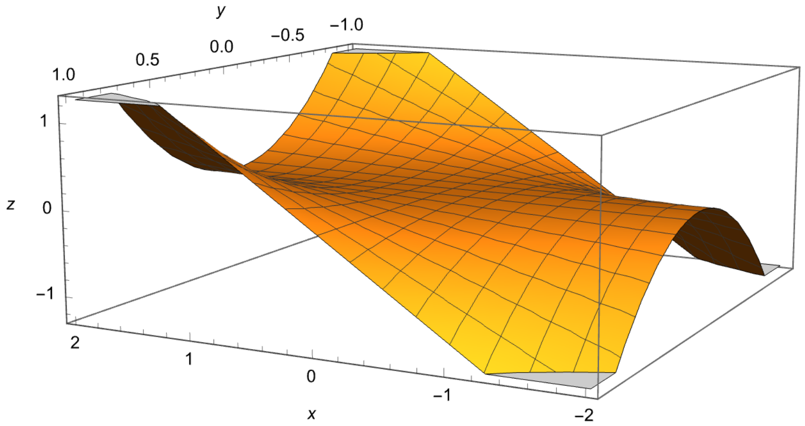
Anells de polinomis en diverses variables

Primavera 2025

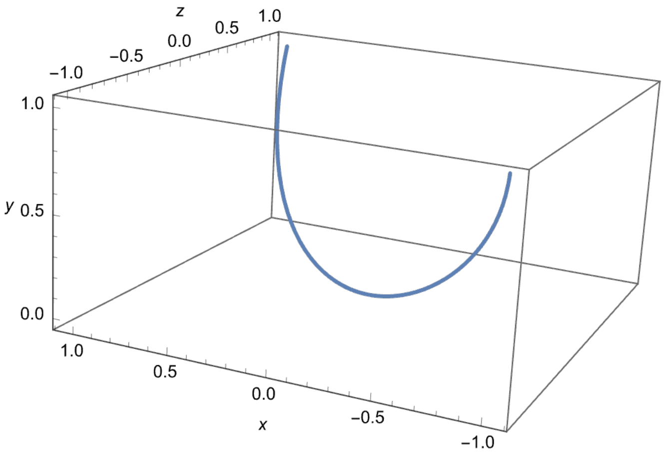
Laboratori 1 (plot, factor, solve)

Usarem "Plot", i les seves variacions, per dibuixar corbes i superfícies:

In[ ]:=
```
Plot3D[ x * y^2, {x, -2, 2}, {y, -1, 1}, Axes → True, AxesLabel → {x, y, z}]
```
Out[ ]=

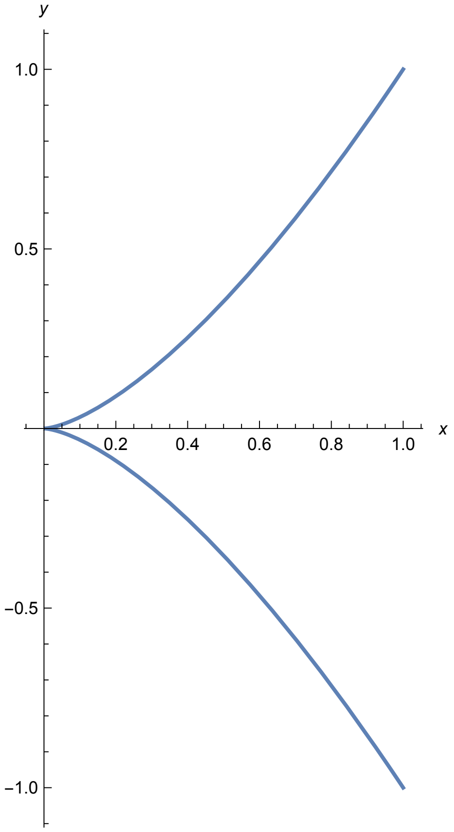


In[ ]:=
```
ParametricPlot3D[{t, t^2, t^3}, {t, -1, 1}, Axes → True, AxesLabel → {x, y, z}]
```
Out[ ]=



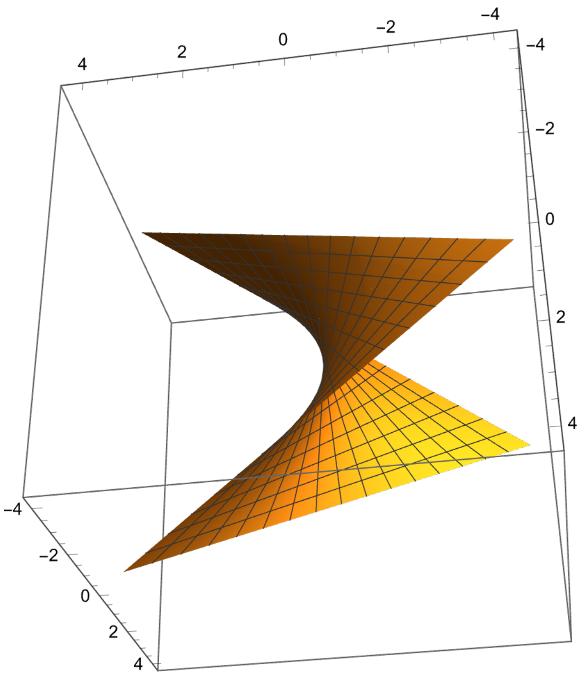
In[ ]:=
```
ParametricPlot[{t^2, t^3}, {t, -1, 1}, Axes → True, AxesLabel → {x, y}]
```

