Anells de polinomis en diverses variables

Primavera 2025

Laboratori 4: Bases de Gröbner

Teorema : Sigui G una base de Gröbner d ' un ideal I, aleshores :

f ∈ I sí i només si el reste de dividir f per G és zero

```
In[1]:=
GroebnerBasis[{x^3 - 2 * x * y, x^2 * y - 2 * y^2 + x},
 {x, y}, MonomialOrder → DegreeLexicographic]
```

Out[1]= $\{-x + 2 y^2, x y, x^2\}$

```
In[2]:= GroebnerBasis[{x^3 - 2 * x * y, x^2 * y - 2 * y^2 + x},
 {x, y}, MonomialOrder → DegreeReverseLexicographic]
```

Out[2]= $\{-x + 2 y^2, x y, x^2\}$

```
In[3]:=
GroebnerBasis[{x^3 - 2 * x * y, x^2 * y - 2 * y^2 + x},
 {x, y}, MonomialOrder → Lexicographic]
```

Out[3]= $\{y^3, x - 2 y^2\}$

```
In[4]:=
GroebnerBasis[{x^3 - 2 * x * y, x^2 * y - 2 * y^2 + x}, {x, y}]
```

Out[4]= $\{y^3, x - 2 y^2\}$

Ex 1: Sigui I l'ideal generat per:

```
In[5]:= G := {x * z - y^2, x^3 - z^2}
```

Detemineu si f pertany a I:

```
In[6]:= f := -4 * x^2 * y^2 * z^2 + y^6 + 3 * z^5
```

```
In[7]:= base = GroebnerBasis[G, {x, y, z}]
PolynomialReduce[f, base, {x, y, x}]
```

Out[7]= $\{y^6 - z^5, -y^2 + x z, x y^4 - z^4, x^2 y^2 - z^3, x^3 - z^2\}$

Out[8]= $\{\{-3, -4 y^4 - 4 x y^2 z, 0, 0, 0\}, 0\}$

Ex 2: Calculeu una base de Gröbner de l'ideal generat per

In[12]:=
```
G := {x^5 + y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1}
```

usant "Lex" i "DegRevLex" amb x>y>z.

In[13]:=
```
baselex = GroebnerBasis[G, {x, y, z}, MonomialOrder → Lexicographic]
basedegrevlex =
 GroebnerBasis[G, {x, y, z}, MonomialOrder → DegreeReverseLexicographic]
```

