Chapter 8: Controllability and Observability

We saw the basic idea of controllability and observability from the examples at the end of Chapter 3.

From those examples, a linear system is "controllable" if we can find inputs to make the state vector go to zero from any initial condition.

The system is "observable" if we can always find an unknown initial condition given the history of inputs and outputs.

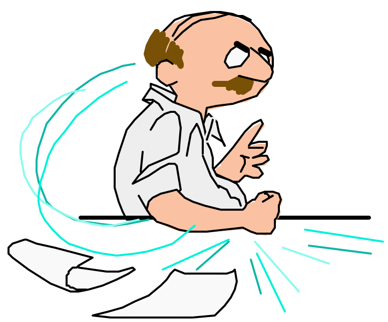
In this chapter, we consider these concepts for continuous-time systems and discuss their implications and importance.

Some Definitions:

<u>Definition</u>: A linear system is **controllable** at t_0 if it is possible to find an input function $u(t)$, defined over the time of interest, that will transfer the initial state $x(t_0)$ to the origin in finite time.

If this is true regardless of the initial time and initial condition, the system is said to be **completely controllable**.

Determining whether or not a system is controllable will be important before one wastes a lot of time trying to design a controller.



Definition: A linear system is **observable** at t_0 if $x(t_0)$ can be determined from the output function recorded over finite time, $y(t), \quad t_0 \le t \le t_1 < \infty$, assuming the input $u(t)$ is known also.

If this is true regardless of the initial time and initial state, the system is **completely observable**.

Controllability will depend on the **A** and **B** matrices.

Observability will depend on the **A** and **C** matrices.

Controllability and Observability Criteria (Tests):

We resort to the convenient ability to find the Jordan Form of systems using the modal matrix.

Suppose, for now, that we can *diagonalize* a system; (i.e., the *A*-matrix has a complete set of *n* linearly independent eigenvectors.

$$x = M\bar{x}$$

$$\dot{x} = M\dot{\bar{x}}$$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \longrightarrow \begin{aligned} \dot{\bar{x}} &= M^{-1}AM\bar{x} + M^{-1}Bu \\ y &= CM\bar{x} + Du \end{aligned}$$

diagonal

$$\dot{\bar{x}} = \boxed{\bar{A}}\bar{x} + \bar{B}u$$
$$y = \bar{C}\bar{x} + \bar{D}u$$

Then we have a set of *n* **decoupled** first order equations, each of which looks like:

$$\dot{\bar{x}}_i = \lambda_i \bar{x}_i + \bar{b}_i u(t)$$

where $\bar{b}_i$ is the *i* th row of the matrix $\bar{B} = M^{-1}B$.

So if this *i* th row is entirely *zero*, then the input cannot affect the *i* th state equation, and that ***mode*** will not be controllable.

A diagonalizable LTI system is (completely) controllable iff there exists only one Jordan block per eigenvalue, and there are no zero rows of $\bar{B} = M^{-1}B$.

What if the system is not diagonalizable, but can be transformed into a Jordan Form?

We can still use this form to determine controllability by inspection. Recall that the Jordan form shows how the system modes are coupled into a chain within the Jordan ***blocks***. It is apparent that the input applied to the "bottom" mode will influence the chained modes through the '1' above the matrix diagonal.

A system in Jordan form that has only ***one*** Jordan block for each eigenvalue is completely controllable iff the row of the matrix $\overline{B} = M^{-1}B$ corresponding to the **bottom row** of ***each*** Jordan block is nonzero.

IMPORTANT:

When we have MIMO equations, and the possibility of more than one Jordan block per eigenvalue, the collection of rows of the B-matrix that correspond to the last rows of Jordan blocks *of the same eigenvalue* in the A-matrix must be ***linearly independent***. Last rows corresponding to Jordan blocks resulting from ***different*** eigenvalues need not be linearly independent of rows from **other** eigenvalues.

Of course for SISO systems, this condition just reduces to the non-zero row/column requirement.

Example:

$$
\overline{A} = \begin{bmatrix}
-5 & 0 & & & \cdots & & & 0 \\
0 & -3 & 1 & & & & & \\
 & & -3 & 0 & & & & \\
 & & & 4 & 0 & & & \vdots \\
\vdots & & & \ddots & 3 & 0 & & \\
 & & & & & 0 & 0 & 0 \\
 & & & & & & -2 & 1 \\
0 & & & \cdots & & & 0 & -2
\end{bmatrix}
\quad
\overline{B} = \begin{bmatrix}
\boxed{1} \\ 0 \\ \boxed{4} \\ \boxed{0} \\ \boxed{1} \\ \boxed{2} \\ 1 \\ \boxed{0}
\end{bmatrix}
$$

$$
\overline{C} = \begin{bmatrix} 1 & 1 & 0 & -5 & 1 & 2 & 2 & 1 \end{bmatrix}
$$

The system given by these matrices is ***not*** controllable, because the indicated rows of the *B*-matrix are **not** all nonzero.