Out[ ]=



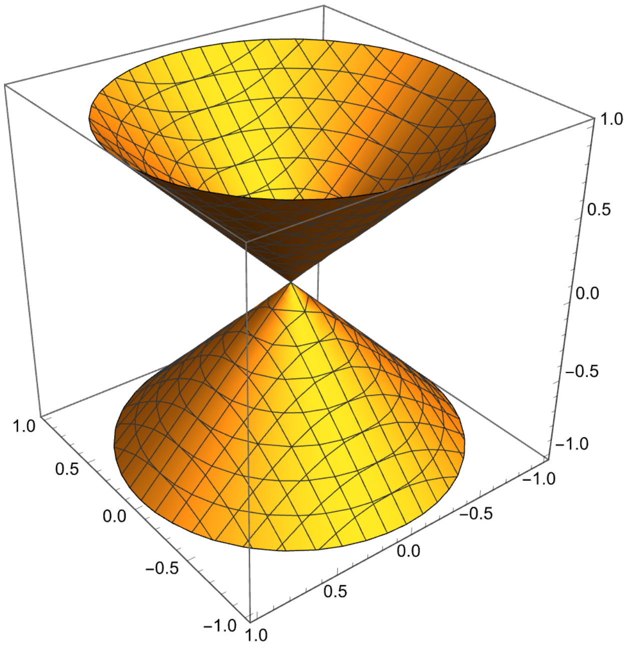
In[ ]:=
```
ParametricPlot3D[{x * y, x - y, x + y}, {x, -2, 2}, {y, -2, 2}]
```

Out[ ]=



Usarem el ContourPlot per dibuixar conjunts definits per funcions implícites :

In[ ]:=
```
ContourPlot3D[x^2 + y^2 - z^2 == 0, {x, -1, 1}, {y, -1, 1}, {z, -1, 1}]
```
Out[ ]=



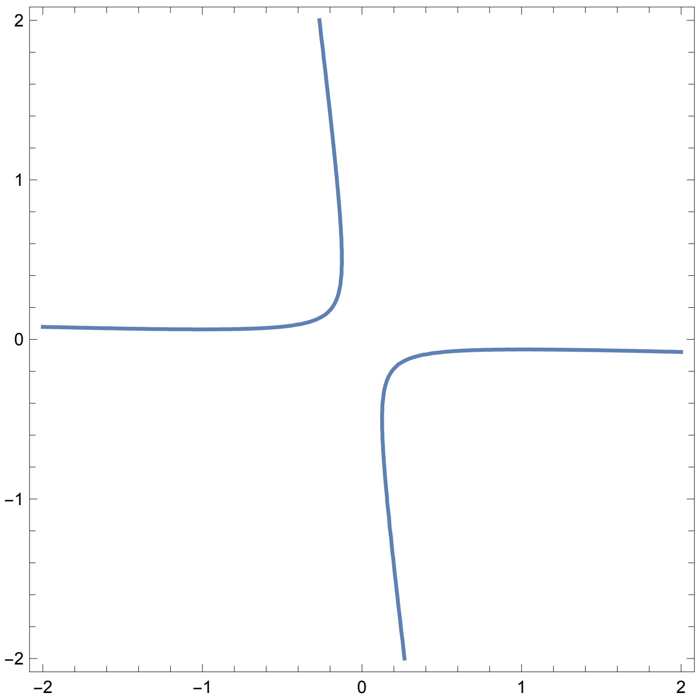
Ex 1 : Dibuxeu en el pla real els següents conjunts algebraics :

a) V (x^2 + 4 y^2 + 2 x - 16 y + 1)

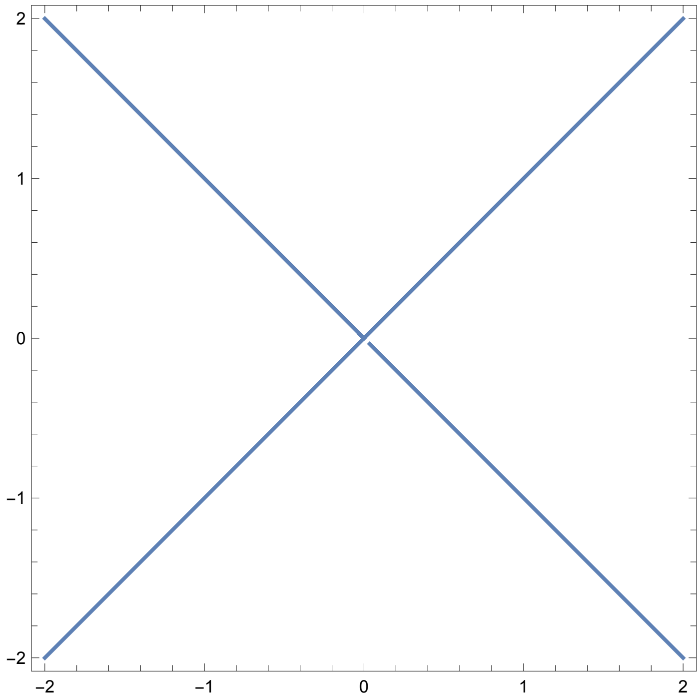
b) V (x^2 - y^2)

c) V (2 x + y - 1, 3 x - y + 2)

