The Egyptian Tangram

mmaca

November 1, 2022

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The Egyptian Tangram

A square dissection firstly proposed as a tangram in:

Luna-Mota, C. (2019) *"El tangram egipci: diari de disseny"* Nou Biaix, 44

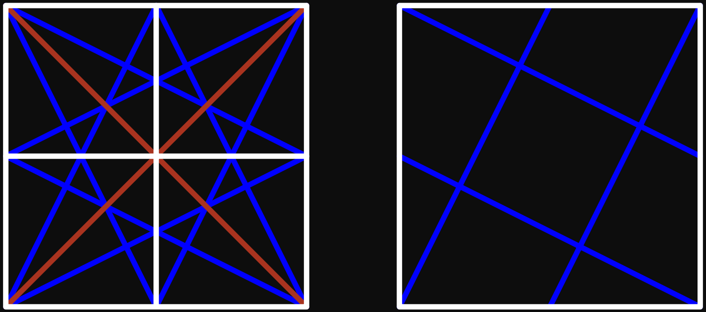
Design process

The Egyptian Tangram inspiration comes from the study of two other 5-piece tangrams...

The “Five Triangles” & “Greek-Cross” tangrams

Design process

...and their underlying grids



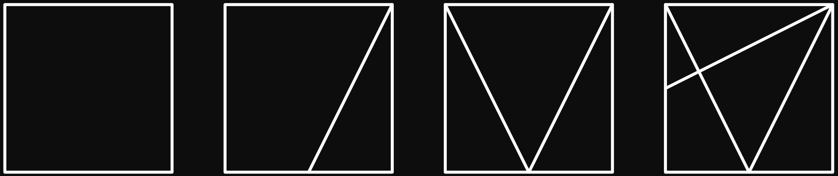
The “Five Triangles” & “Greek-Cross” underlying grids

Design process

This simple *cut* let us build five interesting figures...

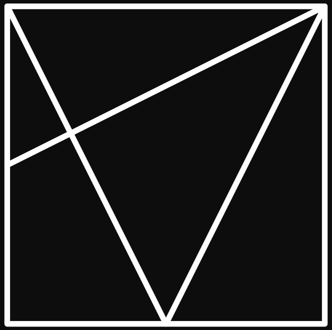
Design process

...so it looked like a good starting point for
our heuristic incremental design process:



Take a square and keep adding "the most interesting straight cut"
until you have a dissection with five or more pieces.

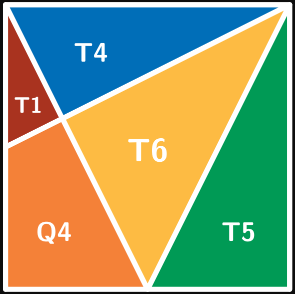
Design process



Straight cuts simplify creating an Egyptian Tangram from a square:

1. Connect the lower midpoint with the upper corners
2. Connect the left midpoint with the top right corner

Promising features

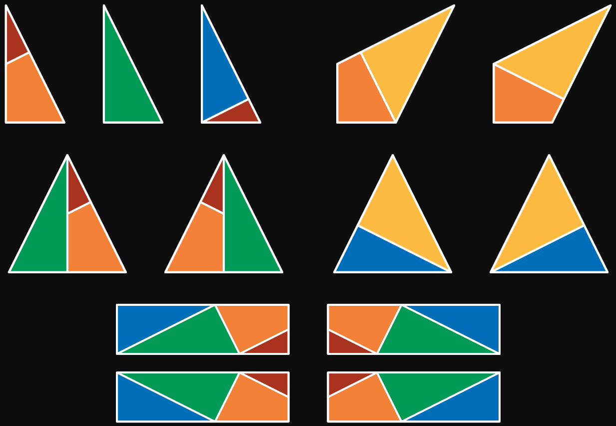


- Just five pieces
- All pieces are different
- All pieces are asymmetric
- Areas are integer and not *too different*
- All sides are multiples of 1 or $\sqrt{5}$
- All angles are linear combinations of 90° and $\alpha = \arctan\left(\frac{1}{2}\right) \approx 26,565^\circ$

Name	Area	Sides	Angles
T1	1	$1,\ 2,\ \sqrt{5}$	$90,\ \alpha,\ 90-\alpha$
T4	4	$2,\ 4,\ 2\sqrt{5}$	$90,\ \alpha,\ 90-\alpha$
T5	5	$\sqrt{5},\ 2\sqrt{5},\ 5$	$90,\ \alpha,\ 90-\alpha$
T6	6	$3,\ 4,\ 5$	$90,\ 90-2\alpha,\ 2\alpha$
Q4	4	$1,\ 3,\ \sqrt{5},\ \sqrt{5}$	$90,\ 90-\alpha,\ 90,\ 90+\alpha$

Promising features

Although all pieces are asymmetric and different, they often combine to make symmetric shapes



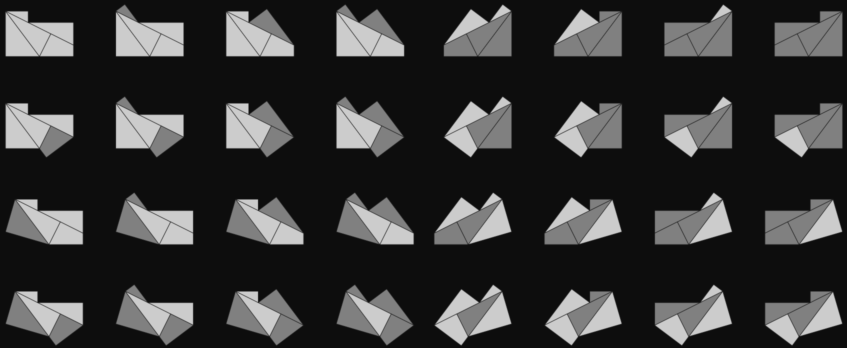
Promising features

This means that it is rare for an Egyptian Tangram figure to have a unique solution

There are three different solutions for the square and, in all three cases, two corners of the square are built as a sum of acute angles!

Promising features

The asymmetry of the pieces also implies that each solution belongs to one of these equivalence classes:

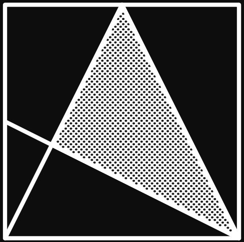


You cannot transform one of these figures into another without flipping a piece

Historical precedents

It turns out that this figure is not new...

Detemple, D. & Harold, S. (1996) *"A Round-Up of Square Problems"*



Problem 3

...but, to the best of our knowledge, nobody used it before **as a tangram**

Historical precedents

The name is not new either...

This dissection is often called “Egyptian Puzzle” or “Egyptian Tangram”

...but there is a good reason to consider our dissection the real “Egyptian Tangram”

(even if it was designed in Catalonia)

Why we called it the *Egyptian* Tangram?

The smallest pieces of the Chinese and Greek-Cross tangrams can be used to build all the other pieces...

...but you cannot do the same with the Egyptian Tangram because of T6

Why we called it the *Egyptian* Tangram?

Initially, T6 was considered as *the leftover piece* that results from cutting all these 1:2:$\sqrt{5}$ triangles from the borders of the square.

But it turned out to be a very well known triangle...

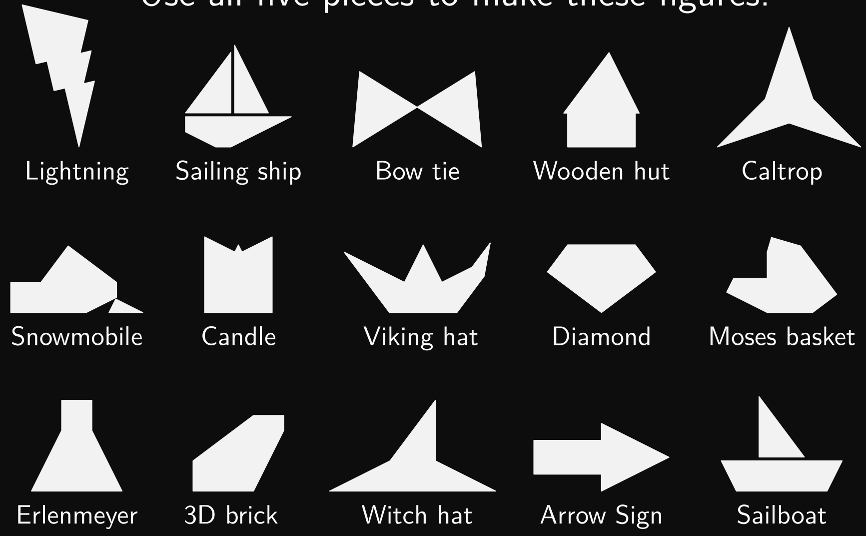
...an **Egyptian** Triangle (3:4:5)

and, hence, the name of this tangram

Puzzles & Activities

Realistic figures

Use all five pieces to make these figures:



Realistic figures

Use all five pieces to make these figures:

Gnome

Handmaid

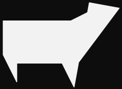
Mountain range

Fish tail

Teddy bear

Cat

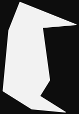
Dromedary



Cow

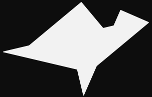
Snail

Fennec Fox

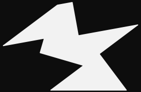


Penguin

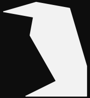
Calf



Sea Turtle



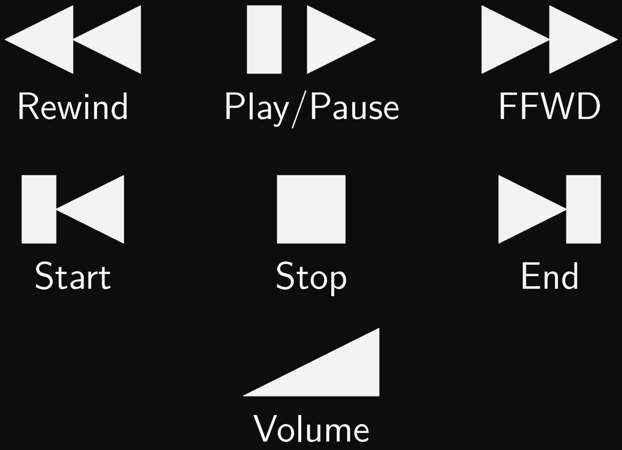
Duck



Crow

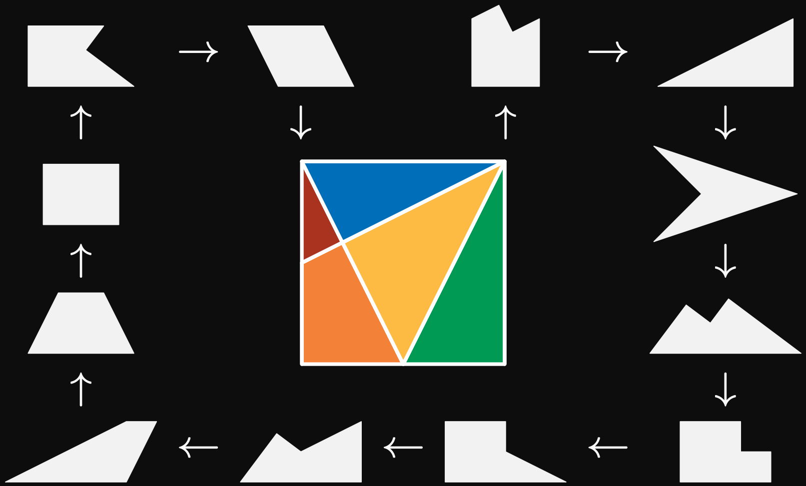
Remote control symbols

Use all five pieces to make these symbols:



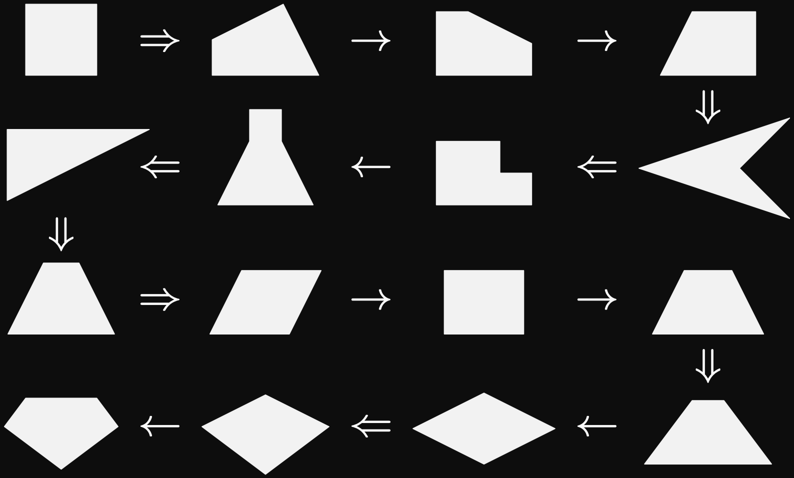
Geometric figures

Complete the cycle moving a different piece each time



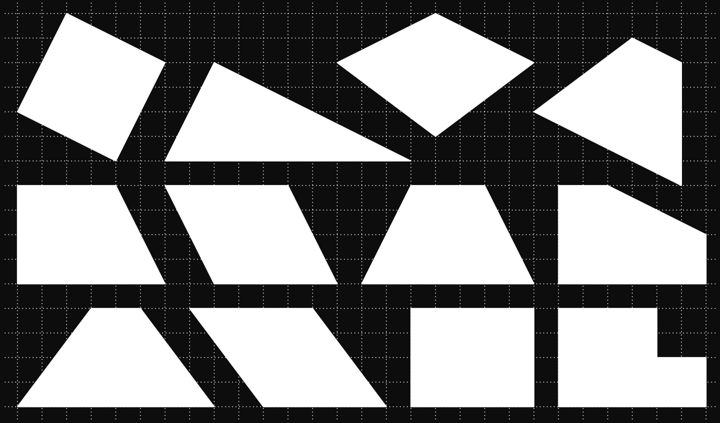
Geometric figures

Use all five pieces to make these figures:



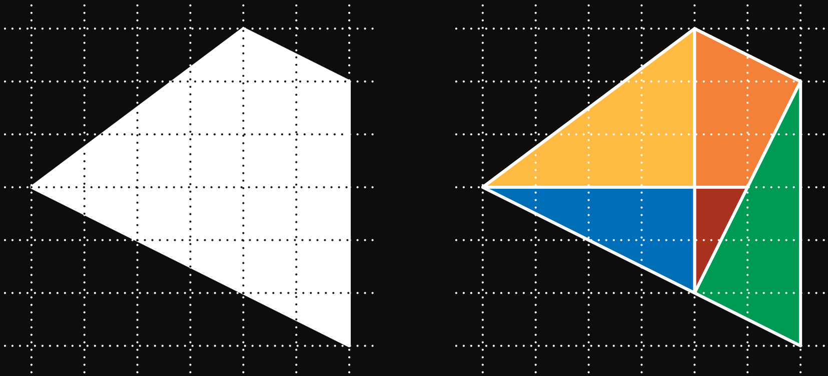
Complete the path moving just one or two pieces at a time

Geometric figures



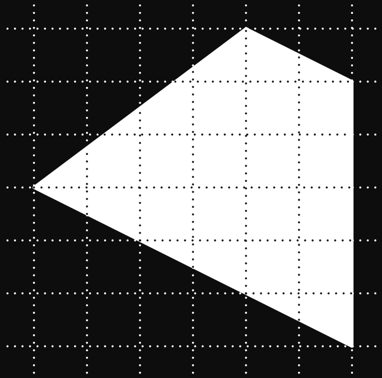
You can draw many of these geometric figures
with all their vertices lying on a square grid...

Geometric figures



...and then try to find a solution that also has the vertices of all 5 pieces lying on the same grid.

Geometric figures

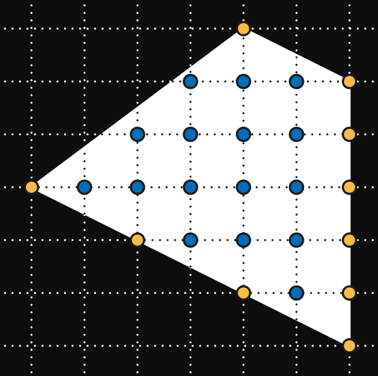


Pythagorean Theorem:

$$
\begin{aligned}
\text{Top} &= \sqrt{1^2 + 2^2} = \sqrt{5} \\
\text{Left} &= \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \\
\text{Right} &= \sqrt{5^2 + 0^2} = \sqrt{25} = 5 \\
\text{Bottom} &= \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5} \\
\\
\text{Perimeter} &= 10 + 4\sqrt{5}
\end{aligned}
$$

You could use the Pythagorean theorem
to compute the perimeter of these figures...

Geometric figures



Pick's Theorem:

lattice points in the interior $= 16$

lattice points on the boundary $= 10$

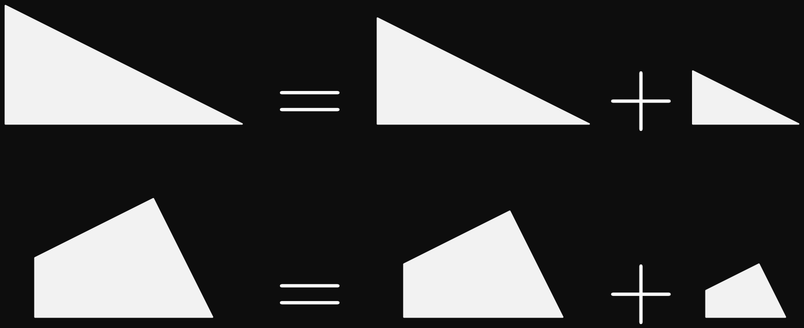
$$\text{Area} = \text{interior} + \frac{\text{boundary}}{2} - 1$$

$$= 16 + \frac{10}{2} - 1 = 20$$

...and Pick's theorem to compute their area.

Sum of similar figures

Use all 5 pieces to make the single figure in the LHS, then use them to make the two figures on the RHS

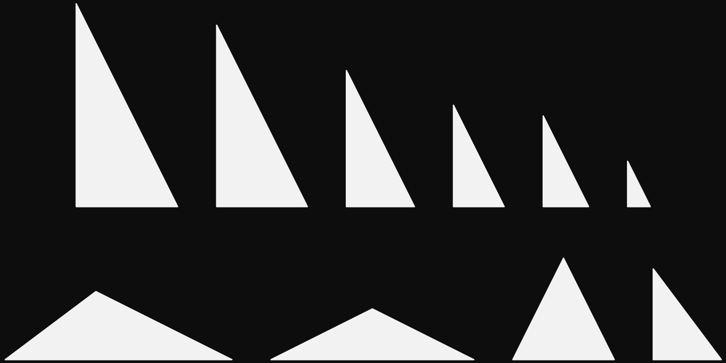


In both equations, the figures are similar and areas are in ratio 5 : 4 : 1

Triangles

Could you prove that there are just 10 triangles you can make with one or more pieces of the Egyptian Tangram?

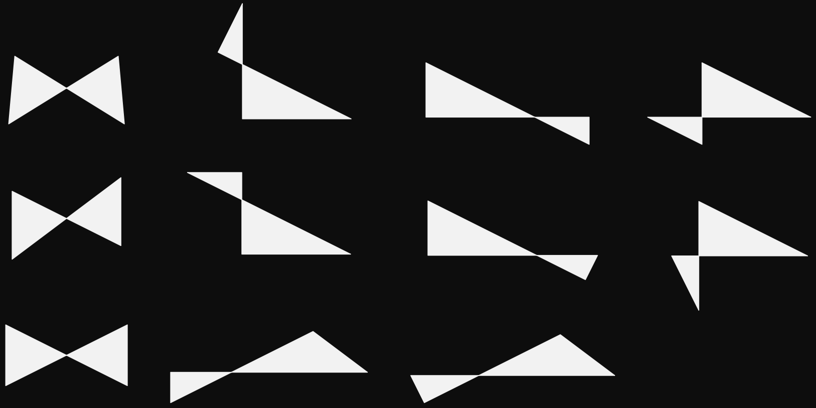
How many solutions could you find for each figure?



Top row areas: 20, 16, 9, 5, 4, 1   Bottom row areas: 15, 10, 10, 6

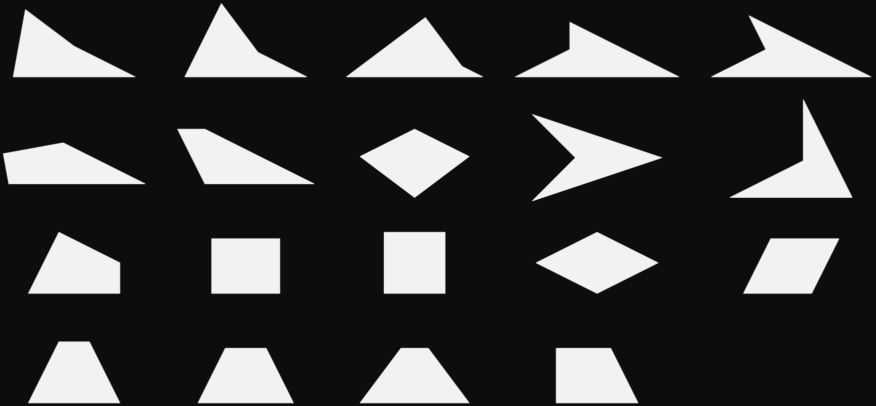
Quadrilaterals

Could you prove that there are just 11 **complex quadrilaterals** you can make with all five pieces of the Egyptian Tangram?



Quadrilaterals

Simple quadrilaterals: Not self-intersecting

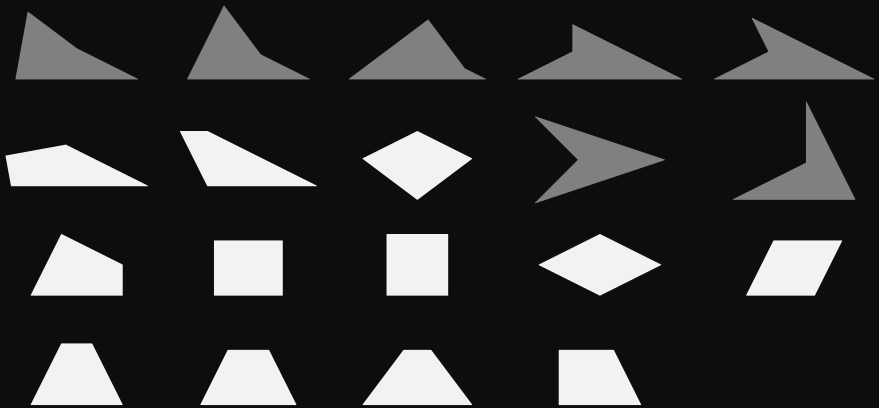


All simple quadrilaterals tile the plane!

$$\alpha + \beta + \gamma + \delta = 2\pi$$

Quadrilaterals

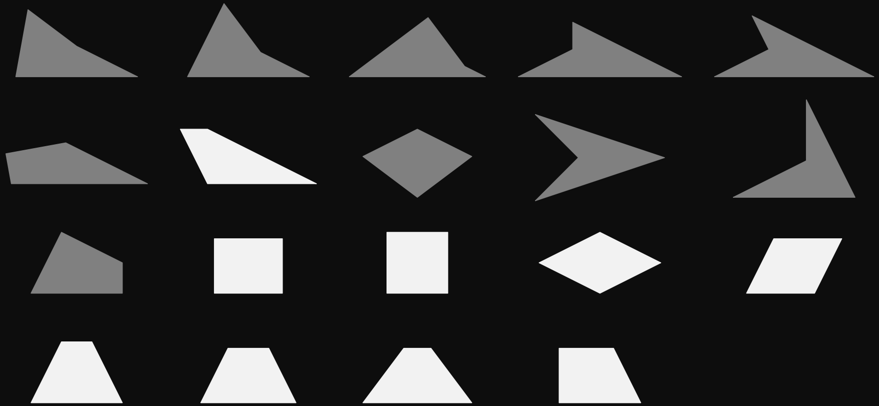
Convex quadrilaterals: All internal angles are smaller than π



Law of Cosines: $p^2q^2 = a^2c^2 + b^2d^2 - 2abcd\cos(\alpha+\gamma)$

Quadrilaterals

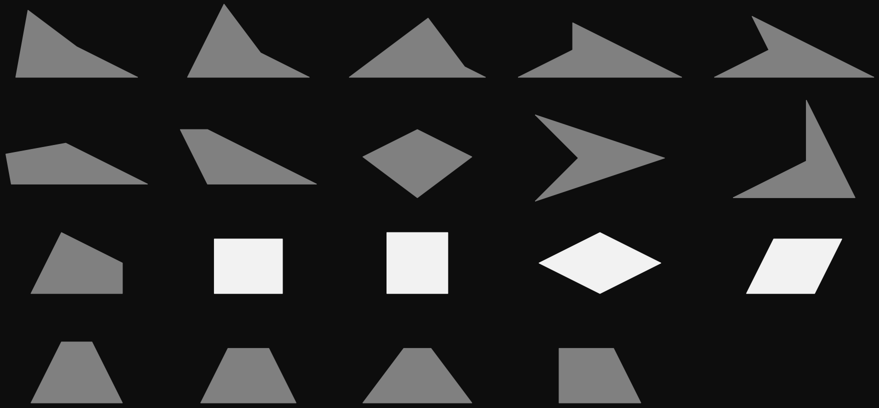
Trapeziums (UK) / **Trapezoids** (US): One pair of parallel sides



