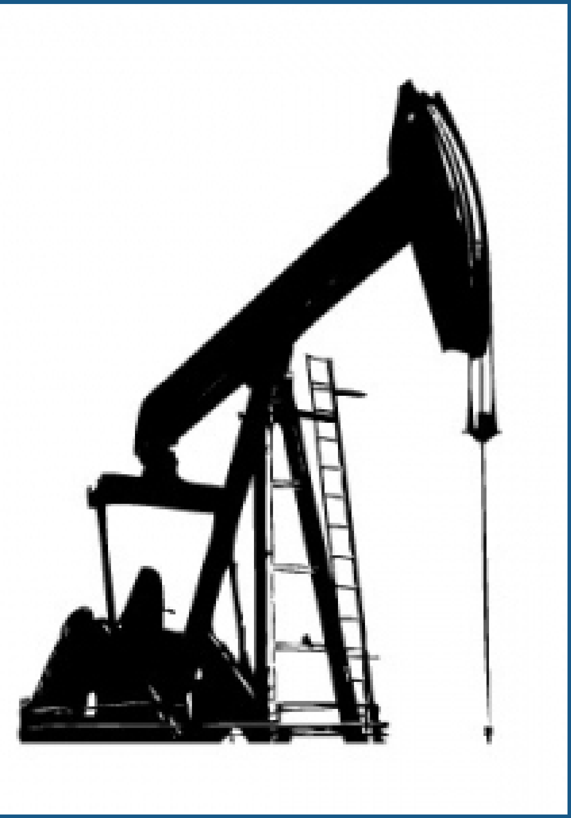
ROD PUMPING

Modern Methods of Design, Diagnosis and Surveillance



. . . wave equation in the form used in rod pumping

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2} - c\frac{\partial y(x,t)}{\partial t} + g$$

by

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THE WAVE EQUATION AS APPLIED TO ROD PUMPING

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2.1 Derivation of Wave Equation

The purpose of this chapter is to establish the wave equation as the fundamental equation of rod pumping. The wave equation is used to model the elastic behavior of the rod string. This famous equation arises from Newtonian dynamics and Hooke's law of elasticity. Its derivation is shown in many texts but usually without consideration of friction (Reference 1).

Since downhole friction in a rod pumping system cannot be ignored, a derivation is given here which includes friction. Figure 2.1 shows the forces acting on a rod element of length Δx with measured depth x increasing downward. Velocities and forces are also considered positive when oriented downward. The term $\rho\, \mathrm{A}\, \Delta x \,/\, 144 g_c$ represents rod mass. Presuming that the element is moving downward, a friction force $c' \Delta x \partial y(x,t) \,/\, \partial t$ acts in the upward direction.

The forces $F(x,t)$ and $F(x+\Delta x,t)$ are axial forces along the rod. Newton's law states that unbalanced forces on the rod element cause an acceleration of the element. Use of this law requires that velocities and accelerations be referred to a fixed coordinate system, say relative

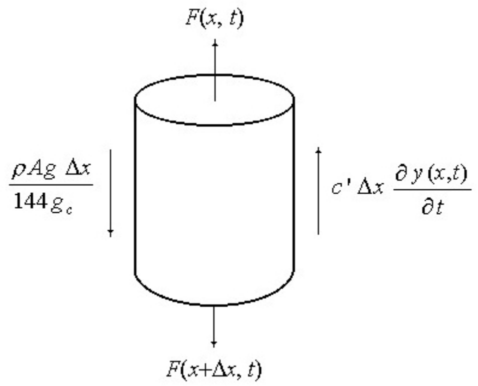

Figure 2.1. Forces on rod element.

to the well casing. Thus

$$\frac{\rho A \Delta x}{144\, g_c}\frac{\partial^2 y\ (x,t)}{\partial t^2} = F\,(x+\Delta x,t) - F\,(x,t) - c'\Delta x \frac{\partial y\,(x,t)}{\partial t} + \frac{\rho A g \Delta x}{144\, g_c}.$$

Using the definition of the partial derivative and passing to the limit as $\Delta x \to 0$, we obtain

$$\frac{\rho\, A}{144\, g_c}\ \frac{\partial^2 y\ (x,t)}{\partial\, t^2} \ = \ \frac{\partial\, F\,(x,t)}{\partial\, x} \ - \ c'\ \frac{\partial\, y\,(x,t)}{\partial\, t} \ + \ \frac{\rho\, A\, g}{144\, g_c} \quad .$$

Introducing the one-dimensional form of Hooke's law,

$$F\,(x,t) \ = \ E\,A\ \frac{\partial\, y\,(x,t)}{\partial\, x} \quad ,$$

we obtain the wave equation in the form used in rod pumping,

$$\frac{\partial^2 y\,(x,t)}{\partial\, t^2} \ = \ v^2\ \frac{\partial^2 y\,(x,t)}{\partial\, x^2} \ - \ c\ \frac{\partial\, y\,(x,t)}{\partial\, t} + g \quad , \tag{2.1}$$

in which

$$v \ = \ \sqrt{\frac{144\, E\, g_c}{\rho}} \tag{2.2}$$

and

$$c \ = \ \frac{144\, c'\, g_c}{\rho\, A} \quad . \tag{2.3}$$

2.2 Modeling of Downhole Friction

In Equation 2.1, friction along the rod string is presumed to be proportional to local velocity relative to the casing (Reference 2). This is not strictly correct. In reality, friction is a complicated blend of Coulomb (rod–tubing drag) and fluid friction effects. Fluid friction depends on relative velocity between rods and fluid (not rods and casing). Coulomb friction is not considered to be velocity-dependent at all. The role of the friction term is to simulate removal of energy from the rod string. Experience shows that the same steady-state dynamometer card results regardless of the manner in which the pumping equipment is started from rest. Downhole friction damps out the initial transients in only a few strokes of the pumping system, and the dynamometer cards begin to repeat. We want the simulated pumping system to exhibit this same behavior. In spite of its theoretical shortcomings, the friction law postulated in Equation 2.1 has been found to be useful and precise enough for practical purposes, at least in vertical wells. The friction law in Equation 2.1 is called equivalent friction. The criterion for equivalence is that the coefficient c is chosen to remove as much energy as do the real frictional forces.

