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Introduction to Logic

Second Edition

Michael Genesereth
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SYNTHESIS LECTURES ON COMPUTER SCIENCE

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Introduction to Logic

Second Edition

Michael Genesereth and Eric Kao
Stanford University

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ABSTRACT

This book is a gentle but rigorous introduction to Formal Logic. It is intended primarily for use at the college level. However, it can also be used for advanced secondary school students, and it can be used at the start of graduate school for those who have not yet seen the material.

The approach to teaching logic used here emerged from more than 20 years of teaching logic to students at Stanford University and from teaching logic to tens of thousands of others via online courses on the World Wide Web. The approach differs from that taken by other books in logic in two essential ways, one having to do with content, the other with form.

Like many other books on logic, this one covers logical syntax and semantics and proof theory plus induction. However, unlike other books, this book begins with Herbrand semantics rather than the more traditional Tarskian semantics. This approach makes the material considerably easier for students to understand and leaves them with a deeper understanding of what logic is all about.

In addition to this text, there are online exercises (with automated grading), online logic tools and applications, online videos of lectures, and an online forum for discussion. They are available at

http://logic.stanford.edu/intrologic/.

KEYWORDS

Formal Logic, Symbolic Logic, Propositional Logic, Relational Logic, deduction, reasoning, Artificial Intelligence

Contents

Preface

This book is a first course in Formal Logic. It is intended primarily for use at the college level. However, it can also be used for advanced secondary school students, and it can be used at the start of graduate school for those who have not yet seen the material.

There are just two prerequisites. The book presumes that the student understands sets and set operations, such as union, intersection, and so forth. It also presumes that the student is comfortable with symbolic manipulation, as used, for example, in solving high-school algebra problems. Nothing else is required.

The approach to teaching Logic used here emerged from more than 10 years of experience in teaching the logical foundations of Artificial Intelligence and more than 20 years of experience in teaching Logic for Computer Scientists. The result of this experience is an approach that differs from that taken by other books in Logic in two essential ways, one having to do with content, the other with form.

The primary difference in content concerns that semantics of the logic that is taught. Like many other books on Logic, this one covers first-order syntax and first-order proof theory plus induction. However, unlike other books, this book starts with Herbrand semantics rather than the more traditional Tarskian semantics.

In Tarskian semantics, we define an interpretation as a universe of discourse together with a function (1) that maps the object constants of our language to objects in a universe of discourse and (2) that maps relation constants to relations on that universe. We define variable assignments as assignments to variables. We define the semantics of quantified expressions as variations on variable assignments, saying, for example, that a universally quantified sentence is true for a given interpretation if and only if it is true for every variation of the given variable assignment. It is a mouthful to say and even harder for students to understand.

In Herbrand semantics, we start with the object constants, function constants, and relation constants of our language; we define the Herbrand base (i.e. the set of all ground atoms that can be formed from these components); and we define a model to be an arbitrary subset of the Herbrand base. That is all. In Herbrand semantics, an arbitrary logical sentence is logically equivalent to the set of all of its instances. A universally quantified sentence is true if and only if all of its instances are true. There are no interpretations and no variable assignments and no variations of variable assignments.

Although both approaches ultimately end up with the same deductive mechanism, we get there in two different ways. Deciding to use Herbrand semantics was not an easy to choice to make. It took years to get the material right and, even then, it took years to use it in teaching Logic. Although there are some slight disadvantages to this approach, experience suggests that the advantages significantly outweigh those disadvantages. This approach is considerably easier for

students to understand and leaves them with a deeper understanding of what Logic is all about. That said, there are some differences between Herbrand semantics and Tarskian semantics that some educators and theoreticians may find worrisome.

First of all, Herbrand semantics is not compact—there are infinite sets of sentences that are inconsistent while every finite subset is consistent. The upshot of this is that there are infinite sets of sentences where we cannot demonstrate unsatisfiability with a finite argument within the language itself. Fortunately, this does not cause any practical difficulties, since in all cases of practical interest we are working with finite sets of premises.

