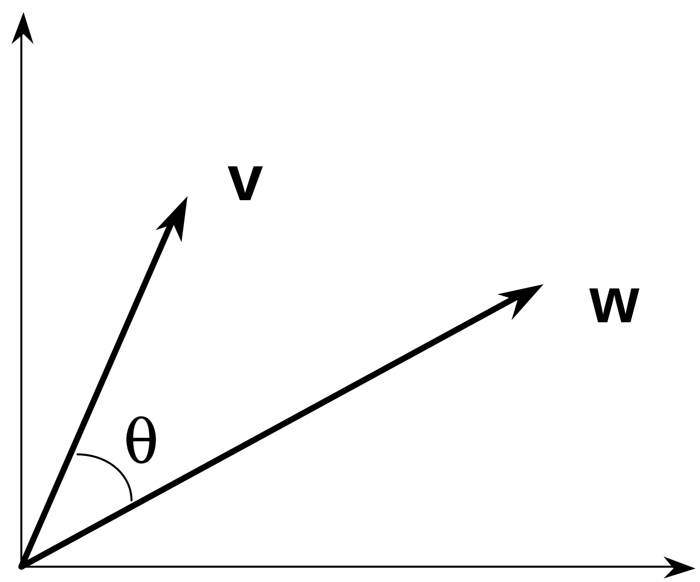
Chapter 2: Vectors and Vector Spaces

For the most part, **vector spaces we consider conform to the undergraduate idea of vectors in 2-D and 3-D: Cartesian n-space with the conventional vector and scalar operations:**



Geometric Interpretation

A vector is viewed as an n-tuple of numbers

Inner (Dot) Product: $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \mathbf{w} = \mathbf{w}^T \mathbf{v} = \sum_{i=1}^{n} w_i v_i$

$$= |v||w|\cos(\theta)$$

v & w are *orthogonal* iff $\langle \mathbf{v}, \mathbf{w} \rangle = 0$

"magnitude" or "norm"

$$\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2}$$

Outer product : $\mathbf{v}\rangle\langle\mathbf{w} = \mathbf{v}\mathbf{w}^T$

Cross (vector) product: $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin(\theta)$

Only defined in 3-D (??)

However we do need the rigorously defined concepts of vectors and spaces in order to generalize into different dimensions (even *infinite* dimensions):

First, a FIELD (see sec. 2.2.1):

A field F is a set of two or more elements for which the operations of addition, multiplication, and division are defined, and for which the following holds:

1. If $a \in F$ and $b \in F$, then $(a+b)$ and $(b+a) \in F$
2. $(ab) = (ba) \in F$
3. There exists a unique element $0 \in F$ such that $a + 0 = a$ and $0(a) = 0$
4. If $b \neq 0$, then $(a/b) \in F$

continued ☞

5. **There exists a unique identity element** $\mathbf{1 \in F}$ **such that**

   $\mathbf{1(a) = (a)1 = (a/1) = a}$

6. **For every** $\mathbf{a \in F}$ **there is a unique negative element** $\mathbf{-a \in F}$ **such that** $\mathbf{a + (-a) = 0.}$

7. **Associative , commutative, and distributive laws hold.**

<u>Examples</u>: Are these fields?

1. The set {0,1}? No 1+1=2 $2 \notin F$
2. The set of all matrices of the form $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ (x, y are real)? Yes Special Matrix
3. The set of all polynomials in s? No (Division)
4. The set of real numbers? Yes
5. The set of integers? No (Division)

Now for the definition of a Linear Vector Space: (*x*, *y* and *z* are considered vectors in the space)

A linear vector space *X* is a set of elements called vectors defined over a scalar field *F*, which satisfies the following rules:

1. If $x + y = v$, then $v \in X$
2. $x + y = y + x$
3. $(x + y) + z = x + (y + z)$
4. There exists a vector $0 \in X$ such that $x + 0 = 0 + x = x$
5. For every $x \in X$, there is a unique vector $y \in X$ such that $x + y = 0 \quad (y = -x)$.

