

The Egyptian Tangram



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mmaca

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The Egyptian Tangram

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A square dissection firstly proposed as a tangram in:

Luna-Mota, C. (2019) *"El tangram egipci: diari de disseny"* Nou Biaix, 44

Origins

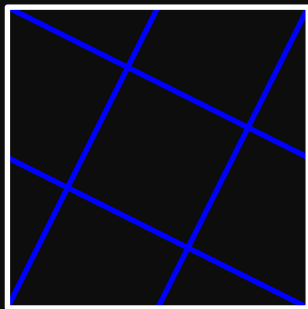
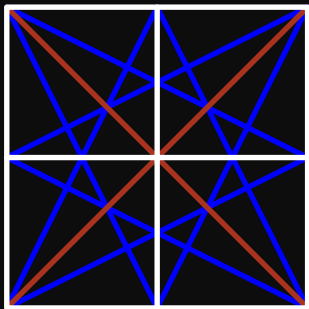
The Egyptian Tangram inspiration comes from the study of two other 5-piece tangrams...



The “Five Triangles” & “Greek-Cross” tangrams

Origins

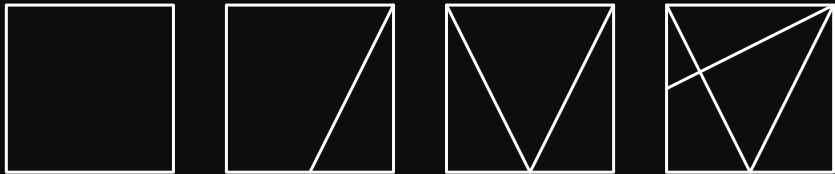
...and their underlying grids



The “Five Triangles” & “Greek-Cross” underlying grids

Design Process

The Egyptian Tangram is the result of an heuristic incremental design process:



Take a square and keep adding “the most interesting straight cut” until you have a dissection with 5 or more pieces.

Design Process



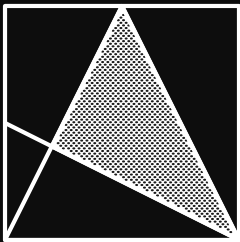
To make an Egyptian Tangram from a square:

1. Connect the midpoint of the lower side with the upper corners.
2. Connect the midpoint of the left side with the top right corner.

Antecedents

It turns out that this figure is not new...

Detemple, D. & Harold, S. (1996) *"A Round-Up of Square Problems"*



Problem 3

...but, to the best of our knowledge,
nobody used it before **as a tangram**

Antecedents

The name is not new either...



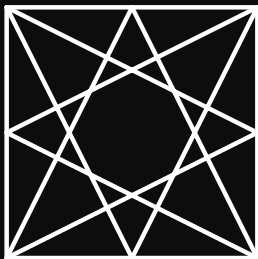
This dissection is often called “Egyptian Puzzle” or “Egyptian Tangram”

...but there is a good reason to consider
our dissection the real “Egyptian Tangram”

(even if it was designed in Barcelona)

Antecedents

The underlying grid is also a well known figure:



Brunés, T. (1967) *"The Secrets of Ancient Geometry – and Its Use"*

Bankoff, L. & W. Trigg, C. (1974) *"The Ubiquitous 3:4:5 Triangle"*

The pieces



- Just five pieces
- All pieces are different
- All pieces are asymmetric
- Areas are integer and not *too different*
- All sides are multiples of 1 or $\sqrt{5}$
- All angles are linear combinations of 90° and $\alpha = \arctan\left(\frac{1}{2}\right) \approx 26,565^\circ$

Name	Area	Sides	Angles
T1	1	1, 2, $\sqrt{5}$	90° , α , $90^\circ - \alpha$
T4	4	2, 4, $2\sqrt{5}$	90° , α , $90^\circ - \alpha$
T5	5	$\sqrt{5}$, $2\sqrt{5}$, 5	90° , α , $90^\circ - \alpha$
T6	6	3, 4, 5	90° , $90^\circ - 2\alpha$, 2α
Q4	4	1, 3, $\sqrt{5}$, $\sqrt{5}$	90° , $90^\circ - \alpha$, 90° , $90^\circ + \alpha$

The pieces

Although all pieces are asymmetric and different, they often combine to make symmetric shapes



The pieces

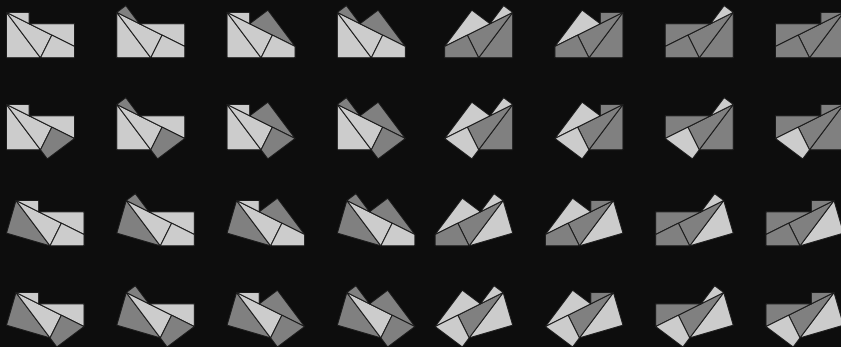
This means that it is very rare for an Egyptian Tangram figure to have a unique solution



There are three different solutions for the square and, in all three cases, two of the corners of the square are built as a sum of acute angles!

The pieces

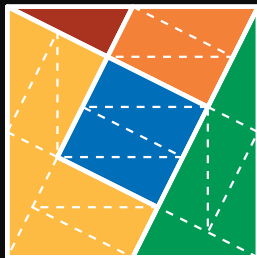
The assymetry of the pieces also implies that each solution belongs to one of these equivalence classes:



You cannot transform one of these figures into another without flipping a piece

Why we called it the *Egyptian* Tangram?

The smallest pieces of the Chinese and Greek-Cross tangrams can be used to build all the other pieces...



...but you cannot do the same with
the Egyptian Tangram because of T6

Why we called it the *Egyptian* Tangram?

Initially, T6 was considered as the *leftover* piece that results from cutting all these $1:2:\sqrt{5}$ triangles from the borders of the square.

But it turned out to be a very well known triangle...



