A Comparative Study of Goodness-of-Fit Tests for the Geometric Distribution and Application to Discrete Time Reliability

Cyril Bracquemond[1], **Emmanuelle Crétois**[2], **Olivier Gaudoin**[1]

[1]INP Grenoble - [2]University Pierre Mendes France

Tour IRMA, BP 53, 38041 Grenoble Cedex 9, France

Cyril.Bracquemond@imag.fr, Emmanuelle.Cretois@imag.fr, Olivier.Gaudoin@imag.fr

Abstract

This paper presents and compares several goodness-of-fit (GOF) tests for the geometric distribution. General families of GOF tests for discrete distributions are described : tests based on the empirical distribution function, on the empirical generating function, smooth tests and tests based on the generalized Smirnov transformation. These general methods are developed to provide GOF tests for the geometric distribution. The powers of these tests are compared through a simulation study. An application to real discrete reliability data is presented, which aims to detect ageing of systems.

Key Words : Model selection; nonparametric tests; EDF tests; lifetime analysis; ageing; discrete Weibull distribution.

1 Introduction

Fitting a probability model to observed data is an important statistical problem from both theory and application points of view. Goodness-of-fit (GOF) techniques have been widely studied and an extensive litterature is available (see for example the books by D'Agostino-Stephens 1986, Rayner-Best 1989, Greenwood-Nikulin 1996). Most work in this field deals with fitting continuous distributions. Rather few studies had been done for discrete distributions.

In this paper, we are interested in GOF tests for the geometric distribution. This work originates from a discrete reliability problem. In many reliability studies, clock time is not the best scale to describe lifetime. This is the case for example when an equipment operates on demand and the observation is the number of demands successfully completed before failure. It is also the case when lifetimes are number of cycles or number of months before failure or when lifetimes come from the grouping of continuous data. In all these situations, lifetimes are discrete random variables over the set $\mathbb{N}^*$ of positive integers. An important problem in reliability is to assess the ageing of systems. When systems are not ageing, their lifetimes are geometrically distributed. Then, testing the fit of the geometric distribution is testing the ageing of systems.

Several kinds of GOF tests for discrete distributions have been recently proposed. Some of them are based on the empirical distribution function, other on the empirical generating function. It is also possible to apply Neyman smooth tests. These families of tests are described in sections 2 to 4. We will not use the chi-square tests here, since they are known to have low power, especially when sample sizes are small, which is the case in reliability. In section 5, we present the idea of a test based on the generalized Smirnov transformation, proposed by Nikulin (1992). Since our purpose is to study GOF tests for the geometric distribution, which is a distribution over $\mathbb{N}^*$, all the tests are described for distributions over $\mathbb{N}^*$, even if they can be used for more general discrete distributions. In section 6, the proposed GOF tests are developed for the case of the geometric distribution. A comparative power study is done in section 7. The chosen alternatives are the type I discrete Weibull distribution and the shifted zero-inflated Poisson distribution. Finally, the tests are applied to real data in section 8.

The general framework for GOF tests is as follows. The observed data, denoted $K_1, ..., K_n$, are supposed to be n independent random variables with the same distribution over $\mathbb{N}^*$. Let F be the true unknown cumulative distribution function (CDF) of this distribution. Let $\mathcal{F} = \{F(.;\theta),\ \theta \in \Theta\}$ be a parametric family of CDF. Two kinds of GOF tests are possible :

Case 1 : Testing the fit of $K_1, ..., K_n$ to the completely specified distribution with CDF F_0, is testing :

$$\mathcal{H}_0 : \text{“}F = F_0\text{”} \text{ vs } \mathcal{H}_1 : \text{“}F \neq F_0\text{”}.$$

Case 2 : Testing the fit of $K_1, ..., K_n$ to the parametric family $\mathcal{F}$ is testing :

$$\mathcal{H}_0 : \text{“}F \in \mathcal{F}\text{”} \text{ vs } \mathcal{H}_1 : \text{“}F \notin \mathcal{F}\text{”}.$$

We will assume in the following that $F_0 = F(., \theta_0)$, where θ_0 is known.

In all the paper, the notation $X \rightsquigarrow \mathcal{L}$ means that the distribution of the random variable X is $\mathcal{L}$. $X_n \xrightarrow{d} \mathcal{L}$ means that the sequence of random variables $\{X_n\}_{n\geq 1}$ converges in distribution to $\mathcal{L}$ when n tends to infinity.

$\mathcal{U}[0,1]$ is the uniform distribution over $[0,1]$, $\mathcal{N}(0,1)$ is the standard normal distribution, $\mathcal{G}(p)$ is the geometric distribution with parameter p, $\mathcal{E}(\lambda)$ is the exponential distribution with parameter λ and χ^2_J is the chi-square distribution with J degrees of freedom. $\lfloor X \rfloor$ is the integer part of X.

2 Tests based on the empirical distribution function

In this section, we present GOF tests based on the empirical distribution function (EDF) $\mathbb{F}_n$ of the sample $K_1, ..., K_n$, defined by :

$$\forall k \in \mathbb{N}^*, \ \mathbb{F}_n(k) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{K_i \leq k\}}$$

The principle of EDF tests in case 1 is to reject $\mathcal{H}_0$ if $\mathbb{F}_n$ and F_0 are significantly different. The most famous statistics which measure the distance between these two functions are :

- the Kolmogorov-Smirnov statistic : $KS_n = \sqrt{n} \max_{k \in \mathbb{N}^*} |\mathbb{F}_n(k) - F_0(k)|$ (1)
- the Cramér-von Mises statistic : $W_n^2 = n \sum_{k=1}^{\infty} [\mathbb{F}_n(k) - F_0(k)]^2 p_0(k)$ (2)
- the Anderson-Darling statistic : $A_n^2 = n \sum_{k=1}^{\infty} \frac{[\mathbb{F}_n(k) - F_0(k)]^2 p_0(k)}{F_0(k)(1 - F_0(k))}$ (3)

where $\forall k \in \mathbb{N}^*, p_0(k) = F_0(k) - F_0(k-1)$ is the probability under $\mathcal{H}_0$ that the observed random variable is equal to k.

The distribution of the Kolmogorov-Smirnov statistic (1) under $\mathcal{H}_0$ has been studied by Schmid (1958), Conover (1972), Pettitt-Stephens (1977), Horn (1977) and Wood-Altavela (1978). These last authors gave the asymptotic distribution of KS_n under $\mathcal{H}_0$, using functionals of the discrete empirical process $\{\sqrt{n}\,(\mathbb{F}_n(k) - F_0(k))\}_{k \geq 1}$. The main difference between the continuous and the discrete cases is that the distribution of KS_n under $\mathcal{H}_0$ does not depend on F_0 for continuous distributions but depends on F_0 for discrete distributions.

The Cramér-von Mises and Anderson-Darling statistics (2) and (3) have been studied much more recently. Choulakian, Lockhart and Stephens (1994) gave the asymptotic distribution of these statistics under $\mathcal{H}_0$ for data grouped in cells, as a weighted sum of independent chi-square variables.

For testing the fit to a parametric family of distributions, the natural idea is to estimate the unknown parameter. Let $\hat{\theta}_n$ be the maximum likelihood estimator of θ. The EDF test statistics become :

$$\begin{aligned}
\widehat{KS}_n &= \sqrt{n} \max_{k \in \mathbb{N}^*} |\mathbb{F}_n(k) - F(k, \hat{\theta}_n)| & (4)\\
\widehat{W}_n^2 &= n \sum_{k=1}^{\infty} [\mathbb{F}_n(k) - F(k; \hat{\theta}_n)]^2 p(k; \hat{\theta}_n) & (5)\\
\widehat{A}_n^2 &= n \sum_{k=1}^{\infty} \frac{[\mathbb{F}_n(k) - F(k; \hat{\theta}_n)]^2 p(k; \hat{\theta}_n)}{F(k; \hat{\theta}_n)(1 - F(k; \hat{\theta}_n))} & (6)
\end{aligned}$$

Henze (1996) proved the convergence of the discrete estimated empirical process $\{\sqrt{n}(\mathbb{F}_n(k) - F(k; \hat{\theta}_n))\}_{k \geq 1}$ to a gaussian sequence with covariance depending on the tested distribution and on the true value of θ. Then, the asymptotic distribution of the test statistics under $\mathcal{H}_0$ depends on the unknown θ. So Henze recommended the use of parametric bootstrap to perform the tests in practice. On the other hand, Spinelli-Stephens (1997) used the method proposed by Choulakian et al to provide a table of quantiles for EDF tests to the family of Poisson distributions.

