The Egyptian Tangram



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mmaca

December 26, 2020

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A square dissection firstly proposed as a tangram in:

Luna-Mota, C. (2019) "El tangram egipci: diari de disseny" Nou Biaix, 44

Origins

The Egyptian Tangram inspiration comes from the study of two other 5-piece tangrams...

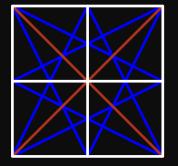


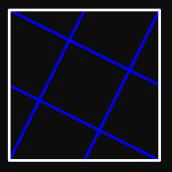


The "Five Triangles" & "Greek-Cross" tangrams

Origins

...and their underlying grids





The "Five Triangles" & "Greek-Cross" underlying grids

Design Process

The Egyptian Tangram is the result of an heuristic incremental design process:



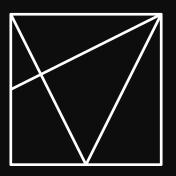






Take a square and keep adding "the most interesting straight cut" until you have a dissection with 5 or more pieces.

Design Process



To make an Egyptian Tangram from a square:

- 1. Connect the midpoint of the lower side with the upper corners.
- 2. Connect the midpoint of the left side with the top right corner.

Antecedents

It turns out that this figure is not new...

Detemple, D. & Harold, S. (1996) "A Round-Up of Square Problems"



...but, to the best of our knowledge, nobody used it before as a tangram

Antecedents

The name is not new either...



This dissection is often called "Egyptian Puzzle" or "Egyptian Tangram"

...but there is a good reason to consider our dissection the real "Egyptian Tangram"

(even if it was designed in Barcelona)

Antecedents

Its underlying grid is also a well known figure:



Brunés, T. (1967) "The Secrets of Ancient Geometry – and Its Use"

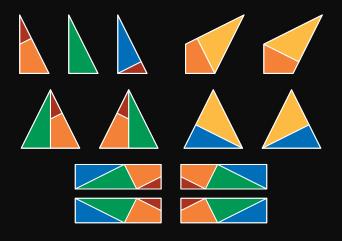
Bankoff, L. & W. Trigg, C. (1974) "The Ubiquitous 3:4:5 Triangle"



- Just five pieces
- All pieces are different
- All pieces are asymmetric
- Areas are integer and not too different
- All sides are multiples of 1 or $\sqrt{5}$
- All angles are linear combinations of 90° and $\alpha = \arctan(\frac{1}{2}) \approx 26,565^{\circ}$

Name	Area	Sides	Angles
T1	1	1, 2, $\sqrt{5}$	90, α , 90 – α
T4	4	2, 4, $2\sqrt{5}$	90, α , 90 – α
Т5	5	$\sqrt{5}$, $2\sqrt{5}$, 5	90, α , 90 – α
Т6	6	3, 4, 5	90, $90-2\alpha$, 2α
Q4	4	1, 3, $\sqrt{5}$, $\sqrt{5}$	90, $90-\alpha$, 90 , $90+\alpha$

Although all pieces are asymmetric and different, they often combine to make symmetric shapes

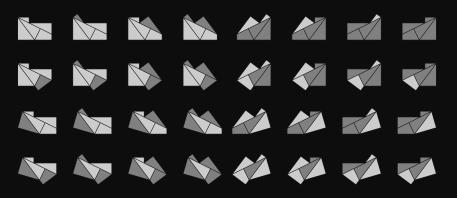


This means that it is very rare for an Egyptian Tangram figure to have a unique solution



There are three different solutions for the square and, in all three cases, two of the corners of the square are built as a sum of acute angles!

The asymmetry of the pieces also implies that each solution belongs to one of these equivalence classes:



You cannot transform one of these figures into another without flipping a piece

Why we called it the Egyptian Tangram?

The smallest pieces of the Chinese and Greek-Cross tangrams can be used to build all the other pieces...







...but you cannot do the same with the Egyptian Tangram because of T6

Why we called it the *Egyptian* Tangram?

Initially, T6 was considered as the *leftover piece* that results from cutting all these $1:2:\sqrt{5}$ triangles from the borders of the square.

But it turned out to be a very well known triangle...