It is convenient to introduce a non-dimensional factor λ by means of

$$c = \frac{\pi\, v \lambda}{2L} \tag{2.4}$$

The non-dimensional factor varies over narrow limits and is easier to remember than the dimensional quantities c and c'. The dimensional quantity c' and the non-dimensional factor are related through

$$c' = \frac{\pi\, v \lambda\, \rho\, A}{288 g_c L} \quad . \tag{2.5}$$

To get an impression of the magnitude of downhole friction in a vertical well, consider

Example 2.1
Using the dimensional quantity above, determine the damping force on 2000 ft of 0.875 in. rods moving at a uniform speed of 10 ft/sec given the basic information:

$\lambda = 0.1$ (a commonly used non-dimensional damping value for low-viscosity pumping installations)

v = 16000 ft /sec (velocity of stress waves in the rod material)

ρ = 490 lb_m / ft^3 (density of steel rods)

A = 0.6012 $in.^2$ (area of 0.875 in. rod)

g_c = 32.17 lb_m ft /lb_f sec^2 (gravitational conversion constant)

L= 4000 ft (pump depth) ★

Solution.

Using Equation 2.5, convert from the nondimensional damping format to obtain the dimensional coefficient

$$c' = \frac{\pi\,(16000)\,(0.1)\,(490)\,(0.6012)}{288\,(32.17)\,(4000)} = 0.0399 \text{ lb / ft / ft / sec .}$$

Expressed in words, the dimensional quantity means that a frictional force of 0.0399 lb_f is exerted per foot of rod length per foot per second of rod velocity. Thus, for 2000 ft of 0.875 in. rods moving at an average velocity of 10 ft/sec, the frictional force would be 798 lb, i.e., [0.0399 (2000) (10) = 798]. Average velocity is used for illustrative purposes only. The wave equation predicts local velocity so that the appropriate damping force is applied at each infinitesimal rod increment according to its velocity. ▲

2.3 Speed of Stress Wave Propagation in Rod Material

The wave equation models the elastic behavior of the rod string. Mechanical stresses are propagated along the rod string at a certain speed. An event which occurs at the downhole pump does not manifest itself immediately at the surface. A finite interval of time is required for the stress wave to propagate the length of the rod string. In simulating the sucker rod system, the appropriate speed must be calculated and used in the model. The speed of stress wave travel is given by Equation 2.2. This is implied by the units of v, i.e., ft/sec. Another indicator that ν represents velocity is derived from one of the oldest solutions to the undamped wave equation, i.e., D'Alembert's solution. This 18^{th} century scientist showed that

$$y\,(x,t) \quad = \quad f\,(x+v\,t) \quad + \quad g\,(x-v\,t) \tag{2.6}$$

is a solution to the undamped wave equation (obtained by setting c= 0 in Equation 2.1). The arbitrary function f represents a wave traveling upward and g represents a wave traveling downward. If we presume at t_0 that the f wave front is at x_0, its value is $f(x_0+vt_0)$. At a later time $t_1 = t_0 + \Delta t$, the f wave front has moved upward to position $x_1 = x_0 - \Delta x$. Thus, f now has the value $f(x_0 - \Delta x + v(t_0 + \Delta t))$. In order that the magnitude of f be the same at t_0 and t_1, i.e., the position on the wave front is maintained as the wave travels upward, we must have $-\Delta x + v\Delta t = 0$. Thus, $v = \Delta x/\Delta t$, which implies that the wave progresses at velocity v. Similar manipulations can be applied to the downward-moving wave g.

It is instructive to compute propagation velocity in a practical case from Equation 2.2. For steel rods (E = 30000000 psi and ρ = 490 lb$_m$ /ft^3),

$$v \quad = \quad \sqrt{\frac{144\,(30000000)\,(32.17)}{490}} \quad = \quad 16841 \text{ ft/sec} .$$

Actual measurements indicate that velocity is about 16000 ft/sec in steel rods. Thus the value computed above is too fast. Although not intuitively obvious, Equation 2.2 applies to rods without couplings. Consideration of the added mass of the couplings in the computations diminishes the velocity. Since the distance between couplings is small in relation to the length of the rod string, the effect of the couplings can be approximated by increasing the density sufficiently to make a rod without couplings as massive as a rod of the same body diameter with couplings. Thus

$$\rho = \frac{144\,g_c w}{gA} \quad ,$$

which leads to an alternate formula for propagation velocity,

$$v \;=\; \frac{d}{2}\sqrt{\frac{\pi\, E\, g}{w}}. \tag{2.7}$$

An example will clarify the use of Equation 2.7.

Example 2.2
Determine the speed of stress wave propagation in 1 in. rods that weigh 2.904 lb/ft. Assume that the local acceleration of gravity is 32 ft/sec^2. ★

Solution.
From Equation 2.7,

$$v \;=\; \frac{1}{2}\sqrt{\frac{\pi\,(30000000)\,(\,32)}{2.904}} \;=\; 16014 \text{ ft/sec.}$$

This value better agrees with actual measurements of velocity. These measurements can be made by deliberately causing the pump to "hit down." The round trip time of the stress wave so created is measured and velocity is calculated from the distance traveled. Equation 2.7 also applies to rod materials other than steel. ▲

