The Egyptian Tangram

mmaca

November 24, 2020

The Egyptian Tangram

The Egyptian Tangram

A square dissection firstly proposed as a tangram in:

Luna-Mota, C. (2019) *"El tangram egipci: diari de disseny"* Nou Biaix, 44

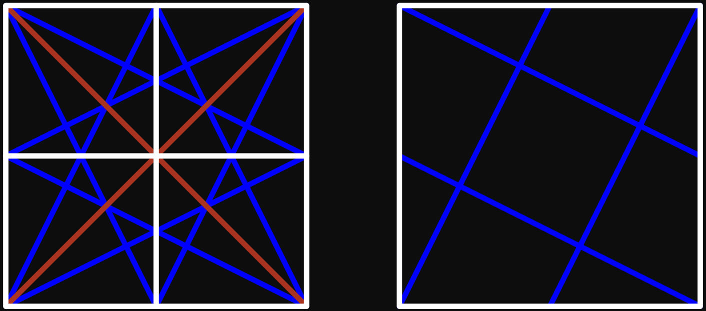
Origins

The Egyptian Tangram inspiration comes from the study of two other 5-piece tangrams...

The “Five Triangles” & “Greek-Cross” tangrams

Origins

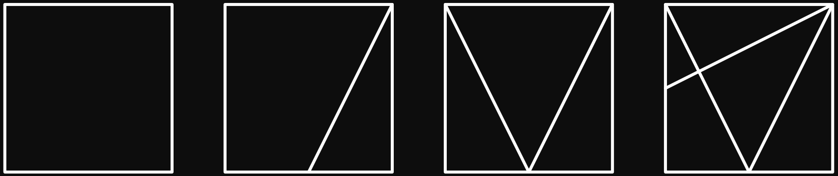
...and their underlying grids



The “Five Triangles” & “Greek-Cross” underlying grids

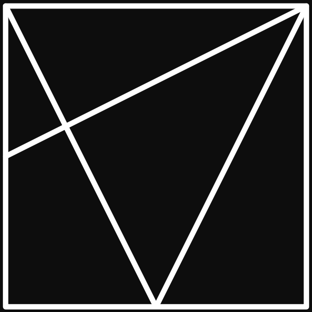
Design Process

The Egyptian Tangram is the result of
an heuristic incremental design process:



Take a square and keep adding “the most interesting straight cut”
until you have a dissection with 5 or more pieces.

Design Process



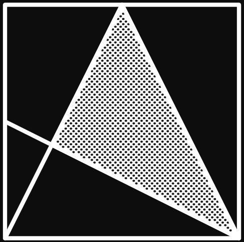
To make an Egyptian Tangram from a square:

1. Connect the midpoint of the lower side with the upper corners.
2. Connect the midpoint of the left side with the top right corner.

Antecedents

It turns out that this figure is not new...

Detemple, D. & Harold, S. (1996) *"A Round-Up of Square Problems"*



Problem 3

...but, to the best of our knowledge, nobody used it before **as a tangram**

Antecedents

The name is not new either...

This dissection is often called "Egyptian Puzzle" or "Egyptian Tangram"

...but there is a good reason to consider
our dissection the real "Egyptian Tangram"

(even if it was designed in Barcelona)

Antecedents

The underlying grid is also a well known figure:

Brunés, T. (1967) *"The Secrets of Ancient Geometry – and Its Use"*

Bankoff, L. & W. Trigg, C. (1974) *"The Ubiquitous 3:4:5 Triangle"*

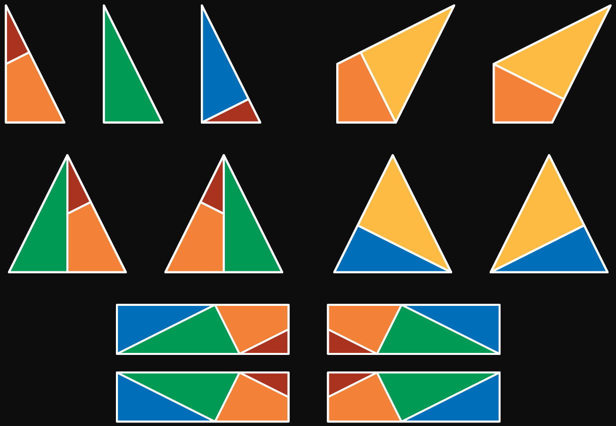
The pieces

- Just five pieces
- All pieces are different
- All pieces are asymmetric
- Areas are integer and not *too different*
- All sides are multiples of 1 or $\sqrt{5}$
- All angles are linear combinations of 90° and $\alpha = \arctan\left(\frac{1}{2}\right) \approx 26{,}565^\circ$

Name	**Area**	**Sides**	**Angles**
T1	1	$1,\ 2,\ \sqrt{5}$	$90,\ \alpha,\ 90-\alpha$
T4	4	$2,\ 4,\ 2\sqrt{5}$	$90,\ \alpha,\ 90-\alpha$
T5	5	$\sqrt{5},\ 2\sqrt{5},\ 5$	$90,\ \alpha,\ 90-\alpha$
T6	6	$3,\ 4,\ 5$	$90,\ 90-2\alpha,\ 2\alpha$
Q4	4	$1,\ 3,\ \sqrt{5},\ \sqrt{5}$	$90,\ 90-\alpha,\ 90,\ 90+\alpha$

The pieces

Although all pieces are asymmetric and different, they often combine to make symmetric shapes



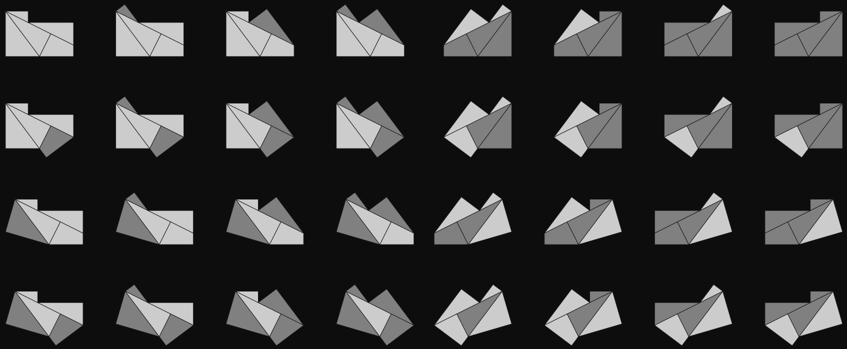
The pieces

This means that it is very rare for an Egyptian Tangram figure to have a unique solution

There are three different solutions for the square and, in all three cases, two of the corners of the square are built as a sum of acute angles!

The pieces

The assymetry of the pieces also implies that each solution belongs to one of these equivalence classes:



You cannot transform one of these figures into another without flipping a piece

Why we called it the *Egyptian* Tangram?

The smallest pieces of the Chinese and Greek-Cross tangrams can be used to build all the other pieces...

...but you cannot do the same with the Egyptian Tangram because of T6

Why we called it the *Egyptian* Tangram?

Initially, T6 was considered as the *leftover* piece that results from cutting all these 1:2:$\sqrt{5}$ triangles from the borders of the square.

But it turned out to be a very well known triangle...