...the **Egyptian** Triangle (3:4:5) and, hence, the name of this tangram

Puzzles & Activities

Realistic figures

Use all five pieces to make these figures:



Lightning



Sailing ship



Bow tie



Wooden hut



Caltrop



Snowmobile



Candle



Viking hat



Diamond



Moses basket



3D brick



Witch hat

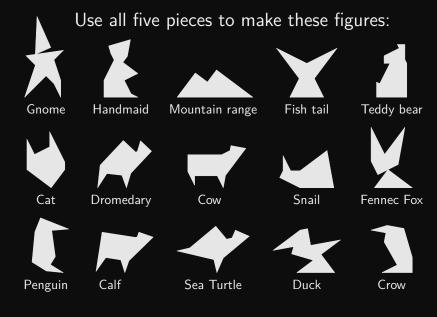


Arrow Sign



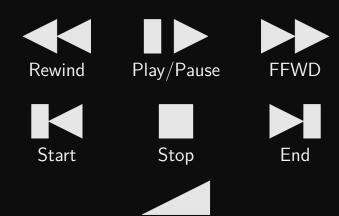
Sailboat

Realistic figures



Remote control symbols

Use all five pieces to make these symbols:

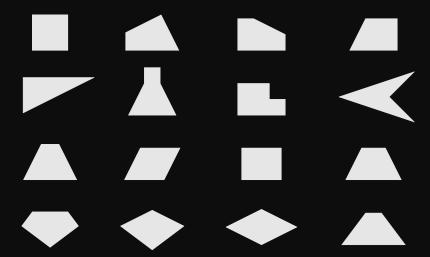


Volume

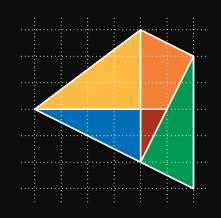
Use all five pieces to make these figures:

Complete the path moving just one or two pieces at a time

All of these figures, but one, have multiple solutions:



Could you find which is the only figure with unique solution?

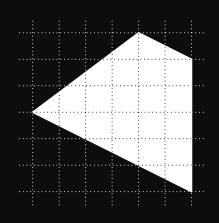


All the figures from the previous page, but one, can be drawn with their vertices lying on this lattice.

Moreover, they can be drawn with the vertices of all 5 pieces lying on the same lattice.

These conditions simplify the task of finding perimeters and areas.

Could you find which is the only figure that requires a finer-grained lattice?

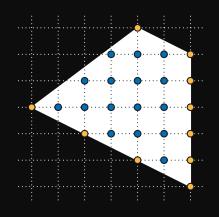


Pythagorean Theorem:

Top
$$=\sqrt{1^2+2^2}=\sqrt{5}$$

Left $=\sqrt{3^2+4^2}=\sqrt{25}=5$
Right $=\sqrt{5^2+0^2}=\sqrt{25}=5$
Bottom $=\sqrt{3^2+6^2}=\sqrt{45}=3\sqrt{5}$
Perimeter $=10+4\sqrt{5}$

Could you use the Pythagorean theorem to compute the perimeter of these figures?



Pick's Theorem:

lattice points on the boundary =10

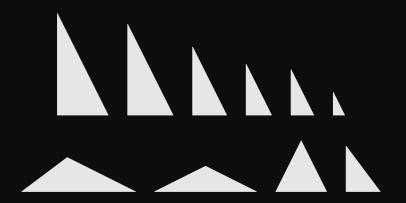
Area = interior +
$$\frac{\text{boundary}}{2}$$
 - 1 = $16 + \frac{10}{2} - 1 = 20$

Could you use Pick's theorem to compute the area of these figures?

Triangles

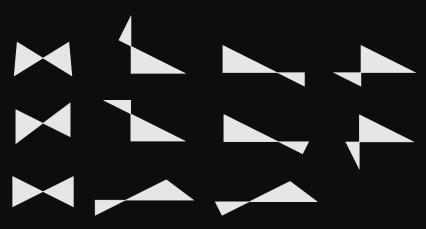
Could you prove that there are just 10 triangles you can make with one or more pieces of the Egyptian Tangram?

How many solutions could you find for each figure?

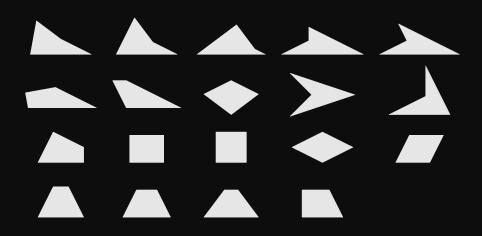


Top row areas: 20, 16, 9, 5, 4, 1 Bottom row areas: 15, 10, 10, 6

Could you prove that there are just 11 **complex quadrilaterals** you can make with all five pieces of the Egyptian Tangram?