Computing the Kolmogorov-Smirnov statistic in practice is easy, since $\forall k \geq K_{(n)} = \max(K_1, \ldots, K_n)$, $\mathbb{F}_n(k) = 1$ and F is increasing. Then, (4) becomes :

$$\widehat{KS}_n = \sqrt{n} \max_{k \leq K_{(n)}} |\mathbb{F}_n(k) - F(k, \hat{\theta}_n)| \tag{7}$$

Similarly, to compute $\widehat{W}_n^2$ and $\widehat{A}_n^2$ in practice, the sums in (5) and (6) must be finite. The truncated statistics can be written as :

$$\begin{aligned}
\widehat{W}_n^2 &= n \sum_{k=M_l}^{M_u} [\mathbb{F}_n(k) - F(k; \hat{\theta}_n)]^2 p(k; \hat{\theta}_n) & (8)\\
\widehat{A}_n^2 &= n \sum_{k=M_l}^{M_u} \frac{[\mathbb{F}_n(k) - F(k; \hat{\theta}_n)]^2 p(k; \hat{\theta}_n)}{F(k; \hat{\theta}_n)(1 - F(k; \hat{\theta}_n))} & (9)
\end{aligned}$$

A first kind of truncation was proposed by Henze (1996) :

$$\begin{aligned}
M_l &= 1\\
M_u &= \min\{k \geq K_{(n)}; (1 - F(k; \hat{\theta}_n))^3 \leq 10^{-4}/n\}
\end{aligned}$$

Another one was proposed by Spinelli-Stephens (1997) :

$\mathbb{F}_n(M_l) - \mathbb{F}_n(M_l - 1) = 0$ and $\forall k < M_l$, $(F(k; \hat{\theta}_n) - F(k-1; \hat{\theta}_n)) < 10^{-3}/n$,

$\mathbb{F}_n(M_u) - \mathbb{F}_n(M_u - 1) = 0$ and $\forall k > M_u$, $(F(k; \hat{\theta}_n) - F(k-1; \hat{\theta}_n)) < 10^{-3}/n$.

The values of the statistics are not significantly changed by the truncations.

A test based on a statistic $\widehat{T}_n$ such as $\widehat{KS}_n$, $\widehat{W}_n^2$, or $\widehat{A}_n^2$ consists in rejecting the hypothesis $\mathcal{H}_0$ at level α if $\widehat{T}_n$ is greater than a critical value c_α. Since the asymptotic distribution of these statistics under $\mathcal{H}_0$ depends on the unknown parameter θ, the critical value has to be estimated from the data. This is done by parametric bootstrap.

This method has three steps. First, compute $\hat{\theta}_n$. Then, simulate a large number N of samples from $F(k;\hat{\theta}_n)$ and compute $\widehat{T}_n$ for each sample. Finally, order the obtained values of the statistic. c_α is estimated by the empirical quantile of order $(1-\alpha)$ of these values. Henze showed the validity of the method : when n and N tend to infinity, the significance level of the test tends to α.

3 Tests based on the empirical generating function

The probability generating function φ of the random variable K is defined for t in $]-1,1]$ as $\varphi(t;\theta)=\mathbb{E}\left(t^K\right)$. Then, the empirical generating function (EGF) of the sample $K_1,\ldots,K_n$ is defined as :

$$\forall\ t\in\,]-1,1],\ \varphi_n(t)=\frac{1}{n}\sum_{i=1}^{n}t^{K_i} \tag{10}$$

Kocherlakota and Kocherlakota (1986) proposed to build GOF tests based on the comparison of the EGF to a completely specified or estimated generating function. They showed that, for $t_0\in\,]-1,1]$:

$$\sqrt{n}\,\frac{\varphi_n(t_0)-\varphi(t_0;\theta)}{\varphi(t_0^2;\theta)-\varphi^2(t_0;\theta)}\xrightarrow{d}\mathcal{N}(0,1) \tag{11}$$

Then, it is possible to use this statistic to build a GOF test in case 1. In case 2, θ is replaced by its maximum likelihood estimator $\hat{\theta}_n$. The same authors proved that, under usual regularity conditions :

$$\widehat{KK}_n=\frac{\varphi_n(t_0)-\varphi(t_0;\hat{\theta}_n)}{\sigma_{(t_0;\hat{\theta}_n)}}\xrightarrow{d}\mathcal{N}(0,1) \tag{12}$$

where $\sigma^2_{(t_0;\theta)}=\dfrac{1}{n}\left(\varphi(t_0^2;\theta)-\varphi^2(t_0;\theta)\right)-\displaystyle\sum_{i=1}^{\dim\theta}\sum_{j=1}^{\dim\theta}\sigma_{ij}\frac{\partial\varphi(t_0;\theta)}{\partial\theta_i}\frac{\partial\varphi(t_0;\theta)}{\partial\theta_j}$ and σ_{ij} denotes the (i,j)th element of the inverse of the Fisher information matrix $J(\theta)$.

The hypothesis $\mathcal{H}_0$ is rejected at level α if $|\widehat{KK}_n|$ is greater than the quantile of order $(1-\alpha/2)$ of the standard normal distribution. For the Poisson distribution, the authors concluded from simulations that a good choice for t_0 is a small value, and proposed $t_0=0.125$.

The main drawback of the test from Kocherlakota and Kocherlakota is its dependence on the choice of t_0. To solve this problem, Rueda, Pérez-Abreu, and O'Reilly (1991) proposed a statistic which looks like a Cramér-von Mises one and allows to take into account all values of t.

Let us consider the estimated empirical process associated with the generating function:

$$\left\{\sqrt{n}(\varphi_n(t) - \varphi(t;\hat{\theta}_n))\right\}_{t\in]-1,1]} \tag{13}$$

The authors have shown that, when $\dim\theta = 1$, this process converges to a gaussian process with mean zero and covariance function $C_\theta(s,t)$, where :

$$C_\theta(s,t) = \varphi(st;\theta) - \varphi(s;\theta)\varphi(t;\theta) + (J(\theta) - 2)\frac{\partial\varphi(s;\theta)}{\partial\theta}\frac{\partial\varphi(t;\theta)}{\partial\theta} \tag{14}$$

The test statistic is then :

$$\widehat{RPO}_n = n\int_0^1 (\varphi_n(t) - \varphi(t;\hat{\theta}_n))^2 dt \tag{15}$$

Under $\mathcal{H}_0$, the distribution of $\widehat{RPO}_n$ depends on the true value of θ. So the critical values of this test are calculated by the bootstrap method described in section 2.

The method proposed by Baringhaus and Henze (1992) consists in characterizing the probability generating function of the tested distribution as the root of a differential equation. Then, an empirical differential equation can be used to build a test. For example, if the differential equation is written as $\psi(t;\varphi;\varphi';\theta) = 0$, the test statistic is :

$$\widehat{BH}_n = n\int_0^1 [\psi(t;\varphi_n;\varphi_n';\hat{\theta}_n)]^2 dt \tag{16}$$

where φ_n' is the derivative of the EGF.

$\mathcal{H}_0$ will be rejected for large value of $\widehat{BH}_n$. Since the distribution of $\widehat{BH}_n$ under $\mathcal{H}_0$ is unknown, bootstrap is needed here again.

4 Neyman smooth tests

The idea of smooth GOF tests, described by Rayner and Best (1987, 1989), is to imbed the tested distribution into a larger family defined by its probability density function :

$$g_J(k;\beta,\theta) = C(\beta,\theta)\exp\left\{\sum_{j=1}^{J}\beta_j h_j(k;\theta)\right\}(F(k;\theta) - F(k-1;\theta)) \tag{17}$$

where the h_j, $j \in \{1,\ldots,J\}$ are the first orthonormal polynomials for the tested distribution and $C(\beta,\theta)$ is a normalizing constant.

