Homework 1
Optimization

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Show that the following functions are log-concave[1], i.e.

Definition 1 *f is log-concave if*

$$f(\theta x + (1-\theta y)) \geq f(x)^{\theta} f(y)^{1-\theta} \tag{1}$$

or equivalently

$$\log f(\theta x + (1-\theta)y) \geq \theta \log f(x) + (1-\theta)\log f(y) \tag{2}$$

$$\forall x, y \in dom f; 0 \leq \theta \leq 1$$

1. Logistic function

$$\frac{e^x}{(1+e^x)} \quad \mathrm{dom} f = \mathbb{R} \tag{3}$$

$$\begin{gathered} f(\theta x + (1-\theta)y) = \frac{e^{\theta x + (1-\theta)y}}{1+e^{\theta x + (1-\theta)y}} \geq \frac{e^{\theta x + (1-\theta)y}}{(1+e^x)^{\theta}(1+e^y)^{1-\theta}} = f(x)^{\theta} f(y)^{1-\theta} \\ 1 + e^{\theta x} e^{(1-\theta)y} \leq (1+e^x)^{\theta}(1+e^y)^{1-\theta} \\ \theta \log e^x + (1-\theta)\log e^y \leq \theta \log(1+e^x) + (1-\theta)\log(1+e^y) \end{gathered} \tag{4}$$

Which holds since $\log(e^x) \leq \log(e^x + c)$, $\forall\, c \geq 0$

□

2. Harmonic mean :

$$f(x) = \frac{1}{\sum_i^n x_i} \quad \mathrm{dom} f = \mathbb{R}^n_{++} \tag{5}$$

Recall $\min x_i \leq H(x) \leq n \min x_i$. With $H(x) = nf(x)$ the conventional Harmonic mean. Then

$$\begin{gathered} f(\theta x + (1-\theta y)) \geq f(x)^{\theta} f(y)^{1-\theta} \\ n \min\{\ \theta x_i + (1-\theta)y_i\} \geq \min\{x_i\}^{\theta} \min\{y_i\}^{1-\theta} \end{gathered} \tag{6}$$

Let x_k, y_k be the minimums respectively.

$$\theta x_k + (1-\theta)y_k \geq x_k^{\theta} y_k^{1-\theta} \tag{7}$$

Which holds $\forall\, x_k, y_k \in \mathbb{R}^n_{++}$ and $0 \leq \theta \leq 1$

□

[1]From *Boyd, S., & Vandenberghe, L. (2002). Convex Optimization* problem 3.49

3. Product over sum:

$$f(x) = \frac{\prod_{i=1}^n x_i}{\sum_{i=1}^n x_i} \quad \mathrm{dom} f = \mathbb{R}^n_{++} \tag{8}$$

$$\begin{aligned} \log f(x) &= \log \prod_{i=1}^n x_i - \log \sum_{i=1}^n x_i \\ &= \sum_{i=1}^n \log x_i - \log \sum_{i=1}^n x_i \end{aligned} \tag{9}$$

$$\begin{aligned} f(\theta x + (1-\theta)y) &\geq f(x)^\theta f(y)^{1-\theta} \\ \frac{\prod_{i=1}^n \theta x + (1-\theta)y}{\sum_{i=1}^n \theta x + (1-\theta)y} &\geq \left(\frac{\prod_{i=1}^n x_i}{\sum_{i=1}^n x_i}\right)^\theta \left(\frac{\prod_{i=1}^n y_i}{\sum_{i=1}^n y_i}\right)^{1-\theta} \\ &\geq \frac{\prod x_i^\theta y_i^{1-\theta}}{(\sum x_i)^\theta (\sum y_i)^{1-\theta}} \end{aligned} \tag{10}$$

Which holds by the previous result

□

4. Determinant over trace:

$$f(X) = \frac{\det X}{\operatorname{tr} X} \quad \mathrm{dom} f = \mathbb{S}^n_{++} \tag{11}$$

Since $X \in \mathbb{S}^n_{++}$ we can always find Λ the associated diagonal matrix with eigenvalues $\lambda_k > 0$ and we may write the invariants:

$$f(X) = f(\Lambda) = \frac{\det \Lambda}{\operatorname{tr} \Lambda} = \frac{\prod \lambda_i}{\sum \lambda_i} \tag{12}$$

$$\log f(x) = \sum \log \lambda_i - \log \sum \lambda_i \tag{13}$$

Which is the previous result

□