Generalized Eigenvectors and Jordan Form

We have seen that an $n \times n$ matrix A is diagonalizable precisely when the dimensions of its eigenspaces sum to n. So if A is not diagonalizable, there is at least one eigenvalue with a geometric multiplicity (dimension of its eigenspace) which is strictly less than its algebraic multiplicity. In this handout, we will discuss how one can make up for this deficiency of eigenvectors by finding what are called generalized eigenvectors, which can in turn be used to find the Jordan form of the matrix A.

First consider the following non-diagonalizable system.

Example 1. $\diamond$ The matrix

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

has characteristic polynomial $(\lambda - 3)^2$, so it has only one eigenvalue $\lambda = 3$, and the corresponding eigenspace is $E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. Since $\dim(E_3) = 1 < 3$, the matrix A is not diagonalizable. Nevertheless, it is still possible to solve the system

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}$$

without much difficulty. Writing out the two equations

$$\frac{dy_1}{dt} = 3y_1 + y_2$$
$$\frac{dy_2}{dt} = 3y_2$$

we see that the second equation has general solution $y_2 = c_1 e^{3t}$. Plugging this into the first equation gives

$$\frac{dy_1}{dt} = 3y_1 + c_1 e^{3t}$$

This is a first order linear equation that can be solved by the method of integrating factors. Its solution is $y_1 = c_1 t e^{3t} + c_2 e^{3t}$. Thus the general solution of the system is

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 t e^{3t} + c_2 e^{3t} \\ c_2 e^{3t} \end{bmatrix} = c_1 e^{3t}\left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + c_2 e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\diamond$

The matrix A in the previous example is said to be in Jordan form.

Definition 1. A 2×2 matrix of the form

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

for some number λ is called a **Jordan form** matrix.

The following theorem generalizes the calculations of the previous example.

Theorem 1. The general solution of the system

$$\frac{d\mathbf{y}}{dt} = J\mathbf{y}, \qquad J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

is

$$\mathbf{y} = c_1 e^{\lambda t} \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + c_2 e^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In general, suppose A is a 2×2 matrix with a single repeated eigenvalue λ with $\dim(E_\lambda) = 1$. Then, as we have seen before, making the change of variables $\mathbf{y} = C\mathbf{x}$ transforms the system

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}$$

into the system

$$\frac{d\mathbf{x}}{dt} = C^{-1}AC\mathbf{x}.$$

We now want to find a matrix C such that the coefficient matrix $C^{-1}AC$ for the new system is a Jordan form matrix J. That is, we want $AC = CJ$. Writing

$$C = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix}, \qquad J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

we have

$$AC = \begin{bmatrix} | & | \\ A\mathbf{v}_1 & A\mathbf{v}_2 \\ | & | \end{bmatrix}, \qquad CJ = \begin{bmatrix} | & | \\ \lambda\mathbf{v}_1 & \lambda\mathbf{v}_2 + \mathbf{v}_1 \\ | & | \end{bmatrix}$$

Therefore the columns of C must satisfy

$$\begin{aligned} A\mathbf{v}_1 &= \lambda\mathbf{v}_1 \\ A\mathbf{v}_2 &= \lambda\mathbf{v}_2 + \mathbf{v}_1 \end{aligned}$$

Thus the vector $\mathbf{v}_1$ is an eigenvector with eigenvalue λ. Rewriting these equations

$$\begin{aligned} (A - \lambda I)\mathbf{v}_1 &= \mathbf{0} \\ (A - \lambda I)\mathbf{v}_2 &= \mathbf{v}_1 \end{aligned}$$

it follows that $(A - \lambda I)^2\mathbf{v}_2 = (A - \lambda I)\mathbf{v}_1 = \mathbf{0}$. Thus the vector $\mathbf{v}_2$ must be in $\ker(A - \lambda I)^2$.