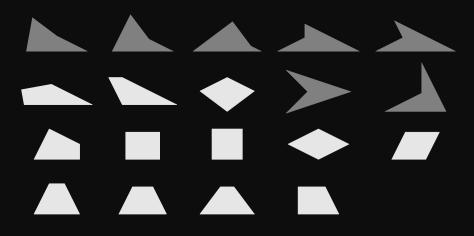


Simple quadrilaterals: Not self-intersecting



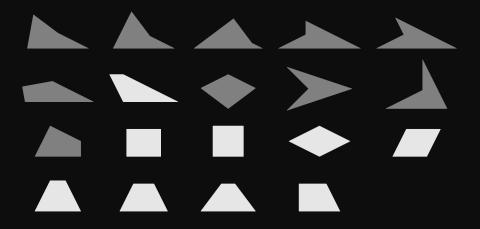
All simple quadrilaterals tile the plane!

Convex quadrilaterals: All internal angles are smaller than π



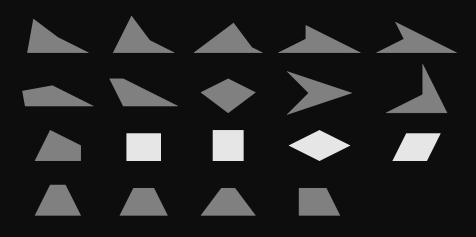
Law of Cosines: $p^2q^2 = a^2c^2 + b^2d^2 - 2abcd\cos(\alpha + \gamma)$

Trapeziums (UK) / Trapezoids (US): One pair of parallel sides



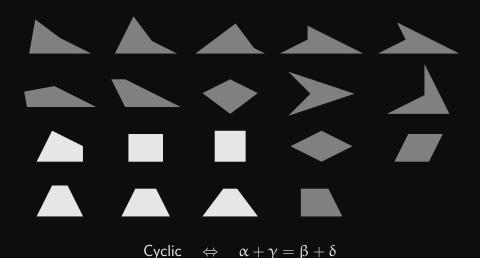
 $Trapezium/Trapezoid \Leftrightarrow Diagonals cut each other in the same ratio$

Parallelograms: Two pairs of parallel sides

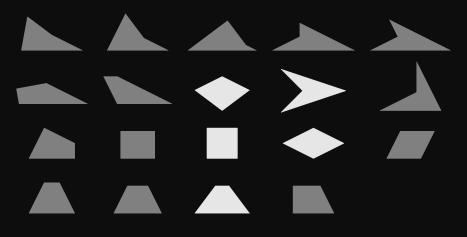


Parallelogram \Leftrightarrow Diagonals bisect each other \Leftrightarrow $a^2 + b^2 + c^2 + d^2 = p^2 + q^2$

Cyclic quadrilaterals: All vertices lie on a circle

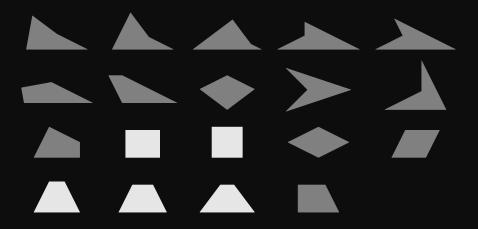


Tangential quadrilaterals: All sides are tangent to a circle



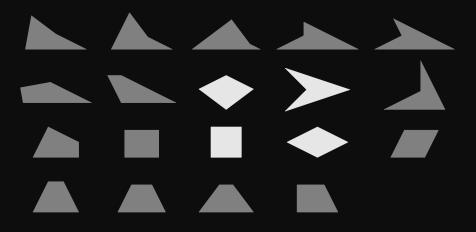
 $\overline{\mathsf{Tangential}} \quad \Leftrightarrow \quad \overline{a+c} = b+d$

Isosceles Trapezoids: Two pairs of adjacent angles are equal



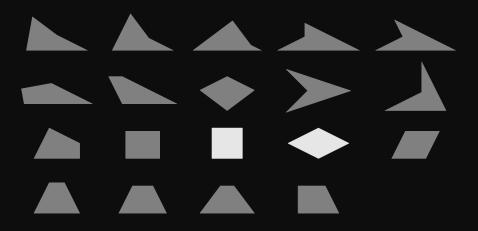
Isosceles trapezoids \Leftrightarrow Cyclic quadrilaterals with equal diagonals