Exactly analogous reasoning works for observability also. That is; we require that each mode be passed to the output through the *C*-matrix, either directly, or through the "chain" of modes in a Jordan block.

A diagonalizable LTI system with only a single Jordan block per eigenvalue is observable iff the matrix $\overline{C} = CM$ has no zero ***columns***.

An LTI system in Jordan form is observable iff the columns of the matrix $\overline{C} = CM$ corresponding to the **first columns** of each Jordan block for a particular eigenvalue are **linearly independent**.

NOTE that we must be looking at Jordan (or diagonal) forms for these to apply; un-transformed systems might have zero rows (columns), and still be controllable (observable).

Example:

$$\bar{A} = \left[\begin{array}{c|cc|c|c|c|cc}
-3 & 0 & & & \cdots & & & 0 \\
\hline
0 & -3 & 1 & & & & & \\
 & & -3 & 0 & & & & \\
\hline
 & & & 4 & 0 & & & \vdots \\
\hline
\vdots & & & \ddots & 4 & 0 & & \\
\hline
 & & & & & 0 & 0 & 0 \\
\hline
 & & & & & & -2 & 1 \\
0 & & & \cdots & & & 0 & -2
\end{array}\right] \quad \bar{B} = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} \boxed{1} & \boxed{1} & 0 & \boxed{-5} & \boxed{1} & \boxed{2} & \boxed{2} & 1 \end{bmatrix}$$

NOT LINEARLY INDEPENDENT

This system is <u>not</u> observable

This controllability/observability criterion is called "Modal Controllability/Observability." It is geometrically the requirement that ***the mapping of the input into the state-space by the B-matrix should span the entire space***; i.e., ***have an influence in all directions.*** (Similarly for observability).

There is another type of canonical form that allows one to determine the controllability (observability) of a system by inspection. These are the ***controllable canonical form*** and the ***observable canonical form***:

Controllable Canonical Form:

$$A=\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \quad B=\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$C = [\text{ arbitrary }]$

Given a system in this form, provided that the ***last entry*** of the *B*-matrix is nonzero, the system will be controllable.

Note that the coefficients in this bottom row are the negative, reverse order coefficients of the original D.E. (and hence, the characteristic polynomial, too)

Observable Canonical Form:

$$A = \begin{bmatrix} -a_{n-1} & 1 & & & \\ -a_{n-2} & 0 & 1 & & \\ \vdots & & 0 & \ddots & \\ -a_1 & & & \ddots & 1 \\ -a_0 & & & & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \text{arbitrary} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & & \cdots & 0 \end{bmatrix}$$

Given a system in this canonical form, provided that the ***first entry*** of the *C*-matrix is nonzero, the system will be observable.

For the previous two tests to be useful, and to show why they are true, we need a much more general test for controllability/observability. This will be useful for systems with *arbitrary* eigenvalues/vectors.

Theorem: An LTI system is completely controllable iff the matrix

$$P \stackrel{\Delta}{=} \begin{bmatrix} B & AB & A^2 B & \cdots & A^{n-1} B \end{bmatrix}$$

has ***full rank***; i.e., rank n.

This is called the ***controllability matrix***.

Theorem: An LTI system is completely observable iff the matrix:

$$Q \triangleq \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has ***full rank***; i.e., rank *n*.

This is called the ***observability matrix***.

These two tests are the ones most often used to determine controllability and observability.

PROOF: (For the controllability part only. Observability is proven the same way, with mostly changes in notation).

Recall the solution to the LTI state equations:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Given an arbitrary initial condition, we would like, after finite time t_1, to get to the *zero* state. So we require that

$$\left.\int_{t_0}^{t_1} e^{A(t-\tau)}Bu(\tau)d\tau = -e^{A(t_1-t_0)}\underbrace{x(t_0)}_{\text{known}} \overset{\Delta}{=} \xi_1 \quad\right\} \quad x(t)\big|_{t=t_1} = 0$$

ξ_1 will be arbitrary (because the initial condition $x(t_0)$ is arbitrary).