Then, testing the fit of F is testing $\mathcal{H}_0$: "$\beta = 0$" vs "$\beta \neq 0$". This can be done with the score statistic :

$$\widehat{S}_J = \left(\frac{1}{\sqrt{n}} \sum_{j=1}^{J} \sum_{i=1}^{n} h_j(K_i; \widehat{\theta}_n) \right)^t M_{0,\widehat{\theta}_n}^{-1} \left(\frac{1}{\sqrt{n}} \sum_{j=1}^{J} \sum_{i=1}^{n} h_j(K_i; \widehat{\theta}_n) \right) \tag{18}$$

where $M_{\beta,\theta} = I_{\beta,\beta} - I_{\beta,\theta} I_{\theta,\theta}^{-1} I_{\theta,\beta}$ and $I = \begin{pmatrix} I_{\theta,\theta} & I_{\theta,\beta} \\ I_{\beta\theta} & I_{\beta,\beta} \end{pmatrix}$ is the Fisher information matrix of the parametric family of densities $g_J(k; \beta, \theta)$. $M_{0,\widehat{\theta}_n}$ is assumed to be invertible. Under $\mathcal{H}_0$, $\widehat{S}_J \xrightarrow{d} \chi^2_J$.

5 Tests based on the generalized Smirnov transformation

Nikulin (1992) proposed to use the generalized Smirnov transformation (GST) to build GOF tests for discrete or continuous distributions. Hocine (1997) applied this idea for testing the fit of some completely specified discrete distributions.

The generalized Smirnov transformation is defined as (Greenwood-Nikulin 1996):

Proposition 1 *(GST) : Let X be a real random variable with CDF F_0 defined as $F_0(x) = P(X \leq x)$ and let F_0^- be defined as $F_0^-(x) = P(X < x)$. Let U be a random variable independent of X and uniformly distributed over $[0,1]$. Then, the random variable $T = F_0^-(X) + [F_0(X) - F_0^-(X)]U$ is uniformly distributed over $[0,1]$.*

For a discrete random variable K over $\mathbb{N}^*$, $\forall k \in \mathbb{N}^*$, $F_0^-(k) = F_0(k-1)$. Let $p_0(k) = P(K = k) = F_0(k) - F_0(k-1)$. The GST in this case can be written as :

$$T = F_0(K-1) + p_0(K)U \rightsquigarrow \mathcal{U}[0,1] \tag{19}$$

The GST for a whole sample $K_1, \ldots, K_n$ is :

Proposition 2 : *Let $K_1, \ldots, K_n$ be n independent random variables over $\mathbb{N}^*$ with the same CDF F_0. Let $U_1, \ldots, U_n$ be independent random variables independent of $K_1, \ldots, K_n$ and uniformly distributed over $[0,1]$. Then, the random variables $T_i = F_0(K_i - 1) + p_0(K_i)U_i$ are independent and uniformly distributed over $[0,1]$.*

Then, testing the fit of $K_1, \ldots, K_n$ to the completely specified distribution with CDF F_0, is testing the fit of $T_1, \ldots, T_n$ to the uniform distribution over $[0,1]$. Several GOF tests for the uniform distribution are available, for example in D'Agostino-Stephens (1986).

In case 2, the true CDF F_0 is replaced by the estimated CDF $F(., \hat{\theta}_n)$. For $i \in$

$\{1, \ldots, n\}$, let us define :

$$\widehat{T}_i = F(K_i - 1; \hat{\theta}_n) + p(K_i; \hat{\theta}_n) U_i \tag{20}$$

With Slutsky's theorem (Pollard 1984), it is easy to show that :

$$\forall i \in \{1, \ldots, n\}, \quad \widehat{T}_i \xrightarrow{d} \mathcal{U}[0,1] \tag{21}$$

However, testing the fit of $K_1, \ldots, K_n$ to the family $\mathcal{F}$ by testing the fit of $\widehat{T}_1, \ldots, \widehat{T}_n$ to the uniform distribution over $[0,1]$ is not possible. There is the same problem as for EDF tests for continuous distributions : the asymptotic distribution of the GOF test statistic when θ is replaced by $\hat{\theta}_n$ is not the same as it is when θ is known. So the asymptotic distribution of the test statistic under $\mathcal{H}_0$ has to be studied for each family of distributions as it is for continuous distributions.

6 Application to the Geometric Distribution

In this section, all the above presented GOF tests are developed for the case of the geometric distribution. At our knowledge, none of these tests, except the Neyman smooth test, have been used yet for the geometric distribution.

The geometric distribution with parameter $p \in]0,1[$, $\mathcal{G}(p)$, is characterized by its CDF:

$$\forall k \in \mathbb{N}^*, \; F(k;p) = 1 - (1-p)^k \tag{22}$$

or by its probability generating function :

$$\forall t \in]-1,1], \; \varphi(t;p) = \frac{pt}{1-(1-p)t} \tag{23}$$

The maximum likelihood estimator $\hat{p}_n$ of p for a geometric sample $K_1, \ldots, K_n$ is

$$\hat{p}_n = \frac{n}{\sum_{i=1}^{n} K_i} \tag{24}$$

For the EDF tests, we will use the statistics in (7, 8, 9) and the parametric bootstrap.

6.1 Test from Kocherlakota and Kocherlakota

Proposition 3 : *The Kocherlakota and Kocherlakota statistic for the geometric distribution is given by :*

$$\widehat{KK}_n = \sqrt{n} \frac{[1-(1-\hat{p}_n)t_0^2]\,[1-(1-\hat{p}_n)t_0]^3}{t_0(1-t_0)^2(1-\hat{p}_n)\sqrt{\hat{p}_n}} \left\{ \frac{1}{n} [1-(1-\hat{p}_n)t_0] \sum_{i=1}^{n} t_0^{K_i} - \hat{p}_n t_0 \right\} \tag{25}$$

Proof : Since $\dim\theta = 1$, the expression for $\sigma^2_{(t_0;p)}$ is :

$$\begin{aligned}
\sigma^2_{(t_0;p)} &= \frac{1}{n}\left(\varphi(t_0^2;p) - \varphi^2(t_0;p)\right) - J^{-1}(p)\left(\frac{\partial\varphi(t_0;p)}{\partial p}\right)^2 \\
&= \frac{1}{n}\left(\frac{pt_0^2}{1-(1-p)t_0^2} - \frac{p^2t_0^2}{\left[1-(1-p)t_0\right]^2}\right) - J^{-1}(p)\left(\frac{\partial\varphi(t_0;p)}{\partial p}\right)^2
\end{aligned}$$

The partial derivative is easy to compute :

$$\frac{\partial}{\partial p}\varphi(t;p) = \frac{t(1-t)}{[1-(1-p)t]^2}$$

and the Fisher information is $J(p) = \dfrac{n}{p^2(1-p)}$.
Then, we obtain :

$$\begin{aligned}
\sigma^2_{(t_0;p)} &= \frac{1}{n}\left(\frac{pt_0^2}{1-(1-p)t_0^2} - \frac{p^2t_0^2}{\left[1-(1-p)t_0\right]^2}\right) - \frac{p^2(1-p)t_0^2(1-t_0)^2}{n\left[1-(1-p)t_0\right]^4} \\
&= \frac{pt_0^2(1-p)^2(1-t_0)^4}{n[1-(1-p)t_0^2][1-(1-p)t_0]^4}
\end{aligned}$$

Now, we plug $\sigma^2_{(t_0;\hat{p}_n)}$ into the general formula (12) and we obtain the desired result.

■

Remark : Kocherlakota and Kocherlakota suggested to choose $t_0 = 0.125$ in the case of the Poisson distribution. There is no reason that this choice should also be a good one for the geometric distribution. In fact, it seems that a value such as $t_0 = 0.5$ should be better, in order to avoid a too small denominator in $\widehat{KK}_n$. Some simulation results will be shown in section 7, which confirm this feeling.

