

# The Egyptian Tangram



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**mmaca**

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# The Egyptian Tangram

# The Egyptian Tangram

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A square dissection firstly proposed as a tangram in:

Luna-Mota, C. (2019) *"El tangram egipci: diari de disseny"* Nou Biaix, 44

# Design process

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The Egyptian Tangram inspiration comes from the study of two other 5-piece tangrams...

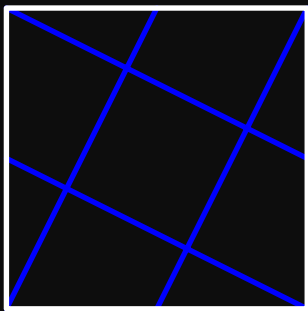
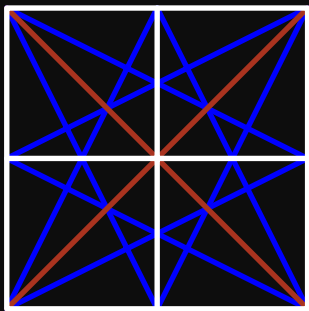


The “Five Triangles” & “Greek-Cross” tangrams

# Design process

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...and their underlying grids



The "Five Triangles" & "Greek-Cross" underlying grids

# Design process

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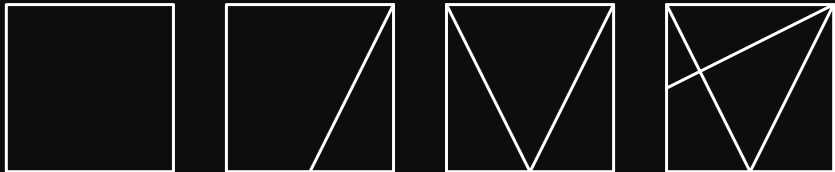
This simple *cut* let us build five interesting figures...



# Design process

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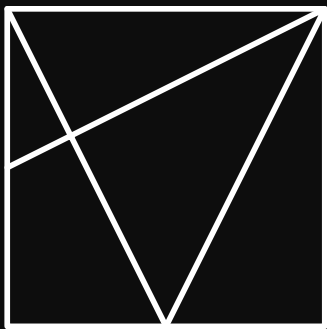
...so it looked like a good starting point for our heuristic incremental design process:



Take a square and keep adding “the most interesting straight cut” until you have a dissection with five or more pieces.

# Design process

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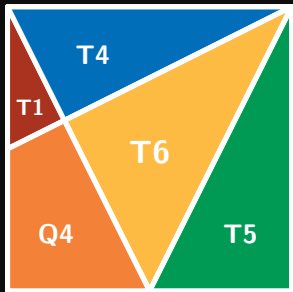


Straight cuts simplify creating an Egyptian Tangram from a square:

1. Connect the lower midpoint with the upper corners
2. Connect the left midpoint with the top right corner



# Promising features



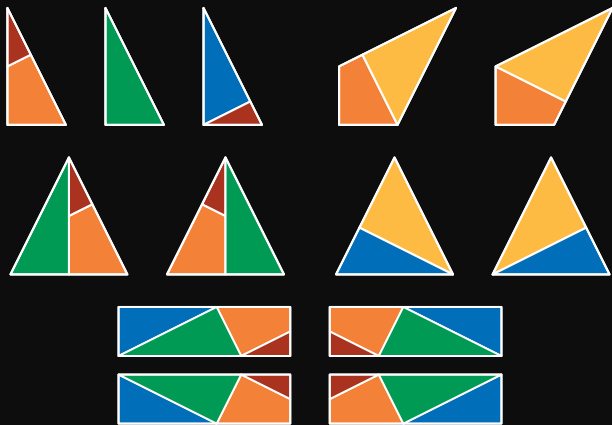
- Just five pieces
- All pieces are different
- All pieces are asymmetric
- Areas are integer and not *too different*
- All sides are multiples of 1 or  $\sqrt{5}$
- All angles are linear combinations of  $90^\circ$  and  $\alpha = \arctan\left(\frac{1}{2}\right) \approx 26,565^\circ$

Name	Area	Sides	Angles
T1	1	1, 2, $\sqrt{5}$	$90^\circ$ , $\alpha$ , $90^\circ - \alpha$
T4	4	2, 4, $2\sqrt{5}$	$90^\circ$ , $\alpha$ , $90^\circ - \alpha$
T5	5	$\sqrt{5}$ , $2\sqrt{5}$ , 5	$90^\circ$ , $\alpha$ , $90^\circ - \alpha$
T6	6	3, 4, 5	$90^\circ$ , $90^\circ - 2\alpha$ , $2\alpha$
Q4	4	1, 3, $\sqrt{5}$ , $\sqrt{5}$	$90^\circ$ , $90^\circ - \alpha$ , $90^\circ$ , $90^\circ + \alpha$

# Promising features

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Although all pieces are asymmetric and different, they often combine to make symmetric shapes



# Promising features

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This means that it is rare for an Egyptian Tangram figure to have a unique solution

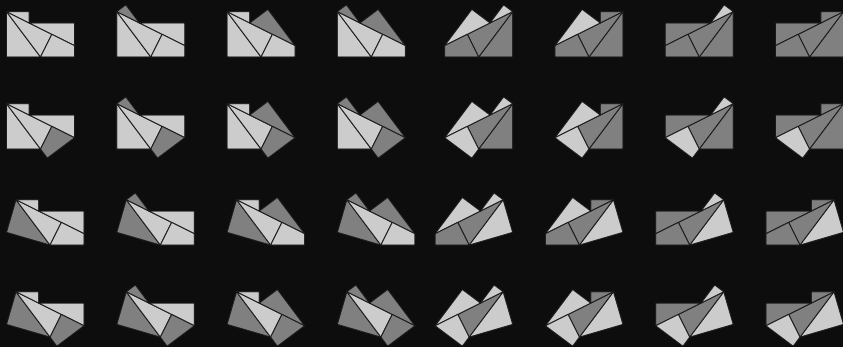


There are three different solutions for the square and, in all three cases, two corners of the square are built as a sum of acute angles!

# Promising features

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The asymmetry of the pieces also implies that each solution belongs to one of these equivalence classes:



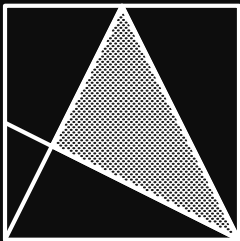
You cannot transform one of these figures into another without flipping a piece

# Historical precedents

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It turns out that this figure is not new...

Detemple, D. & Harold, S. (1996) *"A Round-Up of Square Problems"*



Problem 3

...but, to the best of our knowledge,  
nobody used it before **as a tangram**

# Historical precedents

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The name is not new either...



This dissection is often called “Egyptian Puzzle” or “Egyptian Tangram”

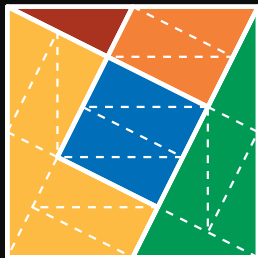
...but there is a good reason to consider  
our dissection the real “Egyptian Tangram”

(even if it was designed in Catalonia)

# Why we called it the *Egyptian* Tangram?

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The smallest pieces of the Chinese and Greek-Cross tangrams can be used to build all the other pieces...



...but you cannot do the same with  
the Egyptian Tangram because of T6

# Why we called it the *Egyptian* Tangram?

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Initially, T6 was considered as *the leftover piece* that results from cutting all these  $1:2:\sqrt{5}$  triangles from the borders of the square.

But it turned out to be a very well known triangle...



...an **Egyptian** Triangle (3:4:5)  
and, hence, the name of this tangram



# Puzzles & Activities

# Realistic figures

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Use all five pieces to make these figures:



Lightning



Sailing ship



Bow tie



Wooden hut



Caltrop



Snowmobile



Candle



Viking hat



Diamond



Moses basket



Erlenmeyer



3D brick



Witch hat



Arrow Sign



Sailboat

# Realistic figures

---

Use all five pieces to make these figures:



Gnome



Handmaid



Mountain range



Fish tail



Teddy bear



Cat



Dromedary



Cow



Snail



Fennec Fox



Penguin



Calf



Sea Turtle



Duck



Crow

# Remote control symbols

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Use all five pieces to make these symbols:



Rewind



Play/Pause



FFWD



Start



Stop



End

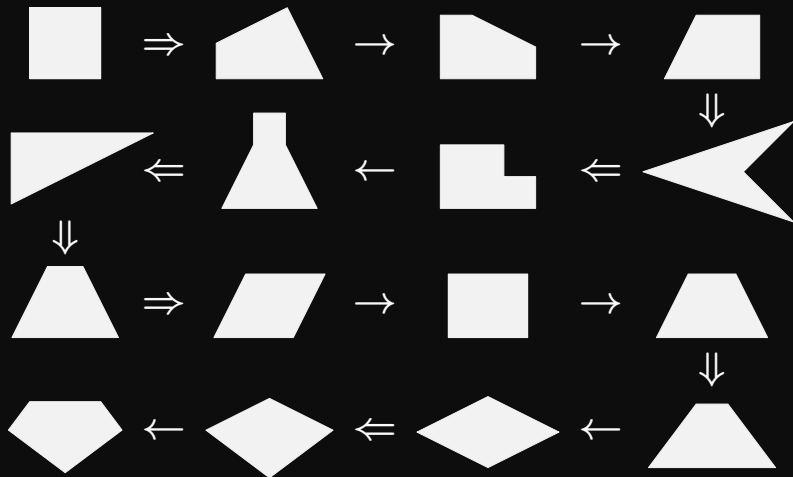


Volume

# Geometric figures

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Use all five pieces to make these figures:

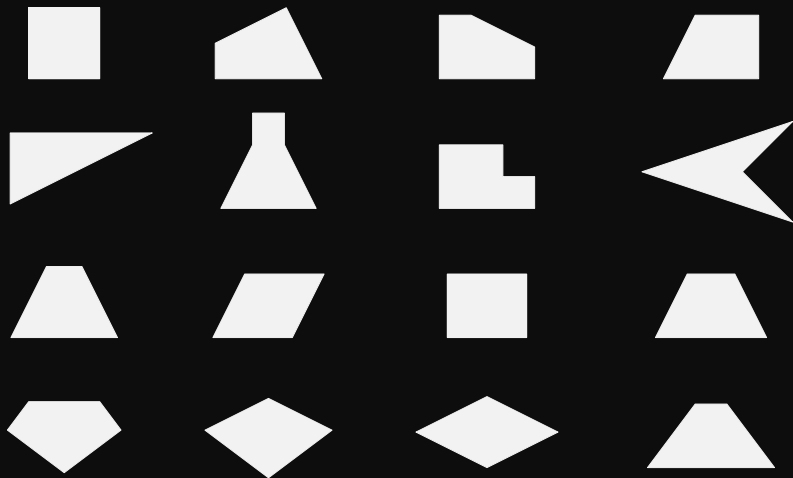


Complete the path moving just one or two pieces at a time

# Geometric figures

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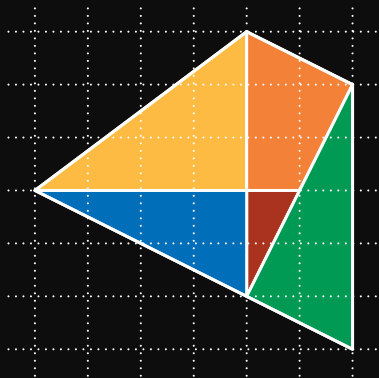
All of these figures, but one, have multiple solutions:



Could you find which is the only figure with unique solution?