...the **Egyptian** Triangle (3:4:5)
and, hence, the name

Puzzles & Activities

Realistic figures

Use all five pieces to make these figures:



Lightning



Sailing ship



Bow tie



Wooden hut



Caltrop



Snowmobile



Candle



Viking hat



Diamond



Moses basket



Erlenmeyer



3D brick



Witch hat



Arrow Sign



Sailboat

Realistic figures

Use all five pieces to make these figures:



Gnome



Handmaid



Mountain range



Fish tail



Teddy bear



Cat



Cow



Crab claw



Snail



Fennec Fox



Penguin



Calf



Sea Turtle



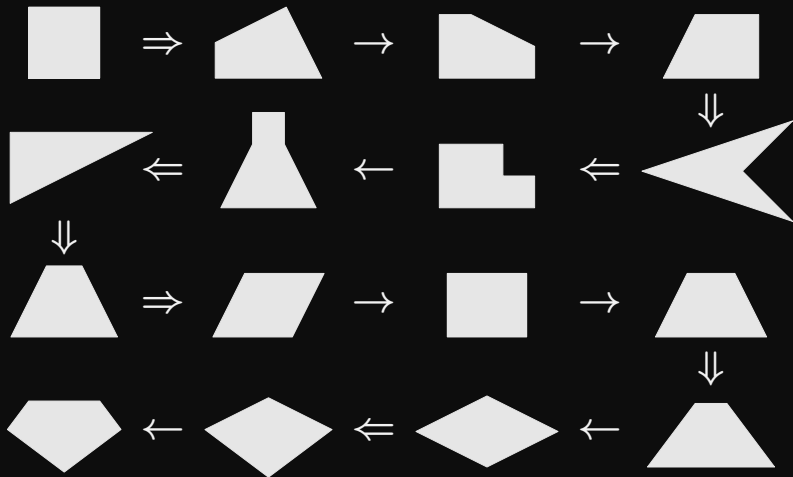
Duck



Crow

Geometric figures

Use all five pieces to make these figures:



Complete the path moving just one or two pieces at a time

Triangles

Could you prove that there are just 10 triangles you can make with one or more pieces of the Egyptian Tangram?

How many solutions could you find for each figure?

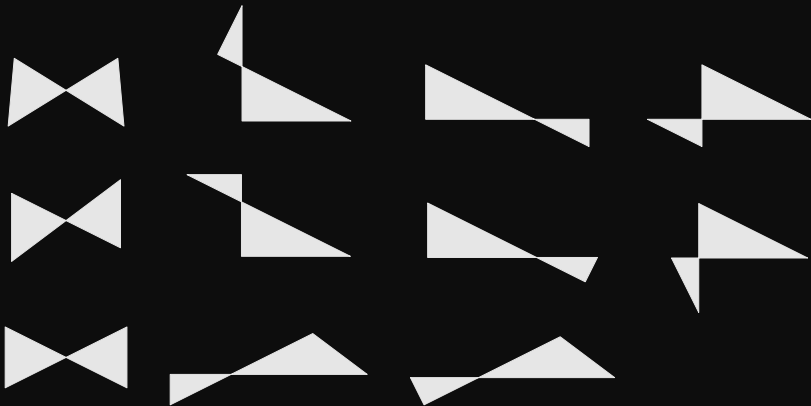


Top row areas: 20, 16, 9, 5, 4, 1

Bottom row areas: 15, 10, 10, 6

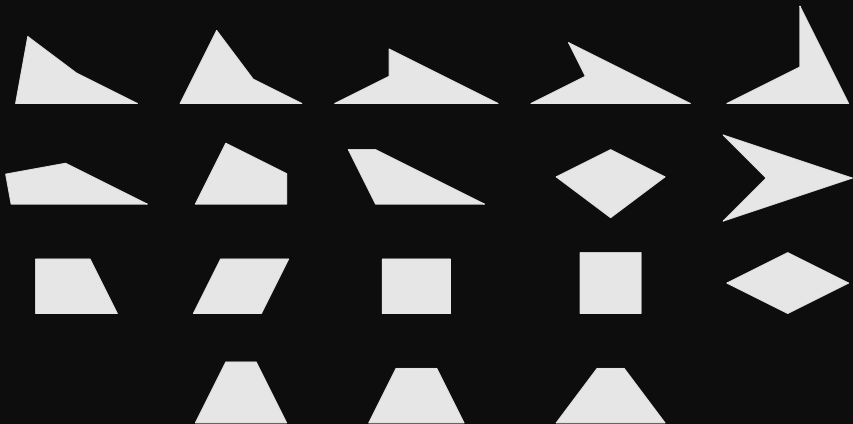
Quadrilaterals

Could you prove that there are just 11 **complex quadrilaterals** you can make with all five pieces of the Egyptian Tangram?



Quadrilaterals

Simple quadrilaterals: Not self-intersecting

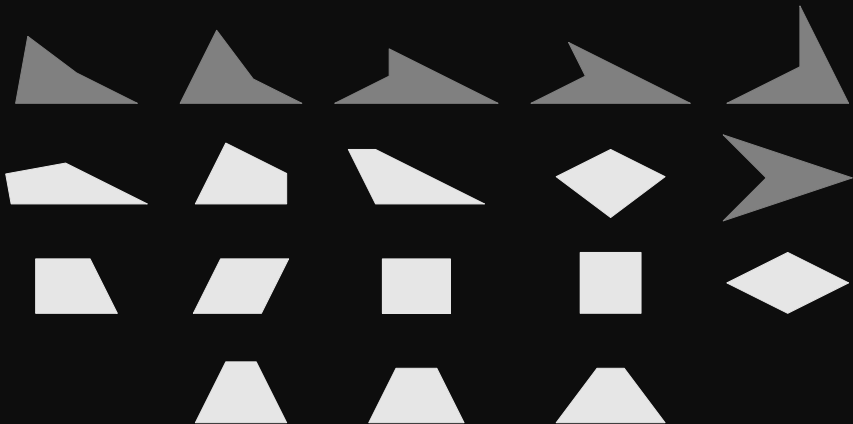


All simple quadrilaterals tile the plane!

$$\alpha + \beta + \gamma + \delta = 2\pi$$

Quadrilaterals

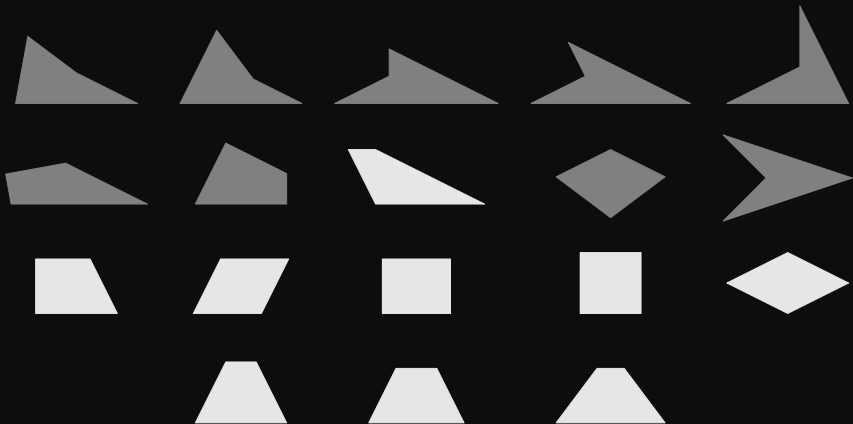
Convex quadrilaterals: All internal angles are smaller than π