6.2 Test from Rueda, Pérez-Abreu and O'Reilly

Proposition 4 : *The Rueda et al. test statistic for the geometric distribution is* $\widehat{RPO}_n = RPO_n(\hat{p}_n)$*, where :*

$$\begin{aligned}
RPO_n(p) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{K_i+K_j+1} + \frac{np(p+1)}{(1-p)^2} + \frac{2np^2\ln p}{(1-p)^3} \\
&+ 2p\sum_{i=1}^{n}\left[\left(\sum_{j=1}^{K_i+1}\frac{1}{(1-p)^j(K_i+2-j)}\right) + \frac{\ln p}{(1-p)^{K_i+2}}\right] \qquad (27)
\end{aligned}$$

Proof :

$$\begin{aligned}
RPO(p) &= \int_0^1 n\left(\varphi_n(t)-\varphi(t;p)\right)^2 dt \\
&= \int_0^1 n\left(\frac{1}{n}\sum_{i=1}^n t^{K_i}-\frac{pt}{1-(1-p)t}\right)^2 dt \\
&= \int_0^1 n\left[\frac{1}{n^2}\left(\sum_{i=1}^n t^{K_i}\right)^2-\frac{2}{n}\frac{pt}{1-(1-p)t}\sum_{i=1}^n t^{K_i}+\frac{p^2t^2}{[1-(1-p)t]^2}\right]dt \\
&= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n\frac{1}{K_i+K_j+1}-2\sum_{i=1}^n\int_0^1\frac{pt^{K_i+1}}{1-(1-p)t}dt+n\int_0^1\frac{p^2t^2}{[1-(1-p)t]^2}dt
\end{aligned}$$

Let $A=2\sum_{i=1}^n\int_0^1\frac{pt^{K_i+1}}{1-(1-p)t}dt$ and $B=n\int_0^1\frac{p^2t^2}{[1-(1-p)t]^2}dt$.

After some calculation, we obtain :

$$\begin{aligned}
A &= 2p\sum_{i=1}^n\left[\left(\sum_{j=1}^{K_i+1}\frac{1}{(1-p)^j(K_i-j+2)}\right)+\frac{\ln p}{(1-p)^{K_i+2}}\right] \\
B &= \frac{np(p+1)}{(1-p)^2}+\frac{2np^2\ln p}{(1-p)^3}.
\end{aligned}$$

Then the proposition is proved. ■

6.3 Test from Baringhaus and Henze

Proposition 5 : *The Baringhaus and Henze test statistic for the geometric distribution is $\widehat{BH}_n=BH_n(\hat{p}_n)$, where :*

$$\begin{aligned}
BH_n(p) &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n\frac{K_iK_j}{K_i+K_j-1}-\frac{2}{n^2p}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\frac{K_i}{K_i+K_j+K_k-2} \\
&+ \frac{1}{p^2n^3}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n\frac{1}{K_i+K_j+K_k+K_l-3}. \qquad (28)
\end{aligned}$$

Proof : It is easy to prove that the probability generating function for the geometric distribution is the only root of the differential equation :

$$y'(t;p)-\frac{y^2(t;p)}{pt^2}=0 \qquad (29)$$

Then :

$$
\begin{aligned}
BH_n(p) &= n\int_0^1 \left(\varphi_n'(t) - \frac{\varphi_n^2(t)}{pt^2}\right)^2 dt \\
&= n\int_0^1 \left[\frac{1}{n}\sum_{i=1}^{n} K_i t^{K_i-1} - \frac{1}{pt^2}\left(\sum_{i=1}^{n} t^{K_i}\right)^2\right]^2 dt \\
&= \frac{1}{n}\int_0^1 \left(\sum_{i=1}^{n} K_i t^{K_i-1}\right)^2 dt - \frac{2}{p}\int_0^1 \frac{1}{t^2}\left(\sum_{i=1}^{n} K_i t^{K_i-1}\right)\left(\sum_{i=1}^{n} t^{K_i}\right)^2 dt \\
&+ \frac{n}{p^2}\int_0^1 \frac{1}{t^4}\left(\sum_{i=1}^{n} t^{K_i}\right)^4 dt
\end{aligned}
$$

After some computation, the above terms can be written as :

$$
\begin{aligned}
\int_0^1 \left(\sum_{i=1}^{n} K_i t^{K_i-1}\right)^2 dt &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{K_i K_j}{K_i + K_j - 1} \\
\int_0^1 \frac{1}{t^2}\left(\sum_{i=1}^{n} K_i t^{K_i-1}\right)\left(\sum_{i=1}^{n} t^{K_i}\right)^2 dt &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{K_i}{K_i + K_j + K_k - 2} \\
\int_0^1 \frac{1}{t^4}\left(\sum_{i=1}^{n} t^{K_i}\right)^4 dt &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \frac{1}{K_i + K_j + K_k + K_l - 3}
\end{aligned}
$$

Then the proposition is proved. ■

The computation of the RPO and BH statistics is not easy and needs some time. Then, the bootstrap method which consists in repeating a large number of times this computation, makes that these tests are very slow in practice.

6.4 Neyman smooth test

Rayner and Best (1989) showed, by means of a power study and comparison with the test of Vit (1974), that the best number of orthonormal polynomial to take in the smooth test for the geometric distribution is 4. Moreover, $M_{0,p} = Id$. Therefore, the best smooth test statistic for the geometric distribution is :

$$
\widehat{S}_4 = \frac{1}{n}\sum_{j=2}^{5}\left(\sum_{i=1}^{n} h_j(K_i;\hat{p}_n)\right)^2 \tag{30}
$$

where the h_j's are a particular case of Meixner polynomials :

$$
h_1(x;p) = \frac{px-1}{\sqrt{1-p}} \qquad\qquad h_2(x;p) = \frac{x^2p^2 + p(p-4)x + 2}{2(1-p)}
$$

$$
\begin{aligned}
h_3(x;p) &= \frac{x^3p^3 - x^2(9p^2 - 3p^3) + x(2p^3 - 9p^2 + 18p) - 6}{6\sqrt{1-p}(1-p)} \\
h_4(x;p) &= \frac{x^4p^4 + x^3(6p^4 - 16p^3) + x^2(72p^2 - 48p^3 + 11p^4)}{24(1-p)^2} \\
&+ \frac{x(6p^4 - 32p^3 + 72p^2 - 96p) + 24}{24(1-p)^2} \\
h_5(x;p) &= \frac{x^5p^5 + x^4(10p^5 - 25p^4) + x^3(35p^5 - 150p^4 + 200p^3)}{120\sqrt{1-p}(1-p)^2} \\
&+ \frac{x^2(50p^5 - 275p^4 + 600p^3 - 600p^2)}{120\sqrt{1-p}(1-p)^2} \\
&+ \frac{x(24p^5 - 150p^4 + 400p^3 - 600p^2 + 600p) - 120}{120\sqrt{1-p}(1-p)^2}
\end{aligned}
$$

Under $\mathcal{H}_0$, $\widehat{S}_4 \xrightarrow{d} \chi^2_4$.

For small samples, a correction is proposed by Rayner and Best. $\mathcal{H}_0$ will be rejected at level α if the observed value of $\widehat{S}_4$ is greater than the quantile of order $(1-\alpha)$ of the χ^2_4 distribution multiplied by a corrective quantity. For $\alpha = 0.05$, this quantity is $1 + 3.643/n - 2.314/\sqrt{n} - 0.447/\sqrt{n(1-\hat{p}_n)}$, where $n(1-\hat{p}_n)$ is replaced by n if it is smaller than 1.

6.5 Tests based on the generalized Smirnov transformation

In section 5, we pointed out the fact that GOF tests based on the GST need a particular study for each family of distribution. For the geometric family, we will use the fact that if $X \rightsquigarrow \mathcal{E}(\lambda)$, then $K = \lfloor X \rfloor + 1 \rightsquigarrow \mathcal{G}(1-e^{-\lambda})$. Conversely, it is possible with the GST to build an exponential sample from a geometric sample.

The GST (19) applied to the geometric distribution says that if $K \rightsquigarrow \mathcal{G}(p)$ and $U \rightsquigarrow \mathcal{U}[0,1]$, independent of K, then :

$$T = 1 - (1-pU)(1-p)^{K-1} \rightsquigarrow \mathcal{U}[0,1] \tag{31}$$

Using the usual transformation of the uniform distribution into the exponential distribution, we have :

$$X = K - 1 + \frac{\ln(1-pU)}{\ln(1-p)} \rightsquigarrow \mathcal{E}(-\ln(1-p)) \tag{32}$$

It is clear that $K = \lfloor X \rfloor + 1$.