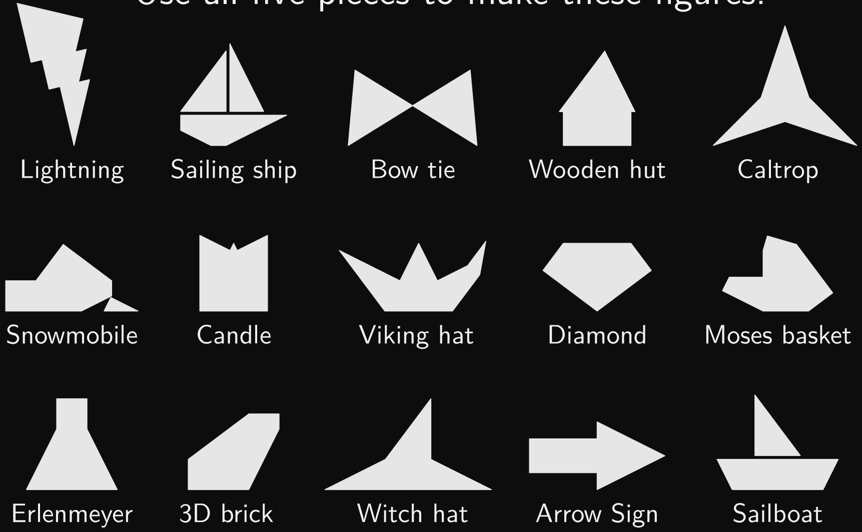
...the **Egyptian** Triangle (3:4:5)

and, hence, the name

Puzzles & Activities

Realistic figures

Use all five pieces to make these figures:



Realistic figures

Use all five pieces to make these figures:

Gnome

Handmaid

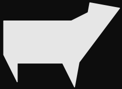
Mountain range

Fish tail

Teddy bear

Cat

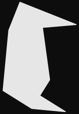
Dromedary



Cow

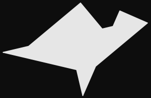
Snail

Fennec Fox

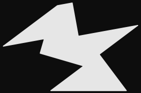


Penguin

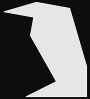
Calf



Sea Turtle



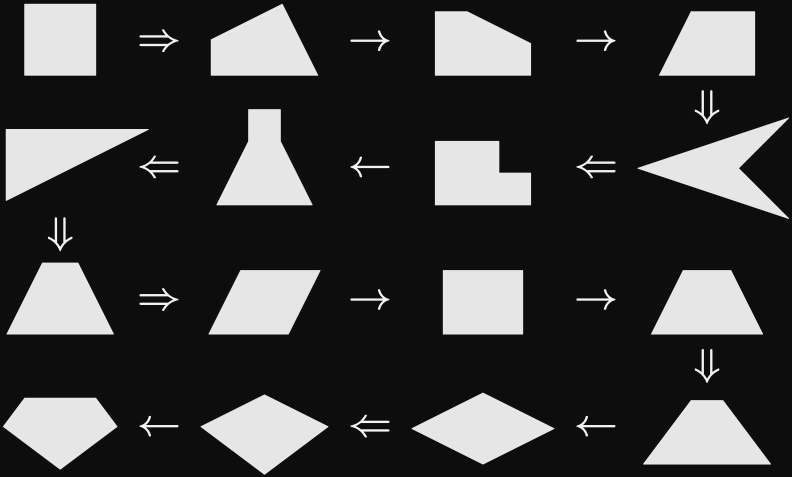
Duck



Crow

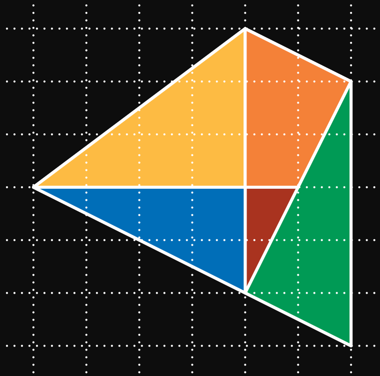
Geometric figures

Use all five pieces to make these figures:



Complete the path moving just one or two pieces at a time

Geometric figures



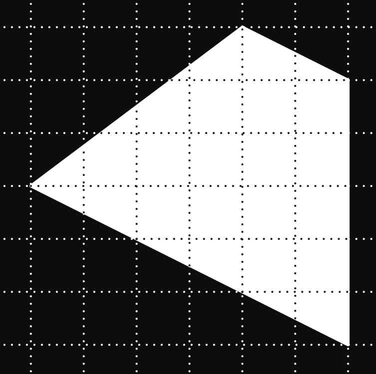
All the figures from the previous page, but one, can be drawn with their vertices lying on this lattice.

Moreover, their solutions can be drawn with the vertices of all 5 pieces lying on the same lattice.

These conditions simplify the task of finding perimeters and areas.

Could you find which is the only figure that requires a finer-grained lattice?

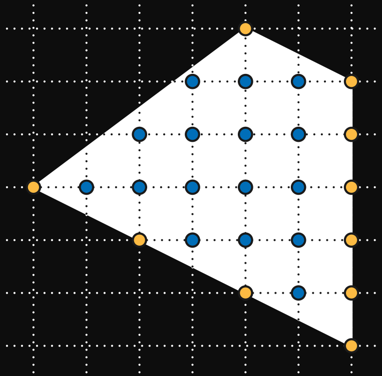
Geometric figures



$$\begin{aligned}
\text{Top} &= \sqrt{1^2+2^2} = \sqrt{5} \\
\text{Left} &= \sqrt{3^2+4^2} = \sqrt{25} = 5 \\
\text{Bottom} &= \sqrt{3^2+6^2} = \sqrt{45} = 3\sqrt{5} \\
\text{Right} &= 5 \quad \text{(just count them!)} \\[1em]
\text{Perimeter} &= \text{sum of all sides} \\
&= 10 + 4\sqrt{5}
\end{aligned}$$

Could you use the Pythagorean theorem
to compute the perimeter of these figures?

Geometric figures



Pick's Theorem:

lattice points in the interior $= 16$

lattice points on the boundary $= 10$

$$\text{Area} = \text{interior} + \frac{\text{boundary}}{2} - 1$$

$$= 16 + \frac{10}{2} - 1 = 20$$

Could you use Pick's theorem
to compute the area of these figures?

Triangles

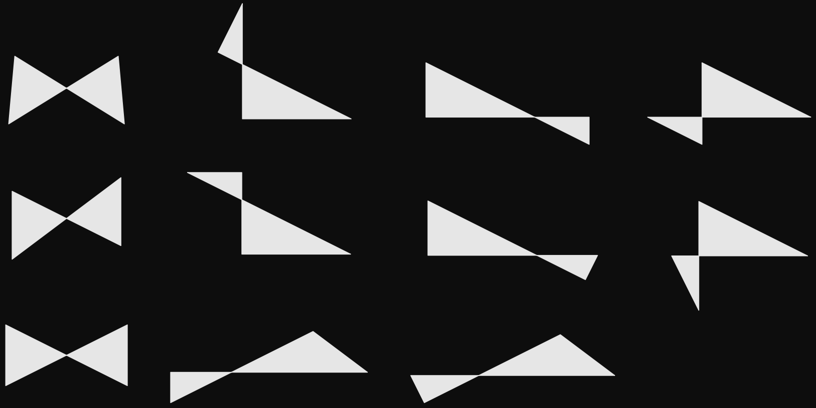
Could you prove that there are just 10 triangles you can make with one or more pieces of the Egyptian Tangram?

How many solutions could you find for each figure?

Top row areas: 20, 16, 9, 5, 4, 1 Bottom row areas: 15, 10, 10, 6

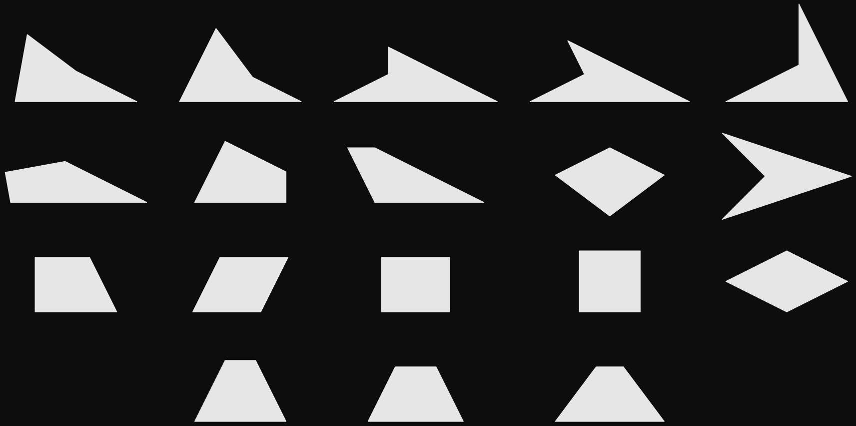
Quadrilaterals

Could you prove that there are just 11 **complex quadrilaterals** you can make with all five pieces of the Egyptian Tangram?



