5.1 Least Squares Global Localization

Least Squares Positioning is a well-known algorithm for estimating the robot localization x given a set of known landmarks in a map. Least Squares is akin to find the best pose $\hat{x}$ by solving a system of equations of the form:

$$z_{m\times1} = H_{m\times n} \cdot x_{n\times1}$$

where:

- n is the length of the pose ($n = 3$ in our case, position plus orientation),
- m represents the number of observations, and
- H is the matrix that codifies the observation model relating the measurement z with the robot pose x.

This simple concept, nevertheless, has to be modified in order to be used in real scenarios:

5.1.1 Pseudo-inverse

Generally, to solve an equation system, we only need as many equations as variables. In the robot localization problem, each observation z sets an equation, while the variables are the components of the state/pose, x.

In such a case, where $n = m$, a direct attempt to this problem exists:

$$x = H^{-1}z$$

So a unique solution exists if H is invertible, that is, H is a square matrix with $det(H) \neq 0$.

However, in real scenarios typically there are available more observations than variables. An approach to address this could be to drop some of the additional equations, but given that observations z are inaccurate (they have been affected by some noise), we may use the additional information to try to mitigate such noise. However, by doing that H is no a squared matrix anymore, hence not being invertible.

Two tools can help us at this point. The first one is the utilization of **Least Squares** to find the closest possible $\hat{x}$, i.e. the one where the the error ($e = Hx - z$) is minimal:

$$\hat{x} = \arg\min_x e^T e = [(z - Hx)^T(z - Hx)] = \arg\min_x ||z - Hx||^2$$

which has a close form solution using the **pseudo-inverse** of a matrix:

$$\hat{x} = \underbrace{(H^T H)^{-1} H^T}_{\textit{pseudo-inverse } (H^+)} z$$

The **pseudo-inverse**, in contrast to the normal inverse operation, can be used in non-square matrices!

In [1]:

```
#%matplotlib widget
#%matplotlib inline

# IMPORTS

import math

import numpy as np
from numpy import linalg
import matplotlib
matplotlib.use('TkAgg')
import matplotlib.pyplot as plt
import scipy
from scipy import stats

import sys
sys.path.append("..")
from utils.PlotEllipse import PlotEllipse
from utils.DrawRobot import DrawRobot
from utils.tcomp import tcomp
from utils.tinv import tinv, jac_tinv1 as jac_tinv
from utils.Jacobians import J1, J2
```

ASSIGNMENT 1: Playing with a robot in a corridor

The following code illustrates a simple scenario where a robot is in a corridor looking at a door, which is placed at the origin of the reference system (see Fig.1). The robot is equipped with a laser scanner able to measure distances, and takes a number of observations z. The robot is placed 3 meters away from the door, but this information is unknown for it. **Your goal is** to estimate the position of the robot in this 1D world using such measurements.

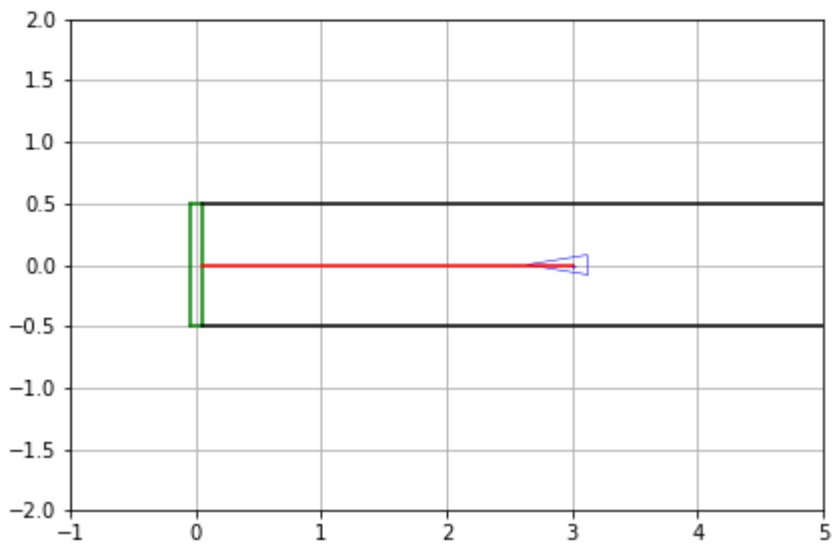


Fig. 1: Simple 1D scenario with a robot equipped with a laser scanner measuring distance to a door.