As said in section 5, in case 1, testing that $K_1, \ldots, K_n$ are geometrically distributed with parameter p known is testing that $T_1, \ldots, T_n$ are uniformly distributed over $[0,1]$. It is equivalent to test that $X_1, \ldots, X_n$ are exponentially distributed with parameter $-\ln(1-p)$, where, $\forall i \in \{1, \ldots, n\}$:

$$X_i = K_i - 1 + \frac{\ln(1 - U_i p)}{\ln(1-p)} \tag{33}$$

In case 2, when p is unknown, the natural idea is to test the fit of the :

$$\widehat{X}_i = K_i - 1 + \frac{\ln(1 - U_i \hat{p}_n)}{\ln(1-\hat{p}_n)} \tag{34}$$

to the exponential family (we will assume that $\hat{p}_n < 1$ since the GOF test has no interest if $\hat{p}_n = 1$). This can be done, for example, with the (continuous) Anderson-Darling statistic.

This test is valid only if the asymptotic distribution of the test statistic is the same as the distribution of the corresponding statistic used for testing the exponential family. It will be true if the empirical process of the $\widehat{X}_i$'s converges to the same gaussian process as the empirical process of a usual exponential sample.

Since this result has not been theoretically proved yet, a comparison has been done by Monte Carlo simulations between the empirical quantiles obtained for the Anderson-Darling statistic based on a geometric sample and the theoretical quantiles of the asymptotic distribution of the Anderson-Darling statistic for an exponential distribution. The difference is very small, so it seems that the proposed test is valid and can be applied.

A GOF test of $K_1, \ldots, K_n$ to the geometric distribution is therefore simply a GOF test of $\widehat{X}_1, \ldots, \widehat{X}_n$ to the exponential distribution. The Anderson-Darling test has been chosen, and the test statistic will be denoted $\widehat{A}^2_{GST}$, but other tests are possible. The advantage of these tests is that they are well known and they do not need bootstrap.

The same method can be easily used to test the fit of other discrete distributions closely linked to usual continuous distributions, for example the type I discrete Weibull distribution defined in next section.

7 Simulation study

Eight tests are available for the geometric distribution : three EDF tests $\widehat{KS}_n$, $\widehat{W}^2_n$, $\widehat{A}^2_n$, three EGF tests $\widehat{KK}_n$, $\widehat{RPO}_n$, $\widehat{BH}_n$, the Neyman smooth test $\widehat{S}_4$ and the test based on the generalized Smirnov transformation $\widehat{A}^2_{GST}$. For the EDF tests, only $\widehat{W}^2_n$ and $\widehat{A}^2_n$ have been considered, since simulations have shown that they are always more powerful than $\widehat{KS}_n$.

The results are quickly obtained for $\widehat{KK}_n$, $\widehat{S}_4$ and $\widehat{A}^2_{GST}$. Because of the bootstrap, simulations are much longer for $\widehat{W}^2_n$, $\widehat{A}^2_n$, $\widehat{RPO}_n$ and $\widehat{BH}_n$. So we have used a cluster of PC's to fasten the computations.

The first study consists in checking the significance level of the tests. For asymptotic tests, this study is also useful for assessing the minimal sample size for applying the test correctly. For $p \in \{0.0001, 0.001, 0.01, 0.1, 0.3, 0.5, 0.7\}$, 2000 samples of size $n \in \{20, 40, 60, 80, 100\}$ of the $\mathcal{G}(p)$ distribution have been simulated. All tests are applied at the 5% significance level. In the bootstrap, 500 replications are used. For the $\widehat{W}^2_n$ and $\widehat{A}^2_n$ tests, the Henze stopping criterion has been choosed. Henze (1996) did a short similar study for the $\widehat{KS}_n$ test.

Table 1 gives the percentage of samples for which the geometric hypothesis is rejected. It can be seen that, for all tests and any sample size, the empirical significance level is very close to the nominal, especially for $\widehat{W}^2_n$, $\widehat{A}^2_n$ and $\widehat{A}^2_{GST}$. Then, all the tests can be used in practice.

The second step is a power study. We want to know which of these tests can reject the best the geometric hypothesis when data do not come from a geometric distribution. For that, two alternative distributions have been chosen.

The first one is the type I discrete Weibull distribution $\mathcal{W}(q, \beta)$, introduced by Nakagawa and Osaki (1975) in order to be the discrete counterpart of the widely used continuous Weibull distribution. This distribution is defined for $q \in]0, 1[$ and $\beta > 0$ by:

$$\forall k \in \mathbb{N}^*, \;\; P(K = k) = q^{(k-1)^\beta} - q^{k^\beta} \tag{35}$$

For $\beta < 1$, the distribution has a decreasing failure rate, so, if K is a system lifetime, the system is improving. For $\beta > 1$, the distribution has an increasing failure rate, so the system is ageing. For $\beta = 1$, the type I discrete Weibull distribution reduces to the geometric distribution.

The chosen values of β are 0.8 and 1.2, for which the type I discrete Weibull distribution is not too far from a geometric distribution. The sample sizes are the same as before and the values of q are $\{0.3, 0.5, 0.7, 0.9, 0.99, 0.999, 0.9999\}$. Then, the order of magnitude of simulated data varies from the unity to thousands. For each alternative, 2000 samples are simulated. The significance level of the tests is 5%. Tables 2 and 3 give the percentage of samples for which the geometric hypothesis is rejected.

The second alternative distribution is the zero-inflated Poisson (ZIP) distribution, which is a Poisson distribution with a Dirac mass w on zero. Since we are interested in distributions defined over $\mathbb{N}^*$, the ZIP distribution has to be shifted. Therefore, we consider the shifted zero-inflated Poisson distribution defined, for $w \in [0,1]$ and $\lambda > 0$ by:

$$\forall k \in \mathbb{N}^*,\ P(K=k) = w\mathbb{1}_{\{k=1\}} + (1-w)\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}\mathbb{1}_{\{k>1\}} \tag{36}$$

From the reliability point of view, the mass in 1 means that there are some design faults in the system which will lead it to fail at the first use. For w large enough, the mode of the distribution will be 1, as it is for the geometric distribution.

The chosen values of λ are 1 and 3. The sample sizes are the same as before and the values of w are $\{0.0, 0.1, 0.3, 0.5, 0.7, 0.8, 0.9\}$. For each alternative, 2000 samples are simulated. The significance level of the tests is 5%. Tables 4 and 5 give the percentage of samples for which the geometric hypothesis is rejected.

From these tables, it is clear that we can not conclude that one of these tests is universally the best. The quality of the tests depend on the real data distribution, on the parameters of this distribution, and on the sample size. A test can be very powerful in some situations and of very bad quality in other ones. The following remarks try to highlight the most interesting conclusions that can be drawn from this power study.

For the Weibull distribution with $\beta = 0.8$, the most powerful tests are clearly the EGF tests, especially $\widehat{BH}_n$ and $\widehat{RPO}_n$. $\widehat{S}_4$ and $\widehat{A}^2_{GST}$ have a satisfying power. When q is close to 1, the EDF tests and $\widehat{KK}_n$ have an extremely low power. A possible explanation for that lies in the order of magnitude of the data : when $q = 0.9999$, the sample mean is around 100000. It seems that numerical problems occur in the computation of the test statistics when data are too large. On the contrary, when q is very small, most observations are equal to 1. Then, data are highly discrete and that is why the $\widehat{A}^2_{GST}$ test, which generates continuous data from original discrete data, is less powerful than the others. Except for the problems mentioned, it seems that the power is, for all the tests, an increasing function of q.

For the Weibull distribution with $\beta = 1.2$, the EGF are still the most powerful for q small, but the difference with the EDF tests is much less than for $\beta = 0.8$. For q large, $\widehat{A}^2_{GST}$ is the best test. The low power problem for the EDF tests when q is very large is no longer true, but it is still the case for $\widehat{KK}_n$ and now also for $\widehat{RPO}_n$. $\widehat{S}_4$ is clearly the less powerful of all tests.

For the shifted ZIP distribution, the main feature is that, for all tests, the power is high when w is small or large, but is very low when w is medium (near 0.3-0.5). For w small, the ZIP distribution is close to a Poisson distribution, which is not similar to a geometric distribution. When w is close to 1, nearly all data are equal to 1, so this is also far from a (non-trivial) geometric distribution. It is for medium values of w that the shifted ZIP distribution is the closest to the geometric distribution. Then, the comparison between the tests is interesting mainly for these values. The EDF tests $\widehat{W}_n^2$ and $\widehat{A}_n^2$ are the most powerful. The EGF tests are just after for $\lambda = 1$ but they are the less powerful for $\lambda = 3$. $\widehat{S}_4$ is less powerful than the EGF tests for $\lambda = 1$, but it is better for $\lambda = 3$. $\widehat{A}_{GST}^2$ is the worse test for $\lambda = 1$ but is just after the EDF tests for $\lambda = 3$. In fact, when λ is small, most data are equal to 1 or 2, so $\widehat{A}_{GST}^2$ will be more powerful for larger values of λ.