Quadrilaterals

Simple quadrilaterals: Not self-intersecting

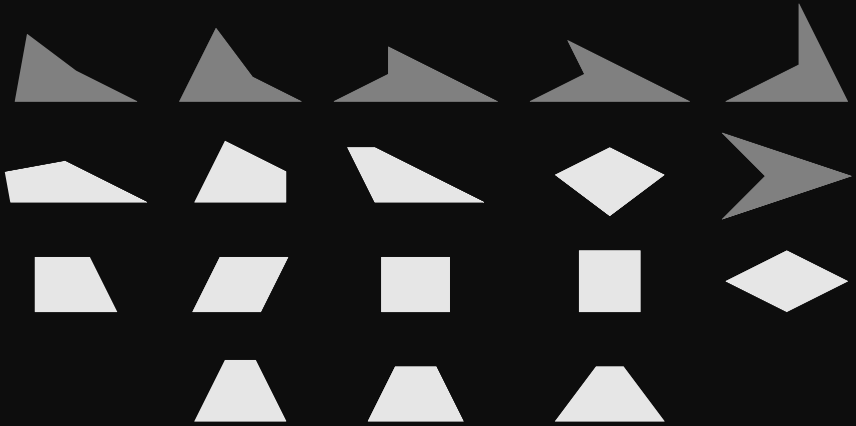


All simple quadrilaterals tile the plane!

$$\alpha + \beta + \gamma + \delta = 2\pi$$

Quadrilaterals

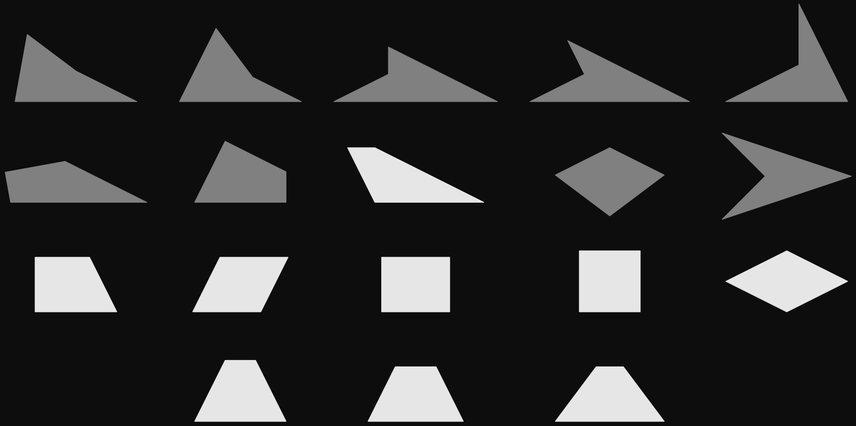
Convex quadrilaterals: All internal angles are smaller than π



Law of Cosines: $p^2 q^2 = a^2 c^2 + b^2 d^2 - 2abcd\cos(\alpha + \gamma)$

Quadrilaterals

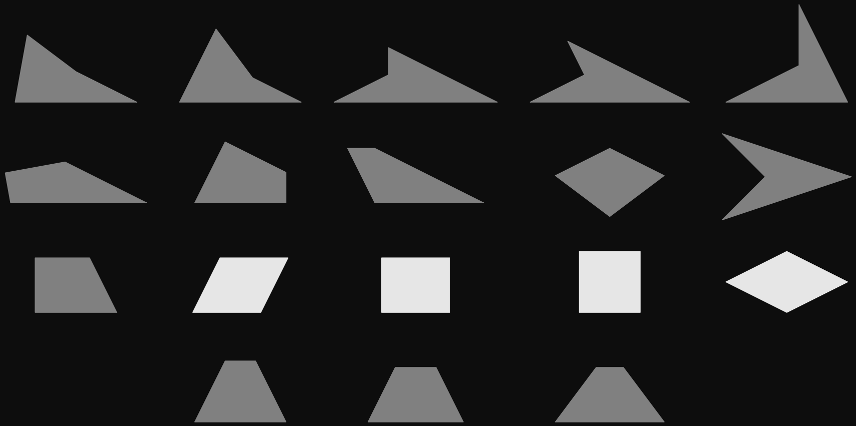
Trapeziums (UK) / Trapezoids (US): One pair of parallel sides



Trapezium/Trapezoid $\Leftrightarrow$ Diagonals cut each other in the same ratio

Quadrilaterals

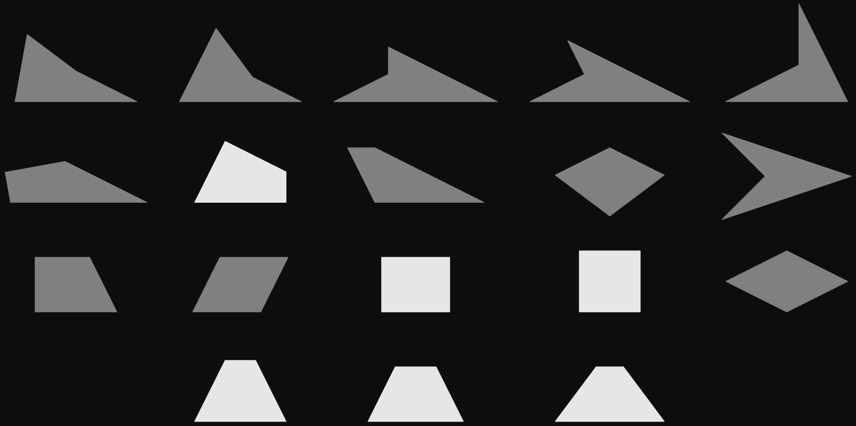
Parallelograms: Two pairs of parallel sides



Parallelogram $\Leftrightarrow$ Diagonals bisect each other $\Leftrightarrow$ $a^2 + b^2 + c^2 + d^2 = p^2 + q^2$

Quadrilaterals

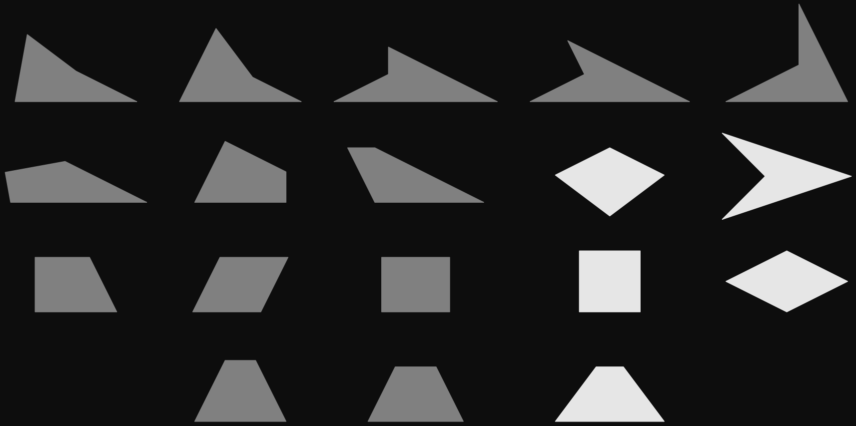
Cyclic quadrilaterals: All vertices lie on a circle



$$\text{Cyclic} \quad \Leftrightarrow \quad \alpha + \gamma = \beta + \delta$$

Quadrilaterals

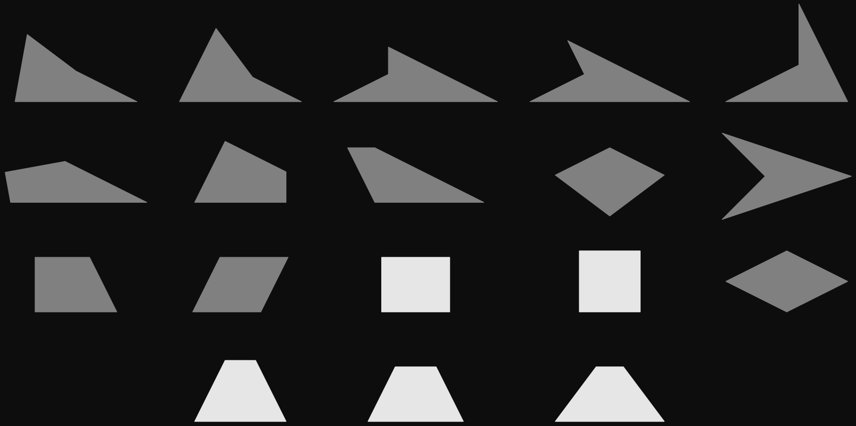
Tangential quadrilaterals: All sides are tangent to a circle