Darts & Kites: Two pairs of adjacent sides are equal



 $Darts/Kites \Leftrightarrow Tangential \ quadrilaterals \ with \ perpendicular \ diagonals$

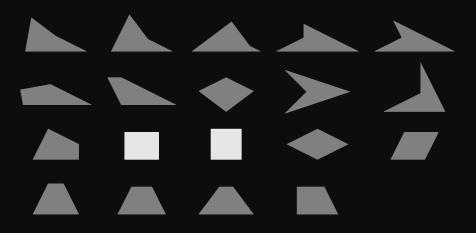
Rhombi: All sides are equal



Rhombi \Leftrightarrow Parallelograms with perpendicular diagonals

Quadrilaterals

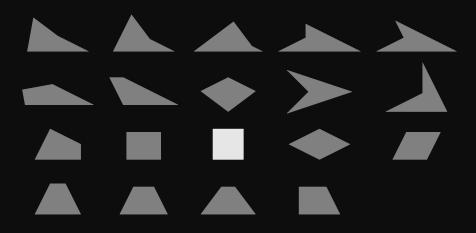
Rectangles: All angles are equal



Rectangles \Leftrightarrow Parallelograms with equal diagonals

Quadrilaterals

Squares: Regular quadrilaterals



Among all quadrilaterals, squares maximize the Area: Perimeter ratio

The three solutions of the square

Could you prove that there are just three different solutions for the square?



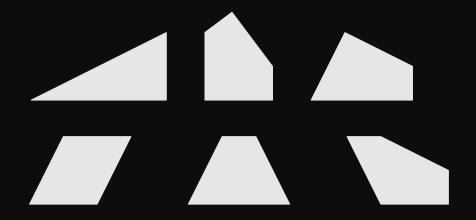




What's the area of this square? What's its perimeter? How many times do you find $\sqrt{5}$ in the Egyptian Tangram pieces?

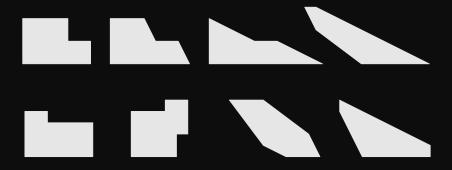
Figures with seven solutions

Could you find seven different solutions for each of these figures?



Figures with unique solutions

Could you prove that there is only one solution for each of these figures?



Both figures use all 5 pieces...



Both figures use all 5 pieces...



Both figures use all 5 pieces...



Both figures use all 5 pieces...



Both figures use all 5 pieces...



Both figures use all 5 pieces...





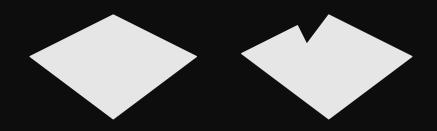
Both figures use all 5 pieces...



Both figures use all 5 pieces...



Both figures use all 5 pieces...



Both figures use all 5 pieces...



Both figures use all 5 pieces...



Both figures use all 5 pieces...





Both figures use all 5 pieces...





Both figures use all 5 pieces...





Both figures use all 5 pieces...





Both figures use all 5 pieces...



Both figures use all 5 pieces...





Both figures use all 5 pieces...



Missing rectangle paradox

Both figures use all 5 pieces...



Missing rectangle paradox

Both figures use all 5 pieces...



Missing rectangle paradox

Both figures use all 5 pieces...





Missing square paradox

Both figures use all 5 pieces...



Where is the missing square?

Missing square paradox

Both figures use all 5 pieces...



Where is the missing square?

Missing square paradox

Both figures use all 5 pieces...

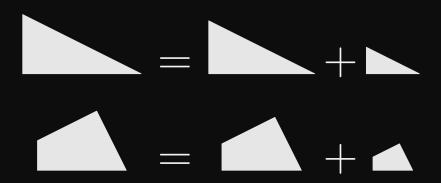




Where is the missing square?

Sum of similar figures

Use all 5 pieces to make the single figure in the LHS, then use them to make the two figures on the RHS



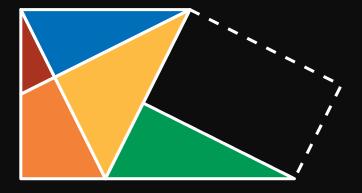
In both equations, the figures are similar and areas are in ratio 5:4:1

The dashed rectangle proportions are $1:\varphi$



where $\phi=rac{1+\sqrt{5}}{2}$ is the golden ratio

The dashed rectangle proportions are $1:\varphi$



where $\phi=\frac{1+\sqrt{5}}{2}$ is the golden ratio

The dashed rectangle proportions are $1:\phi$



where $\overline{\phi}=rac{1+\sqrt{5}}{2}$ is the golden ratio

The dashed rectangles proportions are $1:\phi$



where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

There are 4 golden rectangles hidden in this figure



Could you spot them?

