Research Statement

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My research interests are dynamical systems and ergodic theory. I have focused on dynamics of translation surfaces, their strata and interval exchange transformations. I will explain my recent works in two sections. The first section describes my work on the real Rel flow on spaces of translation surfaces, and the second describes my work on determining the slow entropy of interval exchange transformations. Both sections intersect: the geodesic flow on a translation surface, interval exchange transformation and the real Rel flow have zero entropy. Moreover, the first return map of the geodesic flow on a translation surface to a transverse segment is an interval exchange transformation.

A dynamical system is a pair (X, T), where X is a space (topological or measurable) and $T : X \to X$ is a map. A flow is a family of dynamical systems (X, T_t) parameterized by $t \in \mathbb{R}$. The family of maps $T_t : X \to X$ satisfy $T_{t+s} = T_t \circ T_s$.

1 Translation Surfaces

A **translation surface** is a finite collection of polygons with each edge identified by translation to another edge. The surface is flat everywhere except at a finite collection of points. Those points are called singularities.

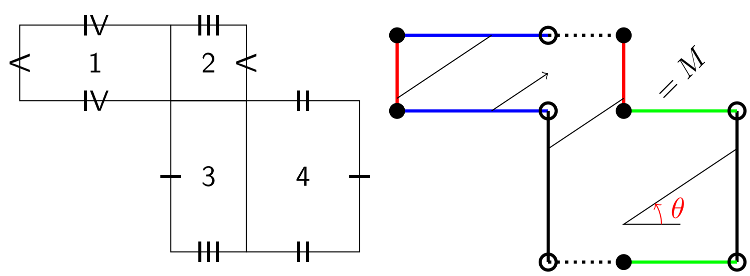

Figure 1: On the left, sides of 4 rectangles are identified to produce the surface M on the right. On the surface M, we can draw straight lines with inclination θ; these are the trajectories of the geodesic flow in the direction θ.

A **singularity** is a point for which all small curves around it make an angle of the form $2(n+1)\pi$ for $n \in \{1, 2, 3, \dots\}$. The number n is the **order** of the singularity. In Figure 1, the surface M has two singularities labeled $\circ$ and $\bullet$. A small closed curve around one of these singularities makes a 4π angle because they have an "extra" 2π-angle of rotation around them.

Let $\mathcal{H}(n_1, \dots, n_k)$ be the set of translation surfaces with k singularities where the i-th singularity has order n_i. These collections are called **strata**, with a natural topology and an orbifold structure.

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1.1 The Real Rel Flow

We consider a dynamical system on the spaces $\mathcal{H}$. The points are now translation surfaces. The **real Rel flow** denoted by $Rel_t : \mathcal{H} \to \mathcal{H}$ is a flow that deforms $M \in \mathcal{H}$. It fixes one of the singularities and moves the position of some of the other singularities horizontally at a fixed rate for the time t. Denote $Rel_t M$ the resulting surface. The lengths and directions of all closed geodesics remain unchanged along a trajectory. In Figure 2, we illustrate an example of this flow in $\mathcal{H}(1,1)$.

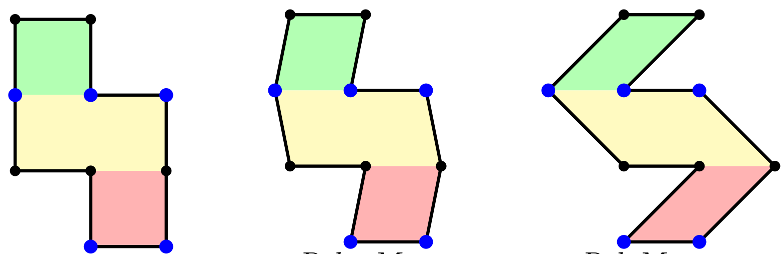

Figure 2: The surface $M \in \mathcal{H}(1,1)$, and different times of real Rel flow Rel_t on M.