In[ ]:=
```
ContourPlot[x^2 + 4 y^2 + 2 x 16 y + 1 == 0, {x, -2., 2.}, {y, -2., 2.}]
```



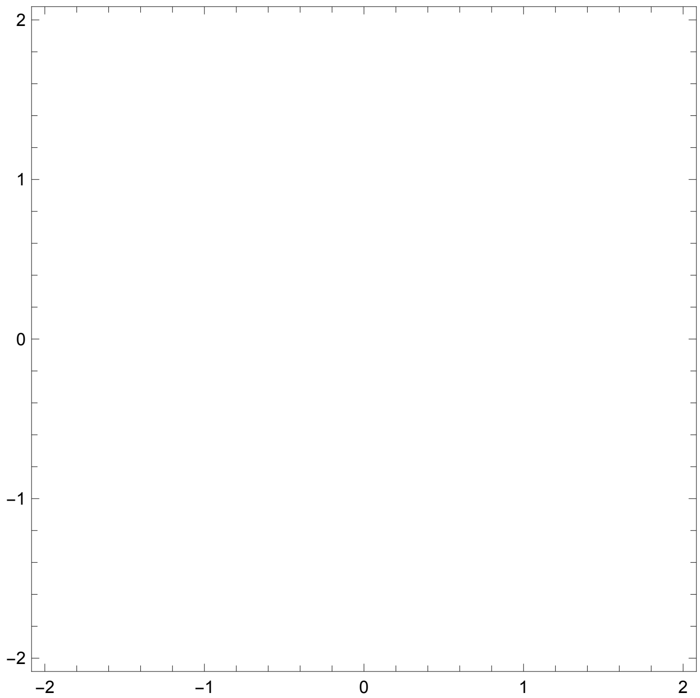
In[ ]:=
```
ContourPlot[x^2 - y^2 == 0, {x, -2., 2.}, {y, -2., 2.}]
```



In[ ]:=

```
ContourPlot[2 x + y - 1 == 0 && 3 x - y + 2 == 0, {x, -2, 2}, {y, -2, 2}]
```

Out[ ]=



Ex 2 : Dibuixeu en l' espai real els següents conjunts algebraics :

a) V (x^2 + 4 y^2 + z^2 - 1)

b) V (x*z^2 - x*y)

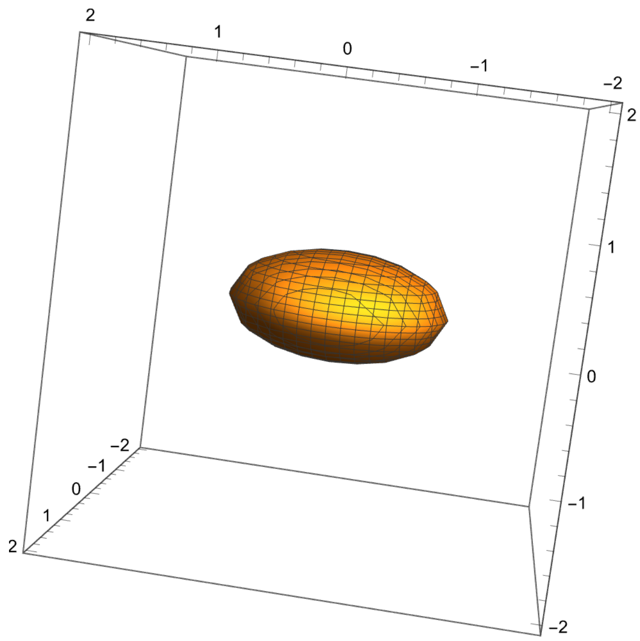
c) V (x^4 - z*x, x^3 - y*x)

d) V (x^2 + y^2 + z^2 - 1, x^2 + y^2 + (z - 1)^2 - 1)

In[ ]:=

```
ContourPlot3D[x^2 + 4 y^2 + z^2 - 1 == 0,
 {x, -2., 2.}, {y, -2., 2.}, {z, -2., 2.}]
```