Tangential $\quad\Leftrightarrow\quad a + c = b + d$

Quadrilaterals

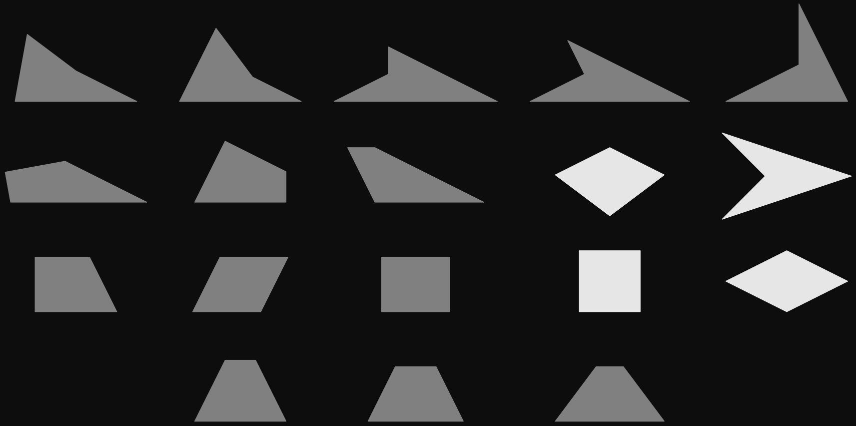
Isosceles Trapezoids: Two pairs of adjacent angles are equal



Isosceles trapezoids $\Leftrightarrow$ Cyclic quadrilaterals with equal diagonals

Quadrilaterals

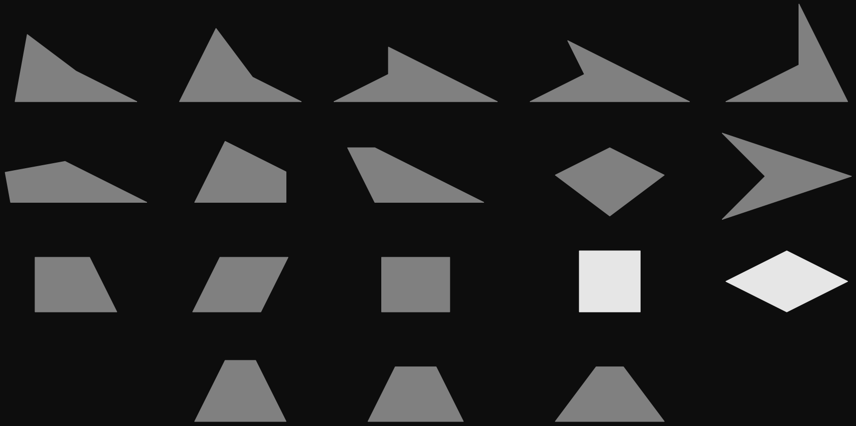
Darts & Kites: Two pairs of adjacent sides are equal



Darts/Kites $\Leftrightarrow$ Tangential quadrilaterals with perpendicular diagonals

Quadrilaterals

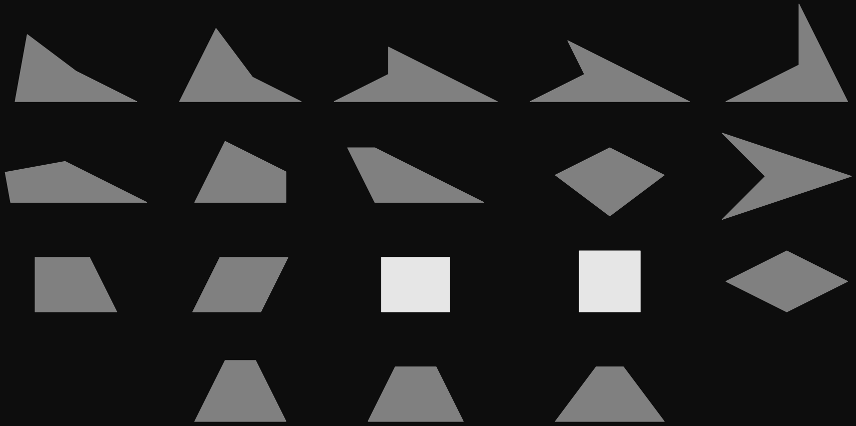
Rhombi: All sides are equal



Rhombi $\Leftrightarrow$ Parallelograms with perpendicular diagonals

Quadrilaterals

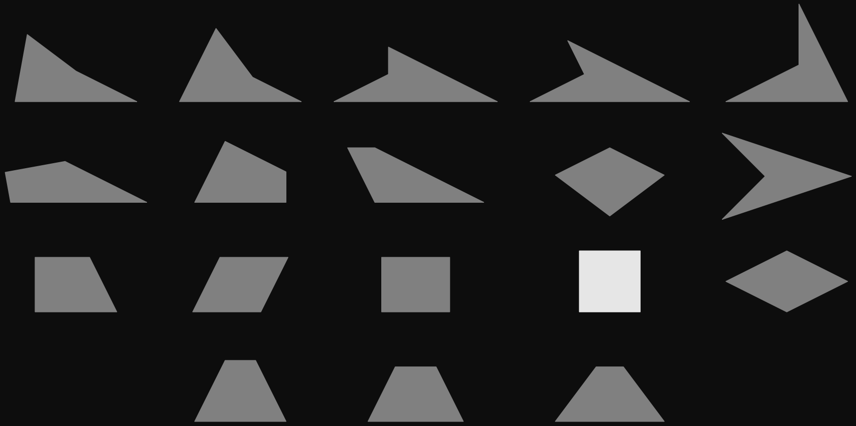
Rectangles: All angles are equal



Rectangles $\Leftrightarrow$ Parallelograms with equal diagonals

Quadrilaterals

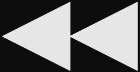
Squares: Regular quadrilaterals



Among all quadrilaterals, squares maximize the *Area* : *Perimeter* ratio

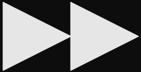
Remote control symbols

Use all five pieces to make these symbols:

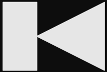


Rewind

Play/Pause

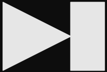


FFWD



Start

Stop



End

Volume

The three solutions of the square

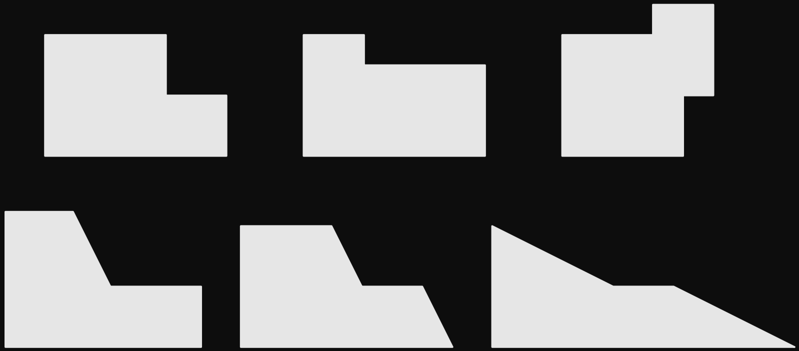
Could you prove that there are just
three different solutions for the square?

What's the area of this square? What's its perimeter?

How many times do you find $\sqrt{5}$ in the Egyptian Tangram pieces?

Figures with unique solutions

These figures are conjectured to have unique solutions:



Could you prove it?

Missing triangle paradox

Both figures use all 5 pieces...

Where is the missing triangle?

Missing triangle paradox

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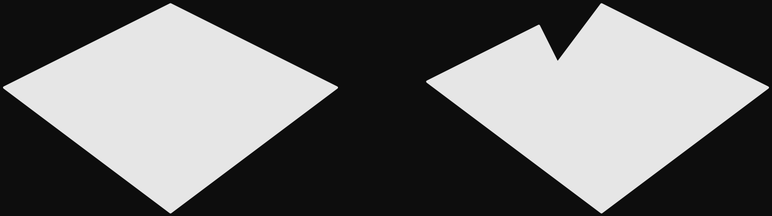

Where is the missing triangle?