The following code cell shows the dimensions of all the actors involved in LS-positioning. Complete it for computing the robot pose x from the available information. *Recall* [`np.linalg.inv()` (https://numpy.org/doc/stable/reference/generated/numpy.linalg.inv.html)](https://numpy.org/doc/stable/reference/generated/numpy.linalg.inv.html).

In [2]:
```python
# Set the robot pose to unknown
x = np.vstack(np.array([None]))

# Sensor measurements to the door
z = np.vstack(np.array([3.7,2.9,3.6,2.5,3.5]))

# Observation model
H = np.ones(np.array([5,1]))

print ("Dimensions:")
print ("Pose x:           " + str(x.shape))
print ("Observations z: " + str(z.shape))
print ("Obs. model H:    " + str(H.shape))
print ("H.T@H:            " + str((H.T@H).shape))
print ("inv(H.T@H):       " + str((np.linalg.inv(H.T@H)).shape))
print ("H.T@z :           " + str((H.T@z).shape))

# Do Least Squares Positioning
x = np.linalg.inv(H.T @ H) @ H.T @ z

print('\nLS-Positioning')
print('x = ' + str(x[0]))
```

```
Dimensions:
Pose x:          (1, 1)
Observations z: (5, 1)
Obs. model H:    (5, 1)
H.T@H:           (1, 1)
inv(H.T@H):      (1, 1)
H.T@z :          (1, 1)

LS-Positioning
x = [3.24]
```

Expected output

```
x = [3.24]
```

5.1.2 Weighted measurements

In cases where multiple sensors affected by different noise profiles are used, or in those where the robot is using a sensor with a varying error (*e.g.* typically radial laser scans are more accurate while measuring distances to close objects), it is interesting to weight the contribution of such measurements while retrieving the robot pose. For example, we are going to consider a sensor whose accuracy drops the further the observed landmark is. Given a *covariance* matrix Q describing the error in the measurements, the equations above are rewritten as:

$$\hat{x} = \arg\min_x e^T Q^{-1} e = [(Hx - z)^T Q^{-1} (Hx - z)]$$

$$\hat{x} \leftarrow (H^T Q^{-1} H)^{-1} H^T Q^{-1} z \qquad \text{(1. Best estimation)}$$

$$\Sigma_{\hat{x}} \leftarrow (H^T Q^{-1} H)^{-1} \qquad \text{(2. Uncertainty of the estimation)}$$

Example with three measurements having different uncertainty ($\sigma_1^2, \sigma_2^2, \sigma_3^2$):

$$e^T Q^{-1} e = [e_1\ e_2\ e_3] \begin{bmatrix} 1/\sigma_1^2 & 0 & 0 \\ 0 & 1/\sigma_2^2 & 0 \\ 0 & 0 & 1/\sigma_3^2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{e_1^2}{\sigma_1^2} + \frac{e_2^2}{\sigma_2^2} + \frac{e_3^2}{\sigma_3^2} = \sum_{i=1}^{m} \frac{e_i^2}{\sigma_i^2}$$

ASSIGNMENT 2: Adding growing uncertainty

We have new information! The manufacturer of the laser scanner mounted on the robot wrote an email telling us that the device is considerably more inaccurate for further distances. Concretely, such uncertainty is characterized by $\sigma^2 = e^z$ (the laser is not so accurate, being polite).

With this new information, implement the computation of the weighted LS-positioning so you can compare the previously estimated position with the new one.

In [3]:
```
# Sensor measurements to the door
z = np.vstack(np.array([3.7,2.9,3.6,2.5,3.5]))

# Uncertainty of the measurements
Q = np.eye(5)*np.exp(z)

# Observation model
H = np.ones(np.array([5,1]))

# Do Least Squares Positioning
x = np.linalg.inv(H.T @ H) @ H.T @ z

# Do Weighted Least Squares Positioning
x_w = np.linalg.inv(H.T @ np.linalg.inv(Q) @ H) @ H.T @ np.linalg.in

print('\nLS-Positioning')
print('x = ' + str(x[0]))

print('\nWeighted-LS-Positioning')
print('x = ' + str(np.round(x_w[0],2)))
```

```
LS-Positioning
x = [3.24]

Weighted-LS-Positioning
x = [3.01]
```

Expected output

```
LS-Positioning
x = [3.24]

Weighted-LS-Positioning
x = [3.01]
```