Law of Cosines: $p^2q^2 = a^2c^2 + b^2d^2 - 2abcd \cos(\alpha + \gamma)$

Quadrilaterals

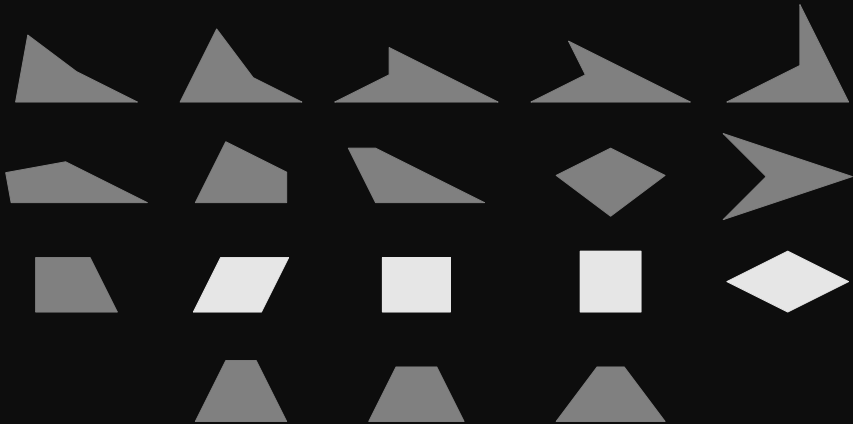
Trapeziums (UK) / Trapezoids (US): One pair of parallel sides



Trapezium/Trapezoid \Leftrightarrow Diagonals cut each other in the same ratio

Quadrilaterals

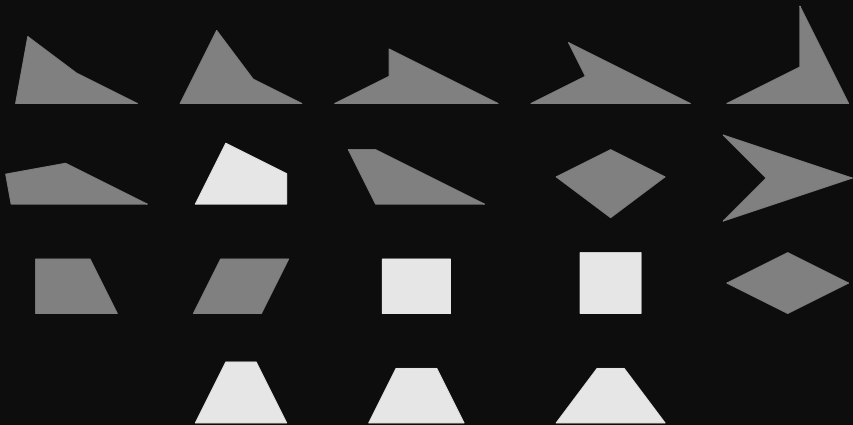
Parallelograms: Two pairs of parallel sides



$$\text{Parallelogram} \Leftrightarrow \text{Diagonals bisect each other} \Leftrightarrow a^2 + b^2 + c^2 + d^2 = p^2 + q^2$$

Quadrilaterals

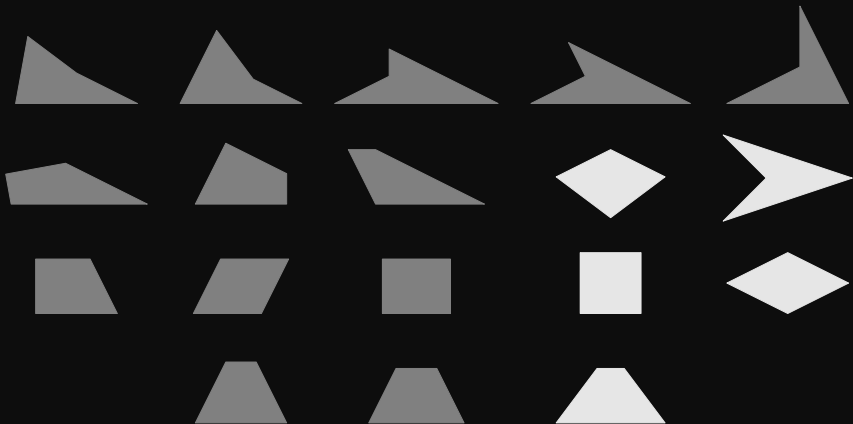
Cyclic quadrilaterals: All vertices lie on a circle



$$\text{Cyclic} \Leftrightarrow \alpha + \gamma = \beta + \delta$$

Quadrilaterals

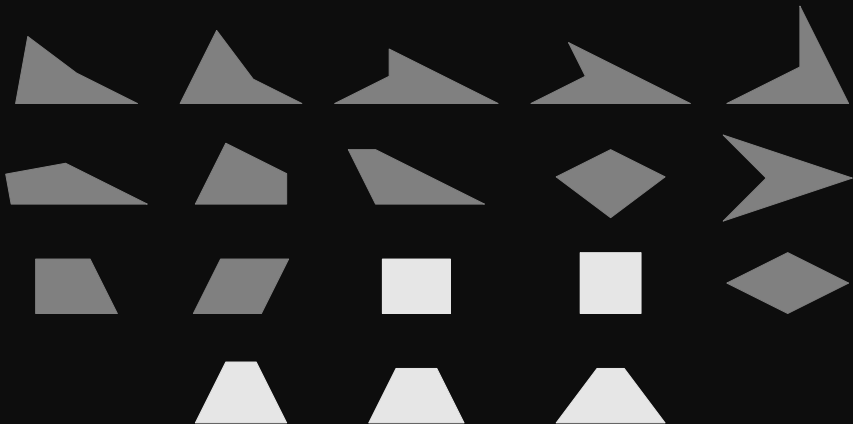
Tangential quadrilaterals: All sides are tangent to a circle



$$\text{Tangential} \Leftrightarrow a + c = b + d$$

Quadrilaterals

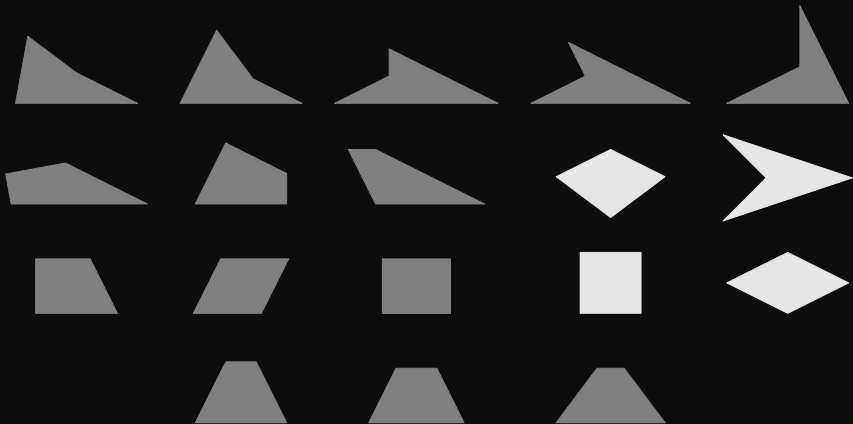
Isosceles Trapezoids: Two pairs of adjacent angles are equal



