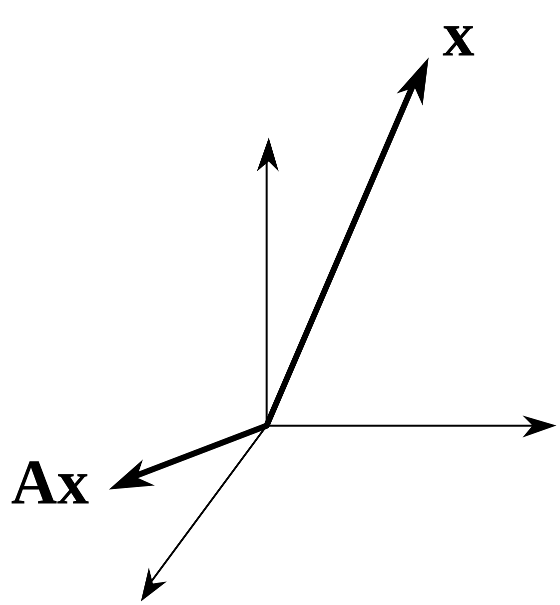
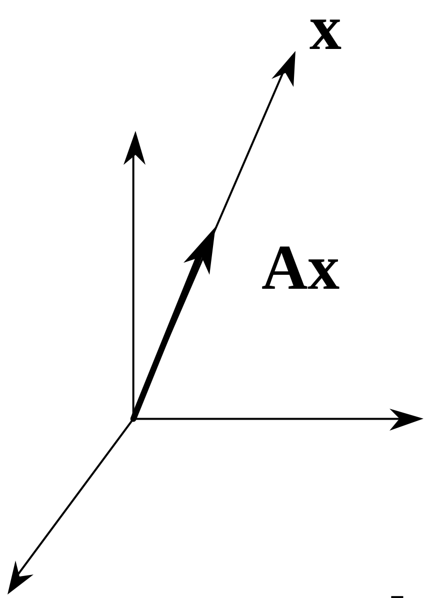
Chapter 4: Eigenvalues and Eigenvectors

Throughout this discussion, we will be dealing with transformations from a space into itself, so all matrices will be square.

Recall that a linear transformation is a rule that makes one vector into another vector.



In special situations, when *A* operates on *x*, we get a scaled version of *x* back again:



So we have $Ax = \lambda x$

This only happens for special x **'s and special** λ **'s.**

When $Ax = \lambda x$

λ **"Eigenvalue"**

x **"Eigenvector"**

For an *n* x *n* matrix A, there are:

- ***n* eigenvalues, some of which may be complex and/or repeated.**
- **At least one eigenvector *corresponding* to each *distinct* eigenvalue. These will be complex if the eigenvalues are complex.**
- **Sometimes repeated eigenvalues have associated with them *generalized eigenvectors*.**

How to find *eigenvalues and eigenvectors*:

$$Ax = \lambda x$$

$$\downarrow$$

$$(A - \lambda I)x = 0$$

We know from the previous chapter that this system of linear equations will have a solution (other than zero) whenever

$$x \in N(A - \lambda I)$$

For this to happen, the *degeneracy* of $A - \lambda I$ must be at least one. Equivalently, the rank of the matrix $A - \lambda I$ must be *less than full (n)*.

When the rank of a square matrix is deficient, then the determinant of that matrix is zero.

So $$|A-\lambda I| = 0 \quad (= |\lambda I - A|)$$

If we do this computation and solve for λ **then we get the eigenvalues. We will always get an *n* th order polynomial in** λ**, so its *n* roots will be the eigenvalues.**

$$\lambda_i, \ \ i = 1, \dots, n$$

We then solve the linear equation $(A-\lambda I)x = 0$ **to get the corresponding *eigenvectors*.**

<u>Example:</u> **Find the eigenvalues and eigenvectors of**

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$|A-\lambda I| = \begin{bmatrix} 3-\lambda & 0 & 2 \\ 0 & 3-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{bmatrix} = (3-\lambda)[(3-\lambda)(1-\lambda)-4]+2[-2(3-\lambda)]$$

$$= -\lambda^3 + 7\lambda^2 - 7\lambda - 15$$

$$= (5-\lambda)(3-\lambda)(-1-\lambda) = 0$$

This happens to be the *characteristic equation* of the system; the eigenvalues are the *poles*!

