El cálculo de Secuentes

H_1
H_2
H_3

$$\frac{H}{P}$$

C

1° $H \rightsquigarrow P$

2° $P \rightsquigarrow H$

3° $H \rightsquigarrow Z$
$P \rightsquigarrow Z$

Derivaciones

$$\underbrace{\Gamma \vdash A}_{\text{Secuente}}$$

"de gamma se deriva A"

$\Gamma \cup \{P\}$ se escribe Γ, P

Γ es un conjunto de fórmulas (Hipótesis) — contexto

"lo que sabemos que es verdadero"

A es la conclusión "lo que queremos demostrar"

Reglas Derechas

Hipótesis

$$\frac{}{\Gamma, A \vdash A}$$

Conjunción

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

Implicación

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

Disyunción

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

Reglas Izquierdas

Conjunción

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

Implicación

$$\frac{\Gamma, A \to B \vdash A}{\Gamma, A \to B \vdash B}$$

Disyunción

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

Árbol de derivación (Demostración)

$$\dfrac{\dfrac{}{p \land q \to r, p, q \vdash p}\,\text{Hip} \qquad \dfrac{}{p \land q \to r, p, q \vdash q}\,\text{Hip}}{p \land q \to r, p, q \vdash p \land q}\,(\land R)$$

$$\dfrac{p \land q \to r, p, q \vdash p \land q}{p \land q \to r, p, q \vdash r}\,(\to L)$$

$$\dfrac{p \land q \to r, p, q \vdash r}{p \land q \to r, p \vdash q \to r}\,(\to R)$$

$$\dfrac{p \land q \to r, p \vdash q \to r}{p \land q \to r \vdash p \to q \to r}\,(\to R)$$

$$\dfrac{p \land q \to r \vdash p \to q \to r}{\vdash (p \land q \to r) \to p \to q \to r}\,(\to R)$$

$$\Gamma = \{\, p \to q \vee r,\ q \to r,\ r \to s \,\}$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\Gamma, p \vdash p}\,\text{Hip}}{\Gamma, p \vdash q \vee r}\,(\to L)
\qquad
\dfrac{\dfrac{}{\Gamma, p, q \vdash q}\,\text{Hip} \qquad \dfrac{}{\Gamma, p, r \vdash q}}{\Gamma, p, q \vee r \vdash q}\,(\vee L)
}{\Gamma, p \vdash q}\,(\text{cut})
}{\Gamma, p \vdash r}\,(\to L)
}{\Gamma, p \vdash s}\,(\to L)
}{\Gamma \vdash p \to s}\,(\to R)
$$

Regla de Corte "demuestra lo que quieres usar"

$$\frac{\Gamma \vdash A \qquad \Gamma, A \vdash C}{\Gamma \vdash C}\ \text{cut}$$

Debilitamiento

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\ (\text{weak})$$

$$\Gamma = \{p \to q \vee r,\ q \to r,\ r \to s\}$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Hip}
\UnaryInfC{$\Gamma, p \vdash p$}
\RightLabel{$(\to L)$}
\UnaryInfC{$\Gamma, p \vdash q \vee r$}
\AxiomC{}
\RightLabel{Hip}
\UnaryInfC{$\Gamma, p, q \vdash q$}
\RightLabel{$(\to L)$}
\UnaryInfC{$\Gamma, p, q \vdash r$}
\RightLabel{$(\to L)$}
\UnaryInfC{$\Gamma, p, q \vdash s$}
\AxiomC{}
\RightLabel{Hip}
\UnaryInfC{$\Gamma, p, r \vdash r$}
\RightLabel{$(\to L)$}
\UnaryInfC{$\Gamma, p, r \vdash s$}
\RightLabel{$(\vee L)$}
\BinaryInfC{$\Gamma, p, q \vee r \vdash s$}
\RightLabel{cut}
\BinaryInfC{$\Gamma, p \vdash s$}
\RightLabel{$(\to R)$}
\UnaryInfC{$\Gamma \vdash p \to s$}
\end{prooftree}
$$

