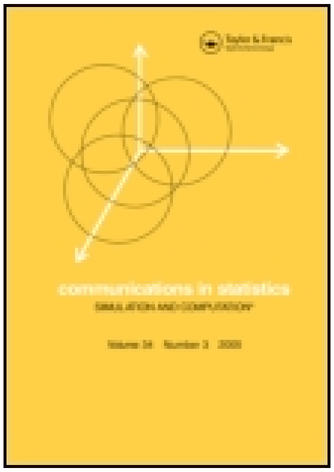

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ABSTRACT

This article gives power comparisons of some tests of fit for the Geometric distribution. These tests include a Chernoff–Lehmann X^2 test, some smooth tests, a Kolmogorov–Smirnov test, and an Anderson–Darling test. This article suggests that a good test of fit analysis is provided by a data dependent Chernoff–Lehmann X^2 test with class expectations greater than unity, and its components. These data dependent statistics involve arithmetically simple parameter estimation, convenient approximate distributions, and provide a fairly complete assessment of how well the data agrees with a Geometric distribution. The power comparisons indicate also that the best performed single statistic is the Anderson–Darling statistic.

Key Words: Diagnostic test; Dispersion test; Empirical distribution function; Partition of X^2.

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1. INTRODUCTION

The Geometric distribution is one of the better known discrete probability distributions and has many useful applications. See, for example, Johnson et al. (1992, p. 203). These applications include the description of runs of a species in transect surveys of plant populations and inventory demand distributions as discussed in Law and Kelton. (1991, p. 366). Introductory statistics courses sometimes introduce the Geometric distribution as the probability distribution for the number of coin tosses to obtain the first ‘‘head’’. The Geometric probability function is

$$f(x;q) = (1-q)q^x, \quad x = 0, 1, 2, \ldots, \text{ in which } 0 < q < 1.$$

Suppose we wish to test the hypothesis that n observations come from a Geometric distribution F, against the general alternative, not F. A common approach is to form $k+1$ classes from the data with associated counts N_j, $j = 0, 1, \ldots, k$, and use the familiar Pearson goodness of fit test based on the statistic

$$X^2 = \sum_{j=0}^{k} (N_j - np_j)^2/(np_j),$$

where p_j is the probability under F of an observation lying in the jth class, $j = 0, 1, \ldots, k$, and where $n = N_0 + \cdots + N_k$. Throughout this article we use X^2 to denote statistics of this form, noting that there are many ways in which the classes may be constructed, and many ways in which the estimation of the parameter, when unknown, may be performed.

We focus on the important case where the unknown q is estimated by

$$\hat{q} = \bar{Z}/(1+\bar{Z}),$$

where $\bar{z}$ is the mean of the ungrouped data $z_1, \ldots, z_n$, and $\bar{Z}$ is the corresponding random variable. The estimator $\hat{q}$ is thus the maximum likelihood estimator based on the *ungrouped* data. It is known that X^2 based on $\hat{q}$, X^2_{CL} say, is a Chernoff and Lehmann (1954) statistic and does not have an asymptotic χ^2 distribution; the distribution is asymptotically a linear combination of χ^2_1 variables. For Geometric testing the distribution of X^2_{CL} is bounded between χ^2_{k-1} and χ^2_k. See Kimber (1987) for an easy to read discussion that supplements the original Chernoff and Lehmann (1954) article. Note that Snedecor and Cochran (1989, Example 11.7.2, p. 205), as do other authors, use just the χ^2_{k-1} distribution.

The X^2 Geometric test will lose information if too much pooling is done. On the other hand, the χ^2 approximation to the null distribution of

X^2 may be poor unless at least some pooling is done. Clearly, choice of classes is a problem for the X^2 Geometric test. Further, the X^2 test may also have low power because it is an omnibus test: it has some power against many alternatives, but may not have substantial power against any particular alternative.