So the three eigenvalues are: $\lambda_1 = 5, \quad \lambda_2 = 3, \quad \lambda_3 = -1$

Now find the eigenvector associated with λ_1 **:**

$$(A-\lambda I)\big|_{\lambda=\lambda_1=5} x_1 = (A-5I)x_1 = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & -2 \\ 2 & -2 & -4 \end{bmatrix} x_1 = 0$$

How would you solve this?

☞

Using your favorite method; e.g., Gaussian elimination, echelon forms, etc., we can get:

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Because the equation $Ax = \lambda x$ **can be multiplied on both sides by an arbitrary constant and still be true, any scalar multiple of an eigenvector is still an eigenvector. We often express eigenvectors with such a constant:**

$$x_1 = \begin{bmatrix} c \\ -c \\ c \end{bmatrix}$$

Better yet, we normalize all eigenvectors so that they all have length 1:

$$x_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad \text{(corres. to } \lambda_1 = 5\text{)}$$

Identical procedures give us the other two eigenvectors:

$$x_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \quad \text{(corres. to } \lambda_2 = 3\text{)}$$

$$x_3 = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \quad \text{(corres. to } \lambda_3 = -1\text{)}$$

Pause here to explore an application for the eigenvalues/vectors: Consider the previous example.

Note that the eigenvectors are linearly independent. They therefore form a basis for a 3-D vector space.

no input

Suppose we had a (homogeneous) system:

$$\dot{x} = Ax$$

What will happen if we use the eigenvectors as the basis of the vector space that *x* belongs to?

Let $\bar{x}$ denote the state vector in the new basis (x denotes the old state vector). The relationship between x and $\bar{x}$ is:

$$x = M\bar{x}$$

where M is a matrix whose columns are the *new* basis vectors.

When M is formed by columns that are eigenvectors, it is called a *modal matrix*.

Proceeding, $\quad x = M\bar{x}$ so $\dot{x} = M\dot{\bar{x}}$ (substitute)

$$\dot{x} = Ax \longrightarrow M\dot{\bar{x}} = AM\bar{x}$$

☞

$$\boxed{\dot{\overline{x}} = M^{-1}AM\overline{x}}$$

Recall that $M^{-1}AM$ **is called a *similarity transform* on** A**.**

$$M^{-1}AM = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & -2 \\ 2 & -2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}}_{\text{(happens to be orthonormal)}}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Highly significant!!

$$\triangleq \overline{A}$$

These are the eigenvalues of A!

The modal matrix has *diagonalized* the system.

"New" system is

$$\begin{bmatrix}\dot{\bar{x}}_1\\ \dot{\bar{x}}_2\\ \dot{\bar{x}}_3\end{bmatrix} = \bar{A}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 5\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix} = \begin{bmatrix}-\bar{x}_1\\ 3\bar{x}_2\\ 5\bar{x}_3\end{bmatrix}$$

These are three "decoupled" first order linear differential equations.

A more general system would transform as:

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned} \;\longrightarrow\; \boxed{\begin{aligned}x &= M\bar{x}\\ \dot{x} &= M\dot{\bar{x}}\end{aligned}} \;\longrightarrow\; \begin{aligned}\dot{\bar{x}} &= \overbrace{M^{-1}AM}^{\bar{A}}\bar{x} + \overbrace{M^{-1}B}^{\bar{B}}u\\ y &= \underbrace{CM}_{\bar{C}}\bar{x} + \underbrace{D}_{\bar{D}}u\end{aligned}$$

Something different happens when we have one or more eigenvalues that are *repeated* (multiple roots) of the characteristic equation.