One significant deficiency of Herbrand semantics vis a vis Tarksian semantics is that with Herbrand semantics there are restrictions on the cardinality of the worlds that can be axiomatized. Since there is no external universe, the cardinality of the structures that can be axiomatized is equal to the number of ground terms in the language. (To make things easy, we can always choose a countable language. We can even choose an uncountable language, though doing so would ruin some of the nice properties of the logic. On the positive side, it is worth noting that in many practical applications we do not care about uncountable sets. Although there are uncountably many real numbers, remember that there are only countably many floating point numbers.) More significantly, recall that the Lowenheim-Skolem Theorem for Tarskian semantics assures us that even with Tarskian semantics we cannot write sentences that distinguish models of different infinite cardinalities. So, it is unclear whether this restriction has any real significance for the vast majority of students.

Herbrand semantics shares most important properties with Tarskian semantics. In the absence of function constants, the deductive calculus is complete for all finite axiomatizations. In fact, the calculus derives the exact same set of sentences. When we add functions, we lose this nice property. However, we get some interesting benefits in return. For one, it is possible with Herbrand semantics (with functions) to finitely axiomatize arithmetic. As we know from Godel, this is not possible in a first-order language with Tarskian semantics. The downside is that we lose completeness. However, it is nice to know that we can at least define things, even though we cannot prove them. Moreover, as mentioned above, we do not actually lose any consequences that we are able to deduce with Tarskian semantics.

That's all for what makes the content of this book different from other books. There is also a difference in form. In addition to the text of the book in print and online, there are also online exercises (with automated grading), some online Logic tools and applications, online videos of lectures, and an online forum for discussion.

The online offering of the course began with an experimental version early in the 2000s. While it was moderately successful, we were at that time unable to combine the online materials and tools and grading program with videos and an online forum, and so we discontinued the experiment. Recently, it was revived when Sebastian Thrun, Daphne Koller, and Andrew Ng created technologies for comprehensive offering online courses and began offering highly successful online courses of their own. With their technology and the previous materials, it was easy to create a comprehensive online course in Logic. And this led to completion of this book.

Thanks also to Pat Suppes, Jon Barwise, John Etchemendy, David-Barker Plummer, and others at the Stanford Center for the Study of Language and Information for their pioneering work on online education in Logic. *Language, Proof, and Logic* (LPL) in particular is a wonderful introduction to Logic and is widely used around the world. Although there are differences between that volume and this one in theory (especially semantics) and implementation (notably the use here of browser-based exercises and applications), this volume is in many ways similar to LPL. In particular, this volume shamelessly copies the LPL tactic of using online worlds (like Tarski's World) as a teaching tool for Logic.

And thanks as well to the thousands of students who over the years have had to endure early versions of this material, in many cases helping to get it right by suffering through experiments that were not always successful. It is a testament to the intelligence of these students that they seem to have learned the material despite multiple bumbling mistakes on our part. Their patience and constructive comments were invaluable in helping us to understand what works and what does not.

Finally, we need to acknowledge the enormous contributions of a former graduate student—Tim Hinrichs. He is a co-discoverer of many of the results about Herbrand semantics, without which this book would not have been written.

Michael Genesereth and Eric Kao
July 2013

CHAPTER 1

Introduction

1.1 LOGIC

Logic is one of the oldest intellectual disciplines in human history. It dates back to Aristotle; it has been studied through the centuries; and it is still a subject of active investigation today.

We use Logic is just about everything we do. We use it in our personal lives. We use it in our professional activities. We use the language of Logic to state observations, to define concepts, and to formalize theories. We use logical reasoning to derive conclusions from these bits of information. We use logical proofs to convince others of these conclusions.

And we are not alone! Logic is increasingly being used by computers—to prove mathematical theorems, to validate engineering designs, to diagnose failures, to encode and analyze laws and regulations and business rules.

This chapter is an overview of Logic as presented in this book. We start with an introduction to the key concepts of Logic using sentences written in English. We then talk about the value of using a formal language for expressing logical information, and we talk about the formal rules for manipulating sentences expressed in this language. After this, we discuss how formalization of this sort enables automation of logical reasoning, and we summarize the current state of Logic technology and its applications. Finally, we conclude with a reading guide for the remainder of the book.

