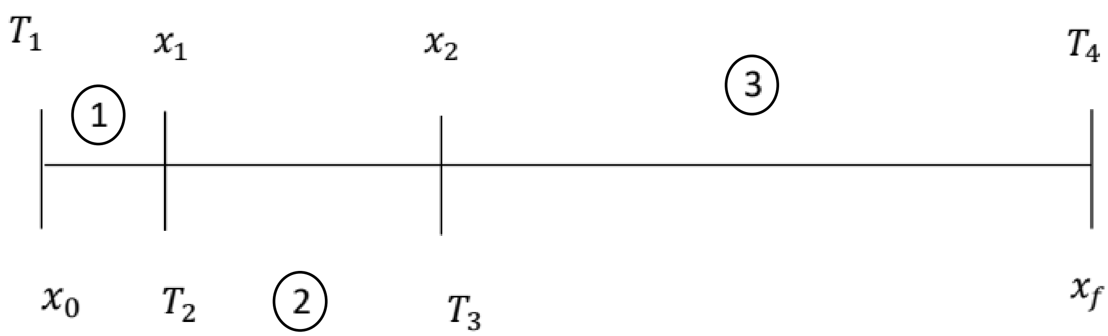
Método de los elementos finitos para el problema de transferencia de calor en una dimensión con funciones de forma lineales y con peso de Galerkin

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = -Q, \qquad k \wedge Q = kte$$

Mallado:

Asumimos que la respuesta está en los nodos:

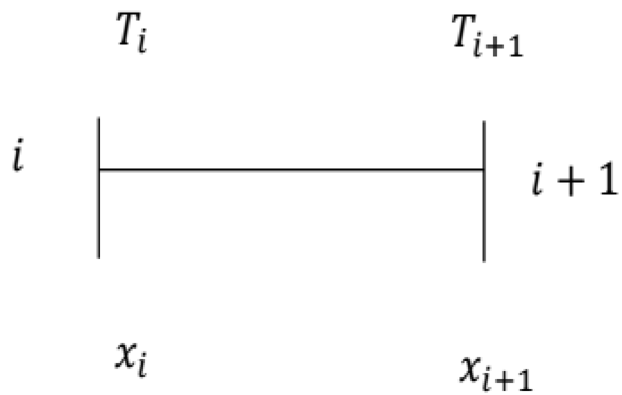


Interpolación:

Utilizaremos funciones de forma lineales para una dimensión.

$$\hat{T} \approx N_i T_i + N_{i+1} T_{i+1}$$

Para cada elemento i:



$$N_i = \frac{x_{i+1} - x}{x_{i+1} - x_i}$$

$$N_{i+1} = \frac{x - x_i}{x_{i+1} - x_i}$$

$$T = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$$\hat{T} \approx \mathbf{NT}, \qquad \mathbf{N}_{(x)}$$

Discretización:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = -Q$$

$$\frac{d}{dx}\left(k\frac{d\hat{T}}{dx}\right) \approx -Q \;\equiv\; \frac{d}{dx}\left(k\frac{d\mathbf{N}_{(x)}}{dx}\right)\mathbf{T} \approx -Q$$

Calculo Residual:

$$\frac{d}{dx}\left(k\frac{d\mathbf{N}_{(x)}}{dx}\right)\mathbf{T} + \mathrm{Q} = \xi$$

Método de los residuos ponderados:

$$\int_{\Omega} \xi_i w_i \, d\Omega = 0$$

$$\int_{\Omega} \boldsymbol{W} \left[\frac{d}{dx}\left(k\frac{d\mathbf{N}}{dx}\right)\mathbf{T} + \mathrm{Q}\right] d\Omega = 0$$

