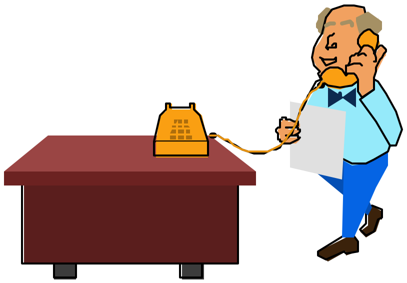
ECE 801: Linear State Space Systems



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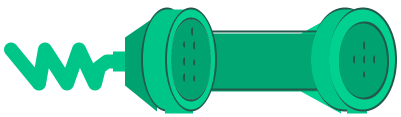

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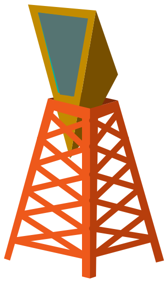
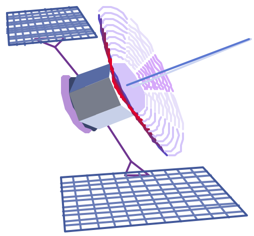
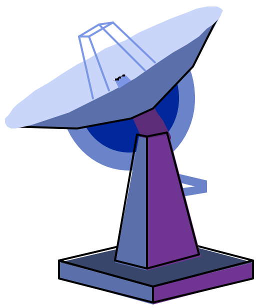
CLASS TIME: See the Schedule
OFFICE HOURS: To be announced.







Coverage of text:

- ☑ **Chapter 1: Background, Modeling, Intro. to State Variables**
- ☑ **Chapter 2: Vector Spaces**
- ☑ **Chapter 3: Linear Operators**
- ☑ **Chapter 4: Eigenvalues and Eigenvectors**
- ☑ **Chapter 5: Functions of Vectors and Matrices**
- ☑ **Chapter 6: Solutions to State Equations**
- ☑ **Chapter 7: Stability**
- ☑ **Chapter 8: Controllability and Observability**

☑ Chapter 9: Realizations

☑ Chapter 10: Feedback and Observers

☒ Chapter 11: Optimal Control and Estimation

Other Suggested Texts:

The MathWorks, Inc., " The Student Edition of MATLAB," Prentice-Hall, version 5, 1997

Brogan, W. R., “Modern Control Theory”, 3e, Prentice-Hall, 1991.

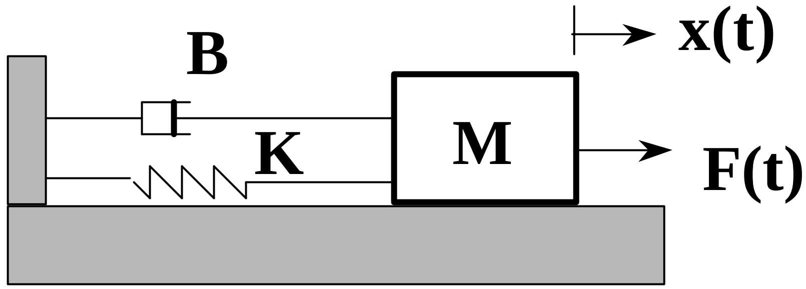
C.-T. Chen, "Linear System Theory and Design," Holt, Rinehart and Winston, 1984.

W. M. Wonham, "Linear Multivariable Control; A Geometric Approach," 3rd Ed., Springer-Verlag, 1985.

T. Kailath, "Linear Systems," Prentice-Hall, 1980.

Chapter 1 The Concept of "STATE"

First, an intuitive example:



Differential Equation (from F=Ma):

$$M\ddot{x}(t) + B\dot{x}(t) + Kx(t) = F(t)$$

Define "state variables and control ":

$$x_1 = x(t) \qquad u(t) = F(t)$$

$$x_2 = \dot{x}(t)$$

Now take derivatives:

$$\dot{x}_1 = \dot{x} = x_2$$

$$\dot{x}_2 = \ddot{x} = u/M - B/M\, x_2 - K/M\, x_1$$

Use vector-matrix form: $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **"State Vector"**