5.1.3 Non-linear Least Squares

Until now we have assumed that $\hat{x}$ can be solved as a simple system of equations, i.e. H is a matrix. Nevertheless, typically observation models are non-linear, that is: $z = h(x)$, so the problem now becomes:

$$\hat{x} = \arg\min_{x} ||z - h(x)||^2$$

No close-form solutions exists for this new problem, but we can approximate it iteratively:

$$\textit{(Recall) Taylor expansion: } h(x) = h(x_0 + \delta) = h(x_0) + J_{h_0}\delta$$

$$||z - h(x)||^2 \cong ||\underbrace{z - h(x_0)}_{\textit{error vector } e} - J_{h_o}\delta||^2 = ||e - J_{h_0}\delta||^2 \leftarrow \delta \textit{ is unknown, } J_e = -J_{h_0}$$

So we can define the equivalent optimization problem:

$$\delta = \arg\min_{\delta} ||e + J_e\delta||^2 \rightarrow \underbrace{\delta}_{nx1} =$$

$$-\underbrace{(J_e^T J_e)^{-1}}_{nxn} \underbrace{J_e^T}_{nxm} \underbrace{e}_{mx1} \textit{ (}\delta \textit{ that makes the previous squared norm minimum)}$$

The weighted form of the δ computation results:

$$\delta = (J_e^T Q^{-1} J_e)^{-1} J_e^T Q^{-1} e$$

Where:

- Q is the measurement covariance (*weighted measurement*)
- J_e is the negative of the Jacobian of the observation model at $\hat{x}$, also known as $\nabla h_{\hat{x}}$
- e is the error of z against $h(\hat{x})$ (computed using the map information).

As commented, there is no closed-form solution for the problem, but we can iteratively approximate it using the **Gauss-Newton algorithm**:

$$\begin{aligned}
&\hat{x} \leftarrow (\ldots) && \text{(1. Initial guess)} \\
&\delta \leftarrow (J_e^T Q^{-1} J_e)^{-1} J_e^T Q^{-1} e && \text{(2. Evaluate delta/increment)} \\
&\hat{x} \leftarrow \hat{x} - \delta && \text{(3. Update estimation)} \\
&\textit{if } \delta > \textit{tolerance} \rightarrow \textit{goto (1.)} && \\
&\textit{else } \rightarrow \textit{return } \hat{x} && \text{(4. Exit condition)}
\end{aligned}$$

LS positioning in practice

Suppose that a mobile robot equipped with a range sensor aims to localize itself in a map consisting of a number of landmarks by means of Least Squares and Gauss-Newton optimization.

For that, **you are provided with** the class `Robot` that implements the behavior of a robot that thinks that is placed at `pose` (that's its initial guess, obtained by composing odometry commands), but that has a real position `true pose`. In addition, the variable `cov` models the uncertainty of its movement, and `var_d` represents the variance (noise) of the range measurements. Take a look at it below.

In [4]:
```
class Robot(object):
    """ Simulate a robot base and positioning.

        Attrs:
            pose: Position given by odometry (in this case true_pose
            true_pose: True position, selected by the mouse in this
            cov: Covariance for the odometry sensor. Used to add noi
            var_d: Covariance (noise) of each range measurement
    """
    def __init__(self,
                 pose: np.ndarray,
                 cov: np.ndarray,
                 desv_d: int = 0):
        # Pose related
        self.true_pose = pose
        self.pose = pose + np.sqrt(cov)@np.random.randn(3, 1)
        self.cov = cov

        # Sensor related
        self.var_d = desv_d**2

    def plot(self, fig, ax, **kwargs):
        DrawRobot(fig, ax, self.pose, color='red', label="Pose estim
        DrawRobot(fig, ax, self.true_pose, color="blue", label="Real
```

ASSIGNMENT 3a: Computing distances from the robot to the landmarks

Implement the following function to simulate how our robot observes the world. In this case, the landmarks in the map act as beacons: the robot can sense how far away they are without any information about angles. The robot uses a range sensor with the following observation model:

$$z_i = [d_i] = h(m_i, x) = \left[\sqrt{(x_i - x)^2 + (y_i - y)^2}\right] + w_i$$

where m_i stands for the i^{th} landmark, and w_i is a noise added by the sensor.

Consider two scenarios in the function implementation:

- The measurment is carried out with an ideal sensor, so no noise nor uncertainty exists (`cov_d = 0`).

