Chapter 6: Solutions to State Equations

We know how to solve scalar linear differential equations, but what about the state-space equations:

$$\dot{x} = Ax + Bu, \qquad x(t_0) = x_0$$
$$y = Cx + Du$$

Actually, we need only to consider $\dot{x} = Ax + Bu$ because finding y will then be a simple matter of matrix multiplication.

Brogan starts out with the scalar case, but we'll go directly to the vector equations:

Recall the technique of *integrating factor* in the solution of linear differential equations:

$$\dot{x} - Ax = Bu$$

Multiplying this equation by e^{-At} will result in the left-hand side becoming a "perfect" differential:

$$e^{-At}\left[\dot{x} - Ax = Bu\right]$$

$$e^{-At}\dot{x} - e^{-At}Ax = e^{-At}Bu$$

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu(t)$$

Now multiply both sides by dt and integrate over a dummy variable τ from t_0 to t .

$$e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

Move initial condition term to RHS and multiply through by e^{At}

$$x(t) = e^{At}e^{-At_0}x(t_0) + e^{At}\int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Note that if matrix *B* were a function of time, this would become simply

$$\boxed{x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B(\tau)u(\tau)d\tau}$$

(but if *A* were a function of time, we run into bigger problems.)

If we wanted to compute $y(t)$, we would simply get:

$$y(t) = Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Again, *C* and *D* could be functions of time without complicating matters too much. If *A* is time-varying, we must be more careful choosing a proper integrating factor. The matrix exponential will no longer work.

Again, the importance of the matrix exponential e^{At} arises. We'll summarize the several ways to compute it shortly.

System modes and modal decompositions:

This is a very powerful representation of a system's solutions, used widely in large-scale systems and infinite-dimensional systems, which are often represented by partial differential equations rather than ordinary differential equations. It underscores the importance of a basis of the state-space.

Let the set $\{\xi_i\}$ be the set of *n* linearly independent eigenvectors, including, if necessary, generalized eigenvectors, corresponding to eigenvalues λ_i of the **constant** matrix *A*. Because this set forms a basis of the state-space, we can write

$$x(t) = \sum_{i=1}^{n} q_i(t)\xi_i$$

$q_i(t)$ denotes the scalar coefficients

x_i denotes the the eigenvectors

$$x(t) = \sum_{i=1}^{n} q_i(t)\xi_i$$

For some time-varying coefficients

$$q_i(t), \qquad i = 1, \ldots, n$$

We can easily do the same for the term $B(t)u(t)$:

$$B(t)u(t) = \sum_{i=1}^{n} \beta_i(t)\xi_i$$

$b_i(t)$ denotes the scalar coefficients

x_i eigenvectors

Substituting these expansions into the state-equations

$$\dot{x} = Ax + Bu$$

Gives . . . . .

$$\sum_{i=1}^{n} \dot{q}_i(t)\xi_i = \sum_{i=1}^{n} q_i(t)A\xi_i + \sum_{i=1}^{n} \beta_i(t)\xi_i$$

We have implicitly assumed in this step that we have *n* linearly independent eigenvectors. If this is not the case, relatively minor complications arise.

Re-arranging, $$\sum_{i=1}^{n} \left(\dot{q}_i(t) - q_i(t)A - \beta_i(t)\right)\xi_i = 0$$

These coefficients must **all** be zero, so

$$\dot{q}_i(t) = q_i(t)\lambda_i + \beta_i(t) \quad \text{for} \quad i = 1, \ldots, n$$

Recall $\boxed{Ax_i = l_i x_i}$ } Eigenvalue/Eigenvector Problem

$$\dot{q}_i(t) = q_i(t)\lambda_i + \beta_i(t) \quad \text{for} \quad i = 1, \ldots, n$$

This is a set of *n* **de-coupled** equations (if we had used any generalized eigenvectors, some would still be coupled, but only to one other equation).

Jordan Form

The terms $q_i(t)\xi_i$ are called ***system modes***, and are equivalent to the "new" state variables $\bar{x}(t)$ that we obtained in the past example where we "diagonalized" the system using the modal matrix. Recall that if *M* is the modal matrix, we can define new variables.

$$x = Mq$$

such that

$$\dot{q} = M^{-1}AMq + M^{-1}Bu$$
$$y = CMq + Du$$

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$$y = CMq + Du$$

where $M^{-1}AM = J$ is the *Jordan form* of the *A*-matrix (diagonal if there are n eigenvectors).

Because these equations are decoupled, the solutions to the state equations are particularly simple. We can find the solutions $q(t)$ and then change them back to the original variables $x(t)$ by un-doing the transformation afterward. That is,

$$J = M^{-1}AM$$

Initial Conditions

$$q(t) = e^{J(t-t_0)}q(t_0) + \int_{t_0}^{t} e^{J(t-\tau)}M^{-1}B(\tau)u(\tau)d\tau, \boxed{q(t_0) = M^{-1}x(t_0)}$$

after which

$$x(t) = Mq(t)$$

In ***diagonal*** form, the computation of e^{Jt} is particularly easy:

$$e^{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} t} = e^{\begin{bmatrix} \lambda_1 t & & 0 \\ & \ddots & \\ 0 & & \lambda_n t \end{bmatrix}}$$

and

$$e^{At} = Me^{Jt}M^{-1}$$

Whenever two matrices *A* and *J* are similar, we can compute our function of *J* and perform the reverse-similarity transform afterward. That is,

if $J = M^{-1}AM$,

then $f(A) = Mf(J)M^{-1}$ and $f(J) = M^{-1}f(A)M$

Note that $\hat{A}$ and A are similar if

$\hat{A} = M^{-1}AM$ for some orthonormal M

Often in large-scale or infinite-dimensional systems, some modes are negligible and are discarded after modal expansion, thus reducing the size of the system. For example, when a beam vibrates, we have an infinite number of terms in a series expansion of its displacement function, but only the first few (2 - 5) may dominate.