Trapezium/Trapezoid $\Leftrightarrow$ Diagonals cut each other in the same ratio

Quadrilaterals

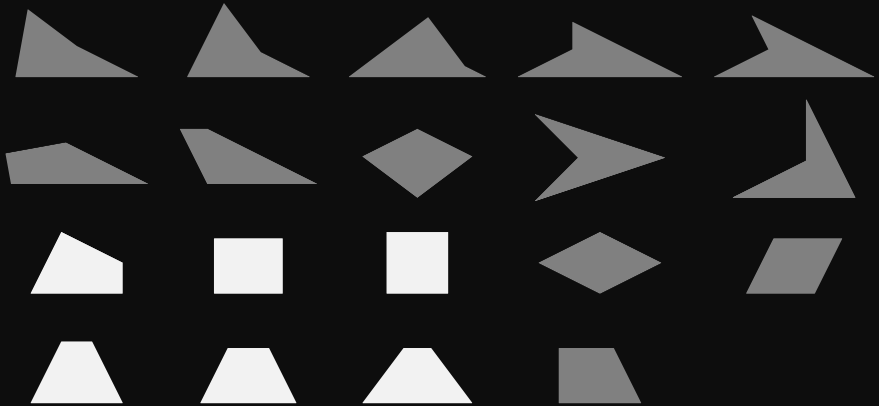
Parallelograms: Two pairs of parallel sides



Parallelogram $\Leftrightarrow$ Diagonals bisect each other $\Leftrightarrow$ $a^2 + b^2 + c^2 + d^2 = p^2 + q^2$

Quadrilaterals

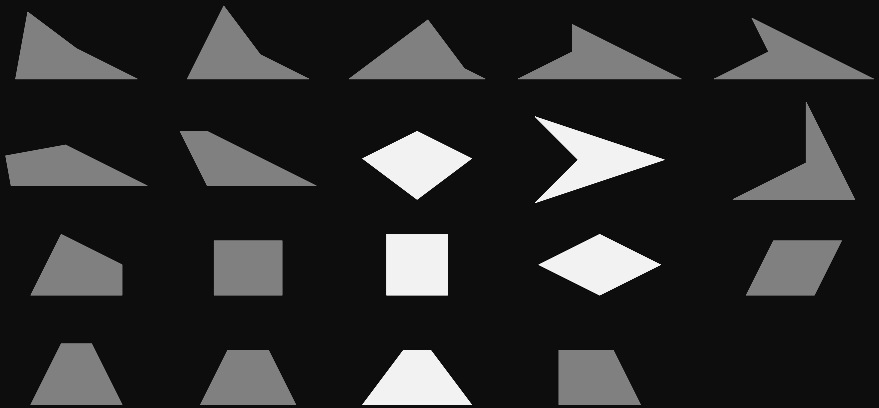
Cyclic quadrilaterals: All vertices lie on a circle



$$\text{Cyclic} \quad \Leftrightarrow \quad \alpha + \gamma = \beta + \delta$$

Quadrilaterals

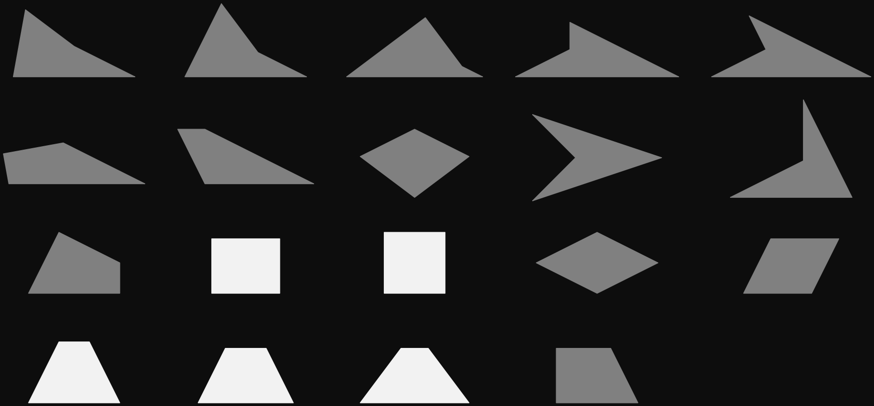
Tangential quadrilaterals: All sides are tangent to a circle



Tangential $\quad\Leftrightarrow\quad a + c = b + d$

Quadrilaterals

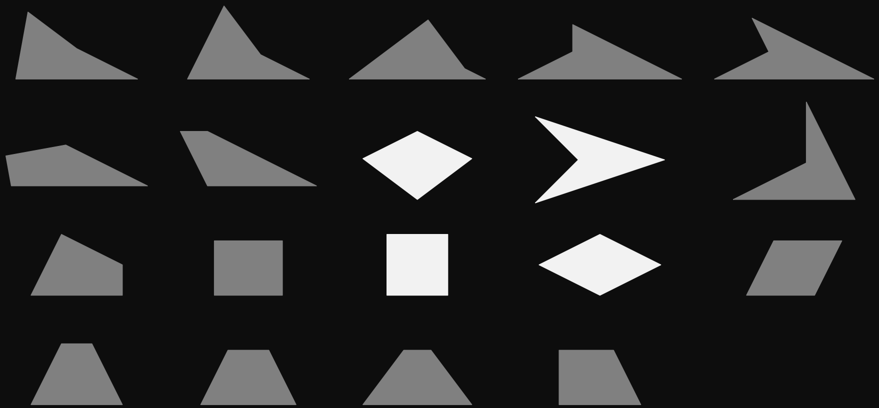
Isosceles Trapezoids: Two pairs of adjacent angles are equal



Isosceles trapezoids $\Leftrightarrow$ Cyclic quadrilaterals with equal diagonals

Quadrilaterals

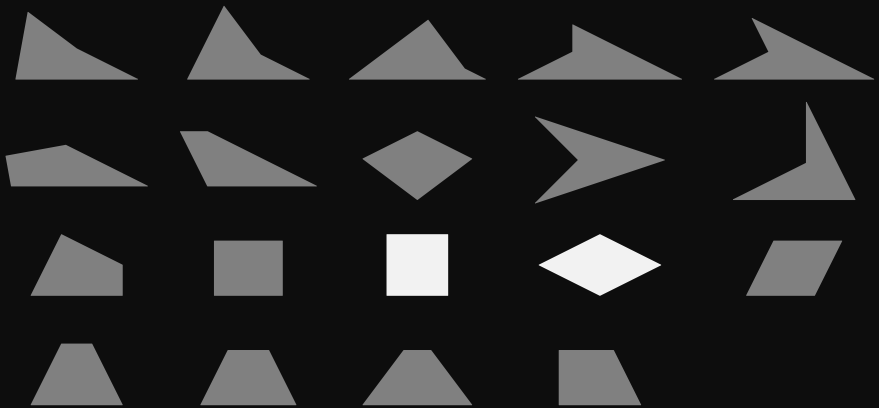
Darts & Kites: Two pairs of adjacent sides are equal



Darts/Kites $\Leftrightarrow$ Tangential quadrilaterals with perpendicular diagonals

Quadrilaterals

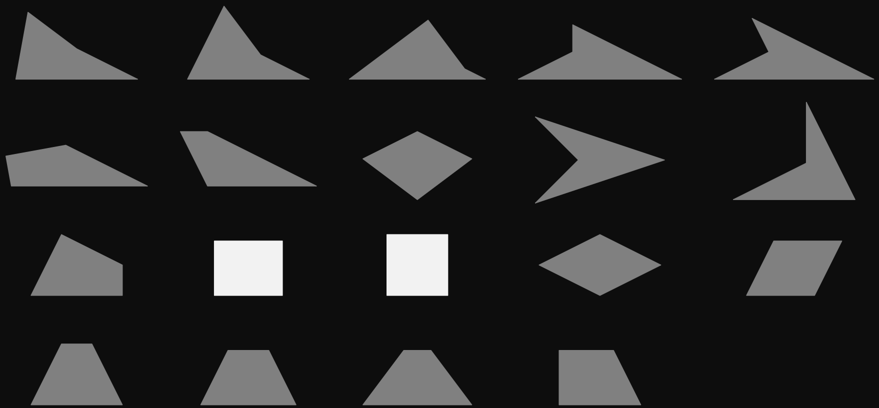
Rhombi: All sides are equal



Rhombi $\Leftrightarrow$ Parallelograms with perpendicular diagonals

Quadrilaterals

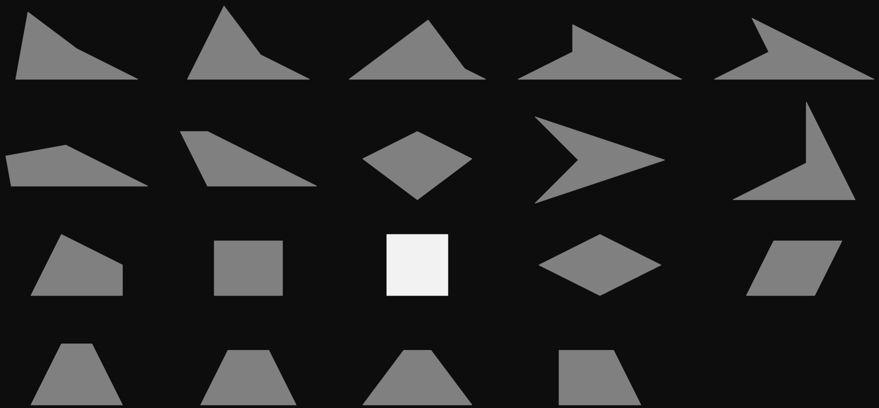
Rectangles: All angles are equal



Rectangles $\Leftrightarrow$ Parallelograms with equal diagonals

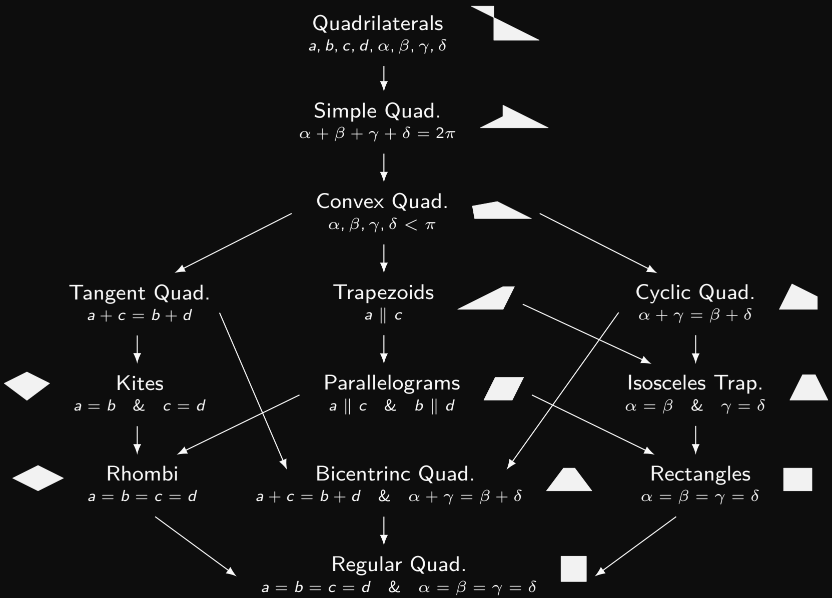
Quadrilaterals

Squares: Regular quadrilaterals



Among all quadrilaterals, squares maximize the *Area* : *Perimeter* ratio

Quadrilaterals



The three solutions of the square

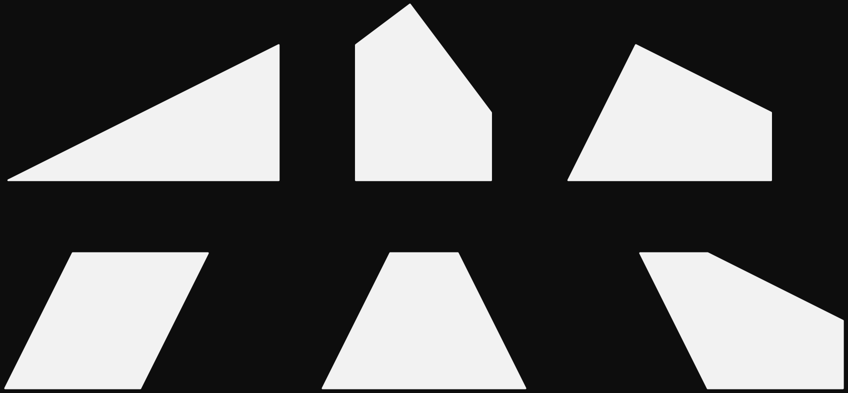
Could you prove that there are just three different solutions for the square?

What is the area of this square? What is its perimeter?

How many times do you find $\sqrt{5}$ in the Egyptian Tangram pieces?

Figures with seven solutions

Could you find seven different solutions
for each of these figures?

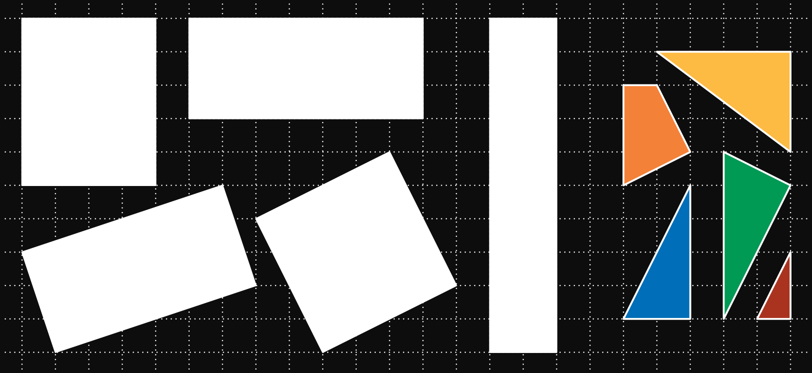


Figures with unique solutions

Could you prove that there is only one solution for each of these figures?

