A deep learning method for high dimensional PDE's

An application to crowd motion

by

Carlos Daniel Contreras Quiroz

Advisor: Mauricio Junca

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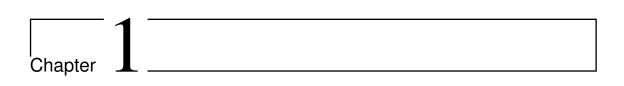An application to crowd motion

Carlos Daniel Contreras Quiroz

Abstract

Nada

Acknowledgements

A mi lulú y mi pancita.

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Chapter 1

Introduction

Chapter 2

Backward stochastic differential equations and PDEs

When addressing deterministic optimal control problems of dynamical systems, there are two approaches, one involving Bellman's dynamic programming principle, and the other relying on the Pontryagin's maximum principle. The former approach leads to a partial differential equation, the Hamilton-Jacobi-Bellman equation, to be solved for the value function and the optimal control of the process. The latter leads to a system of ordinary differential equations, one equation forward in time for the state and one backward in time for its adjoint.

The stochastic version of these problems is solved by methods analogous to those of the deterministic case. However, there are issues with desirable mathematical properties of solutions when we state them extending directly the ones proposed by deterministic methods. That is the case of the stochastic version of the Pontryagin's maximum principle, in which the backward differential equation cannot be stated directly as an SDE with terminal condition, as the solution is not guaranteed to be adapted to the filtration generated by the brownian motion.

The theory of backward stochastic differential equations (BSDEs) emerged in Bismut's [1] early work, and later generalized by Pardoux and Peng[2], as an attempt to formalize the application of the stochastic maximum principle. Here we give an introduction and compilation of results about them based on [3, 4, 5, 6], including its relation with a certain class of nonlinear parabolic partial differential equations, which will be the main tool for the method explained in the following chapters.

2.1 Backward stochastic differential equations

2.1.1 Motivation

Let's introduce the necessity for a different formulation of stochastic differential equations through an example. In the usual setting for a stochastic differential equation (SDE), we

specify the evolution of a $\mathbb{R}^d$-valued stochastic process X_t through its dynamics and an initial value $x_0 \in \mathbb{R}^d$ (possibly random), in the form

$$X_t = x_0 + \int_0^t \mu(t, X_t)dt + \int_0^t \sigma(t, X_t)dW_t, \tag{2.1}$$

or equivalently,

$$
\begin{aligned}
dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\
X_0 &= x_0,
\end{aligned}
\tag{2.2}
$$

where W_t is a m-dimensional Brownian motion process and the stochastic integral is defined in the Ito sense.

We know that, under some Lipschitz and boundedness conditions for the drift μ and the volatility σ, the equation with initial condition (2.2) has a unique solution which is adapted with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_t$ generated by W_t.

Now, what happens if we consider the problem (2.2) with a terminal condition at time $T > 0$? Consider, for instance, the particular case with $\mu(t, X_t) = \sigma(t, X_t) = 0$, and a square integrable $\mathcal{F}_T$-measurable random variable $\xi \in L^2(0, T)$ for which we try to solve the problem of finding a process Y_t such that

$$
\begin{aligned}
dY_t &= 0 \\
Y_t(T) &= \xi.
\end{aligned}
\tag{2.3}
$$

This equation has a unique solution given by $Y(t) = \xi$, which is not necessarily $\mathcal{F}_t$−measurable for every $0 \leq t \leq T$, and therefore (2.3) may not have solution in the usual SDE sense.

Despite this, we can try to solve this problem reinterpreting the solution to (2.3) based on the following representation theorem.

Theorem 2.1.1 *Martingale representation theorem [7] Let $(M_t)_{0 \leq t \leq T}$ be a continuous $\mathbb{R}^d$-valued square-integrable martingale with respect to $\mathcal{F}_t$, the augmented filtration generated by a m-dimensional Brownian motion $(W_t)_t$. Then, there is a unique $\mathbb{R}^{d \times m}$-valued $\mathcal{F}_t$-adapted stochastic process $f(s)$, with $\mathbb{E}[\int_0^T |f|^2 dt] < \infty$, such that*

$$M_t = M_0 + \int_0^t f(s)dW_s \quad for \quad t \in [0, T], \tag{2.4}$$

where the uniqueness is interpreted in the mean squared norm.

We can intend to enforce the solution Y_t to be $\mathcal{F}_t$−measurable for every $0 \leq t \leq T$ by taking its conditional expectation with respect to the evolving σ-algebra

$$Y(t) := \mathbb{E}[\xi | \mathcal{F}_t], \tag{2.5}$$

which satisfies the terminal condition $Y(T) = \xi$, since ξ is $\mathbb{F}_T$-measurable. Thus, as a consequence of the Martingale respresentation theorem 2.1.1, we conclude that there exist an square-integrable $\mathcal{F}_t$-measurable process Z_t such that

$$Y_t = Y_0 + \int_0^t Z_s dW_s \quad for \quad t \in [0, T] \tag{2.6}$$

2.1.2 An existence and uniqueness theorem

Let be $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $T > 0$ a fixed horizon time. We consider a d-dimensional Brownian motion $W = (W_t)_{t \in [0,T]}$ and let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ be the corresponding natural augmented filtration (i.e with the completeness and right continuity conditions).

Denote by $\mathbb{S}^2(0, T)$ the set of real-valued progressively measurable processes Y_t such that
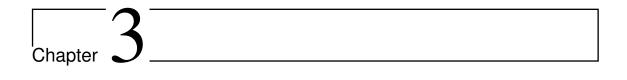
$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] < \infty, \tag{2.7}$$

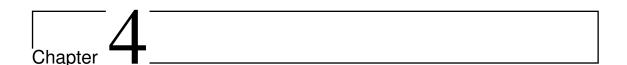and by $\mathbb{H}^2(0, T)^d$ the set of $\mathbb{R}$-valued progressively measurable processes Z_t such that

$$\mathbb{E} \left[\int_0^T |Z_t|^2 dt \right] < \infty, \tag{2.8}$$
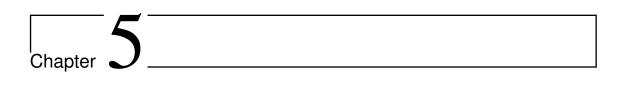
Definition 2.1.1 *Una fola sapo*

2.1.3 Forward-Backward stochastic differential equations

2.2 The Feynman-Kac formulas

2.2.1 The linear Feynman-Kac formula
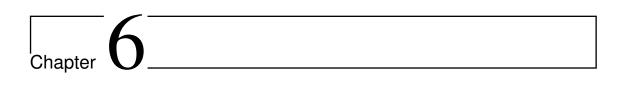
2.2.2 The non-linear Feynman-Kac formula

Chapter 3

The Deep BSDE method

Chapter 4

Crowd motion modeling

Chapter **5**

An application

Chapter **6**

Conclusion

Appendix A

Neural Networks

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