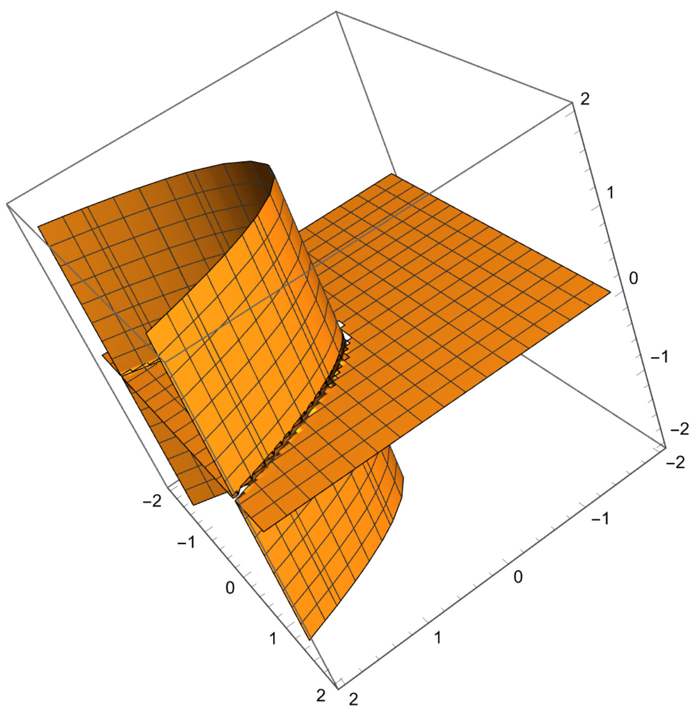
Out[ ]=



In[ ]:=

```
ContourPlot3D[x * z^2 - x * y == 0, {x, -2., 2.}, {y, -2., 2.}, {z, -2., 2.}]
```

Out[ ]=

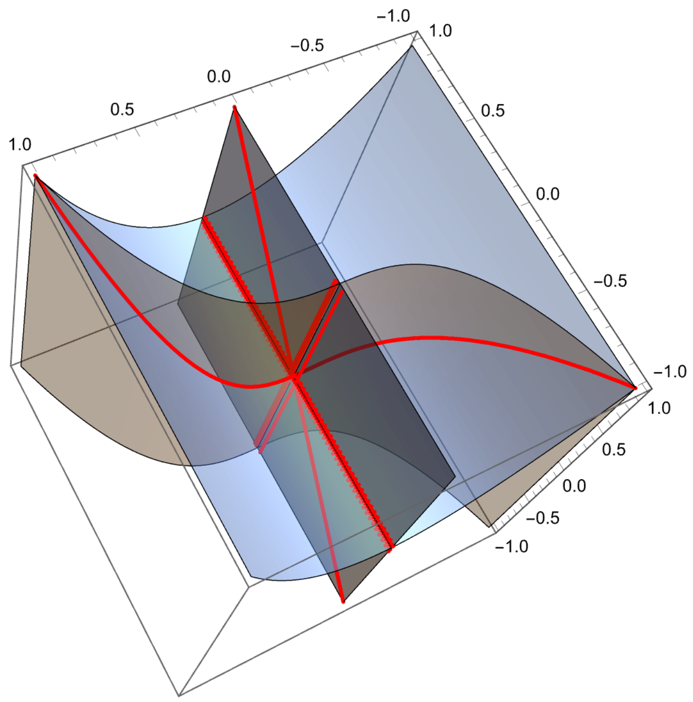


Para resolver el apartado (3) he intentado utilizar el método descrito en clase, pero como no he conseguido que pinte el plano de la intersección he buscado una alternativa menos elegante pero funcional.

In[24]:=
```
g := x^4 - z*x;
h := x^3 - y * x ;
```

In[97]:=
```
ContourPlot3D[
 {g == 0 , h == 0}, {x, -1., 1.}, {y, -1., 1.}, {z, -1., 1.},
 MeshFunctions → {Function[{x, y, z, f}, h - g]},
 MeshStyle → {Red, Thick}, Mesh → {{0}},
 ContourStyle → Opacity[0.4], PlotPoints → 60]
```