Phase Portraits

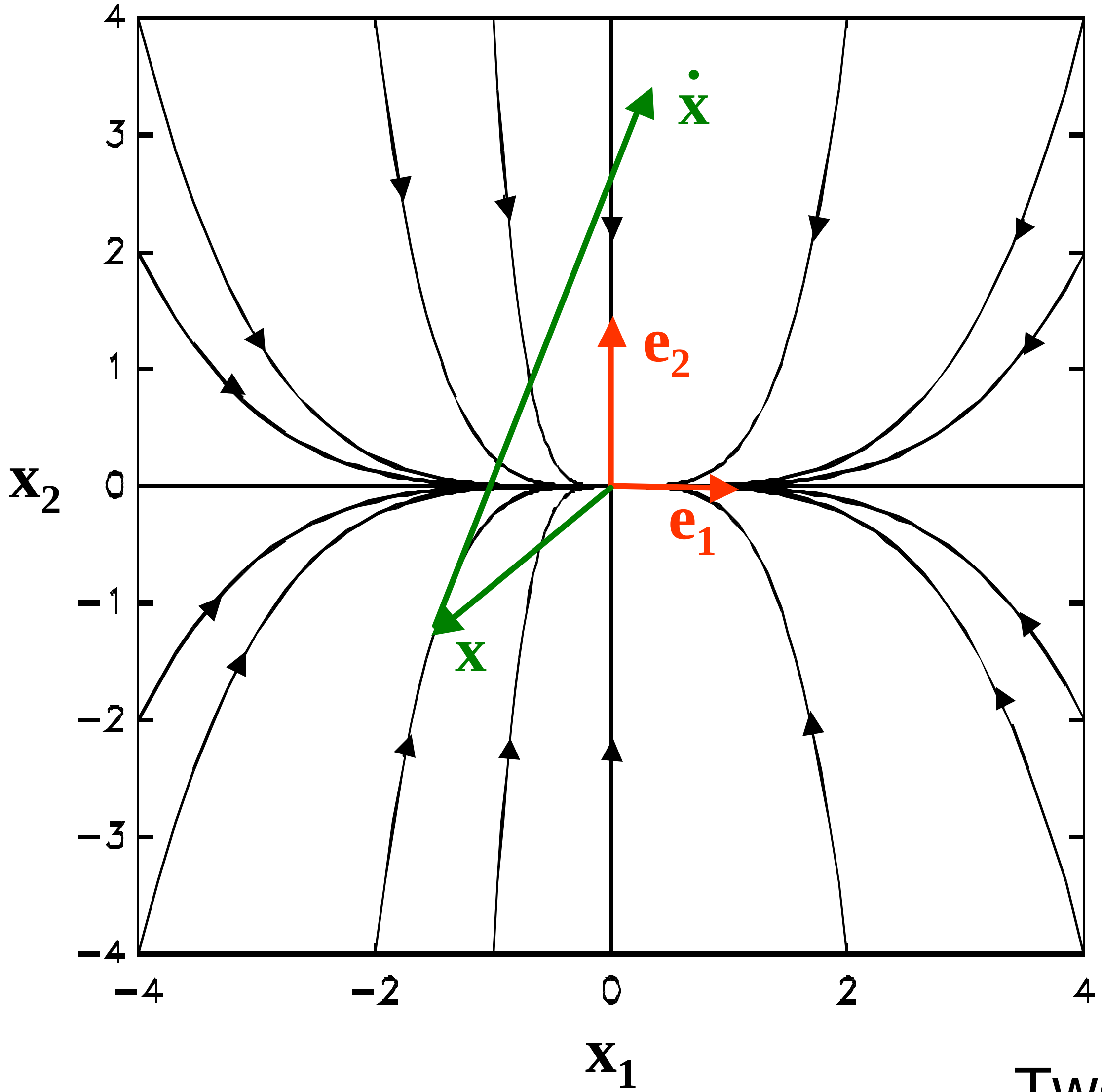
Consider the homogeneous system

$$\dot{x} = Ax$$

A *phase portrait* is a graphical depiction of the solutions to this equation, starting from a variety of initial conditions. By sketching a few such solutions ("trajectories"), the general behavior of a system can be easily understood.

Phase portraits can be constructed qualitatively, from knowledge of the eigenvalues and eigenvectors, and are often used for nonlinear system analysis as well.

Some examples:



Stable Node

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

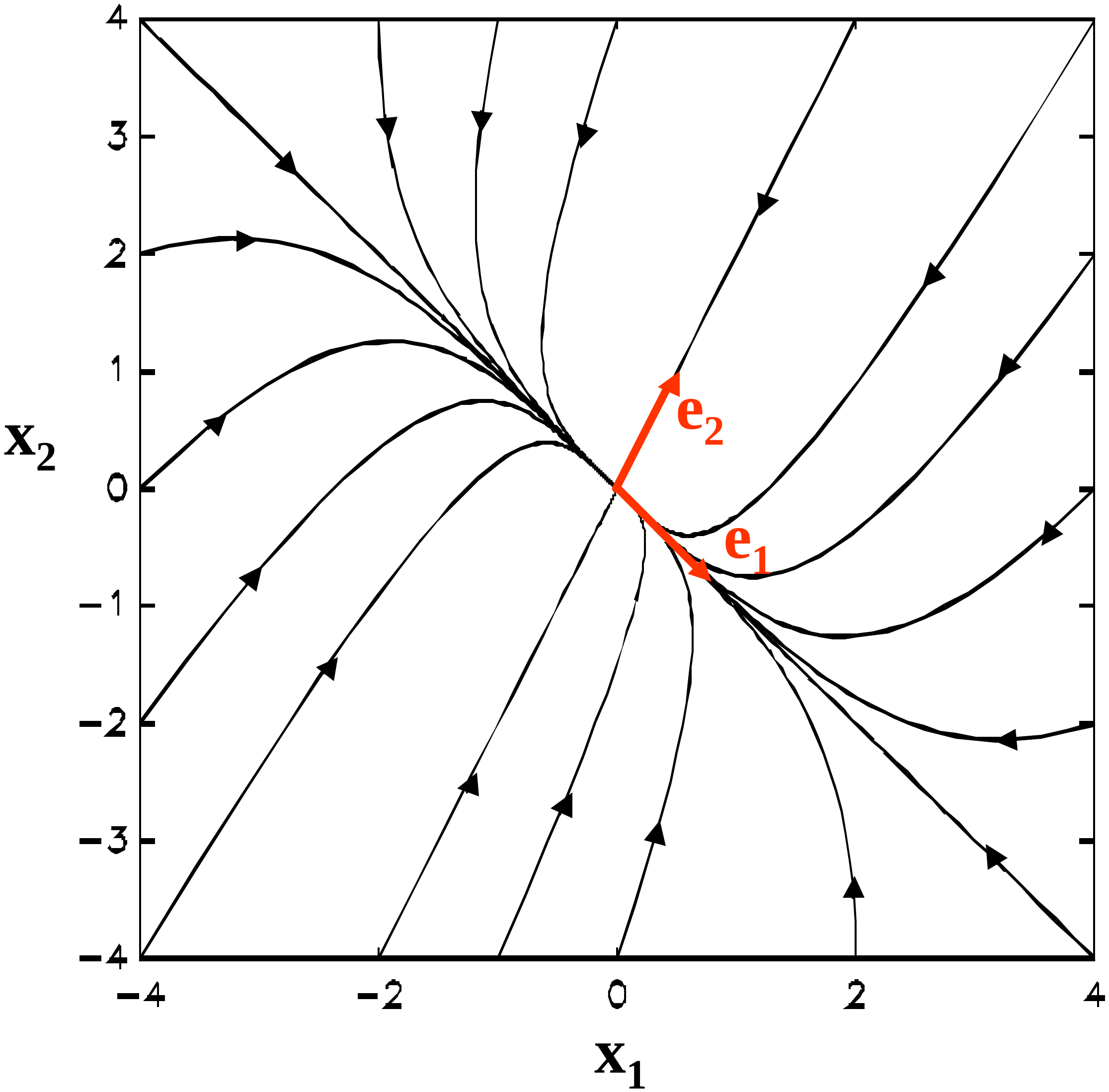
$$\dot{x} = Ax = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$l = \{-1 \quad -4\}$$