Figures without solution

Could you explain why some of these rectangles can not be made with these 5 pieces?



Missing triangle paradox

Both figures use all 5 pieces...

Where is the missing triangle?

Missing triangle paradox

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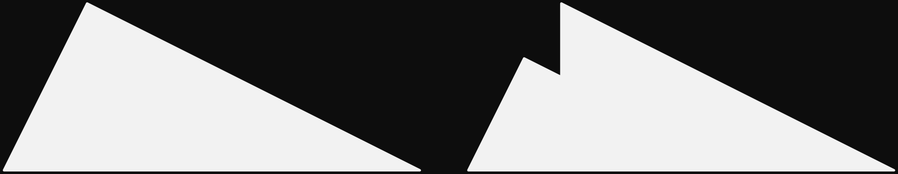

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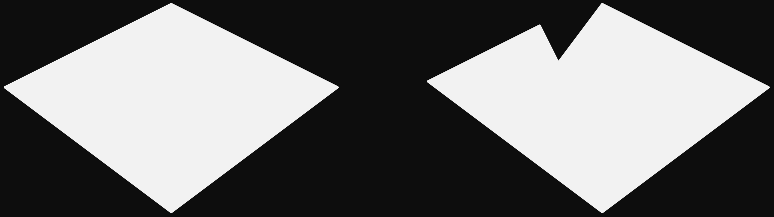

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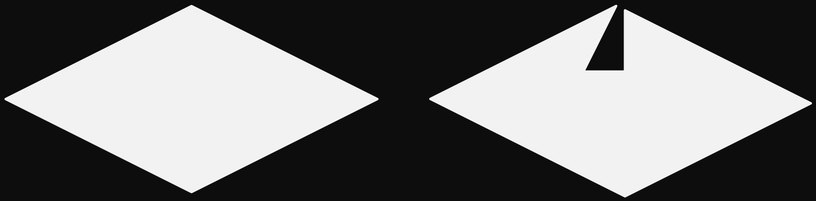

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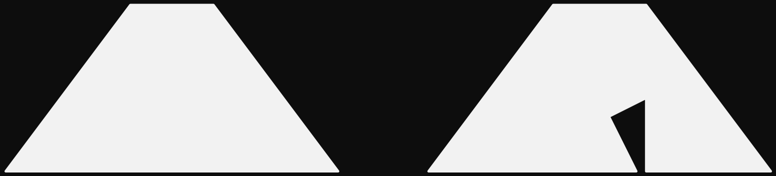

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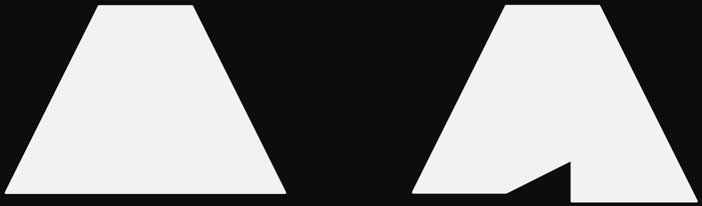

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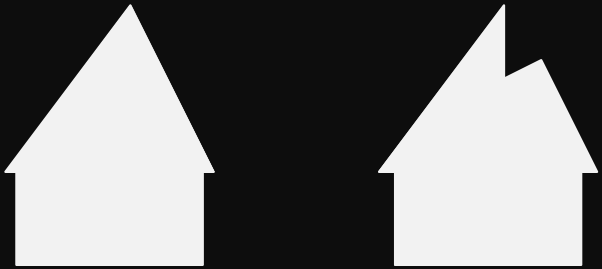

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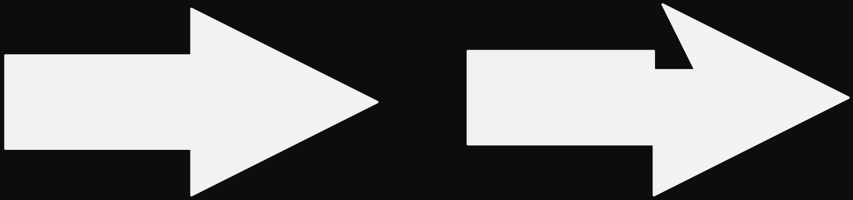
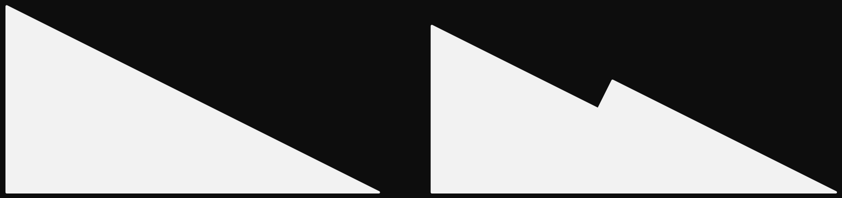


Where is the missing triangle?

Missing rectangle paradox

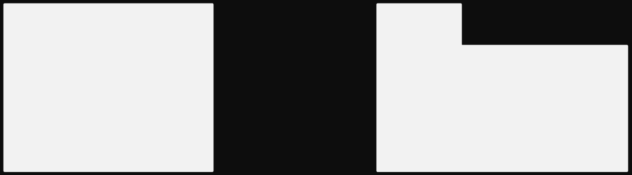
Both figures use all 5 pieces...



Where is the missing rectangle?

Missing rectangle paradox

Both figures use all 5 pieces...



Where is the missing rectangle?

Missing rectangle paradox

Both figures use all 5 pieces...

Where is the missing rectangle?

Missing square paradox

Both figures use all 5 pieces...

Where is the missing square?

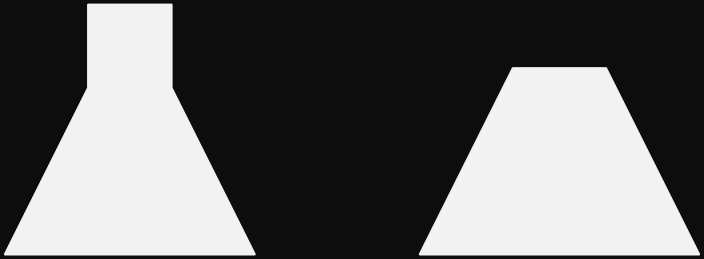
Missing square paradox

Both figures use all 5 pieces...

Where is the missing square?

Missing square paradox

Both figures use all 5 pieces...



Where is the missing square?

Golden Rectangles

The dashed rectangle proportions are $1{:}\varphi$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

Golden Rectangles

The dashed rectangle proportions are 1:φ

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Golden Rectangles

The dashed rectangles proportions are $1{:}\varphi$

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Golden Rectangles

There are 4 golden rectangles hidden in this figure

Could you spot them?

Golden Rectangles

There are 4 golden rectangles hidden in this figure

Could you spot them?

Golden Rectangles

There are 4 golden rectangles hidden in this figure

Could you spot them?

Golden Rectangles

There are 5 golden rectangles hidden in this figure

Could you spot them?

Golden Rectangles

You can find golden rectangles of 14 different types

Type	Proportions		
A	$3-\sqrt{5}$	:	$2\sqrt{5}-4$
B	$\sqrt{5}-1$	:	$3-\sqrt{5}$
C	2	:	$\sqrt{5}-1$
D	$2\sqrt{5}-2$	:	$6-2\sqrt{5}$
E	$5-\sqrt{5}$	:	$3\sqrt{5}-5$
F	$1+\sqrt{5}$	:	2
G	4	:	$2\sqrt{5}-2$

Type	Proportions		
H	$2\sqrt{5}$	:	$5-\sqrt{5}$
I	$3+\sqrt{5}$	:	$1+\sqrt{5}$
J	6	:	$3\sqrt{5}-3$
K	$2+2\sqrt{5}$	:	4
L	$5+\sqrt{5}$	:	$2\sqrt{5}$
M	$1+3\sqrt{5}$	:	$7-\sqrt{5}$
N	$4+2\sqrt{5}$	:	$3+\sqrt{5}$

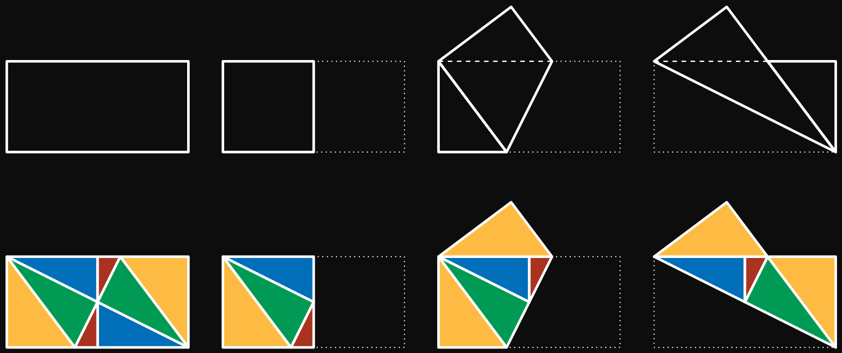
Could you build an example of each type?

Simplified Tangrams

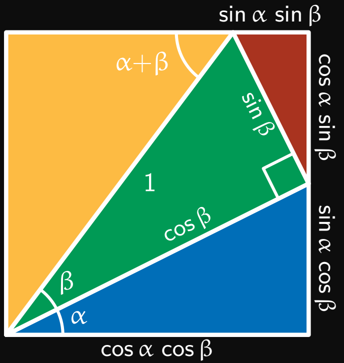
The Egyptian Four–Triangle–Tangram

T1, T4, T5 & T6 appear naturally

when you fold a 2 : 1 rectangle



The Egyptian Four–Triangle–Tangram

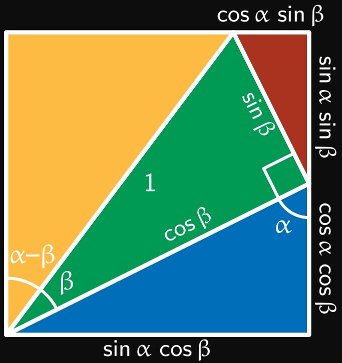


You can use this figure to prove these identities:

$$\sin(\alpha+\beta) = \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta$$

The Egyptian Four–Triangle–Tangram

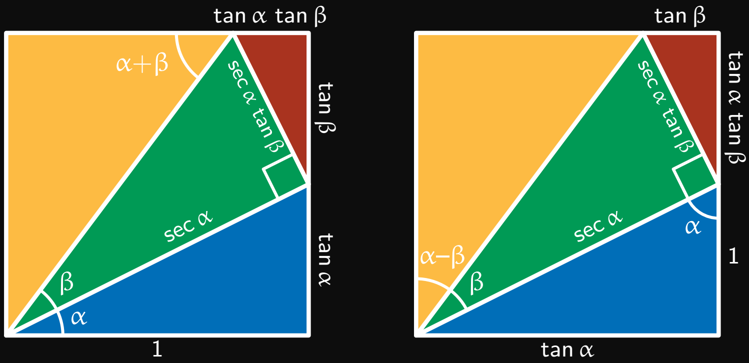


You can use this figure to prove these identities:

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

The Egyptian Four–Triangle–Tangram

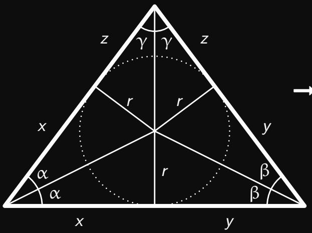


You can use these figures to prove these identities:

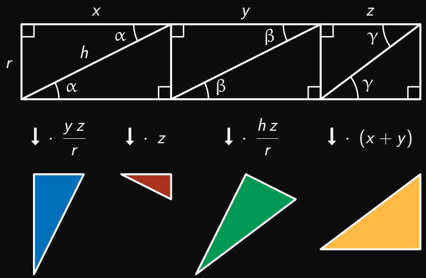
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,\tan\beta} \qquad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\,\tan\beta}$$