There are 4 golden rectangles hidden in this figure



Could you spot them?

Golden Rectangles

There are 4 golden rectangles hidden in this figure



Could you spot them?

Golden Rectangles

There are 5 golden rectangles hidden in this figure



Could you spot them?

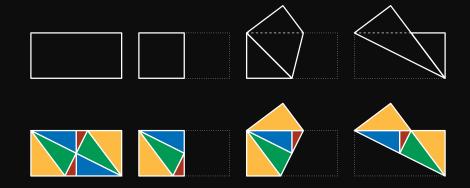
Golden Rectangles

You can find golden rectangles of 14 different types

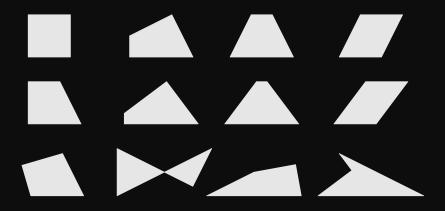
Туре	Proportions	Туре	Proportions	
Α	$3-\sqrt{5}$: $2\sqrt{5}-4$	н	$2\sqrt{5} : 5-\sqrt{5}$	
В	$\sqrt{5} - 1 : 3 - \sqrt{5}$	I	$3+\sqrt{5}$: $1+\sqrt{5}$	
С	2 : $\sqrt{5}-1$	J	$6 : 3\sqrt{5}-3$	
D	$2\sqrt{5}-2 : 6-2\sqrt{5}$	K	$2+2\sqrt{5}$: 4	
E	$5 - \sqrt{5}$: $3\sqrt{5} - 5$	L	$5+\sqrt{5}$: $2\sqrt{5}$	
F	$1+\sqrt{5}$: 2	M	$1+3\sqrt{5}$: $7-\sqrt{5}$	
G	4 : $2\sqrt{5}-2$	Ν	$4+2\sqrt{5}$: $3+\sqrt{5}$	

Could you build an example of each type?

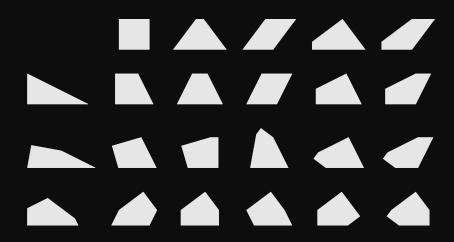
T1, T4, T5 & T6 appear naturally when you fold a 2:1 rectangle



You can make 12 quadrilaterals using T1, T4, T5 & T6



You can make 23 convex figures using T1, T4, T5 & T6



You can find golden rectangles of 7 different types using just T1, T4, T5 & T6

Type	Proportions			
Α	$3 - \sqrt{5}$		$2\sqrt{5}-4$	
C	2		$\sqrt{5}-1$	
F	$1 + \sqrt{5}$		2	
G	4		$2\sqrt{5}-2$	
Н	$2\sqrt{5}$		$5 - \sqrt{5}$	
K	$2+2\sqrt{5}$		4	
L	$5 + \sqrt{5}$		$2\sqrt{5}$	

Could you build an example of each type?

The Egyptian Three-Triangle-Tangram

Given any right triangle with sides: $a \le b \le c...$

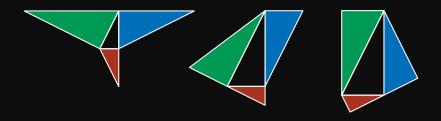


...you can make a rectangle with three similar triangles: (a^2, ab, ac) , (ba, b^2, bc) & (ca, cb, c^2) and compare the top (a^2+b^2) and the bottom (c^2) sides of the rectangle to proof Pythagoras

(T1, T4 & T5 verify this relationship for: $a=1,\ b=2$ & $c=\sqrt{5}$)

The Egyptian Three-Triangle-Tangram

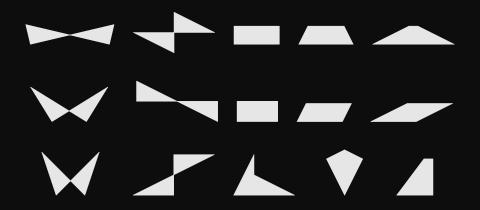
Since
$$area(T1) + area(T4) = area(T5)$$
 ...