# Geometric figures

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All the figures from the previous page, but one, can be drawn with their vertices lying on this lattice.

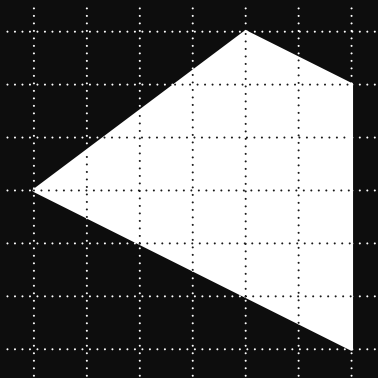
Moreover, they can be drawn with the vertices of all 5 pieces lying on the same lattice.

These conditions simplify the task of finding perimeters and areas.

Could you find which is the only figure that requires a finer-grained lattice?

# Geometric figures

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## Pythagorean Theorem:

$$\text{Top} = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\text{Left} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\text{Right} = \sqrt{5^2 + 0^2} = \sqrt{25} = 5$$

$$\text{Bottom} = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$$

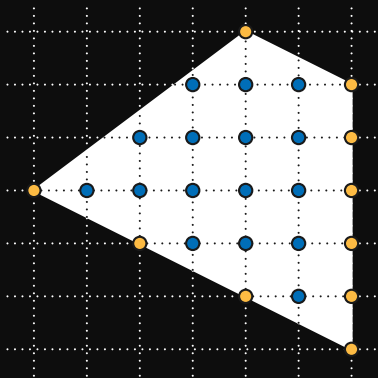
$$\text{Perimeter} = 10 + 4\sqrt{5}$$

Could you use the Pythagorean theorem to compute the perimeter of these figures?



# Geometric figures

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## Pick's Theorem:

lattice points in the interior = 16

lattice points on the boundary = 10

$$\begin{aligned}\text{Area} &= \text{interior} + \frac{\text{boundary}}{2} - 1 \\ &= 16 + \frac{10}{2} - 1 = 20\end{aligned}$$

Could you use Pick's theorem  
to compute the area of these figures?

# Triangles

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Could you prove that there are just 10 triangles you can make with one or more pieces of the Egyptian Tangram?

How many solutions could you find for each figure?



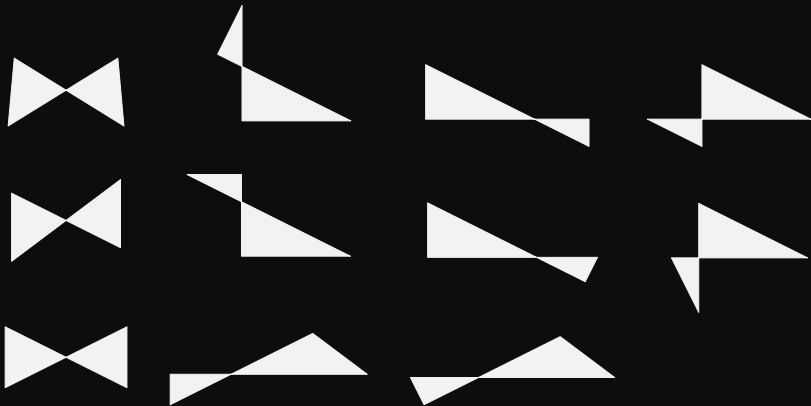
Top row areas: 20, 16, 9, 5, 4, 1

Bottom row areas: 15, 10, 10, 6

# Quadrilaterals

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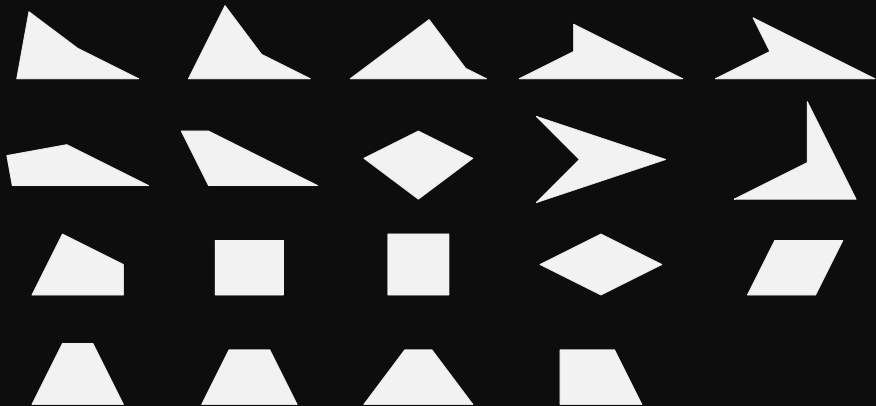
Could you prove that there are just 11 **complex quadrilaterals** you can make with all five pieces of the Egyptian Tangram?



# Quadrilaterals

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Simple quadrilaterals: Not self-intersecting



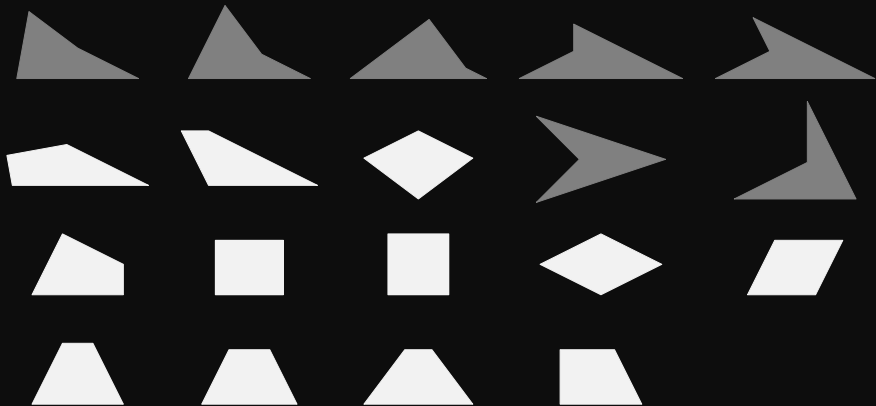
All simple quadrilaterals tile the plane!

$$\alpha + \beta + \gamma + \delta = 2\pi$$

# Quadrilaterals

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**Convex quadrilaterals:** All internal angles are smaller than  $\pi$

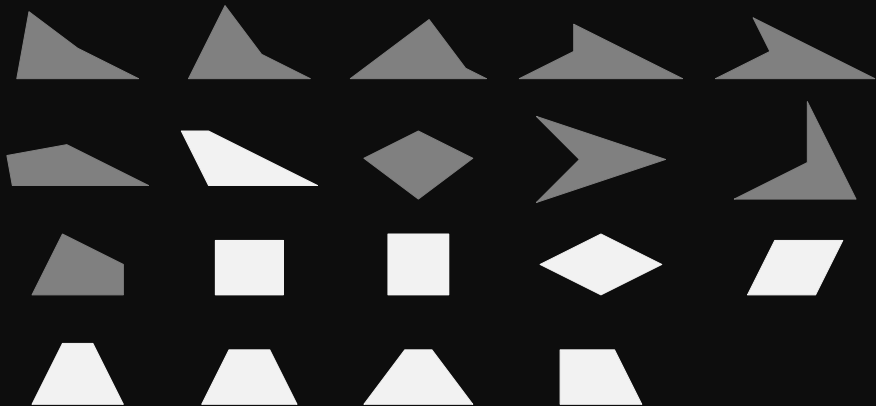


Law of Cosines:  $p^2q^2 = a^2c^2 + b^2d^2 - 2abcd \cos(\alpha + \gamma)$

# Quadrilaterals

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Trapeziums (UK) / Trapezoids (US): One pair of parallel sides

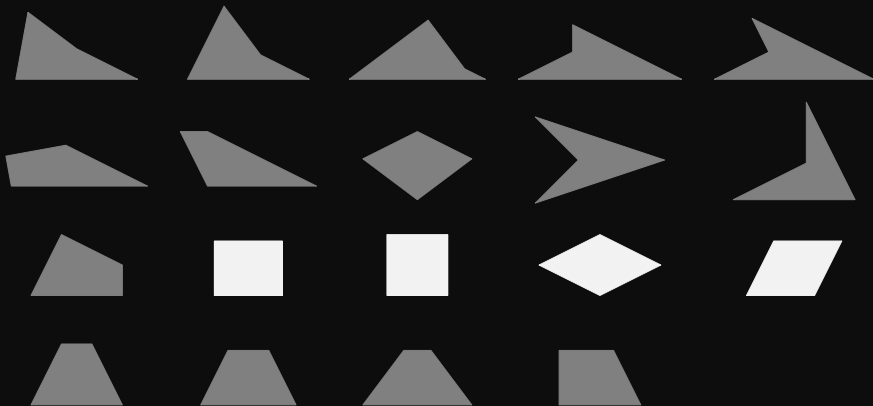


Trapezium/Trapezoid  $\Leftrightarrow$  Diagonals cut each other in the same ratio

# Quadrilaterals

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**Parallelograms:** Two pairs of parallel sides

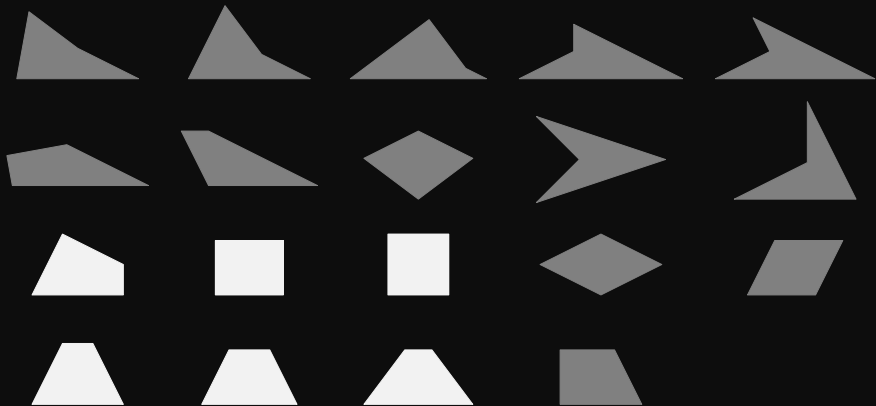


Parallelogram  $\Leftrightarrow$  Diagonals bisect each other  $\Leftrightarrow a^2 + b^2 + c^2 + d^2 = p^2 + q^2$