Negación

Swap

$$f(x, g(h(a), y)) = x$$

$$a = x$$

swap

$$x = a$$

Swap

$$x = f(x, g(h(a), y))$$

$$f(x, y) = a \quad ✗$$

$$f(x, y) = g(x) \quad ✗$$

Negación

- Lógica minimal
- lógica intuicionista
- Lógica clasica

Lógica minimal

$$\neg A =_{def} A \to \bot$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A,\ A \to \bot \vdash_m A}\ (\text{Hip})}{A,\ A \to \bot \vdash_m \bot}\ (\to L)}{A,\ \neg A \vdash_m \bot}\ (\text{def } \neg)}{A \vdash_m \neg A \to \bot}\ (\to R)}{A \vdash_m \neg\neg A}\ (\text{def } \neg)}{\vdash_m A \to \neg\neg A}\ (\to R)$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Hip)}
\UnaryInfC{$(A \vee B) \to \bot, A \vdash_m A$}
\RightLabel{$(\vee R)$}
\UnaryInfC{$(A \vee B) \to \bot, A \vdash_m A \vee B$}
\RightLabel{$(\to L)$}
\UnaryInfC{$(A \vee B) \to \bot, A \vdash_m \bot$}
\RightLabel{$(\neg)$}
\UnaryInfC{$\neg(A \vee B), A \vdash_m \bot$}
\RightLabel{$(\to R)$}
\UnaryInfC{$\neg(A \vee B) \vdash_m A \to \bot$}
\RightLabel{$(\neg)$}
\UnaryInfC{$\neg(A \vee B) \vdash_m \neg A$}
\AxiomC{}
\RightLabel{Hip}
\UnaryInfC{$(A \vee B) \to \bot, B \vdash_m B$}
\RightLabel{$(\vee R)$}
\UnaryInfC{$(A \vee B) \to \bot, B \vdash_m A \vee B$}
\RightLabel{$(\to L)$}
\UnaryInfC{$(A \vee B) \to \bot, B \vdash_m \bot$}
\RightLabel{$(\to R)$}
\UnaryInfC{$(A \vee B) \to \bot \vdash_m B \to \bot$}
\UnaryInfC{$\neg(A \vee B) \vdash_m \neg B$}
\BinaryInfC{$\neg(A \vee B) \vdash_m \neg A \wedge \neg B$}
\RightLabel{$(\to R)$}
\UnaryInfC{$\vdash_m \neg(A \vee B) \to \neg A \wedge \neg B$}
\end{prooftree}
$$

$$
\begin{array}{cl}
\hline & \text{Hip} \\
A \to B \to \bot ,\ B,\ A \vdash B & \\
\hline & (\to R) \\
A \to B \to \bot ,\ B,\ A \vdash A \to B & \\
\hline & (\to L) \\
A \to B \to \bot ,\ B,\ A \vdash \bot & \\
\hline & (\to R) \\
A \to B \to \bot ,\ B \vdash_m A \to \bot & \\
\hline & (\neg) \\
A \to \neg B ,\ B \vdash_m \neg A & \\
\hline & (\to R) \\
A \to \neg B \vdash_m B \to \neg A & \\
\hline & (\to R) \\
\vdash_m (A \to \neg B) \to (B \to \neg A) &
\end{array}
$$

Lógica Intuicionista (minimal + nuevas reglas)

ex-falso

$$\frac{}{\Gamma, \bot \vdash C}$$

Explosión

$$\frac{}{\Gamma, A, \neg A \vdash C}$$

$$
\dfrac{\dfrac{\dfrac{\overline{\Gamma, A, A \to \bot \vdash_i A}\ \text{Hip}}{\Gamma, A, A \to \bot \vdash_i \bot}\ (L \to)}{\Gamma, A, \neg A \vdash_i \bot}\ (\neg)}{\Gamma, A, \neg A \vdash_i C}\ \text{Ex falso}
$$

Silogismo Disyuntivo

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\neg A , A \vdash_i B}\,\text{Exp}
      \qquad
      \dfrac{}{\neg A , B \vdash_i B}\,\text{Hip}
    }{\neg A , A \vee B \vdash_i B}\,(L\vee)
  }{\neg A \vdash_i A \vee B \rightarrow B}\,(R\rightarrow)
}{\vdash_i \neg A \rightarrow A \vee B \rightarrow B}\,(\rightarrow R)
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A ,\ A \vdash_i B}\ \text{exp –} \qquad \dfrac{}{B ,\ A \vdash_i B}\ \text{Hip}}{\neg A \lor B ,\ A \vdash_i B}\ (L\lor)}{\neg A \lor B \vdash_i A \to B}\ (R\to)}{\vdash_i \neg A \lor B \to (A \to B)}\ (R\to)
$$

Todo lo derivable en minimal es también derivable en intuicionista

Pero no todo lo derivable en intuicionista es derivable en minimal

Lógica Clásica

Tercero excluido

$$A \vee \neg A$$

$$\dfrac{\dfrac{\dfrac{A \vdash_i \bot}{\vdash_i A \to \bot}\,(R\to)}{\vdash_i \neg A}\,(\neg)}{\vdash_i A \vee \neg A}\,(R\vee)$$

