

# The Egyptian Tangram



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**mmaca**

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# The Egyptian Tangram

# The Egyptian Tangram

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A square dissection firstly proposed as a tangram in:

Luna-Mota, C. (2019) *"El tangram egipci: diari de disseny"* Nou Biaix, 44

# Origins

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The Egyptian Tangram inspiration comes from the study of two other 5-piece tangrams...

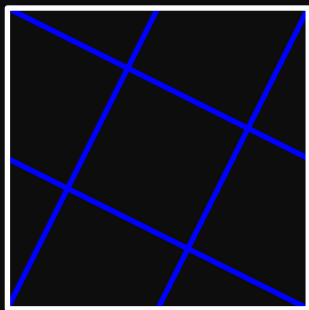
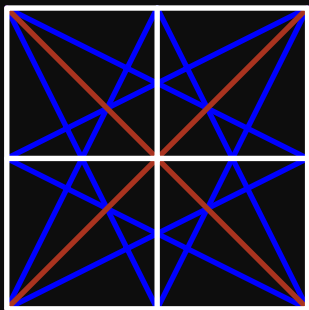


The “Five Triangles” & “Greek-Cross” tangrams

# Origins

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...and their underlying grids

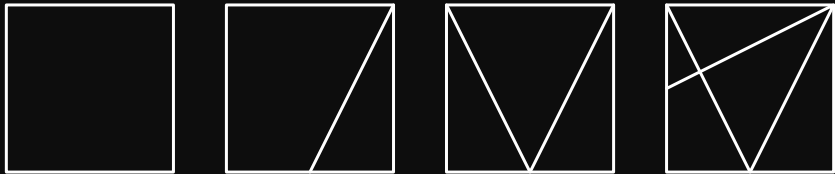


The “Five Triangles” & “Greek-Cross” underlying grids

# Design Process

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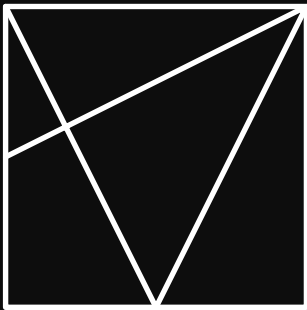
The Egyptian Tangram is the result of an heuristic incremental design process:



Take a square and keep adding “the most interesting straight cut” until you have a dissection with 5 or more pieces.

# Design Process

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To make an Egyptian Tangram from a square:

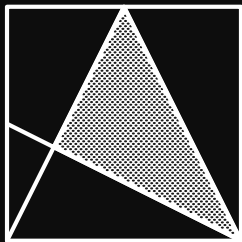
1. Connect the midpoint of the lower side with the upper corners.
2. Connect the midpoint of the left side with the top right corner.

# Antecedents

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It turns out that this figure is not new...

Detemple, D. & Harold, S. (1996) *"A Round-Up of Square Problems"*



Problem 3

...but, to the best of our knowledge,  
nobody used it before **as a tangram**



# Antecedents

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The name is not new either...



This dissection is often called “Egyptian Puzzle” or “Egyptian Tangram”

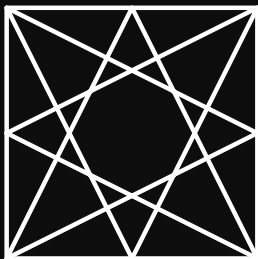
...but there is a good reason to consider  
our dissection the real “Egyptian Tangram”

(even if it was designed in Barcelona)

# Antecedents

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The underlying grid is also a well known figure:



Brunés, T. (1967) *"The Secrets of Ancient Geometry – and Its Use"*

Bankoff, L. & W. Trigg, C. (1974) *"The Ubiquitous 3:4:5 Triangle"*

# The pieces



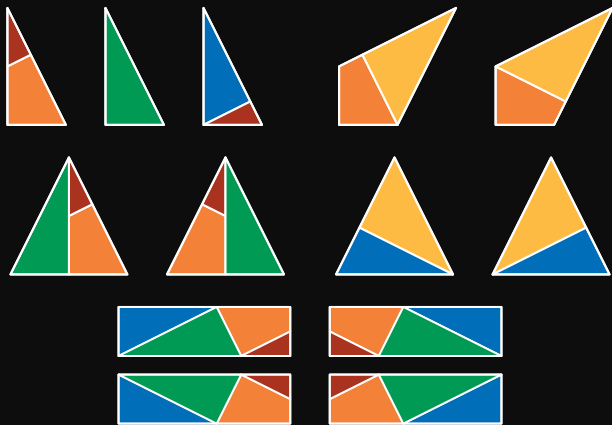
- Just five pieces
- All pieces are different
- All pieces are asymmetric
- Areas are integer and not *too different*
- All sides are multiples of 1 or  $\sqrt{5}$
- All angles are linear combinations of  $90^\circ$  and  $\alpha = \arctan\left(\frac{1}{2}\right) \approx 26,565^\circ$

Name	Area	Sides	Angles
T1	1	1, 2, $\sqrt{5}$	$90^\circ$ , $\alpha$ , $90^\circ - \alpha$
T4	4	2, 4, $2\sqrt{5}$	$90^\circ$ , $\alpha$ , $90^\circ - \alpha$
T5	5	$\sqrt{5}$ , $2\sqrt{5}$ , 5	$90^\circ$ , $\alpha$ , $90^\circ - \alpha$
T6	6	3, 4, 5	$90^\circ$ , $90^\circ - 2\alpha$ , $2\alpha$
Q4	4	1, 3, $\sqrt{5}$ , $\sqrt{5}$	$90^\circ$ , $90^\circ - \alpha$ , $90^\circ$ , $90^\circ + \alpha$

# The pieces

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Although all pieces are asymmetric and different, they often combine to make symmetric shapes



# The pieces

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This means that it is very rare for an Egyptian Tangram figure to have a unique solution

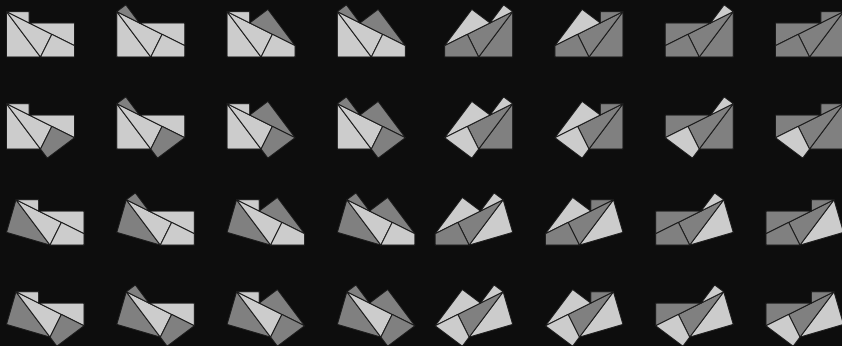


There are three different solutions for the square and, in all three cases, two of the corners of the square are built as a sum of acute angles!

# The pieces

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The assymetry of the pieces also implies that each solution belongs to one of these equivalence classes:

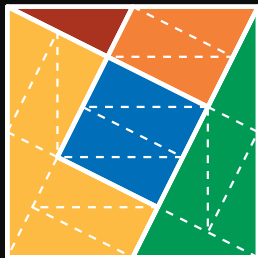


You cannot transform one of these figures into another without flipping a piece

# Why we called it the *Egyptian* Tangram?

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The smallest pieces of the Chinese and Greek-Cross tangrams can be used to build all the other pieces...