Although none of the tests can be chosen as the best of all, some recommendations can be done. Among the EGF tests, $\widehat{BH}_n$ is the best. Among the EDF tests, $\widehat{A}_n^2$ is slightly the best. Only $\widehat{BH}_n$, $\widehat{S}_4$ and $\widehat{A}_{GST}^2$ never gave extremely bad results, but $\widehat{S}_4$ was sometimes largely less powerful than the others. So, finally, we recommend the use of $\widehat{BH}_n$, $\widehat{A}_n^2$ and $\widehat{A}_{GST}^2$. $\widehat{A}_{GST}^2$ should not be use when data are very small (lots of 1 or 2), but it has satisfying power otherwise. Moreover, it is the easiest to use since it does not need bootstrap.

Finally, table 6 compares the power of the $\widehat{KK}_n$ test for several values of t_0. It appears that the value 0.125 proposed in the case of the Poisson distribution should be replaced by 0.5 for the geometric distribution. This value has been chosen for the simulations in tables 1 to 5.

8 Application to real data and future work

In this section, the GOF tests presented are applied to two real sets of data.

The first data set presented in table 7 consists in numbers of thousands of demands before failure of electromechanical devices during mechanical reliability trials. Testing the fit of the geometric distribution is testing the ageing of these devices.

The second data set presented in table 8 consists in numbers of inspections between discovery of defects in an industrial process (Xie and Goh, 1993).

The seven GOF tests are applied to these data sets at significance level 5%. Table 9 gives the value of the statistics and the conclusion of the tests : R if the geometric distribution is rejected, A otherwise.

It appears that the results are the same for all tests : the geometric distribution is rejected for the electromechanical data and not rejected for the inspection data. The result for the electromechanical data means that these devices are ageing. So the lifetimes of these devices must be fitted with an ageing distribution, such as the type I Weibull discrete distribution.

Then, GOF tests for this kind of distributions are needed. The EGF tests can not be used for the discrete Weibull distribution because this distribution has no explicit probability generating function. Since the discrete Weibull distribution is closely linked to the continuous Weibull distribution, it is possible to use the generalized Smirnov transformation in this case. Then the application of this transformation for testing the fit of other discrete distributions than the geometric is a promising area of future research.

References

Baringhaus, L., and Henze, N. (1992), "A Goodness-of-Fit Test for the Poisson Distribution based on the Empirical Generating Function," *Statistics and Probability letters*, 13, 269-274.

Choulakian, V., Lockart, R.A., and Stephens, M.A. (1994), "Cramér-Von Mises Tests for Discrete Distributions," *Canadian Journal of Statistics*, 22, 125-137.

Conover, W.J. (1972), "A Kolmogorov Goodness-of-Fit Test for Discontinuous Distributions," *Journal of the American Statistical Association*, 67, 339, 591-596.

D'Agostino, R.B., and Stephens, M.A. (1986), *Goodness-of-Fit Techniques,* Marcel Dekker.

Greenwood, P.E., and Nikulin, M.S. (1996), *A Guide to Chi-Squared Testing,* Wiley Series in Probability and Statistics.

Henze, N. (1996), "Empirical Distribution Function Goodness-of-Fit Tests for Discrete Models," *Canadian Journal of Statistics*, 24, 1, 81-93.

Hocine, M.E. (1997), "Tests Basés sur les Processus Empiriques pour les Lois Discontinues," *Actes des XXIXèmes Journées de Statistique*, Carcassonne, 454-455. (In French)

Horn, S.D. (1977), "Goodness-of-Fit for Discrete Data," *Biometrics*, 33, 237-248.

Kocherlakota, S., and Kocherlakota, K. (1986), "Goodness-of-Fit Tests for Discrete Distributions," *Communications in Statistics, Theory and Methods*, 15, 815-829.

Nakagawa, T., and Osaki, S. (1975), "The Discrete Weibull Distribution," *IEEE Transactions on Reliability*, R-24, 5, 300-301.

Nikulin, M.S. (1992), "Gihman Statistic and Goodness-of-Fit Tests for Grouped Data," *C.R. Math. Rep. Acad. Sci. Canada*, 14, 4, 151–156.

Pettitt, A.N., and Stephens, M.A. (1977), "The Kolmogorov-Smirnov Goodness-of-Fit Statistics with Discrete and Grouped Data," *Technometrics*, 19, 205-210.

Pollard, D. (1984), *Convergence of Stochastic Processes, Springer Series in Statistics*-Springer-Verlag, 1984.

Rayner, J.C.W., and Best, D.J. (1987), "Goodness-of-Fit for Grouped Data using Components of Pearson's χ^2," *Computational Statistics and Data Analysis*, 5, 53-57.

Rayner, J.C.W., and Best, D.J. (1989), *Smooth Tests of Goodness-of-Fit,"* Oxford University Press.

Rueda, R., Pérez-Abreu, V., and O'Reilly, F. (1991), "Goodness-of-Fit Test for the Poisson Distribution based on the Probability Generating Function," *Communications in Statistics, Theory and Methods*, 20, 3093-3110.

Schmid, P. (1958), "On the Kolmogorov and Smirnov Limit Theorems for Discontinuous Distribution Functions," *Annals of Mathematical Statistics*, 29, 1011–1027.

Spinelli, J.J., and Stephens, M.A. (1997), "Cramér-Von Mises Tests of Fit for the Poisson Distribution," *Canadian Journal of Statistics*, 25, 257-268.

Vit, P. (1974), "Testing for Homogeneity; the Geometric Distribution," *Biometrika*, 61, 565–568.

Wood, C.L., and Altavela, M.M. (1978), "Large Sample Results for Kolmogorov-Smirnov Statistics for Discrete Distributions," *Biometrika*, 65, 235-239.

Xie, M., and Goh, T.N. (1993), "SPC of a Near Zero-Defect Process Subject to Random Shock," *Quality and Reliability Engineering International*, 9, 89-93.

Table 1: Empirical Significance Level for the Geometric Distribution

p	$\widehat{W}_n^2$	$\widehat{A}_n^2$	$\widehat{KK}_n$	$\widehat{RPO}_n$	$\widehat{BH}_n$	$\widehat{S}_4$	$\widehat{A}_{GST}^2$
				n=20			
0.0001	5.7	5.2	4.3	5.1	5.6	5.7	5.6
0.001	5.6	5.2	4.5	4.4	6.5	5.8	5.6
0.01	5.5	5.1	4.3	4.4	5.0	5.7	5.5
0.1	4.8	4.5	4.3	5.0	4.5	6.1	5.7
0.3	5.0	5.0	5.0	4.9	4.9	5.7	5.3
0.5	5.7	4.6	3.5	5.1	5.0	5.7	5.2
0.7	4.7	4.0	3.0	3.0	3.2	5.7	5.3
p				n=40			
0.0001	4.5	5.0	4.6	5.3	4.2	5.8	5.8
0.001	4.4	4.9	5.4	5.1	3.4	5.8	5.8
0.01	4.5	5.5	4.6	4.4	4.6	5.9	5.8
0.1	4.4	5.2	5.0	4.8	5.0	6.1	6.0
0.3	4.7	5.1	5.3	6.0	6.0	6.4	5.5
0.5	5.6	4.5	4.7	5.0	4.7	5.5	5.4
0.7	5.2	5.4	4.4	3.8	4.6	7.0	5.1
p				n=60			
0.0001	4.7	5.2	4.7	5.8	5.9	6.1	4.7
0.001	4.1	4.7	3.9	6.0	6.0	6.4	4.6
0.01	5.1	4.4	3.9	6.7	6.7	6.4	4.7
0.1	5.3	4.7	5.5	5.6	5.8	6.2	4.6
0.3	5.6	4.8	5.3	5.4	4.6	6.6	4.4
0.5	5.1	4.7	5.0	4.2	4.0	6.5	4.3
0.7	4.0	5.2	3.5	5.1	5.1	5.3	5.3
p				n=80			
0.0001	5.5	5.2	4.0	5.4	5.8	5.5	5.1
0.001	5.0	4.8	4.1	5.2	5.6	5.5	5.1
0.01	4.8	4.7	4.0	4.9	5.3	5.5	5.2
0.1	5.5	4.8	5.1	5.3	5.0	5.4	4.5
0.3	5.6	4.8	5.4	5.0	6.0	6.2	4.4
0.5	5.0	4.9	4.1	5.4	4.6	5.5	4.7
0.7	5.2	4.8	4.5	4.2	5.0	6.1	5.0
p				n=100			
0.0001	5.5	4.9	3.8	5.7	5.8	5.0	4.8
0.001	5.2	5.2	3.9	4.8	5.7	5.0	4.8
0.01	5.9	6.2	4.0	5.0	4.2	5.0	4.6
0.1	6.0	5.8	4.4	5.2	5.8	5.3	4.5
0.3	5.9	5.0	4.8	6.8	6.4	5.5	5.3
0.5	5.3	5.1	5.0	5.0	5.2	5.5	5.4
0.7	5.0	4.7	4.5	4.8	5.0	4.2	4.8