# Quadrilaterals

---

**Cyclic quadrilaterals:** All vertices lie on a circle



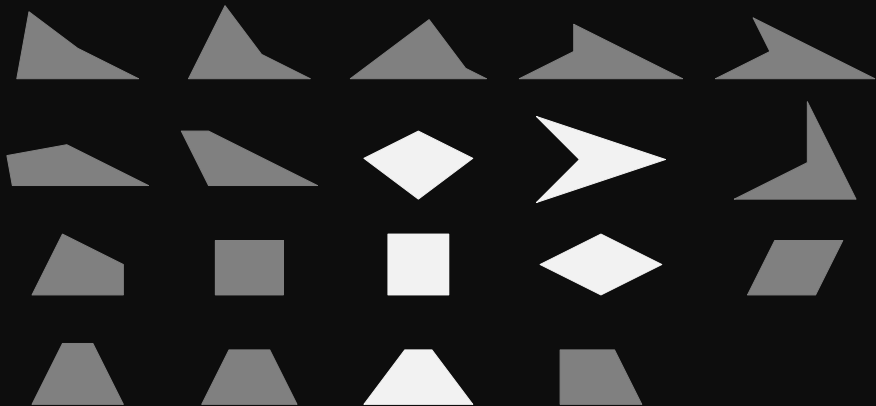
$$\text{Cyclic} \Leftrightarrow \alpha + \gamma = \beta + \delta$$



# Quadrilaterals

---

**Tangential quadrilaterals:** All sides are tangent to a circle

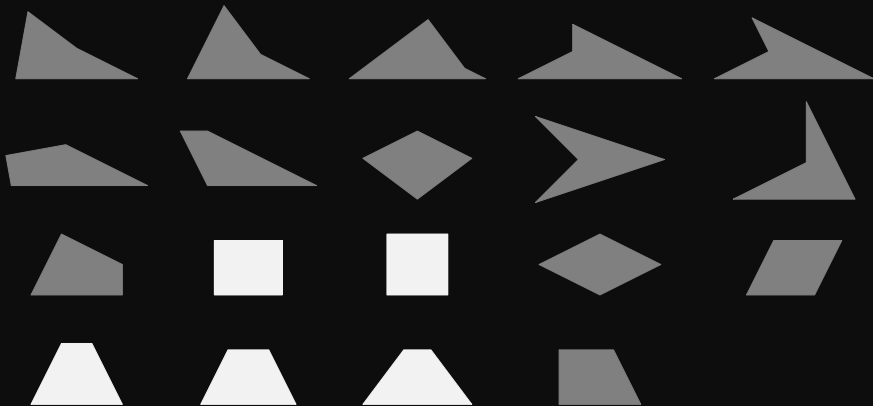


$$\text{Tangential} \iff a + c = b + d$$

# Quadrilaterals

---

**Isosceles Trapezoids:** Two pairs of adjacent angles are equal

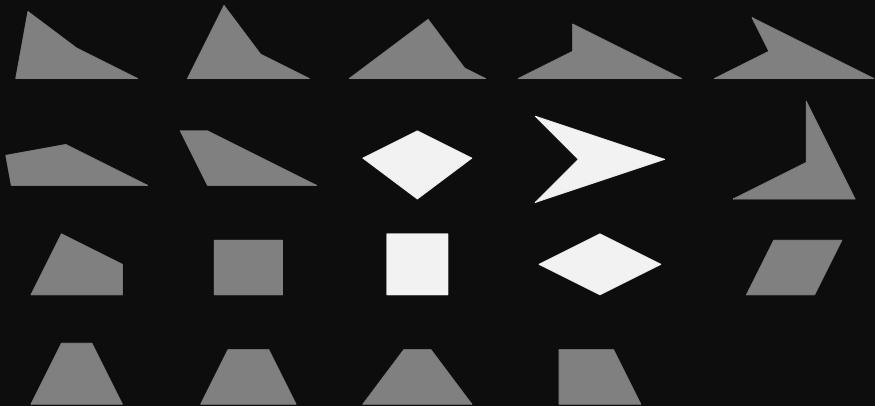


Isosceles trapezoids  $\Leftrightarrow$  Cyclic quadrilaterals with equal diagonals

# Quadrilaterals

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**Darts & Kites:** Two pairs of adjacent sides are equal

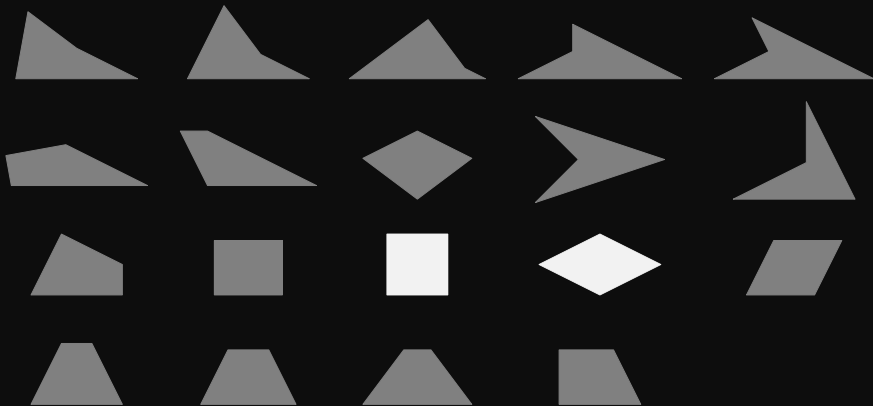


Darts/Kites  $\Leftrightarrow$  Tangential quadrilaterals with perpendicular diagonals

# Quadrilaterals

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**Rhombi:** All sides are equal

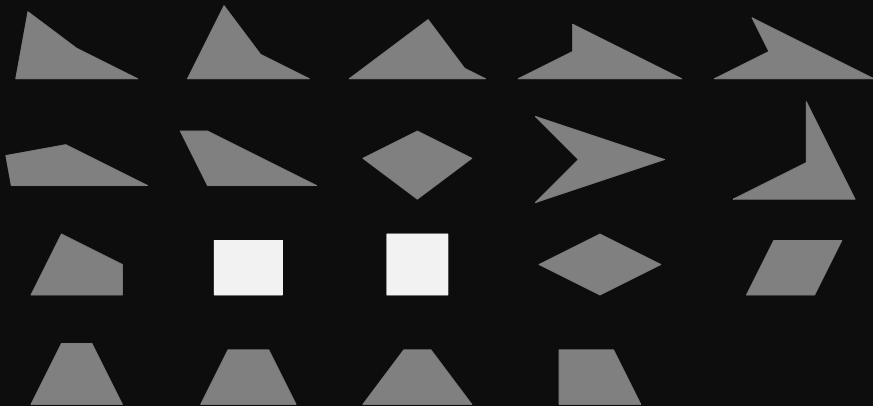


Rhombi  $\Leftrightarrow$  Parallelograms with perpendicular diagonals

# Quadrilaterals

---

**Rectangles:** All angles are equal

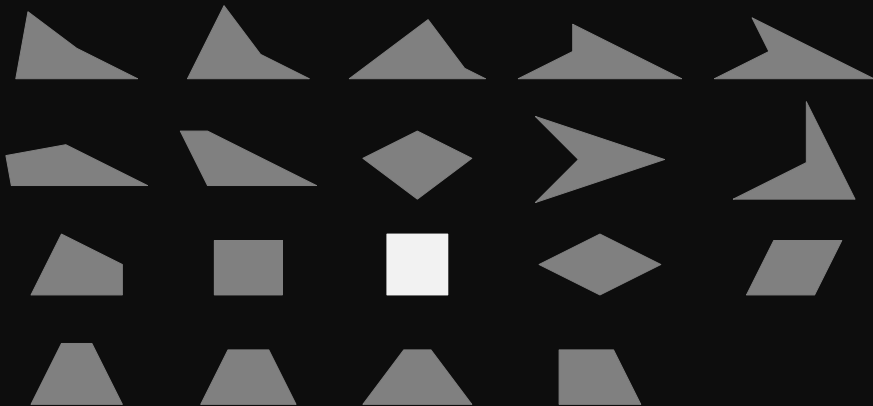


Rectangles  $\Leftrightarrow$  Parallelograms with equal diagonals

# Quadrilaterals

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Squares: Regular quadrilaterals



Among all quadrilaterals, squares maximize the *Area:Perimeter* ratio

# The three solutions of the square

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Could you prove that there are just three different solutions for the square?



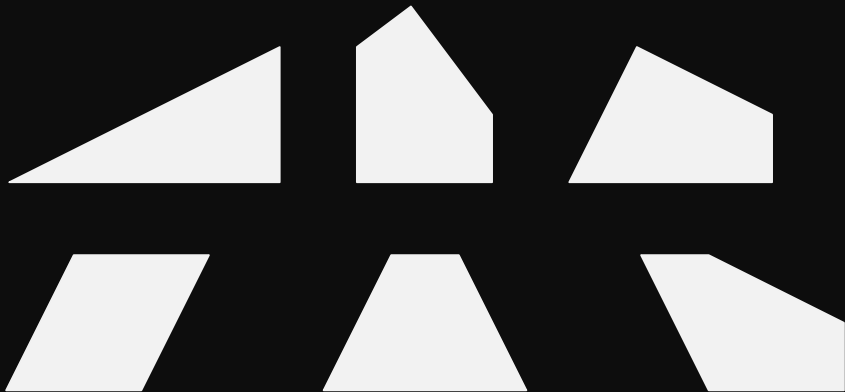
What is the area of this square? What is its perimeter?

How many times do you find  $\sqrt{5}$  in the Egyptian Tangram pieces?

# Figures with seven solutions

---

Could you find seven different solutions for each of these figures?





# Figures with unique solutions

---

Could you prove that there is only one solution for each of these figures?



# Missing triangle paradox

---

Both figures use all 5 pieces...



Where is the missing triangle?

# Missing triangle paradox

---

Both figures use all 5 pieces...



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Where is the missing triangle?

# Missing triangle paradox

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Both figures use all 5 pieces...



Where is the missing triangle?



# Missing triangle paradox

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Both figures use all 5 pieces...

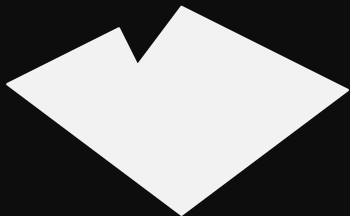
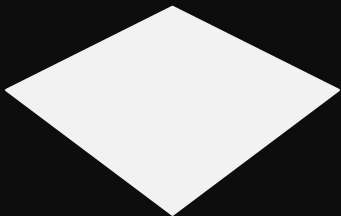


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# Missing triangle paradox

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Both figures use all 5 pieces...

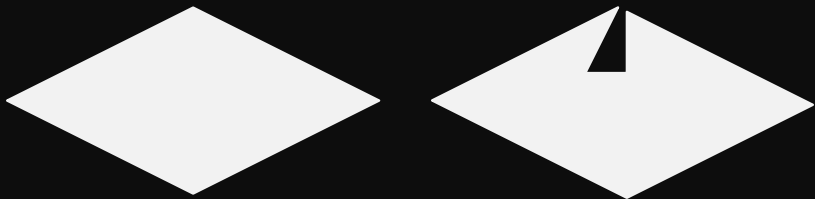


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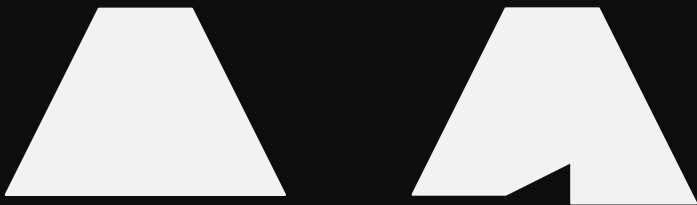


Where is the missing triangle?