Isosceles trapezoids \Leftrightarrow Cyclic quadrilaterals with equal diagonals

Quadrilaterals

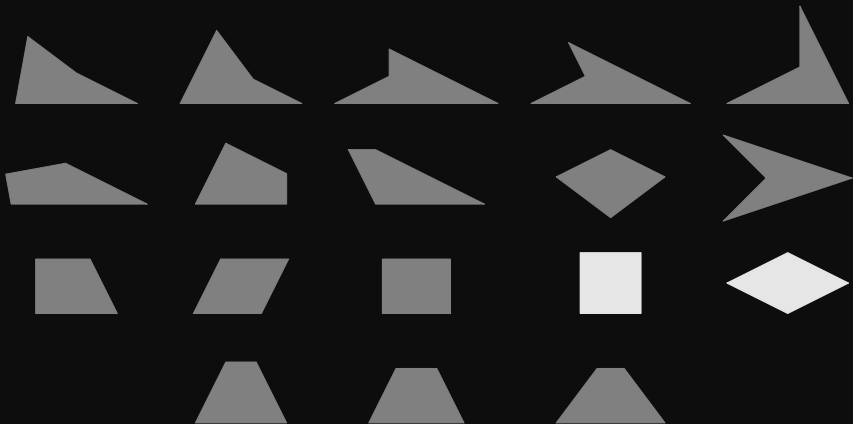
Darts & Kites: Two pairs of adjacent sides are equal



Darts/Kites \Leftrightarrow Tangential quadrilaterals with perpendicular diagonals

Quadrilaterals

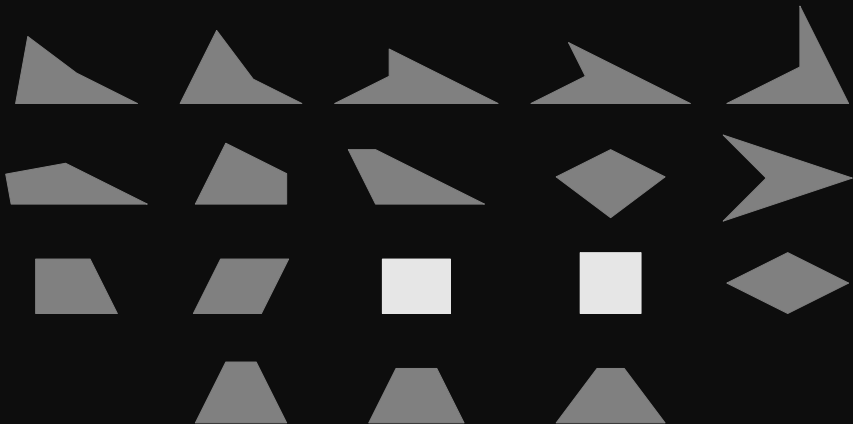
Rhombi: All sides are equal



Rhombi \Leftrightarrow Parallelograms with perpendicular diagonals

Quadrilaterals

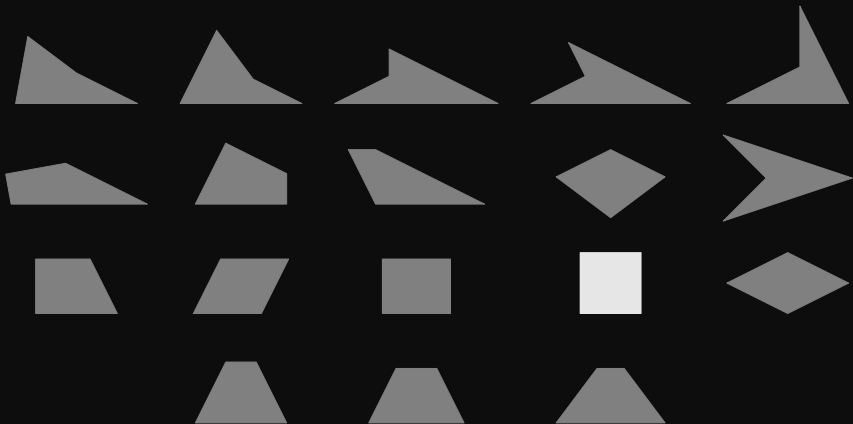
Rectangles: All angles are equal



Rectangles \Leftrightarrow Parallelograms with equal diagonals

Quadrilaterals

Squares: Regular quadrilaterals



Among all quadrilaterals, squares maximize the *Area:Perimeter* ratio

Remote control symbols

Use all five pieces to make these symbols:



Rewind



Play/Pause



FFWD



Start



Stop



End



Volume

The three solutions of the square

Could you prove that there are just three different solutions for the square?



What's the area of this square? What's its perimeter?

How many times do you find $\sqrt{5}$ in the Egyptian Tangram pieces?

Figures with unique solutions

These figures are conjectured to have unique solutions:



Could you prove it?

Missing triangle paradox

Both figures use all 5 pieces...



Where is the missing triangle?

Missing triangle paradox

Both figures use all 5 pieces...



Where is the missing triangle?

Missing triangle paradox

Both figures use all 5 pieces...



Where is the missing triangle?

Missing triangle paradox

Both figures use all 5 pieces...



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Where is the missing triangle?

Missing triangle paradox

Both figures use all 5 pieces...



Where is the missing triangle?

Missing triangle paradox

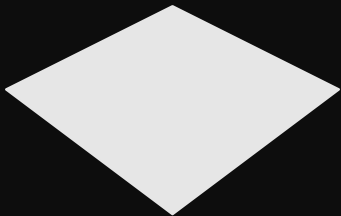
Both figures use all 5 pieces...



Where is the missing triangle?

Missing triangle paradox

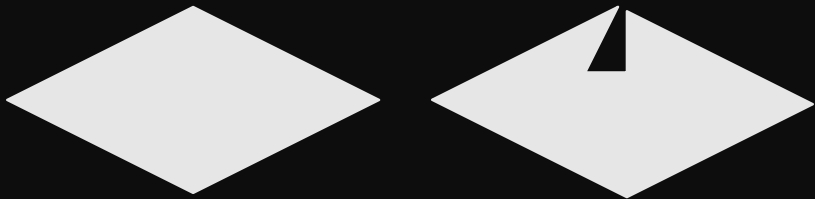
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Missing triangle paradox

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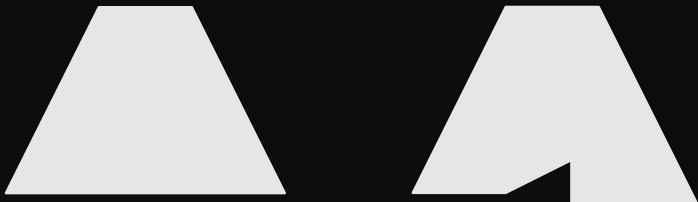
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Missing triangle paradox

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Missing triangle paradox

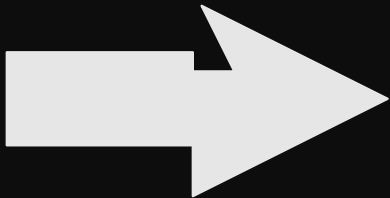
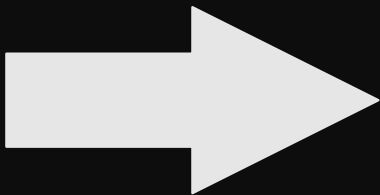
Both figures use all 5 pieces...



