Next Up Previous
Next: The Newton Polynomial Interpolation **Up:** Interpolation and Extrapolation **Previous:** Polynomial Interpolation

The Lagrange Interpolation

The polynomial $P_n(x)$ that fits a set of $n+1$ node points $\{x_i,\; y_i=f(x_i),\;\; i=0,\cdots,n\}$ can also be obtained by the *Lagrange interpolation*:

$$L_n(x)=\sum_{i=0}^n y_i l_i(x) \tag{15}$$

where $l_i(x)$ are the Lagrange basis polynomials of degree n that span the space of all nth degree polynomials:

$$l_i(x)=\prod_{j=0,\;j\ne i}^n \frac{x-x_j}{x_i-x_j}=\frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} \tag{16}$$

Note that when $x=x_j,\;\;(j=0,\cdots,n)$, we get

$$l_i(x_j)=\delta_{ij}=\left\{\begin{array}{ll}1 & i=j\\ 0 & i\ne j\end{array}\right. \tag{17}$$

This polynomial $L_n(x)$ passes through all $n+1$ points:

$$L_n(x_j)=\sum_{i=0}^n y_i l_i(x_j)=\sum_{i=0}^n y_i\delta_{ij}=y_j=f(x_j),\qquad (j=0,\cdots,n) \tag{18}$$

but it is only an approximation of $f(x)$ at any other point $x\ne x_j\;(j=0,\cdots,n)$.

Specially, when $f(x)=1$, i.e., $y_0=\cdots=y_n=1$, we get an important property of the Lagrange basis polynomials:

$$\sum_{i=0}^n l_i(x)=1 \tag{19}$$

Due to the uniqueness of the polynomial interpolation, $L_n(x)=P_n(x)$, and they have the same error function as in Eq. (12):

$$f(x)-L_n(x)=R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}l(x) \tag{20}$$

The computational complexity for calculating one of the n basis polynomials $l_i(x),\;\;(i=0,\cdots,n)$ is $O(n)$ and the complexity for $L_n(x)$ is $O(n^2)$ for each x. If there are $m\gg n$ samples for x, then the total complexity is $O(n^2m)$.

To reduce the computational complexity, we express the numerator of $l_i(x)$ based on the (n+1)th degree polynomial $l(x)=\prod_{j=0}^n(x-x_j)$ defined in Eq. (7) as

$$\prod_{j=0,\;j\neq i}^n (x-x_j)=\frac{l(x)}{x-x_i} \tag{21}$$

Then the denominator of $l_i(x)$ can be found by evaluating $l(x)$ at $x=x_i$:

$$\prod_{j=0,\;j\neq i}^n (x_i-x_j)=\lim_{x\rightarrow x_i}\prod_{j=0,\;j\neq i}^n (x-x_j)=\lim_{x\rightarrow x_i}\frac{l(x)}{x-x_i}=l'(x_i) \tag{22}$$

Here the second to the last expression is an indeterminate form $0/0$ which leads to the last equality due to L'Hôpital's rule. Now the Lagrange basis polynomial can be expressed as

$$l_i(x)=\frac{l(x)}{(x-x_i)l'(x_i)}=l(x)\frac{w_i}{x-x_i} \tag{23}$$

where $w_i=1/l'(x_i)$ is the *barycentric weight*, and the Lagrange interpolation can be written as:

$$L_n(x)=\sum_{i=0}^n y_i l_i(x)=l(x)\sum_{i=0}^n y_i\frac{w_i}{x-x_i} \tag{24}$$

We see that the complexity for calculating $L_n(x)$ for each of the m samples of x is $O(n)$ (both for $l(x)$ and the summation), and the total complexity for all m samples is $O(nm)$.

Example: Approximate function $y=f(x)=x\,\sin(2x+\pi/4)+1$ by a polynomial p_n of degree $n=3$, based on the following $n+1=4$ points:

i	0	1	2	3
x_i	-1	0	1	2
$y_i=f(x_i)$	1.937	1.000	1.349	-0.995

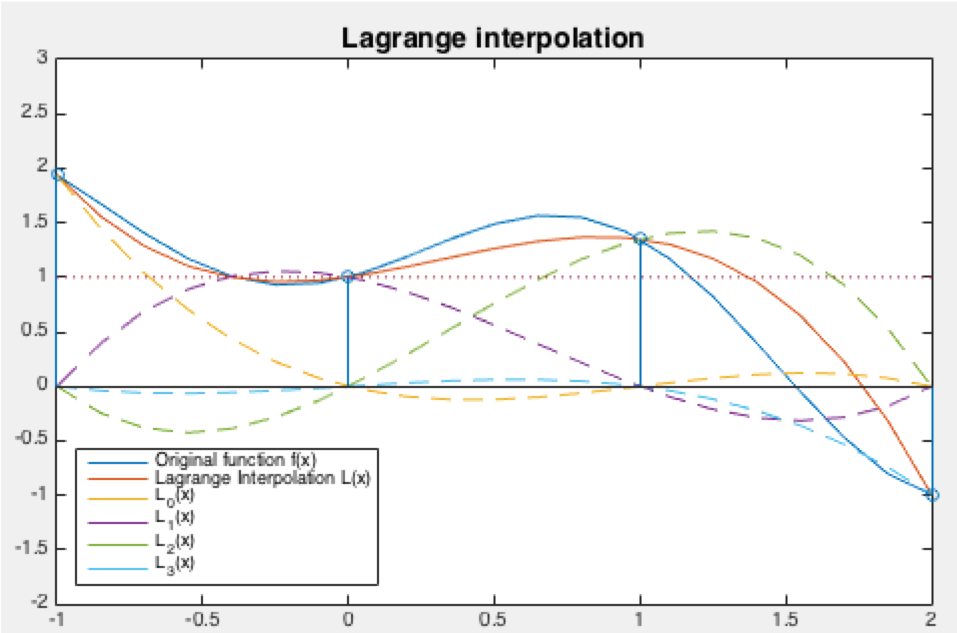
Based on these points, we construct the Lagrange polynomials as the basis functions of the polynomial space (instead of the power functions in the previous example):

$$\begin{aligned} l_0(x)&=\frac{(x-0)(x-1)(x-2)}{-6}=\frac{x^3-3x^2+2x}{-6}\\ l_1(x)&=\frac{(x+1)(x-1)(x-2)}{2}=\frac{x^3-2x^2-x+2}{2}\\ l_2(x)&=\frac{(x+1)(x-0)(x-2)}{-2}=\frac{x^3-x^2-2x}{-2}\\ l_3((x)&=\frac{(x+1)(x-0)(x-1)}{6}=\frac{x^3-x}{6}\end{aligned}$$

Note that indeed $l_0(x)+l_1(x)+l_2(x)+l_3(x)=1$. The interpolating polynomial can be obtained as a weighted sum of these basis functions:

$$L_3(x)=1.937\;l_0(x)+1.0\;l_1(x)+1.349\;l_2(x)-0.995\;l_3(x)=1.0+0.369\,x+0.643\,x^2-0.663\,x^3$$

which is the same as $P_3(x)$ previously found based on the power basis functions, with the same error $\epsilon = 0.3063$.



The Matlab code that implements the Lagrange interpolation (both methods) is listed below:

```
function [v L]=LI(u,x,y)  % Lagrange Interpolation
    % vectors x and y contain n+1 points and the corresponding function values
    % vector u contains all discrete samples of the continuous argument of f(x)
    n=length(x);     % number of interpolating points
    k=length(u);     % number of discrete sample points
    v=zeros(1,k);    % Lagrange interpolation
    L=ones(n,k);     % Lagrange basis polynomials
    for i=1:n
        for j=1:n
            if j~=i
                L(i,:)=L(i,:).*(u-x(j))/(x(i)-x(j));
            end
        end
        v=v+y(i)*L(i,:);
    end
end

function [v L]=LInew(u,x,y)  % Lagrange interpolation
    % u: data points; (x,y) sample points
    n=length(x);     % number of sample points
    m=length(u);     % number of data points
    L=ones(n,m);     % Lagrange basis polynomials
    v=zeros(1,m);    % interpolation results
    w=ones(1,m);     % omega(x)
    dw=ones(1,n);    % omega'(x_i)
    for i=1:n
        w=w.*(u-x(i));
        for j=1:n
            if j~=i
                dw(i)=dw(i)*(x(i)-x(j));
            end
        end
    end
    for i=1:n
        L(i,:)=w./(u-x(i))/dw(i);
        v=v+y(i)*L(i,:);
```

```
    end
end
```

Next Up Previous

Next: The Newton Polynomial Interpolation **Up:** Interpolation and Extrapolation **Previous:** Polynomial Interpolation