$$M = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{e}_1 \quad \mathbf{e}_2 \end{array} \Bigg\} \text{Eigen-vectors}$$

Show how you go from $x(0) \rightarrow 0$

Two invariant subspace you can ride $\mathbf{e}_1$ or $\mathbf{e}_2$ line to the origin



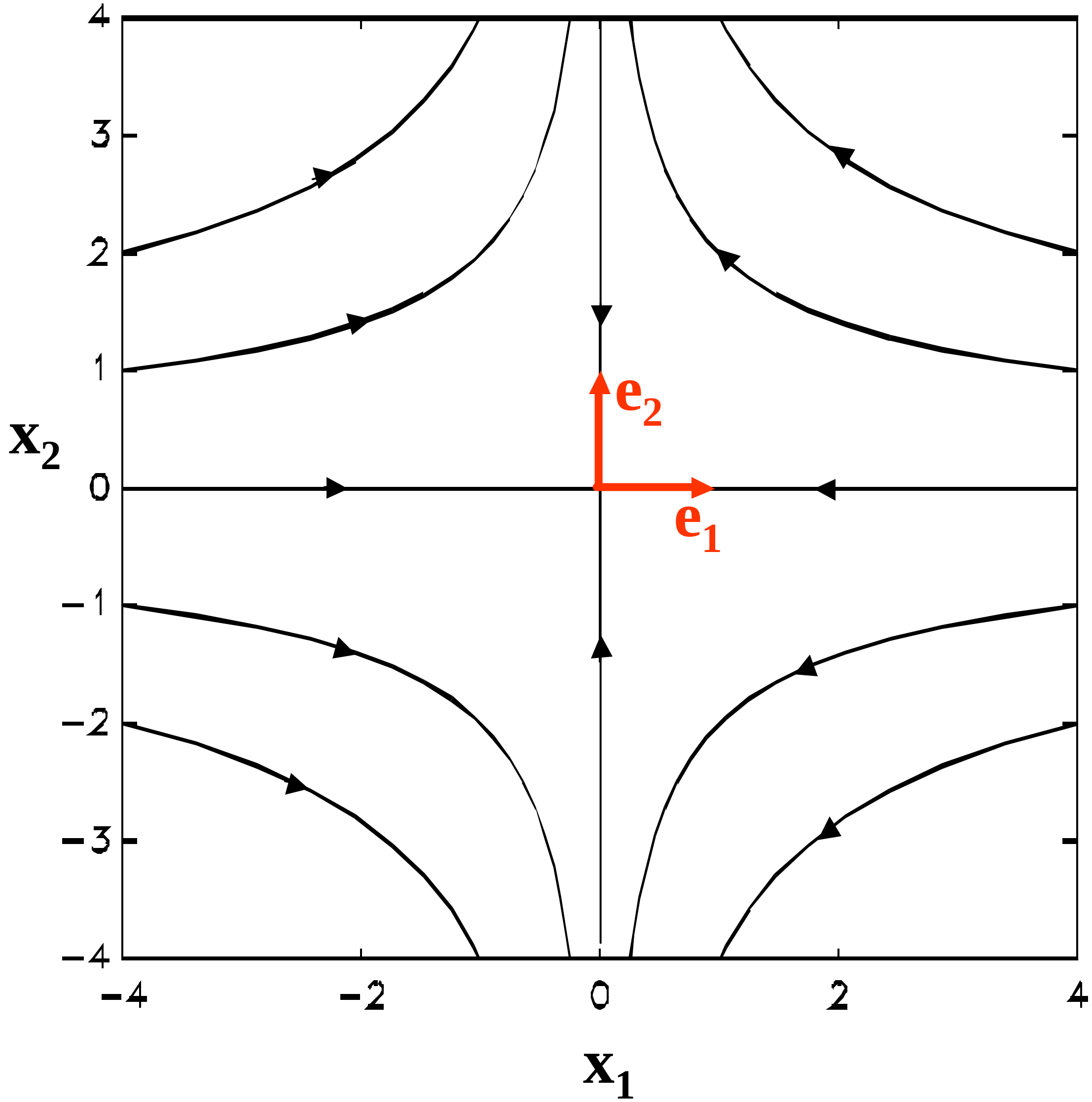
Stable Node

$$A = \begin{bmatrix} -2 & -1 \\ -2 & -3 \end{bmatrix}$$

$$\left.\lambda = \{-1 \quad -4\}\right\} \text{evals}$$

$$\left.M = \begin{array}{c} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \\ \begin{matrix} \mathbf{e}_1 & \mathbf{e}_2 \end{matrix} \end{array}\right\} \text{evectors}$$

