Name: `Solution` Score: /100

This exam is closed-book.

- You must show ALL of your work for full credit.
  - Please read the questions carefully.
  - Please check your answers carefully.
- Calculators may NOT be used.
  - Please leave fractions as fractions, etc.
  - I do not want the decimal equivalents.
- Cell phones and other electronic communication devices must be turned off and stowed under your desk.
- Please do not write on the backs of the exam or additional pages.
  - The instructor will grade only one side of each page.
  - Extra paper is available from the instructor.
- Please write your name on every page that you would like graded.

1	2	3	4	5	6	7	8

1. You are given the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

(a) (3 points) State the Cayley Hamilton Theorem.

`Solution:`
`Every matrix satisfies its own characteristic equation.`

(b) (7 points) Use the Cayley Hamilton Theorem to find $\mathbf{A}^{-1}$.

`Solution:`
`We find the characteristic polynomial.`

$$|sI - \mathbf{A}| = \begin{bmatrix} s & -1 \\ 2 & s-3 \end{bmatrix} = s^2 - 3s + 2 = (s-1)(s-2).$$

`The characteristic equation is thus`

$$s^2 - 3s + 2 = 0,$$

so

$$A^2 - 3A + 2I = 0.$$

`Rearranging,`

$$I = \frac{1}{2}\left(3A - A^2\right) = A\left(\frac{3}{2}I - \frac{1}{2}A\right).$$

`Thus`

$$A^{-1} = \begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix} - \begin{bmatrix} 0 & 1/2 \\ -1 & 3/2 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ 1 & 0 \end{bmatrix}.$$

`Check:`

$$AA^{-1} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3/2 & -1/2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. (4 points) State the condition for realizability of a linear constant-coefficient continuous-time SISO system.

`Solution:`
`A linear constant-coefficient continuous-time SISO system is realizable iff its transfer function is rational and proper.`

3. (11 points) Provide the formula for the solution of a linear constant-coefficient continuous-time (MIMO) system. Label the zero-input response (ZIR) and the zero-state response (ZSR).

`Solution:` `For the linear time-invariant(LTI) system`

$$\dot{x}(t) = Ax(t) + Bu(t)$$

`with initial condition` $x(0)$`, the solution is`

$$x(t) = \underbrace{e^{At}x(0)}_{\text{ZIR}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{ZSR}}$$

4. State Transition Matrices

   (a) (5 points) The state transition matrix is the solution of what initial value problem (IVP)?

   `Solution:`
   `The state transition matrix` $\Phi(t, t_0)$ `is the unique solution of the initial value problem`

   $$\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\Phi(t, t_0)$$

   `with initial condition`

   $$\Phi(t_0, t_0) = I.$$

   (b) (10 points) Give the formula for the state transition matrix of a linear time-varying continuous-time (MIMO) system. Be sure to define all of the variables you use.

   `Solution:`
   `The state transition matrix is given by`

   $$\Phi(t, \tau) = X(t)X^{-1}(\tau),$$

   `where` $X(t)$ `is a fundamental matrix of` $\dot{x}(t) = A(t)x(t)$, `that is a matrix whose columns form a set of linearly independent solutions of`

   $$\dot{x}(t) = A(t)x(t).$$

5. You are given the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

   (a) (3 points) Find the characteristic equation of $\mathbf{A}$.

   Solution:
   The characteristic polynomial is

$$|sI - A| = \left| \begin{bmatrix} s & -1 \\ 2 & s-3 \end{bmatrix} \right| = s^2 - 3s + 2$$

   so the characteristic equation is

$$s^2 - 3s + 2 = 0.$$

   (b) (7 points) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

   Solution:
   The characteristic equation can be factored as

$$s^2 - 3s + 2 = (s-1)(s-2) = 0,$$

   so we can label the eigenvalues $\lambda_I = 1$ and $\lambda_{II} = 2$.
   We obtain the eigenvectors x_I and x_{II} as follows:

$$\mathbf{A}x_I = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_I = \begin{bmatrix} x_2 \\ -2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_I x_I$$

   Thus $x_2 = x_1$ is required for convenience we can choose $x_I = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\mathbf{A}x_{II} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_{II} = \begin{bmatrix} x_2 \\ -2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = \lambda_{II} x_{II}$$

   Thus $x_2 = 2x_1$ is required for convenience we can choose $x_{II} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Continue to use

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

(c) (2 points) Find a matrix $\mathbf{Q}$ that can be used to transform the matrix $\mathbf{A}$ to its Jordan form $\mathbf{J}$.

