Complex Numbers and Fields

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Complex Numbers

Complex numbers are mathematical entities widely used in physics and engineering. Most likely, the first time a student needs to learn about complex numbers is when solving polynomial equations.

For instance, let us try to solve the following quadratic equations:

Complex Numbers

$$x^2 + 1 = 0 \tag{1}$$

Since $a = 1, b = 0$ and $c = 1$, we know that

$$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)}, \text{ i.e. } x = \pm\frac{\sqrt{-4}}{2}$$

Complex Numbers

$$x^2 - 3x + 18 = 0 \tag{2}$$

In this case, $a = 1, b = -3$ and $c = 18$, then

$$x = \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(18)}}{2(1)}, \text{ i.e. } x = \frac{-3 \pm \sqrt{-63}}{2}$$

Complex Numbers

In both cases, we need to calculate the square root of a negative number.

We are in a difficult situation because we want to solve Eqs. (1,2) **but we know that any real number raised to the second power must be a non-negative real number**, i.e.

$$\forall x \in \mathbb{R} \Rightarrow x^2 \geq 0$$

So, if we want to solve polynomial equations which involve computing square roots of negative numbers, we need to go beyond the real number system. An answer to this problem is the complex number system.

Complex Numbers

Imaginary unit. We define the imaginary unit $i = \sqrt{-1}$. With this new tool, we can now write

$$\pm\frac{\sqrt{-4}}{2} \text{ as } \pm\frac{\sqrt{4(-1)}}{2} = \pm\frac{\sqrt{4}\sqrt{-1}}{2} = \pm i$$

Complex Numbers

The set of complex numbers $\mathbb{C}$. Let $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

Any number of the form

$$z = a + bi$$

is known as a **complex number**, where $i = \sqrt{-1}$ is the imaginary unit and a, b are the real and imaginary parts of z, respectively. The set of all complex numbers is denoted by $\mathbb{C}$.

Complex Numbers

Addition, subtraction and multiplication of complex numbers.

Let $z_1, z_2 \in \mathbb{C}$, where $z_1 = a + bi$ and $z_2 = c + di$. We define addition, subtraction and multiplication in $\mathbb{C}$ as follows:

Complex Numbers

Addition in $\mathbb{C}$.
Let $z_1, z_2 \in \mathbb{C}$, where $z_1 = a + bi$ and $z_2 = c + di$. We define addition in $\mathbb{C}$ as follows:

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$$

Note that $z_1 + z_2 \in \mathbb{C}$.

Complex Numbers

Subtraction in $\mathbb{C}$.
Let $z_1, z_2 \in \mathbb{C}$, where $z_1 = a + bi$ and $z_2 = c + di$. We define addition, subtraction and multiplication in $\mathbb{C}$ as follows:

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$$

Note that $z_1 - z_2 \in \mathbb{C}$.

Complex Numbers

Multiplication in $\mathbb{C}$ (1/2).

Let $z_1, z_2 \in \mathbb{C}$, where $z_1 = a + bi$ and $z_2 = c + di$. We define addition, subtraction and multiplication in $\mathbb{C}$ as follows:

$$z_1 z_2 = (a+bi)(c+di) = ac+adi+bci+bdi^2 = ac-bd+(ad+bc)i$$

Note that $i = \sqrt{-1} \Rightarrow i^2 = -1$. Also, notice that multiplication of complex numbers does resemble multiplication of binomials, with the additional feature of $i^2 = -1$.

Complex Numbers

Multiplication in $\mathbb{C}$ (2/2).