- The measurement comes from a real sensor affected by a given noise (`cov_d != 0`). We are going to consider that the range sensor is more accurate measuring distances to close landmarks than to far away ones. To implement this, consider that the noise grows with the root of the distance to the landmark, so the resultant uncertainty can be retrieved by:

$$\sigma_{\text{dist}} = \sigma\sqrt{z}$$

that is, `np.sqrt(z)*np.sqrt(cov_d)` . Recall that the sensor noise is modeled as a gaussian distribution, so you have to define such distribution and take samples from it using the [`stats.norm()` (https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.html)](https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.html) and [`rvs()` (https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.rv_continuous.rvs.html)](https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.rv_continuous.rvs.html) functions.

In [5]:
```
def distance(pose: np.ndarray, m: np.ndarray, cov_d: int = 0) -> np.
    """ Get observations for every landmark in the map.

    In this case our observations are range only.
    If cov_d > 0 then add gaussian noise with var_d covariance

    Args:
        pose: pose (true or noisy) of the robot taking observation
        m: Map containing all landmarks
        cov_d: Covariance of the sensor

    Returns
        z: numpy array containing distances to all obs. It has shape
    """
    z = np.sqrt(np.power(m[0]-pose[0],2)+np.power(m[1]-pose[1],2)) #

    if cov_d > 0:
        z += stats.norm(loc=0, scale=np.sqrt(z)*np.sqrt(cov_d)).rvs(

    return z
```

Try your brand new function with the following code:

In [6]:
```
pose = np.vstack([2, 2, 0.35])
m = np.array([[-5,-15],[20,56],[54,-18]]).T
cov_d = 0

# Compute distances from the sensor to the landmarks
z = distance(pose,m,cov_d)

# Now consider a noisy sensor
cov_d = 0.5
np.random.seed(seed=0)
z_with_noise = distance(pose,m,cov_d)

# Show the results
print('Measurements without noise:' + str(z))
print('Measurements with noise:   ' + str(z_with_noise))
```

```
Measurements without noise:[18.38477631 56.92099788 55.71355311]
Measurements with noise:   [23.73319805 59.05577186 60.87928514]
```

Expected output

```
Measurements without noise:[18.38477631 56.92099788 55.71355
311]
Measurements with noise:   [23.73319805 59.05577186 60.87928
514]
```

ASSIGNMENT 3b: Implementing the algorithm

Finally, we get to implement the Least Squares algorithm for localization. We ask you to complete the gaps in the following function, which:

- Starts by initializing the Jacobian of the observation function (`Jh`) and takes as initial guess (`xEst`) the position at which the robot thinks it is as given by its odometry (`R1.pose`).
- Then, it enters into a loop until convergence is reached, where:
  1. The distances `zEst` to each landmark from the estimated position `xEst` are computed. Recall that the map (landmarks positions) are known (`w_map`).
  - The error is computed by substracting to the obsevations provided by the sensor `z` the distances `zEst` computed at the previous point. Then, the residual error is computed as $e_{residual} = \sqrt{e_x^2 + e_y^2}$.
  - The Jacobian of the observation model is evaluated at the estimated robot pose (`xEst`). This Jacobian has two columns and as many rows as observations to the landmarks:

$$jH = \begin{bmatrix} \frac{-1}{d_1}(x_1 - x) & \frac{-1}{d_1}(y_1 - y) \\ \frac{-1}{d_2}(x_2 - x) & \frac{-1}{d_2}(y_2 - y) \\ \cdots & \cdots \\ \frac{-1}{d_n}(x_n - x) & \frac{-1}{d_n}(y_n - y) \end{bmatrix}$$

    being $xEst = [x, y]$, $[x_i, y_i]$ the position of the i^{th} landmark in the map, and d the distance previously computed from the robot estimated pose $xEst$ to the landmarks. The jacobian of the error `jE` is just `-jH` .
  - Computes the increment δ (`incr`) and substract it to the estimated pose (`xEst`). *Note: recall that* $\delta = (J_e^T Q^{-1} J_e)^{-1} J_e^T Q^{-1} e$