Out[97]=



In[75]:=
```
S = Reduce[g == 0, {x, y, z}]
R = Reduce[h == 0, {x, y, z}]
```

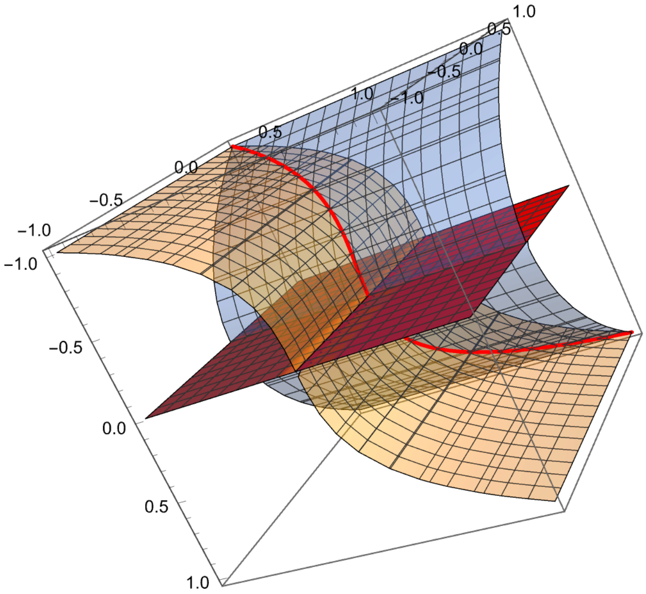
Out[75]=

x == 0 || z == x^3

Out[76]=

x == 0 || y == x^2

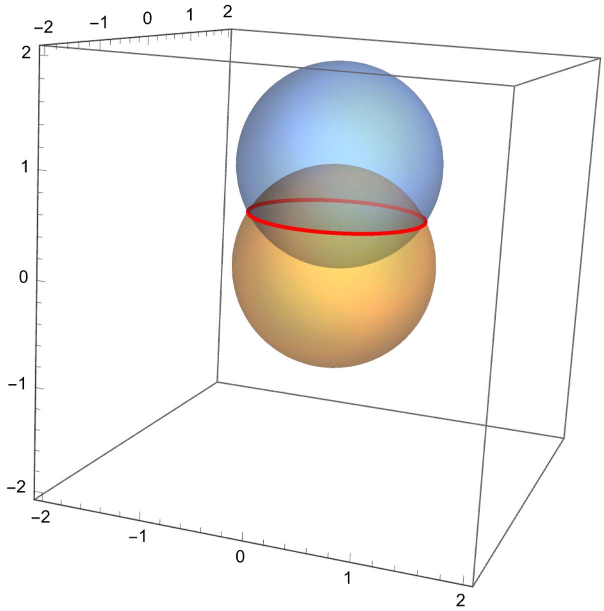
```
p1 = ContourPlot3D[x == 0, {x, -1., 1.},
   {y, -1., 1.}, {z, -1., 1.}, ContourStyle → Red,];
p2 = ParametricPlot3D[{x, x^2, x^3}, {x, -1., 1.}, PlotStyle → Red];
p3 = ContourPlot3D[{g == 0 , h == 0}, {x, -1., 1.},
   {y, -1., 1.}, {z, -1., 1.}, ContourStyle → Opacity[0.4]];
Show[p1, p2, p3]
```



In[101]:=

```
g := x^2 + y^2 + z^2 - 1;
h := x^2 + y^2 + (z - 1) ^2 - 1;
ContourPlot3D[
 {g == 0 , h == 0}, {x, -2., 2.}, {y, -2., 2.}, {z, -2., 2.},
 MeshFunctions → {Function[{x, y, z, f}, h - g]},
 MeshStyle → {Red, Thick}, Mesh → {{0}},
 ContourStyle → Opacity[0.4], PlotPoints → 60]
```

Out[103]=



Podem factoritzar polinomis sobre els racionals o enters:

In[ ]:=
```
Factor[x * y^3 - x * z^2 + x * y * z]
```
Out[ ]=

$x\left(y^3 + y\,z - z^2\right)$

In[ ]:=
```
FactorList[x^2 * y * t - y^2 * x * z - x * y * z * t]
```
Out[ ]=

{{1, 1}, {x, 1}, {y, 1}, {t x - t z - y z, 1}}

In[ ]:=
```
IrreduciblePolynomialQ[x * y - 2]
```
Out[ ]=

True

In[ ]:=
```
IrreduciblePolynomialQ[x * y - 2, Modulus → 2]
```
Out[ ]=

False

In[ ]:=
```
Factor[x * y - 2, Modulus → 2]
```
Out[ ]=

x y

In[ ]:=
```
PolynomialGCD [x * y * z^2, x^2 * y^2 * z, x^2 * y * z^3]
```
Out[ ]=

x y z

Podem resoldre sistemes equacions :

In[ ]:=
```
S := {x^2 + y + z == 1, x + y^2 + z - 1 == 0, x + y * z^2 - 1 == 0}
```
In[ ]:=
```
T := Solve[S, {x, y, z}]
```
In[ ]:=
```
T
```
Out[ ]=

$\{\{x \to 1, y \to 0, z \to 0\},$
$\{x \to -1 + (-1)^{1/3} - 2\,(-1)^{2/3}, y \to (-1)^{1/3}, z \to 2 - (-1)^{1/3} + (-1)^{2/3}\},$
$\{x \to -1 + 2\,(-1)^{1/3} - (-1)^{2/3}, y \to -(-1)^{2/3}, z \to 2 - (-1)^{1/3} + (-1)^{2/3}\},$
$\{x \to \frac{1}{2}\left(1 + \sqrt{5}\right), y \to \frac{1}{2}\left(1 - \sqrt{5}\right), z \to -1\}, \{x \to \frac{1}{2}\left(1 - \sqrt{5}\right), y \to \frac{1}{2}\left(1 + \sqrt{5}\right), z \to -1\},$
$\{x \to \frac{1}{3}\left(3 - 5\,\boxed{0.814\ldots} + 10\,\boxed{0.814\ldots}^2 + 4\,\boxed{0.814\ldots}^3 - 21\,\boxed{0.814\ldots}^4 + \right.$
$\left. 17\,\boxed{0.814\ldots}^5 + \boxed{0.814\ldots}^6 - 13\,\boxed{0.814\ldots}^7 + \boxed{0.814\ldots}^8 + 3\,\boxed{0.814\ldots}^9\right),$
$y \to \boxed{0.814\ldots}, z \to -\frac{1}{3}\,\boxed{0.814\ldots}\left(-5 + 13\,\boxed{0.814\ldots} + 4\,\boxed{0.814\ldots}^2 - \right.$
$21\,\boxed{0.814\ldots}^3 + 17\,\boxed{0.814\ldots}^4 + \boxed{0.814\ldots}^5 -$
$\left. 13\,\boxed{0.814\ldots}^6 + \boxed{0.814\ldots}^7 + 3\,\boxed{0.814\ldots}^8\right)\}, \{x \to$
$\frac{1}{3}\left(3 - 5\,\boxed{-1.75\ldots - 0.496\ldots\,i} + 10\,\boxed{-1.75\ldots - 0.496\ldots\,i}^2 + 4\,\boxed{-1.75\ldots - 0.496\ldots\,i}^3 - \right.$
$21\,\boxed{-1.75\ldots - 0.496\ldots\,i}^4 + 17\,\boxed{-1.75\ldots - 0.496\ldots\,i}^5 + \boxed{-1.75\ldots - 0.496\ldots\,i}^6 -$
$\left. 13\,\boxed{-1.75\ldots - 0.496\ldots\,i}^7 + \boxed{-1.75\ldots - 0.496\ldots\,i}^8 + 3\,\boxed{-1.75\ldots - 0.496\ldots\,i}^9\right),$