...but you cannot do the same with  
the Egyptian Tangram because of T6

# Why we called it the *Egyptian* Tangram?

---

Initially, T6 was considered as the *leftover* piece that results from cutting all these  $1:2:\sqrt{5}$  triangles from the borders of the square.

But it turned out to be a very well known triangle...



...the **Egyptian** Triangle (3:4:5)  
and, hence, the name



# Puzzles & Activities

# Realistic figures

---

Use all five pieces to make these figures:



Lightning



Sailing ship



Bow tie



Wooden hut



Caltrop



Snowmobile



Candle



Viking hat



Diamond



Moses basket



Erlenmeyer



3D brick



Witch hat



Arrow Sign



Sailboat

# Realistic figures

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Use all five pieces to make these figures:



Gnome



Handmaid



Mountain range



Fish tail



Teddy bear



Cat



Cow



Crab claw



Snail



Fennec Fox



Penguin



Calf



Sea Turtle



Duck

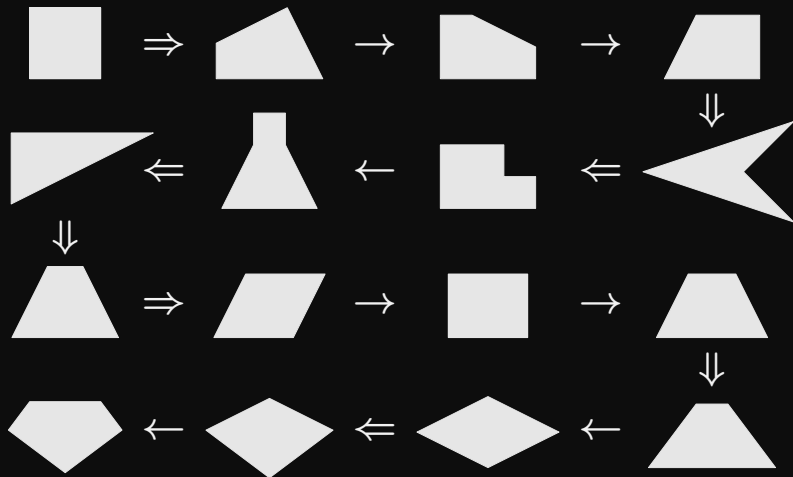


Crow

# Geometric figures

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Use all five pieces to make these figures:



Complete the path moving just one or two pieces at a time

# Triangles

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Could you prove that there are just 10 triangles you can make with one or more pieces of the Egyptian Tangram?

How many solutions could you find for each figure?



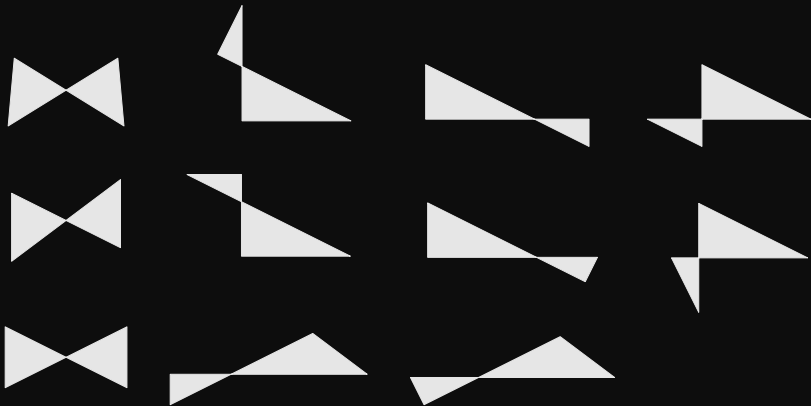
Top row areas: 20, 16, 9, 5, 4, 1

Bottom row areas: 15, 10, 10, 6

# Quadrilaterals

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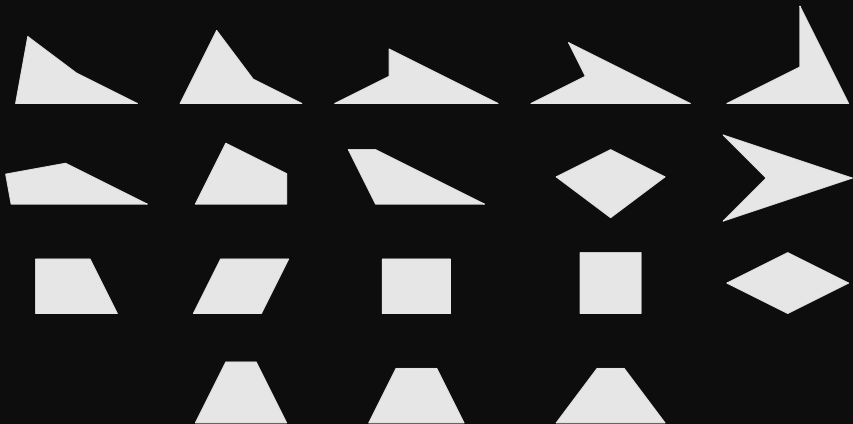
Could you prove that there are just 11 **complex quadrilaterals** you can make with all five pieces of the Egyptian Tangram?



# Quadrilaterals

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Simple quadrilaterals: Not self-intersecting



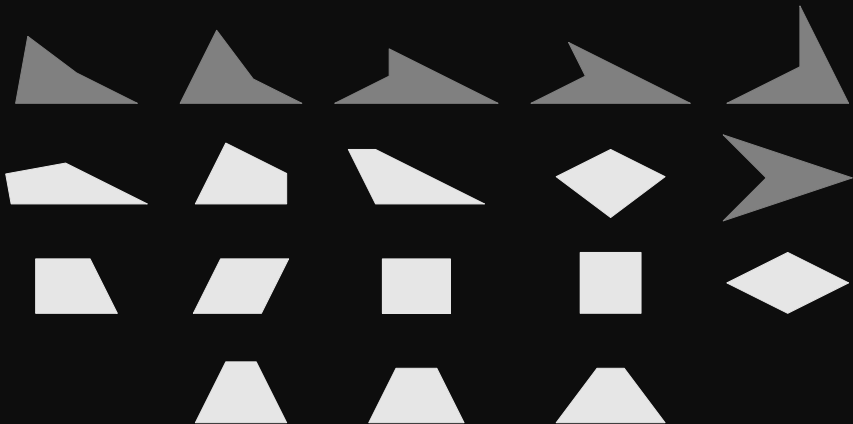
All simple quadrilaterals tile the plane!

$$\alpha + \beta + \gamma + \delta = 2\pi$$

# Quadrilaterals

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**Convex quadrilaterals:** All internal angles are smaller than  $\pi$



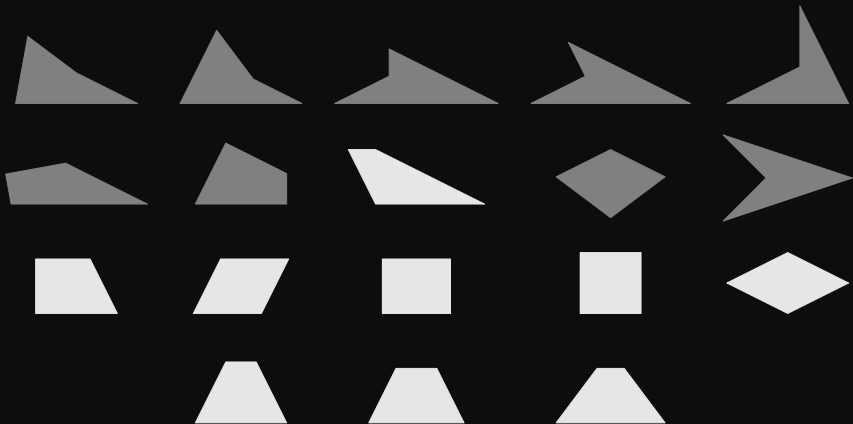
Law of Cosines:  $p^2q^2 = a^2c^2 + b^2d^2 - 2abcd \cos(\alpha + \gamma)$