In [7]:
```python
def LeastSquaresLocalization(R1: Robot,
                             w_map: np.ndarray,
                             z: np.ndarray,
                             nIterations=10,
                             tolerance=0.001,
                             delay=0.5) -> np.ndarray:
    """ Pose estimation using Gauss-Newton for least squares optimiz

        Args:
            R1: Robot which pose we must estimate
            w_map: Map of the environment
            z: Observation received from sensor

            nIterations: sets the maximum number of iterations (defa
            tolerance: Minimum error difference needed for stopping

            delay: Wait time used to visualize the different iterati

        Returns:
            xEst: Estimated pose

    """

    iteration = 0

    # Initialization of useful variables
    incr = np.ones((2, 1)) # Delta
    jH = np.zeros((w_map.shape[1], 2)) # Jacobian of the observation
    xEst = R1.pose #Initial estimation is the odometry position (usu

    # Let's go!
    while linalg.norm(incr) > tolerance and iteration < nIterations:
        #if plotting:
        plt.plot(xEst[0], xEst[1], '+r', markersize=1+math.floor((it
        # Compute the predicted observation (from xEst) and their re

        # 1) TODO: Compute distance to each landmark from xEst (esti
        #
        zEst = distance(xEst, w_map)

        # 2) TODO: error = difference between real observations and
        e = z - zEst
        residual = np.sqrt(e.T@e) #residual error = sqrt(x^2+y^2)

        # 3) TODO: Compute Jacobians with respect (x,y) (slide 13)
        # The jH is evaluated at our current guest (xEst) -> z_p

        jH = np.array([
            (-1/zEst[i]) *
            np.hstack([
                w_map[0,i]-xEst[0,0],
                w_map[1,i]-xEst[1,0]
            ])
            for i in range(w_map.shape[1])
        ])
        jE = -jH

        # The observation variances Q grow with the root of the dist
        Q = np.diag(R1.var_d*np.sqrt(z))
```

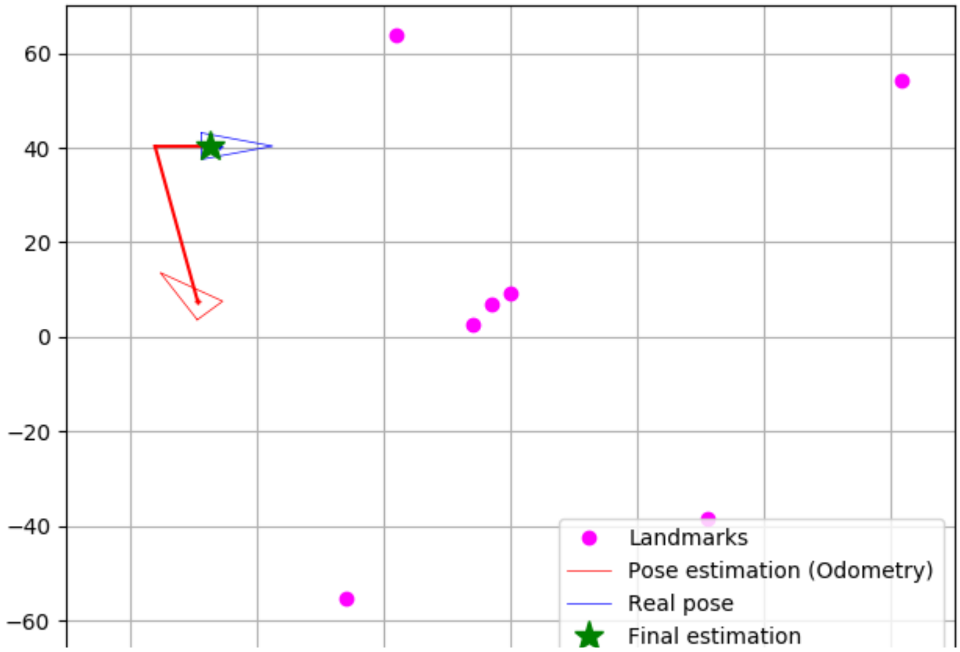
```
        # 4) TODO: Solve the equation --> compute incr
        invQ = np.linalg.inv(Q)
        incr = np.linalg.inv(jE.T @ invQ @ jE) @ jE.T @ invQ @ e

        plt.plot([xEst[0, 0], xEst[0, 0]-incr[0]], [xEst[1, 0], xEst
        xEst[0:2, 0] -= incr

        print ("Iteration :" + str(iteration))
        print ("  delta :    " + str(incr))
        print ("  residual: " + str(residual))

        iteration += 1

        plt.pause(delay)

    plt.plot(xEst[0, 0], xEst[1, 0], '*g', markersize=14, label="Fin

    return xEst
```