Missing triangle paradox

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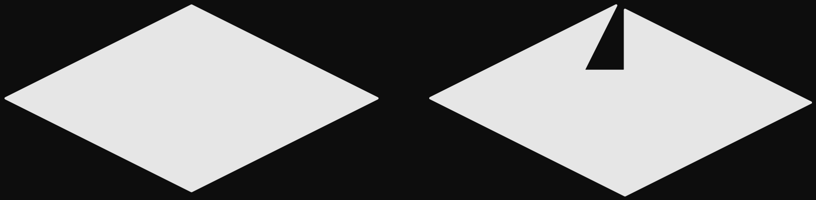

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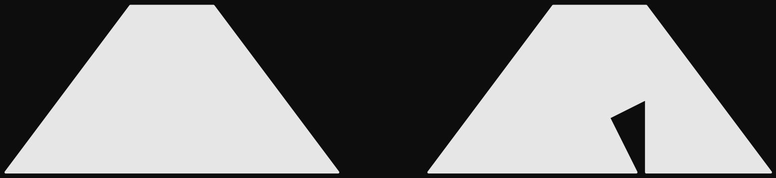

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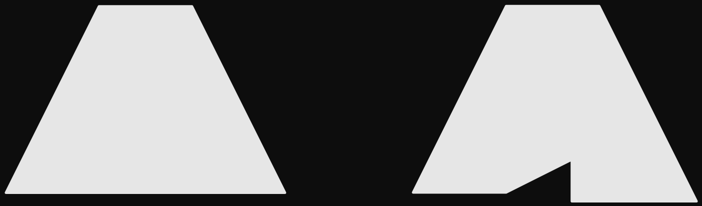

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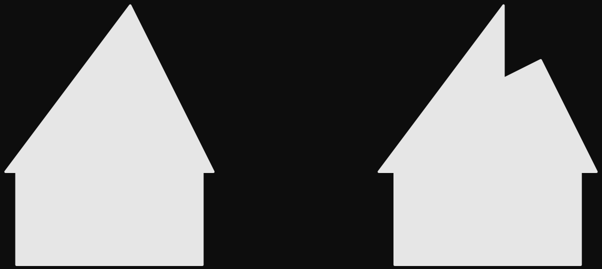

Where is the missing triangle?

Missing triangle paradox

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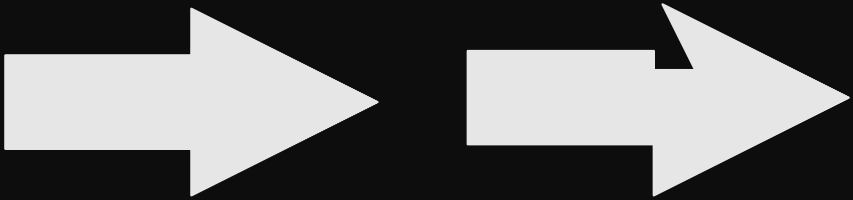


Where is the missing triangle?

Missing square paradox

Both figures use all 5 pieces...

Where is the missing square?

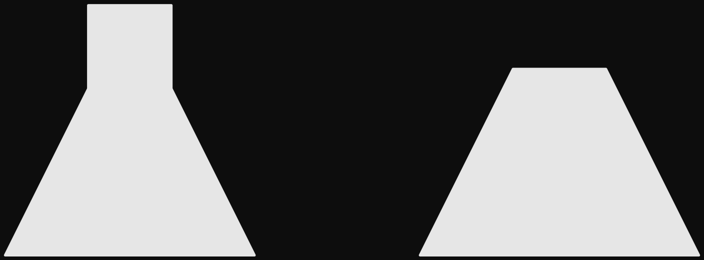
Missing square paradox

Both figures use all 5 pieces...

Where is the missing square?

Missing square paradox

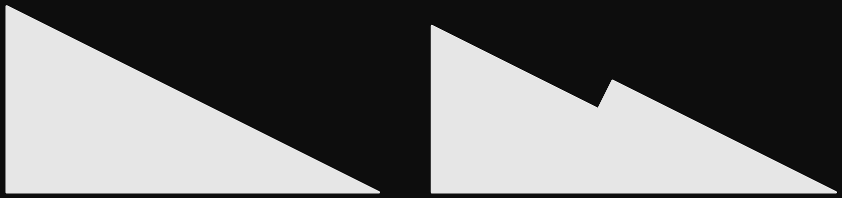
Both figures use all 5 pieces...



Where is the missing square?

Missing rectangle paradox

Both figures use all 5 pieces...



Where is the missing rectangle?

Missing rectangle paradox

Both figures use all 5 pieces...

Where is the missing rectangle?

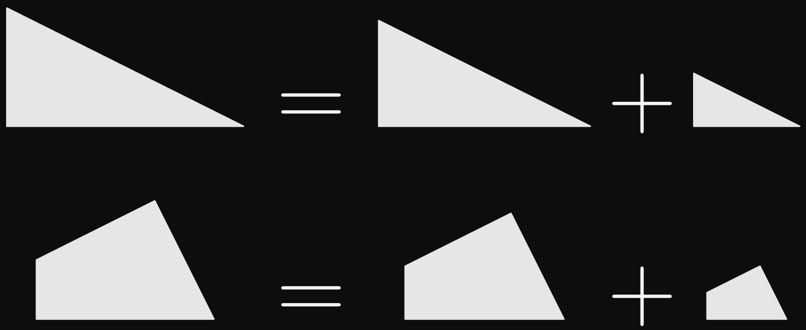
Missing rectangle paradox

Both figures use all 5 pieces...

Where is the missing rectangle?

Sum of similar figures

Use all 5 pieces to make the single figure in the LHS, then use them to make the two figures on the RHS

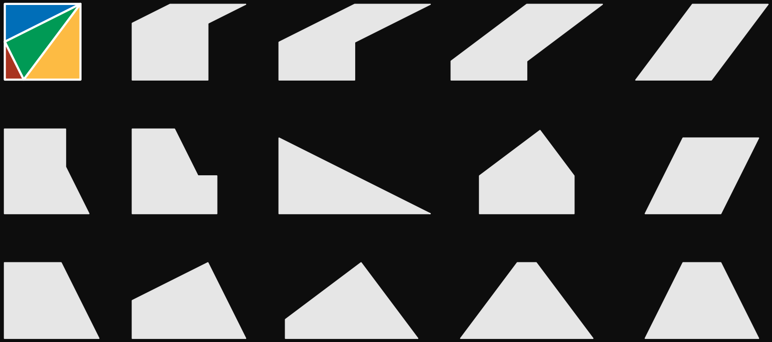


In both equations, the figures are similar and areas are in ratio 5 : 4 : 1

The Egyptian Four–Triangle–Tangram

You can make these figures using just T1, T4, T5 & T6

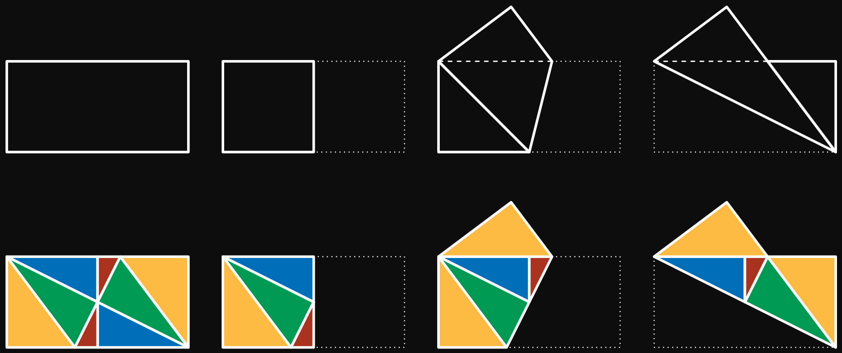
(the four triangles of the Egyptian Tangram)



