1. Cayley-Hamilton para encontrar e^{At} (com autovalores distintos)

■ **Example**

Find $e^{\mathbf{A}t}$ for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

Solution: The characteristic equation is $s^2 + 3s + 2 = 0$, and the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -2$. From Eq. (5)

$$e^{\mathbf{A}t} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A}$$

From Eq. (6), for $\lambda_1 = -1$ and $\lambda_2 = -2$

$$\begin{aligned} e^{-t} &= \alpha_0 - \alpha_1 \\ e^{-2t} &= \alpha_0 - 2\alpha_1, \end{aligned}$$

or $\alpha_0 = (2e^{-t} - e^{-t})$ and $\alpha_1 = (e^{-t} - e^{-2t})$. Then

$$\begin{aligned} e^{\mathbf{A}t} &= (2e^{-t} - e^{-t})\mathbf{I} + (e^{-t} - e^{-2t})\mathbf{A} \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \end{aligned}$$

2. Forma de Jordan p/ encontrar e^{At} (com autovalores múltiplos)

1.8 Example:

Compute e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$.

Eigenvalues:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda + 2 \end{vmatrix} = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$$

$$\Rightarrow \qquad \lambda_1 = \lambda_2 = -1.$$

Eigenvector:

$$(1I - A)v_1 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} v_1 = 0, \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Generalized eigenvector:

$$v_{11} = v_1, \ \ (A - 1I)v_{12} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} v_{12} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = v_{11}, \quad \Rightarrow \quad v_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Jordan form:

$$T = [v_{11} \ \ v_{12}] = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Matrix exponential:

$$\begin{aligned} e^{At} &= Te^{J_2(-t)}T^{-1}, \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \\ &= \begin{bmatrix} (t+1)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}. \end{aligned}$$

$$\boxed{\begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{k-1}}{(k-1)!}e^{\lambda t} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!}e^{\lambda t} \\ & & & \ddots & te^{\lambda t} \\ 0 & & & & e^{\lambda t} \end{bmatrix}}$$