6. For every $x \in X$ and scalar $a \in F$, the product ax gives another vector $y \in X$. (a may be the unit scalar so that $ax = 1 \cdot x = x \cdot 1 = x$)

7. For any scalars $a, b \in F$, and for any vector $x \in X$, $a(bx) = (ab)x$.

8. Scalar multiplication is distributive; i.e.,

$$(a+b)x = ax + bx$$

$$a(x+y) = ax + ay$$

<u>EXAMPLES</u>: Are these vector spaces?

1. The set of all n-tuples of scalars from *F*, over *F*? Yes (for examples, n-dimensional vectors of reals (over **R**), or n-dimensional complex numbers (over **C)**).
2. The set of all complex numbers, over reals? Yes
3. The set of all reals, over complex nos.? No (RN*CN=CN)
4. The set of all m x n matrices, over **R**? Over **C**? Yes, Yes
5. The set of all piecewise continuous functions of time, over **R**? Yes
6. The set of all polynomials in s of order less than n, with real coefficients, over **R**? Yes (Not a field, but it is a vector space, no division requirement)

Linear Dependence and Independence:

Consider a set of vectors from linear vector space $X:\{x_1, x_2, \ldots x_n\}$. If there exists a set of n scalars a_i, ***not all of which are zero***, such that the linear combination $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$ then the set $\{x_i\}$ is **linearly dependent**.

A set of vectors which is not linearly dependent is linearly independent. If the linear combination $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$ implies that **each** $a_i = 0$, then the set $\{x_i\}$ is a linearly **in**dependent set.

In other notation, the set $\{x_i\}$ is linearly independent if the equation $X\boldsymbol{a} = 0$ implies $\boldsymbol{a} = 0$, where

$$X = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}^T$$

<u>EXAMPLE</u>: Consider the three vectors from R ^{3}:

$$x_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 \\ 7 \\ 8 \end{bmatrix}$$

This is a linearly dependent set because we can choose the set of *a*'s as $a_1 = -1, \quad a_2 = 2, \quad a_3 = -1$ **(not all are zero) such that**

$$\sum_{i=1}^{3} a_i x_i = -x_1 + 2x_2 - x_3 = 0$$

When we have a set of *n* n-tuples, a convenient test is to form a determinant from them. If it is zero, the set is linearly dependent.

$$\begin{vmatrix} 2 & 1 & 0 \\ -1 & 3 & 7 \\ 0 & 4 & 8 \end{vmatrix} = 0$$

<u>EXAMPLE</u>: The set R(s) of all rational polynomial functions in *s*, is a vector space over the field of reals, and over the field of rational transfer functions. Consider the two vectors of the space *ordered pairs* of such functions:

$$x_1 = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s+2} \end{bmatrix} \qquad x_2 = \begin{bmatrix} \frac{s+2}{(s+1)(s+3)} \\ \frac{1}{s+3} \end{bmatrix}$$

If our field is the set of all real numbers, is this a linearly independent set? YES. One can verify that if a_1 and a_2 **are *real* numbers, then setting**

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = a_1 x_1 + a_2 x_2 = a_1 \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s+2} \end{bmatrix} + a_2 \begin{bmatrix} \frac{s+2}{(s+1)(s+3)} \\ \frac{1}{s+3} \end{bmatrix}$$

will imply that $a_1 = a_2 = 0$.

If however the field is taken as the set of all rational polynomial functions in s, then we have a different result. With some tinkering, we can choose

$$a_1 = 1 \quad \text{and} \quad a_2 = -\frac{s+3}{s+2}$$

giving

$$\sum a_i x_i = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s+2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{s+1} \\ -\frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

so now the vectors are linearly Dependent !!

In the first case, there might in some circumstances exist a value for s such that the linear combination equals zero. This does *not* result in linear dependence because the combination is not "identically" equal to zero. This is true if the vector's elements are functions of time, too. The combination might be zero for a particular instant in time, even if the vectors are linearly independent.

From our intuitive sense of vectors:

☞ **Two linearly independent vectors in 2-D are non-collinear.**

☞ **Three linearly independent vectors in 3-D are non-coplanar.**

<u>Two LEMMAS:</u>

If we have a set of linearly dependent vectors, and we add another vector to that set, the result will also have to be linearly dependent.