The Egyptian Four–Triangle–Tangram

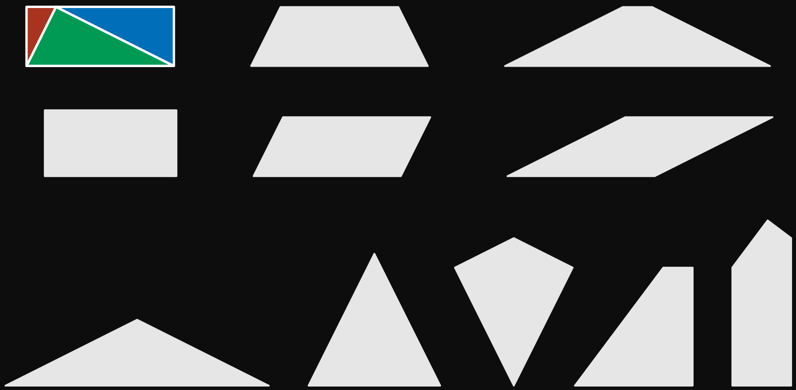
T1, T4, T5 & T6 appear naturally

when you fold a 2 : 1 rectangle



The Egyptian Three–Triangle–Tangram

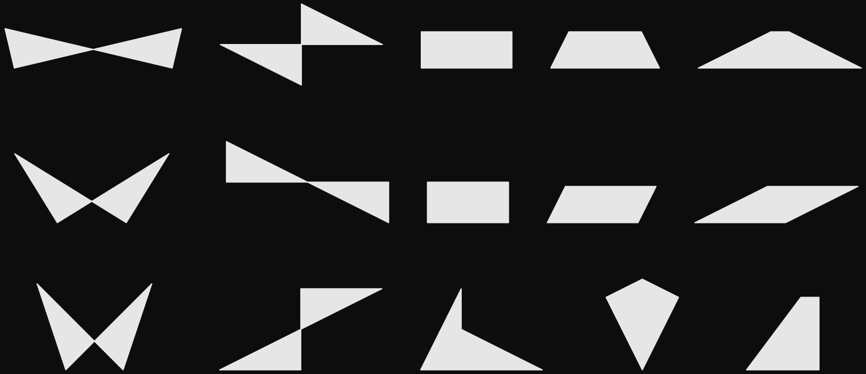
You can make 11 convex figures using just T1, T4 & T5



See: Brügner, G. (1984) *"Three–Triangle–Tangram"*, Bit, 24

The Egyptian Three–Triangle–Tangram

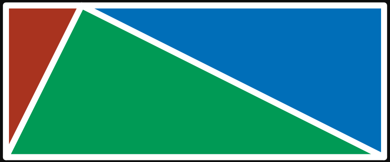
You can make 15 quadrilaterals using just T1, T4 & T5



See: Brügner, G. (1984) *"Three–Triangle–Tangram"*, Bit, 24

The Egyptian Three–Triangle–Tangram

Given any right triangle with sides: $a \leq b \leq c$...

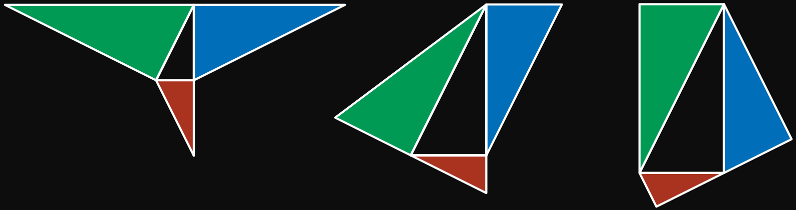


...you can make a rectangle with three similar triangles:
(a^2, ab, ac), (ba, b^2, bc) & (ca, cb, c^2) and compare the top (a^2+b^2) and the bottom (c^2) sides of the rectangle to proof Pythagoras

(T1, T4 & T5 verify this relationship for: $a = 1$, $b = 2$ & $c = \sqrt{5}$)

The Egyptian Three–Triangle–Tangram

Since $\quad \text{area}(T1) + \text{area}(T4) = \text{area}(T5) \quad \ldots$



...you can verify three cases of Pythagoras' theorem

(and these particular cases turn out to be the T1, T4 & T5 right triangles!)

Mathematical Properties

Golden Rectangles

The dashed rectangle proportions are 1:φ

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

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φ and $\sqrt{5}$ are irrational

This is a golden rectangle, which means that $\frac{\text{base}}{\text{height}} = \varphi$, the golden ratio.

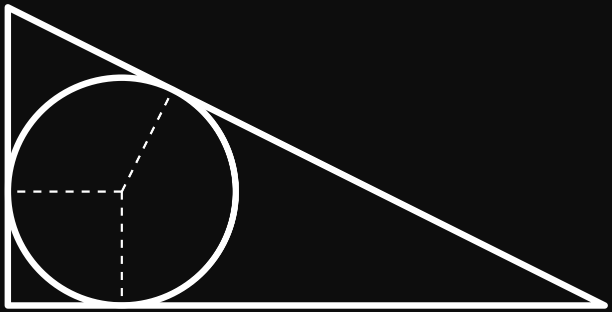
If we remove a square, what remains is also a golden rectangle: $\frac{\text{height}}{\text{base-height}} = \varphi$

If we assume that $\varphi = \frac{b}{h}$, with b and h coprime integers, then $\varphi = \frac{h}{b-h}$ is an equivalent fraction, with a smaller integer numerator and a smaller integer denominator, which is absurd. Therefore, our initial assumption must be false.

And, since $\varphi = \frac{1+\sqrt{5}}{2}$ is irrational, $2\varphi - 1 = \sqrt{5}$ must be irrational too.

The 1:2:$\sqrt{5}$ incenter

If the inradius of a 1:2:$\sqrt{5}$ triangle is 1...



...its shorter leg measures $\varphi + 1 = \varphi^2 = \frac{3+\sqrt{5}}{2}$

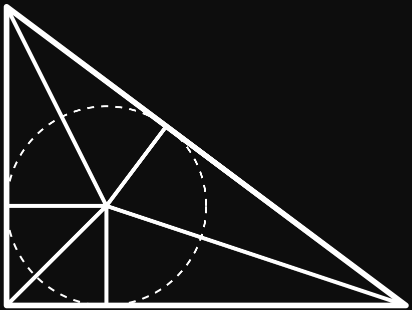
The 3:4:5 incenter

If we overlay T6 and T1 as shown in the figure...

...a T1 vertex lies on the incenter of T6

Dissecting 3:4:5

You can use this dissection of T6 to prove that...

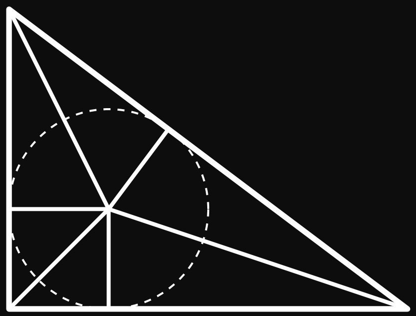


$$\pi = \mathrm{arctan}(1) + \mathrm{arctan}(2) + \mathrm{arctan}(3)$$

(consider the sum of the angles touching the incenter of T6 and divide by 2)

Dissecting 3:4:5

You can use this dissection of T6 to prove that...

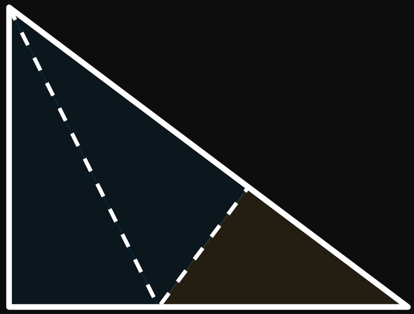


$$\frac{\pi}{2} = \mathrm{arctan}\left(\frac{1}{1}\right) + \mathrm{arctan}\left(\frac{1}{2}\right) + \mathrm{arctan}\left(\frac{1}{3}\right)$$

(consider the sum of the angles touching the vertices of T6 and divide by 2)

Dissecting 3:4:5

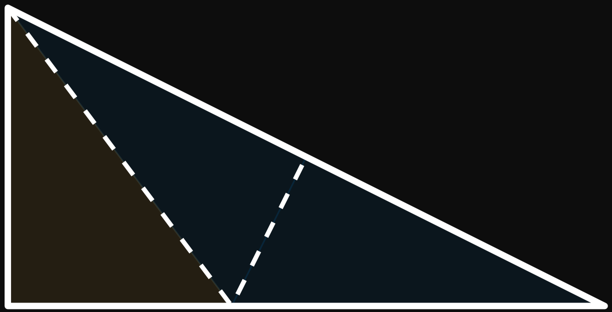
You can dissect a 3:4:5 triangle into...



...a 3:4:5 triangle and
two congruent 1:2:$\sqrt{5}$ triangles

Dissecting $1{:}2{:}\sqrt{5}$

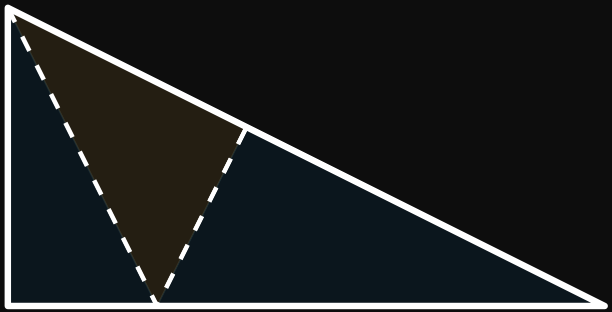
You can dissect a $1{:}2{:}\sqrt{5}$ triangle into...