$$y \to (-1.75\ldots - 0.496\ldots\, i),\ z \to -\frac{1}{3}(-1.75\ldots - 0.496\ldots\, i)$$
$$\Big(-5 + 13(-1.75\ldots - 0.496\ldots\, i) + 4(-1.75\ldots - 0.496\ldots\, i)^2 -$$
$$21(-1.75\ldots - 0.496\ldots\, i)^3 + 17(-1.75\ldots - 0.496\ldots\, i)^4 + (-1.75\ldots - 0.496\ldots\, i)^5 -$$
$$13(-1.75\ldots - 0.496\ldots\, i)^6 + (-1.75\ldots - 0.496\ldots\, i)^7 + 3(-1.75\ldots - 0.496\ldots\, i)^8\Big)\Big\},$$

$$\Big\{x \to \frac{1}{3}\Big(3 - 5(-1.75\ldots + 0.496\ldots\, i) + 10(-1.75\ldots + 0.496\ldots\, i)^2 +$$
$$4(-1.75\ldots + 0.496\ldots\, i)^3 - 21(-1.75\ldots + 0.496\ldots\, i)^4 +$$
$$17(-1.75\ldots + 0.496\ldots\, i)^5 + (-1.75\ldots + 0.496\ldots\, i)^6 -$$
$$13(-1.75\ldots + 0.496\ldots\, i)^7 + (-1.75\ldots + 0.496\ldots\, i)^8 + 3(-1.75\ldots + 0.496\ldots\, i)^9\Big),$$
$$y \to (-1.75\ldots + 0.496\ldots\, i),\ z \to -\frac{1}{3}(-1.75\ldots + 0.496\ldots\, i)$$
$$\Big(-5 + 13(-1.75\ldots + 0.496\ldots\, i) + 4(-1.75\ldots + 0.496\ldots\, i)^2 -$$
$$21(-1.75\ldots + 0.496\ldots\, i)^3 + 17(-1.75\ldots + 0.496\ldots\, i)^4 + (-1.75\ldots + 0.496\ldots\, i)^5 -$$
$$13(-1.75\ldots + 0.496\ldots\, i)^6 + (-1.75\ldots + 0.496\ldots\, i)^7 + 3(-1.75\ldots + 0.496\ldots\, i)^8\Big)\Big\},$$

$$\Big\{x \to \frac{1}{3}\Big(3 - 5(0.341\ldots - 0.506\ldots\, i) + 10(0.341\ldots - 0.506\ldots\, i)^2 +$$
$$4(0.341\ldots - 0.506\ldots\, i)^3 - 21(0.341\ldots - 0.506\ldots\, i)^4 +$$
$$17(0.341\ldots - 0.506\ldots\, i)^5 + (0.341\ldots - 0.506\ldots\, i)^6 -$$
$$13(0.341\ldots - 0.506\ldots\, i)^7 + (0.341\ldots - 0.506\ldots\, i)^8 + 3(0.341\ldots - 0.506\ldots\, i)^9\Big),$$
$$y \to (0.341\ldots - 0.506\ldots\, i),\ z \to -\frac{1}{3}(0.341\ldots - 0.506\ldots\, i)$$
$$\Big(-5 + 13(0.341\ldots - 0.506\ldots\, i) + 4(0.341\ldots - 0.506\ldots\, i)^2 -$$
$$21(0.341\ldots - 0.506\ldots\, i)^3 + 17(0.341\ldots - 0.506\ldots\, i)^4 + (0.341\ldots - 0.506\ldots\, i)^5 -$$
$$13(0.341\ldots - 0.506\ldots\, i)^6 + (0.341\ldots - 0.506\ldots\, i)^7 + 3(0.341\ldots - 0.506\ldots\, i)^8\Big)\Big\},$$