"State Equations" $\dot{X} = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix} X + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u$

"Output equation" $y(t) = x = x_1$

or

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} X$$

"State-Space Form"

$$\dot{X} = AX + Bu$$

$$Y = CX + Du$$

$$\dot{X} = \underbrace{\begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix}}_{A} X + \underbrace{\begin{bmatrix} 0 \\ 1/M \end{bmatrix}}_{B} u$$

$$y = \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{C} X + \overbrace{0}^{D} u$$

These equations, along with the initial conditions of the system (two of them!) are two first-order linear differential equations which provide exactly the same information as the original 2nd order linear differential equation.

But this "state variable description is not unique.

***Another* state-variable description:**

Let $\bar{x}_1 = x + \dot{x}$

$\bar{x}_2 = \dot{x}$

Then

$$\dot{\bar{x}}_1 = \dot{x} + \ddot{x} = \bar{x}_2 + {}^{1}\!/\!_{M}\, u - {}^{B}\!/\!_{M}\, \dot{x} - {}^{K}\!/\!_{M}\, x$$

$$= \bar{x}_2 + {}^{1}\!/\!_{M}\, u - {}^{B}\!/\!_{M}\, \bar{x}_2 - {}^{K}\!/\!_{M}\, (\bar{x}_1 - \bar{x}_2)$$

$$\dot{\bar{x}}_2 = \ddot{x} = {}^{1}\!/\!_{M}\, u - {}^{B}\!/\!_{M}\, \bar{x}_2 - {}^{K}\!/\!_{M}\, (\bar{x}_1 - \bar{x}_2)$$

So

$$\dot{\overline{X}} = \begin{bmatrix} -K/M & 1 + K/M - B/M \\ -K/M & K/M - B/M \end{bmatrix} \overline{X} + \begin{bmatrix} 1/M \\ 1/M \end{bmatrix} u$$

$$y = [1 \quad -1]\overline{X}$$

This state-variable representation, with the initial conditions, is *also* perfectly equivalent to the original 2nd order D.E.!

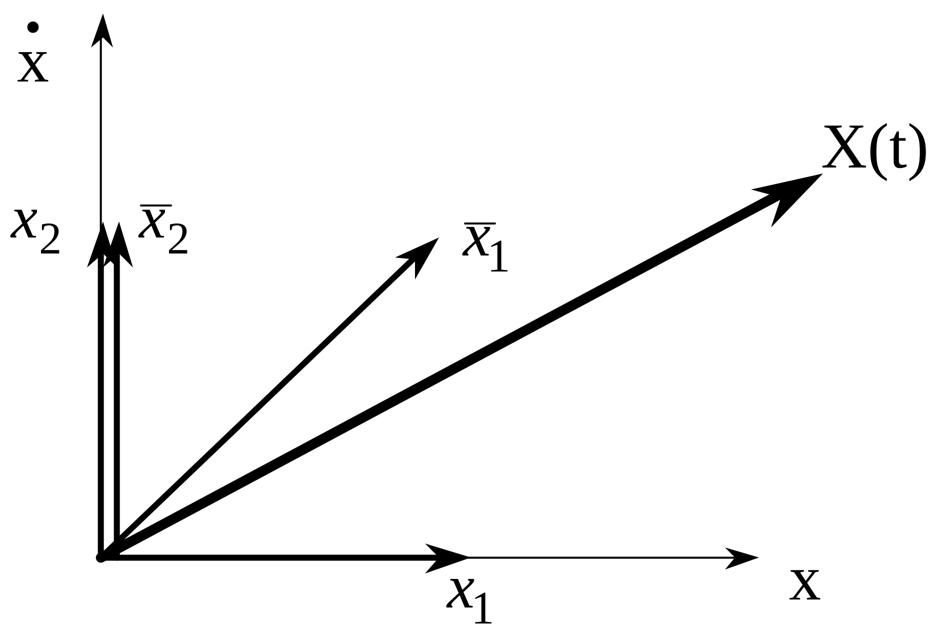
What's the difference?*

Are there advantages to one representation over the other?