...you can verify three cases of Pythagoras' theorem (and these particular cases turn out to be the T1, T4 & T5 right triangles!)

The Egyptian Three–Triangle–Tangram

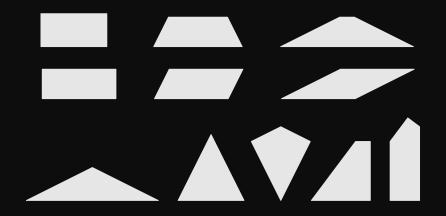
You can make 15 quadrilaterals using just T1, T4 & T5



See also: Brügner, G. (1984) "Three-Triangle-Tangram", Bit, 24

The Egyptian Three-Triangle-Tangram

You can make 11 convex figures using just T1, T4 & T5



See also: Brügner, G. (1984) "Three-Triangle-Tangram", Bit, 24

The Egyptian Three–Triangle–Tangram

You can find golden rectangles of 5 different types using just T1, T4 & T5

Туре	Proportions				
С	2		$\sqrt{5} - 1$		
F	$1 + \sqrt{5}$		2		
G	4		$2\sqrt{5}-2$		
Н	$2\sqrt{5}$		$5 - \sqrt{5}$		
K	$2+2\sqrt{5}$		4		

Could you build an example of each type?

Properties

Mathematical

φ and $\sqrt{5}$ are irrational





This is a golden rectangle, which means that $\frac{base}{height} = \varphi$, the golden ratio.

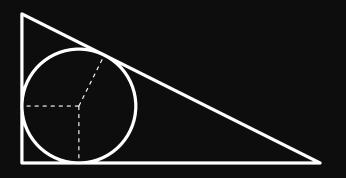
If we remove a square, what remains is also a golden rectangle: $\frac{\text{height}}{\text{base-height}} = \phi$

If we assume that $\varphi = \frac{b}{h}$, with b and h coprime integers, then $\varphi = \frac{h}{b-h}$ is an equivalent fraction, with a smaller integer numerator and a smaller integer denominator, which is absurd. Therefore, our initial assumption must be false.

And, since $\phi = \frac{1+\sqrt{5}}{2}$ is irrational, $2\phi - 1 = \sqrt{5}$ must be irrational too.

The $1:2:\sqrt{5}$ incenter

If the inradius of a 1:2: $\sqrt{5}$ triangle is 1...



...its shorter leg measures $\varphi + 1 = \varphi^2 = \frac{3+\sqrt{5}}{2}$

The 3:4:5 incenter

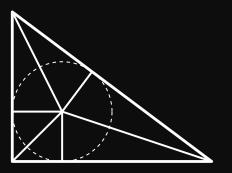
If we overlay T6 and T1 as shown in the figure...



...a T1 vertex lies on the incenter of T6

Dissecting 3:4:5

You can use this dissection of T6 to prove that...

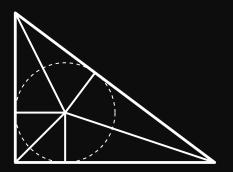


$$\pi = \arctan(1) + \arctan(2) + \arctan(3)$$

(consider the sum of the angles touching the incenter of T6 and divide by 2)

Dissecting 3:4:5

You can use this dissection of T6 to prove that...

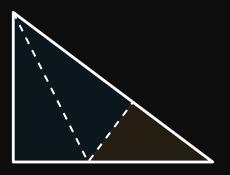


$$rac{\pi}{2} = \arctanig(rac{1}{1}ig) + \arctanig(rac{1}{2}ig) + \arctanig(rac{1}{3}ig)$$

(consider the sum of the angles touching the vertices of T6 and divide by 2)

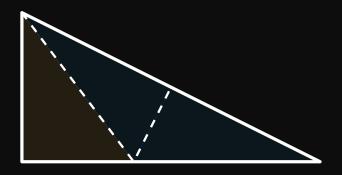
Dissecting 3:4:5

You can dissect a 3:4:5 triangle into...



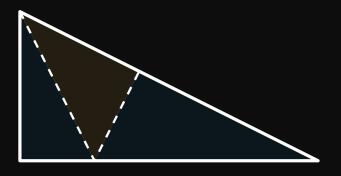
...a 3:4:5 triangle and two congruent 1:2: $\sqrt{5}$ triangles

You can dissect a $1:2:\sqrt{5}$ triangle into...