# Missing triangle paradox

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# Missing triangle paradox

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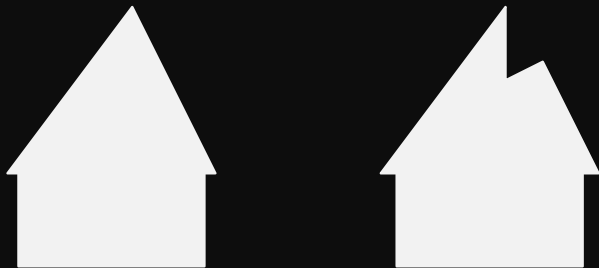


Where is the missing triangle?

# Missing triangle paradox

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Both figures use all 5 pieces...

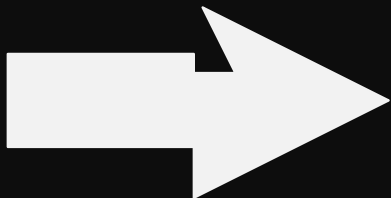
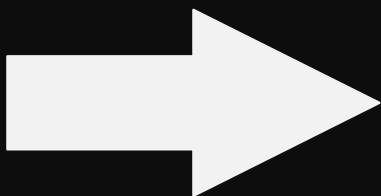


Where is the missing triangle?

# Missing triangle paradox

---

Both figures use all 5 pieces...



Where is the missing triangle?

# Missing rectangle paradox

---

Both figures use all 5 pieces...

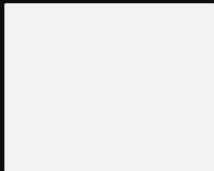


Where is the missing rectangle?

# Missing rectangle paradox

---

Both figures use all 5 pieces...



Where is the missing rectangle?

# Missing rectangle paradox

---

Both figures use all 5 pieces...



Where is the missing rectangle?

# Missing square paradox

---

Both figures use all 5 pieces...

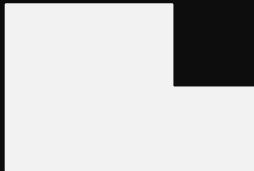


Where is the missing square?

# Missing square paradox

---

Both figures use all 5 pieces...



Where is the missing square?



# Missing square paradox

---

Both figures use all 5 pieces...



Where is the missing square?

# Sum of similar figures

---

Use all 5 pieces to make the single figure in the LHS,  
then use them to make the two figures on the RHS



In both equations, the figures are similar and areas are in ratio 5 : 4 : 1

# Golden Rectangles

---

The dashed rectangle proportions are  $1:\varphi$



where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio

# Golden Rectangles

---

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# Golden Rectangles

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# Golden Rectangles

---

The dashed rectangles proportions are  $1:\varphi$



where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio

# Golden Rectangles

---

There are 4 golden rectangles hidden in this figure



Could you spot them?

# Golden Rectangles

---

There are 4 golden rectangles hidden in this figure



Could you spot them?



# Golden Rectangles

---

There are 4 golden rectangles hidden in this figure



Could you spot them?

# Golden Rectangles

---

There are 5 golden rectangles hidden in this figure



Could you spot them?

# Golden Rectangles

---

You can find golden rectangles of 14 different types

Type	Proportions	Type	Proportions
<b>A</b>	$3 - \sqrt{5} : 2\sqrt{5} - 4$	<b>H</b>	$2\sqrt{5} : 5 - \sqrt{5}$
<b>B</b>	$\sqrt{5} - 1 : 3 - \sqrt{5}$	<b>I</b>	$3 + \sqrt{5} : 1 + \sqrt{5}$
<b>C</b>	$2 : \sqrt{5} - 1$	<b>J</b>	$6 : 3\sqrt{5} - 3$
<b>D</b>	$2\sqrt{5} - 2 : 6 - 2\sqrt{5}$	<b>K</b>	$2 + 2\sqrt{5} : 4$
<b>E</b>	$5 - \sqrt{5} : 3\sqrt{5} - 5$	<b>L</b>	$5 + \sqrt{5} : 2\sqrt{5}$
<b>F</b>	$1 + \sqrt{5} : 2$	<b>M</b>	$1 + 3\sqrt{5} : 7 - \sqrt{5}$
<b>G</b>	$4 : 2\sqrt{5} - 2$	<b>N</b>	$4 + 2\sqrt{5} : 3 + \sqrt{5}$

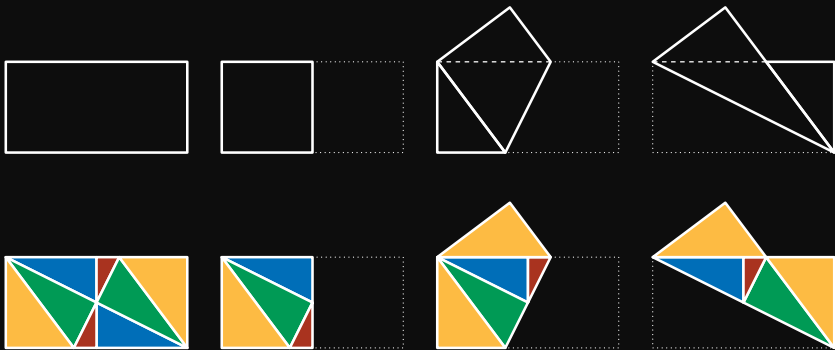
Could you build an example of each type?

# Simplified Tangrams

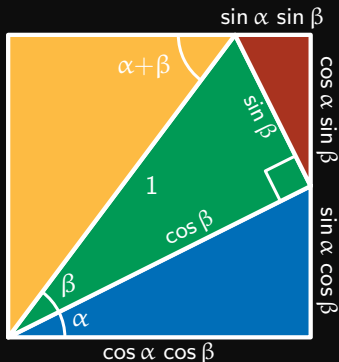
# The Egyptian Four-Triangle-Tangram

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T1, T4, T5 & T6 appear naturally  
when you fold a 2:1 rectangle



# The Egyptian Four-Triangle-Tangram

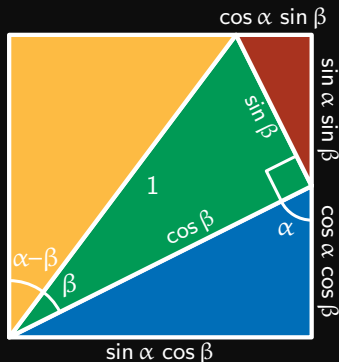


You can use this figure to prove these identities:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

# The Egyptian Four-Triangle-Tangram

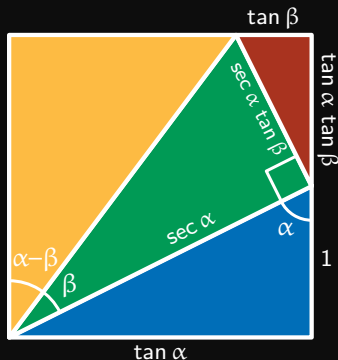
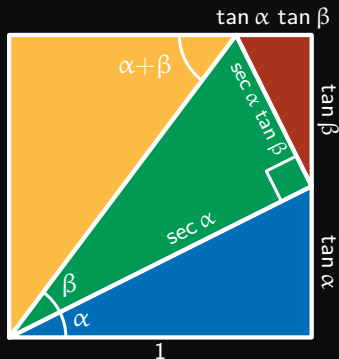


You can use this figure to prove these identities:

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

# The Egyptian Four-Triangle-Tangram



You can use these figures to prove these identities:

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

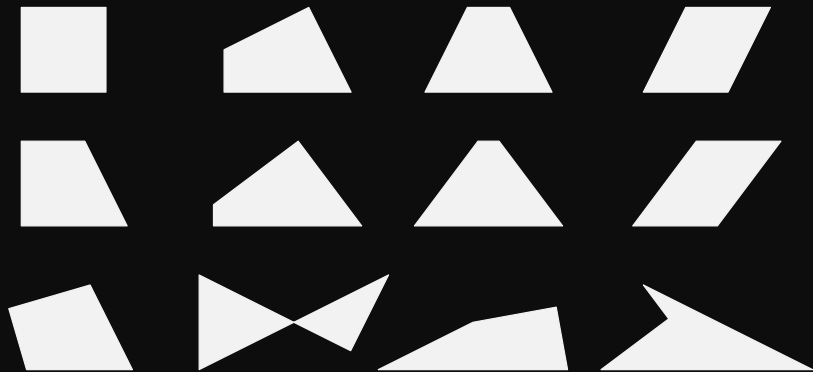
$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$



# The Egyptian Four-Triangle-Tangram

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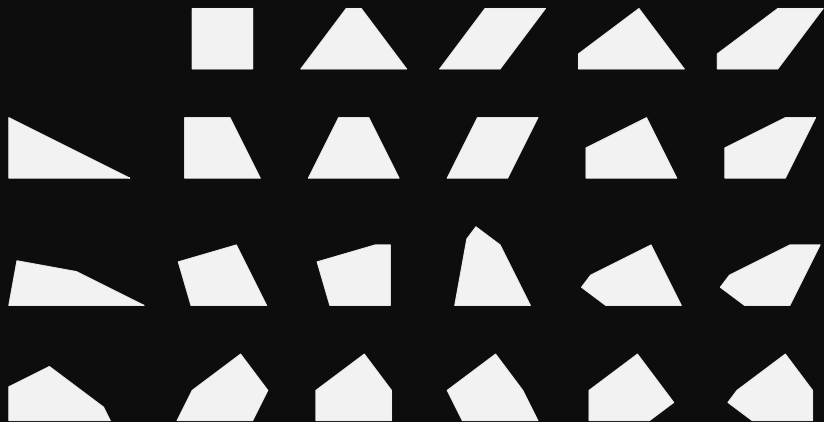
You can make 12 quadrilaterals using T1, T4, T5 & T6



# The Egyptian Four-Triangle-Tangram

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You can make 23 convex figures using T1, T4, T5 & T6



# The Egyptian Four-Triangle-Tangram

---

You can find golden rectangles of 7 different types using just T1, T4, T5 & T6

Type	Proportions
<b>A</b>	$3 - \sqrt{5} : 2\sqrt{5} - 4$
<b>C</b>	$2 : \sqrt{5} - 1$
<b>F</b>	$1 + \sqrt{5} : 2$
<b>G</b>	$4 : 2\sqrt{5} - 2$
<b>H</b>	$2\sqrt{5} : 5 - \sqrt{5}$
<b>K</b>	$2 + 2\sqrt{5} : 4$
<b>L</b>	$5 + \sqrt{5} : 2\sqrt{5}$

Could you build an example of each type?

# The Egyptian Three–Triangle–Tangram

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Given any right triangle with sides:  $a \leq b \leq c$



you can draw three similar triangles:  $(a, b, c)$ ,  $(x, h, a)$  &  $(h, y, b)$

and use them to prove the **Altitude Theorem**:  $h^2 = x \cdot y$

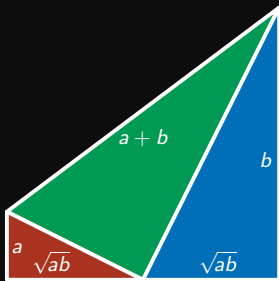
and the **Leg Theorems**:  $a^2 = x \cdot c$  &  $b^2 = y \cdot c$

(T1, T4 & T5 verify this relationship for:  $a = \sqrt{5}$ ,  $b = 2\sqrt{5}$  &  $c = 5$ )

# The Egyptian Three-Triangle-Tangram

---

Since  $\frac{a}{\sqrt{ab}} = \frac{\sqrt{ab}}{b}$ , these three right triangles are similar...