Definition 2. A nonzero vector $\mathbf{v}$ which satisfies $(A-\lambda I)^p\mathbf{v} = \mathbf{0}$ for some positive integer p is called a **generalized eigenvector** of A with eigenvalue λ.

The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a **generalized eigenvector chain**, as the following diagram illustrates:

$$\mathbf{v}_2 \quad \underset{A-\lambda I}{\longrightarrow} \quad \mathbf{v}_1 \quad \underset{A-\lambda I}{\longrightarrow} \quad \mathbf{0}$$

Therefore, to find the columns of the matrix C that puts A in Jordan form, we must find a chain of generalized eigenvectors, as follows:

- Find a nonzero vector $\mathbf{v}_2$ in $\ker(A-\lambda I)^2$ that is *not* in $\ker(A-\lambda I)$.
- Set $\mathbf{v}_1 = (A-\lambda I)\mathbf{v}_2$.

Having found such vectors, we can then write down the solution of the system $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$. From above, we know that the solution of the system $\frac{d\mathbf{x}}{dt} = J\mathbf{x}$ is

$$\mathbf{x} = c_1 e^{\lambda t}\left(t\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) + c_2 e^{\lambda t}\begin{bmatrix}1\\0\end{bmatrix}$$

so multiplying this by C we get

$$\mathbf{y} = C\mathbf{x} = c_1 e^{\lambda t}\left(t\mathbf{v}_1 + \mathbf{v}_2\right) + c_2 e^{\lambda t}\mathbf{v}_1.$$

In summary, we have the following result.

Theorem 2. Let A be a 2×2 matrix and suppose $\mathbf{v}_2 \to \mathbf{v}_1 \to \mathbf{0}$ is a chain of generalized eigenvectors of A with eigenvalue λ. Then the general solution of the system $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$ is

$$\mathbf{y}(t) = c_1 e^{\lambda t}\left(t\mathbf{v}_1 + \mathbf{v}_2\right) + c_2 e^{\lambda t}\mathbf{v}_1.$$

Example 2. $\diamond$ Let

$$A = \begin{bmatrix}2 & 1\\-1 & 4\end{bmatrix}$$

The characteristic polynomial of A is $\lambda^2 - 6\lambda + 9 = (\lambda-3)^2$, so $\lambda = 3$ is the only eigenvalue of A. Next, we compute

$$A - 3I = \begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix}, \qquad (A-3I)^2 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

Now we choose $\mathbf{v}_2$ to be any vector in $\ker(A-3I)^2$ that is not in $\ker(A-3I)$. One such vector is $\mathbf{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$. With this choice, we than have $\mathbf{v}_1 = (A-3I)\mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$. The solution of the system $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$ is therefore

$$\mathbf{y} = c_1 e^{3t}\left(t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) + c_2 e^{3t}\begin{bmatrix}1\\1\end{bmatrix}.$$

$\diamond$

For larger systems it turns out that any chain of generalized eigenvectors of the coefficient matrix can be used to write down a linearly independent set of solutions.

Theorem 3. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a chain of generalized eigenvectors of A with eigenvalue λ. That is,

$$\mathbf{v}_k \quad \xrightarrow[A-\lambda I]{} \quad \mathbf{v}_{k-1} \quad \xrightarrow[A-\lambda I]{} \quad \mathbf{v}_{k-2} \quad \xrightarrow[A-\lambda I]{} \quad \cdots \quad \xrightarrow[A-\lambda I]{} \quad \mathbf{v}_2 \quad \xrightarrow[A-\lambda I]{} \quad \mathbf{v}_1 \quad \xrightarrow[A-\lambda I]{} \quad \mathbf{0}.$$

Then

$$\begin{aligned}
\mathbf{y}_1(t) &= e^{\lambda t}\mathbf{v}_1 \\
\mathbf{y}_2(t) &= e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1) \\
\mathbf{y}_3(t) &= e^{\lambda t}\left[\mathbf{v}_3 + t\mathbf{v}_2 + \frac{t^2}{2!}\mathbf{v}_1\right] \\
&\vdots \\
\mathbf{y}_k(t) &= e^{\lambda t}\left[\mathbf{v}_k + t\mathbf{v}_{k-1} + \frac{t^2}{2!}\mathbf{v}_{k-2} + \cdots + \frac{t^{k-2}}{(k-2)!}\mathbf{v}_2 + \frac{t^{k-1}}{(k-1)!}\mathbf{v}_1\right]
\end{aligned}$$

is a linearly independent set of solutions of $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$.