2.4 Separating Solutions of the Wave Equation into Static and Dynamic Parts

The g term in Equation 2.1 makes that equation non-homogeneous. A change of variable will be made which makes the equation homogeneous and at the same time reveals facts about buoyant rod weight and static stretch. For a rod string with only one interval, assume solutions to the wave equation of the form

$$y\,(x,t) \;=\; u(x,t) \;+\; s\,(x) \quad , \tag{2.8}$$

in which $u(x,t)$ represents the dynamic (time-varying) solution and $s(x)$ represents the static solution (independent of time) which models rod weight, static stretch, and buoyancy. $y(x,t)$ represents the total solution which includes both dynamic and static effects. By differentiating Equation 2.8 and using Hooke's law, the various forces in the rod string are computed. $F(x,t)$ is given by

$$F(x,t) \quad = \quad EA\,\frac{\partial\, y\,(x,t)}{\partial x} \tag{2.9}$$

and represents the total load in the rods, both dynamic and static. $D(x,t)$ is the dynamic (time-varying) force,

$$D(x,t) \quad = \quad EA\,\frac{\partial\, u\,(x,t)}{\partial x} \quad , \tag{2.10}$$

and $B(x)$ is the static force in the rod as it hangs motionless in fluid supporting its own buoyant weight,

$$B(x) \quad = \quad EA\,\frac{ds\,(x)}{dx}. \tag{2.11}$$

We proceed by differentiating Equation 2.8 and substituting into the wave equation:

$$\frac{\partial^2 u\,(x,t)}{\partial t^2} \quad = \quad v^2\left[\frac{\partial^2 u\,(x,t)}{\partial x^2} \quad + \quad \frac{d^2 s\,(x)}{dx^2}\right] \quad - \quad c\,\frac{\partial\, u\,(x,t)}{\partial t} \quad + \quad g.$$

The above equation can be made homogeneous by requiring

$$v^2\,\frac{d^2 s\,(x)}{dx^2} \quad + \quad g \quad = \quad 0,$$

which when integrated becomes

$$\frac{ds\,(x)}{dx} \quad = \quad -\frac{gx}{v^2} \quad + \quad \alpha \quad = \quad -\frac{wx}{EA} \quad + \quad \alpha$$

and

$$s\,(x) \quad = \quad -\frac{gx^2}{2v^2} \quad + \quad \alpha\, x \quad + \quad \beta \quad = \quad -\frac{wx^2}{2EA} \quad + \quad \alpha\, x \quad + \quad \beta \quad ,$$

wherein α and β are constants of integration. We choose $\beta = 0$ to make the coordinate systems for $z(x,t)$ and $u(x,t)$ have the same origin. The buoyant force acts at the bottom of the rod string and is found by multiplying pressure at the bottom of the rods by their cross-sectional area. Pressure is presumed to increase linearly with depth according to

$$p(x) \quad = \quad p_t \ + \ \nabla x \quad , \tag{2.12}$$

in which the tubing pressure gradient is defined by

$$\nabla = 0.433\,G. \tag{2.13}$$

Applying the buoyant force at the bottom of the rod string implies

$$EA\,\frac{ds\,(L)}{dx} \quad = \quad EA\,\left[\frac{-w\,L}{EA} + \alpha\right] \quad = \quad -A\,p\,(L).$$

Thus, $\alpha = \frac{w\,L - A\,p(L)}{EA}$. The single-taper formulas for static rod stretch and rod load are

$$s\,(x) \quad = \quad \frac{x}{EA}\left[w\left(L - \frac{x}{2}\right) \ - \ A\,p\,(L)\right] \tag{2.14}$$

and

$$B\,(x) \quad = \quad w\,(L - x) \ - \ A\,p\,(L). \tag{2.15}$$

$B(0)$ is the buoyant weight of the rod string as it hangs at rest in fluid. $s(L)$ is the corresponding rod stretch. This process yields a homogeneous version of the wave equation that is easier to solve:

$$\frac{\partial^2 u(x,t)}{\partial\, t^2} \quad = \quad v^2\,\frac{\partial^2 u(x,t)}{\partial\, x^2} \quad - \quad c\,\frac{\partial\, u(x,t)}{\partial\, t} \quad . \tag{2.16}$$

In accordance with Equation 2.8, the solution to the full wave equation (Equation 2.1) is obtained by adding solutions to Equations 2.14 and 2.16.

As an illustration, consider a single taper rod string consisting of 4000 ft of 0.875 in. rods which weigh 2.224 lb/ft. Let tubing head pressure be 50 psi with the rods immersed in fluid with specific gravity of unity. Table 2.1 gives computed values of static rod stretch and static rod load using Equations 2.14 and 2.15. The buoyant weight of the rod string as it hangs statically in fluid is 7825 lb. Static stretch of the rod string as it hangs in fluid under its own weight and the force of buoyancy is 0.749 ft.

Table 2.1. Static rod stretch and rod load versus depth.

Measured Depth, x (ft)	Static Rod Stretch, $s(x)$ (ft)	Static Rod Load, $B(x)$ (lb)
0	0	7825
1000	0.372	5601
2000	0.621	3377
3000	0.747	1153
4000	0.749	–1071

Figure 2.2 shows details of the static and dynamic solutions to the wave equation. Imagine that a coordinate system is established by placing marks in the casing 1000 ft apart with the origin at the surface. This is the coordinate system labeled x in Figure 2.2. Now imagine that 4000 ft of rods are screwed together and laid horizontally on a frictionless surface without stretching them. Also draw marks 1000 ft apart on the unstretched rod string. Hang the rods in the tubing (filled

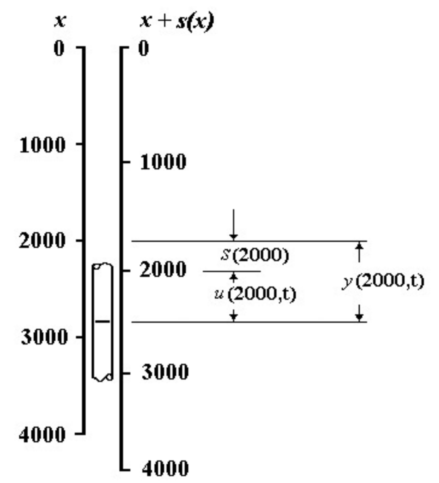

Figure 2.2. Position of rod segment with respect to linear and stretched coordinate systems.

with well fluid) with the zero mark coinciding with the zero mark in the casing. The rods will stretch downward under their own weight and the effects of buoyancy. This establishes the coordinate system labeled $x + s(x)$ in Figure 2.2. The function s (2000) shows the distance that the 2000 ft mark on the rods stretches down from its companion mark on the casing. Because of hanging weight, the marks on the rods between zero and 1000 ft will be 1000.372 ft apart. Near the bottom where the buoyancy force is pushing upward and hanging weight is much diminished, the marks at 3000 ft and 4000 ft are only 1000.002 ft apart. The coordinate system labeled $x + s(x)$ is distorted to show the stretching effect. A small segment of rod near its 2000 ft mark (not labeled) is shown in motion at a particular time t. Figure 2.2 shows that u (2000,t) is measured with respect to a stretched coordinate system which is established with the rods hanging statically in fluid. y (2000,t) is measured with respect to the linear coordinate system fixed in the casing.