<u>Example</u>: Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & 3 & 5 \end{bmatrix}$$

$0 = |A - \lambda I| = (1-\lambda)^2(5-\lambda)$ **So** $\lambda_1 = 5, \quad \lambda_2 = \lambda_3 = 1$

(1 is an eigenvalue of *algebraic multiplicity* 2)

Find the eigenvectors: ☞

Eigenvector corresponding to $\lambda_1 = 5$:

$$(A-5I)x = \begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & 0 \\ -3 & 3 & 0 \end{bmatrix} x = 0 \longrightarrow x_1 = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \left(\text{or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

Now for the eigenvector(s?) corresponding to $\lambda = 1$:

$$(A-I)x = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ -3 & 3 & 4 \end{bmatrix} x = 0 \longrightarrow x_2 = \begin{bmatrix} a \\ 0 \\ \frac{3}{4}a \end{bmatrix}$$

rank=2, so there will be only 1 nontrivial solution.

Three eigenvalues, but only two eigenvectors??

Without three linearly independent eigenvectors, we cannot diagonalize. We can do the next best thing by using "*generalized eigenvectors*."

There are three common ways to compute them:

I. "Bottom-up": For repeated eigenvalue λ_i, find all solutions x_i to:

$$(A - \lambda_i I)x_i = 0$$

(these will be the regular eigenvectors)

Then for each of these x_i's, solve the equation

$$(A - \lambda_i I)x_{i+1} = x_i$$

If you can find x_{i+1}'s that are *linearly independent* of all previous vectors x_i, then the new vectors are generalized eigenvectors.

If the x_{i+1}'s are not linearly independent of previously found vectors, continue on by solving:

$$(A-\lambda_i I)x_{i+2} = x_{i+1}$$

and checking for linear independence. Continue this process until a complete set of *n* vectors are available.

Returning to the example:

$$(A-I)x = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ -3 & 3 & 4 \end{bmatrix} x = 0 \qquad x_1 = \begin{bmatrix} a \\ 0 \\ \frac{3}{4}a \end{bmatrix}$$

Solve: $$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ -3 & 3 & 4 \end{bmatrix} x_2 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \longrightarrow x_2 = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$

This vector is linearly independent of the previous eigenvectors, so it is a generalized eigenvector.

Now form the modal matrix using the two regular eigenvectors and the generalized eigenvector:

Put this in the order you found them

$$M = \begin{bmatrix} 0 & 4 & 5 \\ 0 & 0 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

generalized

regular

Compute similarity transformation:

$$\overline{A} = M^{-1}AM = \begin{bmatrix} -\tfrac{3}{4} & \tfrac{3}{8} & 1 \\ \tfrac{1}{4} & -\tfrac{5}{8} & 0 \\ 0 & \tfrac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 & 4 & 5 \\ 0 & 0 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overline{A} = \left[\begin{array}{c|cc} 5 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

This is called a Jordan (canonical) form. There are two "Jordan blocks." In "block form", this is a "block-diagonal" matrix. A Jordan *block* has the general form:

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & & \cdots & 0 & \lambda \end{bmatrix}$$

for the repeated eigenvalue λ

NOTE that just because an eigenvalue is repeated doesn't mean we will need generalized eigenvectors.

<u>Example</u>:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

For triangular matrices, the eigenvalues will lie on the diagonal.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda) = 0$$

Find eigenvectors: **For** $\lambda = 2$:

$$(A - 2I)x = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = 0 \longrightarrow \quad x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $\lambda = 1$:

$$(A - I)x = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x = 0 \qquad x_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The rank deficiency of this matrix is TWO, so the dimension of its null space is TWO, so there are TWO linearly independent vectors such that the equality holds.