Two Invariant Subspaces



Saddle Point

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

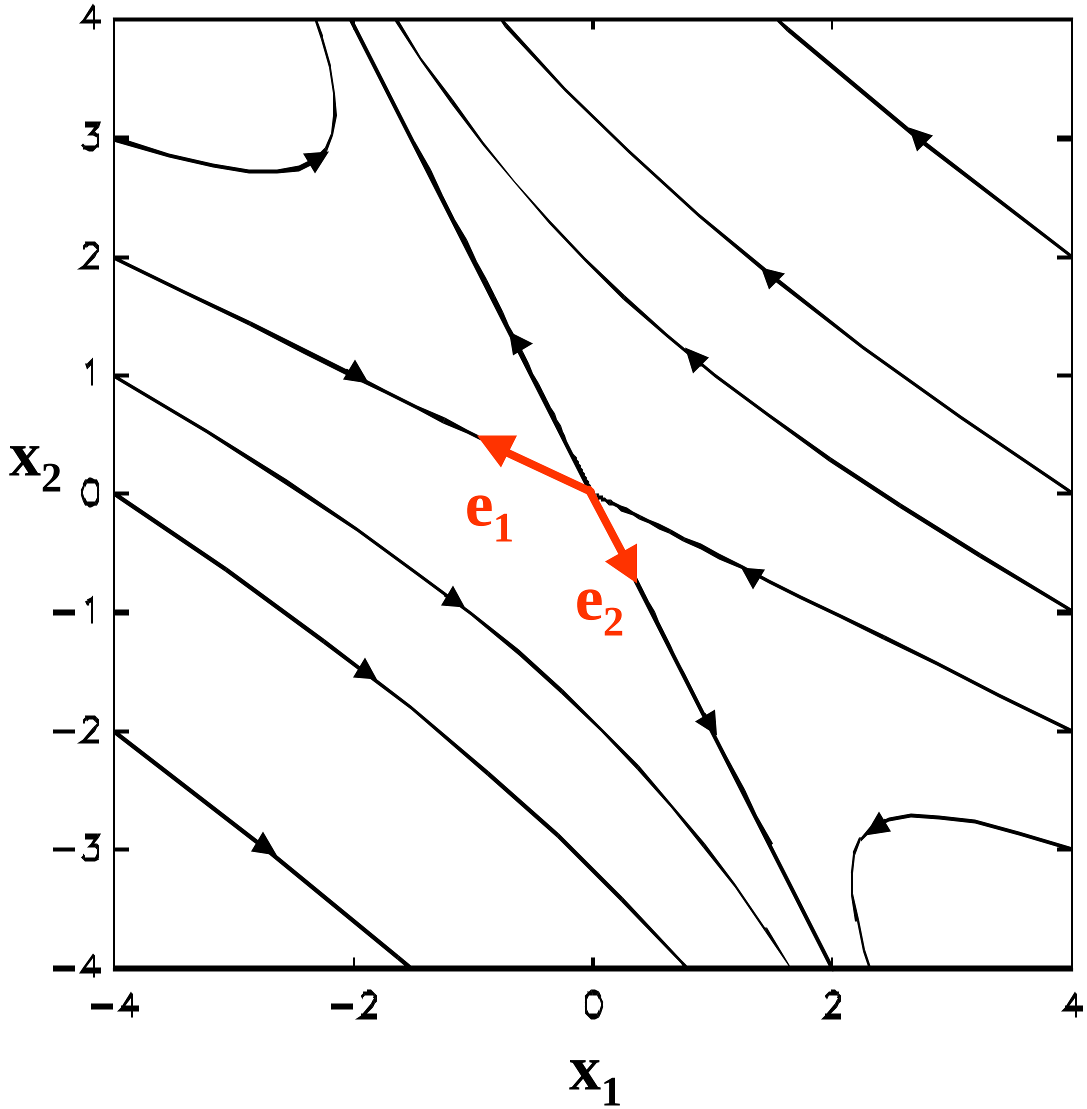
stable

unstable

$$\lambda = \{-2 \quad 1\} \quad \text{evals}$$

$$M = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{e_1} \quad \mathbf{e_2} \end{array} \quad \text{evectors}$$

Two Invariant Subspaces



Saddle Point

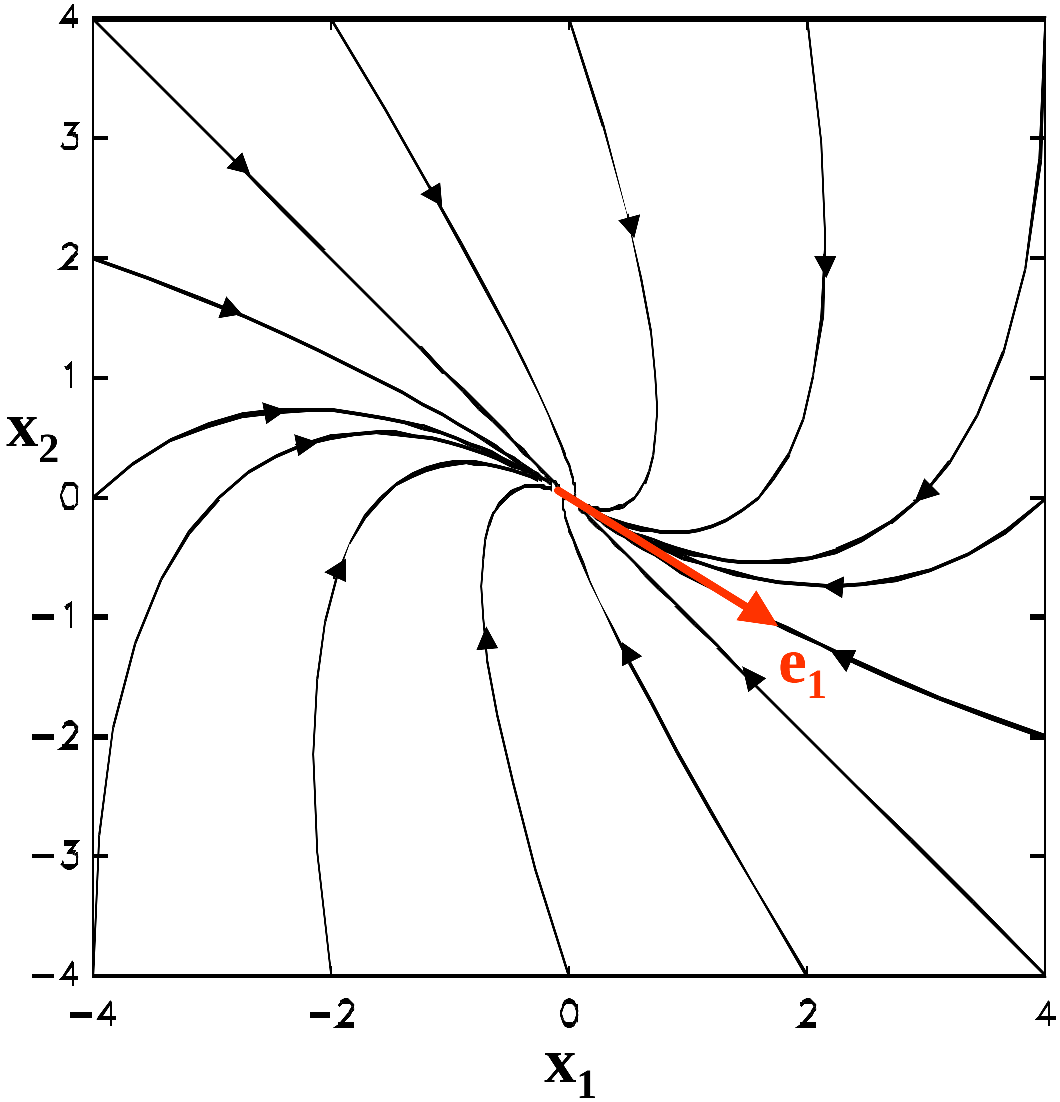
$$A = \begin{bmatrix} -3 & -2 \\ 2 & 2 \end{bmatrix}$$