Solution:
The columns of $\mathbf{Q}$ should be the eigenvalues of $\mathbf{A}$, so we can choose

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

We expect that the eigenvalues will appear in the same order in the diagonal matrix $\mathbf{J}$. Otherwise we will know that we've made a mistake.

(d) (8 points) Using the matrix $\mathbf{Q}$ you found in the previous part, find $\mathbf{J}$.

Solution:
First we need $\mathbf{Q}^{-1}$. By any valid method we obtain

$$\mathbf{Q}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Before we go any further, we should verify that our $\mathbf{Q}^{-1}$ is correct:

$$\mathbf{Q}\mathbf{Q}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now we are ready to find $\mathbf{J}$.

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

We see that the eigenvalues occur in the same order as their corresponding eigenvectors did in $\mathbf{Q}$.

Continue to use

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

6. (15 points) Find e^{At}.

7. (10 points) Find e^{At} using a different method than you used in the previous problem.

`Solution: Any two valid methods are acceptable here. One thing that is not acceptable, as was mentioned several times in class, is to present two different solutions and not explicitly state that one of them must be wrong.`

`In the previous problems we have calculated` $\mathbf{Q}^{-1}$`,` $\mathbf{J}$`, and` $\mathbf{Q}$`, so the simplest solution is`

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{Q}e^{\mathbf{J}t}\mathbf{Q}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2e^t - e^{2t} & -e^t + e^{2t} \\ 2e^t - 2e^{2t} & -e^t + 2e^{2t} \end{bmatrix}. \end{aligned}$$

`A second approach is to equate` $f(\mathbf{A})$ `and` $h(\mathbf{A})$ `on the eigenvalues of` $\mathbf{A}$`. With`

$$f(\lambda) = e^{\lambda t} \qquad \text{and} \qquad h(\lambda t) = \beta_0 + \beta_1\lambda$$

`we proceed as follows.`

$$\begin{aligned} f(\lambda_I) &= e^{\lambda_I t} = e^t = \beta_0 + \beta_1 = \beta_0 + \beta_1\lambda_I = h(\lambda_I), \\ f(\lambda_{II}) &= e^{\lambda_{II} t} = e^{2t} = \beta_0 + 2\beta_1 = \beta_0 + \beta_1\lambda_{II} = h(\lambda_{II}). \end{aligned}$$

`Thus, subtracting the first equation from the second yields`

$$\beta_1 = e^{2t} - e^t.$$

`Then substituting the values of` β_1 `into the first equation we find that`

$$\beta_0 = e^t - (e^{2t} - e^t) = 2e^t - e^{2t}.$$

`Accordingly`

$$\begin{aligned} e^{At} &= \beta_0\mathbf{I} + \beta_1\mathbf{A} \\ &= \begin{bmatrix} 2e^t - e^{2t} & 0 \\ 0 & 2e^t - e^{2t} \end{bmatrix} + \begin{bmatrix} 0 & -e^t + e^{2t} \\ -2(-e^t + e^{2t}) & 3(-e^t + e^{2t}) \end{bmatrix} \\ &= \begin{bmatrix} 2e^t - e^{2t} & -e^t + e^{2t} \\ 2e^t - 2e^{2t} & -e^t + 2e^{2t} \end{bmatrix} \end{aligned}$$

8. (15 points) Find A^k.

   `Solution:`
   `In earlier problems we have calculated` $\mathbf{Q}^{-1}$`,` $\mathbf{J}$`, and` $\mathbf{Q}$`, so the simplest solution is`

$$\begin{aligned}
\mathbf{A}^k &= \underbrace{\mathbf{QJQ}^{-1}\mathbf{QJQ}^{-1}\cdots\mathbf{QJQ}^{-1}}_{k} \\
&= \mathbf{QJ}^k\mathbf{Q}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2^k \\ 1 & 2^{k+1} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2-2^k & -1+2^k \\ 2-2^{k+1} & -1+2^{k+1} \end{bmatrix}.
\end{aligned}$$