Tercero excluido

$$\frac{}{\Gamma \vdash A \vee \neg A}\ (TER)$$

$$\frac{\Gamma, A \vee \neg A \vdash C}{\Gamma \vdash C}\ (TEL)$$

Reducción al absurdo

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}\ (RA)$$

Eliminación de la doble negación

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}\ (\neg\neg E)$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A}\,\text{Hip} \qquad \dfrac{}{\Gamma \vdash B}\,\text{Hip}}{\Gamma \vdash A \wedge B} \qquad \dfrac{}{\Gamma, A \wedge B \vdash \neg A \vee \neg B}\,\text{exp}}{\Gamma = \neg(A \wedge B), A, B \vdash \neg A \vee \neg B}\,\text{cut}$$

$$\dfrac{\dfrac{}{\neg(A \wedge B), A, \neg B \vdash \neg B}\,(\text{Hip})}{\neg(A \wedge B), A, \neg B \vdash \neg A \vee \neg B}\,(\text{Rv})$$

$$\dfrac{\neg(A \wedge B), A, B \vdash \neg A \vee \neg B \qquad \neg(A \wedge B), A, \neg B \vdash \neg A \vee \neg B}{\neg(A \wedge B), A, B \vee \neg B \vdash \neg A \vee \neg B}\,(\text{Lv})$$

$$\dfrac{\neg(A \wedge B), A, B \vee \neg B \vdash \neg A \vee \neg B}{\neg(A \wedge B), A \vdash \neg A \vee \neg B}\,(\text{TEL})$$

$$\dfrac{\dfrac{}{\neg(A \wedge B), \neg A \vdash \neg A}\,(\text{Hip})}{\neg(A \wedge B), \neg A \vdash \neg A \vee \neg B}\,(\text{Rv})$$

$$\dfrac{\neg(A \wedge B), A \vdash \neg A \vee \neg B \qquad \neg(A \wedge B), \neg A \vdash \neg A \vee \neg B}{\neg(A \wedge B), A \vee \neg A \vdash_c \neg A \vee \neg B}\,(\text{Lv})$$

$$\dfrac{\neg(A \wedge B), A \vee \neg A \vdash_c \neg A \vee \neg B}{\neg(A \wedge B) \vdash_c \neg A \vee \neg B}\,(\text{TEL})$$

$\vdash_m (p \wedge q \to r) \to p \to q \to r$

1.- $\vdash_m (p \wedge q \to r) \to p \to q \to r$ $(\to R)$

2.- $(p \wedge q \to r) \vdash_m p \to q \to r$ $(\to R)$

3.- $(p \wedge q \to r), p \vdash q \to r$ $(\to R)$

4.- $p \wedge q \to r, p, q \vdash r$ $(\to L)$

5.- $p \wedge q \to r, p, q \vdash p \wedge q$ $(\wedge R)$

6.- $p \wedge q \to r, p, q \vdash p ; p \wedge q \to r, p, q \vdash q$ (Hip)

7.- $p \wedge q \to r, p, q \vdash q$ (Hip)

Tacticas (basado en metas)

$$\Gamma \vdash A$$

$$\delta_1 ; \delta_2 ; \ldots \delta_n$$

δ_i es una meta

$$\Gamma \vdash A \rhd \Gamma' \vdash A'$$

$(\to R)$ intro

$$\Gamma \vdash A \to B ; S \quad \rhd \quad \Gamma, A \vdash B ; S$$

$(\wedge R)$ split

$$\Gamma \vdash A \wedge B ; S \rhd \Gamma \vdash A ; \Gamma \vdash B ; S$$

$(\vee R)$ left

$\Gamma \vdash A \vee B ; S \rhd \Gamma \vdash A ; S$

$(\vee R)$ right

$\Gamma \vdash A \vee B ; S \rhd \Gamma \vdash B ; S$

$(\to L)$ apply

$$\Gamma, A \to B \vdash B ; S \rhd \Gamma, A \to B \vdash A ; S$$

$(\wedge L)$ destruct

$$\Gamma, A \wedge B \vdash C ; S \rhd \Gamma, A, B \vdash C ; S$$

($\vee$L) destruct

$$\Gamma, A \vee B \vdash C ; S \rhd \Gamma, A \vdash C ; \Gamma, B \vdash C ; S$$

(Hip) trivial

$$\Gamma, A \vdash A ; S \rhd S$$

(cut) assert

$$\Gamma \vdash C ; S \rhd \Gamma \vdash A ; \Gamma, A \vdash C ; S$$

absurd

$$\Gamma \vdash B \,;\, S \rhd \Gamma \vdash A \,;\, \Gamma \vdash \neg A \,;\, S$$

contradict

$$\Gamma, A \vdash B \,;\, S \rhd \Gamma \vdash \neg A \,;\, S$$

exact

$$\Gamma \vdash A \vee \neg A \,;\, S \rhd S$$

$$\Gamma \vdash \neg\neg A \to A \,;\, S \rhd S$$