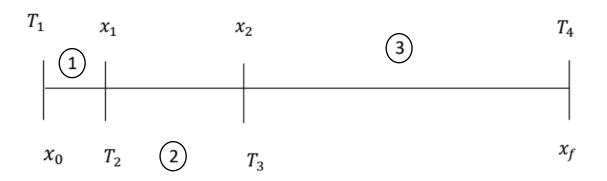
Método de los elementos finitos para el problema de transferencia de calor en una dimensión con funciones de forma lineales y con peso de Galerkin



$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = -Q, \qquad k \wedge Q = kte$$

Mallado:

Asumimos que la respuesta está en los nodos:

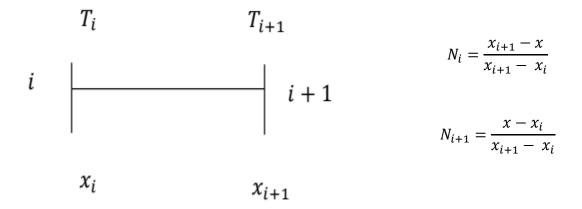


Interpolación:

Utilizaremos funciones de forma lineales para una dimensión.

$$\hat{T} \approx N_i T_i + N_{i+1} T_{i+1}$$

Para cada elemento i:



$$T = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$$\widehat{T} \approx \mathbf{NT}, \qquad \mathbf{N}_{(x)}$$

Discretización:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = -Q$$

$$\frac{d}{dx} \left(k \frac{d\hat{T}}{dx} \right) \approx -Q \equiv \frac{d}{dx} \left(k \frac{d\mathbf{N}_{(x)}}{dx} \right) \mathbf{T} \approx -Q$$

Calculo Residual:

$$\frac{d}{dx}\left(k\frac{d\mathbf{N}_{(x)}}{dx}\right)\mathbf{T} + \mathbf{Q} = \xi$$

Método de los residuos ponderados:

$$\int\limits_{\Omega} \xi_i w_i \, d\Omega = 0$$

$$\int_{\Omega} \mathbf{W} \left[\frac{d}{dx} \left(k \frac{d\mathbf{N}}{dx} \right) \mathbf{T} + \mathbf{Q} \right] d\Omega = 0$$

Realizando la derivada:

$$\frac{d}{dx}[N_i \quad N_{i+1}] = \begin{bmatrix} \frac{d}{dx}N_i & \frac{d}{dx}N_{i+1} \end{bmatrix}$$

$$\frac{d}{dx}N_i = \frac{-1}{x_{i+1} - x_i}$$

$$\frac{d}{dx}N_{i+1} = \frac{1}{x_{i+1} - x_i}$$

$$\int_{\Omega} \mathbf{W} \left[\frac{d}{dx} \left[\frac{-k}{x_{i+1} - x_i} \right] \frac{k}{x_{i+1} - x_i} \right] \mathbf{T} + \mathbf{Q} d\Omega = 0$$

$$\int_{\Omega} \mathbf{W} \left[\frac{d}{dx} \left[\frac{-k}{x_{i+1} - x_i} \right] \frac{k}{x_{i+1} - x_i} \right] \mathbf{T} + \mathbf{Q} \right] d\Omega = 0$$

$$\mathbf{W} = \begin{bmatrix} w_{x_i} \\ w_{x_i+1} \end{bmatrix}$$

$$\int_{\Omega} \begin{bmatrix} w_{x_i} \\ w_{x_i+1} \end{bmatrix} \left[\frac{d}{dx} \left[\frac{-k}{x_{i+1} - x_i} \right] \frac{k}{x_{i+1} - x_i} \right] \mathbf{T} + \mathbf{Q} d\Omega = 0$$

Método de Galerkin:

$$W_i = N_i$$

Forma Fuerte:

$$\int_{x_i}^{x_i+1} \mathbf{N}^{\mathsf{T}} \left[\frac{d}{dx} \left[\frac{-k}{x_{i+1} - x_i} \quad \frac{k}{x_{i+1} - x_i} \right] \mathbf{T} + \mathbf{Q} \right] dx = 0$$

$$\int_{x_i}^{x_i+1} \mathbf{N}^{\mathsf{T}} \frac{d}{dx} \left(\frac{-k}{x_{i+1} - x_i} \quad \frac{k}{x_{i+1} - x_i} \right) \mathbf{T} \, dx + Q \int_{x_i}^{x_i+1} \mathbf{N}^{\mathsf{T}} \, dx = 0$$

Integración por partes:

$$\int_{x_i}^{x_i+1} \mathbf{N}^{\mathsf{T}} \frac{d}{dx} \left(\frac{-k}{x_{i+1} - x_i} \quad \frac{k}{x_{i+1} - x_i} \right) \mathbf{T} \, dx$$

$$\int_{x_i}^{x_i+1} \mathbf{N}^{\mathsf{T}} k \, \frac{d}{dx} \left(\frac{d}{dx} \, \mathbf{N} \right) \mathbf{T} \, dx$$

$$\int u dv = uv - v \int du$$

$$u = \mathbf{N}^{\mathsf{T}}$$

$$dv = k \frac{d}{dx} \left(\frac{d}{dx} \mathbf{N}\right) \mathbf{T} dx$$

$$du = \frac{d}{dx} \mathbf{N}^{\mathsf{T}} dx$$

$$v = k \frac{d\mathbf{N} \mathbf{T}}{dx}$$

$$\int \frac{d}{dx} f(x) \, dx = f(x)$$

$$\mathbf{N}^{\mathsf{T}} k \frac{d\mathbf{N}^{\mathsf{T}}}{dx} \Big|_{\mathsf{\Gamma}} - \int \frac{d}{dx} \mathbf{N}^{\mathsf{T}} k \frac{d\mathbf{N}^{\mathsf{T}}}{dx} dx$$