...and you can use this figure to prove the **AM-GM Inequality**:

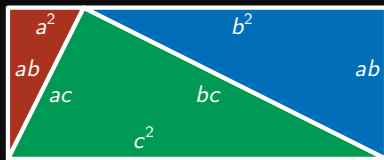
$$\frac{a + b}{2} \geq \sqrt{ab}$$

(T1, T4 & T5 verify this relationship for:  $a = 1$  &  $b = 4$ )

# The Egyptian Three–Triangle–Tangram

---

Given any right triangle with sides:  $a \leq b \leq c$



you can make a rectangle with three similar triangles:

$$(a^2, ab, ac), (ab, b^2, bc) \text{ \& } (ac, bc, c^2)$$

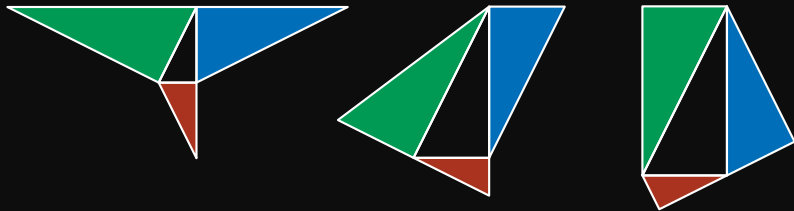
and compare the top ( $a^2 + b^2$ ) and the bottom ( $c^2$ ) sides of the rectangle to prove the **Pythagorean Theorem**

(T1, T4 & T5 verify this relationship for:  $a = 1$ ,  $b = 2$  &  $c = \sqrt{5}$ )

# The Egyptian Three–Triangle–Tangram

---

Since  $\text{area}(T1) + \text{area}(T4) = \text{area}(T5) \dots$

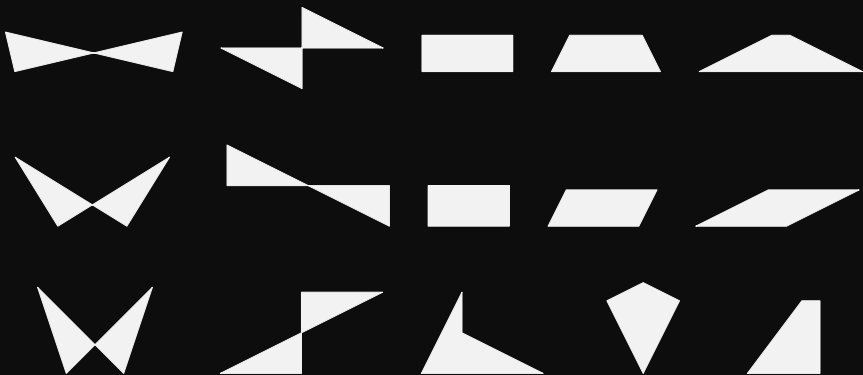


...you can verify 3 cases of the **Pythagorean Theorem**  
(and these particular cases turn out to be the T1, T4 & T5 right triangles!)

# The Egyptian Three-Triangle-Tangram

---

You can make 15 quadrilaterals using just T1, T4 & T5



See also: Brügger, G. (1984) *"Three-Triangle-Tangram"*, Bit, 24



# The Egyptian Three-Triangle-Tangram

---

You can make 11 convex figures using just T1, T4 & T5



See also: Brügger, G. (1984) *"Three-Triangle-Tangram"*, Bit, 24

# The Egyptian Three-Triangle-Tangram

---

You can find golden rectangles of 5 different types using just T1, T4 & T5

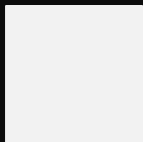
Type	Proportions
C	$2 : \sqrt{5}-1$
F	$1+\sqrt{5} : 2$
G	$4 : 2\sqrt{5}-2$
H	$2\sqrt{5} : 5-\sqrt{5}$
K	$2+2\sqrt{5} : 4$

Could you build an example of each type?

# A Four Q4 Puzzle

---

It is easy to make each of these figures with four copies of Q4:



But... Could you make two squares **simultaneously**?  
Could you make two golden rectangles **simultaneously**?

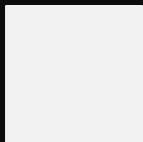


See also: **Make a Square** puzzle by Interlocking Puzzles LLC

# A Four T4 Puzzle

---

It is easy to make each of these figures with four copies of T4:



But... Could you make two squares **simultaneously**?  
Could you make two golden rectangles **simultaneously**?

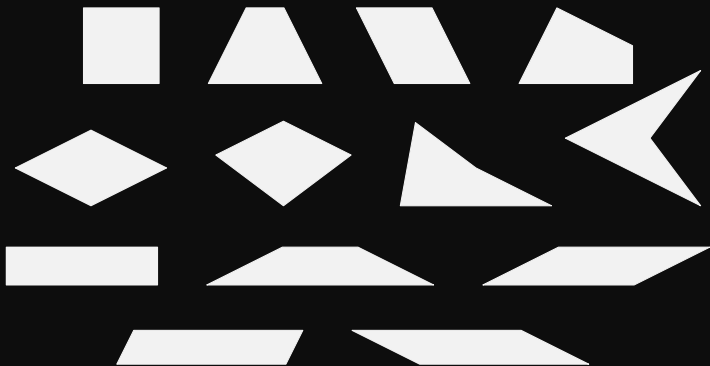


See also: **Four Triangles** by Don Steward

# A Four T4 Puzzle

---

You could also build many other figures with four T4s...



...including 13 different quadrilaterals!

From: **Four Triangles** by Don Steward

# A Five T6 Puzzle

---

Could you make a symmetrical figure with five copies T6?

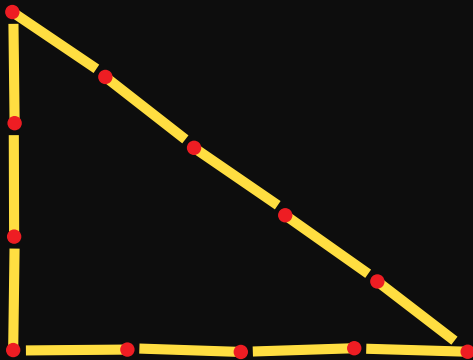


See also: **Curious and Interesting Triangles** by Donald Bell

# Matchsticks Puzzles

---

Since all T6's side lengths are integer...

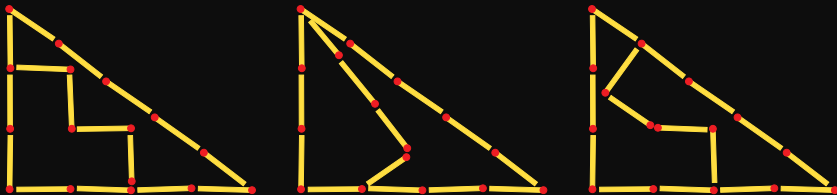


...you could draw it using matchsticks

# Matchsticks Puzzles

---

With 4 matchsticks, it is easy to divide T6 into two polygons with integer side lengths and equal area



But, could you do it using just 2 or 3 matchsticks?



# Mathematical Properties

# $\varphi$ and $\sqrt{5}$ are irrational



This is a **golden rectangle**, which means that  $\frac{\text{base}}{\text{height}} = \varphi$  is the **golden ratio**.

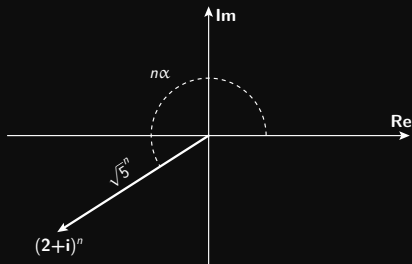
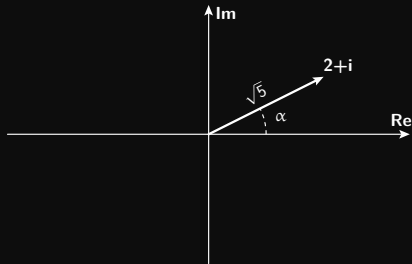
If we remove a square, what remains is also a golden rectangle:  $\frac{\text{height}}{\text{base}-\text{height}} = \varphi$



If we assume that  $\varphi = \frac{b}{h}$ , with  $b$  and  $h$  coprime integers, then  $\varphi = \frac{h}{b-h}$  is an equivalent fraction, with a smaller integer numerator and a smaller integer denominator, which is absurd. Therefore, our initial assumption must be false.

And, since  $\varphi = \frac{1+\sqrt{5}}{2}$  is irrational,  $2\varphi - 1 = \sqrt{5}$  must be irrational too.

# $\arctan(1/2)$ is irrational



$\arctan(\frac{1}{2})$  is not a rational multiple of  $\pi$ .

If it were, then for some integer  $n > 0$ , we would have  $(2+i)^n \in \mathbb{R}$ .

But if we look at the imaginary part of these numbers,  $a_n = \text{Im}((2+i)^n)$ , we can prove that this sequence satisfies the recurrence relation:

$$a_{n+2} = 4a_{n+1} - 5a_n \quad \forall n > 0$$

But  $a_1 = 1$ ,  $a_2 = 4$  and, by induction:

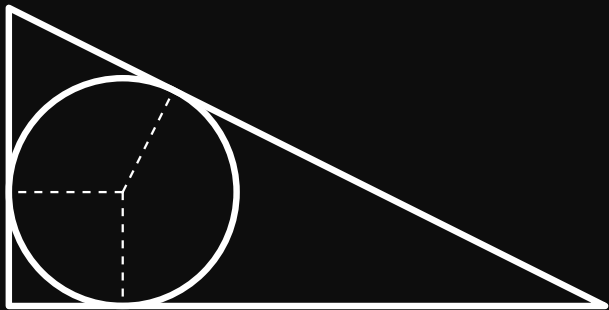
$$a_n \equiv \begin{cases} 1 \pmod{5} & \forall \text{ odd } n > 0 \\ 4 \pmod{5} & \forall \text{ even } n > 0 \end{cases}$$

therefore,  $(2+i)^n \notin \mathbb{R} \quad \forall n > 0$ .

# The $1:2:\sqrt{5}$ incenter

---

If the inradius of a  $1:2:\sqrt{5}$  triangle is 1...



...its shorter leg measures  $\varphi + 1 = \varphi^2 = \frac{3+\sqrt{5}}{2}$

# The 3:4:5 incenter

---

If we overlay T6 and T1 as shown in the figure...

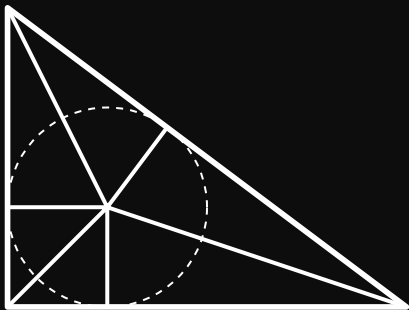


...a T1 vertex lies on the incenter of T6

## Dissecting 3:4:5

---

You can use this dissection of T6 to prove that...