$$\lambda = \{-2 \quad 1\}$$

$$M = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

stable invariant subspace

unstable invariant subspace



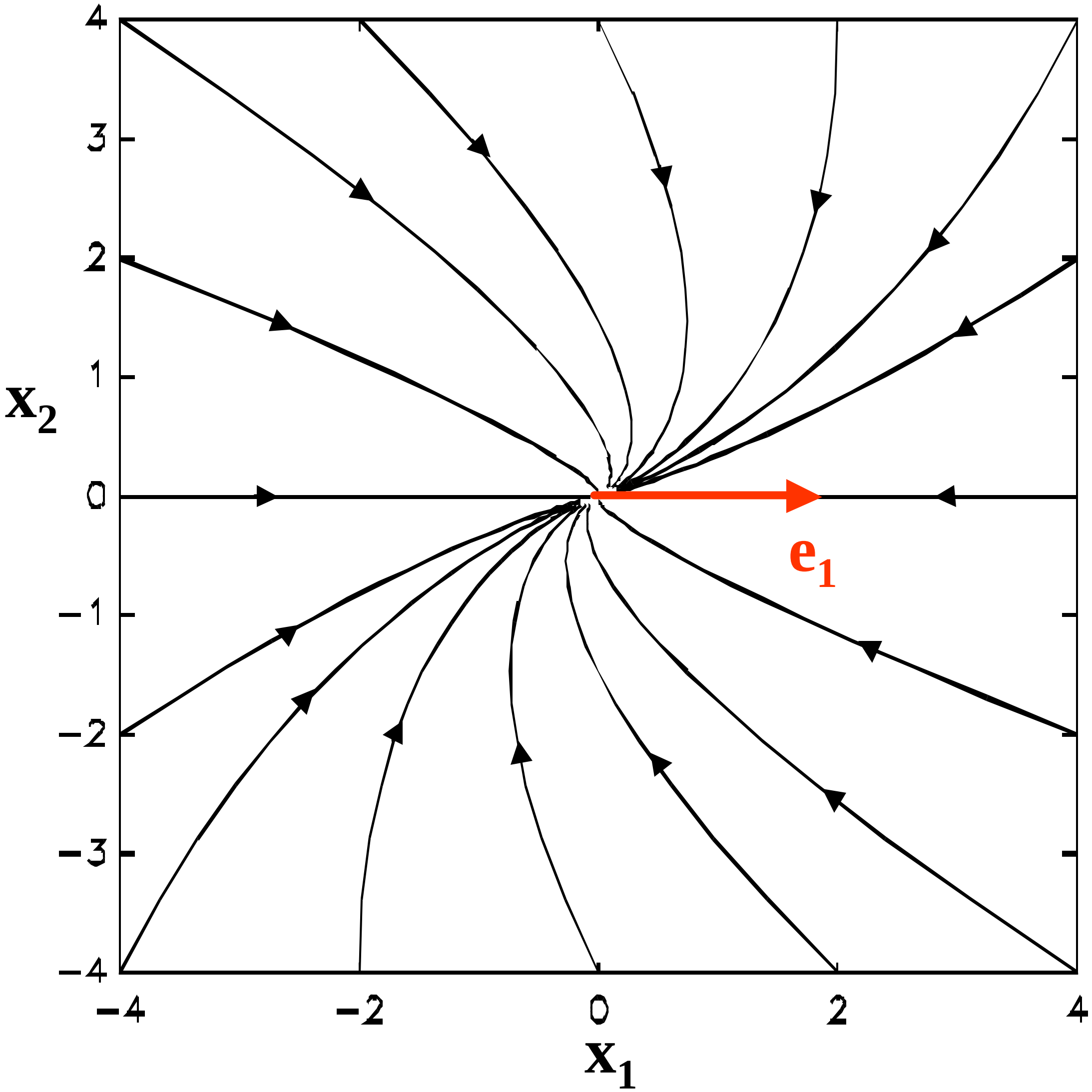
Stable Node

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$\lambda = \underbrace{\{-2 \quad -2\}}_{\text{eval}}$$