# Quadrilaterals

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Trapeziums (UK) / Trapezoids (US): One pair of parallel sides

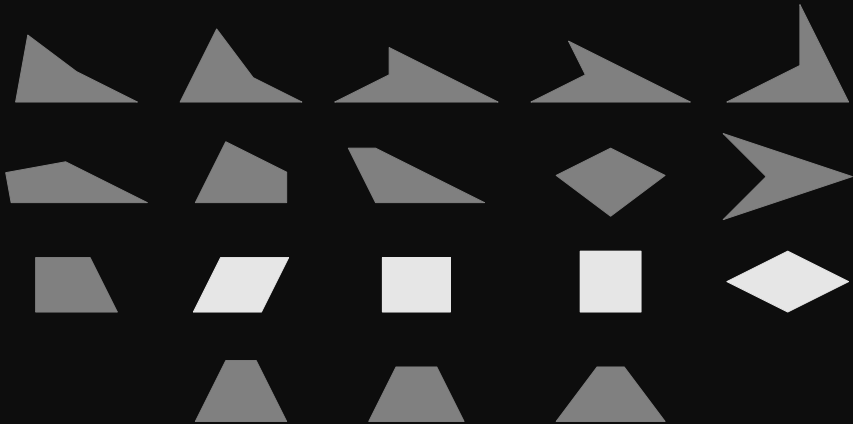


Trapezium/Trapezoid  $\Leftrightarrow$  Diagonals cut each other in the same ratio

# Quadrilaterals

---

Parallelograms: Two pairs of parallel sides

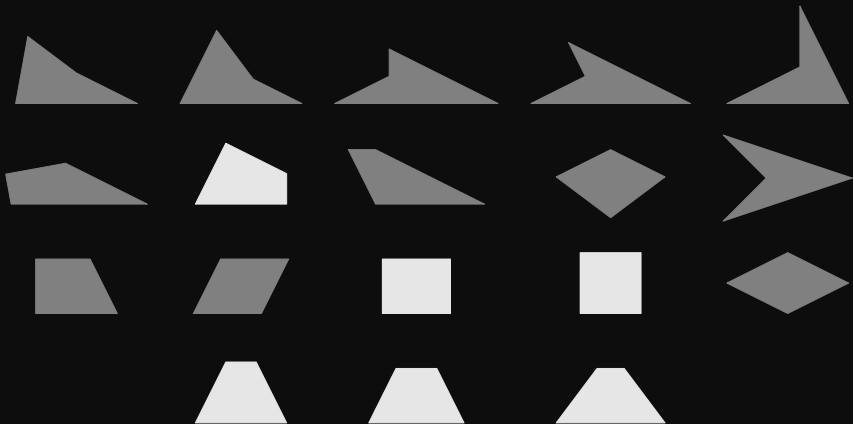


$$\text{Parallelogram} \Leftrightarrow \text{Diagonals bisect each other} \Leftrightarrow a^2 + b^2 + c^2 + d^2 = p^2 + q^2$$

# Quadrilaterals

---

**Cyclic quadrilaterals:** All vertices lie on a circle

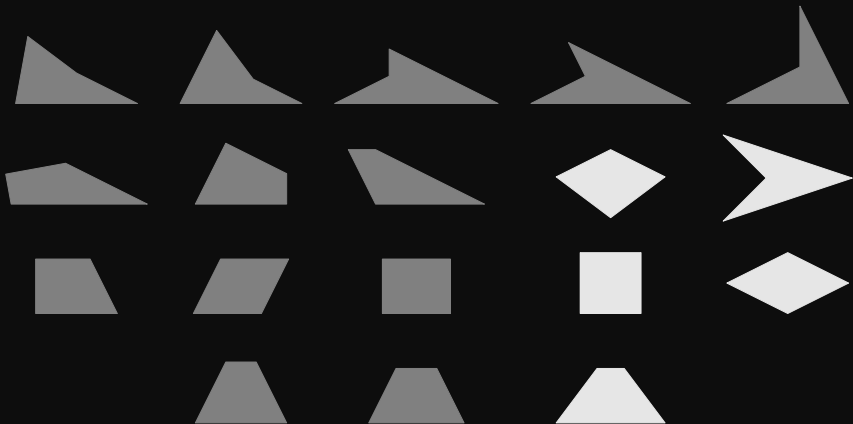


$$\text{Cyclic} \Leftrightarrow \alpha + \gamma = \beta + \delta$$

# Quadrilaterals

---

**Tangential quadrilaterals:** All sides are tangent to a circle

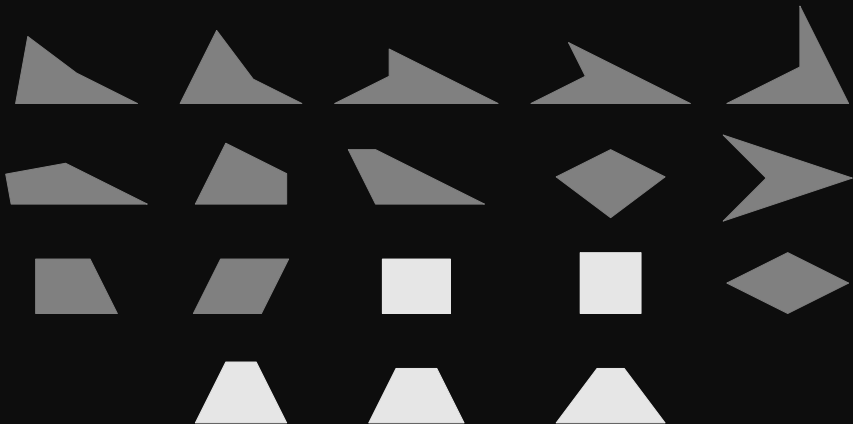


$$\text{Tangential} \Leftrightarrow a + c = b + d$$

# Quadrilaterals

---

**Isosceles Trapezoids:** Two pairs of adjacent angles are equal

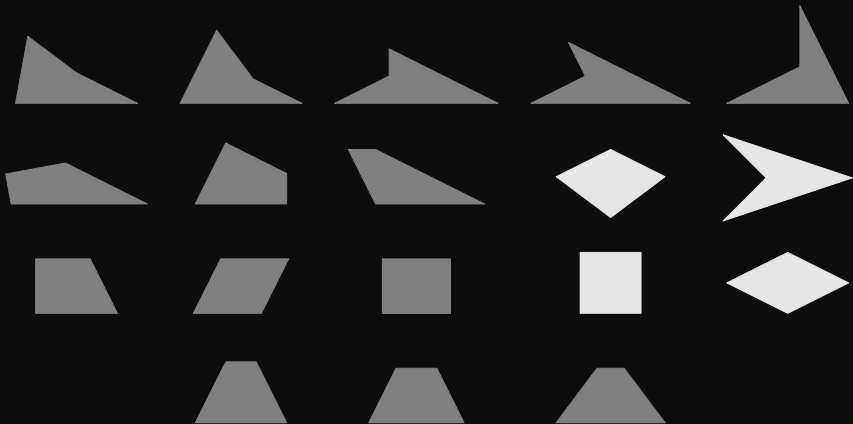


Isosceles trapezoids  $\Leftrightarrow$  Cyclic quadrilaterals with equal diagonals