Altogether, we have three vectors, giving a modal matrix of:

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

And a diagonalized form: $\overline{A} = M^{-1}AM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Another way to compute the generalized eigenvectors is somewhat more "algorithmic":

II. "Top down" method: First we will need the definition of a special integer known as the *index* η_i of the eigenvalue λ_i .

$$\eta_i = \text{smallest } \eta \text{ such that } \; rank\,(A-\lambda_i I)^{\eta} = n - m_i$$

(= matrix size - algebraic multiplicity)

η_i is also the size of the largest Jordan block

Now for the algorithm itself: search for all linearly independent solutions of the equations:

$$(A - \lambda_i I)^{\eta_i} \boldsymbol{x} = 0$$

$$(A - \lambda_i I)^{\eta_i - 1} \boldsymbol{x} \neq 0$$

denote these solutions $\boldsymbol{v}_1^1, \ldots, \boldsymbol{v}_{m_i}^1$. There will be no more than m_i of them (why?).

$$Rank\left(A - l_i I\right)^{h_i} = n - m_i$$

solution = $n - \left(n - m_i\right)$

Now compute a different "chain" of generalized eigenvectors for each $j = 1, \ldots, m_i$:

$$\left.\begin{aligned}
(A-\lambda_i I)\boldsymbol{v}_j^1 &= \boldsymbol{v}_j^2 \\
(A-\lambda_i I)\boldsymbol{v}_j^2 &= \boldsymbol{v}_j^3 \\
&\vdots \\
(A-\lambda_i I)\boldsymbol{v}_j^{\eta_i-1} &= \boldsymbol{v}_j^{\eta_i}
\end{aligned}\right\}$$

“chain” of generalized eigenvectors

$$(A-\lambda_i I)\boxed{\boldsymbol{v}_j^{\eta_i}} = 0$$

REGULAR eigenvector ending each chain

These will be chains of “length” η_i**. If chains of shorter length are needed, start with**

$$(A-\lambda_i I)^{\eta_i-1}\boldsymbol{x} = 0$$

$$(A-\lambda_i I)^{\eta_i-2}\boldsymbol{x} \neq 0$$

etc.

EXAMPLE:

$$A=\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ -3 & 3 & 5\end{bmatrix} \qquad \lambda_1=\lambda_2=1 \qquad \lambda_3=5$$

$$n=3 \qquad m_1=2 \qquad n-m_1=1$$

Consider $\lambda=1$

$$(A-I)=\begin{bmatrix}0 & 2 & 0\\ 0 & 0 & 0\\ -3 & 3 & 4\end{bmatrix} \qquad r(A-I)=2$$

$$(A-I)^2=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -12 & 6 & 16\end{bmatrix} \qquad r(A-I)^2=1$$

$\therefore$ **Index of** $\lambda=1$ **is 2.**

So solve

$$(A-I)^2 x_1 = 0$$
$$(A-I)x_1 \neq 0$$

to get

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

(generalized)

Now generating the "chain":

$$x_2 = (A-I)x_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

The chain stops here, and x_2 **is a *regular* eigenvector, as can be verified by**

$$(A-I)x_2 = 0$$

Note that if there are other *linearly independent* solutions to

$$(A - I)^2 x = 0$$

we can initiate different chains.

Finally, for this example, we note that $x = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ is the regular eigenvector corresponding to $\lambda_3 = 5$, so we get:

reverse order

$$M = \begin{bmatrix} 0 & 4 & 1 \\ 0 & 0 & 2 \\ 1 & 3 & 0 \end{bmatrix} \qquad \overline{A} = M^{-1} A M = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

generalized

regular

ANOTHER EXAMPLE:

$$A=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\qquad \lambda^4=0,\quad m_1=4,\quad n-m_1=0$$

$$r(A-0\cdot I)=r\left(\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\right)=2$$

$$r(A-0\cdot I)^2=r\left(\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\right)=0$$