Now suppose that we have obtained solutions to Equation 2.1 by whatever method for a 4000 ft well. Figure 2.3 shows these solutions in the form of computed dynamometer cards in which dynamic load $D(x, t)$ is plotted versus rod position $u(x, t)$ at both the surface ($x = 0$ ft) and the downhole pump ($x = 4000$ ft). Figure 2.4 shows total loads $F(x, t)$ *versus* $y(x, t)$ for the same well. Note that the downhole pump card is shifted downward by an amount equal to the buoyant force [$B(4000) = -1071lb$] and leftward by the amount s (4000) = 0.749 ft. The surface card is shifted upward by the buoyant rod weight [B(0) = 7825 lb].

Note that rod positions in Figures 2.3 and 2.4 are plotted with increasing values to the reader's left. This is consistent with the previously defined coordinate system which shows rod positions increasing downward. Long-standing dynamometry practice shows increasing rod position (upward motion) to the reader's right. While the convention used in Figures 2.3 and 2.4 is mathematically consistent, we will adhere to the customary practice in displaying solutions. Henceforth in this text, dynamometer cards will show upward rod motion to the right. Furthermore when pump dynamometer cards are shown with their companion surface cards, the lowest rod position will be left-justified, i.e., aligned on the left. This is particularly helpful in showing whether the pump stroke is shorter or longer than the surface stroke.

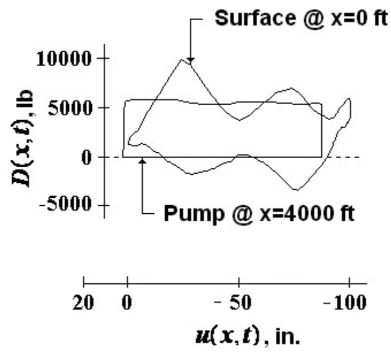

Figure 2.3. Dynamometer cards showing dynamic load and rod position referred to stretched coordinate system.

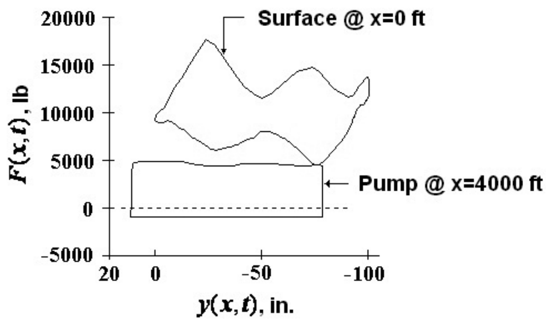

Figure 2.4. Dynamometer cards showing total load and rod position referred to linear coordinate system.

Rod strings usually involve multiple intervals with different diameters, properties or materials. We will in effect solve the wave equation in every interval and then connect the solutions by equating dynamic load and position at each taper point. Henceforth, we will add subscripts to the dependent variables to suggest the interval to which the solutions belong. Thus,

$$\frac{\partial^2 y_i(x,t)}{\partial t^2} = v_i^2 \frac{\partial^2 y_i\ (x,t)}{\partial x^2} - c_i \frac{\partial\, y_i\ (x,t)}{\partial t} + g \tag{2.17}$$

and

$$y_i(x,t) \;=\; u_i(x,t) \;+\; s_i(x) \tag{2.18}$$

pertain to the ith interval in a tapered rod string. We will illustrate the procedure on a two-taper (two-interval) string. We set $i = 1$ to signify the first (top) interval and proceed as before by substituting derivatives of Equation 2.18 into Equation 2.17. We obtain functions for static strain and static rod stretch,

$$\frac{ds_1(x)}{dx} \;=\; -\frac{w_1\,x}{E_1 A_1} \;+\; \alpha_1$$

and

$$s_1(x) \;=\; -\frac{w_1\,x^2}{2\,E_1 A_1} \;+\; \alpha_1 x \;+\; \beta_1 \quad ,$$

where α_1 and β_1 are constants of integration (but different from the single-taper case). α_1 is evaluated knowing that the load at the bottom of the first interval is the buoyant force at the junction point L_1 (as long as $A_1 \neq A_2$) and is supporting the buoyant weight of interval 2 below. Thus,

$$E_1\,A_1\,\alpha_1 \;=\; w_1 L_1 \;+\; w_2 L_2 \;-\; (A_1 - A_2)\,p\,(L_1) \;-\; A_2 p\,(L).$$

The right-hand side of the equation above is the buoyant weight of the two-taper rod string:

$$W_b \;=\; w_1 L_1 \;+\; w_2 L_2 \;-\; (A_1 - A_2)\,p\,(L_1) \;-\; A_2 p\,(L).$$

Thus, $\alpha_1 \;=\; W_b / E_1 A_1$. $\beta_2 = 0$ is chosen to make the coordinate systems for $y_1(x,t)$ and $u_1(x,t)$ have the same origins. The formulas for static rod load and stretch in the first rod interval are

$$B_1(x) \;=\; -w_1 x \;+\; W_b \tag{2.19}$$

and

$$s_1(x) = \frac{1}{E_1 A_1}\left[\frac{-w_1 x^2}{2} + W_b x\right]. \tag{2.20}$$

These relations are valid in the interval $0 \leq \; x \; \leq \; L_1$.