Given these potential problems with the X^2 Geometric tests due to pooling, it is useful to consider other approaches. Another method for testing for the Geometric distribution is to use the smooth tests outlined in Best and Rayner (1989) and in Rayner and Best (1989, Sec. 6.5), where we suggested use of a smooth test of order four. However, this recommendation was not supported by a simulation study, such as that undertaken here. In this article we also assess a Kolmogorov–Smirnov (KS) test, an Anderson-Darling (AD)-type test, and a "diagnostic smooth test" suggested by Henze and Klar (1996). All of these tests other than the X^2 tests have the advantage of not losing information by pooling the data. None of the tests we look at use the lack of memory property of the Geometric distribution discussed, for example, by Johnson et al. (1992, p. 201).

After comparing the various tests we recommend use of the AD test and comprehensive analysis based on a Chernoff–Lehmann X^2 statistic and its associated components.

2. DEFINITIONS

We have already introduced the Chernoff and Lehmann (1954) X^2 statistic X^2_{CL}, and we note that this statistic is recommended in textbooks, sometimes with the wrong asymptotic distribution, and sometimes correctly bounding the asymptotic distribution between χ^2 distributions. This X^2 test with pooling to ensure that the asymptotic distribution is appropriate, could be called the conventional method of testing fit for the Geometric distribution.

There are various rules for choosing the number of classes for the X^2 test. Here we take this number to be as large as possible such that each class has expectation at least unity and call the test statistic the $X^2_{\mathrm{CLE1+}}$ statistic.

As this class construction is data-dependent, it should not be expected that the null distribution of X^2 is χ^2. If the first class, corresponding to the Geometric variable taking the value zero, has expectation less than unity, then the test we are proposing cannot be applied. In such cases more complicated pooling of classes is needed. Douglas (1994) suggests one possible such pooling. He says to use "cells grouped from the smallest up until a fitted frequency of at least unity is achieved, with repetition where necessary.'' Calculations not given here indicate that such an approach may be needed for large q, say $q > 0.9$. Although we do not investigate this approach further here, we expect that components and approximate distributions would behave in similar manner to those we do examine.

The X^2 statistic can be partitioned into useful components. Put

$$\mu = \sum_{j=0}^{k} jp_j \quad \text{and} \quad \mu_r = \sum_{j=0}^{k} (j-\mu)^r p_j \quad \text{for } r = 2, 3, \ldots.$$

If $b = (\mu_4 - \mu_3^2/\mu_2 - \mu_2^2)^{-0.5}$ define the first three orthonormal polynomials by

$$g_0(j) = 1, \quad g_1(j) = (j-\mu)/\sqrt{\mu_2},$$
$$g_2(j) = b\{(j-\mu)^2 - \mu_3(j-\mu)/\mu_2 - \mu_2\}.$$

It is possible to derive $g_3(j), \ldots, g_k(j)$ so that $\{g_r(j)\}$ is a set of orthonormal functions by using the recurrence relations in Emerson (1968). Take

$$V_r = \sum_{j=0}^{k} N_j g_r(j)/\sqrt{n}, \quad r = 1, \ldots, k,$$

and then, as in Lancaster (1953),

$$X^2 = V_1^2 + \cdots + V_k^2.$$

The smooth test statistic S_c of Rayner and Best (1989, p. 97) or Best and Rayner (1989) is defined through components U_r, which in turn are defined for $r = 1, 2, \ldots, c+1$ by

$$U_r = \sum_{j=1}^{n} h_r(Z_j; \hat{q})/\sqrt{n},$$

in which $\{z_j\}$ corresponding to the $\{Z_j\}$ are, as above, the ungrouped data and

$$h_r(z; q) = K \sum_{i=0}^{r} {}^{z}C_{r-i}({}^{r}C_i)^2 i!(r-i)!(-a)^i$$

are Meixner orthonormal polynomials with $a = q/(1-q)$ and $K^{-1} = r!(a^2+a)^{r/2}$. Then S_c is given by

$$S_c = U_2^2 + \cdots + U_{c+1}^2.$$

If the ungrouped maximum likelihood estimator of q, $\hat{q}$, is used, asymptotically the U_r for $r=2,3,4,\ldots$ have the standard normal distribution and are asymptotically mutually independent. Thus asymptotically S_c has the χ_c^2 distribution. The likelihood equation is $U_1=0$, which is why U_1 is excluded from S_c. The U_r often indicate deviations of the data from the Geometric in the rth moment, but may be due to moments up to the 2rth; see Rayner et al. (1995). Comparing observed and expected moments is a well-known method of testing fit. Inglot and Ledwina (1996) suggest an adaptive procedure for making smooth tests more powerful but we do not consider this procedure here.

