Chapter 5: Functions of Vectors and Matrices

$\langle y, Ax \rangle \ \left(= \bar{y}^T Ax\right)$: **"Bilinear Form"**

$\langle x, Ax \rangle \ \left(= \bar{x}^T Ax\right)$: **"Quadratic Form"**

Note that because

$$x^T Ax = \left(x^T Ax\right)^T = x^T A^T x,$$

$$x^T Ax = \frac{1}{2}\left(x^T Ax + x^T A^T x\right) = x^T \left(\frac{A + A^T}{2}\right) x$$

any quadratic form can be written as a quadratic form with a symmetric *A*-matrix. We therefore treat all quadratic forms as is they contained symmetric matrices.

DEFINITIONS: **Let** $Q = \bar{x}^T Ax$

1. ***Q* (or *A*) is *positive definite* iff :** $\langle x, Ax \rangle > 0$ for all $x \neq 0.$
2. ***Q* (or *A*) is *positive semidefinite* if :** $\langle x, Ax \rangle \geq 0$ for all $x \neq 0.$
3. ***Q* (or *A*) is *negative definite* iff:** $\langle x, Ax \rangle < 0$ for all $x \neq 0.$
4. ***Q* (or *A*) is *negative semidefinite* if:** $\langle x, Ax \rangle \leq 0$ for all $x \neq 0.$
5. ***Q* (or *A*) is *indefinite* if:** $\langle x, Ax \rangle > 0$ for some $x \neq 0,$ and
   $\langle x, Ax \rangle < 0$ for other $x \neq 0.$

Tests for definiteness of matrix A in terms of its eigenvalues : λ_i

☞

Matrix *A* is . . .	**If the *real parts* of eigenvalues λ_i of *A* are: . . . .**
1. Positive definite	All > 0
2. Positive semidefinite	All ≥ 0
3. Negative definite	All < 0
4. Negative semidefinite	All ≤ 0
5. Indefinite	Some $\mathrm{Re}(\lambda_i) > 0$, some $\mathrm{Re}(\lambda_i) < 0$.

See book for tests involving *leading principal minors*.

We need to consider functions of matrices before we can solve the state equations in time domain.

Applying a function $f(A)$ **to a matrix** A **is NOT the same thing as applying the function to the matrix entries element-by-element.**

First, define matrix powers:

$$AA = A^2, \ldots \text{ etc.}$$

$$A^0 = I$$

$$A^m A^n = A^{m+n}$$

$$(A^m)^n = A^{mn}$$

$$(A^{-1})^n = A^{-n}$$

Matrix Polynomials:

polynomial form

Matrix Form	**Scalar Form**
$P(A) = c_m A^m + \cdots + c_1 A + c_0 I$	$P(x) = c_m x^m + \cdots + c_1 x + c_0$
$P(A) = c(A - Ia_1)\cdots(A - Ia_m)$	$P(x) = c(x - a_1)\cdots(x - a_m)$

factored form

Convergence of Polynomial Series:

<u>Theorem</u>: Let *A* be an *n* x *n* matrix whose eigenvalues are λ_i **.**
If the infinite series

$$\sigma(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \sum_{i=1}^{\infty} a_k x^k$$

converges for all $x = \lambda_i$**, then . . .**

. . . the series

$$\sigma(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_k A^k + \cdots = \sum_{i=1}^{\infty} a_k A^k$$

converges. This will be important when we want the Taylor series expansions of a function of a matrix.

<u>Theorem</u>: If $f(z)$ is any function (not necessarily a polynomial) whose derivative exists for all z within a circle of the complex plane in which all eigenvalues of matrix A lie, then $f(A)$ can be written as a convergent power series.

<u>Example</u>: Find $\frac{d}{dt}\left(e^{At}\right)$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$

$$\frac{de^{At}}{dt} = A + \frac{2A^2t}{2!} + \frac{3A^3t^2}{3!} + \cdots$$

$$= A\left[I + At + \frac{A^2t^2}{2!} + \cdots\right]$$

$$= Ae^{At} (= e^{At}A)$$

Also note that:

$$\sin(A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \cdots$$

$$\cos(A) = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \cdots$$

. . . etc., same as for expansions of scalar functions.

A much more useful theorem:

<u>Theorem</u>: Let $g(\lambda)$ **be a polynomial of degree n-1 and** $f(\lambda)$ **be ANY function of** λ **. If** $f(\lambda) = g(\lambda)$ **for all eigenvalues of** ***A*** **("on the spectrum of A"), then** $f(A) = g(A)$ **(for** ***A*** **itself.)**

Implication: We can define the matrix-version of a non-polynomial scalar function using a matrix polynomial, if the two functions agree on the spectrum of the matrix!

