrt{2}} \cdot \cos\frac{3\pi}{4}|10\rangle \\
&+ \frac{1}{\sqrt{2}} \cdot \sin\frac{3\pi}{4}|01\rangle - \frac{1}{\sqrt{2}} \cdot \sin\frac{3\pi}{4}|00\rangle
\end{aligned}$$

3. $\hat{\sigma}_z \otimes \hat{\sigma}_x \otimes \hat{H}(|\psi\rangle_1 \otimes |\psi\rangle_2 \otimes |\psi\rangle_3)$
   Solution:

$$\hat{\sigma}_z \otimes \hat{\sigma}_x \otimes \hat{H}(|\psi\rangle_1 \otimes |\psi\rangle_2 \otimes |\psi\rangle_3) = \hat{\sigma}_z|\psi\rangle_1 \otimes \hat{\sigma}_x|\psi\rangle_2 \otimes \hat{H}|\psi\rangle_3$$

We find the following partial results,

$$\begin{aligned}
\hat{\sigma}_z|\psi\rangle_1 &= \big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\big(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\big) \\
&= \frac{1}{\sqrt{2}}\big(|0\rangle\langle 0|0\rangle - |1\rangle\langle 1|0\rangle\big) - \frac{1}{\sqrt{2}}\big(|0\rangle\langle 0|1\rangle - |1\rangle\langle 1|1\rangle\big) \\
&= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{\sigma}_x|\psi\rangle_2 &= \big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\big(\frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|1\rangle\big) \\
&= \frac{1}{2}\big(|0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle\big) + i\frac{\sqrt{3}}{2}\big(|0\rangle\langle 1|1\rangle + |1\rangle\langle 0|1\rangle\big) \\
&= \frac{1}{2}|1\rangle + i\frac{\sqrt{3}}{2}|0\rangle
\end{aligned}$$

$$
\begin{aligned}
\hat{H}|\psi\rangle_3 &= \frac{1}{\sqrt{2}}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\big)\big(\cos\frac{3\pi}{4}|0\rangle + i\sin\frac{3\pi}{4}|1\rangle\big) \\
&= \frac{1}{\sqrt{2}}\cos\frac{3\pi}{4}\big(|0\rangle\langle 0|0\rangle + |0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle - |1\rangle\langle 1|0\rangle\big) \\
&+ \frac{i}{\sqrt{2}}\sin\frac{3\pi}{4}\big(|0\rangle\langle 0|1\rangle + |0\rangle\langle 1|1\rangle + |1\rangle\langle 0|1\rangle - |1\rangle\langle 1|1\rangle\big) \\
&= \frac{1}{\sqrt{2}}\cos\frac{3\pi}{4}\big(|0\rangle + |1\rangle\big) + \frac{i}{\sqrt{2}}\sin\frac{3\pi}{4}\big(|0\rangle - |1\rangle\big) \\
&= \frac{1}{\sqrt{2}}\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big)|0\rangle + \frac{1}{\sqrt{2}}\big(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\big)|1\rangle
\end{aligned}
$$