η_i **= 2 =**

longest chain of eigenvectors

Find *2* linearly independent solutions to

$$(A-\lambda I)^2 x_1 = (A)^2 x_1 = 0$$
$$(A-\lambda I)x_1 = (A)x_1 \neq 0$$

try $x_1 = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$: $\quad (A - 0\cdot I)x_1 = 0$ No

same for $x_1 = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}^T$ No

try

$$x_1 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}: \quad (A - 0\cdot I)\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

generalized **regular**

and also:

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}: \qquad (A - 0 \cdot I) \underset{\textbf{generalized}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} = \underset{\textbf{regular}}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$$

so now,

$$M = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccc} \text{gen} & \text{reg} & \text{gen} & \text{reg} \end{array} \end{array}, \qquad \bar{A} = M^{-1} A M = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

III. "Adjoint" method. This method requires computation of the adjoint of A and must be done "by hand." It is relatively tedious to do, but there is an example in Brogan (page 257).

More discussion of eigenvalues, eigenvectors, generalized eigenvectors, and Jordan forms:

Some facts:

The eigenvectors of a matrix that correspond to distinct eigenvalues are *linearly independent*.

When an eigenvalue is repeated (algebraic multiplicity >1), we don't *always* require generalized eigenvectors. For example, an eigenvalue with algebraic multiplicity *m* may have $p(\leq m)$ **linearly independent *regular* eigenvectors. We then would have to find only *m-p* generalized eigenvectors, for that eigenvalue, to get the modal matrix.**

The *geometric multiplicity* of eigenvalue λ **is defined to be equal to the *rank deficiency* (degeneracy) of the matrix** $A-\lambda I$**. It is the number of linearly independent (regular) eigenvectors we can find associated with the eigenvalue.**

Recall from the example:

$$(A-\lambda I)\big|_{\lambda=1}x=\begin{bmatrix}0 & 2 & 0\\ 0 & 0 & 0\\ -3 & 3 & 4\end{bmatrix}x$$

n=3

rank=2

rank deficiency=1

λ **has *algebraic* multiplicity 2**

λ **has *geometric* multiplicity 1** $\quad n-r(A-lI)=gm$

so we can find one regular eigenvector

When we use the modal matrix to find the Jordan form of a matrix, we will get one Jordan block for each *regular* eigenvector we can find. Similarly, the number of Jordan blocks associated with one repeated eigenvalue will be equal to the geometric multiplicity of that eigenvalue.

The *algebraic* multiplicity of λ will therefore be the sum of the sizes of all the Jordan blocks associated with λ .

(am is the order of root) (obvious)

Example:

The matrix

$$A = \begin{bmatrix} 3 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

has $|A - \lambda I| = (\lambda - 2)^5 \lambda$

So it has eigenvalues: $\lambda_1 = 2,$ algebraic multiplicity 5

$\lambda_2 = 0,$ algebraic multiplicity 1

<u>NOTE</u>: The *A* matrix itself has rank deficiency equal to the geometric multiplicity of any zero eigenvalues. (Why?)

If we compute the rank of

$A - 2I$ $\quad l_1 = 2$

we get 4, for a rank *deficiency* of 6-4=**2**.

$gm = 2$

Therefore, in the Jordan canonical form for this matrix, we will have **2** Jordan blocks for the eigenvalue $\lambda = 2$ (and of course one trivial block corresponding to $\lambda = 0$). We can calculate **2** regular eigenvectors and will need **3** *generalized* eigenvectors.

Because the column of the modal matrix = 0

☞

When we go through the exercise of finding the Jordan form, we get:

$$\left[\begin{array}{ccc|cc|c} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

How could you tell there is a 3x3 block and a 2x2 block, rather than a 4x4 and a 1x1? Number of regular eigenvectors

Recall that the *index* of the eigenvalue is the smallest integer η_i **such that** $rank\,(A-\lambda_i I)^{\eta_i} = n - m_i$.