If we have a set of vectors that is linearly dependent, then one of the vectors can be written as a linear combination of the others in the set.

☠ **This is NOT the definition of linear dependence!**

These dependencies manifest themselves in the degeneracy, or rank deficiency, of the matrix of vectors. We can find relationships between the degeneracies of matrices with the following formulae:

If matrices $A, B,$ and C have ranks $r(A), r(B),$ and $r(C)$, then the relationship $AB = C$ and *Sylvester's inequality* imply ($n =$ no. of cols. of A):

$$\left.r(A) + r(B) - n \le r(C) \le \min(r(A), r(B))\right\} \text{Sylvester's Inequality}$$

or if degeneracies $q(A), q(B),$ and $q(C)$ are used:

$$q(C) \le q(A) + q(B)$$

☞ Bases and Dimensionality

DEFINITION: the DIMENSION of a linear vector space is the largest possible number of linearly independent vectors that can be taken from that space.

DEFINITION; A set of linearly independent vectors in vector space *X* is a basis of *X* iff every vector in *X* can be written as a *unique* linear combination of these vectors.

In an *n*-dimensional vector space, there have to be exactly *n* vectors in any basis set, but there are an infinite number of such sets that qualify as a basis.

BASIS $\longleftrightarrow$ **COORDINATE SYSTEM** } Heuristic View of Basis

<u>THEOREM</u>: In an n-dimensional linear vector space, <u>any</u> set of *n* linearly independent vectors qualifies as a basis set.

PROOF: This statement implies that a vector *x* should be described by any *n* linearly independent vectors, say

$$\{e_1, e_2, \ldots, e_n\}$$

That is, for any vector *x*;

$$x = \text{ linear combination of } e_i\text{'s}$$

$$= \sum_{i=1}^{n} \alpha_i e_i$$

☞

Because the space is n-dimensional, the set $\{x, \quad e_1, \quad e_2, \quad \ldots, \quad e_n\}$ **must be linearly dependent, so that**

$$\alpha_0 x + \alpha_1 e_1 + \ldots + \alpha_n e_n = 0$$

for a set of α **'s not all of which are zero.**

If $\alpha_0 = 0$ **then** $\alpha_1 = \ldots = \alpha_n = 0$**, so we can safely assume** $\alpha_0 \neq 0$**.**

We are therefore allowed to write:

$$x = -\frac{\alpha_1}{\alpha_0} e_1 + \ldots + -\frac{\alpha_n}{\alpha_0} e_n$$

$$\stackrel{\Delta}{=} \beta_1 e_1 + \ldots + \beta_n e_n$$

$$= \sum_{i=1}^{n} \beta_i e_i$$

So we have written vector *x* as a linear combination of the e_i **'s.**

Now we must show that this expression is *unique*:

Suppose there were another set of constants $\{\overline{\beta}_i\}$ **such that**

$$x = \sum_{i=1}^{n} \overline{\beta}_i e_i$$

We already have $x = \sum_{i=1}^{n} \beta_i e_i$ **so**

$$x - x = \sum_{i=1}^{n} \overline{\beta}_i e_i - \sum_{i=1}^{n} \beta_i e_i$$

$$= \sum_{i=1}^{n} (\overline{\beta}_i - \beta_i) e_i = 0$$

But the set $\{e_i\}$ **is known to be a basis; therefore, for this equation to hold, we must have**

☞

$$\bar{\beta}_i - \beta_i = 0$$

for all *i*. (because bases are lin. indep sets).

Therefore $\bar{\beta}_i = \beta_i$ **and this proves uniqueness of the representation**

$$x = \sum_{i=1}^{n} \beta_i e_i$$

The uniqueness result implies that a vector from any space can be identified by its *n* coefficients, once the basis vectors are chosen. In fancy terms, an n-dimensional space is $\underbrace{\textbf{\textit{isomorphic}}}_{\text{one to one and onto mapping}}$ to the Euclidean space of *n*-tuples.