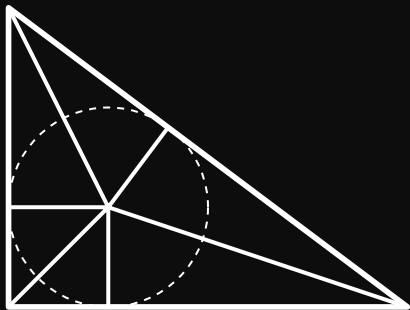
$$\pi = \arctan(1) + \arctan(2) + \arctan(3)$$

(consider the sum of the angles touching the incenter of T6 and divide by 2)

## Dissecting 3:4:5

---

You can use this dissection of T6 to prove that...



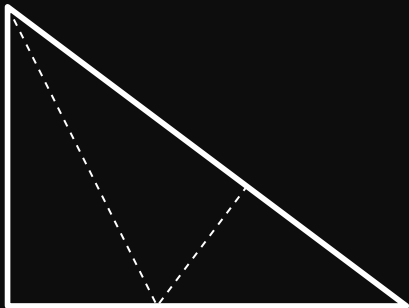
$$\frac{\pi}{2} = \arctan\left(\frac{1}{1}\right) + \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$$

(consider the sum of the angles touching the vertices of T6 and divide by 2)

# Dissecting 3:4:5

---

You can dissect a 3:4:5 triangle into...



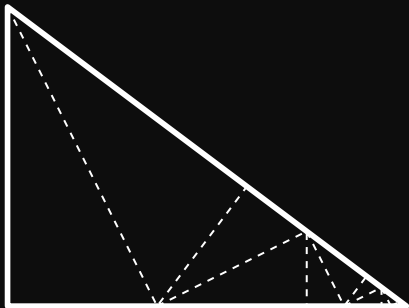
...a 3:4:5 triangle and  
two congruent  $1:2:\sqrt{5}$  triangles



# Dissecting 3:4:5

---

Iterating this dissection of T6 you can prove that...



$$\sum_{n=1}^{\infty} \frac{18}{4^n} = 6$$

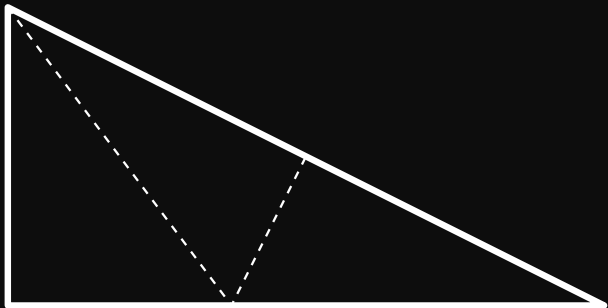
or, equivalently,

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3}$$

# Dissecting $1:2:\sqrt{5}$

---

You can dissect a  $1:2:\sqrt{5}$  triangle into...

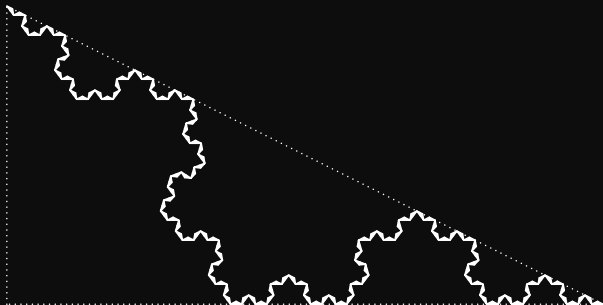


...a  $3:4:5$  triangle and  
two congruent  $1:2:\sqrt{5}$  triangles

# Dissecting $1:2:\sqrt{5}$

---

Removing the 3:4:5 triangle and iterating this dissection...

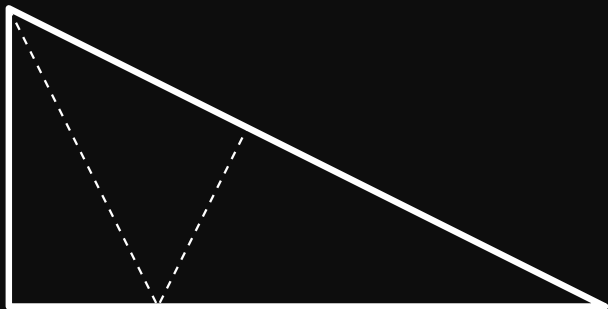


...produces a variant of the **Koch curve** fractal

# Dissecting $1:2:\sqrt{5}$

---

You can dissect a  $1:2:\sqrt{5}$  triangle into...

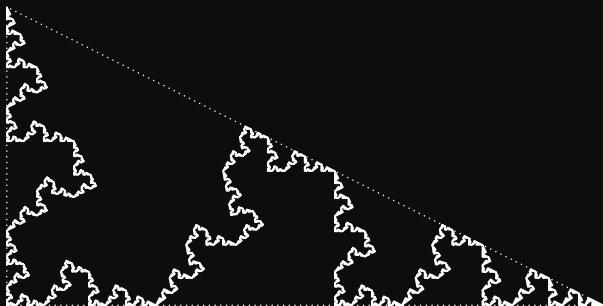


...a  $3:4:5$  triangle and  
two different  $1:2:\sqrt{5}$  triangles

# Dissecting $1:2:\sqrt{5}$

---

Removing the 3:4:5 triangle and iterating this dissection...

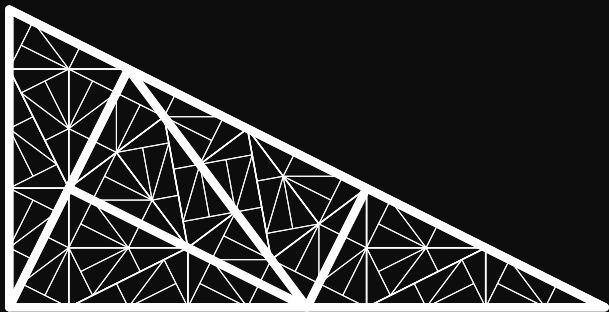


...produces a variant of the **Minkowski sausage** fractal

# Dissecting $1:2:\sqrt{5}$

---

You can dissect a  $1:2:\sqrt{5}$  triangle into...

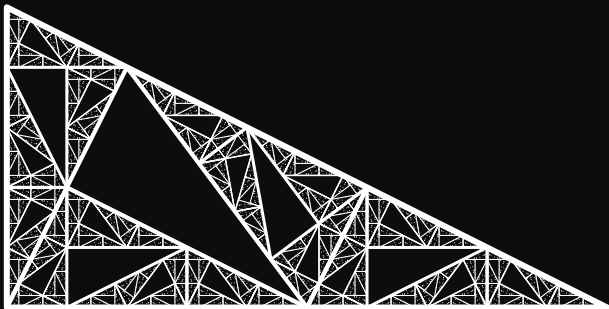


...five congruent  $1:2:\sqrt{5}$  triangles  
and iterate to get the **Pinwheel tiling** of the plane

# Dissecting $1:2:\sqrt{5}$

---

You can dissect a  $1:2:\sqrt{5}$  triangle into...

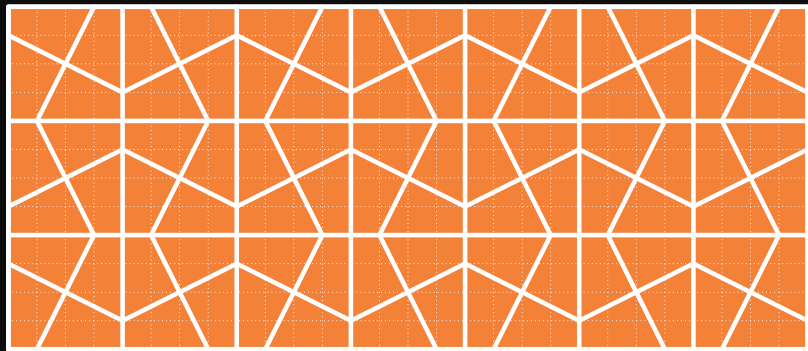


...five congruent  $1:2:\sqrt{5}$  triangles, remove the central one and iterate to get the **Pinwheel fractal**

## Q4 tilings

---

Two copies of Q4 form a pentagon that can be used...



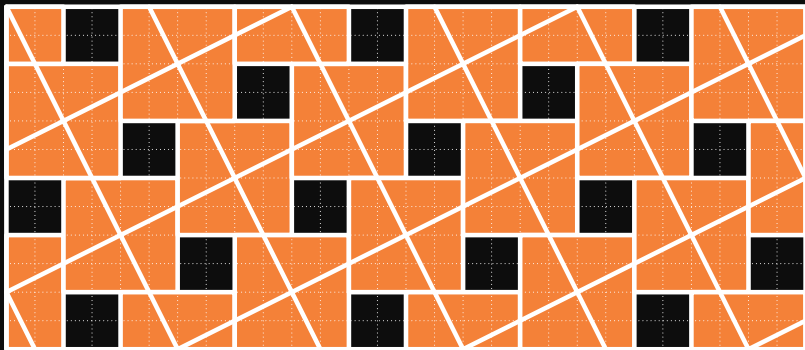
...to make this variant of the **Cairo Tiling** of the plane



# Q4 tilings

---

You can use this **Pythagorean Tiling** to verify...



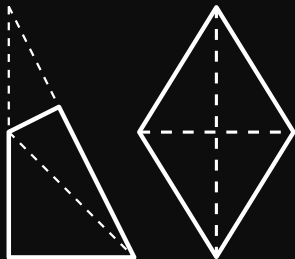
...that T4 satisfies the **Pythagorean Theorem**

# The angles of Q4

---

The angles  $90 - \alpha$  and  $90 + \alpha$  that appear in Q4  
also appear in the **Golden Rhombus**

(a rhombus whose diagonals are in proportion  $1:\varphi$ , with  $\varphi = \frac{1+\sqrt{5}}{2}$ )



$$90 + \alpha = 2 \cdot \arctan(\varphi) = \arctan(1) + \arctan(3)$$

$$90 - \alpha = 2 \cdot \arctan\left(\frac{1}{\varphi}\right) = \arctan(2)$$

The faces of the **rhombic triacontahedron** and  
the **rhombic hexecontahedron** are Golden Rhombi

# The angles of Q4

---

Even though they are NOT similar figures...



...the same angles appear in Q4 and  $T5 \cup T6$

# The perimeter of Q4

---

These three perimeters are in a geometric progression...

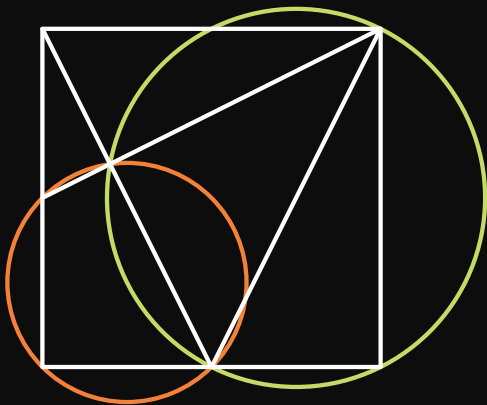


$$\frac{2\sqrt{5} + 4}{\sqrt{5} + 3} = \frac{3\sqrt{5} + 7}{2\sqrt{5} + 4} = \varphi = \frac{1 + \sqrt{5}}{2}$$

# The circumcircles

---

Since opposite angles add to  $\pi$ ...