***Ans: One set of state variables is a "transformed" version of the other. One can consider the two sets of variables** (x_1, x_2), $(\bar{x}_1, \bar{x}_2)$ **as different "coordinate systems" representing the *same* physical process.**

$$\bar{x}_1 = x_1 + x_2 = x + \dot{x}$$

$$\bar{x}_2 = x_2 = \dot{x}$$



Definitions of "State Variables" (Brogan):

Definition 1: The state variables of a system consist of a minimum set of parameters which completely summarize the system's status in the following sense. If at any time t_0, the values of the state variables $x_i(t_0)$ are known, then the output $y(t_1)$ and the values $x_i(t_1)$ can be *uniquely* determined for any time $t_1,\ t_1 > t_0$ provided the input $u_{[t_o, t_1]}$ is also known.

Definition 2: **The state at any time t_0 is a set of the minimum number of parameters $x_i(t_0)$ which allows a *unique* output segment $y_{[t_0, t_1]}$ to be associated with each input segment $u_{[t_o, t_1]}$ for every t_0 and for all $t > t_0$.**

In other words, if the set of variables we choose allows us, along with the initial conditions, to get the same information about output *y* from our "state equations" as we get from the system's overall differential equation, then our variables are state variables.

If the D.E. is *n*th order, there is going to be a set of *n* state variables, and hence, *n* state equations.

Obtaining state variables:

We have already seen that state variables are not uniquely chosen. This suggests many ways to select them from a D.E.

One sure-fire way is as follows:

D.E.: $$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = u(t)$$

Choose state variables:

$$x_1 = y \qquad x_2 = \frac{dy}{dt} \qquad x_3 = \frac{d^2 y}{dt^2} \quad \cdots \quad x_n = \frac{d^{n-1} y}{dt^{n-1}}$$

In discrete-time systems, use successive time-shifts rather than derivatives; i.e., x(k), x(k-1), . . . , x(k-n+1). (Aside Note)

Then in state-variable form:

$$\dot{X} = AX + Bu$$
$$y = CX + Du$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix}_{(n \times n)} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{(n \times 1)}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(1 \times n)} \quad \text{and} \quad D = 0_{\,(1 \times 1)}$$

These particular state variables are called "phase variables".

In this course, all other choices of state variables will be assumed to be linear combinations of these state variables.

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vectors **linear combinations** **matrices**

linear systems **spaces**

coordinate transformations **state variables**

. . . . All is leading to one thing: the need for Vector Spaces and Linear Algebra !!

Appendix A: Matrix Algebra

Matrix equality (matrices of equal size): equality element-by-element.

Matrix Addition:

If $\mathbf{A} = \left[a_{ij} \right]$ and $\mathbf{B} = \left[b_{ij} \right]$, then $\mathbf{A} + \mathbf{B} = \mathbf{C}$ means that $c_{ij} = a_{ij} + b_{ij}$

row index   column index

Matrix Multiplication:

If $\mathbf{A}$ is $(n \times m)$ and $\mathbf{B}$ is $(m \times p)$, then $\mathbf{C} = \mathbf{AB}$ implies that C is $(n \times p)$,

and that $c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$

$\longrightarrow$ **Matrices, in general, do not commute!!!!**

Null and Unit (Identity) Matrices:

$$\boldsymbol{0}_n = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{(n \times n)} \quad \text{and} \quad \boldsymbol{I}_\mathrm{n} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \\ & & & 1 & 0 \\ 0 & & \cdots & 0 & 1 \end{bmatrix}_{(n \times n)}$$

$$\mathbf{0}\mathbf{A} = \mathbf{0} \qquad \text{and} \qquad \mathbf{IA} = \mathbf{A}$$

Associative, Commutative, and Distributive Laws:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

$$\alpha\mathbf{A} = \mathbf{A}\alpha$$

$$\mathbf{A}(\mathbf{B}\mathbf{C}) = (\mathbf{A}\mathbf{B})\mathbf{C}$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{A}\mathbf{B} + \mathbf{A}\mathbf{C}$$

Always assuming compatible dimensions!!