1.2 Recent Results on Real Rel Trajectories that are not Recurrent

One of my most recent results discusses the recurrence of trajectories. The question is whether an orbit of a point will return and approach arbitrarily close to the starting point.

The real Rel orbit of M is **recurrent** if for every open neighborhood U containing M and every real number t_0, there exists t_1 such that $t_1 > t_0$ and $Rel_{t_1} M \in U$. Additionally, a real Rel trajectory of M is **divergent** if for every compact set K there exists t_0 such that $Rel_t M \notin K$ for $t \geq t_0$.

Theorem 1 (Ospina 2024)**.** *There exists a translation surface $M \in \mathcal{H}(1,1)$ such that the real Rel trajectory is not recurrent and not divergent.*

The real Rel flow has been an active area of research in recent years. Mathematicians have proved ergodic theoretical results and topological results. The Masur-Veech measure is a natural measure on $\mathcal{H}(1,1)$ similar to the Lebesgue. This measure is invariant with respect to real Rel flow. In ergodic theory, one of the most basic theorems is of Poincaré recurrence. This theorem states that all but a measure zero set of points is recurrent. In 2018, Pat Hooper and Barak Weiss proved that the Rel leaf (a relative of the flow) of the Arnoux-Yoccoz translation surface is dense in the stratum $\mathcal{H}(n,n)$ for $n \geq 3$. In 2022, Karl Winsor proved that there are dense real Rel trajectories in every connected component of a stratum. None of these results could predict Theorem 1.

The surfaces in Theorem 1 can be described explicitly. In the "slit construction", where one can take two tori T_1 and T_2 and an embedded segment J into each T_i, then identifying opposite sides of the embedded J's we obtain a surface $M = T_1 \#_J T_2 \in \mathcal{H}(1,1)$. The tori T_i can be interpreted as elements in the group $\mathcal{H}(0) := \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$. The matrices $u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ act on $\mathcal{H}(0)$ by left multiplication. A surface $M = (u_{s_1}T)\#_J(u_{s_2}T)$ where, $s_1 \neq s_2$, T is "badly approximable" and J is horizontal satisfies Theorem 1.

Theorem 1 is a classification of the recurrence for real Rel trajectories coming from tremors of the eigenform locus of discriminant 4, denoted here by $\mathcal{E}_4$.

1.3 Future directions

The locus $\mathcal{E}_4 \subset \mathcal{H}(1,1)$ of surfaces of the form $T\#_I T$ was introduced by Curt McMullen in 2007. He described more loci parametrized by the "discriminant". These sets of translation surfaces are denoted by $\mathcal{E}_D$. A natural problem would be the classification of real Rel trajectories coming from these loci. In particular, we can study analogs of Theorem 1:

Question 1. *Given a locus $\mathcal{E}_D \subset \mathcal{H}(1,1)$ of discriminant $D > 4$. Is there a surface $M \in \mathcal{E}_D$ such that $trem_\beta M$ is not Rel recurrent?*

Tremors (denoted as *trem*) and their properties were introduced Jon Chaika, John Smillie and Barak Weiss in 2020. They used tremors as an important tool to prove their results. Tremors are deformations of translation surfaces motivated by geometric and ergodic theoretical properties. These deformations were used again in the proof of Theorem 1. The previous question is interesting because it will test if tremors can be used to extract more properties and behaviors of the real Rel flow.

In unpublished work, I have determined that some orbit closures related to Theorem 1 contain constructions of the form $T\#_I T$, where T is fixed, and the segment I is allowed to vary. This is not all the orbit closure, the remaining is the orbit itself. This remaining piece is similar to the spikes of the "spiky fish" in the work of Jon Chaika, John Smillie and Barak Weiss. In the context of the horocycle flow, they proved:

Theorem 2. *(Chaika-Smillie-Weiss 2020) There exists a translation surface $M \in \mathcal{H}(1,1)$, such that $HD\left(\overline{\left\{\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} M\right\}}\right) \in [5.5, 6)$.*

The spiky fish is the orbit closure of the horocycle flow (a flow defined by a group action in strata), and HD is the Hausdorff dimension.