# Quadrilaterals

---

**Darts & Kites:** Two pairs of adjacent sides are equal

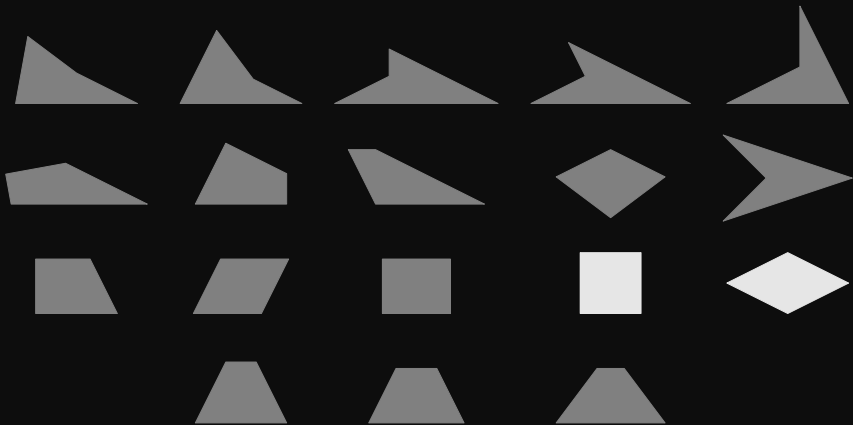


Darts/Kites  $\Leftrightarrow$  Tangential quadrilaterals with perpendicular diagonals

# Quadrilaterals

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**Rhombi:** All sides are equal

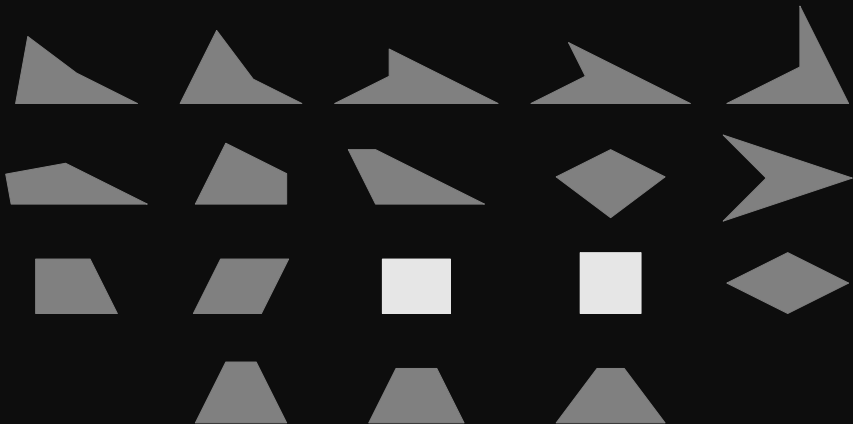


Rhombi  $\Leftrightarrow$  Parallelograms with perpendicular diagonals

# Quadrilaterals

---

**Rectangles:** All angles are equal



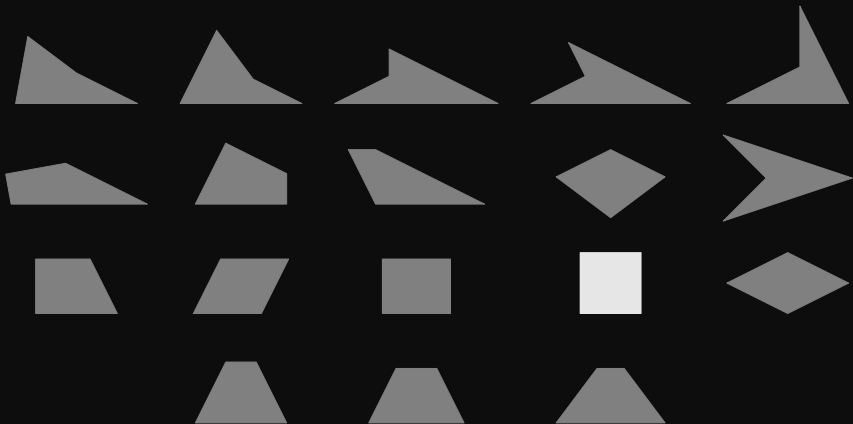
Rectangles  $\Leftrightarrow$  Parallelograms with equal diagonals



# Quadrilaterals

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Squares: Regular quadrilaterals



Among all quadrilaterals, squares maximize the *Area:Perimeter* ratio

# Remote control symbols

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Use all five pieces to make these symbols:



Rewind



Play/Pause



FFWD



Start



Stop



End



Volume

# The three solutions of the square

---

Could you prove that there are just three different solutions for the square?



What's the area of this square? What's its perimeter?

How many times do you find  $\sqrt{5}$  in the Egyptian Tangram pieces?

# Figures with unique solutions

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These figures are conjectured to have unique solutions:



Could you prove it?

# Missing triangle paradox

---

Both figures use all 5 pieces...



Where is the missing triangle?

# Missing triangle paradox

---

Both figures use all 5 pieces...



Where is the missing triangle?

# Missing triangle paradox

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Where is the missing triangle?

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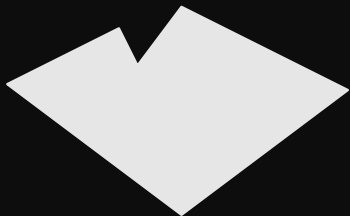
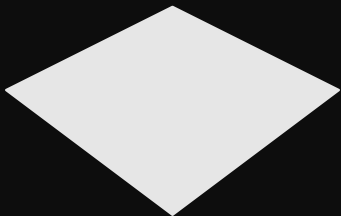


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# Missing triangle paradox

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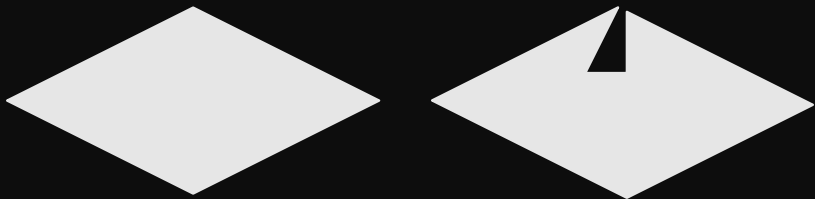


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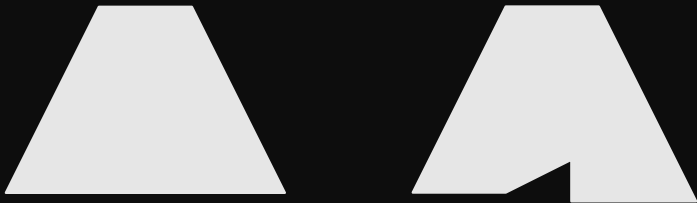


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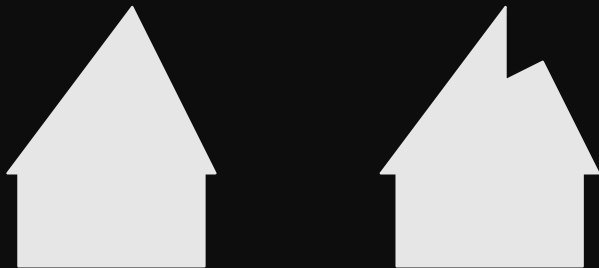


Where is the missing triangle?

