ES728 – Exercício – Transformação de Similaridade

Example 1. Show that $T\colon\ E^2 \to E^2$ defined by $T(x_1, x_2) = (x_1 + 6x_2, 3x_1 + 4x_2)$ has standard matrix

$$\begin{pmatrix} 1 & 6 \\ 3 & 4 \end{pmatrix}$$

Then show that, with respect to the basis $\mathcal{T} = \{(2,-1),(1,1)\}, T$ has a diagonal matrix representation.

Solution For the standard matrix we have

$$T((1,0)) = (1,3) = 1(1,0) + 3(0,1) \qquad \text{so} \qquad (T((1,0)))_{\text{std}} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$T((0,1)) = (6,4) = 6(1,0) + 4(0,1) \qquad \text{so} \qquad (T((0,1)))_{\text{std}} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

and

$$M_{\text{std}} = \begin{pmatrix} 1 & 6 \\ 3 & 4 \end{pmatrix}$$

But with respect to $\mathcal{T}$,

$$T((2,-1)) = (-4,2) = -2(2,-1) + 0(1,1) \qquad \text{so} \qquad [T((1,0))]_{\mathcal{T}} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$T((1,1)) = (7,7) = 0(2,-1) + 7(1,1) \qquad \text{so} \qquad [T((1,1))]_{\mathcal{T}} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$$

and the matrix with respect to $\mathcal{T}$ is

$$M = \begin{pmatrix} -2 & 0 \\ 0 & 7 \end{pmatrix}$$

Theorem 4.3.1. *Let $T\colon\ V \to V$ be a linear transformation with matrix $M_{(\mathcal{S})}$ with respect to a basis $\mathcal{S}$ and with matrix $M_{(\mathcal{T})}$ with respect to a basis $\mathcal{T}$. If P is the transition matrix from basis $\mathcal{T}$ to basis $\mathcal{S}$, then*

$$M_{(\mathcal{T})} = P^{-1} M_{(\mathcal{S})} P$$

The relation $M_{(\mathcal{T})} = P^{-1} M_{(\mathcal{S})} P$ is important enough to be given a name.

Definition 4.3.1. Two $n \times n$ matrices A and B are **similar** if there exists an invertible matrix P such that $B = P^{-1}AP$.

Example 2. In Example 1, denote the standard basis by $\mathcal{S}$. Illustrate Theorem 4.3.1 for T, $\mathcal{S}$, and $\mathcal{T}$, as given in Example 1.

Solution The standard matrix, as before, is

$$M_{(\mathcal{S})} = \begin{pmatrix} 1 & 6 \\ 3 & 4 \end{pmatrix}$$

Now calculate the transition matrix from $\mathcal{T}$ to $\mathcal{S}$:

$$\begin{aligned} (2,-1) &= 2(1,0) + (-1)(0,1) \\ (1,1) &= 1(1,0) + 1(0,1) \\ P = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} &\Rightarrow P^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \end{aligned}$$

Then

$$\begin{aligned} P^{-1}M_{(\mathcal{S})}P = P^{-1}\begin{pmatrix} 1 & 6 \\ 3 & 4 \end{pmatrix}P &= \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} -4 & 7 \\ 2 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 \\ 0 & 7 \end{pmatrix} = M_{(\mathcal{T})} \end{aligned}$$