Question 2. *Are there any real Rel orbit closures with non-integer Hausdorff dimensions?*

The real Rel flow played a crucial role in classifying horocycle orbit closures in the work of Matt Bainbridge, John Smillie and Barak Weiss in 2022 for the stratum $\mathcal{H}(1,1)$. So, instead of the real Rel flow, let's consider the horocycle flow on strata of translation surfaces. A question close to my work and Theorem 1 is:

Question 3. *Is there a translation surface whose horocycle flow is not recurrent?*

The strata of translation surfaces are similar to homogeneous spaces, which quotients of the form G/Γ where G is a semisimple Lie group and Γ is a lattice. Homogenous spaces carry flows similar to the horocycle flow and are called *unipotent flows*. This question is interesting because in the context of homogeneous dynamics, Question 3 has a negative answer. It is known due to rigidity results by Marina Ratner. She proved that every unipotent orbit is recurrent if the quotient G/Γ has a finite volume.

A reason to study this question is that on strata of translation surfaces, the real Rel flow Rel_t commutes with the horocycle flow u_s:

$$Rel_t \circ u_s = u_s \circ Rel_t.$$

Both flows are **re-normalized** by the geodesic flow $g_r = \begin{pmatrix}e^r & 0\\ 0 & e^{-r}\end{pmatrix}$:

$$g_r \circ Rel_t = Rel_{e^r t} \circ g_r \qquad g_r \circ u_s = u_{e^{2r}s} \circ g_r.$$

The framework of re-normalization has been very fruitful to unravel properties of either the real Rel flow and the horocycle flow. These equations suggest that behaviors of the real Rel flow could be expected for the horocycle flow and vice-versa. Eventhough there is a big difference between the real Rel flow and the horocycle flow:

- There are divergent real Rel orbits. An example was presented in the work of Patrick Hooper and Barak Weiss in 2018.
- Every horocycle orbit is non-divergent by the work of Yair Minsky and Barak Weiss in 2002.

The work Minksy-Weiss proves that no horocycle orbit is divergent. They showed that the time a horocycle orbit will visit a compact set has a positive proportion. However, this does not guarantee that all the horocycle orbits will be recurrent. The examples I found to prove Theorem 1 are non-divergent, we could wonder if there are examples to answer Question 3 in the afirmative.

2 Zero Entropy Systems

A measure preserving system is a triple (X, T, μ) where $T : X \to X$ is measurable map and $\mu(T^{-1}A) = \mu(A)$ for any measurable set $A \subseteq X$.

How do we determine if two measure preserving systems (X, T, μ) and (Y, S, ν) are the same? Two such systems are measurably equivalent (the *same* from the point of view of the measures), if there is an invertible map

$$\pi : X \to Y \quad \text{such that} \quad \pi \circ T = S \circ \pi \text{ and } \mu(\pi^{-1}(B)) = \nu(B)$$

for all measurable sets $B \subset Y$. We can rule out measurably equivalence by computing the entropy of the systems and showing that they are different. Even if two systems have the same entropy, the systems might not be the same since entropy is an invariant that is constant in equivalence classes. In the early years of the theory, Ornstein and Sinai established that metric entropy is a complete invariant for Bernoulli systems. This result established entropy as one of the most important invariants of a dynamical system.