The Egyptian Four–Triangle–Tangram

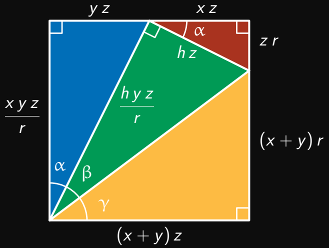
You can use T1, T4, T5 & T6 to prove **Heron's formula**:



$$\Rightarrow \quad \text{AREA} = (x+y+z)\,r$$



$$\downarrow \quad \alpha + \beta + \gamma = \frac{\pi}{2}$$



$$\Rightarrow \quad \text{AREA} = (x+y+z)\,r = \frac{xyz}{r}$$

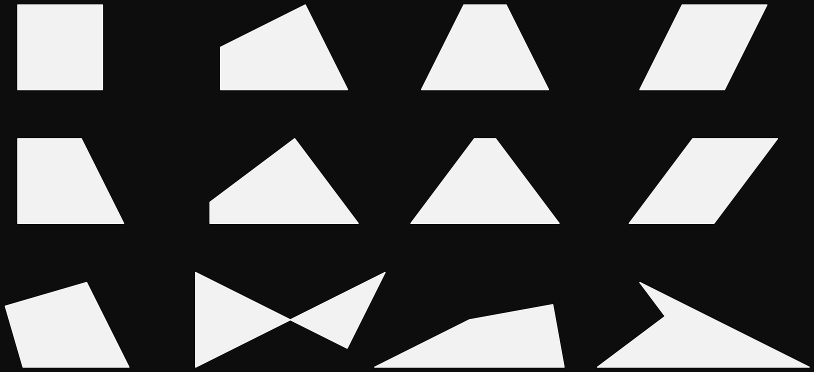
$$\Rightarrow \quad \text{AREA}^2 = (x+y+z)\,xyz$$

$$\left\{\begin{array}{lcl} s &=& x+y+z \\ a &=& y+z \\ b &=& x+z \\ c &=& x+y \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{lcl} s &=& \frac{a+b+c}{2} \\ x &=& s-a \\ y &=& s-b \\ z &=& s-c \end{array}\right\}$$

$$\Rightarrow \quad \text{AREA} = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

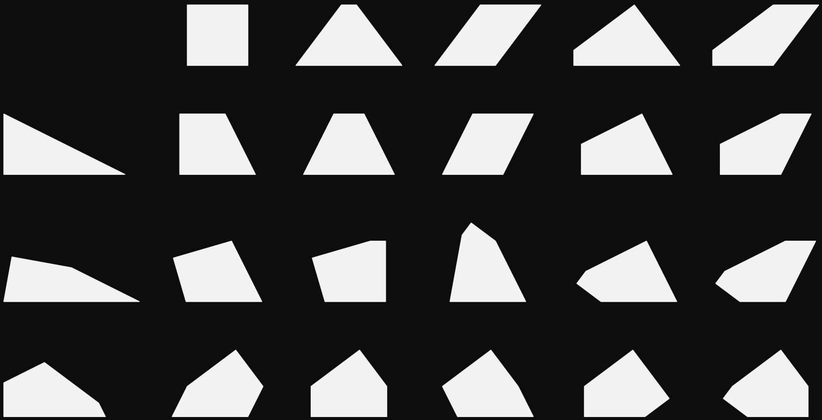
The Egyptian Four–Triangle–Tangram

You can make 12 quadrilaterals using T1, T4, T5 & T6



The Egyptian Four–Triangle–Tangram

You can make 23 convex figures using T1, T4, T5 & T6



The Egyptian Four–Triangle–Tangram

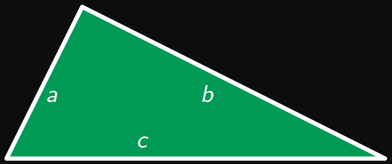
You can find golden rectangles of 7 different types using just T1, T4, T5 & T6

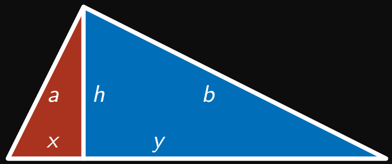
Type	**Proportions**		
A	$3-\sqrt{5}$	:	$2\sqrt{5}-4$
C	2	:	$\sqrt{5}-1$
F	$1+\sqrt{5}$	:	2
G	4	:	$2\sqrt{5}-2$
H	$2\sqrt{5}$	:	$5-\sqrt{5}$
K	$2+2\sqrt{5}$	:	4
L	$5+\sqrt{5}$	:	$2\sqrt{5}$

Could you build an example of each type?

The Egyptian Three–Triangle–Tangram

Given any right triangle with sides: $a \leq b \leq c$





you can draw three similar triangles: (a, b, c), (x, h, a) & (h, y, b)

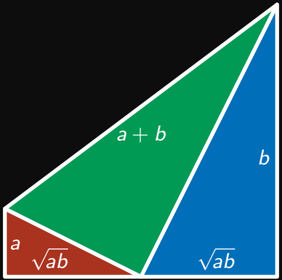
and use them to prove the **Altitude Theorem**: $h^2 = x \cdot y$

and the **Leg Theorems**: $a^2 = x \cdot c$ & $b^2 = y \cdot c$

(T1, T4 & T5 verify this relationship for: $a = \sqrt{5}$, $b = 2\sqrt{5}$ & $c = 5$)

The Egyptian Three–Triangle–Tangram

Since $\frac{a}{\sqrt{ab}} = \frac{\sqrt{ab}}{b}$, these three right triangles are similar...



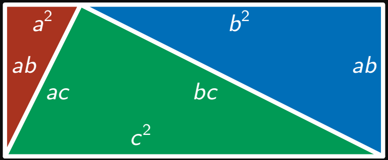
...and you can use this figure to prove the **AM-GM Inequality**:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

(T1, T4 & T5 verify this relationship for: $a = 1$ & $b = 4$)

The Egyptian Three–Triangle–Tangram

Given any right triangle with sides: $a \leq b \leq c$



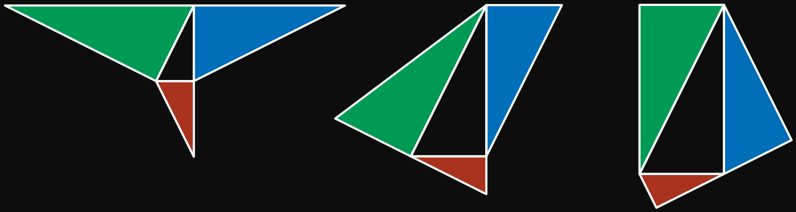
you can make a rectangle with three similar triangles:
(a^2, ab, ac), (ab, b^2, bc) & (ac, bc, c^2)

and compare the top $(a^2 + b^2)$ and the bottom (c^2) sides of the rectangle to prove the **Pythagorean Theorem**

(T1, T4 & T5 verify this relationship for: $a = 1$, $b = 2$ & $c = \sqrt{5}$)

The Egyptian Three–Triangle–Tangram

Since area(T1) + area(T4) = area(T5) ...

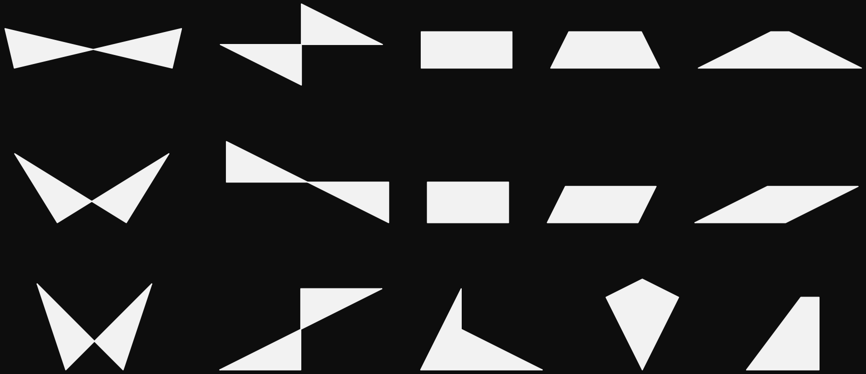


...you can verify 3 cases of the **Pythagorean Theorem**

(and these particular cases turn out to be the T1, T4 & T5 right triangles!)

The Egyptian Three–Triangle–Tangram

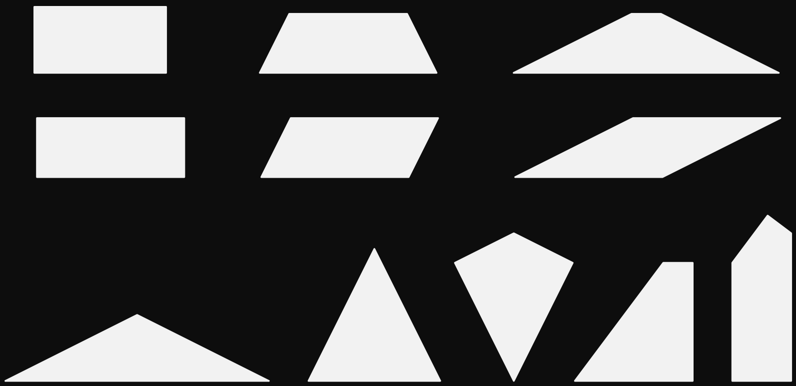
You can make 15 quadrilaterals using just T1, T4 & T5



See also: Brügner, G. (1984) *"Three–Triangle–Tangram"*, Bit, 24

The Egyptian Three–Triangle–Tangram

You can make 11 convex figures using just T1, T4 & T5



See also: Brügner, G. (1984) *"Three–Triangle–Tangram"*, Bit, 24

The Egyptian Three–Triangle–Tangram

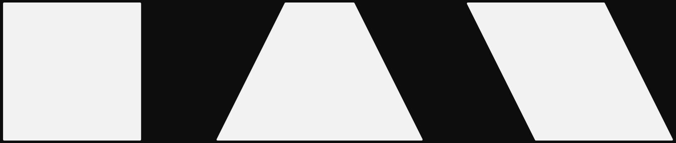
You can find golden rectangles of 5 different types using just T1, T4 & T5

Type	Proportions
C	$2 : \sqrt{5}-1$
F	$1+\sqrt{5} : 2$
G	$4 : 2\sqrt{5}-2$
H	$2\sqrt{5} : 5-\sqrt{5}$
K	$2+2\sqrt{5} : 4$

Could you build an example of each type?

A Four Q4 Puzzle

It is easy to make each of these figures with four copies of Q4:



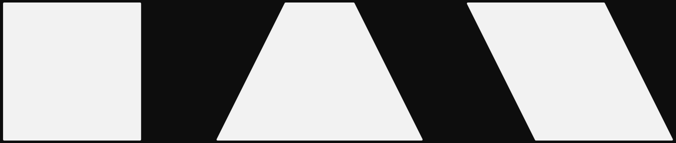
But... Could you make two squares **simultaneously**?

Could you make two golden rectangles **simultaneously**?

See also: **Make a Square** puzzle by Interlocking Puzzles LLC

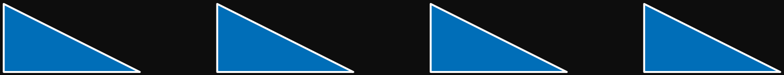
A Four T4 Puzzle

It is easy to make each of these figures with four copies of T4:



But... Could you make two squares **simultaneously**?

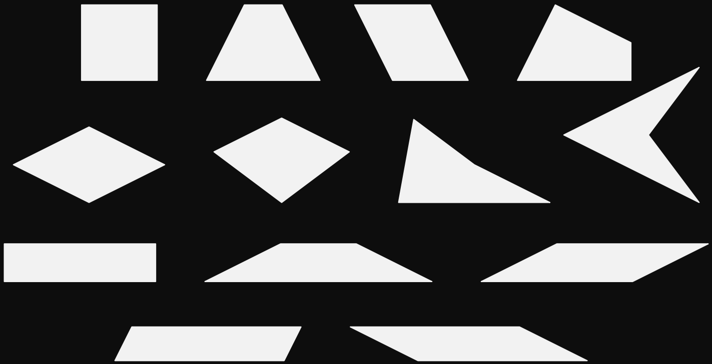
Could you make two golden rectangles **simultaneously**?



See also: **Four Triangles** by Don Steward

A Four T4 Puzzle

You could also build many other figures with four T4s...



...including 13 different quadrilaterals!

See also: **Four Triangles** by Don Steward

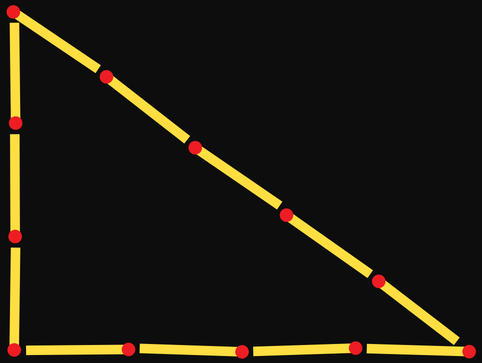
A Five T6 Puzzle

Could you make a symmetrical figure with five copies T6?

See also: **Curious and Interesting Triangles** by Donald Bell

Matchsticks Puzzles

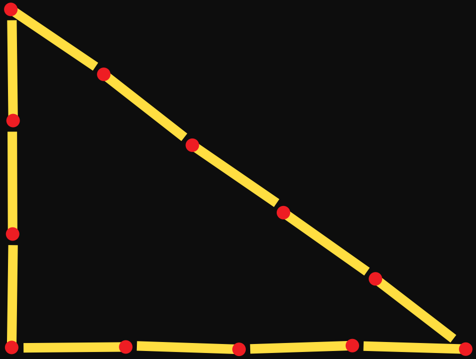
Since all T6's side lengths are integer...