For this matrix and eigenvalue λ_i, $\eta_i = 3$ **, and *this will be the size of the largest Jordan block associated with eigenvalue*** λ_i **.**

A Geometric Interpretation of All This Stuff:

<u>Definition</u>: Let X_1 **be a subspace of linear vector space** X **. This subspace is said to be *A-invariant* if for every vector,** $x \in X_1$ $Ax \in X_1$.

<u>Definition</u>: The set of all (regular) eigenvectors corresponding to an eigenvalue λ_i **forms a basis of a *subspace* of** X **, called the *eigenspace* of** λ_i **. This also happens to be the *null space* of a transformation defined as** $A - \lambda_i I$.

<u>Theorem</u>: The eigenspace of λ_i **is *A*-invariant and has dimension equal to the *degeneracy* of** $A - \lambda_i I$.

Proof: Denote the eigenspace of λ_i as N_i. We have already seen that the number of eigenvectors we will find is equal to q_i, the rank deficiency of $A-\lambda_i I$. So we will have a basis of N_i consisting of q_i vectors, so the dimension of N_i is q_i.

Now if we take a vector $x \in N_i$, we can expand it in the basis of eigenvectors e_i as

$$x = \sum_{i=1}^{q_i} a_i e_i ,$$

where a_i's are coefficients. Then applying operator A:

$$Ax = A\sum_{i=1}^{q_i} a_i e_i = \sum_{i=1}^{q_i} a_i (Ae_i) = \sum_{i=1}^{q_i} a_i (\lambda_i e_i) = \sum_{i=1}^{q_i} (a_i \lambda_i) e_i \in N_i$$

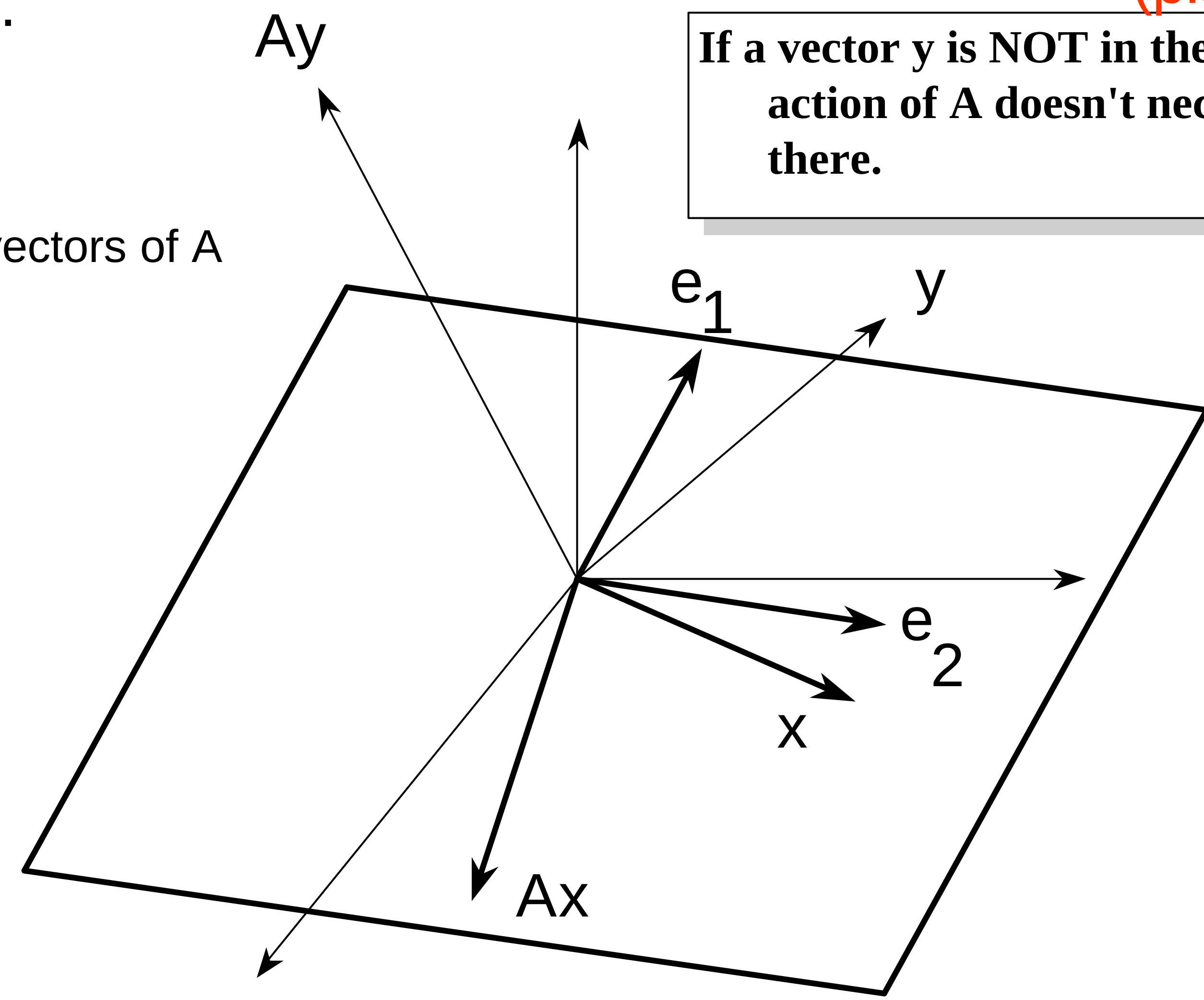
So Ax is in N_i by virtue of it being a linear combination of the basis vectors. ■