Once the basis $\{e_i\}$ **is chosen, the set of numbers** $\beta = [\beta_1 \quad \beta_2 \quad \cdots \quad \beta_n]^T$ **is called the *representation* of *x* in** $\{e_i\}$**.**

<u>DEFINITION</u>: A set of vectors X is spanned by a set of vectors $\{x_i\}$ if every $x \in X$ can be written as a linear combination of the x_i 's . Equivalently, the x_i 's span *X*.

Notation: $X = sp\{x_i\}$

Note: The set $\{x_i\}$ is not necessarily a basis. It may be linearly dependent. For example, five non-collinear vectors in 2-D suffice to *span* the 2-D Euclidean space, but as a set they are not a basis.

<u>EXAMPLES</u>:

1. Let X be the space of all vectors written $x = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$ **such that** $x_1 = x_2 = \cdots = x_n$ **.**

One legal basis for this space is the single vector

$$v = [1 \quad 1 \quad \cdots \quad 1]^T$$

This is therefore a *one* dimensional space, despite it consisting of vectors containing *n* "components".

2. Consider the space of all polynomials in *s* of degree less than 4 with real coefficients. One basis for this space is the set

$$e_1 = 1 \quad e_2 = s \quad e_3 = s^2 \quad e_4 = s^3$$

In this basis, the vector

$$x = 3s^3 + 2s^2 - 2s + 10$$

can be written as

$$x = 3e_4 + 2e_3 + (-2)e_2 + 10e_1$$

So the vector *x* has the *representation* $x = \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ \left[10\right. & -2 & 2 & \left.3\right]^T \end{array}$ in the $\{e_i\}$ basis.

We can write **another** basis for this space too:

$$\{f_i\} = \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \\ \{s^3 - s^2, & s^2 - s, & s - 1, & 1\} \end{array}$$

$$\{f_i\} = \{\overset{f_1}{s^3 - s^2}, \quad \overset{f_2}{s^2 - s}, \quad \overset{f_3}{s - 1}, \quad \overset{f_4}{1}\}$$

In this different basis, the same vector has a different representation:

$$x = 3(\overset{f_1}{s^3 - s^2}) + 5(\overset{f_2}{s^2 - s}) + 3(\overset{f_3}{s - 1}) + 13\overset{f_4}{}$$

or

$$x = [3 \quad 5 \quad 3 \quad 13]^T$$

Changing the Basis of a Vector:

First we take a vector *x* and write its expansion in two different bases, $\{v_i\}$ and $\{v_i'\}$:

scalars

$$x = \sum_{j=1}^{n} x_j v_j = \sum_{i=1}^{n} \hat{x}_i \hat{v}_i$$

But since the basis vectors v_j **are in the same space, they themselves can be expanded into the** $\{\hat{v}_i\}$ **basis:**

$$\boxed{v_j = \sum_{i=1}^{n} b_{ij} \hat{v}_i}$$

Substituting in here,

$$\sum_{j=1}^{n} x_j \left[\sum_{i=1}^{n} b_{ij} \hat{v}_i \right] = \sum_{i=1}^{n} \hat{x}_i \hat{v}_i$$ **and simplifying;**

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}x_j - \hat{x}_i\right)\hat{v}_i = 0$$

Because $\{\hat{v}_i\}$ is a basis, for this to be true, we must have:

$$\boxed{\hat{x}_i = \sum_{j=1}^{n} b_{ij}x_j}$$

Otherwise, one of the $\hat{v}_i$ could be written in terms of the others

This is how we get the components of a vector in a new basis from the components of the old basis. The *b*'s above come from our knowledge of how the two bases are related.

Notice that this relationship can also be written as a matrix; that is,

$$[x]_{\hat{v}} = [B][x]_v$$

In words, the representation of *x* in the *new* basis are equal to the representation of *x* in the *old* basis, times a matrix with elements taken from the expansion of the *old* basis in terms of the *new* basis.

EXAMPLE: **Consider the space** $\mathbf{R}^2$ **and the two bases:**

$$\{e_1, e_2\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix}\right\} \text{ and } \{\bar{e}_1, \bar{e}_2\} = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \quad \begin{bmatrix}-1\\0\end{bmatrix}\right\}$$

Now let a vector x be represented by $x = \begin{bmatrix}2\\2\end{bmatrix}$ in the $\{e_i\}$ basis.