Find condition on the LHS so the RHS can be solved for

From the Cayley-Hamilton theorem, it will be possible to write:

$$e^{A(t_1-\tau)} = \sum_{i=1}^{n} \gamma_i(\tau) A^{n-i}$$

substitute

Now if we define

$$\Gamma_i \stackrel{\Delta}{=} \int_{t_0}^{t_1} \gamma_i(\tau) u(\tau) d\tau$$

then we can write

$$\xi_1 = \int_{t_0}^{t_1} e^{A(t_1-\tau)} B u(\tau) d\tau = \int_{t_0}^{t_1} \left[\sum_{i=1}^{n} \gamma_i(\tau) A^{n-i} \right] B u(\tau) d\tau$$

expand

$$= \underbrace{A^{n-1} B \int_{t_0}^{t_1} \gamma_1(\tau) u(\tau) d\tau + A^{n-2} B \int_{t_0}^{t_1} \gamma_2(\tau) u(\tau) d\tau + \cdots + B \int_{t_0}^{t_1} \gamma_n(\tau) u(\tau) d\tau}_{\text{use definition of } \Gamma}$$

or equivalently,

$$\xi_1 = \left(B\Gamma_n + AB\Gamma_{n-1} + \cdots + A^{n-1}B\Gamma_1\right) = P\begin{bmatrix}\Gamma_1 \\ \vdots \\ \Gamma_n\end{bmatrix}$$

For us to be sure that $\xi(t_1)$ lies in the *range space of* P, P should have rank *n*. ■

P is the controllability matrix

A more general proof results from the consideration of *time-varying* systems; the general approach starts out the same as the previous proof:

For the time-varying case, we have as the solution for the state at time t_1:

$$x(t_1) = \Phi(t_1, t_0)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

Again, the vector $\xi_1 \overset{\Delta}{=} \Phi(t_1, t_0)x(t_0)$ will be an arbitrary constant vector if the initial condition itself is arbitrary.

We use this property in the proof of the following theorem:

Theorem: The time-varying system given by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

is completely controllable iff the matrix $G_c(t_0, t_1)$ defined below is ***positive definite*** for every t_0 and every $t_1 > t_0$.

$$G_c(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^*(\tau)\Phi^*(t_0, \tau)d\tau$$

This matrix is called the ***Controllability Grammian***. Its being positive definite also implies that the columns of $\Phi(t_0, \tau)B(\tau)$ are *linearly independent* time functions. It also happens to be the solution to the Lyapunov equation: $AG_c + G_cA^* = -BB^*$

PROOF: We will prove sufficiency and necessity separately.

Sufficiency: If G is positive definite, then G^{-1} exists. We are allowed to (somewhat magically) choose

$$u(t) = -B^*(t)\Phi^*(t_0,t)G^{-1}(t_0,t_1)x(t_0)$$

Input is a free variable

as an input which we substitute into the solution for $x(t_1)$:

$$x(t_1) = \Phi(t_1,t_0)x(t_0) - \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^*(\tau)\Phi^*(t_0,\tau)G^{-1}\underbrace{(t_0,t_1)x(t_0)}_{\text{no } \tau \text{ dependence}}d\tau$$

factor out $\Phi(t_1,t_0)$ from the left:

$$\Phi(t_1,t_0)\{x(t_0)-\int_{t_0}^{t_1}\Phi(t_0,\tau)B(\tau)B^*(\tau)\Phi^*(t_0,\tau)d\tau G^{-1}(t_1,t_0)$$

$$\cdot\, x(t_0)d\tau\}$$

$$\Phi\left(t,t_0\right)\Phi\left(t_0,t\right)=\Phi\left(t,t\right)$$

$$=\Phi(t_1,t_0)\left\{x(t_0)-G(t_1,t_0)G^{-1}(t_1,t_0)x(t_0)\right\}$$

use def of G

$$=\Phi(t_1,t_0)\left\{x(t_0)-x(t_0)\right\}$$

$$=0$$

Thus, the control we chose suffices to get us to the zero state. This implies that the system <u>is</u> controllable.

Necessity: Now we suppose the system is controllable and show that G must be positive definite.