...a 3:4:5 triangle and
two congruent $1{:}2{:}\sqrt{5}$ triangles

Dissecting 1:2:$\sqrt{5}$

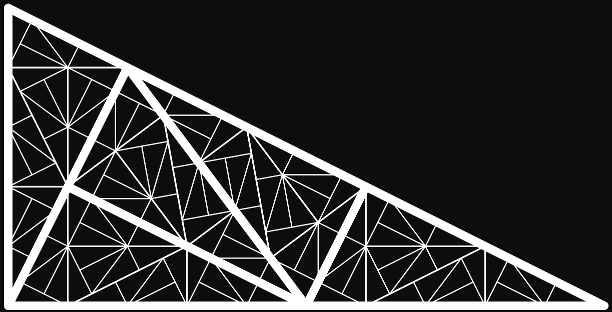
You can dissect a 1:2:$\sqrt{5}$ triangle into...



...a 3:4:5 triangle and
two different 1:2:$\sqrt{5}$ triangles

Dissecting $1{:}2{:}\sqrt{5}$

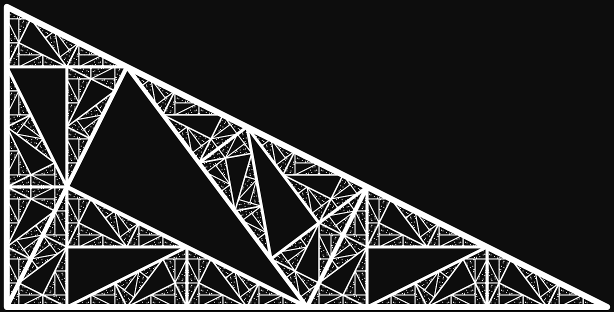
You can assemble a $1{:}2{:}\sqrt{5}$ triangle aggregating...



...five congruent $1{:}2{:}\sqrt{5}$ triangles
and iterate to get the **Pinwheel tiling** of the plane

Dissecting $1{:}2{:}\sqrt{5}$

You can dissect a $1{:}2{:}\sqrt{5}$ triangle into...

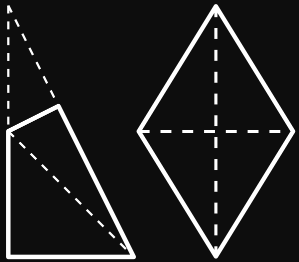


...five congruent $1{:}2{:}\sqrt{5}$ triangles, remove the central one and iterate to get the **Pinwheel fractal**

The angles of Q4

The angles $90-\alpha$ and $90+\alpha$ that appear in Q4 also appear in the Golden Rhombus

(a rhombus whose diagonals are in proportion $1:\varphi$, with $\varphi = \frac{1+\sqrt{5}}{2}$)



$$90+\alpha = 2\cdot\arctan(\varphi) = \arctan(1)+\arctan(3)$$

$$90-\alpha = 2 \cdot \arctan\left(\frac{1}{\varphi}\right) = \arctan(2)$$

The faces of the rhombic triacontahedron and

the rhombic hexecontahedron are Golden Rhombi

The angles of Q4

Even though they are NOT similar figures...

...the same angles appear in Q4 and T5 ∪ T6

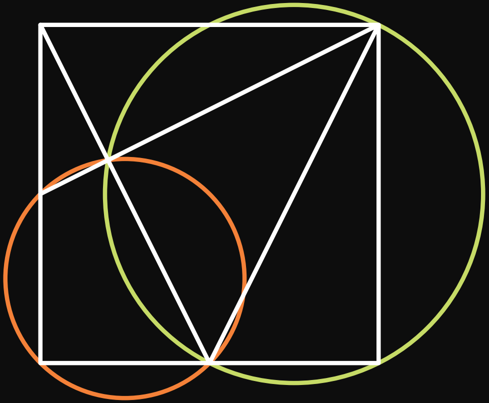
The perimeter of Q4

These three perimeters are in a geometric progression...

$$\frac{2\sqrt{5}+4}{\sqrt{5}+3} = \frac{3\sqrt{5}+7}{2\sqrt{5}+4} = \varphi = \frac{1+\sqrt{5}}{2}$$

The circumcircles

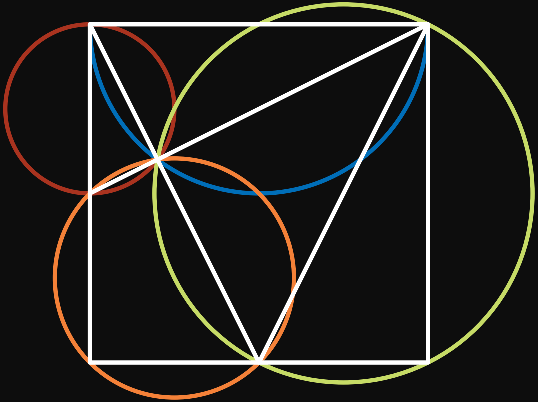
Since opposite angles add to π...



...Q4 and T5 $\cup$ T6 are cyclic quadrilaterals

The circumcircles

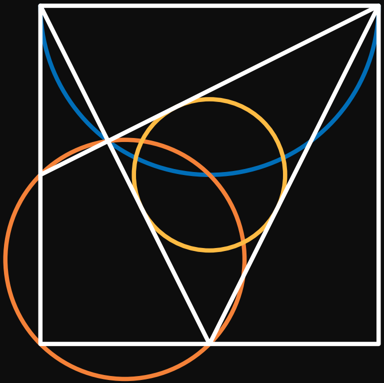
All circumcircles pass through a common point...



...and $C(T5 \cup T6)$ passes through the center of $C(Q4)$ and $C(T4)$

The circumcircles

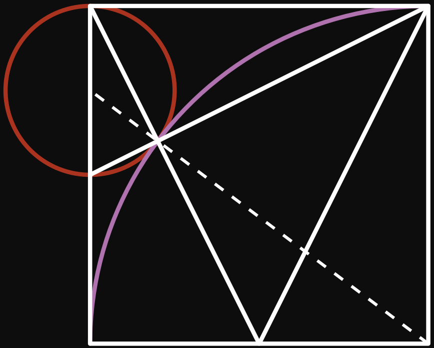
These cirmcumcircles intersect at the square's center...



...which happens to be T6's incenter

Tangent circles

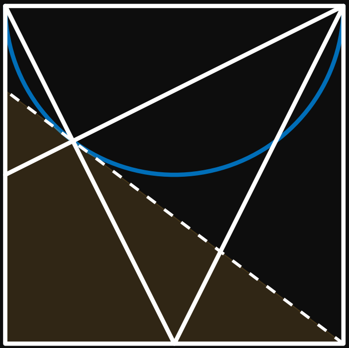
These three points are aligned...



...and these two circles are tangent

Tangent circles

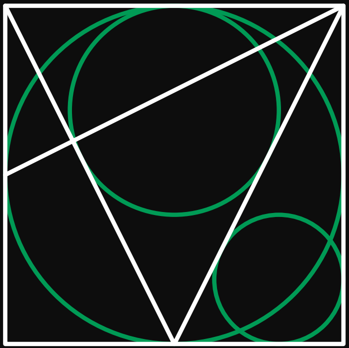
The line is tangent to this circle...