one eigenvector : $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

One Invariant Subspace



Stable Node

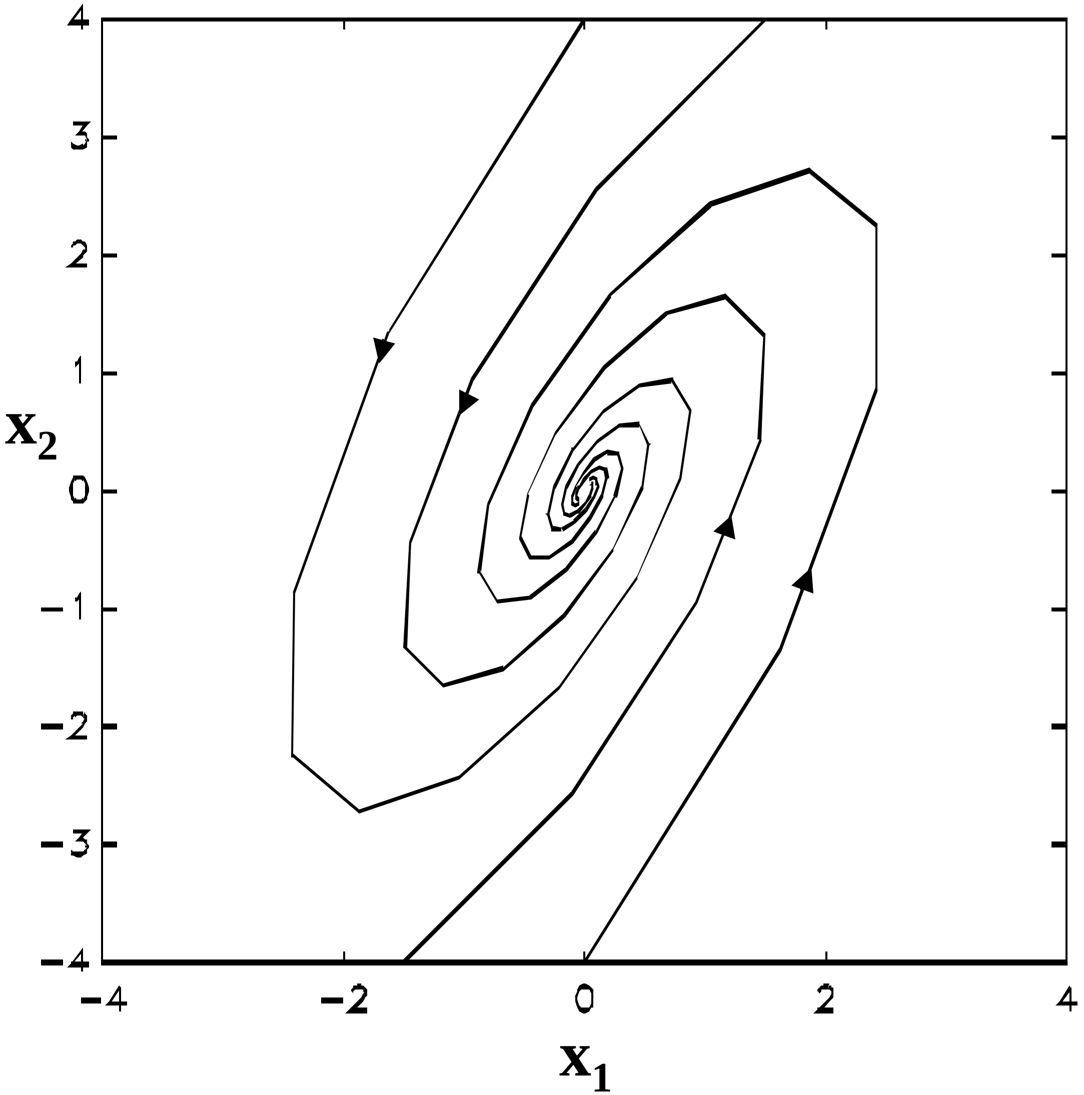
$$A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$ Prev. Example

(Jordan Form of $\begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$)