$$\Big\{x \to \frac{1}{3}\Big(3 - 5(0.341\ldots + 0.506\ldots\, i) + 10(0.341\ldots + 0.506\ldots\, i)^2 +$$
$$4(0.341\ldots + 0.506\ldots\, i)^3 - 21(0.341\ldots + 0.506\ldots\, i)^4 +$$
$$17(0.341\ldots + 0.506\ldots\, i)^5 + (0.341\ldots + 0.506\ldots\, i)^6 -$$
$$13(0.341\ldots + 0.506\ldots\, i)^7 + (0.341\ldots + 0.506\ldots\, i)^8 + 3(0.341\ldots + 0.506\ldots\, i)^9\Big),$$
$$y \to (0.341\ldots + 0.506\ldots\, i),\ z \to -\frac{1}{3}(0.341\ldots + 0.506\ldots\, i)$$
$$\Big(-5 + 13(0.341\ldots + 0.506\ldots\, i) + 4(0.341\ldots + 0.506\ldots\, i)^2 -$$
$$21(0.341\ldots + 0.506\ldots\, i)^3 + 17(0.341\ldots + 0.506\ldots\, i)^4 + (0.341\ldots + 0.506\ldots\, i)^5 -$$
$$13(0.341\ldots + 0.506\ldots\, i)^6 + (0.341\ldots + 0.506\ldots\, i)^7 + 3(0.341\ldots + 0.506\ldots\, i)^8\Big)\Big\}\Big\}$$

In[ ]:=
```
eqns = {1 == 2 a y, y == 9 + 2 x}
```
Out[ ]=
$\{1 == 2\,a\,y,\ y == 9 + 2\,x\}$

In[ ]:=
```
Solve[eqns, {x, y}]
```
Out[ ]=
$$\left\{\left\{x \to -\frac{-1 + 18\,a}{4\,a},\ y \to \frac{1}{2\,a}\right\}\right\}$$

Ex3 : Executeu les següents comandes i esbrineu què fan:
a) Eliminate[{a*x^2 + b*x*y + c*y^2 == 0, x + y == 1}, x]
b) Reduce[x^2 < 1 && y - Sqrt[x] == 0, {x, y}]
c) Roots[x^3 - x - 7 == 0, x]
d) NSolve[-(1 + x) + 2 Sqrt[1 + x^2] - 3 (1 + x^3)^{1/3} == 0, x]
e) Solve[x^2 +a*x+b==0, x, Reals]
f) SolveAlways[(a-b)*x^2+(a^2- b+1)*x+a-c==0, x]

In[ ]:=
```
Eliminate[{a * x ^ 2 + b * x * y + c * y ^ 2 == 0, x + y == 1}, x]
```
Out[ ]=
$a - 2\,a\,y + b\,y + a\,y^2 - b\,y^2 + c\,y^2 == 0$

In[ ]:=
```
Reduce[x ^ 2 < 1 && y - Sqrt[x] == 0, {x, y}]
```
Out[ ]=
$-1 < x < 1\ \&\&\ y == \sqrt{x}$

In[ ]:=
```
Roots[x ^ 3 - x - 7 == 0, x]
```
Out[ ]=
$$x == \frac{1}{3}\left(\frac{189}{2} - \frac{3\sqrt{3957}}{2}\right)^{1/3} + \frac{\left(\frac{1}{2}\left(63 + \sqrt{3957}\right)\right)^{1/3}}{3^{2/3}}\ ||$$
$$x == -\frac{1}{6}\left(1 + i\sqrt{3}\right)\left(\frac{189}{2} - \frac{3\sqrt{3957}}{2}\right)^{1/3} - \frac{\left(1 - i\sqrt{3}\right)\left(\frac{1}{2}\left(63 + \sqrt{3957}\right)\right)^{1/3}}{2 \times 3^{2/3}}\ ||$$
$$x == -\frac{1}{6}\left(1 - i\sqrt{3}\right)\left(\frac{189}{2} - \frac{3\sqrt{3957}}{2}\right)^{1/3} - \frac{\left(1 + i\sqrt{3}\right)\left(\frac{1}{2}\left(63 + \sqrt{3957}\right)\right)^{1/3}}{2 \times 3^{2/3}}$$

In[ ]:=
```
NSolve[- (1 + x) + 2 Sqrt[1 + x ^ 2] - 3 (1 + x ^ 3) ^ {1 / 3} == 0, x]
```
Out[ ]=
$\{\{x \to 0.500206 - 0.865906\,i\},\ \{x \to 0.500206 + 0.865906\,i\},\ \{x \to -0.800121\}\}$

In[ ]:=
```
Solve[x ^ 2 + a * x + b == 0, x, Reals]
```
Out[ ]=
$$\left\{\left\{x \to \boxed{-\frac{a}{2} - \frac{1}{2}\sqrt{a^2 - 4\,b}\ \text{ if }\ b < \frac{a^2}{4}}\right\},\ \left\{x \to \boxed{-\frac{a}{2} + \frac{1}{2}\sqrt{a^2 - 4\,b}\ \text{ if }\ b < \frac{a^2}{4}}\right\}\right\}$$

In[ ]:=
```
SolveAlways[(a - b) * x ^ 2 + (a ^ 2 - b + 1) * x + a - c == 0, x]
```
Out[ ]=
$$\left\{\left\{a \to (-1)^{1/3},\ b \to (-1)^{1/3},\ c \to (-1)^{1/3}\right\},\ \left\{a \to -(-1)^{2/3},\ b \to -(-1)^{2/3},\ c \to -(-1)^{2/3}\right\}\right\}$$

Treball personal :