We define a Kolmogorov–Smirnov (*KS*) statistic for the Geometric distribution following the approach of Henze (1996) for the Poisson distribution. We note that *KS* statistics are well known for providing tests of fit. First put $m=\max(z_1,\ z_2,\ldots,\ z_n)$ and $\hat{p}_m=1-\hat{p}_0-\hat{p}_1-\cdots-\hat{p}_{m-1}$. Now for $j=0,1,2,\ldots,m$, let $R_j=N_0+N_1+\cdots+N_j-n(\hat{p}_0+\hat{p}_1+\cdots+\hat{p}_j)$. Then put

$$KS=\max(|R_0|,|R_1|,\ldots,|R_m|).$$

Following Henze and Klar (1996) we define a modified dispersion test statistic

$$S_1^*=nS_1\Big/\sum_{j=1}^{n}h_2^2(Z_j;\hat{q}).$$

Henze and Klar (1996) point out that S_1^* can always be used to *diagnose* dispersion alternatives; the Geometric null hypothesis is accepted if and only if the data are consistent with the Geometric dispersion. On the other hand, significance of $S_1=U_2^2$ can sometimes be due to moments other than the second.

The Anderson–Darling statistic is becoming a popular choice for testing fit. For example, recent editions of MINITAB use an Anderson–Darling (*AD*) test of normality. Put $H_j=\hat{p}_0+\hat{p}_1+\cdots+\hat{p}_j$, $j=0,1,2,\ldots,m$, and then

$$AD=n\sum_{j=0}^{m}R_j^2\hat{p}_j/\{H_j(1-H_j)\}.$$

The *AD* and *KS* statistics compare empirical and theoretical distribution functions. Spinelli and Stephens (1997) show that a similar Anderson–Darling statistic does well for testing for the Poisson distribution. We follow Spinelli and Stephens (1997) by choosing m so that $p_m<10^{-3}/n$ and $N_m=0$. Notice that this choice of m differs from that used for the *KS* statistic.

It is well known that the Geometric distribution is a special case of the negative binomial distribution. A possibly less well known result is that the Geometric distribution is a special case of the grouped data Exponential distribution. Spinelli (2001) gives details. This means tests of fit for the grouped Exponential can also be used for the Geometric. Thus as part of our power comparisons we include two tests from Gulati and Neus (2002). Define

$$SW1 = \sqrt{n}\sum_{j=0}^{m-1} \left|R_j\right| \Psi_1(j)$$

in which $\Psi_1(j) = 1/\sqrt{\{H_j(1 - H_j)\}}$. Also define

$$SW2 = \sqrt{n}\sum_{j=0}^{m-1} \left|R_j\right| \Psi_2(j)$$

where $\Psi_2(j) = (n/2 - j)^2+$ in which $x+ = 1$ if $x = 0$ and $x+ = x$ if $x > 0$.

3. POWER COMPARISONS

Good power comparisons need accurate critical values so that no tests are unfairly advantaged. Critical values for $\alpha = 0.05$ and for each of the statistics defined above were calculated for each of 25,000 random samples of n Geometric values generated using the inversion algorithm of Evans et al. (1993, p. 84). A value of the test statistic was calculated for each of these 25,000 samples and the values of the test statistics were ordered, with the 23,750th taken as the critical value. We took $q = 0.4$, 0.5, 0.75. The critical values are given in Table 1. Notice that none of the smooth test critical values are close to their asymptotic values. Rayner and Best (1989, p. 97) give corrected critical values of S_4 that allows the asymptotic value to be used. The tests based on V_2^2 and V_3^2 have critical values close to their asymptotic values for much of the range of q. For $q < 0.1$ there may be problems with the approximation, but such values seem rare with actual data. Unpresented results show that the same is true for subsequent V_r^2 and for sums of V_r^2 such as $V_2^2 + \cdots + V_k^2$. The test based on $X^2_{\text{CLE1+}}$ is excluded from Table 1 as different data sets will result in different estimates on the unknown parameters and hence different numbers of classes and different critical values. However from the $NB(1, q)$ rows in Table 2 it appears that the rejection rate under the null hypothesis for this test is consistently close to that specified.