Where is the missing triangle?

Missing triangle paradox

Both figures use all 5 pieces...



Where is the missing triangle?

Missing square paradox

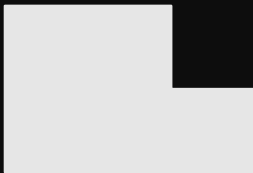
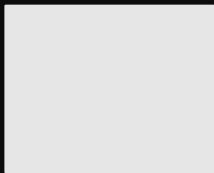
Both figures use all 5 pieces...



Where is the missing square?

Missing square paradox

Both figures use all 5 pieces...



Where is the missing square?

Missing square paradox

Both figures use all 5 pieces...



Where is the missing square?

Missing rectangle paradox

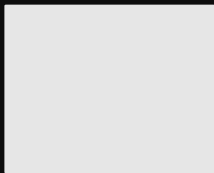
Both figures use all 5 pieces...



Where is the missing rectangle?

Missing rectangle paradox

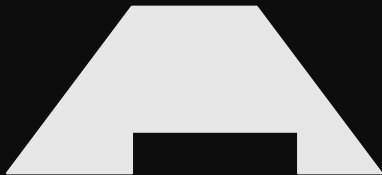
Both figures use all 5 pieces...



Where is the missing rectangle?

Missing rectangle paradox

Both figures use all 5 pieces...



Where is the missing rectangle?

Sum of similar figures

Use all 5 pieces to make the single figure in the LHS,
then use them to make the two figures on the RHS



In both equations, the figures are similar and areas are in ratio 5 : 4 : 1

The Egyptian Four-Triangle-Tangram

You can make these figures using just T1, T4, T5 & T6
(the four triangles of the Egyptian Tangram)



The Egyptian Three-Triangle-Tangram

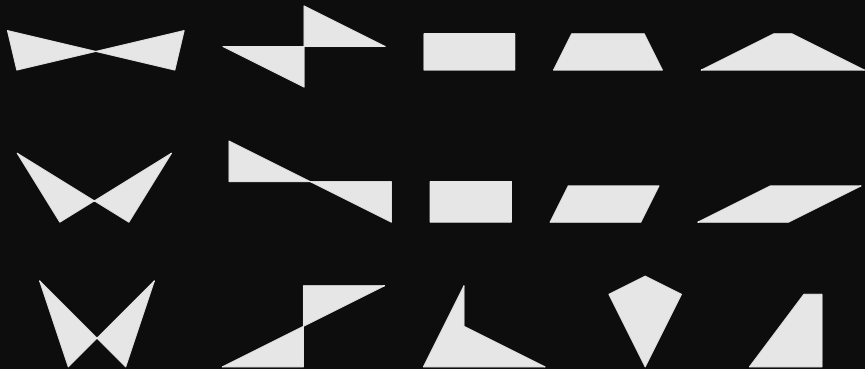
You can make 11 convex figures using just T1, T4 & T5



See: Brügger, G. (1984) *"Three-Triangle-Tangram"*, Bit, 24

The Egyptian Three-Triangle-Tangram

You can make 15 quadrilaterals using just T1, T4 & T5



See: Brügger, G. (1984) *"Three-Triangle-Tangram"*, Bit, 24

The Egyptian Three–Triangle–Tangram

Given any right triangle with sides: $a \leq b \leq c...$

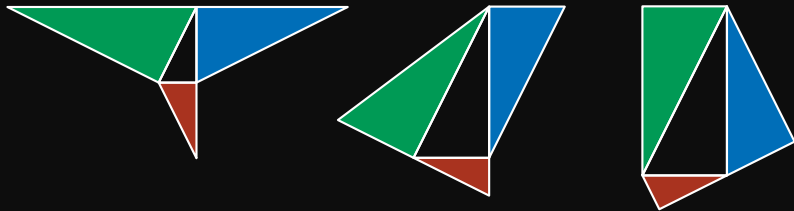


...you can make a rectangle with three similar triangles:
 (a^2, ab, ac) , (ba, b^2, bc) & (ca, cb, c^2) and compare the top $(a^2 + b^2)$
and the bottom (c^2) sides of the rectangle to prove Pythagoras

(T1, T4 & T5 verify this relationship for: $a = 1$, $b = 2$ & $c = \sqrt{5}$)

The Egyptian Three-Triangle-Tangram

Since $\text{area}(T1) + \text{area}(T4) = \text{area}(T5) \dots$



...you can verify three cases of Pythagoras' theorem
(and these particular cases turn out to be the T1, T4 & T5 right triangles!)

Mathematical Properties

Golden Rectangles

The dashed rectangle proportions are $1:\varphi$



where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

Golden Rectangles

The dashed rectangle proportions are $1:\varphi$



where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

Golden Rectangles

The dashed rectangle proportions are $1:\varphi$



where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

Golden Rectangles

The dashed rectangles proportions are $1:\varphi$



where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

φ and $\sqrt{5}$ are irrational



This is a golden rectangle, which means that $\frac{\text{base}}{\text{height}} = \varphi$, the golden ratio.

If we remove a square, what remains is also a golden rectangle: $\frac{\text{height}}{\text{base}-\text{height}} = \varphi$

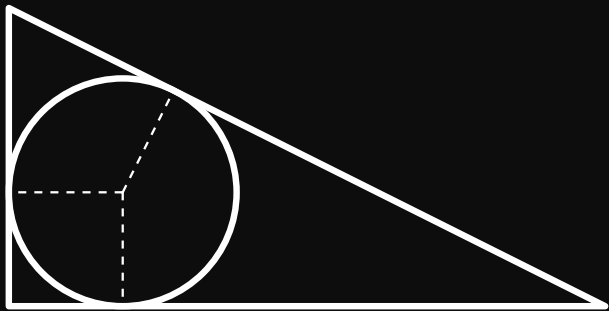


If we assume that $\varphi = \frac{b}{h}$, with b and h coprime integers, then $\varphi = \frac{h}{b-h}$ is an equivalent fraction, with a smaller integer numerator and a smaller integer denominator, which is absurd. Therefore, our initial assumption must be false.

And, since $\varphi = \frac{1+\sqrt{5}}{2}$ is irrational, $2\varphi - 1 = \sqrt{5}$ must be irrational too.

The $1:2:\sqrt{5}$ incenter

If the inradius of a $1:2:\sqrt{5}$ triangle is 1...



...its shorter leg measures $\varphi + 1 = \varphi^2 = \frac{3+\sqrt{5}}{2}$

The 3:4:5 incenter

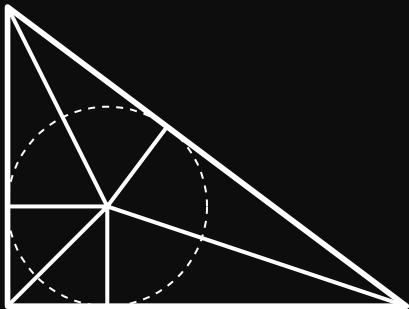
If we overlay T6 and T1 as shown in the figure...



...a T1 vertex lies on the incenter of T6

Dissecting 3:4:5

You can use this dissection of T6 to prove that...