And we can now put everything together,

$$
\begin{aligned}
\hat{\sigma}_z \otimes \hat{\sigma}_x \otimes \hat{H}(|\psi\rangle_1 \otimes |\psi\rangle_2 \otimes |\psi\rangle_3) &= \hat{\sigma}_z|\psi\rangle_1 \otimes \hat{\sigma}_x|\psi\rangle_2 \otimes \hat{H}|\psi\rangle_3 \\
&= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{2}|1\rangle + i\frac{\sqrt{3}}{2}|0\rangle\right) \\
&\otimes \left(\frac{1}{\sqrt{2}}\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big)|0\rangle\right. \\
&\left. + \frac{1}{\sqrt{2}}\big(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\big)|1\rangle\right) \\
&= \left(\frac{1}{2\sqrt{2}}|0\rangle \otimes |1\rangle + \frac{i\sqrt{3}}{2\sqrt{2}}|0\rangle \otimes |0\rangle\right. \\
&\left. + \frac{1}{2\sqrt{2}}|1\rangle \otimes |1\rangle + \frac{i\sqrt{3}}{2\sqrt{2}}|1\rangle \otimes |0\rangle\right) \\
&\otimes \left(\frac{1}{\sqrt{2}}\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big)|0\rangle\right. \\
&\left. + \frac{1}{\sqrt{2}}\big(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\big)|1\rangle\right) \\
&= \frac{1}{4}\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big)|0\rangle \otimes |1\rangle \otimes |0\rangle \\
&+ \frac{1}{4}\big(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\big)|0\rangle \otimes |1\rangle \otimes |1\rangle \\
&+ \frac{i\sqrt{3}}{4}\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big)|0\rangle \otimes |0\rangle \otimes |0\rangle \\
&+ \frac{i\sqrt{3}}{4}\big(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\big)|0\rangle \otimes |0\rangle \otimes |1\rangle \\
&+ \frac{1}{4}\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big)|1\rangle \otimes |1\rangle \otimes |0\rangle \\
&+ \frac{1}{4}\big(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\big)|1\rangle \otimes |1\rangle \otimes |1\rangle \\
&+ \frac{i\sqrt{3}}{4}\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big)|1\rangle \otimes |0\rangle \otimes |0\rangle \\
&+ \frac{i\sqrt{3}}{4}\big(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\big)|1\rangle \otimes |0\rangle \otimes |1\rangle \\
&= \frac{1}{4}\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big)|010\rangle + \frac{1}{4}\big(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\big)|011\rangle \\
&+ \frac{i\sqrt{3}}{4}\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big)|000\rangle + \frac{i\sqrt{3}}{4}\big(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\big)|001\rangle \\
&+ \frac{1}{4}\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big)|110\rangle + \frac{1}{4}\big(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\big)|111\rangle \\
&+ \frac{i\sqrt{3}}{4}\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big)|100\rangle + \frac{i\sqrt{3}}{4}\big(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\big)|101\rangle
\end{aligned}
$$

Exercise 3. Let $\hat{H}$ be the Hadamard operator. Prove that

$$\hat{H}^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle$$

Solution:

We will solve this exercise using mathematical induction. To do so, first let us see that when $n = 1$, we have

$$\begin{aligned}
\hat{H}|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle \\
&= \frac{1}{\sqrt{2}}(|0\rangle\langle 0|0\rangle + |0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle - |1\rangle\langle 1|0\rangle) \\
&= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle,
\end{aligned}$$

which is the same as

$$\begin{aligned}
\frac{1}{\sqrt{2^1}} \sum_{i=0}^{2^1-1} |i\rangle &= \frac{1}{\sqrt{2}} \sum_{i=0}^{1} |i\rangle \\
&= \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) \\
&= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.
\end{aligned}$$

We have thus proven the base case, $n = 1$.

At this point, we use the induction hypothesis, that is, we suppose that

$$\hat{H}^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle.$$

We now have to prove that

$$\hat{H}^{\otimes n+1}|0\rangle^{\otimes n+1} = \frac{1}{\sqrt{2^{n+1}}} \sum_{i=0}^{2^{n+1}-1} |i\rangle.$$

$$
\begin{aligned}
\hat{H}^{\otimes n+1}|0\rangle^{\otimes n+1} &= \hat{H}^{\otimes n} \otimes \hat{H}\big(|0\rangle^{\otimes n} \otimes |0\rangle\big) \\
&= \hat{H}^{\otimes n}|0\rangle^{\otimes n} \otimes \hat{H}|0\rangle \\
&= \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle \otimes \big(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\big) \\
&= \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{2^{n/2}} \cdot \frac{1}{2^{1/2}} \sum_{i=0}^{2^n-1} |i\rangle \otimes \big(|0\rangle + |1\rangle\big) \\
&= \frac{1}{2^{(n+1)/2}} \sum_{i=0}^{2^{n+1}-1} |i\rangle \\
&= \frac{1}{\sqrt{2^{n+1}}} \sum_{i=0}^{2^{n+1}-1} |i\rangle,
\end{aligned}
$$

where in the line in teal we have used both the base case and the induction hypothesis.