Proof. We show that $\mathbf{y}_k$ is a solution. The proof that the others are solutions is similar. Differentiating gives

$$\begin{aligned}
\frac{d\mathbf{y}_k}{dt} = e^{\lambda t}\Bigg[&(\lambda\mathbf{v}_k + \mathbf{v}_{k-1}) + t(\lambda\mathbf{v}_{k-1} + \mathbf{v}_{k-2}) + \frac{t^2}{2!}(\lambda\mathbf{v}_{k-2} + \mathbf{v}_{k-3}) + \cdots \\
&+ \frac{t^{k-2}}{(k-2)!}(\lambda\mathbf{v}_2 + \mathbf{v}_1) + \frac{t^{k-1}}{(k-1)!}(\lambda\mathbf{v}_1)\Bigg].
\end{aligned}$$

Multiplying by A gives

$$A\mathbf{y}_k = e^{\lambda t}\left[(A\mathbf{v}_k) + t(A\mathbf{v}_{k-1}) + \frac{t^2}{2!}(A\mathbf{v}_{k-2}) + \cdots + \frac{t^{k-2}}{(k-2)!}(A\mathbf{v}_2) + \frac{t^{k-1}}{(k-1)!}(A\mathbf{v}_1)\right].$$

The two expressions above are equal, since $A\mathbf{v}_1 = \lambda\mathbf{v}_1$, $A\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1$, and so on.

Next we show that the chain $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is necessarily linearly independent. Suppose

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}. \tag{1}$$

The multiplying by $(A - \lambda I)^{k-1}$ sends $\mathbf{v}_1$ through $\mathbf{v}_k$ to zero, and $\mathbf{v}_k$ to $\mathbf{v}_1$, so we are left with $c_k\mathbf{v}_1 = \mathbf{0}$. Since $\mathbf{v}_1$ is nonzero, this implies $c_k = 0$. Similar reasoning shows that the remaining coefficients must also be zero. Thus the chain of generalized eigenvectors is linearly independent. Now suppose

$$c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t) = \mathbf{0}$$

for all t. Then since $\mathbf{y}_j(0) = \mathbf{v}_j$, at $t = 0$ this equation is precisely equation (1) above, which we just showed implies $c_1 = c_2 = \cdots = c_k = 0$. □

The next theorem guarantees that we can always find enough solutions of this form to generate a fundamental set of solutions.

Theorem 4. Let A be an $n \times n$ matrix, and suppose λ is an eigenvalue of A with algebraic multiplicity m. Then there is some integer $p \leq m$ such that $\dim(\ker(A - \lambda I)^p) = m$.

Example 3. $\diamond$ Let

$$A = \begin{bmatrix} 5 & 1 & -4 \\ 4 & 3 & -5 \\ 3 & 1 & -2 \end{bmatrix}$$

The characteristic polynomial of A is $p_A(\lambda) = (\lambda - 2)^3$. Since

$$A - 2I = \begin{bmatrix} 3 & 1 & -4 \\ 4 & 1 & -5 \\ 3 & 1 & -4 \end{bmatrix} \quad \underset{rref}{\longrightarrow} \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

the eigenspace $E_2 = \ker(A - 2I)$ has dimension 1. Next, since

$$(A - 2I)^2 = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \quad \underset{rref}{\longrightarrow} \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the space $\ker(A - 2I)^2$ has dimension 2. Finally, since

$$(A - 2I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the space $\ker(A - 2I)^3$ has dimension 3, which matches the algebraic multiplicity of the eigenvalue $\lambda = 2$.