Powers based on 10,000 Monte Carlo samples of size $n = 50$ are shown in Table 2. Powers for $n = 20$ give very similar comparisons and

Table 1. Critical values based on 25,000 samples for geometric tests of fit when $n = 50$, $\alpha = 0.05$, and q as shown.

q	$S_1 = U_2^2$	U_3^2	S_2	S_3	S_4	S_1^*
(i) Geometric smooth tests						
0.75	2.794	1.972	4.495	5.717	6.513	4.489
0.5	2.794	1.918	4.499	5.690	6.483	4.194
0.4	2.647	1.867	4.366	5.570	6.383	3.966
χ^2 critical value	3.841	3.841	5.991	7.815	9.488	3.841
q	KS	AD	V_2^2	V_3^2	$SW1$	$SW2$
(ii) Various other Geometric tests						
0.75	6.062	1.323	3.846	3.760	26.067	692.038
0.5	5.005	1.277	3.863	3.814	10.664	47.631
0.4	4.333	1.201	3.889	3.858	7.960	19.985
χ^2 critical value	—	—	3.841	3.841	—	—

Table 2a. Powers (%) based on 10,000 samples for geometric tests of fit when $n = 50$, $q = 0.75$, $\alpha = 0.05$. The mean of all alternatives is 3.0; the null variance is 12.

Alternative	(σ^2)	$S_1 = U_2^2$	U_3^2	S_2	S_3	S_4	S_1^*
(i) Geometric smooth tests							
$BB(1, 2, 9)$	(6)	65	56	64	52	44	85
$BB(0.5, 1, 9)$	(8.4)	8	6	7	6	13	17
$NA(1, 3)$	(12)	5	19	9	17	13	2
$NB(1, 0.25)$	(12)	5	5	5	5	5	5
$NA(0.95, 3.16)$	(12.5)	6	25	11	23	19	2
$NA(0.9, 3.33)$	(13)	8	33	17	32	27	2

Alternative	(σ^2)	KS	AD	$X^2_{\mathrm{CLE1+}}$	V_2^2	V_3^2	$SW1$	$SW2$
(ii) Various other geometric tests								
$BB(1, 2, 9)$	(6)	51	61	53	65	21	19	38
$BB(0.5, 1, 9)$	(8.4)	20	23	52	10	56	10	20
$NA(1, 3)$	(12)	64	66	59	19	59	3	22
$NB(1, 0.25)$	(12)	5	5	5	5	5	5	5
$NA(0.95, 3.16)$	(12.5)	75	77	69	26	65	3	26
$NA(0.9, 3.33)$	(13)	84	85	78	34	71	5	30

Table 2b. Powers (%) based on 10,000 samples for geometric tests of fit when $n = 50$, $q = 0.5, \alpha = 0.05$. The mean of all alternatives is 1.0; the null variance is 2.

Alternative	(σ^2)	$S_1 = U_2^2$	U_3^2	S_2	S_3	S_4	S_1^*
(i) Geometric smooth tests							
DU(0, 2)	(0.67)	99	99	99	99	48	99
BB(1, 2, 3)	(1)	65	65	65	56	98	87
$\{P(0.5) + P(1.5)\}/2$	(1.25)	27	32	31	27	23	56
NB(1, 0.5)	(2)	5	5	5	5	5	5
$\{P(0.0) + P(2.0)\}/2$	(2)	1	10	2	6	15	0
NA(1, 1)	(2)	3	7	4	4	3	3
NA(0.625, 1.6)	(2.6)	21	45	32	39	32	8
NA(0.5, 2)	(3)	42	1	63	73	66	22
NA(0.4, 2.5)	(3.5)	69	86	89	94	92	46