$\lambda = \{-2 \quad -2\}$

one eigenvector : $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

One Invariant Subspace Rotates Space



Stable Focus

$$A = \begin{bmatrix} 2 & -3 \\ 6 & -4 \end{bmatrix}$$

evals

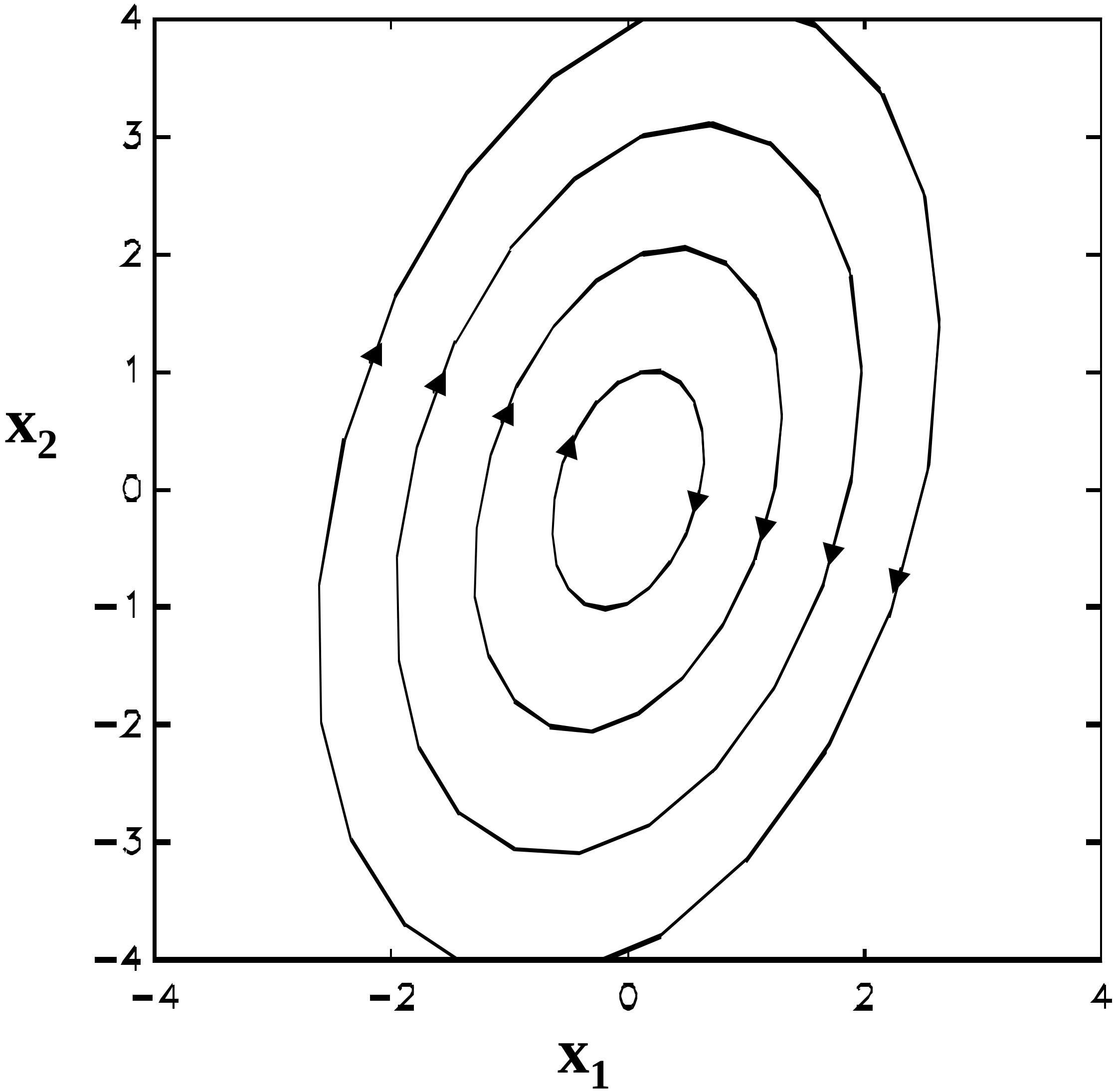
$$\lambda = \{-1 + j3 \quad -1 - j3\}$$

$$M = \begin{bmatrix} -1 + j1 & -1 - j1 \\ j2 & -j2 \end{bmatrix}$$

evectors

geometric interpretation of invariant subspace dissolves

Inward arrows stable
Spirals denote oscillation from imaginary part



Center

$$A = \begin{bmatrix} -1 & 2 \\ -5 & 1 \end{bmatrix}$$

evals

$$\lambda = \{3j \quad -3j\}$$

$$M = \begin{bmatrix} 3+j & 3-j \\ j5 & -j5 \end{bmatrix}$$

evectors

oscillations for all time

Time-varying case: Things get hairy.

To simplify some computations, consider the simpler homogeneous system:

$$\dot{x}(t) = A(t)x(t)$$

(For uniqueness, we ask that the elements of *A(t)* be continuous functions of time). Remember that the matrix exponential is no longer an integrating factor, so we must look for a different one:

It is known that the set of solutions of an *n*th order linear homogeneous differential equation (or a system of n first order equations) forms an *n-dimensional vector space*.

A basis of *n* such solutions can be chosen in a number of different ways, such as choosing a basis of *n* linearly independent *initial condition vectors* and using the resulting solutions. To make things easy, choose:

$$x_1(t_0) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x_2(t_0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \quad \cdots \quad x_n(t_0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

When we stack the resulting solutions together side-by-side, we get the **fundamental solution matrix**:

$$X(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}$$

Obviously, $\underbrace{\dot{X}(t) = A(t)X(t)}_{\text{matrix system}}$ Since $\left.\dot{x} = Ax\right\}$ vector system

And an expansion of the solution of the state vector $x(t)$ into this basis will be

$$x(t) = X(t)x(t_0)$$

So if we know the solution of the system to *n* linearly independent initial conditions, we know it for any by computing $X(t)$.

Now we notice from the identity

$$\left.\frac{dX^{-1}(t)}{dt} = -X^{-1}(t)\frac{dX(t)}{dt}X^{-1}(t)\right\}$$ Matrix Identity

$$\frac{dX^{-1}(t)}{dt} = -X^{-1}(t)\frac{dX(t)}{dt}X^{-1}(t)$$ same Identity

$$\frac{dX^{-1}(t)}{dt} = -X^{-1}(t)A(t)\underbrace{X(t)X^{-1}(t)}_{I}$$ substitute $\dot{X} = AX$

$$= -X^{-1}(t)A(t)$$ new identity

So $X^{-1}(t)$ qualifies as a valid integrating factor for the state equations:

$$\dot{X} = A(t)X(t) + B(t)u(t)$$

$$X^{-1}(t)[\dot{x}(t) - A(t)x(t) = B(t)u(t)]$$ premultiply both sides by X^{-1}

$$X^{-1}(t)\dot{x}(t) - X^{-1}(t)A(t)x(t) = X^{-1}(t)B(t)u(t)$$ rearrange

$$X^{-1}(t)\dot{x}(t) + \underbrace{\frac{dX^{-1}(t)}{dt}}_{\text{substitute new identity}}x(t) = X^{-1}(t)B(t)u(t)$$

Solving

$$X^{-1}(t)\dot{x}(t)+\frac{dX^{-1}(t)}{dt}x(t)=X^{-1}(t)B(t)u(t)$$ same

product rule $$\frac{d}{dt}\left[X^{-1}(t)x(t)\right]=X^{-1}(t)B(t)u(t)$$

$$X^{-1}(t)x(t)-X^{-1}(t_o)x(t_o)=\int_{t_0}^{t}X^{-1}(\tau)B(\tau)u(\tau)d\tau$$ integrate

premultiply by $X(t)$ and simplify

or

$$\boxed{x(t)=X(t)X^{-1}(t_0)x(t_0)+\int_{t_0}^{t}X(t)X^{-1}(\tau)B(\tau)u(\tau)d\tau}$$

This would be great if we knew $X(t)$ all the time, but unfortunately, it is difficult to compute.

$X(t)$ is the solution to $\dot{X}=A(t)X$

State Transition Matrix: Define the **State Transition Matrix** as:

$$\boxed{\Phi(t,\tau) = X(t)X^{-1}(\tau)}$$

This is an *nxn* linear transformation from the state-space into itself. For ***homogeneous*** systems, it relates the state vectors at any two times:

$$x(t) = \Phi(t,\tau)x(\tau)$$

$u = 0$

$u = 0$

From prev. page $X(t) = X(t)X^{-1}(t_0)X(t_0) \overset{t_0=t}{\Longrightarrow} x(t) = \Phi(t,t)X(t)$

(Verify this using $x(t) = X(t)x(t_o)$).