Find the representation for x in the $\{\bar{e}_i\}$ basis:

First we write down the relationship between the two bases:

$$e_1 = 0\bar{e}_1 + -1\bar{e}_2 = [\bar{e}_1 \quad \bar{e}_2]\begin{bmatrix}0\\-1\end{bmatrix} \begin{array}{l} b_{11}\\ b_{21}\end{array}$$

$$e_2 = \tfrac{1}{2}\bar{e}_1 + \tfrac{1}{2}\bar{e}_2 = [\bar{e}_1 \quad \bar{e}_2]\begin{bmatrix}\tfrac{1}{2}\\ \tfrac{1}{2}\end{bmatrix} \begin{array}{l} b_{12}\\ b_{22}\end{array}$$

So $$B = \begin{bmatrix}0 & \tfrac{1}{2}\\ -1 & \tfrac{1}{2}\end{bmatrix}$$

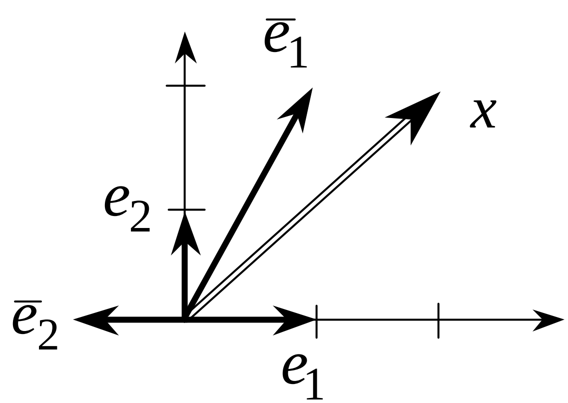
Giving

$$\bar{x}(= x \text{ in the } \{\bar{e}\} \text{ basis}) = Bx = \begin{bmatrix} 0 & \tfrac{1}{2} \\ -1 & \tfrac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Notice that to do the reverse (go from $\{e\}$ to $\{\bar{e}\}$), we would get B^{-1}, which always exists. (Why?) . That is,

Property of Basis

$$x = B^{-1}\bar{x} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$



<u>Some more definitions:</u>

An inner product is defined on a linear vector space as an operation of two vectors, denoted $\langle x, y \rangle$ such that:

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ Complex Conjugate
2. $\langle x, \alpha y_1 + \beta y_2 \rangle = \alpha \langle x, y_1 \rangle + \beta \langle x, y_2 \rangle$
3. $\langle x, x \rangle \geq 0$ for all x and $\langle x, x \rangle = 0$ iff $x = 0$.

Linear Vector Spaces with inner products defined are sometimes called *Hilbert* spaces.

A norm is used to describe a concept of "length" for a vector, and it must follow the rules:

1. $\|x\| = 0$ iff $x = 0$.
2. $\|\alpha x\| = |\alpha| \|x\|$ for any scalar α.
3. $\|x + y\| \le \|x\| + \|y\|$ **(triangle inequality)**

and also

$|\langle x, y \rangle| \le \|x\| \cdot \|y\|$ Cauchy - Schwarz inequality

Linear Vector Spaces with norms defined are sometimes called *Banach* spaces.

The most common norm, the "Euclidean" norm is induced from the inner product:

$$\|x\| = \langle x, x \rangle^{1/2} = \left[\sum_{i=1}^{n} |x_i|^2 \right]^{1/2}$$

A unit vector is a vector whose norm is *one*. Any vector can be made into a unit vector by dividing itself by its norm.