We proceed to the second interval and obtain the equations

$$\frac{ds_2(x)}{dx} \;=\; -\frac{w_2 x}{E_2 A_2} \;+\; \alpha_2$$

and

$$s_2(x) = -\frac{w_2 x^2}{2\,E_2 A_2} + \alpha_2 x + \beta_2.$$

Evaluation of the above constants of integration is left as an exercise for the reader. α_2 is evaluated knowing that the buoyant force is acting beneath the lower interval. β_2 is evaluated by making the position of the bottom of the top interval coincide with the top of the bottom interval, i.e., $s_2(L_1) = s_1(L_1)$. We proceed directly to listing the formulas for static rod load and stretch for the bottom interval in a two-taper string,

$$B_2(x) \;=\; w_2(L - x) \;-\; A_2 p\,(L) \tag{2.21}$$

and

$$\begin{aligned} s_2(x) \;&=\; s_1(L_1) + \frac{1}{E_2 A_2} \\ &\left[\frac{-w_2}{2}\left(x^2 - L_1^2\right) + (x - L_1)\,\left[w_2 L - A_2 p(L)\right]\right]. \end{aligned} \tag{2.22}$$

These relations are valid in the interval $L_1 \quad < \quad x \quad \leq \quad L.$

Table 2.2 illustrates the static rod load and stretch for a two-taper steel string consisting of 1500 ft of 0.875 in. rods which weigh 2.224 lb/ft and 2500 ft of 0.75 in. rods (1.634 lb/ft). The rods are immersed in fluid with specific gravity of one and tubing head pressure is 50 psi. Figure 2.5 shows static rod load versus measured depth. Note the discontinuity in static load at 1500 ft in Table 2.2 and Figure 2.5. This reflects the pressure that acts upward against the annular area formed by the junction between 0.875 in. and 0.75 in. rods. The magnitude of this force is –112 lb. Another buoyancy force of –787 lb acts at the bottom of the rod string.

2.4.1 Rod Buoyancy

In general, the buoyant weight of a rod string with N intervals is

$$W_b \;=\; \sum_{i=1}^{N} w_i L_i \;-\; \sum_{i=1}^{N-1} (A_i - A_{i+1}) p\,(\Lambda_i) \;-\; A_N p\,(L), \tag{2.23}$$

in which

$$L \;=\; \sum_{i=1}^{N} L_i \tag{2.24}$$

Table 2.2. Static rod load and stretch for two-taper rod string.

Measured Depth, x (ft)	Interval, i	Static Rod Stretch, $s_i(x)$ (ft)	Static Rod Load, $B_i(x)$ (lb)
0	1	0	6522
500	1	0.1654	5410
1000	1	0.3000	4298
1500	1	0.4037	3186
1500	2	0.4037	3298
2000	2	0.5127	2481
2500	2	0.5909	1664
3000	2	0.6383	847
3500	2	0.6549	30
4000	2	0.6406	–787

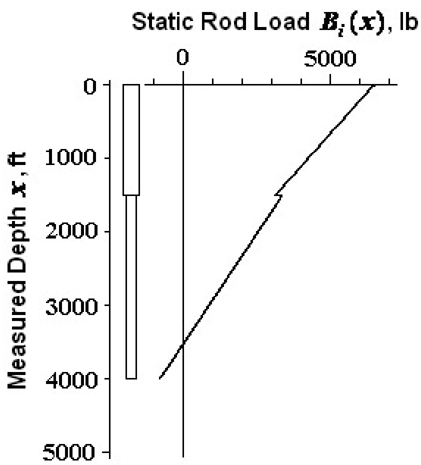

Figure 2.5. Static rod load versus measured depth for two-taper rod string.

and

$$\Lambda_i = \sum_{j=1}^{i} L_j. \tag{2.25}$$

Buoyancy effects are computed as the product of pressure against an area exposed to the pressure. The resulting force acts perpendicular to the exposed area. There is an interesting historical connection that can be discovered from Equation 2.23. Note that the first sum in that formula is the weight of the rod string in a vacuum. The remainder of the equation represents the forces of buoyancy acting at junction points in a multiple-taper string and at the bottom of the string. To illustrate, take a three-taper string and neglect surface tubing pressure. Expand and simplify the terms representing buoyancy in the following sequence:

$$-(A_1 - A_2)p(L_1) - (A_2 - A_3)p(L_1 + L_2) - Ap(L_1 + L_2 + L_3),$$

which becomes

$$-\nabla(A_1 - A_2)L_1 - \nabla(A_2 - A_3)(L_1 + L_2) - \nabla A_3(L_1 + L_2 + L_3)\,,$$

which becomes, when simplified $-\nabla(A_1L_1 + A_2L_2 + A_3L_3)$.

The sum of the products A_iL_i above is the volume of the rod string. The gradient is a measure of fluid weight. Thus, the expression is the weight of fluid displaced by the rod string, and the method (pressure over an area) is shown to be just another way of expressing Archimedes' principle (Reference 3). This states that the submerged weight of an object is less than its weight *in vacuo* by an amount equal to the weight of displaced fluid. The subject of buoyancy seems to be a continuing source of confusion and controversy. Some students of rod pumping object to showing buoyancy forces acting upward beneath the rods because these forces are thought to cause rod buckling and unduly warn of failure. They prefer to reduce rod weight such that no compressive forces will be shown along and beneath the rod string. This point of view is incorrect. Buoyancy forces do act upward and cause compression in the rods. However, buoyancy forces alone do not cause buckling. Archimedes, the mechanical genius of the third century BCE, correctly settled the issue of buoyancy for all time. More about rod buckling is presented in Chapter 8.