Table 2: Power Comparison for Weibull Distribution, $\beta = 0.8$

	$\widehat{W}_n^2$	$\widehat{A}_n^2$	$\widehat{KK}_n$	$\widehat{RPO}_n$	$\widehat{BH}_n$	$\widehat{S}_4$	$\widehat{A}_{GST}^2$
q				n=20			
0.9999	0.8	0.8	1.1	25.6	32.8	27.1	27.5
0.999	0.8	0.8	5.9	22.3	33.0	27.1	27.5
0.99	4.7	6.6	21.8	28.7	30.7	27.2	27.6
0.9	23.9	27.5	23.4	27.8	32.5	26.6	26.1
0.7	17.7	17.3	19.3	20.5	25.7	24.7	21.2
0.5	12.2	11.7	15.4	17.2	20.7	21.5	14.8
0.3	8.9	9.1	10.5	9.5	11.2	16.3	9.5
q				n=40			
0.9999	0.3	0.3	1.5	49.2	52.5	39.4	45.0
0.999	0.4	0.4	9.6	48.4	52.4	39.4	45.0
0.99	38.0	34.7	30.7	45.1	50.8	39.5	44.9
0.9	39.9	43.5	37.4	45.0	50.3	39.1	42.7
0.7	38.2	31.5	34.9	35.9	41.2	36.3	34.0
0.5	23.0	23.3	27.6	25.9	29.3	30.4	22.2
0.3	16.5	15.4	19.5	19.9	21.2	23.2	11.3
q				n=60			
0.9999	0.0	0.0	2.0	65.0	68.5	49.8	57.4
0.999	15.2	1.0	12.5	64.2	69.5	49.8	57.4
0.99	60.0	57.0	31.5	60.7	65.3	49.7	57.5
0.9	54.4	55.6	50.1	56.3	61.1	49.6	50.5
0.7	47.5	44.0	48.3	50.6	56.2	46.0	45.3
0.5	35.5	33.8	38.8	39.2	40.8	38.8	29.3
0.3	24.2	22.5	24.5	29.2	30.0	27.9	13.5
q				n=80			
0.9999	0.1	0.1	3.3	80.2	90.4	58.2	69.1
0.999	49.3	7.3	17.8	78.5	88.2	58.2	69.1
0.99	70.3	72.3	41.4	74.1	85.6	58.2	69.3
0.9	63.5	66.8	70.1	85.3	88.4	58.0	68.1
0.7	58.2	55.7	69.4	71.1	69.3	53.8	56.3
0.5	44.3	43.4	56.5	59.2	64.8	45.6	38.8
0.3	29.4	27.9	32.8	34.4	34.0	33.4	17.8
q				n=100			
0.9999	0.2	0.0	6.3	97.1	98.1	67.2	77.8
0.999	6.9	26.5	31.8	97.0	98.3	67.2	77.7
0.99	76.9	83.3	60.8	96.4	97.5	67.3	77.5
0.9	71.6	76.2	94.0	96.2	97.3	66.5	76.5
0.7	66.6	66.5	94.1	91.2	91.5	61.3	64.8
0.5	52.3	52.8	86.2	85.2	87.1	51.6	44.4
0.3	34.4	34.5	37.9	29.6	22.8	36.7	18.3

Table 3: Power Comparison for Weibull Distribution, $\beta = 1.2$

	$\widehat{W}_n^2$	$\widehat{A}_n^2$	$\widehat{KK}_n$	$\widehat{RPO}_n$	$\widehat{BH}_n$	$\widehat{S}_4$	$\widehat{A}_{GST}^2$
q				n=20			
0.9999	18.3	17.9	0.8	2.3	6.9	8.7	11.1
0.999	17.4	17.4	2.5	1.6	7.7	8.7	11.0
0.99	10.8	10.4	0.4	8.5	6.6	8.9	11.0
0.9	13.6	14.5	11.9	14.9	7.2	9.1	11.0
0.7	14.3	14.0	12.4	12.1	7.3	9.2	9.9
0.5	13.2	11.9	6.7	9.0	7.5	9.1	7.1
0.3	6.5	6.3	3.0	2.7	3.6	5.8	5.6
q				n=40			
0.9999	24.5	24.0	1.0	0.5	17.5	12.8	22.5
0.999	18.0	19.0	1.3	8.0	23.0	12.8	22.6
0.99	17.2	14.1	18.0	33.0	23.5	12.9	22.6
0.9	23.1	23.4	22.0	24.1	16.2	13.0	22.1
0.7	21.2	20.4	22.1	23.2	18.0	12.19	18.5
0.5	18.1	15.9	13.6	15.2	14.2	9.4	10.3
0.3	10.2	10.0	5.4	1.5	1.5	6.7	5.8
q				n=60			
0.9999	30.0	32.7	1.3	2.1	29.5	17.6	33.5
0.999	23.0	24.9	0.7	18.0	33.0	17.6	33.5
0.99	25.2	26.7	29.8	30.2	29.5	17.6	33.1
0.9	29.5	34.5	32.9	35.7	26.8	17.2	31.9
0.7	25.8	30.2	32.0	31.9	25.3	14.7	23.7
0.5	22.2	22.5	20.7	21.2	24.5	10.7	14.2
0.3	13.7	13.2	9.9	12.3	12.6	7.1	7.4
q				n=80			
0.9999	40.1	36.6	1.0	7.2	42.9	24.4	43.5
0.999	34.0	31.5	0.3	28.0	46.6	24.3	43.5
0.99	37.6	39.0	12.3	39.4	47.2	24.4	43.4
0.9	42.9	44.5	42.2	46.5	41.1	23.6	39.9
0.7	36.0	37.6	40.2	34.2	34.5	20.4	28.9
0.5	30.5	26.8	26.9	32.0	28.7	12.8	16.0
0.3	17.3	16.8	13.3	15.4	15.0	7.4	8.0
q				n=100			
0.9999	45.9	39.5	2.5	15.2	65.8	29.8	52.2
0.999	43.2	38.2	0.1	38.6	58.7	29.8	52.2
0.99	48.4	41.8	45.3	48.2	56.5	29.9	52.1
0.9	50.9	51.5	82.5	55.8	51.4	29.1	47.8
0.7	44.8	44.4	77.3	49.6	39.4	23.6	37.0
0.5	36.7	33.8	61.3	34.5	32.0	15.7	21.0
0.3	22.7	20.6	16.3	18.8	17.8	8.6	7.7