We can now form a chain of 3 generalized eigenvectors by choosing a vector $\mathbf{v}_3$ in $\ker(A - 2I)^3$ and defining $\mathbf{v}_2 = (A - 2I)\mathbf{v}_3$ and $\mathbf{v}_1 = (A - 2I)\mathbf{v}_2 = (A - 2I)^2\mathbf{v}_3$. To ensure that $\mathbf{v}_2$ and $\mathbf{v}_1$ are both non-zero, we need $\mathbf{v}_3$ to *not* be in $\ker(A - 2I)^2$ (which in turn implies that $\mathbf{v}_3$ is not in $\ker(A - 2I)$). Since $\ker(A - 2I)^3 = \mathbf{R}^3$, we can choose $\mathbf{v}_3$ to be any vector not in $\ker(A - 2I)^2$. Since the first column of $(A - 2I)^2$ is non-zero, we may choose

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{v}_2 = (A - 2I)\mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{v}_1 = (A - 2I)\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then

$$\mathbf{y}_1(t) = e^{2t}\begin{bmatrix}1\\1\\1\end{bmatrix}$$
$$\mathbf{y}_2(t) = e^{2t}\left(t\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}3\\4\\3\end{bmatrix}\right)$$
$$\mathbf{y}_3(t) = e^{2t}\left(\frac{t^2}{2!}\begin{bmatrix}1\\1\\1\end{bmatrix} + t\begin{bmatrix}3\\4\\3\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

is a fundamental set of solutions of $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$. ◇

There may in general be more than one chain of generalized eigenvectors corresponding to a given eigenvalue. Since the last vector in each chain is an eigenvector, the number of chains corresponding to an eigenvalue λ is equal to the dimension of the eigenspace E_λ.

Example 4. ◇ Let

$$A = \begin{bmatrix}4 & 1 & 1\\-2 & 1 & -2\\1 & 1 & 4\end{bmatrix}.$$

The characteristic polynomial of A is $p_A(\lambda) = (\lambda - 3)^3$. Since

$$A - 3I = \begin{bmatrix}1 & 1 & 1\\-2 & -2 & -2\\1 & 1 & 1\end{bmatrix} \xrightarrow[rref]{} \begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix},$$

the eigenspace $E_3 = \ker(A - 3I)$ has dimension 2, so there will be two chains. Next, since

$$(A - 3I)^2 = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix},$$

the space $\ker(A - 3I)^2$ has dimension 3, which matches the algebraic multiplicity of $\lambda = 3$. Thus one of the chains will have length 2, so the other must have length 1.

We now form a chain of 2 generalized eigenvectors by choosing $\mathbf{v}_2$ in $\ker(A - 3I)^2$ such that $\mathbf{v}_2$ is not in $\ker(A - 3I)$. Since every vector is in $\ker(A - 3I)^2$ and the first column of $A - 3I$ is non-zero,we may again choose

$$\mathbf{v}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \Longrightarrow \quad \mathbf{v}_1 = (A - 3I)\mathbf{v}_2 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$$

To form a basis for $\mathbf{R}^3$, we now need one additional chain of one generalized eigenvector. This vector must be an eigenvector independent from $\mathbf{v}_1$. Since

$$E_3 = \operatorname{span}\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right),$$

and neither of these spanning vectors is itself a scalar multiple of $\mathbf{v}_1$, we may choose either one of them. So let

$$\mathbf{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Then we have the two chains

$$\begin{array}{ccccc} \mathbf{v}_2 & \longrightarrow & \mathbf{v}_1 & \longrightarrow & \mathbf{0} \\ & & \mathbf{w}_1 & \longrightarrow & \mathbf{0}. \end{array}$$

The first chain generates the two solutions

$$\mathbf{y}_1(t) = e^{3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\mathbf{y}_2(t) = e^{3t} \left(t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

and the second chain generates a third solution

$$\mathbf{y}_3(t) = e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and together $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ form a fundamental set of solutions of $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$. $\diamond$