By differentiating it, one can show that:

$$\frac{d\Phi(t,\tau)}{dt} = \frac{d\left[X(t)X^{-1}(\tau)\right]}{dt} = \frac{dX(t)}{dt}X^{-1}(\tau)$$

$$= A(t)X(t)X^{-1}(\tau)$$

$$\frac{d\Phi(t,\tau)}{dt} = A(t)\Phi(t,\tau)$$

state transition matrix
satisfies original system
$\dot{X} = A(t)X$

(Chen uses this last line as the *definition* of $\Phi(t,\tau)$ in his book.) Using our definition,

$$\boxed{\Phi(t,\tau) = X(t)X^{-1}(\tau)}$$

it should be obvious that

$$\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0) = X(t_2)X^{-1}(t_1)(X(t_1))X(t_0)$$

and

$$\Phi^{-1}(t,t_0) = \Phi(t_0,t) \Rightarrow \left[X(t)X^{-1}(t_0)\right]^{-1} = X(t_0)X^{-1}(t)$$

If our system is time-invariant, then it is easy to verify that

$$\Phi(t,\tau) = e^{A(t-\tau)} \Big\} \quad \dot{X} = AX \qquad X(t) = e^{At}$$

by substitution into the definition.

When this is the case, we can compute $\Phi(t,\tau) = e^{A(t-\tau)}$ in many ways:

1. Because $X(s) = (sI - A)^{-1} BU(s) + (sI - A)^{-1} x(t_0)$

and

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

we can compare terms and get:

$$\boxed{e^{A(t-t_0)} = \mathbf{L}^{-1}\left\{(sI - A)^{-1}\right\}\Big|_{t-t_0} = \Phi(t, t_0)}$$

Note that $\Phi(t,\tau) = \Phi(t-\tau, 0)$

whenever A is a constant matrix.

property of
the exponential
function

2. Use the Cayley-Hamilton theorem to express e^{At} as:

$$\boxed{\Phi(t,\tau) = e^{A(t-\tau)} = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}}$$

and find the coefficients from the system of equations found by substituting eigenvalues of *A* in the scalar polynomial:

$$e^{l_i(t-t)} = a_0 + a_1 l_i + \cdots + a_{n-1} l_i^{n-1}$$

ith
eigenvalue

3. First simplify the system by putting it in diagonal form (or Jordan form). Then

$$\boxed{\Phi(t,\tau) = Me^{J(t-\tau)}M^{-1}}$$

4. "Sylvester's Expansion" (explained in Brogan)

5. Taylor series expansion:

$$\boxed{\Phi(t,\tau) = I + A(t-\tau) + \frac{1}{2!}A^2(t-\tau)^2 + \cdots}$$

However when *A=A(t)*, **none** of the choices are good:

1. Computer simulation of $\dot{\Phi}(t,\tau) = A(t)\Phi(t,\tau)$ with $\Phi(\tau,\tau) = I$

2. Define $B(t,\tau) = \int_{\tau}^{t} A(\varsigma)d\varsigma$. Then if $AB = BA$,

$$\Phi(t,\tau) = e^{B(t,\tau)}$$

3. Integral expansions (Peano-Baker series):

$$\Phi(t,t_0) = I_n + \int_{t_0}^{t} A(\tau_0)d\tau_0 + \int_{t_0}^{t} A(\tau_0)\int_{t_0}^{\tau_0} A(\tau_1)d\tau_1 d\tau_0 + \cdots$$

4. Approximate with discrete-time systems.

An Introduction to Discrete-Time Systems:

Consider the continuous-time linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0$$

and suppose it is sampled every *T* seconds to give a discrete-time system. Assume that this sampling speed is much faster than the rate at which *u(t)* changes, we therefore consider it to be constant over any individual sampling period $T = t_{k+1} - t_k$, i.e., $u(t) \approx u(t_k)$, for $t_k \le t \le t_{k+1}$.

Consider time t_k to be an initial condition and use the t_{k+1}. state-transition matrix to find the state vector at time

$$x(t_{k+1}) = \Phi(t_{k+1}, t_k)x(t_k) + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)B(\tau)d\tau u(t_k)$$

This can be further simplified if the system has a constant *A*-matrix, so that:

$$\Phi(t,\tau) = e^{A(t-\tau)}$$

$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}B(\tau)d\tau u(t_k)$$

$$x(t_{k+1}) = \underbrace{\boxed{e^{AT}}}_{A_d}x(t_k) + \underbrace{\boxed{\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}B(\tau)d\tau}}_{B_d}u(t_k)$$

This is the discrete-time approximation to the continuous-time system. If we want the state-transition matrix from a discrete-time system, we can use induction:

(We will give it a new name, $\Psi(k, j)$):

Recall the recursions we obtained in the example that introduced the concept of controllability of a discrete-time system:

$$x(k+1) = Ax(k) + Bu(k)$$

$$x(1) = Ax(0) + Bu(0)$$

$$x(2) = A^2x(0) + ABu(0) + Bu(1)$$

$$x(3) = A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2)$$

$$\vdots$$

$$x(k) = A^k x(0) + A^{k-1}Bu(0) + A^{k-2}Bu(1) + \ldots + Bu(k-1)$$

Or

$$x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-j-1} B(j) u(j)$$

or

$$\left. x(k) = A^k x(0) + \sum_{j=1}^{k} A^{k-j} B(j-1) u(j-1) \right\} \text{change the index}$$

Leading to:

$$\boxed{\Psi(k,j) = A^{k-j}}$$

It may be apparent that in state-variables, discrete-time systems are considerably easier to analyze than continuous-time systems.

If $A = A_d(k)$, that is, a discrete-time, time-varying system, then

$$\boxed{\Psi(k, j) = \prod_{i=j}^{k-1} A_d(i)}$$

Computation of eigenvalues, eigenvectors, and canonical forms for discrete-time systems is exactly the same as for continuous-time systems. The interpretation of eigenvalues in the context of stability properties will be different, but *modal decompositions and diagonalization procedures are exactly the same*:

If *M* is the modal matrix, we will get a diagonalized (or perhaps Jordan) form:

$$\boxed{\begin{gathered} q(k+1) = M^{-1}AMq(k) + M^{-1}Bu(k) \\ y(k) = CMq(k) + Du(k) \\ \text{and} \\ x(k) = Mq(k) \end{gathered}}$$