Stretched coordinate systems for any number of taper intervals can be developed. The associated formulae can be made concise by simulating the multi-interval string as a succession of single-interval strings. A tapered string with N intervals will have $N - 1$ junction points. A junction is a point where two tapers join. Introduce an index j ($j = 0, 1, 2, \ldots N - 1, N$) such that $j = 0$ represents the top of the rod string, $j = 1, 2, \ldots N - 1$ represent the junction points, and $j = N$ represents the bottom of the rod string. In the process previously illustrated for one- and two-taper strings, recall that the constant of integration α_i was evaluated using the buoyant weight of the rods hanging below a junction point j and the buoyant force, if any, acting at that junction. We calculate this force at point j by modifying the formula for buoyant rod weight (Equation 2.23), i.e.,

$$F_j \;=\; -A_N p\,(\Lambda_N) \;+\; \sum_{i=j+1}^{N} w_i L_i \;-\; \sum_{i=j}^{N-1} (A_i - A_{i+1}) p\,(\Lambda_i) \quad (2.26)$$

The force F_j acts at the bottom of an interval at junction points $j = 1, 2, \ldots N - 1$ or at the bottom of the rod string, $j = N$. We employ the formula for buoyant rod weight in an arbitrary interval and evaluate the constant of integration,

$$B_j(x_d) \;=\; -\,w_j\,x_d \;+\; E_j\,A_j\,\alpha_j \quad (2.27)$$

in which x_d ranges from the top ($x_d = 0$) of interval j to the bottom ($x_d = L_j$) of the interval. The discontinuous variable x_d is related to the continuous variable x by means of

$$x_d = x - \Lambda_{j-1} \quad (2.28)$$

with the index j determined from

$$\Lambda_{j-1} \;<\; x \;\leq\; \Lambda_j. \quad (2.29)$$

Figure 2.6 shows a three-taper rod string in which the depth of investigation x falls in interval 3, so $j = 3$. At a given point j, the dummy measured depth is $x_d = L_j$ and

$$B_j(L_j) = \,F_j$$

and the constant of integration becomes

$$\alpha_j = \frac{F_j + w_j L_j}{E_j A_j} \quad ;\, j = 1,\ 2,\ \ldots N. \quad (2.30)$$

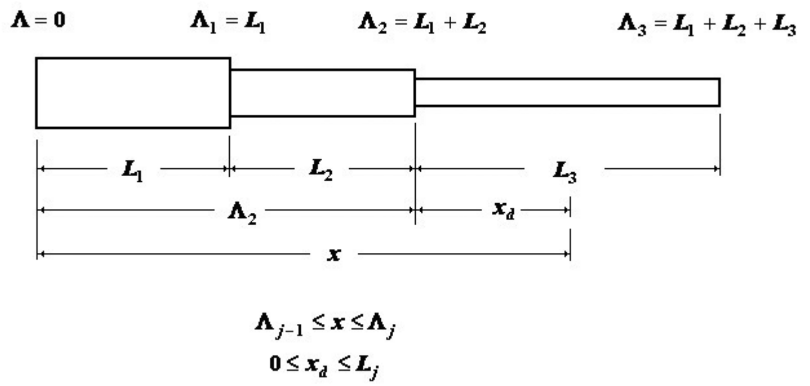

Figure 2.6. Nomenclature of three-taper rod string.

The remaining constant of integration is evaluated at the top ($x_d = 0$) of the interval using

$$s_j(x) = -\frac{w_j\, x^2}{2E_j A_j} + \alpha_j x + \beta_j \tag{2.31}$$

We must make the top of a given interval have the same position as the bottom of the previous interval. This requires

$$\beta_j \;=\; s_{j-1}(L_{j-1}) \quad ;\, j = 1,\ 2,\ ..\ N \tag{2.32}$$

in which the origins of the stretched and linear coordinate systems are made to coincide by defining

$$\begin{aligned} L_0 &= 0 \\ s_0(L_0) &= 0. \end{aligned} \tag{2.33}$$

Using Equation 2.31, we compute recursively the position of the various points j from

$$s_j(L_j) \;=\; -\frac{w_j\, L_j^2}{2E_j A_j} + \;\alpha_j\, L_j + \beta_j;\; j = 1,\ 2,\ ..\ N. \tag{2.34}$$

The formulas for buoyant weight and rod stretch at arbitrary measured depth are

$$B_j(x_d) \;=\; -\,w_j\,x_d \;+\; E_j\,A_j\,\alpha_j \tag{2.35}$$

and

$$s_j(x_d) \;=\; -\frac{w_j\,x_d^2}{2E_jA_j} \;+\; \alpha_j\,x_d \;+\; \beta_j \tag{2.36}$$

Further practice with multi-interval coordinate systems is given in the Exercises that follow.

2.5 Design and Diagnostic Solutions

Two of the fundamental problems in rod pumping are (1) diagnosing problems in existing installations and (2) designing new installations. Both activities can be accomplished by solving boundary value problems based on the wave equation. Figure 2.7 illustrates the diagnostic problem. Solutions to the wave equation which satisfy measured time histories of rod load F(0,t) and position u(0,t) at the surface enable us to solve for time histories of downhole pump load F(L,t) and position y(L,t). From these solutions, downhole operating conditions can be inferred from the surface and correction of operating problems and other optimization activities can be accomplished. Figure 2.8 illustrates the basis for design predictions. Use of the wave equation with simulations of pumping unit kinematics, prime mover characteristics, and assumed downhole pump action lead to prediction of equipment loading, lifting capacity, power requirement, and dynamometer card shape. This sets the stage for most of the remaining text. Dynamometer cards either measured or predicted are the principal tool in pumping well analysis and design.

Closure

The purpose of this chapter is to establish the wave equation as the fundamental equation of rod pumping. There is not much in the chapter of immediate use to the practical oilfield worker. Yet most everyone has seen or heard the wave equation at work. Think of the times

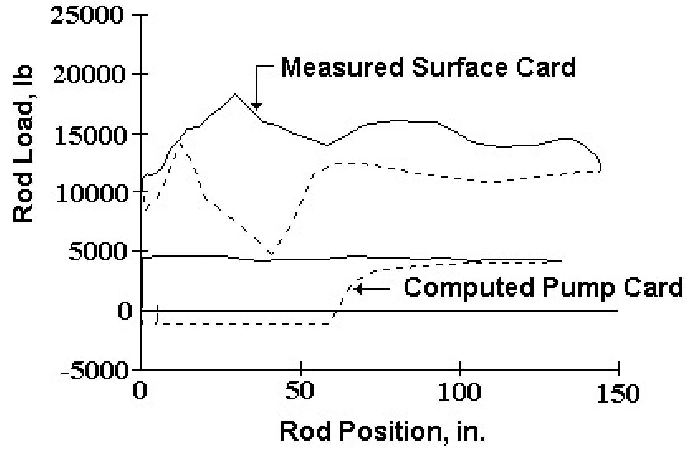

Figure 2.7. Dynamometer cards illustrating diagnostic process starting with surface measurements.