...you can draw it using matchsticks

Matchsticks Puzzles

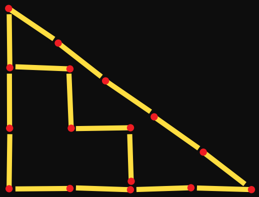
Could you move some matchsticks to get a polygon...

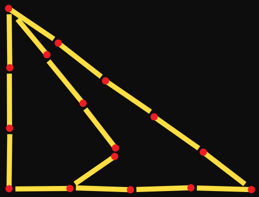


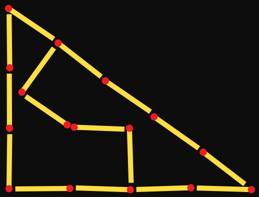
...with integer side lengths and area equal to 5, 4, 3 or 2?

Matchsticks Puzzles

With 4 matchsticks, it is easy to divide T6 into two polygons with integer side lengths and equal area



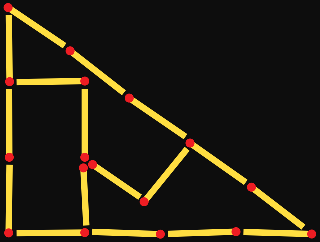




But, could you do it using just 2 or 3 matchsticks?

Matchsticks Puzzles

With 5 matchsticks, it is easy to divide T6 into three polygons with integer side lengths and equal area

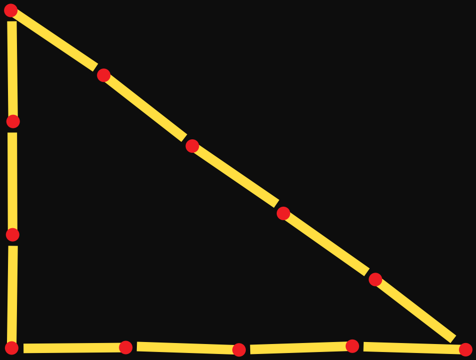


But, could you do it using just 4 matchsticks?

Matchsticks Puzzles

Could you divide T6 into three polygons

with integer side lengths and areas 1, 2 & 3...



...using just 3 matchsticks?

Mathematical Properties

φ and $\sqrt{5}$ are irrational

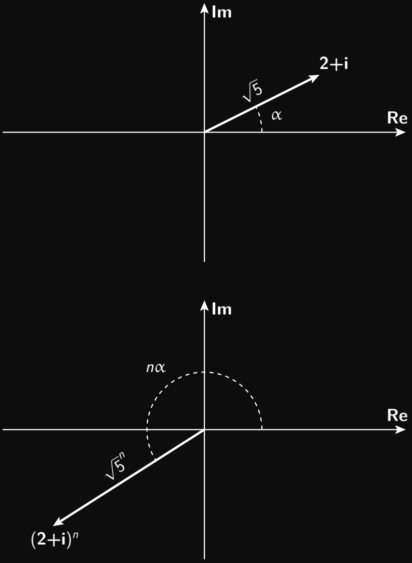
This is a **golden rectangle**, which means that $\frac{\text{base}}{\text{height}} = \varphi$ is the **golden ratio**.

If we remove a square, what remains is also a golden rectangle: $\frac{\text{height}}{\text{base-height}} = \varphi$

If we assume that $\varphi = \frac{b}{h}$, with b and h coprime integers, then $\varphi = \frac{h}{b-h}$ is an equivalent fraction, with a smaller integer numerator and a smaller integer denominator, which is absurd. Therefore, our initial assumption must be false.

And, since $\varphi = \frac{1+\sqrt{5}}{2}$ is irrational,
$2\varphi - 1 = \sqrt{5}$ must be irrational too.

arctan$(1/2)$ is irrational



arctan$(\frac{1}{2})$ is not a rational multiple of π.

If it were, then for some integer $n > 0$, we would have $(2+i)^n \in \mathbb{R}$.

But if we look at the imaginary part of these numbers, $a_n = \mathrm{Im}((2+i)^n)$, we can prove that this sequence satisfies the recurrence relation:

$$a_{n+2} = 4a_{n+1} - 5a_n \qquad \forall\ n > 0$$

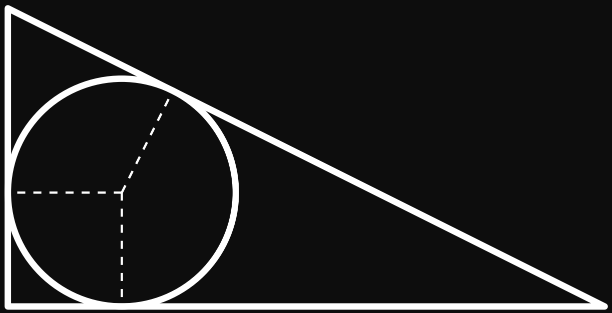
But $a_1 = 1$, $a_2 = 4$ and, by induction:

$$a_n \equiv \begin{cases} 1 \pmod 5 & \forall \text{ odd } n > 0 \\ 4 \pmod 5 & \forall \text{ even } n > 0 \end{cases}$$

therefore, $(2+i)^n \notin \mathbb{R} \quad \forall\ n > 0$.

The 1:2:$\sqrt{5}$ incenter

If the inradius of a 1:2:$\sqrt{5}$ triangle is 1...



...its shorter leg measures $\varphi + 1 = \varphi^2 = \frac{3+\sqrt{5}}{2}$

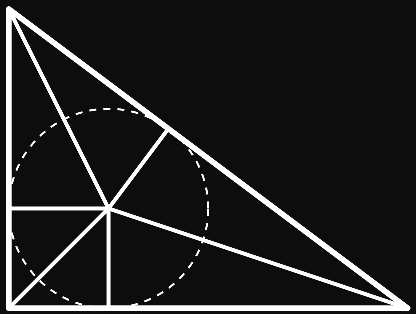
The 3:4:5 incenter

If we overlay T6 and T1 as shown in the figure...

...a T1 vertex lies on the incenter of T6

Dissecting 3:4:5

You can use this dissection of T6 to prove that...

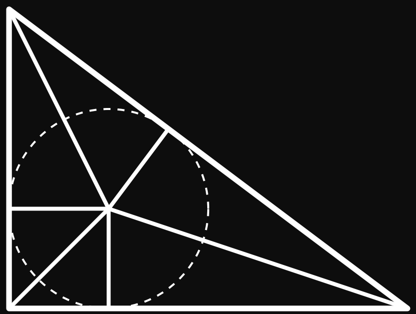


$$\pi = \arctan(1) + \arctan(2) + \arctan(3)$$

(consider the sum of the angles touching the incenter of T6 and divide by 2)

Dissecting 3:4:5

You can use this dissection of T6 to prove that...

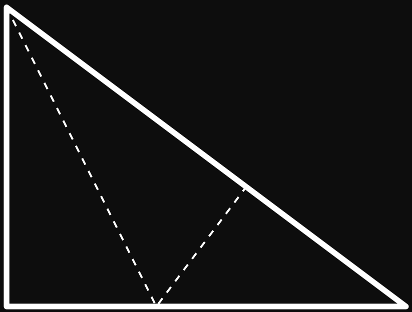


$$\frac{\pi}{2} = \arctan\left(\frac{1}{1}\right) + \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$$

(consider the sum of the angles touching the vertices of T6 and divide by 2)

Dissecting 3:4:5

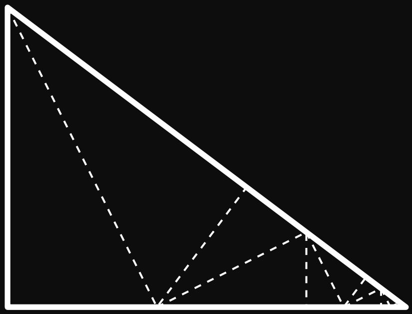
You can dissect a 3:4:5 triangle into...



...a 3:4:5 triangle and
two congruent $1{:}2{:}\sqrt{5}$ triangles

Dissecting 3:4:5

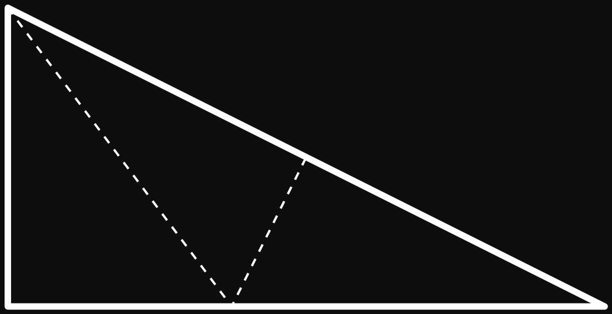
Iterating this dissection of T6 you can prove that...



$$\sum_{n=1}^{\infty} \frac{18}{4^n} = 6 \qquad \text{or, equivalently,} \qquad \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3}$$

Dissecting $1{:}2{:}\sqrt{5}$

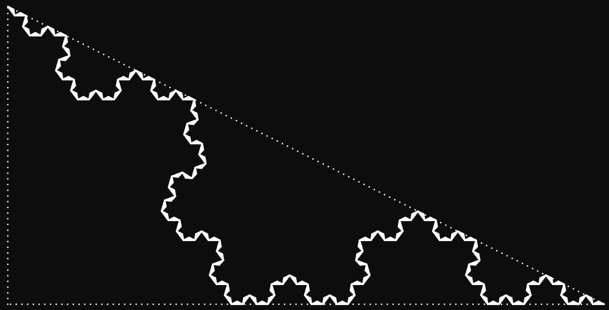
You can dissect a $1{:}2{:}\sqrt{5}$ triangle into...



...a 3:4:5 triangle and
two congruent $1{:}2{:}\sqrt{5}$ triangles

Dissecting $1{:}2{:}\sqrt{5}$

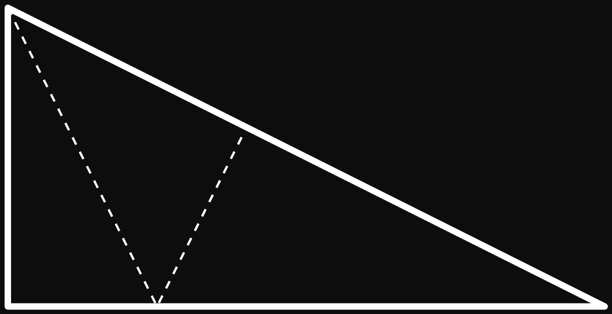
Removing the 3:4:5 triangle and iterating this dissection...



...produces a variant of the **Koch curve** fractal

Dissecting 1:2:$\sqrt{5}$

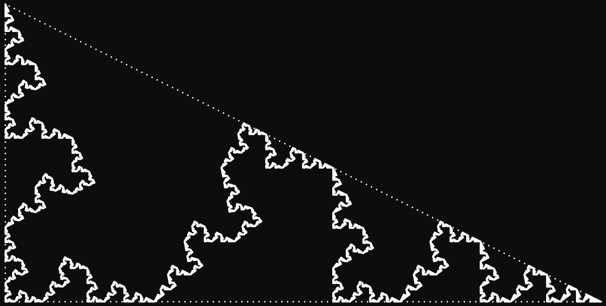
You can dissect a 1:2:$\sqrt{5}$ triangle into...



...a 3:4:5 triangle and
two different 1:2:$\sqrt{5}$ triangles

Dissecting $1{:}2{:}\sqrt{5}$

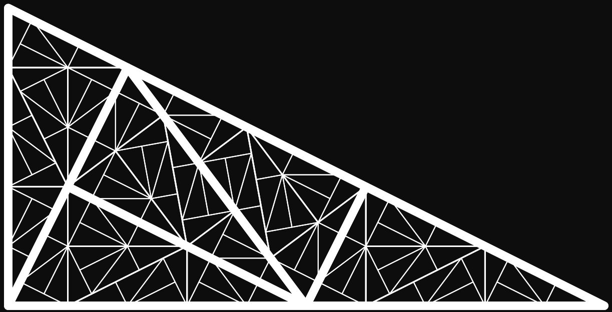
Removing the 3:4:5 triangle and iterating this dissection...



...produces a variant of the **Minkowski sausage** fractal

Dissecting $1{:}2{:}\sqrt{5}$

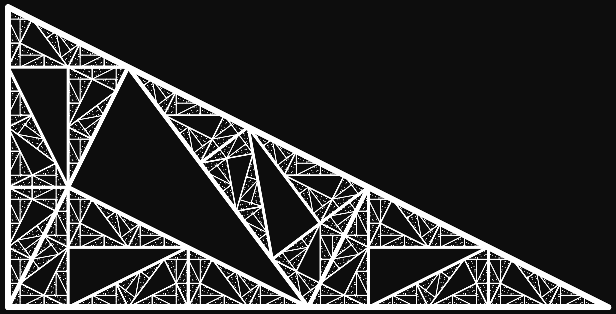
You can dissect a $1{:}2{:}\sqrt{5}$ triangle into...