Table 4: Power Comparison for the Shifted ZIP Distribution, $\lambda = 1$

	$\widehat{W}_n^2$	$\widehat{A}_n^2$	$\widehat{KK}_n$	$\widehat{RPO}_n$	$\widehat{BH}_n$	$\widehat{S}_4$	$\widehat{A}_{GST}^2$
w				n=20			
0.0	44.9	43.5	38.0	39.4	34.3	35.8	26.7
0.1	25.7	24.8	20.5	21.4	17.6	19.8	15.0
0.3	6.3	6.2	3.8	4.6	3.9	5.3	4.9
0.5	3.2	2.9	2.0	2.2	2.8	5.0	5.8
0.7	11.3	6.9	8.1	5.6	11.1	14.2	7.9
0.8	21.1	8.5	13.3	7.8	20.1	22.5	7.0
0.9	47.4	5.9	11.7	5.2	41.3	25.2	5.3
w				n=40			
0.0	70.2	69.1	69.1	70.4	65.9	54.9	50.7
0.1	45.8	44.9	44.5	45.5	41.2	29.6	28.9
0.3	9.2	8.7	7.1	7.8	6.9	4.4	7.0
0.5	5.4	5.7	2.2	2.5	3.8	5.0	4.9
0.7	20.5	20.9	16.8	16.1	18.2	27.5	8.3
0.8	34.7	31.9	26.5	26.8	32.8	40.4	8.0
0.9	44.3	27.7	32.9	25.0	43.1	47.9	6.3
w				n=60			
0.0	85.8	85.7	87.7	88.7	85.7	74.6	71.3
0.1	61.7	61.9	63.9	64.5	59.7	42.9	42.9
0.3	10.7	10.9	11.8	12.0	10.1	4.3	8.5
0.5	6.6	7.6	2.4	3.6	4.8	5.3	5.3
0.7	30.8	31.3	23.6	25.2	28.2	31.2	11.0
0.8	51.6	51.6	43.3	43.2	47.4	48.9	11.6
0.9	58.0	47.2	50.5	45.8	56.0	55.0	8.7
w				n=80			
0.0	94.4	94.4	95.7	96.8	95.0	85.0	85.3
0.1	72.8	75.8	78.8	79.8	74.4	56.5	56.2
0.3	14.6	13.8	16.1	16.8	13.8	5.3	8.7
0.5	8.0	9.0	3.1	3.6	4.8	6.1	6.6
0.7	43.4	44.6	33.2	34.4	38.8	38.5	14.3
0.8	60.6	60.4	53.7	56.0	58.0	58.1	14.8
0.9	67.4	63.2	61.5	58.2	62.4	66.9	10.0
w				n=100			
0.0	98.0	98.0	98.5	99.0	98.4	93.2	92.2
0.1	85.4	86.2	87.8	88.4	86.4	67.4	67.0
0.3	14.6	15.4	20.1	16.6	14.8	7.2	11.5
0.5	7.8	9.4	3.1	3.4	4.8	7.2	6.2
0.7	54.4	56.2	39.1	42.0	47.2	42.6	16.5
0.8	72.2	72.4	64.6	68.0	70.8	66.0	17.5
0.9	70.8	70.6	70.7	66.2	70.4	72.8	12.0

Table 5: Power Comparison for the Shifted ZIP Distribution, $\lambda = 3$

	$\widehat{W}_n^2$	$\widehat{A}_n^2$	$\widehat{KK}_n$	$\widehat{RPO}_n$	$\widehat{BH}_n$	$\widehat{S}_4$	$\widehat{A}_{GST}^2$
w				n=20			
0.0	96.1	96.4	97.0	97.3	92.1	94.2	93.7
0.1	67.6	67.6	66.8	66.6	51.6	63.0	64.4
0.3	12.8	13.1	8.1	7.8	5.2	9.9	14.9
0.5	23.3	22.3	10.0	10.6	19.9	25.9	19.8
0.7	72.1	66.2	52.0	53.1	67.3	73.3	44.0
0.8	80.3	74.9	71.3	70.5	80.6	85.1	48.0
0.9	84.7	56.4	65.6	54.9	80.9	75.6	35.3
w				n=40			
0.0	100	100	100	100	100	100	100
0.1	95.9	96.1	93.6	94.9	91.1	90.5	92.8
0.3	38.3	40.0	13.2	15.0	10.3	12.9	28.7
0.5	58.9	59.4	18.9	18.6	33.1	48.7	40.4
0.7	98.0	98.0	87.3	90.1	95.4	95.3	82.1
0.8	99.1	98.6	97.5	97.5	99.0	98.6	82.7
0.9	95.5	89.7	93.4	89.5	95.0	95.5	61.4
w				n=60			
0.0	100	100	100	100	100	100	100
0.1	99.3	99.2	99.0	99.1	98.2	98.4	99.1
0.3	60.3	61.0	16.5	21.1	16.2	19.4	47.8
0.5	87.2	87.8	27.8	32.3	53.6	68.9	63.5
0.7	100	100	97.3	98.6	99.6	99.7	95.5
0.8	100	100	99.8	100	100	99.9	96.1
0.9	99.2	98.0	98.8	97.8	99.2	98.9	79.9
w				n=80			
0.0	100	100	100	100	100	100	100
0.1	100	100	99.9	100	100	99.9	100
0.3	89.0	90.4	20.7	29.6	22.8	36.1	65.8
0.5	95.2	95.4	36.8	46.2	66.8	84.3	81.4
0.7	100	100	99.7	100	100	100	99.4
0.8	100	100	100	100	100	100	99.7
0.9	99.8	99.6	99.7	99.6	99.8	99.7	91.7
w				n=100			
0.0	100	100	100	100	100	100	100
0.1	100	100	100	100	100	100	100
0.3	89.0	90.4	24.1	29.6	22.8	54.9	80.6
0.5	98.4	98.4	45.1	45.6	74.0	94.4	90.7
0.7	100	100	100	100	100	100	100
0.8	100	100	100	100	100	100	99.8
0.9	100	100	100	100	100	100	95.5

Table 6: Comparison of Different Values of t_0 for the $\widehat{KK}_n$ Statistic

$\mathcal{G}(p)$	$t_0 = 0.125$				$t_0 = 0.333$				$t_0 = 0.5$			
p	n=20	n=40	n=80	n=100	n=20	n=40	n=80	n=100	n=20	n=40	n=80	n=100
0.3	5.2	5.8	5.5	4.2	5.2	5.4	5.4	4.6	5.2	5.3	5.4	4.8
0.1	3.6	4.6	5.2	4.8	4.1	4.6	5.2	4.7	4.3	5.0	5.1	4.4
0.01	6.1	5.8	4.5	4.9	5.3	5.2	4.5	4.5	4.5	4.6	4.0	4.0
$\mathcal{W}(q, 1.2)$	$t_0 = 0.125$				$t_0 = 0.333$				$t_0 = 0.5$			
q	n=20	n=40	n=80	n=100	n=20	n=40	n=80	n=100	n=20	n=40	n=80	n=100
0.7	13.1	21.5	37.8	44.7	13.3	22.5	39.3	47.5	12.4	22.1	39.5	48.4
0.9	7.2	14.8	30.3	38.1	9.6	19.1	38.5	46.0	11.9	22.0	43.6	52.0
0.99	1.0	0.5	0.2	0.1	0.7	0.2	1.7	9.0	0.4	0.2	11.2	17.5
$\mathcal{W}(q, 0.8)$	$t_0 = 0.125$				$t_0 = 0.333$				$t_0 = 0.5$			
q	n=20	n=40	n=80	n=100	n=20	n=40	n=80	n=100	n=20	n=40	n=80	n=100
0.7	17.5	31.9	52.7	61.6	18.2	33.6	56.7	66.4	19.3	34.9	59.1	69.4
0.9	21.1	32.7	51.2	58.6	22.4	35.4	56.5	65.6	23.4	37.4	60.6	70.1
0.99	19.2	33.8	25.1	27.9	19.8	32.6	32.3	37.1	21.8	30.7	36.1	41.4

Table 7: Data Set 1 - Electromechanical Devices Data

12	15	15	15	15	17	18	18	19	20	20	22
22	23	23	24	25	25	25	29	31	32	32	

Table 8: Data Set 2 - Inspection Data

13	5	2	1	2	1	9	1	3	2	1	4	1	4
1	2	29	5	18	14	7	17	3	14	3	11	26	4

Table 9: Application of GOF Tests to Real Data

	$\widehat{W}_n^2$	$\widehat{A}_n^2$	$\widehat{KK}_n$	$\widehat{RPO}_n$	$\widehat{BH}_n$	$\widehat{S}_4$	$\widehat{A}^2_{GST}$	
Data set 1	0.925	4.685	-1.975	0.154	34.939	23.301	5.897	R
Data set 2	0.164	0.835	1.168	0.038	3.571	4.390	1.084	A