Transposes and Symmetry:

If $\mathbf{A} = \left[a_{ij}\right]$ then $\mathbf{A}^T = \left[a_{ji}\right]$. If $\mathbf{A}^T = \mathbf{A}$, **then $\mathbf{A}$ is "symmetric."**

If $\mathbf{A}^T = -\mathbf{A}$, then $\mathbf{A}$ is "skew symmetric."

If $\mathbf{A} = \mathbf{A}^*$ (complex - conjugate transpose), then $\mathbf{A}$ is "Hermitian"

$$(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$$

Determinants: (square matrices only)

If $\mathbf{A}$ and $\mathbf{B}$ are both $(n \times n)$, then $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$

$|\mathbf{A}| = |\mathbf{A}^T|$

If a whole row or column is zero, or if any row or column is a linear combination of another row or column, then $|\mathbf{A}| = 0.$

"Rank", or $r(A)$, is the size of the largest nonzero determinant that can be formed while crossing out rows and columns of A.

If A is *(m x n)*, the rank of A must be $\leq \min(m, n)$ **.**

$q(A) = n - r(A)$ **= "nullity"**

"degeneracy" = "rank deficiency" = $\min(m, n) - r(A)$

If A is *square* and rank-deficient (rank <n), it is "singular," and $|\mathbf{A}| = 0$, otherwise "nonsingular" or "full rank"

Matrix Inverses:

Only square, nonsingular matrices have inverses.

If $\mathbf{A}^{-1} = \mathbf{B}$, then $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$

Sometimes we refer to "left" inverses and "right" inverses, usually for polynomial matrices.

$$(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$$

If $\mathbf{A}^{-1} = \mathbf{A}$, $\mathbf{A}$ is "involutory."

If $\mathbf{A}^{-1} = \mathbf{A}^T$, $\mathbf{A}$ is "orthogonal."

If $\mathbf{A}^{-1} = \overline{\mathbf{A}}^T$, $\mathbf{A}$ is "unitary."

(complex - conjugate transpose)

Trace: (square matrices only)

"trace" of A, or $tr(\mathbf{A})$, is the sum of all the elements on the diagonal.

$$tr(\mathbf{A}) = tr(\mathbf{A}^T)$$

$tr(\mathbf{AB}) = tr(\mathbf{BA})$ **the matrix AB must be square**

$tr(\mathbf{A}+\mathbf{B}) = tr(\mathbf{B}+\mathbf{A})$ **A and B must be square**

Block matrices:

"Block" matrices can be multiplied just as if their individual entries were scalars:

$$\left[\begin{array}{c:c} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \hdashline \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{array}\right]\left[\begin{array}{c:c} \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \hdashline \boldsymbol{B}_3 & \boldsymbol{B}_4 \end{array}\right] = \left[\begin{array}{c:c} \boldsymbol{A}_1\boldsymbol{B}_1 + \boldsymbol{A}_2\boldsymbol{B}_3 & \boldsymbol{A}_1\boldsymbol{B}_2 + \boldsymbol{A}_2\boldsymbol{B}_4 \\ \hdashline \boldsymbol{A}_3\boldsymbol{B}_1 + \boldsymbol{A}_4\boldsymbol{B}_3 & \boldsymbol{A}_3\boldsymbol{B}_2 + \boldsymbol{A}_4\boldsymbol{B}_4 \end{array}\right]$$

Each element in these matrices is a matrix itself.

"Elementary"operations and matrices:

Elementary Operations:

1. **Interchange any two rows or columns.**
2. **Multiply any row or column by a scalar**
3. **Add a multiple of one row (column) to another row (column) without altering the first row (column).**

Elementary Matrix:

Any matrix that can be obtained by applying any number of elementary operations to the identity matrix.