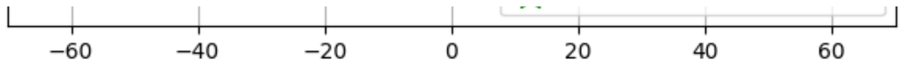
The next cell code launches our algorithm, so **we can try it!**. This is done according to the following steps:

1. The map `w_map` is built. In this case, the map consists of a number of landmarks (`nLandmarks`).
2. The program asks the user to set the true position of the robot (`xTrue`) by clicking with the mouse in the map.
3. A new pose is generated from it, `xOdom`, which represents the pose that the robot thinks it is in. This simulates a motion command from an arbitrary pose that ends up with the robot in `xTrue`, but it thinks that it is in `xOdom`.
4. Then the robot takes a (noisy) range measurement to each landmark in the map.
5. Finally, the robot employs a Least Squares definition of the problem and Gauss-Newton to iteratively optimize such a guess (`xOdom`), obtaining a new (and hopefully better) estimation of its pose `xEst`.

Example

The figure below shows an example of execution of this code (once completed).




−60
−40
−20
0
20
40
60

In [8]:
```python
def main(nLandmarks=7, env_size=140):
    # MATPLOTLIB
    fig, ax = plt.subplots()
    plt.xlim([-90, 90])
    plt.ylim([-90, 90])
    plt.grid()
    plt.ion()
    plt.tight_layout()

    fig.canvas.draw()

    # VARIABLES
    num_landmarks = 7 # number of landmarks in the environment
    env_size = 140 # A square environment with x=[-env_size/2,env_si

    # MAP CREATION AND VISUALIZATION
    w_map = env_size*np.random.rand(2, num_landmarks) - env_size/2 #
    ax.plot(w_map[0, :], w_map[1, :], 'o', color='magenta', label="L

    # ROBOT POSE AND SENSOR INITIALIZATION
    desv_d = 0.5 # standard deviation (noise) of the range measureme
    cov = np.diag([25, 30, np.pi*180])**2 # covariance of the motion
    xStart = np.vstack(plt.ginput(1)).T # get the robot starting poi
    robot_pose=np.vstack([xStart, 0]) # robot_pose

    R1 = Robot(robot_pose, cov, desv_d)
    R1.plot(fig, ax)

    # MAIN
    z = distance(R1.true_pose, w_map, cov_d=R1.var_d) # take (noisy)
    LeastSquaresLocalization(R1, w_map, z) # LS Positioning!

    # PLOTTING RESULTS
    plt.legend()
    fig.canvas.draw()

# RUN
main()
```

```
Iteration :0
  delta :   [62.76198308  2.01922692]
  residual: 97.99999116491452
Iteration :1
  delta :   [-5.06702922 15.08434138]
  residual: 37.57813235267521
Iteration :2
  delta :   [-1.02135505 -0.43253851]
  residual: 18.917617807277903
Iteration :3
  delta :   [-0.02513509 -0.04401566]
  residual: 18.681070995039338
Iteration :4
  delta :   [-0.00126247 -0.00286635]
  residual: 18.681657414395552
Iteration :5
  delta :   [-7.59565833e-05 -1.76843510e-04]
  residual: 18.681764931834394
```

Thinking about it (1)

Having completed this notebook above, you will be able to **answer the following questions**:

- What are the dimensions of the error residuals? Does them depend on the number of observations?

  El error residual es de dimension 1x1, es decir un escalar. No depende del numero de observaciones

- Why is Weighted LS obtaining better results than LS?

  Porque hay que tener en cuenta la precision de los sensores, teniendo mas peso los que son mas precisos

- Which is the minimum number of landmarks needed for localizing the robot? Why?

  Necesitamos como minimo tantas landmarks como elementos haya en nuestra pose, en este caso 3, $[x, y, \theta]$, porque si tuevieramos menos observaciones que elementos, $H^T H$ no seria invertible y no podriamos calcular el error minimo cuadratico

- Play with different “qualities” of the range sensor. Could you find a value for its variance so the LS method fails?

  Para valores altos de la varianza como 2 o 3, la estimacion es muy mala. Y para valores altos donde std > 8, el metodo falla

- Play also with different values for the odometry uncertainty. What does this affect?

  Esto afecta a la estimacion de la pose inicial realizada por la odometria, pero no afecta en gran medida a la estimacion final realizada por el metodo de minimos cuadrados