Out[13]=

$$\{-5 y^2 + 13 y^4 - 10 y^6 + 2 y^8 - y^{10} + y^{12} - 5 z^2 + 20 y^2 z^2 - 30 y^4 z^2 + 20 y^6 z^2 - 5 y^8 z^2 + 3 z^3 - 6 y^4 z^3 + 3 y^8 z^3 + 10 z^4 - 30 y^2 z^4 + 30 y^4 z^4 - 10 y^6 z^4 - 13 z^6 + 20 y^2 z^6 - 7 y^4 z^6 + 5 z^8 - 5 y^2 z^8 + z^9 - z^{10},$$
$$20 y^2 - 32 y^4 + 8 y^6 + 4 y^{10} + 20 y^2 z - 32 y^4 z + 8 y^6 z + 4 y^{10} z + 20 z^2 - 70 y^2 z^2 + 76 y^4 z^2 - 24 y^6 z^2 - 2 y^{10} z^2 + 8 z^3 - 77 y^2 z^3 + 90 y^4 z^3 - 16 y^6 z^3 - 2 y^8 z^3 - 3 y^{10} z^3 - 62 z^4 + 108 y^2 z^4 - 64 y^4 z^4 + 20 y^6 z^4 - 2 y^8 z^4 - 39 z^5 + 106 y^2 z^5 - 78 y^4 z^5 + 10 y^6 z^5 + y^8 z^5 + 77 z^6 + 8 x z^6 - 80 y^2 z^6 + 17 y^4 z^6 - 6 y^6 z^6 + 55 z^7 - 4 x z^7 - 68 y^2 z^7 + 21 y^4 z^7 - 42 z^8 - 10 x z^8 + 24 y^2 z^8 + 2 y^4 z^8 - 31 z^9 + x z^9 + 17 y^2 z^9 + 7 z^{10} + 4 x z^{10} + 7 z^{11} + x z^{11},$$
$$-24 + 24 x + 7 y^2 - 24 x y^2 + 38 y^4 - 8 y^6 - 6 y^8 - 7 y^{10} + 24 z - 24 x z - 87 y^2 z + 24 x y^2 z + 90 y^4 z - 24 y^6 z + 6 y^8 z - 9 y^{10} z - 17 z^2 + 129 y^2 z^2 - 174 y^4 z^2 + 62 y^6 z^2 - 9 y^8 z^2 + 9 y^{10} z^2 - 36 z^3 - 12 x z^3 + 170 y^2 z^3 - 165 y^4 z^3 + 20 y^6 z^3 + 11 y^8 z^3 + 127 z^4 + 12 x z^4 - 302 y^2 z^4 + 226 y^4 z^4 - 60 y^6 z^4 - 3 y^8 z^4 + 68 z^5 - 118 y^2 z^5 + 44 y^4 z^5 + 18 y^6 z^5 - 206 z^6 - 23 x z^6 + 239 y^2 z^6 - 53 y^4 z^6 - 22 z^7 + 21 x z^7 + 13 y^2 z^7 - 6 y^4 z^7 + 93 z^8 + 12 x z^8 - 51 y^2 z^8 + 14 z^9 - 7 x z^9 - 21 z^{10} - 3 x z^{10},$$
$$1 - x - 2 y^2 + y^4 + x y^4 - 2 z^2 + 2 y^2 z^2 + x z^3 + z^4,$$
$$-55 y^2 + 58 y^4 - 4 y^6 + 6 y^8 - 5 y^{10} + 20 y^2 z - 32 y^4 z + 8 y^6 z + 4 y^{10} z - 31 z^2 - 24 x^2 z^2 + 144 y^2 z^2 - 156 y^4 z^2 + 34 y^6 z^2 + 3 y^8 z^2 + 6 y^{10} z^2 + 41 z^3 + 12 x^2 z^3 - 56 y^2 z^3 + 69 y^4 z^3 - 34 y^6 z^3 + 4 y^8 z^3 + 83 z^4 + 12 x z^4 + 12 x^2 z^4 - 192 y^2 z^4 + 131 y^4 z^4 - 20 y^6 z^4 - 2 y^8 z^4 - 90 z^5 - 12 x z^5 + 102 y^2 z^5 - 36 y^4 z^5 + 12 y^6 z^5 - 93 z^6 - x z^6 + 119 y^2 z^6 - 42 y^4 z^6 + 63 z^7 + 14 x z^7 - 48 y^2 z^7 - 4 y^4 z^7 + 55 z^8 - 3 x z^8 - 34 y^2 z^8 - 14 z^9 - 8 x z^9 - 14 z^{10} - 2 x z^{10},$$
$$1 - x^2 + x^2 y^2 - y^4 + x^2 z^2 - z^3, -1 + x^3 + y^2 + z^2\}$$

Out[14]=

$$\{-1 + x^3 + y^2 + z^2, 1 - x^2 + x^2 y^2 - y^4 + x^2 z^2 - z^3,$$
$$1 - x - 2 y^2 + y^4 + x y^4 - 2 z^2 + 2 y^2 z^2 + x z^3 + z^4, -2 + 2 x + y^2 - 2 x y^2 + y^6 + 3 z^2 - 2 x z^2 - 2 x^2 z^2 - 2 y^2 z^2 + 2 x y^2 z^2 - y^4 z^2 + z^3 - x z^3 + x^2 z^3 + y^2 z^3 - z^4 + x z^4 + x^2 z^4 - z^5\}$$

Ex 3: Calculeu una base de Gröbner de l'ideal de l'exercici anterior per DegRevLex amb z>y>x.