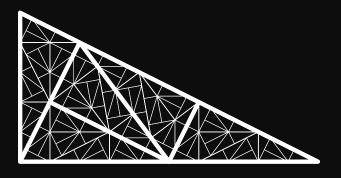
...a 3:4:5 triangle and two congruent 1:2: $\sqrt{5}$ triangles

You can dissect a $1:2:\sqrt{5}$ triangle into...



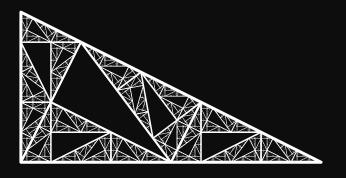
...a 3:4:5 triangle and two different 1:2: $\sqrt{5}$ triangles

You can assemble a $1:2:\sqrt{5}$ triangle aggregating...



...five congruent $1:2:\sqrt{5}$ triangles and iterate to get the **Pinwheel tiling** of the plane

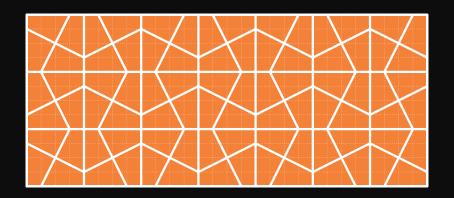
You can dissect a $1:2:\sqrt{5}$ triangle into...



...five congruent $1:2:\sqrt{5}$ triangles, remove the central one and iterate to get the **Pinwheel fractal**

Q4 tilings

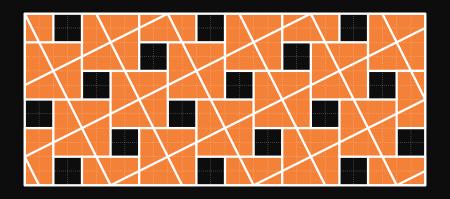
Two copies of Q4 form a pentagon that can be used...



...to make this variant of the Cairo Tiling of the plane

Q4 tilings

You can use this **Pythagorean Tiling** to verify...

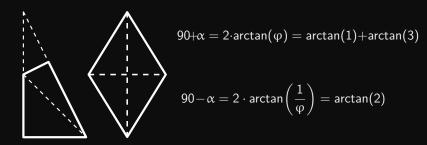


...that T4 satisfies the **Pythagorean Theorem**

The angles of Q4

The angles $90-\alpha$ and $90+\alpha$ that appear in Q4 also appear in the Golden Rhombus

(a rhombus whose diagonals are in proportion $1:\phi$, with $\phi=\frac{1+\sqrt{5}}{2}$)



The faces of the rhombic triacontahedron and the rhombic hexecontahedron are Golden Rhombi

The angles of Q4

Even though they are NOT similar figures...



 $\overline{\dots}$ the same angles appear in Q4 and T5 \cup T6

The perimeter of Q4

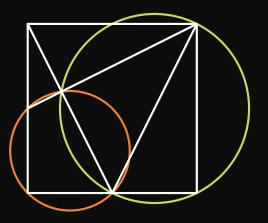
These three perimeters are in a geometric progression...



$$\frac{2\sqrt{5}+4}{\sqrt{5}+3} = \frac{3\sqrt{5}+7}{2\sqrt{5}+4} = \phi = \frac{1+\sqrt{5}}{2}$$

The circumcircles

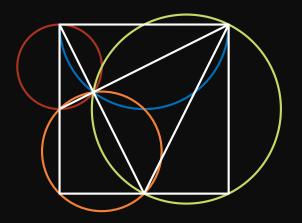
Since opposite angles add to π ...



...Q4 and $\overline{\mathsf{T5}} \cup \mathsf{T6}$ are cyclic quadrilaterals

The circumcircles

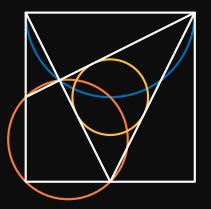
All circumcircles pass through a common point...



...and $C(T5 \, \cup \, T6)$ passes through the center of C(Q4) and C(T4)

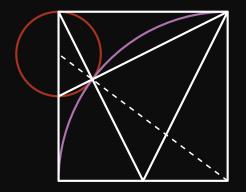
The circumcircles

These cirmcumcircles intersect at the square's center...



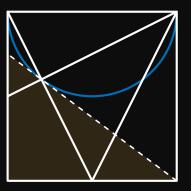
...which happens to be T6's incenter

These three points are aligned...