In particular, note that

$$iz_1 = (0+i)(a+bi) = 0*a+ai+0*bi+bi^2 = 0-b+ai = -b+ai$$

That is,

$$i(a+bi) = -b+ai$$

Complex Numbers

Examples

Let $z_1 = 2 - i$ and $z_2 = -7 + 3i$. Then

$$\begin{aligned} z_1 + z_2 &= (2 - i) + (-7 + 3i) \\ &= (2 - 7) + (-1 + 3)i \\ &= -5 + 2i \end{aligned}$$

Complex Numbers

$$z_1 = 2 - i$$
$$z_2 = -7 + 3i$$

$$\begin{aligned} z_1 - z_2 &= (2 - i) - (-7 + 3i) \\ &= (2 - (-7)) + (-1 - 3)i \\ &= (2 + 7) + (-1 - 3)i \\ &= 9 - 4i \end{aligned}$$

Complex Numbers

$$z_1 = 2 - i$$
$$z_2 = -7 + 3i$$

$$\begin{aligned} z_1 z_2 &= (2 - i)(-7 + 3i) \\ &= (2)(-7) + (2)(3i) + (-1)(-7)i + (-1)(3)i^2 \\ &= -14 + 6i + 7i - 3(-1) \\ &= -14 + 6i + 7i + 3 \\ &= -11 + 13i \end{aligned}$$

Complex Numbers

Compute

$$i^2 z_1 - i^3 z_2$$

$z_1 = 2 - i$ and $z_2 = -7 + 3i$

Since $i = \sqrt{-1}$ then $i^2 = -1$ and $i^3 = i^2 * i = -i$

Then

$$\begin{aligned} i^2 z_1 - i^3 z_2 &= (-1)(2-i) - (-i)(-7+3i) \\ &= (-2+i) + (-1)(-1)((-7)i + (3)i^2) \\ &= (-2+i) + (-7i-3) \\ &= (-2-3) + (i-7i) \\ &= -5 - 6i \end{aligned}$$

Complex Numbers

Complex Conjugation and Modulus.

Let $z \in \mathbb{C}$, where $z = a + bi$. We define the complex conjugate and the modulus of z as follows:

Complex conjugation.

$$\bar{z} = a - bi \text{ is the complex conjugate of } z$$

Modulus.

$$||z|| = \sqrt{a^2 + b^2}$$

Complex Numbers

Example. Let $z = 5 - \pi i$. Then

$$\bar{z} = 5 + \pi i$$

$$||z|| = \sqrt{5^2 + (-\pi)^2} = \sqrt{25 + \pi^2}$$

Complex Numbers

One more example. Let $z = a + bi$. Then

$$z\bar{z} = (a+bi)(a-bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2$$

In other words, $z\bar{z} = ||z||^2$.

Complex Numbers

Complex division. Let $z_1, z_2 \in \mathbb{C}$, where $z_1 = a + bi$ and $z_2 = c + di$. We define the division operation as follows

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Note that $\frac{z_1}{z_2} = \frac{a+bi}{c+di}\frac{c-di}{c-di} = \frac{ac+bci-adi-bdi^2}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$.

Complex Numbers

Euler's formula. $\forall\ \theta \in \mathbb{R} \Rightarrow$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Complex Numbers

Exercises for Euler's formula.

Compute:

a) $e^{i\pi/2}$

b) $e^{i\pi}$

a) $e^{i2\pi}$

Complex Numbers

Exercises for Euler's formula.

a) $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

b) $e^{i\pi} = \cos\pi + i\sin\pi = -1$

a) $e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1$

Complex Numbers

Exercises for Euler's formula.

Prove that

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$$

using the following trigonometric identities:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

Complex Numbers

We know that

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{i\phi} &= \cos\phi + i\sin\phi \\ e^{i(\theta+\phi)} &= \cos(\theta+\phi) + i\sin(\theta+\phi) \end{aligned}$$

Now,

$$\begin{aligned} e^{i\theta}e^{i\phi} &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= \cos\theta\cos\phi + i\sin\theta\cos\phi + i\cos\theta\sin\phi + i^2\sin\theta\sin\phi \\ &= \cos\theta\cos\phi + i\sin\theta\cos\phi + i\cos\theta\sin\phi - \sin\theta\sin\phi \\ &= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \sin\phi\cos\theta) \\ &= \cos(\theta+\phi) + i\sin(\theta+\phi) \\ &= e^{i(\theta+\phi)} \end{aligned}$$