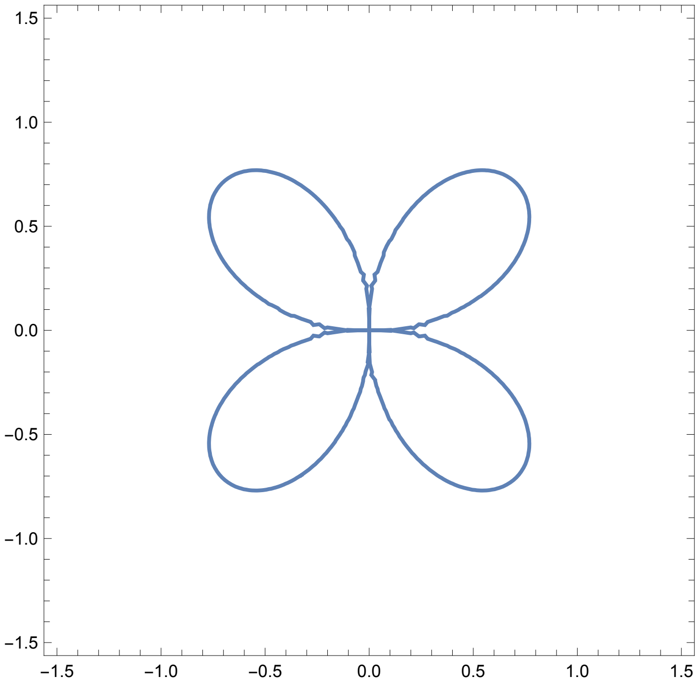
a) Veriqueu mitjançant un “Plot” que la corba amb equació polar r = Sin[2 t] i la corba V ((x^2 + y^2)^3 - 4 x^2*y^2) són la mateixa.

b) Proveu que a) és cert .

In[∘]:=

```
ContourPlot[(x^2 + y^2)^3 - 4 x^2 * y^2 == 0, {x, -1.5, 1.5}, {y, -1.5, 1.5}]
```

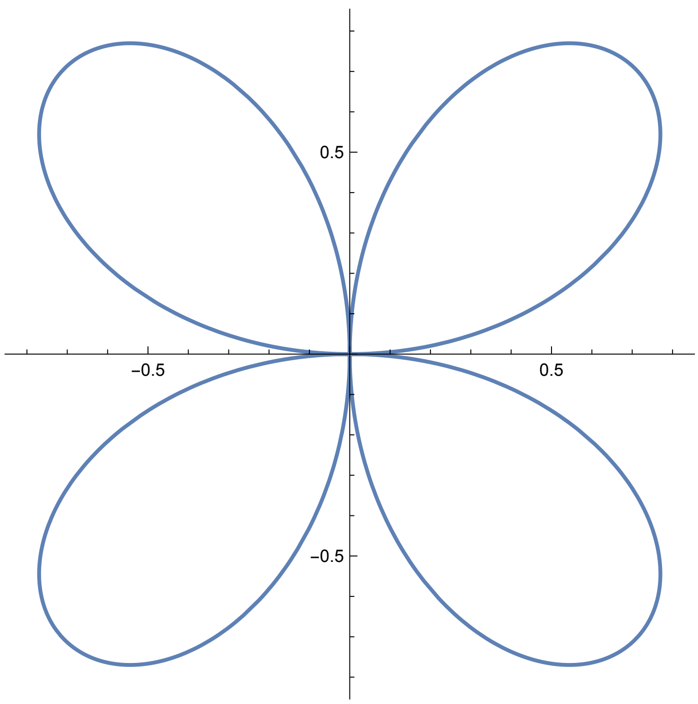
Out[∘]=



In[∘]:=

```
ParametricPlot[{Cos[t] * Sin[2 * t], Sin[t] * Sin[2 t]}, {t, 0, 2 Pi}]
```

Out[∘]=



Veremos que los puntos de $r = \sin 2t$ pertenecen a $(x^2 + y^2)^3 - 4x^2 y^2 = 0$. Tomamos $x = r\cos t$ y $y = r\sin t$, entonces:

$(x^2+y^2)^3 - 4x^2y^2 = (r^2(\cos^2 t + \sin^2 t))^3 - 4r^4\cos^2$
$t \sin^2 t = r^6 - r^4 (2\cos t \sin t)^2 = r^6 - r^4 (\sin 2t)^2 = r^6 - r^6 = 0$

Veamos ahora la implicación contraria:

Sea $(x, y) \in V((x^2+y^2)^3 - 4x^2y^2)$, expresandolo en polares tenemos $x = r\cos t$ y $y = r\sin t$, y substituyendo obtenemos:

$r^6 = 4r^4\cos^2 t \sin^2 t = r^4 \sin^2 2t$

$r^2 = \sin^2 2t \Rightarrow r = |\sin 2t|$

Si tenemos que $\sin 2t < 0$ entonces podemos hacer el canvio $-\cos t = \cos t + \pi$ y $-\sin t = \sin t + \pi$, y obtenemos que el punto tambien pertenece a la curva $r = \sin 2t$:

$x = -\sin 2t \cos t = \sin(2t + 2\pi)\cos(t+\pi)$ y $y = -\sin 2t \sin t = \sin(2t + 2\pi)\sin(t+\pi)$ como queríamos.