Alternative	(σ^2)	KS	AD	$X^2_{\mathrm{CLE1+}}$	V_2^2	V_3^2	$SW1$	$SW2$
(ii) Various other geometric tests								
DU(0, 2)	(0.67)	97	99	99	99	37	78	85
BB(1, 2, 3)	(1)	53	69	60	82	4	21	41
$\{P(0.5) + P(1.5)\}/2$	(1.25)	41	41	22	45	5	3	22
NB(1, 0.5)	(2)	5	5	5	5	5	5	5
$\{P(0.0) + P(2.0)\}/2$	(2)	19	24	32	4	48	1	19
NA(1, 1)	(2)	7	7	10	5	11	3	6
NA(0.625, 1.6)	(2.6)	51	54	40	35	27	16	30
NA(0.5, 2)	(3)	83	87	71	66	35	37	49
NA(0.4, 2.5)	(3.5)	98	99	93	90	38	65	63

are available on request to the first author. Six types of alternative distribution are used.

(i) $BB(a, b, m^*)$: a beta-binomial distribution with parameters a, b, and m^*. Observations from this distribution come from a composite binomial distribution with parameters m^* and p^*, where p^* has a standard beta distribution with parameters a and b.

(ii) $DU(i, j)$: a discrete uniform distribution defined on integers i, $i+1, \ldots, j$. The probability of a value x occurring is $(j - i + 1)^{-1}$, where $i \le x \le j$.

(iii) $B(m^*, p^*)$: a standard binomial distribution with m^* trials and probability of success equal to p^*.

(iv) $NB(k^*, p^*)$: observations from this distribution are the number of trials until k^* successes, each having probability p^* of success.

Table 2c. Powers (%) based on 10,000 samples for geometric tests of fit when $n = 50$, $q = 0.4$, $\alpha = 0.05$. The mean of all alternatives is 2/3; the null variance is 10/9.

Alternative	(σ^2)	$S_1 = U_2^2$	U_3^2	S_2	S_3	S_4	S_1^*
(i) Geometric smooth tests							
$B(4, 0.17)$	(0.56)	69	74	75	69	65	89
$\{P(0.33) + P(1)\}/2$	(0.83)	14	18	18	15	13	37
$\{P(0) + P(1.33)\}/2$	(1.11)	1	7	2	3	5	1
$NA(1, 0.67)$	(1.11)	4	5	4	4	3	3
$NB(1, 0.6)$	(1.11)	5	5	5	5	5	5
$NA(0.67, 1)$	(1.33)	13	21	16	17	13	3
$NA(0.33, 2)$	(2)	72	80	88	90	87	47

Alternative	(σ^2)	KS	AD	$X^2_{\mathrm{CLE1+}}$	V_2^2	V_3^2	$SW1$	$SW2$
(ii) Various other geometric tests								
$B(4, 0.17)$	(0.56)	79	80	60	83	21	23	49
$\{P(0.33) + P(1)\}/2$	(0.83)	27	26	13	26	5	2	14
$\{P(0) + P(1.33)\}/2$	(1.11)	7	7	13	3	21	1	7
$NA(1, 0.67)$	(1.11)	5	5	6	5	8	3	5
$NB(1, 0.6)$	(1.11)	5	5	4	5	5	5	5
$NA(0.67, 1)$	(1.33)	20	21	17	16	15	11	15
$NA(0.33, 2)$	(2)	90	94	85	86	20	71	56

(v) $w_1P(\lambda_1) + w_2P(\lambda_2)$: a Poisson mixture with weights w_1 and w_2. The means of the two Poisson distributions are λ_1 and λ_2.

(vi) $NA(\lambda_1, \lambda_2)$: a Neyman Type A distribution with parameters λ_1 and λ_2.

Random $BB(a, b, m^*)$ values were generated using the random beta distribution generator of Cheng (1978) and the binomial generator of Best (1978, p. 347). Random negative binomials were generated as sums of Geometric random variables that were generated by inversion as described by Evans et al. (1993, p. 84). Random binomial and discrete uniform values were generated as random multinomials following the method of Devroye (1986, p. 558). Random Poisson random variables were generated as in Dagpanur (1988, p. 199). To generate a random Type A variate, a $P(\lambda_1)$ random variate is generated, say i, and then the sum of i $P(\lambda_2)$ random variates is found. Alternatives were chosen so that means equalled the null distribution mean. Thus differences in means cannot mask other differences.