A metric, or concept of distance between two vectors, is also induced by a norm:

$$\rho(x, y) = \|x - y\|$$

The angle q between two vectors can be computed for any dimensional space by the formula:

$$\langle x,\ y\rangle = \|x\|\|y\|\cos\theta$$

A pair of vectors is orthogonal if

$$\langle x, y\rangle = 0$$

A *set* $\{x_i\}$ of vectors is orthogonal if

$$\langle x_i, x_j\rangle = 0 \text{ if } i \neq j \quad \text{and}$$

$$\langle x_i, x_j\rangle \neq 0 \text{ if } i = j$$

A set $\{x_i\}$ of vectors is orthonormal if

$$\langle x_i, x_j\rangle = 0 \text{ if } i \neq j \quad \text{and}$$

$$\langle x_i, x_j\rangle = 1 \text{ if } i = j$$

Gram Schmidt Orthogonalization:

It is computationally convenient if we always use basis sets that are orthonormal. If we have a basis that is *not* orthonormal, we can use this process to orthogonalize the set, then normalize it by making all vectors unit vectors.

Let the sets $\{y_i\}$ **and** $\{v_i\}$ **denote, respectively, the original (non-orthogonal) and the new (orthogonal) basis sets.**

Step 1: Let $v_1 = y_1$

Step 2: Choose v_2 **as** y_2 **with all components along** v_1 **subtracted out.**

component of y_2 **along** v_1

$$v_2 = y_2 - a v_1$$

We don't know *a* yet, so we'll compute it:

Because v_i**'s are orthogonal,**

$$\langle v_1, v_2 \rangle = 0 = \langle v_1, y_2 \rangle - a\langle v_1, v_1 \rangle$$

So $$a = \frac{\langle v_1, y_2 \rangle}{\langle v_1, v_1 \rangle}$$

Step 3: Go on to choose v_3 **as** y_3 **with components along *both* previous** v_i **- basis vectors subtracted out:**

$$v_3 = y_3 - a_1 v_1 - a_2 v_2$$

Again

$$\langle v_1, v_3 \rangle = 0 = \langle v_1, y_3 \rangle - a_1\langle v_1, v_1 \rangle - a_2 \overbrace{\langle v_1, v_2 \rangle}^{\mathbf{0}}$$

$$\langle v_2, v_3 \rangle = 0 = \langle v_2, y_3 \rangle - a_1 \underbrace{\langle v_2, v_1 \rangle}_{\mathbf{0}} - a_2\langle v_2, v_2 \rangle$$

So $$a_1 = \frac{\langle v_1, y_3 \rangle}{\langle v_1, v_1 \rangle} \quad \text{and} \quad a_2 = \frac{\langle v_2, y_3 \rangle}{\langle v_2, v_2 \rangle}$$ **etc.**

Continuing on in the same way,

$$v_i = y_i - \sum_{k=1}^{i-1} \frac{\langle v_k, y_i \rangle}{\langle v_k, v_k \rangle} v_k$$

Then normalize the set by $\quad v_i = \dfrac{v_i}{\|v_i\|}$

If we always use orthonormal bases, we can always find the component of vector *x* along the *j*th basis vector by finding the inner product of *x* with e_j **:**

$$x = \sum_{i=1}^{n} a_i e_i$$

$$\langle e_j, x \rangle = \sum_{i=1}^{n} a_i \langle e_j, e_i \rangle = a_j$$

***j* th component**

Manifolds, Subspaces, and Projections:

In finite dimensions, manifolds and subspaces are equivalent;

A subspace is a *subset* of a vector space that itself qualifies as a vector space.

<u>DEFINITION</u>: A subspace M of vector space X is a subset of X such that if $x, y \in M$ and $z = \alpha x + \beta y$ then $z \in M$.

If $\dim(M) = \dim(X)$, then $M = X$.

If $\dim(M) < \dim(X)$, then M is a *proper* subspace.

☞ **All subspaces must contain the zero vector!!**

A little reflection shows that:

1. **All proper subspaces of $\mathbf{R}^2$ are straight lines that pass through the origin.**
2. **All proper subspaces of $\mathbf{R}^3$ are planes *or lines* that pass through the origin.**
3. **All proper subspaces of $\mathbf{R}^n$ are "surfaces" of dimension *n-1 or less*, and pass through the origin.**

Projections:

Suppose *U* is a proper subspace of *X* (*dim(U)<dim(X)*). **Then for every** $x \in X$ **there is a** $u \in U$ **such that** $\langle x-u,\, y\rangle = 0$ **for every** $y \in U$**. The vector *u* is the orthogonal** ***projection* of *x* into *U*.**

X = R^3