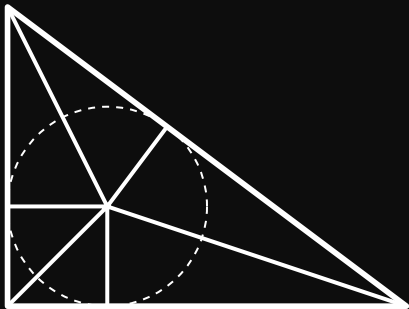


$$\pi = \arctan(1) + \arctan(2) + \arctan(3)$$

(consider the sum of the angles touching the incenter of T6 and divide by 2)

Dissecting 3:4:5

You can use this dissection of T6 to prove that...

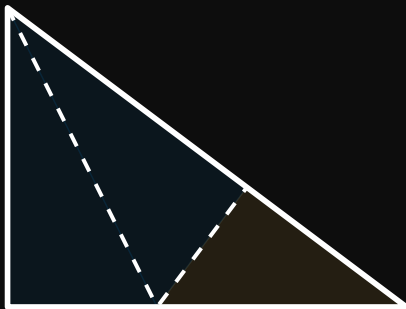


$$\frac{\pi}{2} = \arctan\left(\frac{1}{1}\right) + \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$$

(consider the sum of the angles touching the vertices of T6 and divide by 2)

Dissecting 3:4:5

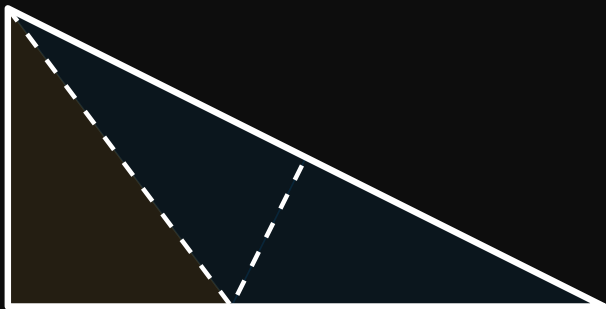
You can dissect a 3:4:5 triangle into...



...a 3:4:5 triangle and
two congruent $1:2:\sqrt{5}$ triangles

Dissecting $1:2:\sqrt{5}$

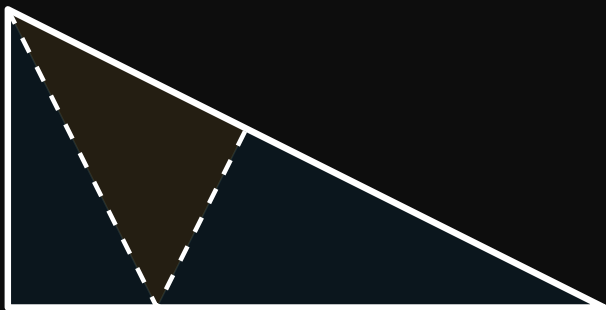
You can dissect a $1:2:\sqrt{5}$ triangle into...



...a $3:4:5$ triangle and
two congruent $1:2:\sqrt{5}$ triangles

Dissecting $1:2:\sqrt{5}$

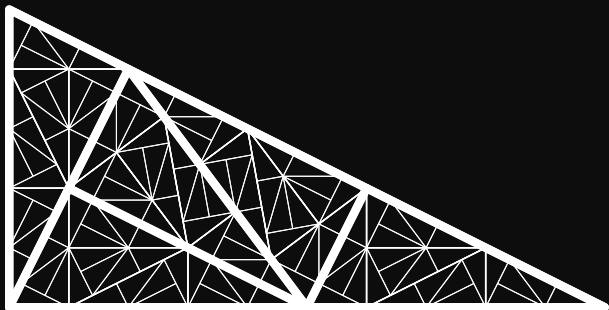
You can dissect a $1:2:\sqrt{5}$ triangle into...



...a 3:4:5 triangle and
two different $1:2:\sqrt{5}$ triangles

Dissecting $1:2:\sqrt{5}$

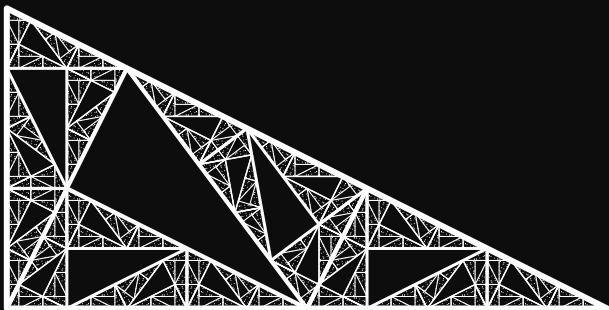
You can assemble a $1:2:\sqrt{5}$ triangle aggregating...



...five congruent $1:2:\sqrt{5}$ triangles
and iterate to get the **Pinwheel tiling** of the plane

Dissecting $1:2:\sqrt{5}$

You can dissect a $1:2:\sqrt{5}$ triangle into...

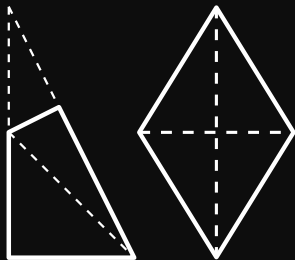


...five congruent $1:2:\sqrt{5}$ triangles, remove the central one and iterate to get the **Pinwheel fractal**

The angles of Q4

The angles $90 - \alpha$ and $90 + \alpha$ that appear in Q4
also appear in the Golden Rhombus

(a rhombus whose diagonals are in proportion $1 : \varphi$, with $\varphi = \frac{1+\sqrt{5}}{2}$)



$$90 + \alpha = 2 \cdot \arctan(\varphi) = \arctan(1) + \arctan(3)$$

$$90 - \alpha = 2 \cdot \arctan\left(\frac{1}{\varphi}\right) = \arctan(2)$$

The faces of the rhombic triacontahedron and
the rhombic hexecontahedron are Golden Rhombi

The angles of Q4

Even though they are NOT similar figures...



...the same angles appear in Q4 and $T5 \cup T6$

The perimeter of Q4

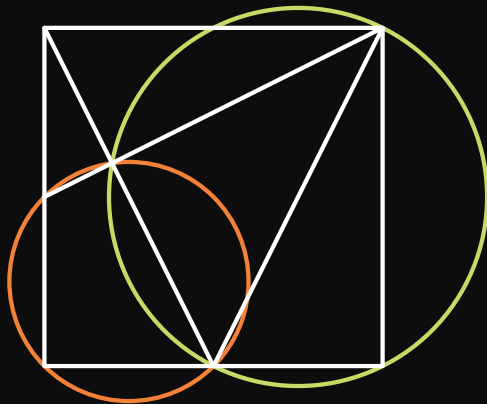
These three perimeters are in a geometric progression...



$$\frac{2\sqrt{5} + 4}{\sqrt{5} + 3} = \frac{3\sqrt{5} + 7}{2\sqrt{5} + 4} = \varphi = \frac{1 + \sqrt{5}}{2}$$

The circumcircles

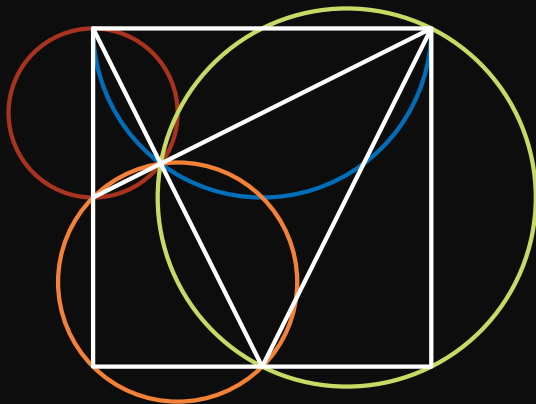
Since opposite angles add to π ...