# Missing triangle paradox

---

Both figures use all 5 pieces...

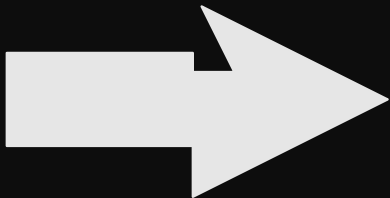
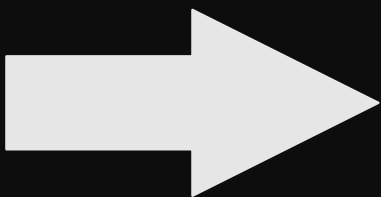


Where is the missing triangle?

# Missing triangle paradox

---

Both figures use all 5 pieces...



Where is the missing triangle?

# Missing square paradox

---

Both figures use all 5 pieces...

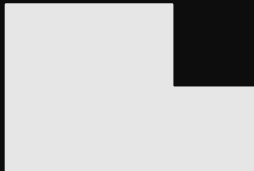
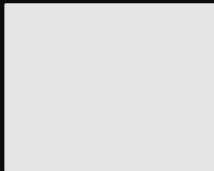


Where is the missing square?

# Missing square paradox

---

Both figures use all 5 pieces...



Where is the missing square?



# Missing square paradox

---

Both figures use all 5 pieces...



Where is the missing square?

# Missing rectangle paradox

---

Both figures use all 5 pieces...

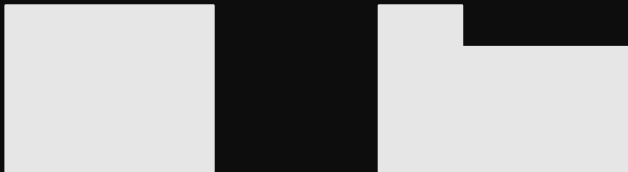


Where is the missing rectangle?

# Missing rectangle paradox

---

Both figures use all 5 pieces...



Where is the missing rectangle?

# Missing rectangle paradox

---

Both figures use all 5 pieces...



Where is the missing rectangle?

# Sum of similar figures

---

Use all 5 pieces to make the single figure in the LHS,  
then use them to make the two figures on the RHS



In both equations, the figures are similar and areas are in ratio 5 : 4 : 1

# The Egyptian Four-Triangle-Tangram

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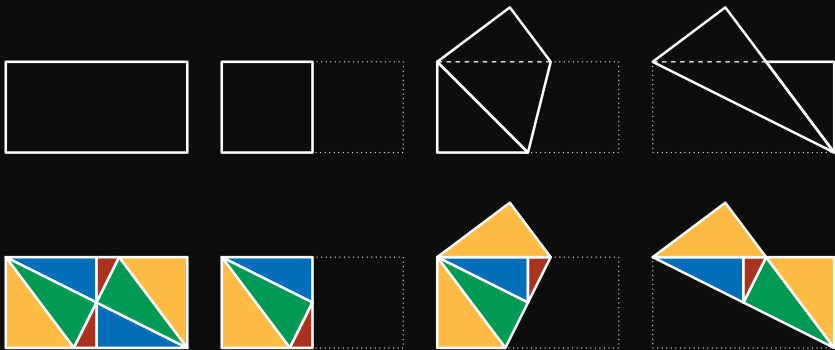
You can make these figures using just T1, T4, T5 & T6  
(the four triangles of the Egyptian Tangram)



# The Egyptian Four-Triangle-Tangram

---

T1, T4, T5 & T6 appear naturally  
when you fold a 2:1 rectangle



# The Egyptian Three-Triangle-Tangram

---

You can make 11 convex figures using just T1, T4 & T5



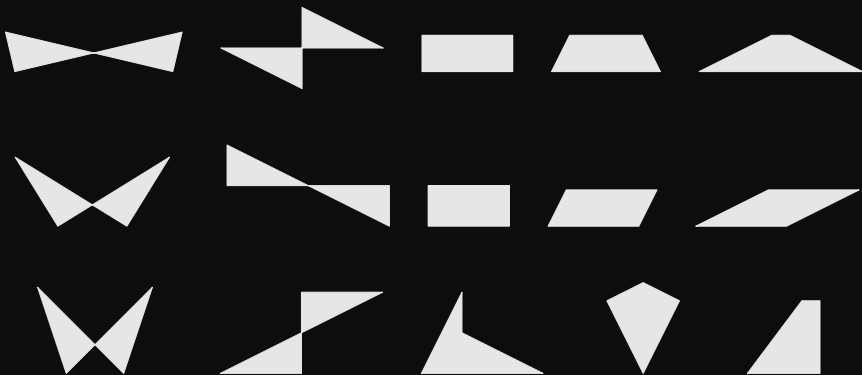
See: Brügger, G. (1984) *"Three-Triangle-Tangram"*, Bit, 24



# The Egyptian Three-Triangle-Tangram

---

You can make 15 quadrilaterals using just T1, T4 & T5



See: Brügger, G. (1984) *"Three-Triangle-Tangram"*, Bit, 24

# The Egyptian Three-Triangle-Tangram

---

Given any right triangle with sides:  $a \leq b \leq c...$



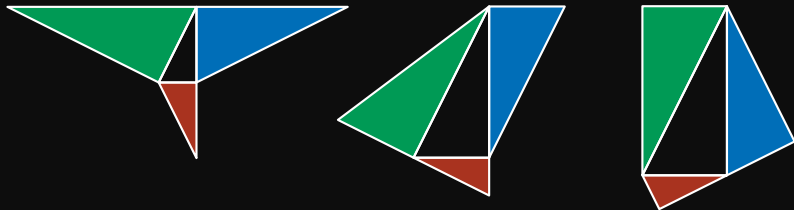
...you can make a rectangle with three similar triangles:  
 $(a^2, ab, ac)$ ,  $(ba, b^2, bc)$  &  $(ca, cb, c^2)$  and compare the top  $(a^2 + b^2)$   
and the bottom  $(c^2)$  sides of the rectangle to prove Pythagoras

(T1, T4 & T5 verify this relationship for:  $a = 1$ ,  $b = 2$  &  $c = \sqrt{5}$ )

# The Egyptian Three-Triangle-Tangram

---

Since  $\text{area}(T1) + \text{area}(T4) = \text{area}(T5) \dots$



...you can verify three cases of Pythagoras' theorem  
(and these particular cases turn out to be the T1, T4 & T5 right triangles!)

# Mathematical Properties

# Golden Rectangles

---

The dashed rectangle proportions are  $1:\varphi$



where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio

# Golden Rectangles

---

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# Golden Rectangles

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The dashed rectangles proportions are  $1:\varphi$



where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio



# $\varphi$ and $\sqrt{5}$ are irrational



This is a golden rectangle, which means that  $\frac{\text{base}}{\text{height}} = \varphi$ , the golden ratio.

If we remove a square, what remains is also a golden rectangle:  $\frac{\text{height}}{\text{base}-\text{height}} = \varphi$

If we assume that  $\varphi = \frac{b}{h}$ , with  $b$  and  $h$  coprime integers, then  $\varphi = \frac{h}{b-h}$  is an equivalent fraction, with a smaller integer numerator and a smaller integer denominator, which is absurd. Therefore, our initial assumption must be false.