you heard a clinking sound while standing near the wellhead. You were hearing the pump "hit down." It was spaced too close to bottom and the clutch on the valve rod was striking the valve rod guide. This induced a large compressive stress at the bottom of the rod string. According to the wave equation, this stress began to travel toward the surface at about 16,000 ft/sec in steel rods. When the stress wave arrived at the surface, you heard the clinking sound. In a deep well, a perceptible travel time is involved, say 0.44 sec in a 7000 ft well. This is why the polished rod has moved well into its upstroke when the clinking sound is heard. Another wave phenomenon that we see occurs while waiting for a traffic signal to turn green. Imagine that you are in the tenth vehicle waiting in line for the light to change. You note the light turn green, yet the car in front of you has not moved. You must remain stationary until the vehicles in front of you begin to move in wave-like fashion. The driver of the second vehicle begins to move after noticing the first car start. Then the driver of the third vehicle follows the second and so forth until the 'wave' reaches you. In rod pumping, the wave equation describes the fact that events at the pump do not manifest themselves immediately at the surface, and *vice versa*. Beginning in Chapter 3, practical oilfield workers will begin to

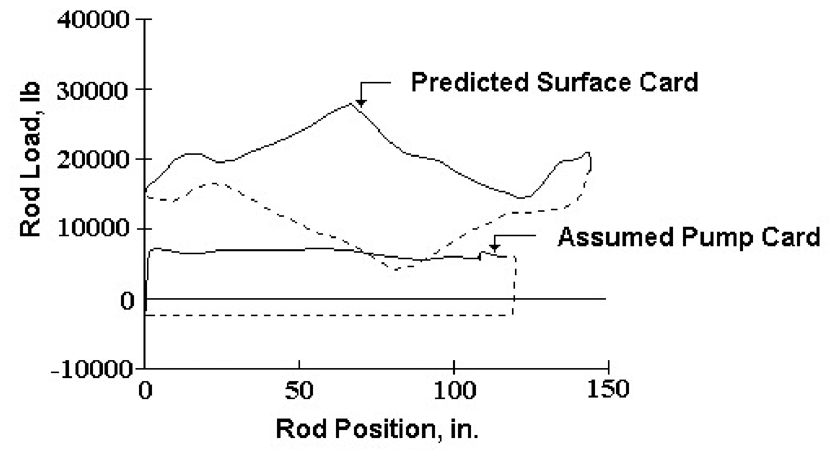

Figure 2.8. Dynamometer cards illustrating design process starting with assumed downhole pump conditions.

see specifically how the wave equation helps them diagnose problems occurring thousands of feet below the surface.

In the derivation of the wave equation, there is no hint of the fact that the linear form loses precision when rod loads become very high. Academic readers might want to re-derive the wave equation using Taylor′s Series in which second- (and maybe third-) order strain terms are retained. This will result in a nonlinear form of the wave equation for which an exact solution cannot be obtained. A good approximate solution is possible using finite differences similar to the treatment in Chapter 7. This could provide more precision in design and diagnostic problems when rod loads are extremely high.

Another problem of possible interest to the academicians is to derive a version of the wave equation for use with fiberglass rods. Fiberglass does not stretch and contract in a perfectly linear fashion as the wave equation of Chapter 2 presumes. In Chapter 2, a stretching force twice as large will cause rod stretch to double. In fiberglass, a doubling of the stretching force will cause rod stretch to increase by a little more than a factor of two. The nonlinear version of the wave equation would improve predictions of pump stroke and rod loads for fiberglass installations.

Problem 10 in the Exercises is comprehensive. Successful completion shows an excellent understanding of buoyancy and the process of splitting the wave equation into dynamic and static parts. The equation for static stretch (or position) that you will derive is the same as in tubular product handbooks in many drilling engineering offices.

Deriving buoyancy for rod pumping from the mathematical process of converting a non-homogeneous equation into a homogeneous one is given in the open literature for the first time in this text. There is a story (probably apocryphal) about Archimedes' (287–212 B.C.) discovery of the principle of buoyancy. It is said that the thought first occurred to him while taking a bath. He was so excited by the revelation that he streaked through the streets of Syracuse stark naked shouting 'Eureka. Eureka.' (I have found it. I have found it). This student–author felt something of Archimedes' thrill by discovering that buoyancy equations fall out of the process of making the wave equation homogeneous.

Exercises

1. In a 1 in. steel rod, what is the tensile force corresponding to a mechanical strain of 0.0011 in./in.?

2. Rod fall problems frequently occur in shallow wells which are producing viscous crude. It has been experimentally determined that terminal falling velocity in a certain well is 0.5 ft/sec. Assume that the rods fall as a rigid body and ignore fluid friction effects within the pump. Use the following information:

   L = 800 ft

   d = 0.875 in. rods

   G = 0.95

   v = 16000 ft/sec

   p_t = 100 psi

   Show that the nondimensional damping factor for this case is $\lambda = 1.9$ (approximately). Note that the calculated damping factor is far higher than the 0.1 value frequently used in low-viscosity fluid.

3. Compute the propagation velocity for 1 in. fiberglass rods which weigh 0.82 lb/ft. Use a modulus of elasticity of 7.2 x 10^6 psi.

4. Derive Equation 2.7 starting with Equation 2.2.

5. Compute the static stretch of 6000 ft of 7 in. casing which is suspended in 12 lb/gallon drilling fluid. The wall area of the casing is 5.749 in.^{2} and it weighs 20 lb/ft.