...Q4 and $T5 \cup T6$ are cyclic quadrilaterals

The circumcircles

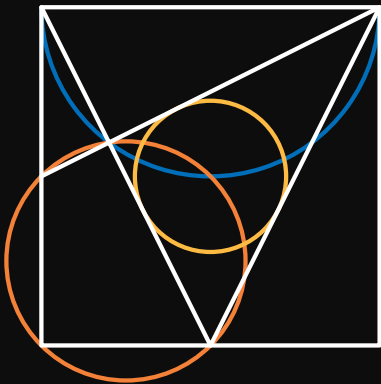
All circumcircles pass through a common point...



...and $C(T5 \cup T6)$ passes through the center of $C(Q4)$ and $C(T4)$

The circumcircles

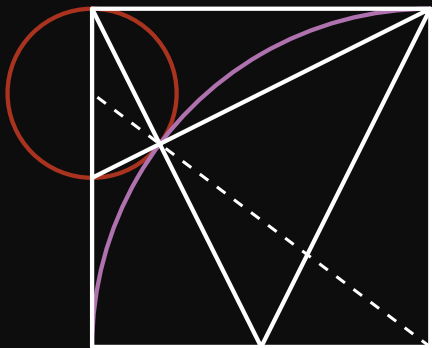
These circumcircles intersect at the square's center...



...which happens to be T6's incenter

Tangent circles

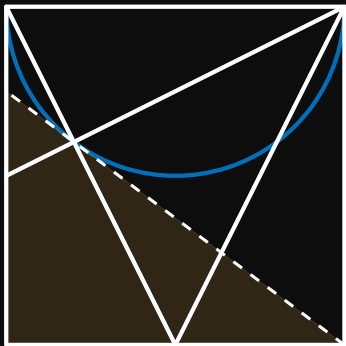
These three points are aligned...



...and these two circles are tangent

Tangent circles

The line is tangent to this circle...



...and the right triangle below is an Egyptian Triangle

Tangent circles

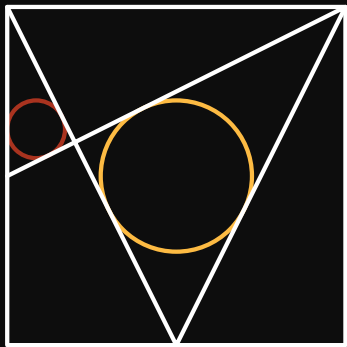
The radius of these three circles are in ratio $1:\varphi:\varphi^2$



where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

Tangent circles

The radius of these two circles are in ratio $1:\varphi^2$



where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

Tangent circles

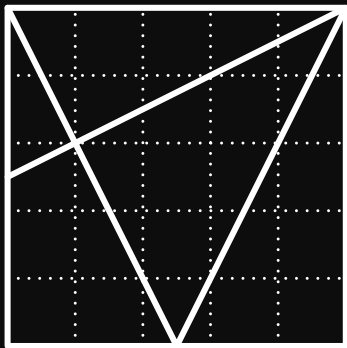
The radius of these two circles are in ratio $1:\varphi^2$



where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

The underlying grid

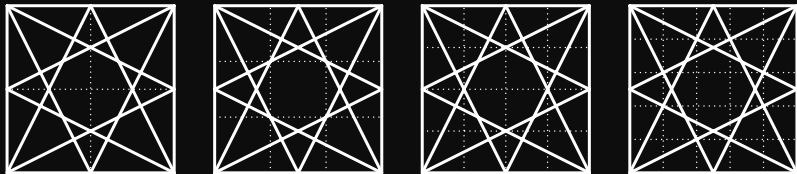
Using the intersection point of the Egyptian Tangram...



...you can divide the square into 5×5 smaller squares!

The underlying grid

Using the intersection points of this figure...

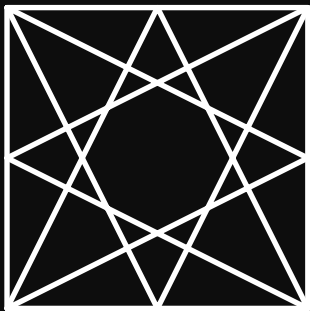


...you can divide the square into:

2×2 , 3×3 , 4×4 or 5×5 smaller squares!

The underlying grid

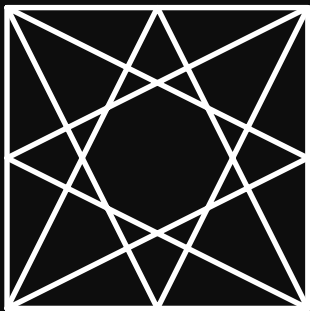
There are 32 egyptian triangles in this figure...



...they come in 4 sizes and there are 8 of each kind

The underlying grid

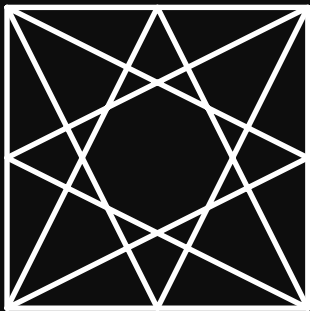
There are 24 $1:2:\sqrt{5}$ triangles in this figure...



...they come in 3 sizes and there are 8 of each kind

The underlying grid

There are 24 other triangles in this figure...



...of 3 different kinds (one of them comes in 2 sizes)

The underlying grid

The relative sizes of these polygons are...