...five congruent $1{:}2{:}\sqrt{5}$ triangles
and iterate to get the **Pinwheel tiling** of the plane

Dissecting $1{:}2{:}\sqrt{5}$

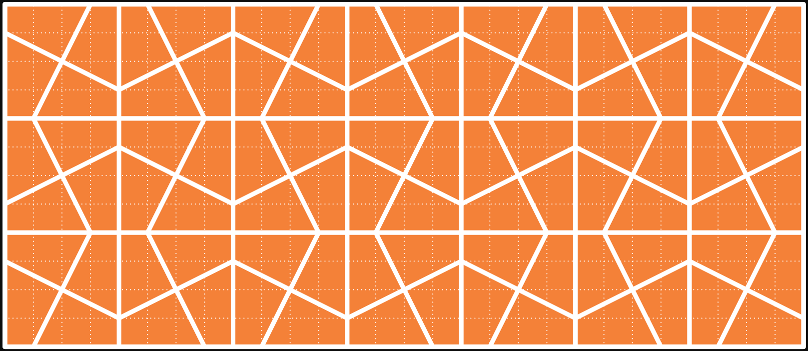
You can dissect a $1{:}2{:}\sqrt{5}$ triangle into...



...five congruent $1{:}2{:}\sqrt{5}$ triangles, remove the central one and iterate to get the **Pinwheel fractal**

Q4 tilings

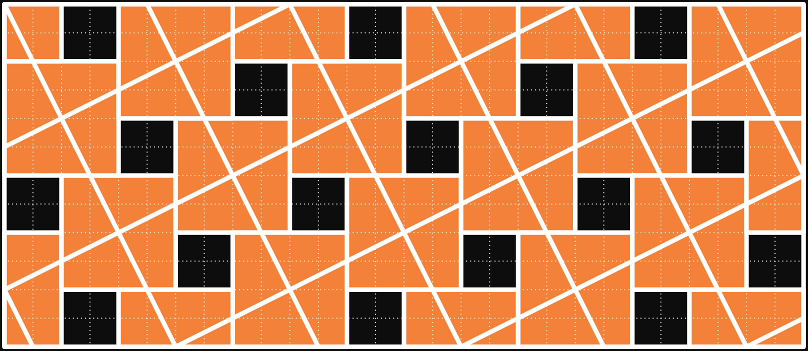
Two copies of Q4 form a pentagon that can be used...



...to make this variant of the **Cairo Tiling** of the plane

Q4 tilings

You can use this **Pythagorean Tiling** to verify...

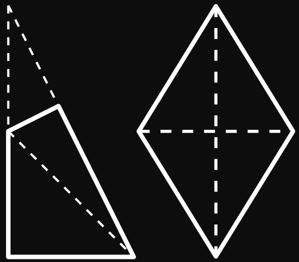


...that T4 satisfies the **Pythagorean Theorem**

The angles of Q4

The angles $90-\alpha$ and $90+\alpha$ that appear in Q4 also appear in the **Golden Rhombus**

(a rhombus whose diagonals are in proportion $1:\varphi$, with $\varphi = \frac{1+\sqrt{5}}{2}$)



$$90+\alpha = 2\cdot\arctan(\varphi) = \arctan(1)+\arctan(3)$$

$$90-\alpha = 2\cdot\arctan\left(\frac{1}{\varphi}\right) = \arctan(2)$$

The faces of the **rhombic triacontahedron** and the **rhombic hexecontahedron** are Golden Rhombi

The angles of Q4

Even though they are NOT similar figures...

...the same angles appear in Q4 and $T5 \cup T6$

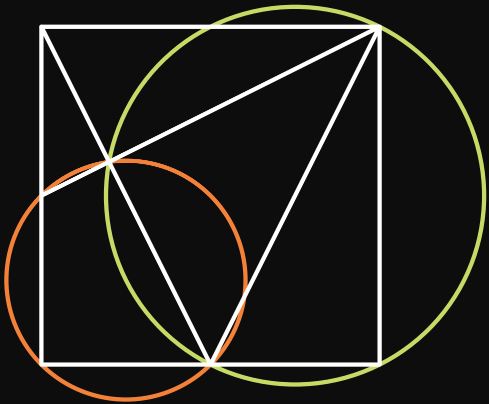
The perimeter of Q4

These three perimeters are in a geometric progression...

$$\frac{2\sqrt{5}+4}{\sqrt{5}+3} = \frac{3\sqrt{5}+7}{2\sqrt{5}+4} = \varphi = \frac{1+\sqrt{5}}{2}$$

The circumcircles

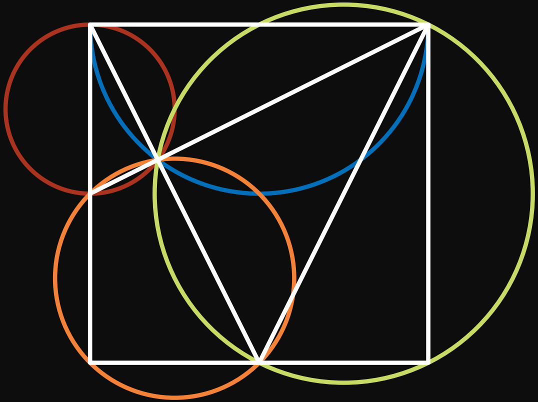
Since opposite angles add to π...



...C(Q4) and C(T5 $\cup$ T6) are cyclic quadrilaterals

The circumcircles

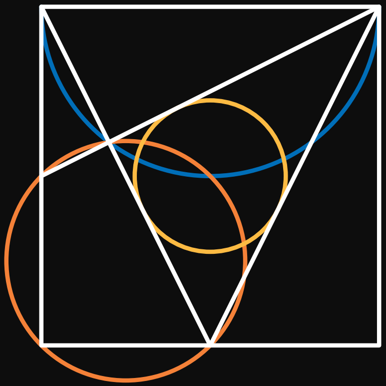
All circumcircles pass through a common point...



...and $C(T5 \cup T6)$ passes through the center of $C(Q4)$ and $C(T4)$

The circumcircles

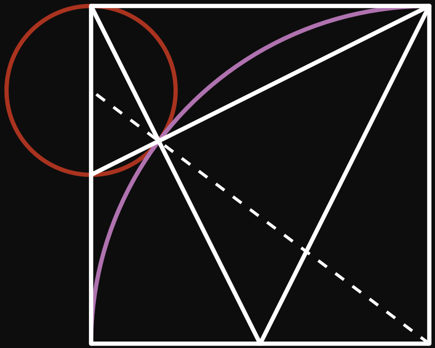
These circumcircles intersect at the square's center...



...which happens to be T6's incenter

Tangent circles

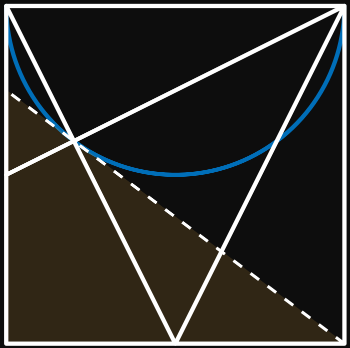
These three points are aligned...



...and these two circles are tangent

Tangent circles

The line is tangent to this circle...

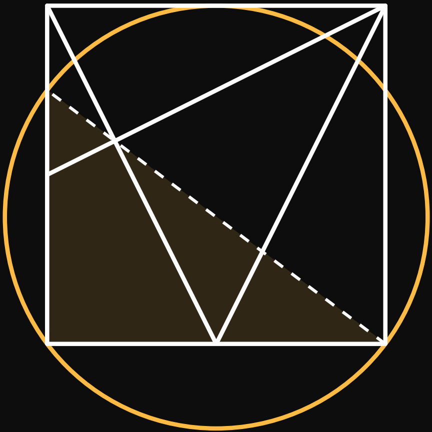


...and the right triangle below is an Egyptian Triangle

Tangent circles

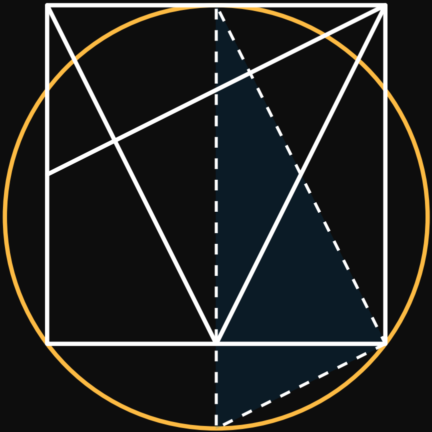
The circumcircle of that Egyptian Triangle...

...is tangent to the top side of the square



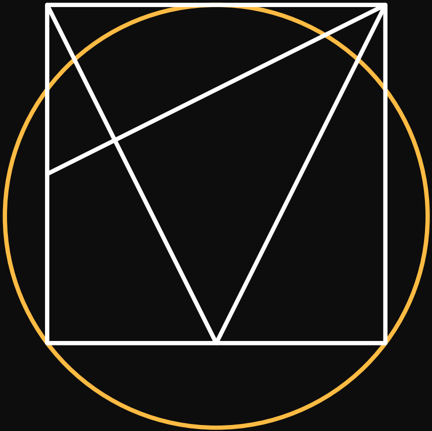
Tangent circles

It is also the circumcircle of this 1:2:$\sqrt{5}$ triangle



Tangent circles

And the ratio $\frac{\text{Square perimeter}}{\text{Circle perimeter}} = \frac{16}{5\pi}$ is very close to 1



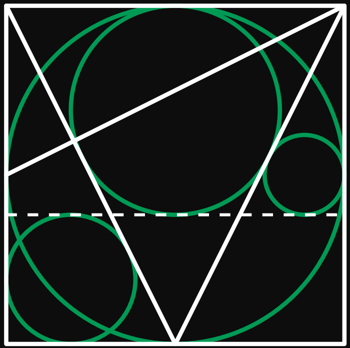
Tangent circles

The radius of these three circles are in ratio $1{:}\varphi^2{:}\varphi^4$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

Tangent circles

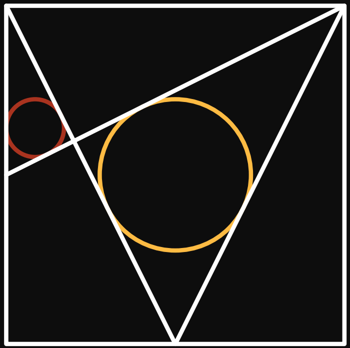
The radius of these four circles are in ratio $1{:}\varphi{:}\varphi^2{:}\varphi^3$



where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

Tangent circles

The radius of these two circles are in ratio $1{:}\varphi^2$



where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

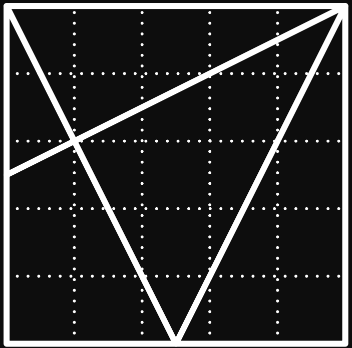
Tangent circles

The radius of these two circles are in ratio $1{:}\varphi^2$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

The underlying grid

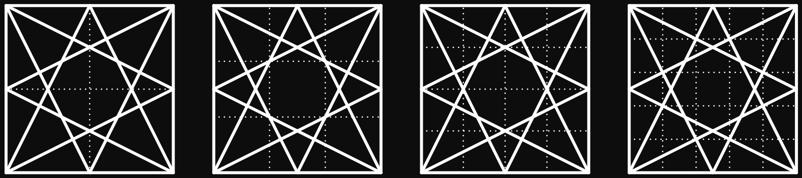
Using the intersection point of the Egyptian Tangram...



...you can divide the square into 5×5 smaller squares!

The underlying grid

Using the intersection points of this figure...



...you can divide the square into:

2×2, 3×3, 4×4 or 5×5 smaller squares!

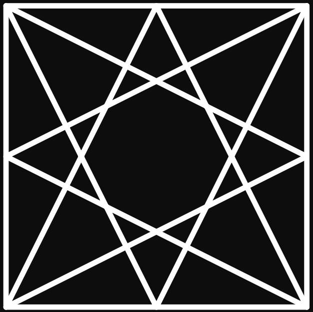
The underlying grid

There are 32 egyptian triangles in this figure...

...they come in 4 sizes and there are 8 of each kind

The underlying grid

There are 24 1:2:$\sqrt{5}$ triangles in this figure...



...they come in 3 sizes and there are 8 of each kind

The underlying grid

There are 24 other triangles in this figure...

...of 3 different kinds (one of them comes in 2 sizes)

The underlying grid

The relative sizes of these polygons are...