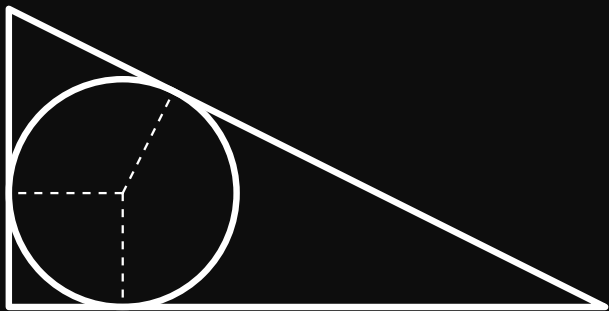
And, since  $\varphi = \frac{1+\sqrt{5}}{2}$  is irrational,  $2\varphi - 1 = \sqrt{5}$  must be irrational too.



# The $1:2:\sqrt{5}$ incenter

---

If the inradius of a  $1:2:\sqrt{5}$  triangle is 1...



...its shorter leg measures  $\varphi + 1 = \varphi^2 = \frac{3+\sqrt{5}}{2}$

# The 3:4:5 incenter

---

If we overlay T6 and T1 as shown in the figure...

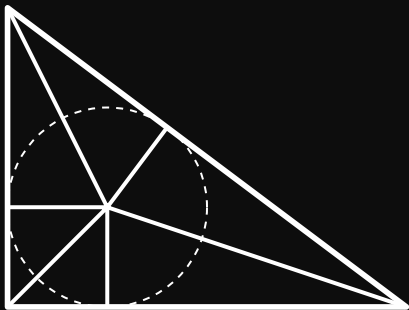


...a T1 vertex lies on the incenter of T6

## Dissecting 3:4:5

---

You can use this dissection of T6 to prove that...



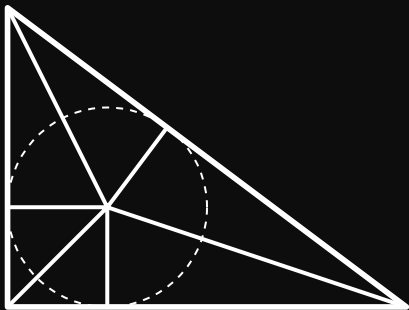
$$\pi = \arctan(1) + \arctan(2) + \arctan(3)$$

(consider the sum of the angles touching the incenter of T6 and divide by 2)

## Dissecting 3:4:5

---

You can use this dissection of T6 to prove that...



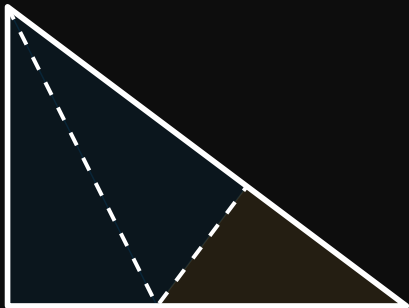
$$\frac{\pi}{2} = \arctan\left(\frac{1}{1}\right) + \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$$

(consider the sum of the angles touching the vertices of T6 and divide by 2)

# Dissecting 3:4:5

---

You can dissect a 3:4:5 triangle into...

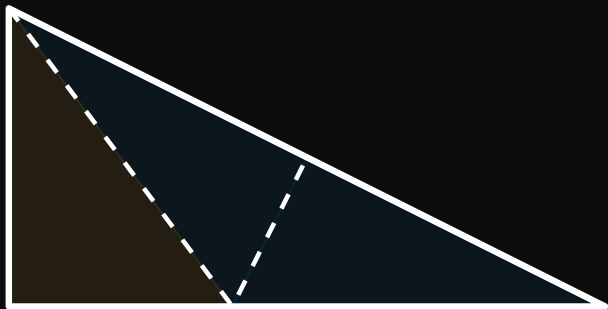


...a 3:4:5 triangle and  
two congruent  $1:2:\sqrt{5}$  triangles

# Dissecting $1:2:\sqrt{5}$

---

You can dissect a  $1:2:\sqrt{5}$  triangle into...

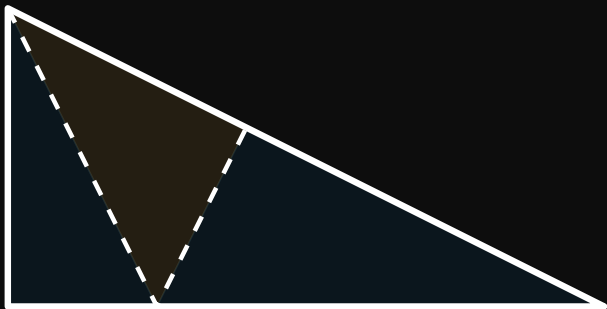


...a  $3:4:5$  triangle and  
two congruent  $1:2:\sqrt{5}$  triangles

# Dissecting $1:2:\sqrt{5}$

---

You can dissect a  $1:2:\sqrt{5}$  triangle into...



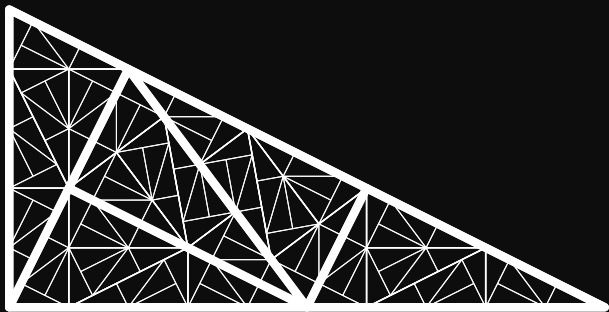
...a  $3:4:5$  triangle and  
two different  $1:2:\sqrt{5}$  triangles



# Dissecting $1:2:\sqrt{5}$

---

You can assemble a  $1:2:\sqrt{5}$  triangle aggregating...

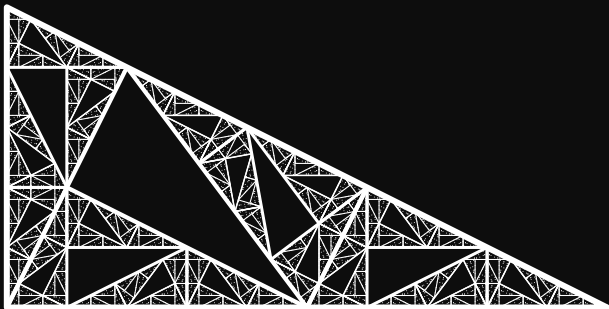


...five congruent  $1:2:\sqrt{5}$  triangles  
and iterate to get the **Pinwheel tiling** of the plane

# Dissecting $1:2:\sqrt{5}$

---

You can dissect a  $1:2:\sqrt{5}$  triangle into...



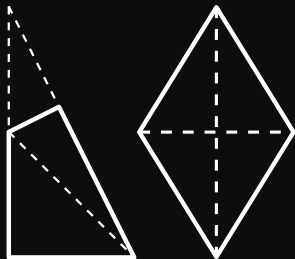
...five congruent  $1:2:\sqrt{5}$  triangles, remove the central one  
and iterate to get the **Pinwheel fractal**

# The angles of Q4

---

The angles  $90 - \alpha$  and  $90 + \alpha$  that appear in Q4  
also appear in the Golden Rhombus

(a rhombus whose diagonals are in proportion  $1 : \varphi$ , with  $\varphi = \frac{1+\sqrt{5}}{2}$ )



$$90 + \alpha = 2 \cdot \arctan(\varphi) = \arctan(1) + \arctan(3)$$

$$90 - \alpha = 2 \cdot \arctan\left(\frac{1}{\varphi}\right) = \arctan(2)$$

The faces of the rhombic triacontahedron and  
the rhombic hexecontahedron are Golden Rhombi

# The angles of Q4

---

Even though they are NOT similar figures...