From Table 2 we see that the diagnostic dispersion statistic S_1^* of Henze and Klar (1996) has good power for underdispersed alternatives but poor power for overdispersed alternatives. This was also the case when testing for the Poisson distribution; see Best and Rayner (1999). If one statistic had to be chosen on the basis of Table 2 then we might opt for *AD* as this statistic performs well against all three types of alternative: that is, underdispersed, equally dispersed and overdispersed alternatives. The *KS* statistic seems almost as good as the *AD* statistic, but most of the other statistics, while good for some alternatives, are poor for others. The tests based on *SW*1 and *SW*2 often have less power than the other tests.

A problem for the tests based on U_r^2 and S_c is that for some alternatives power can be less than the test size so that U_r^2 and S_c produce inconsistent tests; see the mixed Poisson alternatives with dispersion equal to that of the geometric. The diagnostic dispersion statistic of Henze and Klar (1996) also seems to have a problem with this alternative. The statistic S_4 which Best and Rayner (1989) and Rayner and Best (1989, p. 97) propose for testing the Geometric assumption is outperformed by *AD* and *KS*.

We observe that the distribution of V_1^2 is not χ_1^2 if the ungrouped estimator is used. Often $\hat{q}$ is close to the estimator obtained by solving $V_1 = 0$ for q and so V_1 calculated using $q = \hat{q}$ will be close to zero.

Table 2 gives powers for $X^2_{\mathrm{CLE1+}}$, V_2^2, and V_3^2, obtained using critical values from the relevant χ^2 distributions. Observe that either $X^2_{\mathrm{CLE1+}}$ or V_2^2 is as good as or better than *AD* for most alternatives. Of course it is not quite fair to compare a test with the better of two competitor tests. With the alternatives considered here V_2^2 and V_3^2 seem better at giving consistent tests than U_2^2 and U_3^2.

Although we do not pursue the point here, we expect reasonably good approximate powers for X^2 and V_2^2 can be obtained using noncentral χ^2 distributions as in Best and Rayner (1997).

Clearly $X^2_{\mathrm{CLE1+}} - V_1^2$ rather than $X^2_{\mathrm{CLE1+}}$ should be referred to the null asymptotic χ^2_{k-1} distribution, but as V_1^2 should always be small, use of $X^2_{\mathrm{CLE1+}}$ as a first approximation to $X^2_{\mathrm{CLE1+}} - V_1^2$ is reasonable in practice. Of course, if using χ^2_{k-1} as the null distribution, if $X^2_{\mathrm{CLE1+}}$ is not significant at some level the same will be true of $X^2_{\mathrm{CLE1+}} - V_1^2$. Alternatively, using χ^2_{k-1} as the null distribution of $X^2_{\mathrm{CLE1+}}$ will slightly overestimate the true *p*-value. We expect the Chernoff–Lehmann statistic and its components will also provide a good test of fit analysis for other lattice distributions.

4. EXAMPLE

To illustrate tests for Geometric goodness of fit, consider the following example which is similar to one discussed by Law and

Table 3. Statistic values and bootstrap p-values for the inventory data example.

Statistic	$S_1 = U_2^2$	U_3^2	S_2	S_3	S_4	S_1^*
(i) Geometric smooth tests						
Value	3.33	2.50	5.83	6.43	6.43	6.99
p-value	0.036	0.028	0.027	0.037	0.049	0.004

Statistic	KS	AD	$X^2_{\mathrm{CLE1+}}$	V_2^2	V_3^2	$SW1$	$SW2$
(ii) Various other geometric tests							
Value	5.27	1.63	9.33	7.24	0.81	8.80	22.59
p-value	0.043	0.023	0.071	0.005	0.789	0.084	0.086

Kelton (1991, p. 366). Suppose the frequency distribution of items demanded per day from an inventory over a 50 day period is 19 days of zero demand, 15 days of one item demand, 10 days of two items and six days of three items.