Forma débil:

$$\mathbf{N}^{\mathsf{T}} k \frac{d\mathbf{N}^{\mathsf{T}}}{dx} \Big|_{\Gamma} - k \int_{x_i}^{x_{i+1}} \frac{d}{dx} \mathbf{N}^{\mathsf{T}} \frac{d\mathbf{N}}{dx} \mathbf{T} dx + Q \int_{x_i}^{x_{i+1}} \mathbf{N}^{\mathsf{T}} dx = 0$$

Realizando la derivada:

$$\frac{d}{dx}\mathbf{N}^{\mathsf{T}} = \frac{d}{dx} \begin{bmatrix} N_i \\ N_{i+1} \end{bmatrix} = \begin{bmatrix} \frac{dN_i}{dx} \\ \frac{dN_{i+1}}{dx} \end{bmatrix} = \frac{1}{x_{i+1} - x_i} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\frac{d}{dx}\mathbf{N} = \frac{1}{x_{i+1} - x_i} [-1 \quad 1]$$

$$\frac{d}{dx}\mathbf{N}^{\mathsf{T}}\frac{d}{dx}\mathbf{N} = \frac{1}{(x_{i+1} - x_i)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k \int_{x_i}^{x_{i+1}} \frac{1}{(x_{i+1} - x_i)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{T} dx$$

$$= \frac{k}{(x_{i+1} - x_i)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (x_{i+1} - x_i) \mathbf{T} = \frac{k}{(x_{i+1} - x_i)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix}$$

Integrando:

$$Q \int_{x_{i}}^{x_{i+1}} \mathbf{N}^{\mathsf{T}} dx = \frac{Q}{(x_{i+1} - x_{i})} \int_{x_{i}}^{x_{i+1}} \begin{bmatrix} x_{i+1} - x \\ x - x_{i} \end{bmatrix} dx = \frac{Q}{(x_{i+1} - x_{i})} \begin{bmatrix} \int_{x_{i}}^{x_{i+1}} x_{i+1} - x dx \\ \int_{x_{i}}^{x_{i+1}} x_{i+1} - x dx \end{bmatrix}$$

$$\frac{Q}{(x_{i+1} - x_i)} \begin{bmatrix} \int\limits_{x_i}^{x_{i+1}} x_{i+1} - x \, dx \\ \int\limits_{x_i}^{x_{i+1}} x_{i+1} \\ \int\limits_{x_i}^{x_i} x_{i} - x_i \, dx \end{bmatrix} = \frac{Q}{(x_{i+1} - x_i)} \left[\frac{(x_{i+1} - x_i)^2}{2} \right] = \frac{Q(x_{i+1} - x_i)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Sistema de Ecuaciones Lineales:

$$\frac{k}{(x_{i+1} - x_i)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix} = \frac{Q(x_{i+1} - x_i)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \mathbf{N}^{\mathsf{T}} k \frac{d\mathbf{N} \mathbf{T}}{dx} \Big|_{\Gamma}$$

$$k_{2x2}T_{2x1} = b_{2x1}$$

Para cada elemento i:

$$\mathbf{k} = \begin{bmatrix} k_{i1} & k_{i2} \\ k_{i3} & k_{i4} \end{bmatrix}$$
$$\mathbf{b} = \begin{bmatrix} b_{i1} \\ b_{i2} \end{bmatrix}$$

Tabla de conectividades:

Elemento	i	i+1
1	1	2
2	2	3
3	3	4

Ensamblaje:

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{13} & k_{14} + k_{21} & k_{22} & 0 \\ 0 & k_{23} & k_{24} + k_{31} & k_{32} \\ 0 & 0 & k_{33} & k_{34} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12} + b_{21} \\ b_{22} + b_{31} \\ b_{32} \end{bmatrix}$$

$$KT = B$$

Condiciones de contorno

Condiciones de Neumann:

$$\frac{dT}{dx} = T_0 \text{ en } \Gamma_N, \Gamma_N \subseteq \Gamma$$

$$\mathbf{N}^{\mathsf{T}}k\frac{d\mathbf{N}\mathsf{T}}{dx}\Big|_{\Gamma} = \begin{bmatrix} 0\\0\\0\\3 \end{bmatrix}$$

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{13} & k_{14} + k_{21} & k_{22} & 0 \\ 0 & k_{23} & k_{24} + k_{31} & k_{32} \\ 0 & 0 & k_{33} & k_{34} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12} + b_{21} \\ b_{22} + b_{31} \\ b_{32} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

Condiciones de Dirichlet:

$$T = T_0 \text{ en } \Gamma_D, \Gamma_D \subseteq \Gamma$$

$$\begin{bmatrix} 10 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}$$