...the same angles appear in Q4 and  $T5 \cup T6$

# The perimeter of Q4

---

These three perimeters are in a geometric progression...

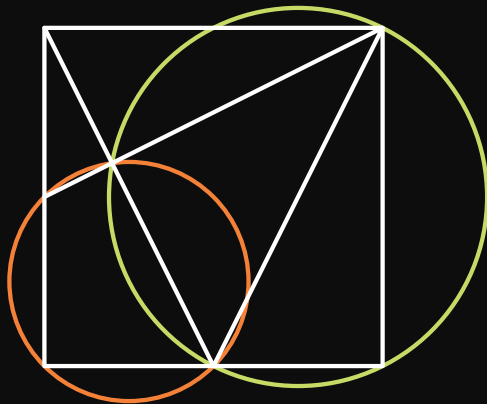


$$\frac{2\sqrt{5} + 4}{\sqrt{5} + 3} = \frac{3\sqrt{5} + 7}{2\sqrt{5} + 4} = \varphi = \frac{1 + \sqrt{5}}{2}$$

# The circumcircles

---

Since opposite angles add to  $\pi$ ...

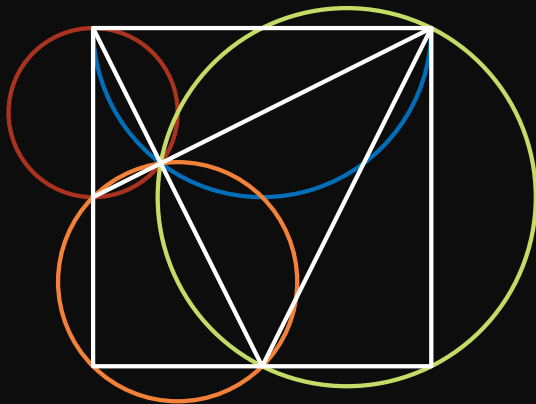


...Q4 and  $T5 \cup T6$  are cyclic quadrilaterals

# The circumcircles

---

All circumcircles pass through a common point...



...and  $C(T5 \cup T6)$  passes through the center of  $C(Q4)$  and  $C(T4)$

# The circumcircles

---

These circumcircles intersect at the square's center...



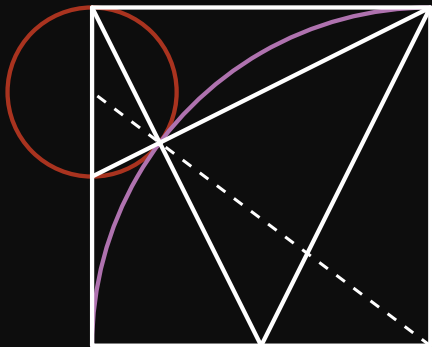
...which happens to be T6's incenter



# Tangent circles

---

These three points are aligned...

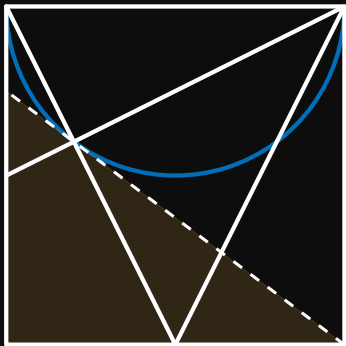


...and these two circles are tangent

# Tangent circles

---

The line is tangent to this circle...



...and the right triangle below is an Egyptian Triangle

# Tangent circles

---

The radius of these three circles are in ratio  $1:\varphi:\varphi^2$

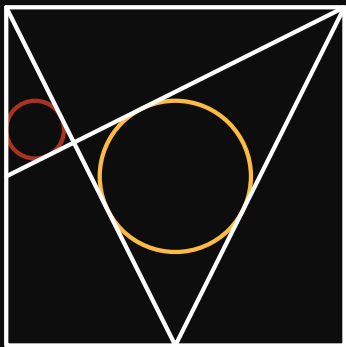


where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio

# Tangent circles

---

The radius of these two circles are in ratio  $1:\varphi^2$



where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio

# Tangent circles

---

The radius of these two circles are in ratio  $1:\varphi^2$

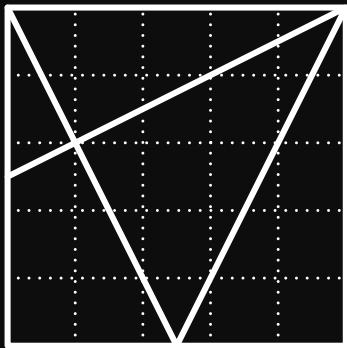


where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio

# The underlying grid

---

Using the intersection point of the Egyptian Tangram...

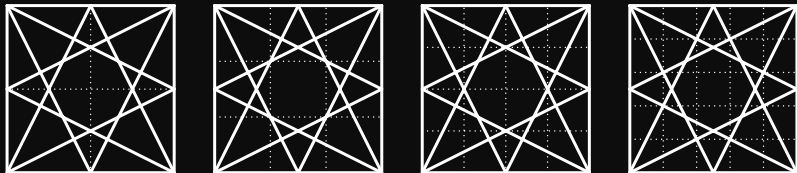


...you can divide the square into  $5 \times 5$  smaller squares!

# The underlying grid

---

Using the intersection points of this figure...



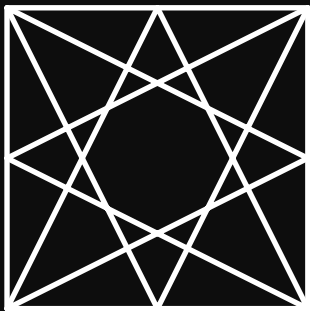
...you can divide the square into:

$2 \times 2$ ,  $3 \times 3$ ,  $4 \times 4$  or  $5 \times 5$  smaller squares!

# The underlying grid

---

There are 32 egyptian triangles in this figure...



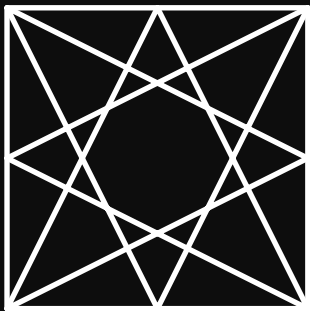
...they come in 4 sizes and there are 8 of each kind



# The underlying grid

---

There are 24  $1:2:\sqrt{5}$  triangles in this figure...

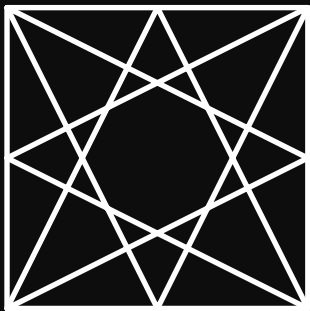


...they come in 3 sizes and there are 8 of each kind

# The underlying grid

---

There are 24 other triangles in this figure...



...of 3 different kinds (one of them comes in 2 sizes)

# The underlying grid

---

The relative sizes of these polygons are...