For these data and for all the statistics considered in the power study, Table 3 gives statistic values and the corresponding p-values calculated using a parametric bootstrap, as explained, for example, in Gulati and Neus (2002). For example, following the results of the previous section to calculate $X^2_{\mathrm{CLE1+}}$ and V_2^2 requires the number of classes to be the largest number such that the expectation of each class is at least unity. Since $\hat{q} = 0.515$ the number of classes will be six. We thus find $X^2_{\mathrm{CLE1+}} = 9.33$ on four degrees of freedom and $V_2^2 = 7.24$ on one degree of freedom. The V_2^2 component gives a highly significant p-value of less than 0.01, suggesting less dispersion ($V_2 = -2.686$) than expected for a Geometric model. Some caution is needed in interpreting this V_2 value; see for example, Rayner et al. (1995). However reference to observed and expected values (see Table 4) of the data also suggests underdispersion of the Geometric model. The dispersion component accounts for most of the Chernoff–Lehmann statistic and so we can say the important differences between the inventory data and the Geometric distribution are related to dispersion. The other tests of fit do not allow such a complete data diagnosis.

From Table 2(b) and knowing $\hat{q}$ is about 0.5, $n = 50$ and the alternative is underdispersed, we expect S_1^*, V_2^2 and AD to do well. In fact they have the smallest p-values. We expect V_3^2 to do badly and $SW1$ and $SW2$ to be mediocre performers. This, too, is the case. The V_3^2, X^2, $SW1$, and $SW2$ tests do not have p-values less than 5%.

For the reader who wishes to calculate V_2 we now give some details. Recall from the previous section that we need to calculate μ, μ_2, μ_3,

Table 4. Observed counts and expected counts under two models: geometric ($50f(j;\ q)$) and geometric dispersion adjusted ($50f^*(j;\ q)$).

j	Observed	$50f(j;\ q)$	$50f^*(j;\ q)$
0	19	24.25	18.3
1	15	12.50	16.2
2	10	6.45	9.9
3	6	3.30	4.2
4	0	1.70	1.2
≥ 5	0	1.80	0.4

and μ_4. We have $\{p_j\} = \{0.485,\ 0.250,\ 0.129,\ 0.066,\ 0.034,\ 0.036\}$ and so

$$\mu = \sum_{j=0}^{5} jp_j = 1.022,\ \mu_2 = \sum_{j=0}^{5} (j - 1.022)^2 p_j = 1.7595,$$

$$\mu_3 = \sum_{j=0}^{5} (j - 1.022)^3 p_j = 3.2800 \text{ and } \mu_4 = \sum_{j=0}^{5} (j - 1.022)^4 p_j = 13.346.$$

As in the previous section, $g_2(j) = b\{(j-\mu)^2 - \mu_3(j-\mu)/\mu_2 - \mu_2\}$, where here

$$b^{-2} = (13.346 - 3.28^2/1.7595 - 1.7595^2) = 4.13569,$$

so that $b = 0.49173$. Thus

$$\begin{aligned} &g_2(0) = 0.58524,\ g_2(1) = -0.84479,\\ &g_2(2) = -1.29136,\ g_2(3) = -0.75448 \text{ and}\\ &V_2 = \sum_{j=0}^{5} N_j g_2(j)/\sqrt{50}\\ &\quad = (19^*0.58524 - 15^*0.84479 - 10^*1.29136 - 6^*0.75448)/\sqrt{50}\\ &\quad = -2.686. \end{aligned}$$

Table 4 compares the observed data with $50f(j;\ q)$, the expected counts under a fitted Geometric model. Since V_2 is highly significant, modifying the model to account for this effect is clearly desirable. The appropriate model could be called Geometric dispersion adjusted, and gives counts $50f^*(j;\ q)$, in which $f^*(j;q) = Cf(j;q)\exp[V_2 g_2(j)/\sqrt{50}]$, where C is a constant such that $\sum_{j=0}^{5} f^*(j;q) = 1$. The alternative model is a significant improvement. Software for calculating X^2, V_2, V_3, and V_4 is available from the first author.

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