So,

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)} \qquad \textit{Q.E.D.}$$

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Fields

Field. A field $(\mathbb{F}, +, \cdot)$ is a set $\mathbb{F}$ with two operations known as addition $(+)$ and multiplication $(\cdot)$ that satisfies the following properties:

Fields

1. **Closure under addition.**
   $\forall\ x, y \in \mathbb{F} \ \Rightarrow \ x + y \in \mathbb{F}$.
2. **Commutativity of addition.**
   $\forall\ x, y \in \mathbb{F} \ \Rightarrow \ x + y = y + x$.
3. **Associativity of Addition.**
   $\forall\ x, y, z \in \mathbb{F} \ \Rightarrow \ x + (y + z) = (x + y) + z$.
4. **Additive identity.**
   $\exists\ 0 \in \mathbb{F}$ such that $\forall\ x \in \mathbb{F} \ \Rightarrow \ x + 0 = 0 + x = x$.
5. **Additive inverses.**
   For each $x \in \mathbb{F}\ \exists\ -x \in \mathbb{F}$ such that $x + (-x) = -x + x = 0$.

Fields

6. **Closure under multiplication.**
   $\forall\ x, y \in \mathbb{F} \ \Rightarrow\ xy \in \mathbb{F}$.
7. **Commutativity of multiplication.**
   $\forall\ x, y \in \mathbb{F} \ \Rightarrow\ xy = yx$.
8. **Associativity of multiplication.**
   $\forall\ x, y, z \in \mathbb{F} \ \Rightarrow\ x(yz) = (xy)z$.
9. **Multiplicative identity.**
   $\exists\ 1 \in \mathbb{F}$ such that $\forall\ x \in \mathbb{F} \ \Rightarrow\ x1 = 1x = x$.
10. **Multiplicative inverses.**
    For each $x \in \mathbb{F} - \{0\}$ $\exists\ x^{-1} \in \mathbb{F}$ such that $xx^{-1} = x^{-1}x = 1$.

Fields

11. **Distributivity of Multiplication over Addition.**
$\forall\, x, y, z \in \mathbb{F} \;\Rightarrow\; x(y+z) = xy + xz$.

$\mathbb{C}$ is a field

Example. Prove that $\mathbb{C}$ is a field.

$\mathbb{C}$ is a field

1. Closure under addition.

$$\begin{aligned} z_1 + z_2 &= (a+bi)+(c+di) \\ &= (a+c)+(b+d)i \\ &= \alpha + \beta i \end{aligned}$$

Since $\mathbb{R}$ is closed under addition, $\alpha, \beta \in \mathbb{R} \Rightarrow \alpha + \beta i \in \mathbb{C}$, i.e. $z_1 + z_2 \in \mathbb{C}$.

$\mathbb{C}$ is a field

2. Commutativity of addition.

$$\begin{aligned}
z_1 + z_2 &= (a+bi) + (c+di) \\
&= (a+c) + (b+d)i \\
&= (c+a) + (d+b)i \qquad \text{(commutativity of } \mathbb{R} \text{ under addition)} \\
&= (c+di) + (a+bi) \\
&= z_2 + z_1
\end{aligned}$$

Therefore $z_1 + z_2 = z_2 + z_1$.

$\mathbb{C}$ is a field

3. Associativity of Addition. In addition to z_1, z_2 as previously introduced, let us define $z_3 = e + fi$. Then,

$$\begin{aligned}
z_1 + (z_2 + z_3) &= (a + bi) + ((c + di) + (e + fi)) \\
&= (a + bi) + ((c + e) + (d + f)i) \\
&= (a + (c + e)) + (b + (d + f))i \\
&= ((a + c) + e) + ((b + d) + f))i \qquad \text{(associativity of } \mathbb{R} \text{ under +)} \\
&= ((a + c) + (b + d)i) + (e + fi) \\
&= (z_1 + z_2) + z_3
\end{aligned}$$

Therefore $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

$\mathbb{C}$ is a field

4. Additive identity.

$$
\begin{aligned}
z_1 + (0 + 0i) &= (a + bi) + (0 + 0i) \\
&= (a + 0) + (b + 0)i \\
&= (0 + a) + (0 + b)i && \text{(commutativity of } \mathbb{R} \text{ under +)} \\
&= (0 + 0i) + (a + bi) \\
&= (0 + 0i) + z_1 \\
&= z_1
\end{aligned}
$$

So, $0 + 0i = 0$ is an additive identity of $\mathbb{C}$ (in fact, it is unique but this is to proved later).