Exploiting the definition of positive definite-ness for matrices, we choose any nonzero vector z and compute

$$\langle z, G_c z \rangle = \int_{t_0}^{t_1} z^* \Phi(t_0, \tau) B(\tau) B^*(\tau) \Phi^*(t_0, \tau) z \, d\tau$$

apply def of G_c

$$= \int_{t_0}^{t_1} \left\| B^*(\tau) \Phi^*(t_0, \tau) z \right\|^2 d\tau \geq 0$$

This tells us that G_c is at least positive *semi*-definite. To show *definite*ness, we can show nonsingularity. ☞

We use proof by contradiction to show nonsingularity. Suppose there *did* exist a nonzero vector z such that:

$$\langle z, G_c z \rangle = 0$$

Take this z and define a new time function

$$f(t) \overset{\Delta}{=} -B^*(t)\Phi^*(t_0, t)z \quad \Big\} \text{ special choice}$$

Taking the squared norm of f and integrating,

$$\int_{t_0}^{t_1} \| f(t) \|^2 dt = \int_{t_0}^{t_1} z^* \Phi(t_0, t) B(t) B^*(t) \Phi^*(t_0, t) z dt$$

$$= \langle z, G_c(t_0, t_1) z \rangle = 0$$

So because its norm is zero over all time intervals regardless of the value of $G_c(t_0, t_1)$, then

$$f(t) \equiv 0$$

for all time and it must also be true that

$$0 = \int_{t_0}^{t_1} f^*(t)u(t)dt$$

for ***any*** $u(t)$.

Substituting in $f(t)$:

$$0 = -\int_{t_0}^{t_1} z^*\Phi(t_0, t)B(t)u(t)dt$$

$$0 = -\int_{t_0}^{t_1} z^* \Phi(t_0, t) B(t) u(t) dt$$ from previous slide

Because we have asserted that the system is totally controllable, we are free to choose the input that transfers the **zero** state to $\Phi(t_1, t_0) z \neq 0$. That is, choose $u(t)$ such that

$$\Phi(t_1, t_0) z = \int_{t_0}^{t_1} \Phi(t_1, t) B(t) u(t) dt$$

or, by multiplying both sides by $\Phi(t_0, t_1)$:

$$z = \int_{t_0}^{t_1} \Phi(t_0, t) B(t) u(t) dt$$ use properties of the state transition matrix

Substituting this into the equation at top of the previous page, we get

contradiction established

$$\int_{t_0}^{t_1} z^* z\,dt = \int_{t_0}^{t_1} \|z(t)\|^2\, dt = 0$$

By assumption z must be non zero

But this would be impossible for nonzero z. We therefore conclude that there is no nonzero z for which $\langle z, Gz \rangle = 0,$ so G is ***positive definite***. ■

As usual, an entirely analogous result holds for the *observability* of a linear system.

Define the **Observability Grammian** as:

$$G_o(t_0,t_1)=\int_{t_0}^{t_1}\Phi^*(\tau,t_0)C^*(\tau)C(\tau)\Phi(\tau,t_0)d\tau$$

<u>Theorem</u>: A linear system is completely observable iff the observability Grammian is **positive definite**.

PROOF: You don't really want me to prove this, do you?

$$\boxed{G_oA+A^*G_o=-C^*C}$$

G_o is the solution to the following Lyapunov equation

Note that Brogan shows an easy way to get the controllability and observability **matrix** tests from these theorems, using the knowledge that the systems are time-invariant.

Otherwise these theorems are not too useful, because they require explicit solutions to the state equations. (Unless we compute the Grammians from the Lyapunov equations.)

Useful for proofs and not calcuations

A couple of other (semi-)useful tests:

Theorem: An n th order LTI system is completely controllable iff for all s:

$$rank\left(\left[sI - A \;\vdots\; B\right]\right) = n$$

Theorem: An n th order LTI system is completely observable iff for all s:

$$rank\left(\begin{bmatrix} sI - A \\ \hdashline C \end{bmatrix}\right) = n$$

These are sometimes called the Hautus tests.

Controllable and Observable Forms and Canonical Decompositions:

Recall the tests that were given for controllability and observability for systems that are *given* in controllable or observable form: the last row of the *B*-matrix (first column of the *C*-matrix) should be non-zero.

The next result shows us why this test is *not* useful unless the systems are **given** in these forms:

The similarity transformation that gives us a controllable canonical form depends on the controllability matrix:

$$P = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

If we take the last row of the inverse of this matrix, call it p:

$$P^{-1} = \begin{bmatrix} \vdots \\ \vdots \\ \hline p \end{bmatrix}$$

And form a new matrix as follows:

$$U^{-1} = \begin{bmatrix} p \\ pA \\ pA^2 \\ \vdots \\ pA^{n-1} \end{bmatrix}$$

Then the inverse of this new matrix is a similarity transformation matrix that changes the system into controllable canonical form.

$$A_c = U^{-1} A U$$

(As usual, a similar result holds for observable canonical form, using the first column of the observability matrix.)