Realizando la derivada:

$$\frac{d}{dx}[N_i \quad N_{i+1}] = \left[\frac{d}{dx}N_i \quad \frac{d}{dx}N_{i+1}\right]$$

$$\frac{d}{dx}N_i = \frac{-1}{x_{i+1} - x_i}$$

$$\frac{d}{dx}N_{i+1} = \frac{1}{x_{i+1} - x_i}$$

$$\int_{\Omega} \boldsymbol{W} \left[\frac{d}{dx}\left[\frac{-k}{x_{i+1} - x_i} \quad \frac{k}{x_{i+1} - x_i}\right]\mathbf{T} + \mathrm{Q}\right] d\Omega = 0$$

$$\int_{\Omega} \boldsymbol{W} \left[\frac{d}{dx}\left[\frac{-k}{x_{i+1} - x_i} \quad \frac{k}{x_{i+1} - x_i}\right]\mathbf{T} + \mathrm{Q}\right] d\Omega = 0$$

$$\mathbf{W} = \begin{bmatrix} w_{x_i} \\ w_{x_i+1} \end{bmatrix}$$

$$\int_{\Omega} \begin{bmatrix} w_{x_i} \\ w_{x_i+1} \end{bmatrix} \left[\frac{d}{dx}\left[\frac{-k}{x_{i+1} - x_i} \quad \frac{k}{x_{i+1} - x_i}\right]\mathbf{T} + \mathrm{Q}\right] d\Omega = 0$$

Método de Galerkin:

$$W_i = N_i$$

Forma Fuerte:

$$\int_{x_i}^{x_i+1} \mathbf{N^T}\left[\frac{d}{dx}\left[\frac{-k}{x_{i+1}-x_i} \quad \frac{k}{x_{i+1}-x_i}\right]\mathbf{T}+\mathrm{Q}\right]dx = 0$$

$$\int_{x_i}^{x_i+1} \mathbf{N^T}\frac{d}{dx}\left(\frac{-k}{x_{i+1}-x_i} \quad \frac{k}{x_{i+1}-x_i}\right)\mathbf{T}\,dx + \mathrm{Q}\int_{x_i}^{x_i+1} \mathbf{N^T}\,dx = 0$$

Integración por partes:

$$\int_{x_i}^{x_i+1} \mathbf{N^T}\frac{d}{dx}\left(\frac{-k}{x_{i+1}-x_i} \quad \frac{k}{x_{i+1}-x_i}\right)\mathbf{T}\,dx + \mathrm{Q}$$

$$\int_{x_i}^{x_i+1} \mathbf{N^T}k\frac{d}{dx}\left(\frac{d}{dx}\boldsymbol{N}\right)\mathbf{T}\,dx + \mathrm{Q}$$

$$\int u dv = uv - v\int du$$

$$u = \mathbf{N^T}$$

$$du = \frac{d}{dx}\mathbf{N^T}dx$$

$$dv = k\frac{d}{dx}\left(\frac{d}{dx}\boldsymbol{N}\right)\mathbf{T}\mathrm{dx}$$

$$v = k\frac{d\mathbf{NT}}{dx}$$

$$\int \frac{d}{dx}f(x)\,dx = f(x)$$

$$\mathbf{N^T}k\frac{d\mathbf{NT}}{dx}\bigg|_{\mathbf{\Gamma}} - \int \frac{d}{dx}\mathbf{N^T}\,k\frac{d\mathbf{NT}}{dx}\,dx$$

Forma débil:

$$\mathbf{N^T}k\,\frac{d\mathbf{NT}}{dx}\bigg|_{\mathbf{\Gamma}} - k\int_{x_i}^{x_{i+1}} \frac{d}{dx}\mathbf{N^T}\ \frac{d\mathbf{N}}{dx}\mathbf{T}\,dx + Q\int_{x_i}^{x_i+1} \mathbf{N^T}\,dx = 0$$