<u>Picture</u>:

e_1
e_2 eigenvectors of A

(plane)

If a vector y is NOT in the subspace, the action of A doesn't necessarily put it there.



Plane (subspace) formed by eigenvectors of A

If a vector *x* starts out in the subspace, it stays in the subspace when *A* acts on it.

Chapter 5: Functions of Vectors and Matrices

$\langle y, Ax \rangle \ \left(= \bar{y}^T Ax\right)$: **"Bilinear Form"**

$\langle x, Ax \rangle \ \left(= \bar{x}^T Ax\right)$: **"Quadratic Form"**

Note that because

$$x^T Ax = \left(x^T Ax\right)^T = x^T A^T x,$$

$$x^T Ax = \frac{1}{2}\left(x^T Ax + x^T A^T x\right) = x^T \left(\frac{A + A^T}{2}\right) x$$

any quadratic form can be written as a quadratic form with a symmetric *A*-matrix. We therefore treat all quadratic forms as is they contained symmetric matrices.

DEFINITIONS: **Let** $Q = \bar{x}^T Ax$

1. ***Q* (or *A*) is *positive definite* iff :** $\langle x, Ax \rangle > 0$ for all $x \neq 0.$
2. ***Q* (or *A*) is *positive semidefinite* if :** $\langle x, Ax \rangle \geq 0$ for all $x \neq 0.$
3. ***Q* (or *A*) is *negative definite* iff:** $\langle x, Ax \rangle < 0$ for all $x \neq 0.$
4. ***Q* (or *A*) is *negative semidefinite* if:** $\langle x, Ax \rangle \leq 0$ for all $x \neq 0.$
5. ***Q* (or *A*) is *indefinite* if:** $\langle x, Ax \rangle > 0$ for some $x \neq 0,$ and
   $\langle x, Ax \rangle < 0$ for other $x \neq 0.$

Tests for definiteness of matrix A in terms of its eigenvalues λ_i:

Matrix *A* is . . .	**If the *real parts* of eigenvalues λ_i of *A* are: . . . .**
1. Positive definite	All > 0
2. Positive semidefinite	All ≥ 0
3. Negative definite	All < 0
4. Negative semidefinite	All ≤ 0
5. Indefinite	Some $\mathrm{Re}(\lambda_i) > 0$, some $\mathrm{Re}(\lambda_i) < 0$.

See book for tests involving *leading principal minors*.