The reason this is not a good test for controllability (or observability) is that in order for us to find the inverse of the controllability matrix, it must *already be known controllable (observable)!!*

However, the transformation to controllable (observable) form will be very useful when we study state-feedback.

Kalman Decompositions: When the system is not necessarily controllable or observable, we can decompose it in order to see exactly which modes are contr./obsv., and which are not.

$$\boxed{P = \mathcal{C}(A,B)}$$

Theorem: Suppose a state-space system is not controllable, shown by its controllability matrix being rank $r(P)\ (<n)$. Construct a similarity transformation matrix from any $r(P)$ linearly independent columns of the controllability matrix P and append to it ***any*** $n-r(P)$ *other* columns such that the resulting $n \times n$ matrix V is nonsingular.

$$V = \left[v_1 \quad \cdots \quad v_{r(P)} \;\middle|\; v_{r(P)+1} \quad \cdots \quad v_n \right] \Bigg\} \text{ nonsingular matrix}$$

Then the similarity-transformed system is of the form

$$\dot{w} = V^{-1}AVw + V^{-1}Bu \qquad (x = Vw)$$
$$y = CVw + Du$$

or in partitioned form:

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + Du$$

None of the state variables w_2 (there are $n - r(P)$ of them) are affected either directly or indirectly by the input, and so are *all uncontrollable*. The others are all *controllable*.

If we perform the analogous procedure by taking $r(Q)$ rows of the *observability* matrix and appending sufficient other rows to get an $n \times n$ nonsingular similarity transformation matrix U, we get:

$Q = \mathcal{O}(A,C)$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u \qquad (z = Ux)$$

$$y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + Du$$

Here, the first $r(Q)$ state variables z_1 are *observable*, and the last $n - r(Q)$ clearly are *not*.

If we now take a system that is neither controllable nor observable, we may perform ***both*** similarity transformations (in either order) (the second being applied to separate **SUB-systems** only) to get the *Kalman decomposition:*

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du$$

Not a matter of applying one transformation after another

Some more geometric interpretations:

Let the range space of the B-matrix be denoted: $\beta = \mathcal{R}(B)$

Let **X** and **Y** be two subspaces of a state-space, and define the ***sum*** of subspaces as

$$\mathbf{X} + \mathbf{Y} = \{x + y : x \in \mathbf{X}, y \in \mathbf{Y}\}$$

Then the ***controllable subspace*** of the state-variable system is:

$$R_c = b + Ab + \cdots + A^{n-1}b$$

If the dimension of this controllable subspace is n, then R_c has the same dimension as the state-space, so the system as a whole is controllable.

The ***<u>un</u>***observable subspace is defined as:

$$R_{\overline{o}} = \bigcap_{i=1}^{n} \mathbf{N}(CA^{i-1})$$

So the system as a whole will be observable if

$$\bigcap_{i=1}^{n} \mathbf{N}(CA^{i-1}) = 0$$

TWO OBSERVATIONS:

The controllable and observable subspaces are just that; ***subspaces***, by the strict definition.

All concepts and definitions involving controllability have an analogous counterpart in observability, but seem to be "backward" in a sense. This relationship is called a "duality" between controllability and observability.

DEFINITION: An uncontrollable system is said to be ***stabilizable*** if all of the uncontrollable modes are already stable, and all of the unstable modes are controllable.

DEFINITION: An unobservable system is said to be ***detectable*** if all of the unobservable modes are already stable, and any unstable modes are all observable.

Remark: Note that if a system is controllable (observable) in one basis, it is controllable (observable) after a similarity transformation also. Consider the controllability matrix of a system after it has been transformed by matrix T:

$$
\begin{aligned}
\bar{P} &= \begin{bmatrix} T^{-1}B & T^{-1}ATT^{-1}B & T^{-1}A^2TT^{-1}B & \cdots & T^{-1}A^{n-1}TT^{-1}B \end{bmatrix} \\
&= \begin{bmatrix} T^{-1}B & T^{-1}AB & T^{-1}A^2B & \cdots & T^{-1}A^{n-1}B \end{bmatrix} \\
&= T^{-1}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \\
&= T^{-1}P
\end{aligned}
$$

Because multiplying any matrix by a nonsingular matrix cannot change the rank, the rank of $\bar{P}$ is the same as the rank of P.