Realizando la derivada:

$$\frac{d}{dx}\mathbf{N^T} = \frac{d}{dx}\begin{bmatrix} N_i \\ N_{i+1} \end{bmatrix} = \begin{bmatrix} \dfrac{dN_i}{dx} \\ \dfrac{dN_{i+1}}{dx} \end{bmatrix} = \frac{1}{x_{i+1} - x_i}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\frac{d}{dx}\mathbf{N} = \frac{1}{x_{i+1} - x_i}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\frac{d}{dx}\mathbf{N^T}\frac{d}{dx}\mathbf{N} = \frac{1}{(x_{i+1} - x_i)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k\int_{x_i}^{x_{i+1}} \frac{1}{(x_{i+1} - x_i)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ \mathbf{T}\mathrm{dx}$$

$$= \frac{k}{(x_{i+1} - x_i)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(x_{i+1} - x_i)\ \mathbf{T} = \frac{k}{(x_{i+1} - x_i)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix}$$

Integrando:

$$Q\int_{x_i}^{x_i+1} \mathbf{N^T}\,dx = \frac{Q}{(x_{i+1} - x_i)}\int_{x_i}^{x_i+1} \begin{bmatrix} x_{i+1} - x \\ x - x_i \end{bmatrix} dx = \frac{Q}{(x_{i+1} - x_i)}\begin{bmatrix} \displaystyle\int_{x_i}^{x_{i+1}} x_{i+1} - x\ dx \\ \displaystyle\int_{x_i}^{x_{i+1}} x - x_i\ dx \end{bmatrix}$$

$$\frac{Q}{(x_{i+1}-x_i)}\begin{bmatrix}\int_{x_i}^{x_{i+1}} x_{i+1}-x\,dx \\ \int_{x_i}^{x_{i+1}} x-x_i\,dx\end{bmatrix} = \frac{Q}{(x_{i+1}-x_i)}\begin{bmatrix}\frac{(x_{i+1}-x_i)^2}{2} \\ \frac{(x_{i+1}-x_i)^2}{2}\end{bmatrix} = \frac{Q(x_{i+1}-x_i)}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

Sistema de Ecuaciones Lineales:

$$\frac{k}{(x_{i+1}-x_i)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_i\\ T_{i+1}\end{bmatrix} = \frac{Q(x_{i+1}-x_i)}{2}\begin{bmatrix}1\\1\end{bmatrix} + \mathbf{N^T}k\frac{d\mathbf{NT}}{dx}\bigg|_{\mathbf{\Gamma}}$$

$$\boldsymbol{k_{2x2}T_{2x1} = b_{2x1}}$$

Para cada elemento i:

$$\boldsymbol{k} = \begin{bmatrix}k_{\boldsymbol{i1}} & k_{\boldsymbol{i2}}\\ k_{\boldsymbol{i3}} & k_{\boldsymbol{i4}}\end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix}b_{i1}\\ b_{i2}\end{bmatrix}$$

Tabla de conectividades:

Elemento	$\boldsymbol{i}$	$\boldsymbol{i+1}$
1	1	2
2	2	3
3	3	4

Ensamblaje:

$$\begin{bmatrix}k_{11} & k_{12} & 0 & 0\\ k_{13} & k_{14}+k_{21} & k_{22} & 0\\ 0 & k_{23} & k_{24}+k_{31} & k_{32}\\ 0 & 0 & k_{33} & k_{34}\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\\T_4\end{bmatrix} = \begin{bmatrix}b_{11}\\ b_{12}+b_{21}\\ b_{22}+b_{31}\\ b_{32}\end{bmatrix}$$

$$\boldsymbol{KT = B}$$

Condiciones de contorno

Condiciones de Neumann:

$$\frac{dT}{dx} = T_0 \text{ en } \Gamma_N, \Gamma_N \subseteq \Gamma$$

$$\mathbf{N^T} k \frac{d\mathbf{NT}}{dx}\bigg|_{\mathbf{\Gamma}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{13} & k_{14} + k_{21} & k_{22} & 0 \\ 0 & k_{23} & k_{24} + k_{31} & k_{32} \\ 0 & 0 & k_{33} & k_{34} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12} + b_{21} \\ b_{22} + b_{31} \\ b_{32} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

Condiciones de Dirichlet:

$$T = T_0 \text{ en } \Gamma_D, \Gamma_D \subseteq \Gamma$$

$$\begin{bmatrix} 10 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}$$