2.1 Interval Exchange Transformations (IETs)

An **IET** is a map from an interval I to itself that is locally isometric and preserves the orientation with finitely many discontinuities. For brevity, we will define 3-IETs. These are maps determined by a vector of positive entries $\vec{\lambda} = (\lambda_A, \lambda_B, \lambda_C)$ such that $\lambda_A + \lambda_B + \lambda_C$ is the length of the interval I. For simplicity assume that $I = [0, \lambda_A + \lambda_B + \lambda_C)$, then a 3-IET is defined by

$$T(x) = \begin{cases} x + \lambda_B + \lambda_C & \text{if } x \in I_A = [0, \lambda_A), \\ x - \lambda_A + \lambda_C & \text{if } x \in I_B = [\lambda_B, \lambda_A + \lambda_B), \\ x - \lambda_B - \lambda_A & \text{if } x \in I_C = [\lambda_A + \lambda_B, \lambda_A + \lambda_B + \lambda_C). \end{cases}$$

A 3-IET takes the subintervals I_A, I_B and I_C and rearranges them according to the permutation $\pi = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$. See Figure 3 for an example of how a 3-IET works.

In general, a d-IET would permute d subintervals according to a permutation of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_d \\ \alpha_{k_1} & \alpha_{k_2} & \alpha_{k_3} & \cdots & \alpha_{k_d} \end{pmatrix}.$$

The subindices $k_1, \ldots, k_d \in \{1, \ldots, d\}$ are all different and $\alpha_1, \ldots, \alpha_d$ are symbols to label the subintervals to be permuted.

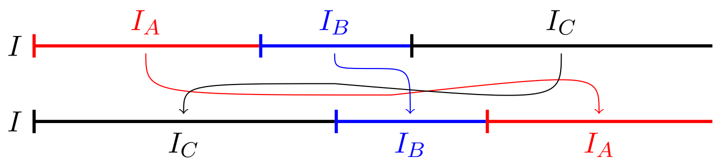

Figure 3: A 3-IET takes three subintervals of I and rearranges them.

2.2 Slow entropy

The class of zero-entropy systems is large, which makes classification questions challenging. For example, the flow in the direction θ on a translation surface has zero entropy. The same is true for the real Rel flow described in the previous section. Also, the entropy of interval exchange transformations is zero. Mathematicians have used various methods, including mixing, joinings, spectral properties, etc., to answer classification questions in the class of zero entropy systems. Entropy comes in two flavors: topological entropy h_{top} and metric entropy h_μ (which depends on the theoretical properties of an invariant measure μ).

A partition $\mathcal{P} = \{P_1, \dots, P_n\}$ of X is a finite collection of disjoint measurable sets whose union is X. For every $x \in X$, let $(x_0, x_1, \dots) \in \{1, \dots, n\}^{\mathbb{N}}$ be

$$x_i = k \quad \text{if} \quad T^i(x) \in P_k.$$

Now, given $N \geq 0$ we have a semi-norm:

$$d_N(x, y) = \frac{|\{0 \leq i \leq N-1 : x_i \neq y_i\}|}{N}.$$

A **Hamming ball** $B_N(x, \epsilon)$ is a ball around x of radius $\epsilon > 0$ using the semi-norm d_N.

Let $S_{N,\mathcal{P}}(\epsilon)$ be the minimum number of Hamming balls of radius $\epsilon > 0$ that cover a subset of X of μ-measure at least $1 - \epsilon$. This means that there are $S_N(\epsilon)$ sets that determine most of the dynamical system up to time N. The growth of $S_N(\epsilon)$ as $N \to \infty$ determines the metric entropy of the system (X, T, μ).

Let $\{a_\chi : \mathbb{N} \to \mathbb{R}\}_{\chi \in \mathbb{R}}$ be a family of functions, that we will call *scale* a_χ. The scale satisfies two conditions:

1. For a fixed χ we have the increasing condition $a_\chi(n) \leq a_\chi(n+1)$ for all $n \in \mathbb{N}$.
2. If $\chi < \chi'$, then $\lim_{n \to +\infty} a_\chi(n)/a_{\chi'}(n) = 0$.