6. Derive the formulas for static load and static stretch of a three-interval rod string.

7. It is possible to experimentally verify the manufacturer-determined modulus for fiberglass rods. The pump is spaced low enough to cause the pump to temporarily hit-down hard enough to create a stress wave whose round-trip time can be measured. Round-trip time for the stress wave is 1.056 sec. The rod string is composed of 5600 ft of 1 in. fiberglass rods (0.82 lb/ft) and 2400 ft of 0.875 in. steel rods (2.224 lb/ft). Show that the modulus of elasticity for 1 in. fiberglass rods made by a certain manufacturer is about 7,060,000 psi. What is the experimentally determined propagation velocity in the fiberglass rods?

8. Knowing that $u(x, t)$ is a solution to Equation 2.16, show that $y(x, t) = u(x, t) + s(x)$ is a solution of Equation 2.1 for a single-taper rod string. This proves the validity of using superposition to compute total rod load (both static and dynamic) in the sucker rod system. Hint: Use Equations 2.8 and 2.14.

9. Determine the buoyant weight and static stretch of the two-taper fiberglass / steel rod string of Exercise 7. The rods are immersed in fluid with specific gravity of 0.9. Tubing head pressure is 60 psi.

10. Use the version of Hooke's law that considers confining pressure,

$$F(x,t) \;=\; EA\frac{\partial\, y\,(x,t)}{\partial\, x} \;+\; A\nu\,[\sigma_r + \sigma_t],$$

to derive another version of the wave equation,

$$\frac{\partial^2 y\ (x,t)}{\partial\, t^2} \quad = \quad v^2 \frac{\partial^2 y\ (x,t)}{\partial\, x^2} \quad - \quad c\frac{\partial\, y\ (x,t)}{\partial\, t} \quad + \quad g\ \left(1 \quad - \quad 2\nu\frac{\rho_f}{\rho_s}\right).$$

Split the wave equation into dynamic and static parts (see Equation 2.8) and show that the part representing static stretch as a function of buoyancy, dead rod weight, and confining pressure is

$$s(x) \quad = \quad -\frac{g(\rho_s - 2\,\nu\,\rho_f)\,x^2}{2Eg_c} \quad + \quad \frac{Lg}{Eg_c}(\rho_s - \rho_f)x.$$

Work Problem 5 again and note that stretch is predicted to be slightly more with the above equation than with Equation 2.14.

Hint: Radial and tangential stresses are each equal to hydrostatic pressure (taken as negative value). Use 0.3 for Poisson's ratio.

Glossary of Terms

Symbol	Description	Units
$\boldsymbol{A}$, $\boldsymbol{A_i}$	cross-sectional area of sucker rod	sq in.
$\boldsymbol{B(x)}$	static force in rods vs depth due to buoyancy and weight	lb
$\boldsymbol{c'}$	dimensional damping factor	lb sec/ft^2
$\boldsymbol{c}$	dimensional damping factor	sec^{-1}
$\boldsymbol{d}$	rod diameter	in.
$\boldsymbol{D(x,t)}$	axial force in rods excluding buoyancy and body weight	lb
$\boldsymbol{E}$, $\boldsymbol{E_i}$	rod modulus of elasticity	psi
$\boldsymbol{F(x,t)}$	total axial force in rod	lb
$\boldsymbol{F_j}$	axial force at rod junction	lb

(continued)

Glossary of Terms (Continued)

Symbol	Description	Units
f, g	arbitrary functions in D′Alembert solution	ft
g_c	gravitational conversion factor	lb_m ft/lb_f sec^2
g	local acceleration of gravity	ft/sec^2
G	specific gravity of tubing fluid	$--$
L, L_i	length of rod interval, pump depth	ft
N	number of rod intervals in tapered string	–
n	summation subscript	–
$p(x)$	tubing pressure vs depth	psi
p_t	tubing head pressure	psi
$s(x)$	static rod position considering weight and buoyancy	ft
t	time	sec
$y(x, t), y_i(x, t$	rod position referred to linear coordinate system	ft
v	propagation velocity of stress waves	ft/sec
W_b	buoyant rod weight	lb
w, w_i	unit weight of rods	lb/ft
x	measured depth along unstrained rod	ft
$u(x, t)$	rod position referred to stretched coordinate system	ft
α	constant of integration	-

(continued)

Glossary of Terms (Continued)

Symbol	Description	Units
β	constant of integration	-
Δx	rod increment (see Figure 2.1)	ft
λ	nondimensional damping coefficient	–
ρ, ρ_s	density of rod material	lb_m/ft^3
ρ_f	density of produced fluid	lb_m/ft^3
Λ_i	depth at rod junction point	ft
∇	pressure gradient of tubing fluid	psi/ft
ν	Poisson's ratio	–
σ_r	radial stress	psi
σ_t	tangential stress	psi

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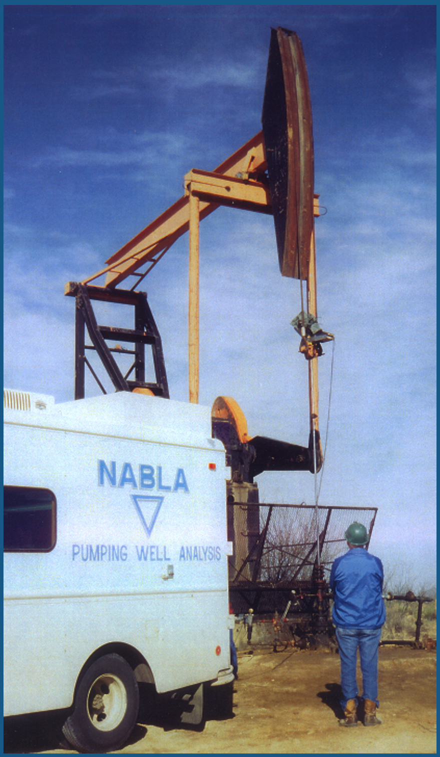
ROD PUMPING

Modern Methods of Design, Diagnosis and Surveillance



Sam seems to wonder why the position transducer has been dislodged from its carrier.

"This book by Dr. S.G. Gibbs is brilliant, a true masterpiece, and a must-have reference for all those working in, or interested in the subject of Artificial Lift, and especially rod pumping."

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President, Theta Oilfield Services, Inc.
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