Small Triangles:	1	**Small Kites:**	3	**Whole Square:**	120
Big Triangles:	6	**Big Kites:**	8	**Octagon:**	20

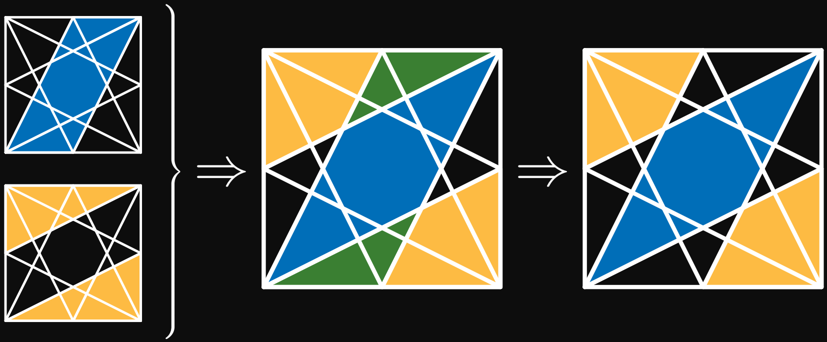
The carpets theorem

Since Area(BLUE) = Area(YELLOW)...

...Area(BLUE−GREEN) = Area(YELLOW−GREEN)

The carpets theorem

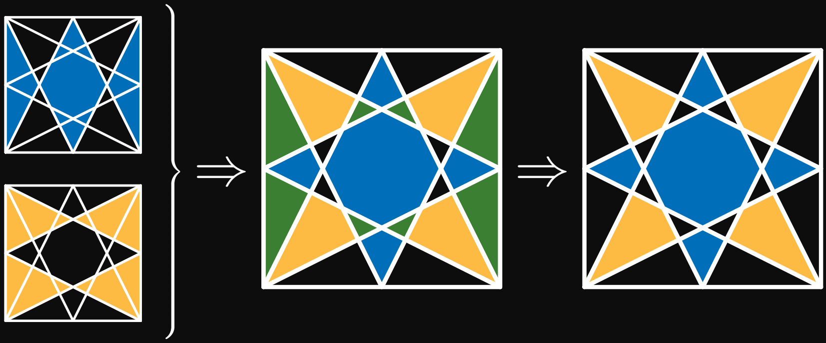
Since Area(BLUE) = Area(YELLOW)...



...Area(BLUE−GREEN) = Area(YELLOW−GREEN)

The carpets theorem

Since Area(BLUE) = Area(YELLOW)...



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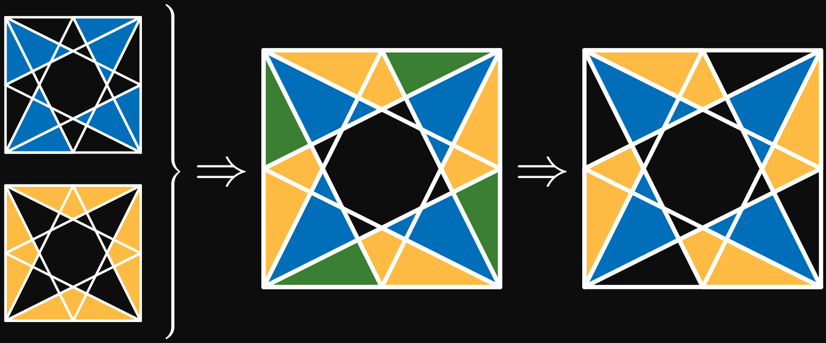

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The carpets theorem

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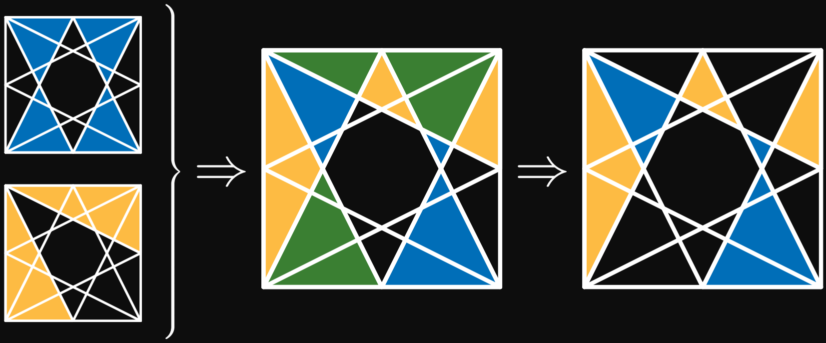
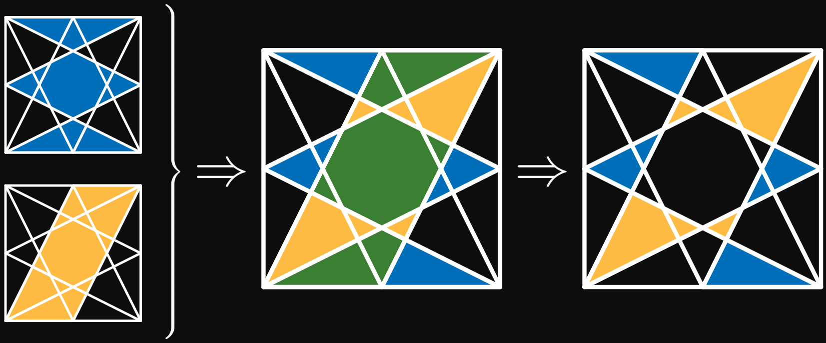


...Area(BLUE−GREEN) = Area(YELLOW−GREEN)

The carpets theorem

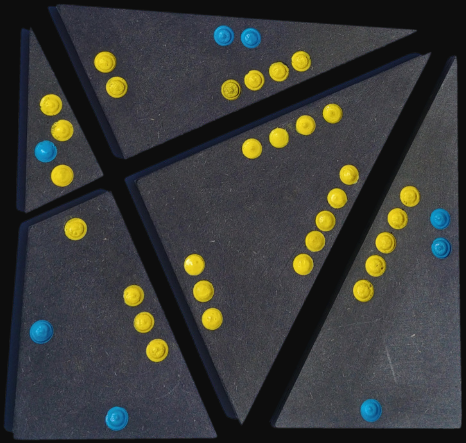
Since Area(BLUE) = Area(YELLOW)...



...Area(BLUE−GREEN) = Area(YELLOW−GREEN)

Divulgation of the Egyptian Tangram

Divulgation of the Egyptian Tangram



Wooden prototype for MMACA's exhibitions (2019)

Divulgation of the Egyptian Tangram

Design diary at Nou Biaix magazine 44 (2019)

100 • noubiaix 44

mmaca

el racó del mmaca

El tangram egipci: diari de disseny

Carlos Luna-Mota
Museu de Matemàtiques de Catalunya, MMACA
carlos.luna@mmaca.cat

Què és un tangram?

El tangram que tots coneixem és un trencaclosques d'origen xinès format per set peces planes, cinc triangles i dos quadrilàters, amb les quals es poden crear un gran nombre de figures [Gardner, 1988].

Figura 1. Tangram xinès.

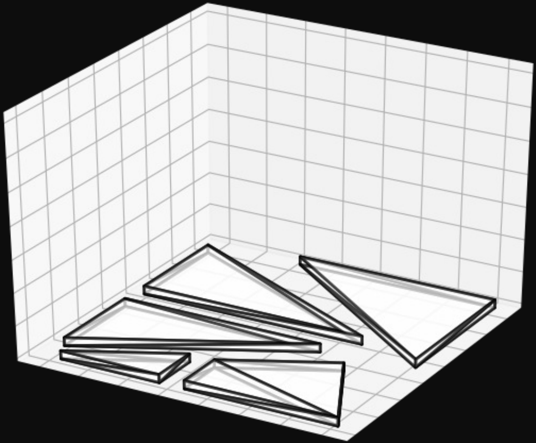
En la seva vessant més lúdica, el tangram s'acompanya d'un llibret on apareixen les siluetes de diverses figures, i el repte consisteix a recobrir completament cadascuna de les siluetes fent servir les set peces sense que sobresurtin de la figura o s'encavalquin entre elles. Sovint, les figures representen objectes o persones amb gran realisme, cosa sorprenent si tenim en compte la simplicitat geomètrica de les peces del tangram [Loyd, 2007].

Figura 2. Tangram xinès: silueta realista i solució.

Divulgation of the Egyptian Tangram

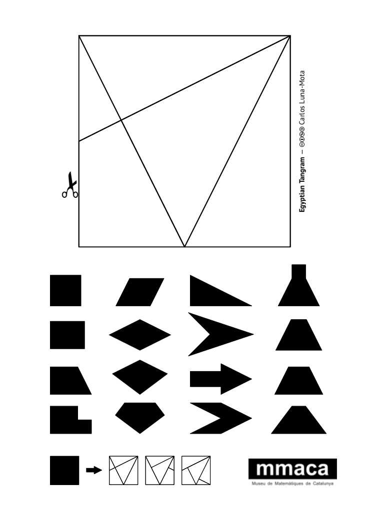
Talk at MMACA (2020)

Divulgation of the Egyptian Tangram



3D printer prototype (2020)

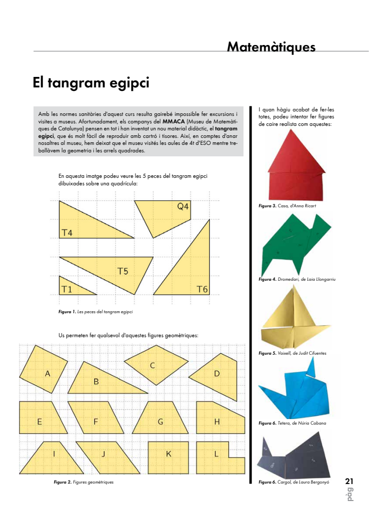
Divulgation of the Egyptian Tangram



Print-&-play flyer for families affected by Covid-19 (2020)

Divulgation of the Egyptian Tangram

The Egyptian Tangram as a high school learning activity (2020)



El tangram egipci

Amb les normes sanitàries d'aquest curs resulta gairebé impossible fer excursions i visites a museus. Afortunadament, els companys del **MMACA** (Museu de Matemàtiques de Catalunya) pensen en tot i han inventat un nou material didàctic, el **tangram egipci**, que és molt fàcil de reproduir amb cartró i tisores. Així, en comptes d'anar nosaltres al museu, hem deixat que el museu visités les aules de 4t d'ESO mentre treballàvem la geometria i les arrels quadrades.

En aquesta imatge podeu veure les 5 peces del tangram egipci dibuixades sobre una quadrícula:

Figura 1. *Les peces del tangram egipci*

Us permeten fer qualsevol d'aquestes figures geomètriques:

Figura 2. *Figures geomètriques*

I quan hàgiu acabat de fer-les totes, podeu intentar fer figures de caire realista com aquestes:

Figura 3. *Casa, d'Anna Ricart*

Figura 4. *Dromedari, de Laia Llongarriu*

Figura 5. *Vaixell, de Judit Cifuentes*

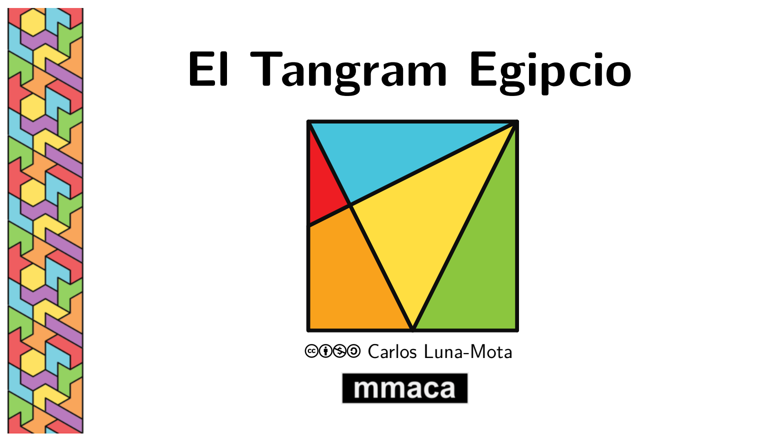
Figura 6. *Tetera, de Núria Cabana*

Figura 6. *Cargol, de Laura Bergonyó*

Divulgation of the Egyptian Tangram

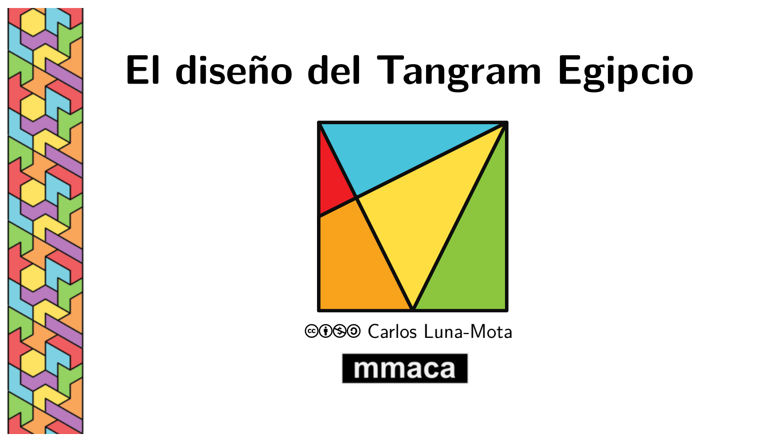
First commercial edition (2021)

Divulgation of the Egyptian Tangram



On-line talks for FUNDAPROMAT and MMACA's 7th anniversary (2021)

Divulgation of the Egyptian Tangram



Talk at the math-teaching conference JAEM 20 (2022)

Ideas? Suggestions? Use examples?

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