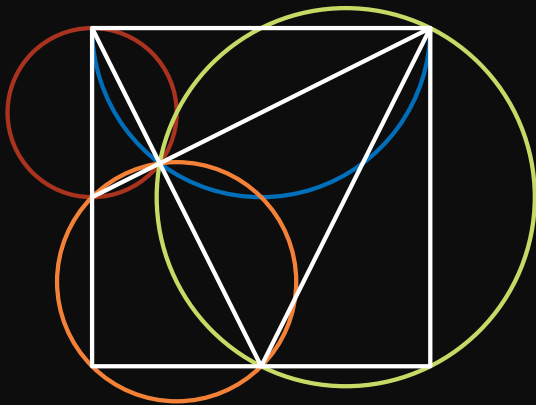


... $C(Q_4)$  and  $C(T_5 \cup T_6)$  are cyclic quadrilaterals

# The circumcircles

---

All circumcircles pass through a common point...

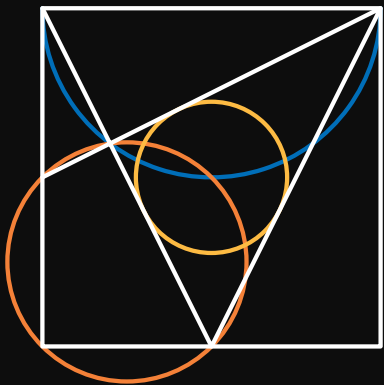


...and  $C(T5 \cup T6)$  passes through the center of  $C(Q4)$  and  $C(T4)$

# The circumcircles

---

These circumcircles intersect at the square's center...

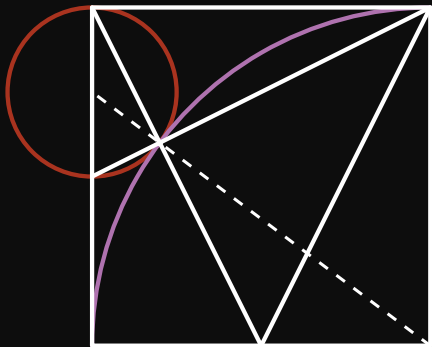


...which happens to be T6's incenter

# Tangent circles

---

These three points are aligned...

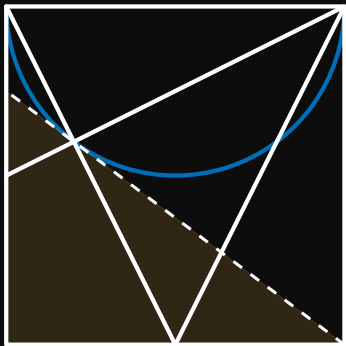


...and these two circles are tangent



# Tangent circles

The line is tangent to this circle...

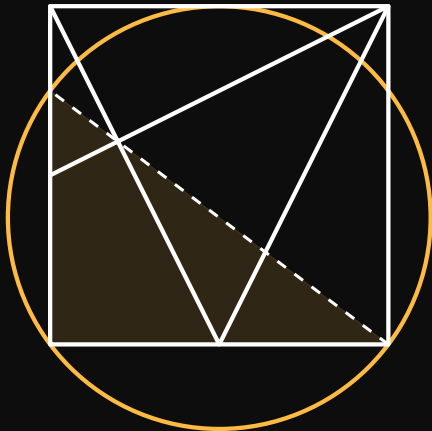


...and the right triangle below is an Egyptian Triangle

# Tangent circles

---

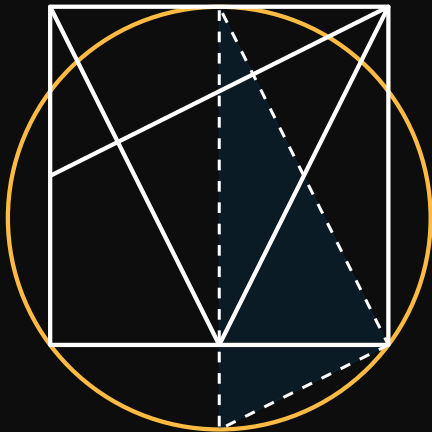
The circumcircle of that Egyptian Triangle...  
...is tangent to the top side of the square



# Tangent circles

---

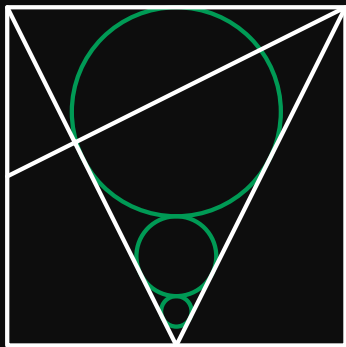
And it is also the circumcircle of this  $1:2:\sqrt{5}$  triangle



# Tangent circles

---

The radius of these three circles are in ratio  $1:\varphi^2:\varphi^4$

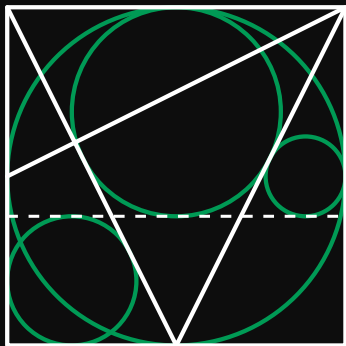


where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio

# Tangent circles

---

The radius of these four circles are in ratio  $1:\varphi:\varphi^2:\varphi^3$

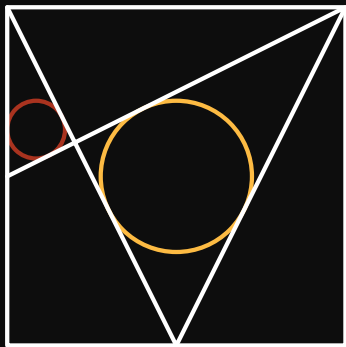


where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio

# Tangent circles

---

The radius of these two circles are in ratio  $1:\varphi^2$



where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio

# Tangent circles

---

The radius of these two circles are in ratio  $1:\varphi^2$

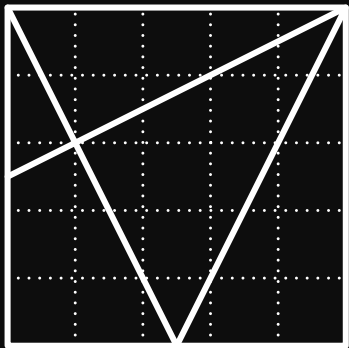


where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio

# The underlying grid

---

Using the intersection point of the Egyptian Tangram...



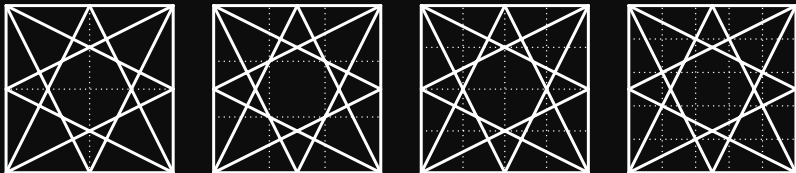
...you can divide the square into  $5 \times 5$  smaller squares!



# The underlying grid

---

Using the intersection points of this figure...



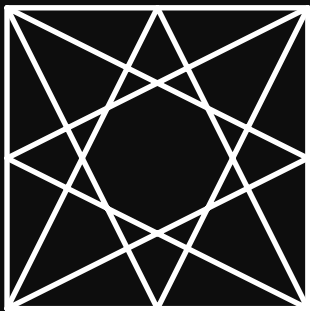
...you can divide the square into:

$2 \times 2$ ,  $3 \times 3$ ,  $4 \times 4$  or  $5 \times 5$  smaller squares!

# The underlying grid

---

There are 32 egyptian triangles in this figure...

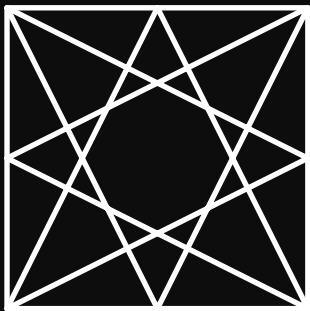


...they come in 4 sizes and there are 8 of each kind

# The underlying grid

---

There are 24  $1:2:\sqrt{5}$  triangles in this figure...

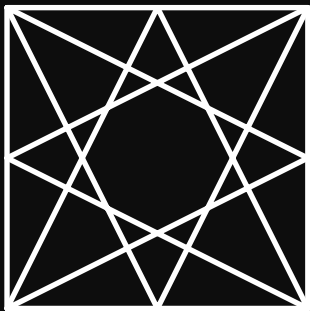


...they come in 3 sizes and there are 8 of each kind

# The underlying grid

---

There are 24 other triangles in this figure...



...of 3 different kinds (one of them comes in 2 sizes)

# The underlying grid

---

The relative sizes of these polygons are...



**Small Triangles:** 1

**Small Kites:** 3

**Whole Square:** 120

**Big Triangles:** 6

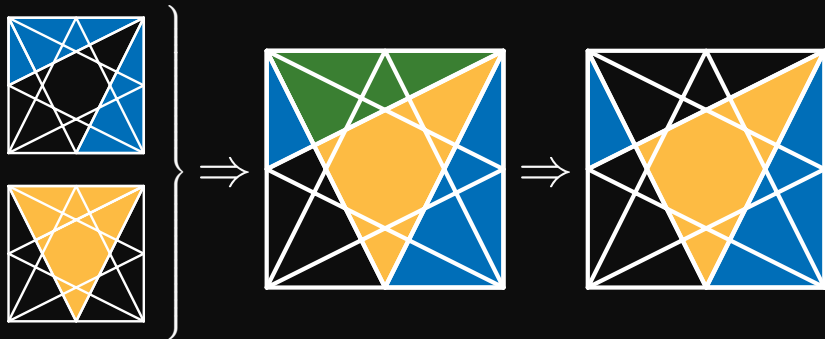
**Big Kites:** 8

**Octagon:** 20

# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$

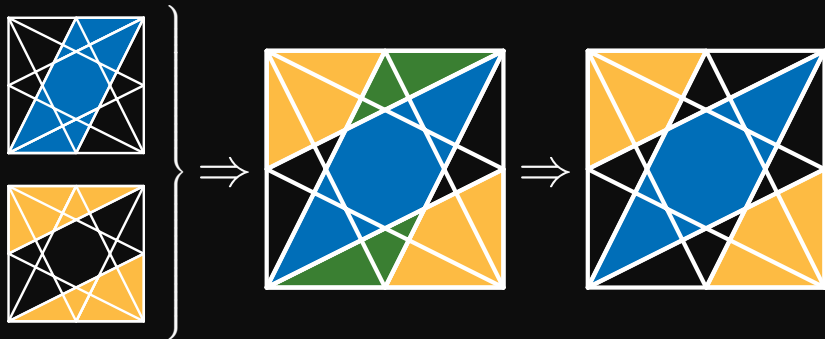


... $\text{Area}(\text{BLUE} - \text{GREEN}) = \text{Area}(\text{YELLOW} - \text{GREEN})$

# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW}) \dots$

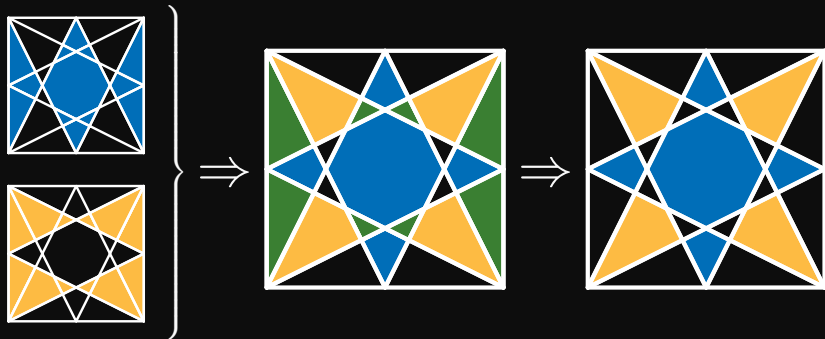


$\dots \text{Area}(\text{BLUE} - \text{GREEN}) = \text{Area}(\text{YELLOW} - \text{GREEN})$

# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$



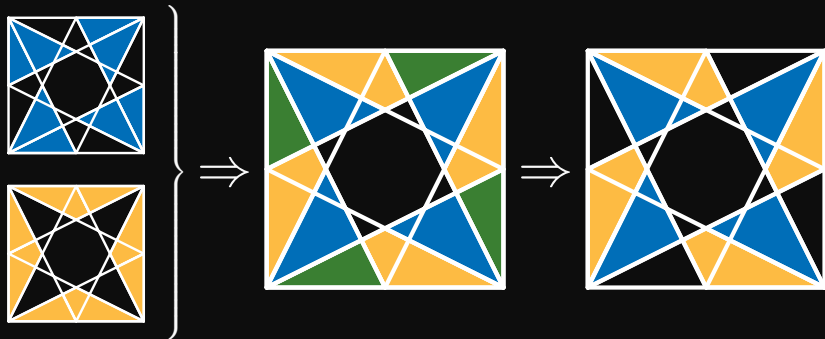
... $\text{Area}(\text{BLUE} - \text{GREEN}) = \text{Area}(\text{YELLOW} - \text{GREEN})$



# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$

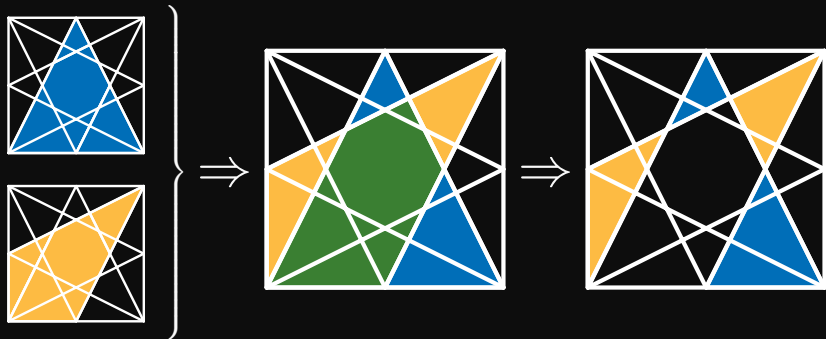


... $\text{Area}(\text{BLUE}-\text{GREEN}) = \text{Area}(\text{YELLOW}-\text{GREEN})$

# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$

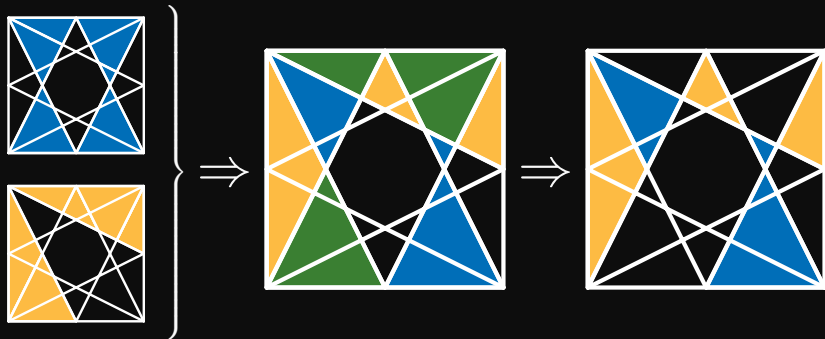


... $\text{Area}(\text{BLUE}-\text{GREEN}) = \text{Area}(\text{YELLOW}-\text{GREEN})$

# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$

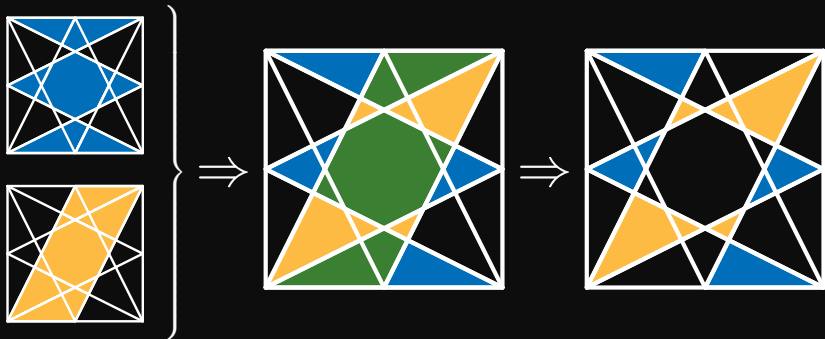


... $\text{Area}(\text{BLUE} - \text{GREEN}) = \text{Area}(\text{YELLOW} - \text{GREEN})$

# The carpets theorem

---

Since  $\text{Area}(\text{BLUE}) = \text{Area}(\text{YELLOW})...$

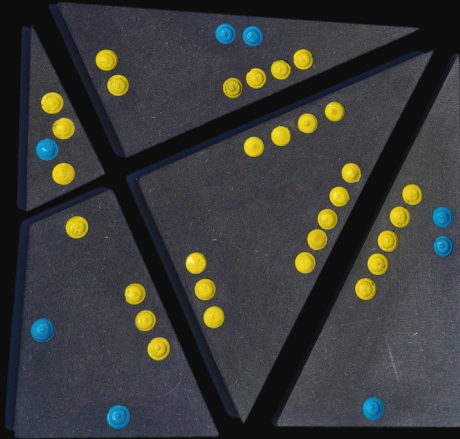


... $\text{Area}(\text{BLUE}-\text{GREEN}) = \text{Area}(\text{YELLOW}-\text{GREEN})$

# Divuligation of the Egyptian Tangram

# Divulgence of the Egyptian Tangram

---



Wooden prototype for MMACA's exhibitions (2019)

# Divulagation of the Egyptian Tangram

Design diary at Nou Biaix  
magazine 44 (2019)

190 • noubiat 44

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el racó del mmaca

## El tangram egipci: diari de disseny

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### Què és un tangram?

El tangram que tots coneixem és un trencaclosques d'origen xinès format per set peces planes, cinc triangles i dos quadrilàters, amb les quals es poden crear un gran nombre de figures [Gardner, 1968].



Figura 1. Tangram xinès.

En la seva vessant més lúdica, el tangram s'acompanya d'un llibret on apareixen les siluetes de diverses figures, i el repte consisteix a recobrir completament cadascuna de les siluetes fent servir les set peces sense que sobresurtin de la figura o s'encavalquin entre elles. Sovint, les figures representen objectes o persones amb gran realisme, cosa sorprenent si tenim en compte la simplicitat geomètrica de les peces del tangram [Lloyd, 2007].



Figura 2. Tangram xinès: silueta realista i solució.

# Divulagation of the Egyptian Tangram

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## Parlem de tangrams!

De la Xina a Egipte passant per Cornellà...



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29 de gener de 2020

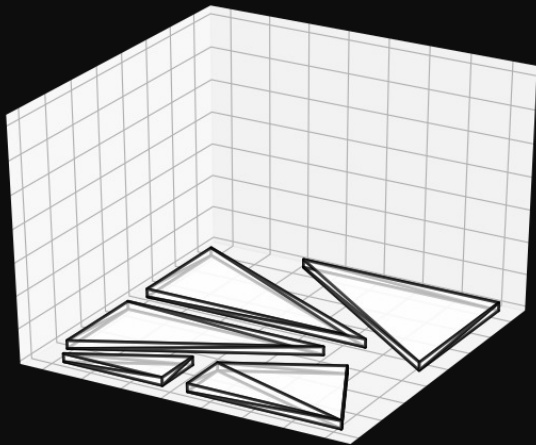
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Talk at MMACA (2020)



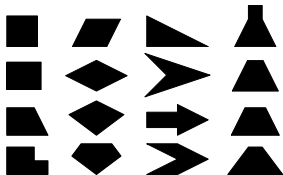
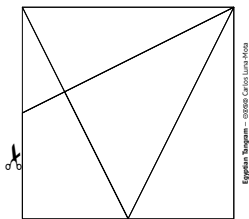
# Divulgence of the Egyptian Tangram

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3D printer prototype (2020)

# Divulagation of the Egyptian Tangram



Print-&-play flyer for families affected by Covid-19 (2020)

# Divulagation of the Egyptian Tangram

## The Egyptian Tangram as a high school learning activity (2020)

### Matemàtiques

#### El tangram egipci

Amb les normes sanitàries d'aquest curs resulta gairebé impossible fer excursions i viatges a museus. Altamentament agraït al company del MNMCA (Museu de Matemàtiques de Catalunya) pensar en tot i han inventat un nou material didàctic, el **tangram egipci**, que és molt fàcil de reproduir amb cartó i fasses. Així, en comptes d'anar trobant el museu, hem descobert que el museu viatja les setes de diàl·log i ens treballa la geometria i les areles quadrades.

En aquesta imatge podreu veure les 5 peces del tangram egipci dibuixades sobre una quadrícula.

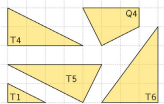


Figure 1. Les peces del tangram egipci

Un pot fer quadrats d'aquestes figures geomètriques:

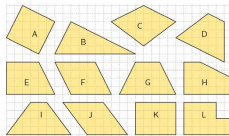


Figure 2. Figures geomètriques

I quan hàgiu acabat de fer-les totes, podeu intentar fer figures de care realitzant amb aquestes:



Figure 3. Casa, d'Anna Riera



Figure 4. Gato, de Sara Llongueres



Figure 5. Piràmide, de Jordi Clotet



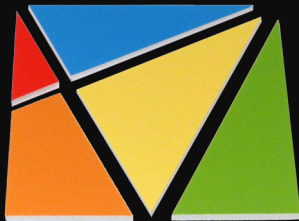
Figure 6. Ocell, de Nària Colomé



Figure 7. Còrnel, de Laura Berenguer

# Divulcation of the Egyptian Tangram

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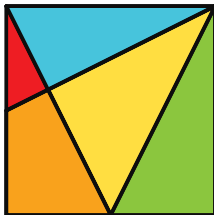
First commercial edition (2021)

# Divulagation of the Egyptian Tangram

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## El Tangram Egipcio



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On-line talks for FUNDAPROMAT and  
MMACA's 7th anniversary (2021)

**Ideas? Suggestions?**  
**Use examples?**

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