$\mathbb{C}$ is a field

5. Additive inverses. For each $z = a + bi$ we define $-z = -a - bi$. So,

$$\begin{aligned}
z + (-z) &= (a + bi) + (-1)(a + bi) \\
&= (a + bi) + (-a - bi) \\
&= (a + (-a)) + (b + (-b))i \\
&= (-a + a) + (-b + b)i && \text{(commutativity of } \mathbb{R} \text{ under addition)} \\
&= (-a - bi) + (a + bi) \\
&= -z + z \\
&= (-a + a) + (-b + b)i \\
&= 0 + 0i
\end{aligned}$$

So, for each $z \in \mathbb{C}$ there is a $-z \in \mathbb{C}$ such that $z + (-z) = -z + z = 0 + 0i$.

$\mathbb{C}$ is a field

6. Closure under multiplication.

$$\begin{aligned} z_1 z_2 &= (a+bi)(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac-bd) + (ad+bc)i \\ &= \alpha + \beta i \end{aligned}$$

Since $\mathbb{R}$ is closed under addition and multiplication, $\alpha, \beta \in \mathbb{R}$ $\Rightarrow \alpha + \beta i \in \mathbb{C}$, i.e. $z_1 z_2 \in \mathbb{C}$.

$\mathbb{C}$ is a field

7. Commutativity of multiplication.

$$
\begin{aligned}
z_1 z_2 &= (a+bi)(c+di) \\
&= ac + adi + bci + bdi^2 \\
&= (ac-bd) + (ad+bc)i \\
&= ca - db + (cb+da)i \\
&= z_2 z_1
\end{aligned}
$$

Therefore $z_1 z_2 = z_2 z_1$.

$\mathbb{C}$ is a field

Associativity of multiplication. In addition to z_1, z_2 as previously introduced, let us define $z_3 = e + fi$. Then,

$$
\begin{aligned}
z_1(z_2z_3) &= (a+bi)((c+di)(e+fi)) \\
&= (a+bi)(ce+cfi+dei+dfi^2) \\
&= (a+bi)((ce-df)+(cf+de)i) \\
&= a(ce-df)+a(cf+de)i+b(ce-df)i+b(cf+de)i^2 \\
&= a(ce-df)-b(cf+de)+(a(cf+de)+b(ce-df))i \\
&= (ace-adf-bcf-bde)+(acf+ade+bce-bdf)i
\end{aligned}
$$

Furthermore,

$\mathbb{C}$ is a field

$$
\begin{aligned}
(z_1 z_2) z_3 &= ((a+bi)(c+di))(e+fi) \\
&= (ac+adi+bci+bdi^2)(e+fi) \\
&= ((ac-bd)+(ad+bc)i)(e+fi) \\
&= (ac-bd)e+(ad+bc)ei+(ac-bd)fi+(ad+bc)fi^2 \\
&= (ace-bde)+(ade+bce)i+(acf-bdf)i-(adf+bcf) \\
&= (ace-bde)+(ade+bce)i+(acf-bdf)i-(adf+bcf) \\
&= (ace-bde-adf-bcf)+(ade+bce+acf-bdf)i
\end{aligned}
$$

Since $(ace-adf-bcf-bde)+(acf+ade+bce-bdf)i = (ace-bde-adf-bcf)+(ade+bce+acf-bdf)i \Rightarrow z_1(z_2 z_3) = (z_1 z_2) z_3$.

$\mathbb{C}$ is a field

Multiplicative identity, **Multiplicative inverses**, and **Distributivity of Multiplication over Addition**.

Left as exercises to the audience (and to be reviewed next lecture).