Exercise 4. Let $\hat{B}$ denote the Bell state circuit. Prove the following:

1. $\hat{B}|00\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$

   Solution:

   Let us call the initial state of the system $|\psi\rangle_0 = |00\rangle = |0\rangle| \otimes |0\rangle$. After applying the Hadamard gate, the first gate of the Bell circuit, we obtain the new state, $|\psi\rangle_1$,

$$
\begin{aligned}
|\psi\rangle_1 &= (\hat{H} \otimes \hat{\mathbb{1}})|\psi\rangle_0 \\
&= (\hat{H} \otimes \hat{\mathbb{1}})(|0\rangle \otimes |0\rangle) \\
&= \hat{H}|0\rangle \otimes \hat{\mathbb{1}}|0\rangle,
\end{aligned}
$$

   where $\hat{\mathbb{1}}$ is the identity operator. As we have seen in the part in orange in Exercise 3,

$$
\hat{H}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle,
$$

which leads us to

$$\begin{aligned}
|\psi\rangle_1 &= (\hat{H} \otimes \hat{\mathbb{1}})(|0\rangle \otimes |0\rangle) \\
&= \hat{H}|0\rangle \otimes \hat{\mathbb{1}}|0\rangle \\
&= \big(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\big) \otimes |0\rangle \\
&= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle.
\end{aligned}$$

Now we apply the CNOT gate, which can be written as

$$\hat{C}_{\text{not}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|.$$

We are in position to calculate the final state of the circuit, $|\psi\rangle_2$,

$$\begin{aligned}
|\psi\rangle_2 &= \hat{C}_{\text{not}}|\psi\rangle_1 \\
&= \big(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|\big)\big(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\big) \\
&= \frac{1}{\sqrt{2}}\big(|00\rangle\langle 00|00\rangle + |01\rangle\langle 01|00\rangle + |10\rangle\langle 11|00\rangle + |11\rangle\langle 10|00\rangle\big) \\
&+ \frac{1}{\sqrt{2}}\big(|00\rangle\langle 00|10\rangle + |01\rangle\langle 01|10\rangle + |10\rangle\langle 11|10\rangle + |11\rangle\langle 10|10\rangle\big) \\
&= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
&= \frac{|00\rangle + |11\rangle}{\sqrt{2}},
\end{aligned}$$

as was asked in the exercise.

2. $\hat{B}|01\rangle = \frac{|01\rangle+|10\rangle}{\sqrt{2}}$

   Solution:

   We will proceed as in the previous exercise, but now the initial state is $|\psi\rangle_0 = |01\rangle = |0\rangle \otimes |1\rangle$. The state $|\psi\rangle_1$, obtained after applying the Hadamard gate to the first qubit is given by

$$\begin{aligned}
|\psi\rangle_1 &= (\hat{H} \otimes \hat{\mathbb{1}})|\psi\rangle_0 \\
&= (\hat{H} \otimes \hat{\mathbb{1}})(|0\rangle \otimes |0\rangle) \\
&= \hat{H}|0\rangle \otimes \hat{\mathbb{1}}|1\rangle \\
&= \big(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\big) \otimes |1\rangle \\
&= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle,
\end{aligned}$$

   where we have used the expression in orange to write $\hat{H}|0\rangle$.

Next, we apply the CNOT gate to obtain the new state, $|\psi\rangle_2$,

$$
\begin{aligned}
|\psi\rangle_2 &= \hat{C}_{\text{not}}|\psi\rangle_1 \\
&= \big(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|\big)\big(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle\big) \\
&= \frac{1}{\sqrt{2}}\big(|00\rangle\langle 00|01\rangle + |01\rangle\langle 01|01\rangle + |10\rangle\langle 11|01\rangle + |11\rangle\langle 10|01\rangle\big) \\
&+ \frac{1}{\sqrt{2}}\big(|00\rangle\langle 00|11\rangle + |01\rangle\langle 01|11\rangle + |10\rangle\langle 11|11\rangle + |11\rangle\langle 10|11\rangle\big) \\
&= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\
&= \frac{|01\rangle + |10\rangle}{\sqrt{2}}.
\end{aligned}
$$