Matrix Calculus:

Matrices can have *functions* (of time, for example) as their individual elements. Differentiation and integration of matrices is done element-by-element; i.e.,

$$\dot{\boldsymbol{A}} = \frac{d\boldsymbol{A}(t)}{dt} = \left[\dot{a}_{ij}(t)\right], \text{ and } \int \boldsymbol{A}(t)dt = \left[\int a_{ij}(t)dt\right]$$

Note that this implies that Laplace transforms are done element-by-element too:

$$\mathcal{L}\begin{Bmatrix} \dot{X} = AX + Bu \\ y = CX + Du \end{Bmatrix} = \begin{Bmatrix} sX(s) = AX(s) + BU(s) + x_0 \\ Y(s) = CX(s) + DU(s) \end{Bmatrix}$$

Taking the first equation:

$$(sI_n - A)X(s) = BU(s) + x_0$$

$$X(s) = (sI_n - A)^{-1}BU(s) + (sI_n - A)^{-1}x_0$$

Substituting into the second equation: $\left(Y(s) = CX(s) + DU(s)\right)$

$$Y(s) = \underbrace{C(sI_n - A)^{-1}BU(s) + DU(s)}_{\text{"zero-state" solution}} + \underbrace{C(sI_n - A)^{-1}x_0}_{\text{"zero-input" solution}}$$

"zero-state" solution **"zero-input" solution**

TRANSFER FUNCTION: Suppose initial conditions all zero:

$$Y(s) = \underbrace{\left[C(sI_n - A)^{-1}B + D\right]}_{\text{transfer function } H(s)} U(s)$$

transfer function $H(s)$

Note: We cannot write $\cancel{Y(s)/U(s) = H(s)}$ **because** $U(s)$

might be a vector!

Some other properties of matrix calculus:

$$\frac{\partial (Ax)}{\partial x} = A$$

$$\frac{\partial (x^{\mathrm{T}} A y)}{\partial y} = x^{\mathrm{T}} A$$

$$\frac{\partial (x^{\mathrm{T}} A y)}{\partial x} = (Ay)^{\mathrm{T}} = y^{\mathrm{T}} A^{\mathrm{T}}$$

$$\frac{\partial (x^{\mathrm{T}} A x)}{\partial x} = x^{\mathrm{T}} A + x^{\mathrm{T}} A^{\mathrm{T}} = x^{\mathrm{T}} \left(A + A^{\mathrm{T}}\right) \left(= 2 x^{\mathrm{T}} A \quad \text{if } \boldsymbol{A} \text{ is symmetric}\right)$$

$$\frac{\partial [y^{\mathrm{T}} A x]}{\partial x} = \frac{\partial [x^{\mathrm{T}} A^{\mathrm{T}} y]}{\partial x}$$

$$\frac{\partial A^{-1}(t)}{\partial t} = -A^{-1} \frac{\partial A}{\partial t} A^{-1}$$

***a*: vector (column)**
***A*: matrix**
***x*: column vector**

If matrices A and B are functions of scalar *t*, but X is not;

$$\frac{\partial(AB)}{\partial t} = \frac{\partial A}{\partial t} B + A \frac{\partial B}{\partial t} \qquad \text{(note the order!)}$$

$$\frac{\partial(XA)}{\partial t} = X \frac{\partial A}{\partial t}$$

If a scalar function *f* is a scalar *function of a vector x*, for example if

$$f(x_1, x_2, x_3) = 2x_1 + x_2 x_3 - \sin(x_3)$$

then

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \dfrac{\partial f}{\partial x_3} \end{bmatrix}$$

(1 x n)

If *f* is a *vector* function of a *vector x*, for example if

$$f(x) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

then

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & & \vdots \\ & & \ddots & \\ \frac{\partial f_m(x)}{\partial x_1} & \cdots & & \frac{\partial f_m(x)}{\partial x_n} \end{bmatrix}$$

***x* or *f* without a subscript indicates a vector quantity**

Note that the derivative of an *m*-dimensional vector with respect to an *n*-dimensional vector is an (*m* x *n*) matrix.

From Now On: Matrices will be denoted in capital, but not necessarily boldface, letters. Their interpretation as matrix quantities should be apparent from the context.