**Small Triangles:** 1

**Small Kites:** 3

**Whole Square:** 120

**Big Triangles:** 6

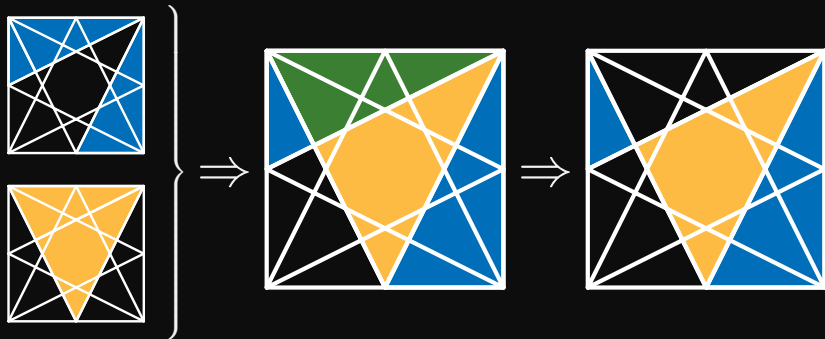
**Big Kites:** 8

**Octagon:** 20

# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW}) \dots$

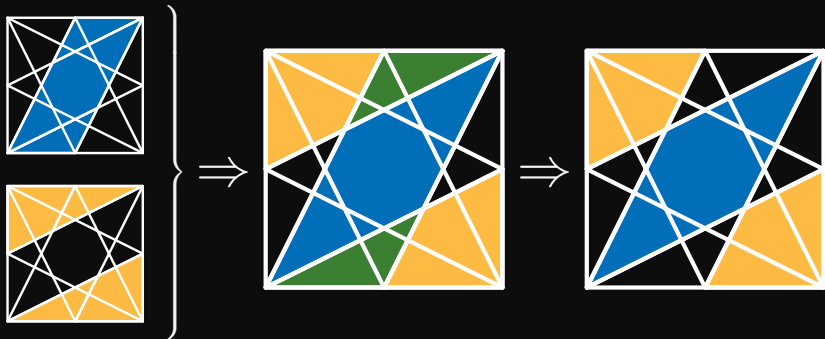


$\dots \text{Area}(\text{BLUE} - \text{GREEN}) = \text{Area}(\text{YELLOW} - \text{GREEN})$

# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$

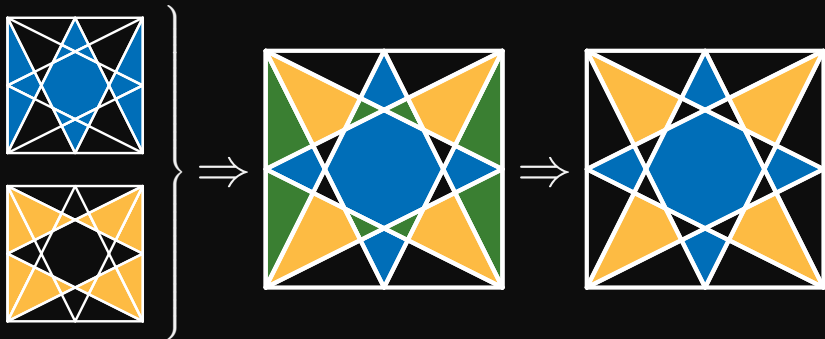


... $\text{Area}(\text{BLUE}-\text{GREEN}) = \text{Area}(\text{YELLOW}-\text{GREEN})$

# The carpets theorem

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Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$

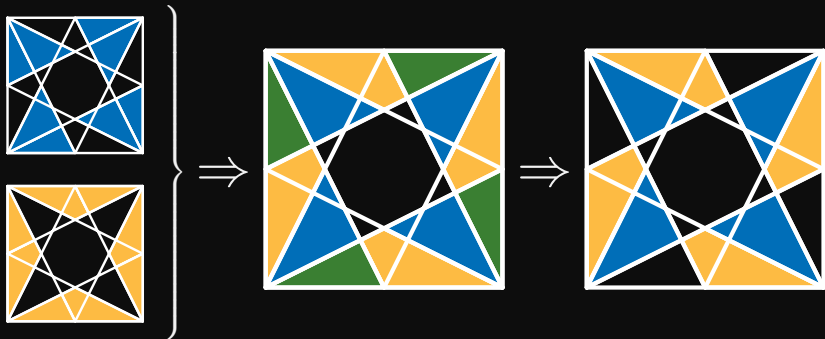


... $\text{Area}(\text{BLUE}-\text{GREEN}) = \text{Area}(\text{YELLOW}-\text{GREEN})$

# The carpets theorem

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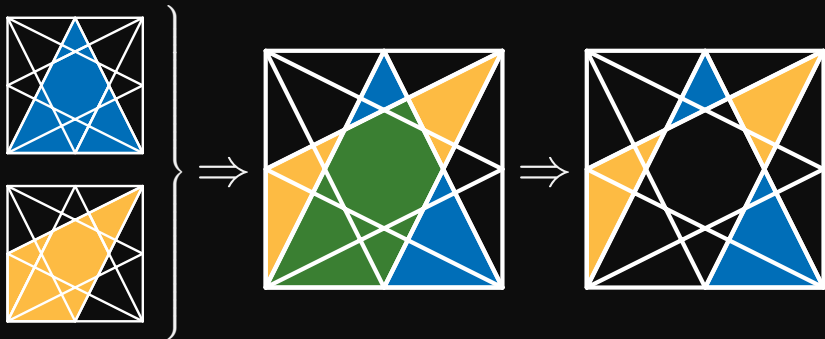


... $\text{Area}(\text{BLUE}-\text{GREEN}) = \text{Area}(\text{YELLOW}-\text{GREEN})$

# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$



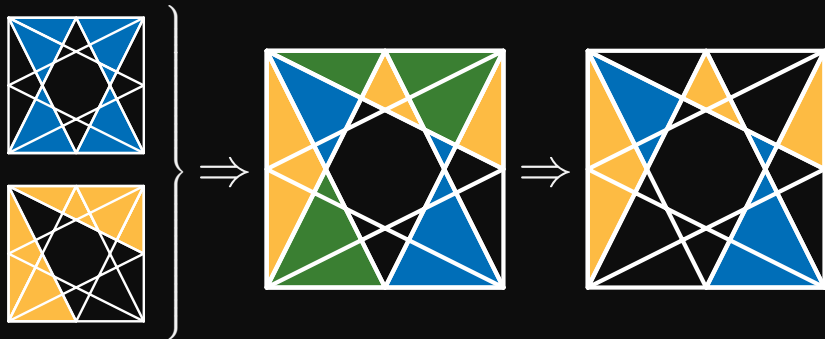
... $\text{Area}(\text{BLUE} - \text{GREEN}) = \text{Area}(\text{YELLOW} - \text{GREEN})$



# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$

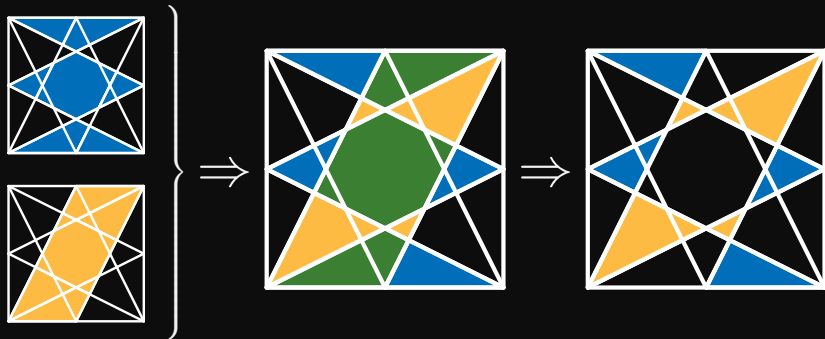


... $\text{Area}(\text{BLUE} - \text{GREEN}) = \text{Area}(\text{YELLOW} - \text{GREEN})$

# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$



... $\text{Area}(\text{BLUE} - \text{GREEN}) = \text{Area}(\text{YELLOW} - \text{GREEN})$

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