<u>Example</u>: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

"spectrum of A"={eigenvalues(A)}= $\sigma(A)$ **={1, 2}**

Let $g(\lambda)$ **be our n-1 order polynomial:** $g(\lambda) = \alpha_0 + \alpha_1 \lambda$

Now suppose we are asked to find $f(A) = A^5$

$$f(\lambda) = \lambda^5$$

So we set $f(\lambda) = g(\lambda)$ for $\lambda = \{1,2\}$

$$\text{find } a_0, a_1 \begin{cases} 1^5 = \alpha_0 + \alpha_1 \cdot 1 \\ 2^5 = 32 = \alpha_0 + \alpha_1 \cdot 2 \end{cases}$$

NOTE: If we had repeated eigenvalues, these equations would not be independent. We could instead use the equation AND its derivatives.

Solving, $\alpha_0 = -30, \quad \alpha_1 = 31$

Using this result:

$$A^5 = -30I + 31A = \begin{bmatrix} 1 & 62 \\ 0 & 32 \end{bmatrix}$$

Alternative way of calculating A

<u>Example</u>: Let $A = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix}$ Find a *closed-form* solution for $\sin(A)$. (Can't use Taylor series) $l_1 = -3, l_2 = -2$

(goes on forever)

See previous theorem

This is similar to an earlier example. Because n=2, any analytic function of *A* can be written as a *first* order matrix polynomial, so

n-1 order

$$\sin(A) = \alpha_0 I + \alpha_1 A$$

Evaluate this expression on the spectrum of *A*:

$$\left.\begin{aligned} \sin(-3) &= \alpha_0 + \alpha_1(-3) \\ \sin(-2) &= \alpha_0 + \alpha_1(-2) \end{aligned}\right\} \quad \begin{aligned} a_1 &= -0.768 \\ a_0 &= -2.45 \end{aligned}$$

Solving,

$$\sin(A) = \begin{bmatrix} \sin(-3) & \sin(-2) - \sin(-3) \\ 0 & \sin(-2) \end{bmatrix} = a_0 I + a_1 A$$

If *A* had repeated eigenvalues, the two equations

$$\sin(-3) = \alpha_0 + \alpha_1(-3)$$
$$\sin(-2) = \alpha_0 + \alpha_1(-2)$$

would be linearly dependent and have no unique solution. Then we could use one of them and use a *derivative* for the other:

$$\frac{d}{d\lambda}\left[\sin(\lambda) = \alpha_0 + \alpha_1\lambda\right]$$

so

$$\cos(\lambda) = \alpha_1$$

This would be the second independent equation.

<u>Cayley-Hamilton Theorem:</u> **Let a system have characteristic polynomial**

$$\left|A - \lambda I\right| = \phi(\lambda)$$

Then

$$\phi(A) = 0$$

That is, every matrix satisfies its own characteristic polynomial.

This theorem, together with the previous one, imply that we never need to consider polynomials of a matrix of order higher than *n-1* (!!)

<u>Example</u>: (Reduction of matrix polynomials to degree n-1 or less). Let

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

and find $$P(A) = A^4 + 3A^3 + 2A^2 + A + I.$$

characteristic equation

$$\Delta(\lambda) = \lambda^2 - 5\lambda + 5 = 0$$

$$A^2 = -5I + 5A$$

So from Cayley-Hamilton,

$$A^4 = A^2 A^2 = (5A - 5I)^2 = 25A^2 - 50A + 25I = 25(5A - 5I) - 50A + 25I$$

$$A^3 = A^2 A = (5A - 5I)A = 5A^2 - 5A = 5(5A - 5I) - 5A$$

$$A^2 = 5A - 5I$$

Now *P(A)* will contain no powers of A higher than 1.

(=n-1)

Some examples of what these theorems allow us to do:

<u>Example</u>: Suppose the characteristic polynomial of a system is

$$\phi(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$$

so

$$\phi(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = 0$$

Noting that c_0 is equal to the product of all the eigenvalues, we know it is nonzero iff matrix *A* is non-singular (no zero eigenvalues), or *A* is invertible. Multiply the above equation through by A^{-1} to get:

$$A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I + c_0 A^{-1} = 0$$

Solving

$$A^{-1} = -\frac{1}{c_0}\left[A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I\right]$$

Easy way for computer to find inverse

<u>Definition</u>: The minimal polynomial of a square matrix *A* is the lowest degree monic polynomial $\phi_m(\lambda)$ which satisfies

$$\phi_m(A) = 0$$

Being minimal affects only powers of repeated terms in characteristic polynomials, for example, if

$$\phi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p},$$

$$\phi_m(\lambda) = (\lambda - \lambda_1)^{\eta_1}(\lambda - \lambda_2)^{\eta_2} \cdots (\lambda - \lambda_p)^{\eta_p}$$

where $\eta_i \leq m_i$

How many ones in super diagonal of Jordan Form

Note that η_i is not necessarily 1, but is rather the *index* of the eigenvalue λ_i.

Another important example of this technique will be in the computation of the matrix exponential:

$$e^{At}$$

We will see in the next chapter how important this matrix will be in the solution of the state variable equations for a system.