...and the right triangle below is an Egyptian Triangle

Tangent circles

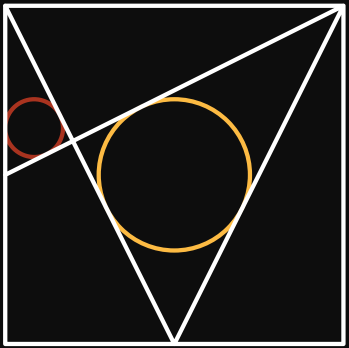
The radius of these three circles are in ratio 1:φ:φ^2



where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

Tangent circles

The radius of these two circles are in ratio $1{:}\varphi^2$



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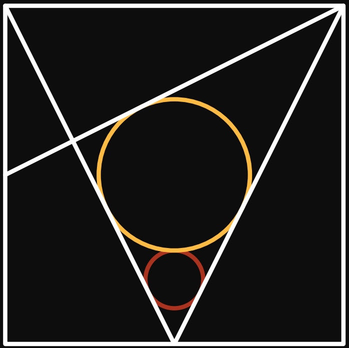
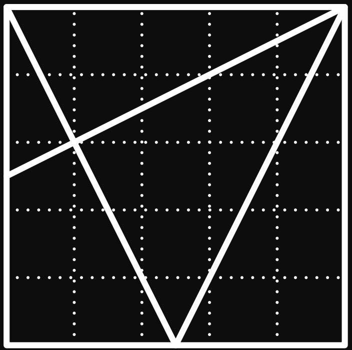


where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

The underlying grid

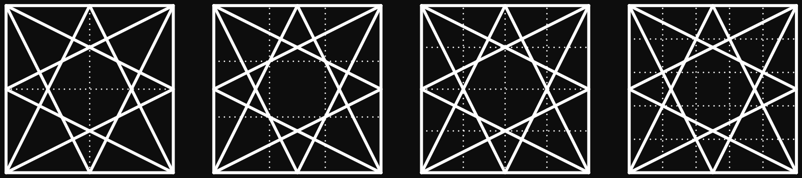
Using the intersection point of the Egyptian Tangram...



...you can divide the square into 5×5 smaller squares!

The underlying grid

Using the intersection points of this figure...



...you can divide the square into:

2×2, 3×3, 4×4 or 5×5 smaller squares!

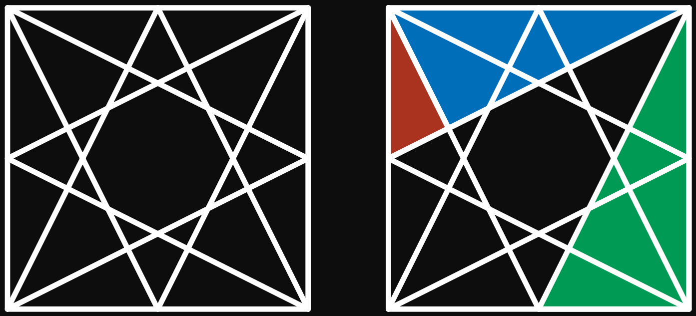
The underlying grid

There are 32 egyptian triangles in this figure...

...they come in 4 sizes and there are 8 of each kind

The underlying grid

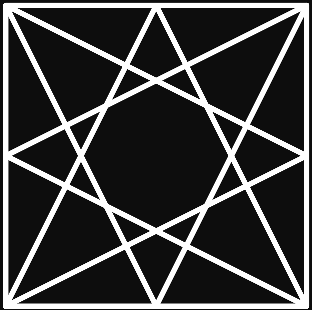
There are 24 1:2:$\sqrt{5}$ triangles in this figure...



...they come in 3 sizes and there are 8 of each kind

The underlying grid

There are 24 other triangles in this figure...



...of 3 different kinds (one of them comes in 2 sizes)

The underlying grid

The relative sizes of these polygons are...

Small Triangles:	1	**Small Kites:**	3	**Whole Square:**	120
Big Triangles:	6	**Big Kites:**	8	**Octagon:**	20

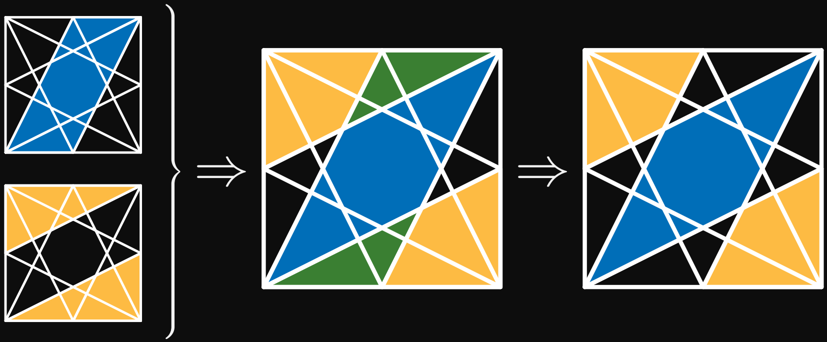
The carpets theorem

Since Area(BLUE) = Area(YELLOW)...

...Area(BLUE−GREEN) = Area(YELLOW−GREEN)

The carpets theorem

Since Area(BLUE) = Area(YELLOW)...



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The carpets theorem

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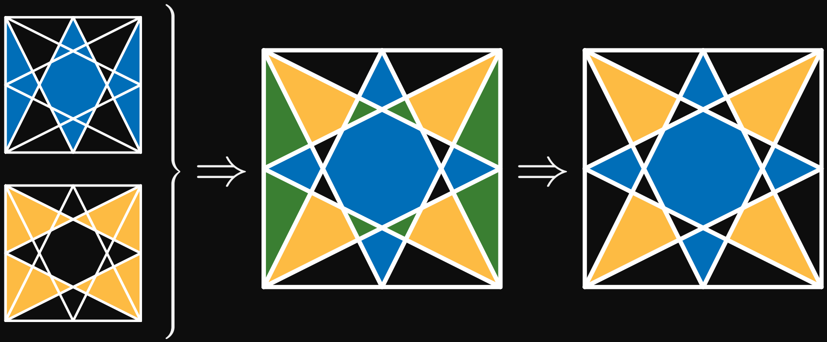

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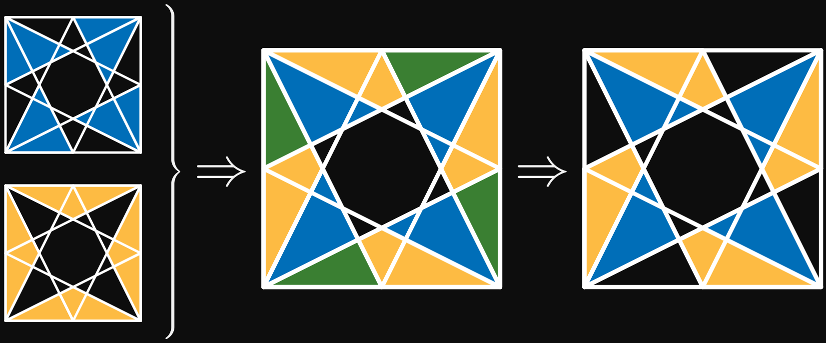

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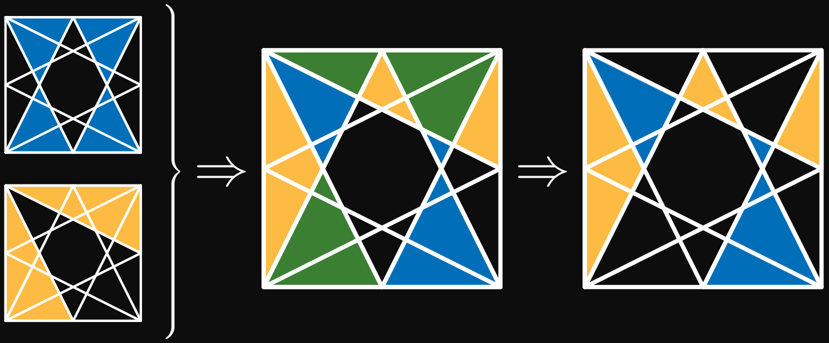

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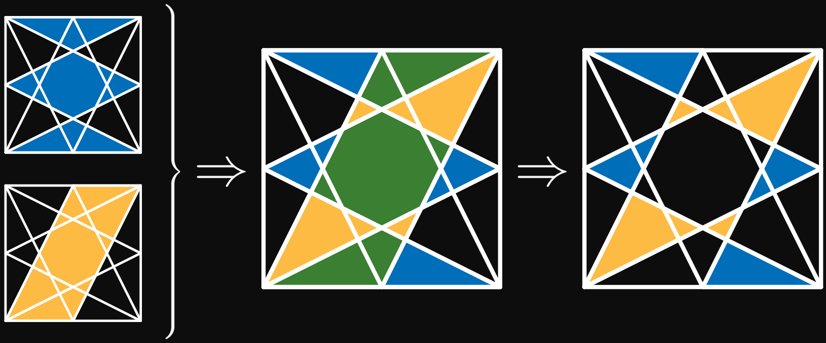

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