The **metric slow entropy** at scale a_χ is given by

$$h_{\mu, a_\chi} = \sup_{\mathcal{P}} \lim_{\epsilon \to 0} \sup_{\chi} \left\{ \chi : \limsup_{N \to \infty} \frac{S_N(\epsilon)}{a_\chi(N)} > 0 \right\}.$$

2.3 Recent Results

An important open problem in the theory is calculating the entropy of interval exchange transformations. Recently, in collaboration with colleagues, we proved:

Theorem 3 (Cheng-Ospina-Vinhage-Zhai 2024)**.** *A set of full Hausdorff dimension of 3-IETs satisfy that $h_{Leb, a_\chi} = 1$ at scale $a_\chi(N) = N^\chi$.*

In the proof of Theorem 3, we chose a parametrization of 3-IETs of the form (α, ξ) that for some constant $c > 0$ satisfies the approximation properties:

$$\|i\alpha - p\| \geq \frac{c}{n} \quad \text{and} \quad \|j\alpha - q - \xi\| \geq \frac{c}{n} \quad \text{for all} \quad 0 < i, j, p, q < n. \tag{1}$$

We used these properties to control the growth of Hamming balls. Analyzing the decay of the Lebesgue measure of "Bowen balls", we estimated the growth of the number $S_N(\epsilon)$ as $n \to \infty$.

2.4 Future Directions

Let $|x - y|$ be the distance in an interval for two points x and y. For $N \in \mathbb{N}$ denote $\bar{d}_n(x, y) = \max_{0 \leq i \leq n-1} |T^i x - T^i y|$, where T is a d-IET (or a map of the interval). A **Bowen ball** $\widetilde{B}_N(x, \epsilon)$ is an ϵ-Ball around x with the metric $\bar{d}_N$. In the case of IETs, the number of Bowen balls that cover the interval is finite. Denote $C_N(\epsilon)$ the minimum number of Bowen balls of radius ϵ that covers the entire interval. We define the **topological slow entropy** at scale a_χ by

$$h_{top,a_\chi} = \lim_{\epsilon \to 0} \sup_{\chi} \left\{ \chi : \limsup_{N \to \infty} \frac{C_N(\epsilon)}{a_\chi(N)} > 0 \right\}.$$

Not all 3-IETs can be parameterized under the conditions of Equation (1). But the proof of Theorem 3 motivates the following question.

Question 4. *Are there any 3-IETs with $h_{top,a_\chi} = 1$ and $h_{\mathrm{Leb},a_\chi} \neq 1$ at scale $a_\chi(N) = N^\chi$?*

Linearly recurrent d-IETs is a class that generalizes the properties in Equation (1). We proved the following:

Theorem 4 (Cheng-Ospina-Vinhage-Zhai 2024)**.** *Suppose that g is an idoc, linearly recurrent d-IET. Then $h_{top,a_\chi} = 1$ with scale $a_\chi(N) = N^\chi$.*

Goodwyn's inequality states that

$$h_{\mu,a_\chi} \leq h_{top,a_\chi}$$

for any scale a_χ. Michael Boshernitzan introduced a condition called property P for uniquely ergodic IETS. Nowadays, it is known as Boshernitzan's criterion for unique ergodicity, in particulars linearly recurrent IET satisfy this property hence uniquely ergodic (Lebesgue measure). Theorem 4 motivates the following:

Question 5. *What is the metric entropy h_{Leb,N^χ} for idoc, linearly recurrent d-IETs with $d \geq 4$?*

Adding the linearly recurrent condition makes this question very specific. Currently, we do not have any results on the slow entropy (metric or topological). Another research direction is to investigate Theorem 4 and Question 5 dropping the linearly recurrent condition.

Jon Chaika and Barak Weiss in 2023 proved that the real Rel flow has zero entropy for many measures (in particular the Masur-Veech measure). It is not known what is the slow entropy of the real Rel flow. Therefore, a question that I would like to answer is:

Question 6. *Is the real Rel flow strongly variational? This means that $h_{\mu,a_\chi} = h_{top,a_\chi}$ for some invariant measure μ.*

The real Rel flow is invariant for many measures μ, but calculating the slow entropy for the Masur-Veech measure would shed light in this direction.