Small Triangles: 1

Small Kites: 3

Whole Square: 120

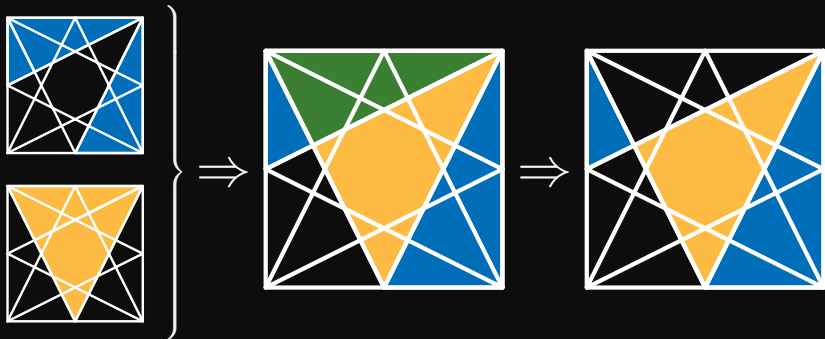
Big Triangles: 6

Big Kites: 8

Octagon: 20

The carpets theorem

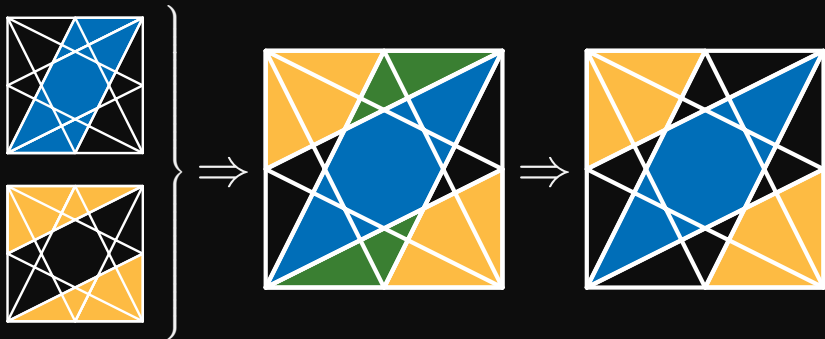
Since $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$



... $\text{Area}(\text{BLUE}-\text{GREEN}) = \text{Area}(\text{YELLOW}-\text{GREEN})$

The carpets theorem

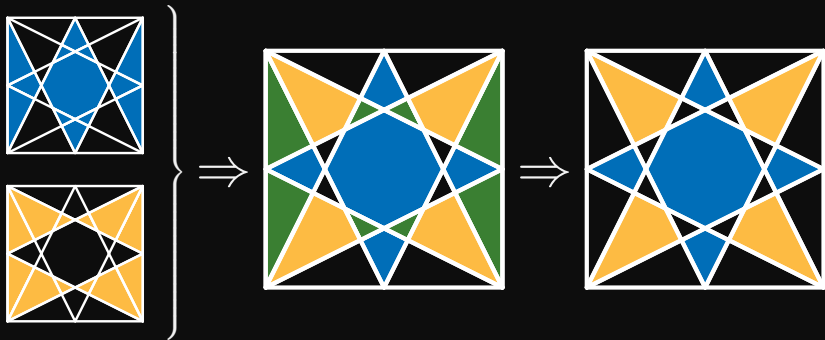
Since $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$



... $\text{Area}(\text{BLUE}-\text{GREEN}) = \text{Area}(\text{YELLOW}-\text{GREEN})$

The carpets theorem

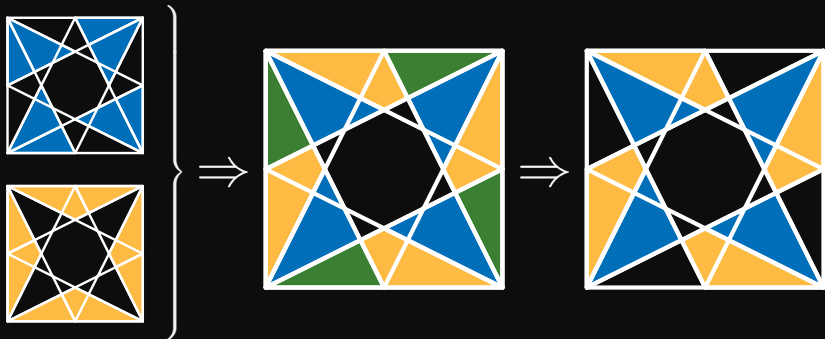
Since $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW}) \dots$



$\dots \text{Area}(\text{BLUE} - \text{GREEN}) = \text{Area}(\text{YELLOW} - \text{GREEN})$

The carpets theorem

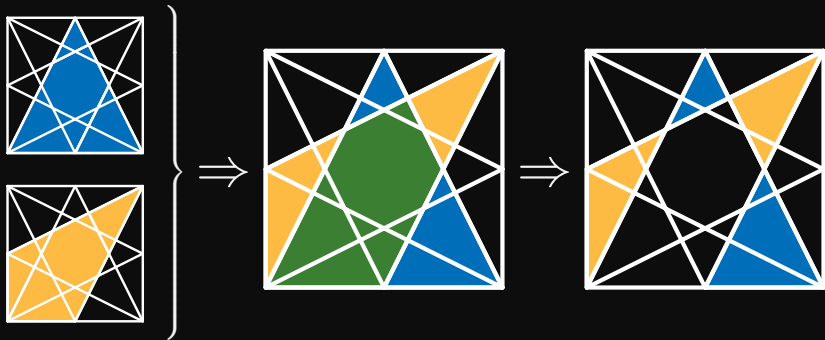
Since $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$



... $\text{Area}(\text{BLUE}-\text{GREEN}) = \text{Area}(\text{YELLOW}-\text{GREEN})$

The carpets theorem

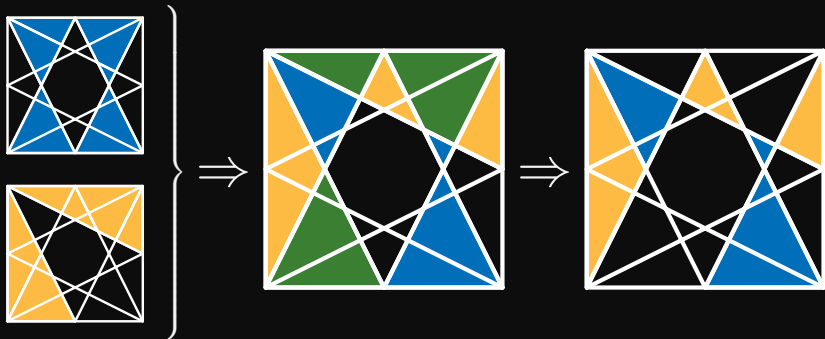
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The carpets theorem

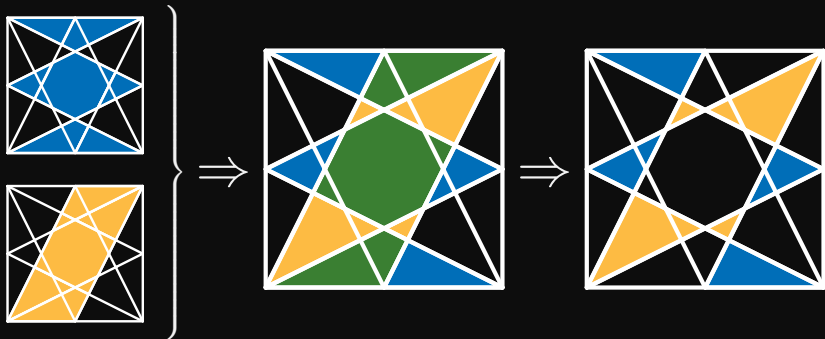
Since $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW}) \dots$



$\dots \text{Area}(\text{BLUE} - \text{GREEN}) = \text{Area}(\text{YELLOW} - \text{GREEN})$

The carpets theorem

Since $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$



... $\text{Area}(\text{BLUE}-\text{GREEN}) = \text{Area}(\text{YELLOW}-\text{GREEN})$

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