Useful Inequalities

For any pair of sorted sequences of real numbers of equal length ($x_0 \leq x_1 \leq \cdots \leq x_n$ and $y_0 \leq y_1 \leq \cdots \leq y_n$) and for any arbitrary permutation σ of their index set:

Rearrangement Inequality: $\boxed{\sum x_i y_{n-i} \;\leq\; \sum x_i y_{\sigma(i)} \;\leq\; \sum x_i y_i}$

with the equality holding if and only if the permutated y_i are equal.

For any pair of finite or infinite sequences of real numbers ($x_1, x_2, \ldots$ and $y_1, y_2, \ldots$) and for any set of non-negative weights $w_i \geq 0$:

Weighted Cauchy-Schwarz Inequality: $\boxed{\left(\sum w_i x_i y_i\right)^2 \;\leq\; \left(\sum w_i x_i^2\right)\left(\sum w_i y_i^2\right)}$

with equality if and only if $\vec{x}$ and $\vec{y}$ are linearly dependent.

For any set of nonnegative weights $w_i \geq 0$ such that $\sum w_i = 1$ and any set $x_1, \ldots, x_n$ of nonnegative real numbers:

Weighted AM-GM Inequality: $\boxed{\min\{x_i\} \;\leq\; \prod x_i^{w_i} \;\leq\; \sum w_i x_i \leq\; \max\{x_i\}}$

with the middle equality holding if and only if all x_i such that $w_i > 0$ are equal.

For any $m \times n$ matrix of nonnegative real numbers $x_{ij} \geq 0 \quad \forall\, i \in \{1, \ldots, m\};\ \forall\, j \in \{1, \ldots, n\}$ and any set of n nonnegative weights $w_j \geq 0$ such that $\sum w_j = 1$:

Generalized Hölder Inequality: $\boxed{\sum_{i=1}^{m}\left(\prod_{j=1}^{n} {x_{ij}}^{w_j}\right) \;\leq\; \prod_{j=1}^{n}\left(\sum_{i=1}^{m} x_{ij}\right)^{w_j}}$

with the equality holding if and only if two colums of the x_{ij} matrix are linearly dependent.

For any function $\phi(x)$ that is convex in a given interval, any set of points $x_1, x_2, \ldots$ of that interval and any set of nonnegative weights $w_i \geq 0$ such that $\sum w_i = 1$:

Weighted Jensen Inequality: $\boxed{\phi\left(\sum w_i x_i\right) \;\leq\; \sum w_i \phi(x_i)}$

A continuous function $\phi(x)$ is convex in a given interval if and only if, for any x_1, x_2 in the interval, it satisfies $\phi\left(\frac{x_1+x_2}{2}\right) \leq \frac{\phi(x_1)+\phi(x_2)}{2}$. In the cases where $\phi''(x)$ exists, $\phi(x)$ is convex if $\phi''(x) \;\geq\; 0$ in the interval.