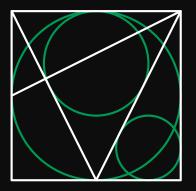
...and these two circles are tangent

The line is tangent to this circle...



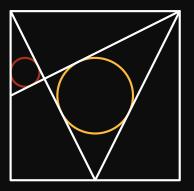
...and the right triangle below is an Egyptian Triangle

The radius of these three circles are in ratio $1:\varphi:\varphi^2$



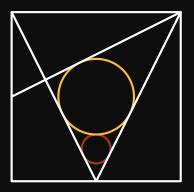
where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

The radius of these two circles are in ratio $1:\phi^2$



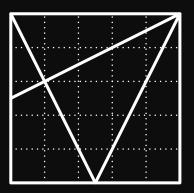
where $\phi=\frac{1+\sqrt{5}}{2}$ is the golden ratio

The radius of these two circles are in ratio $1:\varphi^2$



where $\phi=\frac{1+\sqrt{5}}{2}$ is the golden ratio

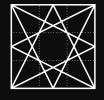
Using the intersection point of the Egyptian Tangram...

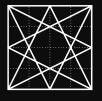


...you can divide the square into 5×5 smaller squares!

Using the intersection points of this figure...









...you can divide the square into:

 2×2 , 3×3 , 4×4 or 5×5 smaller squares!

There are 32 egyptian triangles in this figure...





...they come in 4 sizes and there are 8 of each kind

There are 24 1:2: $\sqrt{5}$ triangles in this figure...





...they come in 3 sizes and there are 8 of each kind

There are 24 other triangles in this figure...





...of 3 different kinds (one of them comes in 2 sizes)

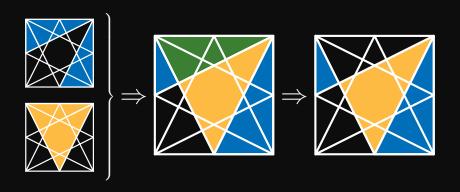
The relative sizes of these polygons are...



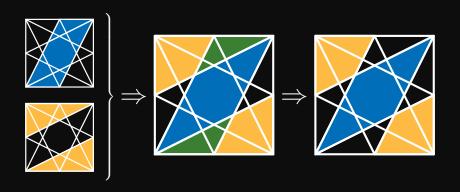
Small Triangles: 1 Small Kites: 3 Whole Square: 120 **Big Triangles:** 6 Big Kites: 8

Octagon: 20

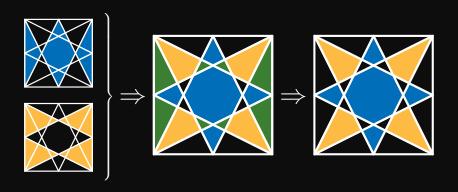
Since Area(BLUE) = Area(YELLOW)...



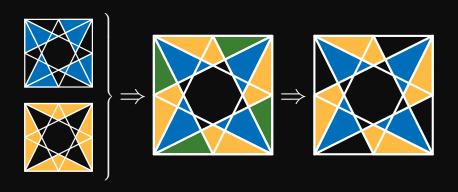
Since Area(BLUE) = Area(YELLOW)...



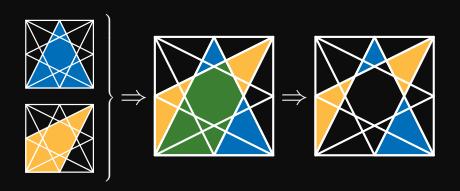
Since Area(BLUE) = Area(YELLOW)...



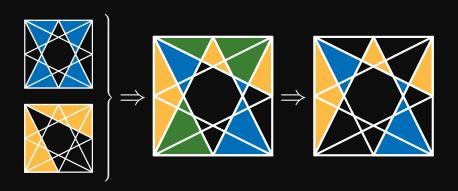
Since Area(BLUE) = Area(YELLOW)...



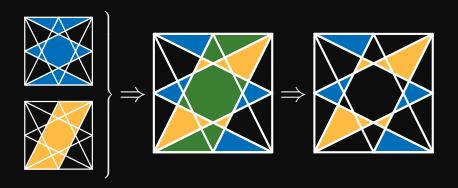
Since Area(BLUE) = Area(YELLOW)...



Since Area(BLUE) = Area(YELLOW)...



Since Area(BLUE) = Area(YELLOW)...



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