Example 5. $\diamond$ Let

$$A = \begin{bmatrix} 5 & 1 & 3 & 2 \\ 0 & 5 & 0 & -3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

The characteristic polynomial of A is $(\lambda - 5)^4$. Since

$$A - 5I = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[rref]{} \begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the eigenspace $E_5 = \ker(A - 5I)$ has dimension 2, and there are two chains. Since

$$(A - 5I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the dimension of the space $\ker(A - 5I)^2$ is 4, matching the algebraic multiplicity of $\lambda = 5$.

To form a chain of length 2, we first choose a vector $\mathbf{v}_2$ in $\ker(A-5I)^2$ which is not in $\ker(A-5I)$. Let

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{v}_1 = (A-5I)\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We now need one more chain of length 2. To find a second chain, we need to choose another vector $\mathbf{w}_2$ in $\ker(A-5I)^2$ which is not in $\ker(A-5I)$, in such a way that $\mathbf{w}_2$ is independent of $\mathbf{v}_2$ and $\mathbf{w}_1 = (A-5I)\mathbf{w}_2$ is independent of $\mathbf{v}_1$. One such choice is

$$\mathbf{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{w}_1 = (A-5I)\mathbf{w}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}.$$

Thus

$$\mathbf{y}_1(t) = e^{5t}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{y}_2(t) = e^{5t}\left(t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$\mathbf{y}_3(t) = e^{5t}\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{y}_4(t) = e^{5t}\left(t\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

is a fundamental set of solutions of $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$. ◇

Jordan Form for $n \times n$ Matrices

Definition 3. A matrix of the form

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}$$

consisting of λ in each entry along the main diagonal, 1 in each entry directly above the main diagonal, and 0 elsewhere, is called an **elementary Jordan block**. A matrix of the form

$$\begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}$$

where each J_i is an elementary Jordan block (with possibly different values of λ) is called a **Jordan form matrix**.

Example 6. $\diamond$ The following are examples of Jordan form matrices:

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\diamond$

Theorem 5. For any $n \times n$ matrix A, there is a matrix C and a Jordan form matrix J such that $C^{-1}AC = J$.

The Jordan matrix J is determined by the number and length of the generalized eigenvector chains for A.

Example 7. $\diamond$ Let A be the matrix in Example 3. Since

$$\begin{aligned} A\mathbf{v}_1 &= 2\mathbf{v}_1 \\ A\mathbf{v}_2 &= \mathbf{v}_1 + 2\mathbf{v}_2 \\ A\mathbf{v}_2 &= \mathbf{v}_2 + 2\mathbf{v}_3, \end{aligned}$$

letting

$$C = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}$$

implies

$$AC = \begin{bmatrix} | & | & | \\ 2\mathbf{v}_1 & \mathbf{v}_1 + 2\mathbf{v}_2 & \mathbf{v}_2 + 2\mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = CJ,$$

so $C^{-1}AC = J$ consists of a single elementary Jordan block. $\diamond$

Example 8. $\diamond$ Let A be the matrix in Example 4. Since

$$\begin{aligned} A\mathbf{v}_1 &= 3\mathbf{v}_1 \\ A\mathbf{v}_2 &= \mathbf{v}_1 + 3\mathbf{v}_2 \\ A\mathbf{w}_1 &= 3\mathbf{w}_1, \end{aligned}$$

letting

$$C = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{w}_1 \\ | & | & | \end{bmatrix}$$

implies

$$AC = \begin{bmatrix} | & | & | \\ 3\mathbf{v}_1 & \mathbf{v}_1 + 3\mathbf{v}_2 & \mathbf{w}_1 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{w}_1 \\ | & | & | \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = CJ,$$

so $C^{-1}AC = J$ consists of 2 elementary Jordan blocks, one 2×2 and one 1×1. $\diamond$