In[15]:=
```
GroebnerBasis[G, {z, y, x}, MonomialOrder → DegreeReverseLexicographic]
```

Out[15]=

$$\{-1 + x^3 + y^2 + z^2, -1 + x^2 - x^2 y^2 + y^4 - x^2 z^2 + z^3\}$$

Ex 4: Per cada enter n considereu l'ideal I generat per

In[36]:=
```
G := {x ^ (n + 1) - y * z ^ (n - 1) * w,
   x * y ^ (n - 1) - z ^ n, x ^ n * z - y ^ n * w},
```

i usant el DegRevLex amb x>y>z>w calculeu una base de Gröbner de I per n baix, p.e. fins n=5.
Comproveu que per cada n=2,...,10,... la base conté el polinomi `z^(n^2+1)-y^(n^2)*w.`
Quants elements té la base de Gröbner?

In[64]:=
```
basis = Table[
  GroebnerBasis[{x^(n+1) - y * z^(n-1) * w, x * y^(n-1) - z^n, x^n * z - y^n * w},
   {x, y, z}, MonomialOrder → DegreeReverseLexicographic], {n, 1, 5}]
numberOfElements = Table[Length[b], {b, basis}]
```

Out[64]=

$$\{\{x - z, -w\,y + z^2\}, \{x\,y - z^2, -w\,y^2 + x^2\,z, x^3 - w\,y\,z, -w\,y^3 + x\,z^3, -w\,y^4 + z^5\},$$
$$\{x\,y^2 - z^3, -w\,y^3 + x^3\,z, x^4 - w\,y\,z^2, -w\,y^5 + x^2\,z^4, -w\,y^7 + x\,z^7, -w\,y^9 + z^{10}\},$$
$$\{x\,y^3 - z^4, -w\,y^4 + x^4\,z, x^5 - w\,y\,z^3, -w\,y^7 + x^3\,z^5, -w\,y^{10} + x^2\,z^9, -w\,y^{13} + x\,z^{13}, -w\,y^{16} + z^{17}\},$$
$$\{x\,y^4 - z^5, -w\,y^5 + x^5\,z, x^6 - w\,y\,z^4, -w\,y^9 + x^4\,z^6,$$
$$-w\,y^{13} + x^3\,z^{11}, -w\,y^{17} + x^2\,z^{16}, -w\,y^{21} + x\,z^{21}, -w\,y^{25} + z^{26}\}\}$$

Out[65]=

{2, 5, 6, 7, 8}

Ex 5.

Siguin ***a, b*** *i* ***c*** *nombres tals que*

a+b+c=3

a^2+b^2+c^2=5

a^3+b^3+c^3=7

a) Sense resoldre les equacions anterior, demostreu que ***a^4+b^4+c^4=9***

(idea: es suficient demostrar que ***a^4+b^4+c^4-9*** *pertany a l'ideal* ***(a+b+c-3, a^2+b^2+c^2-5, a^3+b^3+c^3-7)****).*

b) Demostreu que ***a^5+b^5+c^5*** *no és igual a* ***11.***

c) Calculeu r ***a^n+b^n+c^n*** *per* ***n=5,6,7,...10****.*

In[53]:=
```
base = GroebnerBasis[{a + b + c - 3, a^2 + b^2 + c^2 - 5, a^3 + b^3 + c^3 - 7},
  {a, b, c}, MonomialOrder → DegreeReverseLexicographic]
resFromA = PolynomialReduce[a^4 + b^4 + c^4 - 9, base,
   {a, b, c}, MonomialOrder → DegreeReverseLexicographic][[2]]
resFromB = PolynomialReduce[a^5 + b^5 + c^5 - 9, base,
   {a, b, c}, MonomialOrder → DegreeReverseLexicographic][[2]]
```

Out[53]=

$$\left\{-3+a+b+c,\ 2-3\,b+b^2-3\,c+b\,c+c^2,\ 2+6\,c-9\,c^2+3\,c^3\right\}$$

Out[54]=

0

Out[55]=

$$\frac{2}{3}$$

In[57]:=
```
Table[PolynomialReduce[a^n + b^n + c^n , base, {a, b, c},
   MonomialOrder → DegreeReverseLexicographic][[2]], {n, 5, 10}]
```

Out[57]=

$$\left\{\frac{29}{3},\ \frac{19}{3},\ -\frac{19}{3},\ -\frac{343}{9},\ -\frac{953}{9},\ -\frac{2135}{9}\right\}$$