1.2 ELEMENTS OF LOGIC

Consider the interpersonal relations of a small sorority. There are just four members—Abby, Bess, Cody, and Dana. Some of the girls like each other, but some do not. Figure 1.1 shows one set of possibilities.

	Abby	Bess	Cody	Dana
Abby			✓	
Bess			✓	
Cody	✓	✓		✓
Dana			✓	

Figure 1.1: One state of Sorority World.

Let's assume that we do not know this information ourselves, but we have informants who know the girls and are willing to share what they know. Each informant knows a little about the likes and dislikes of the girls, but no one knows everything. Here is where Logic comes in. By writing logical sentences, each informant can express exactly what he or she knows—no more, no less. For our part, we can combine these sentences into a logical theory; and we can use this theory to draw logical conclusions, including some that may not be known to any one of the informants.

Figure 1.2 shows one such theory. The first sentence is straightforward; it tells us directly that Dana likes Cody. The second and third sentences tell us what is not true, without saying what is true. The fourth sentence says that one condition holds or another but does not say which. The fifth sentence gives a general fact about the girls Abby likes. The sixth sentence expresses a general fact about Cody's likes. The last sentence says something about everyone.

Dana likes Cody.
Abby does not like Dana.
Dana does not like Abby.
Bess likes Cody and Dana.
Abby likes everyone that Bess likes.
Cody likes everyone who likes her.

Figure 1.2: Logical sentences describing Sorority World.

Sentences like these constrain the possible ways the world could be. Each sentence divides the set of possible worlds into two subsets, those in which the sentence is true and those in which the sentence is false. Believing a sentence is tantamount to believing that the world is in the first set. Given two sentences, we know the world must be in the intersection of the set of worlds in which the first sentence is true and the set of worlds in which the second sentence is true. Ideally, when we have enough sentences, we know exactly how things stand.

Unfortunately, this is not always the case. Sometimes, a collection of sentences only partially constrains the world. For example, there are four different worlds that are consistent with the our Sorority World sentences, namely the ones shown in Figure 1.3.

Even though a set of sentences does not determine a unique world, it is often the case that some sentences are true in every world that satisfies the given sentences. A sentence of this sort is said to be a *logical conclusion* from the given sentences. Said the other way around, a set of *premises logically entails* a *conclusion* if and only if every world that satisfies the premises also satisfies the conclusion.

What can we conclude from the bits of information in Figure 1.2? Quite a bit, as it turns out. For example, it must be the case that Bess likes Cody. Also, Bess does not like Dana. There are also some general conclusions that must be true. For example, in this world with just four girls, we can conclude that everybody likes somebody. Also, everyone is liked by somebody.

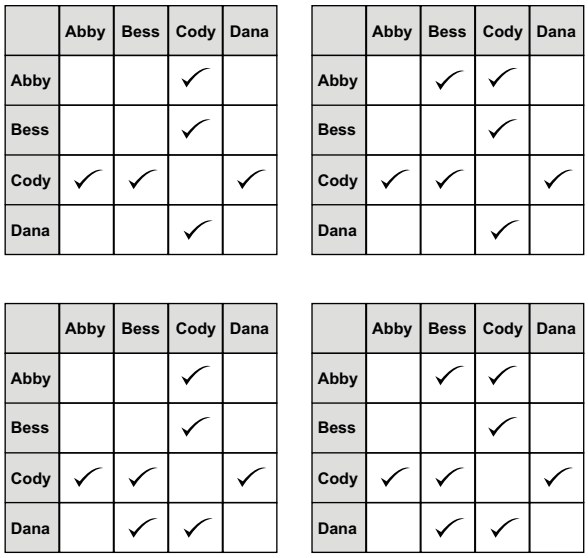


Figure 1.3: Four states of Sorority World.

One way to make this determination is by checking all possible worlds. For example, in our case, we notice that, in every world that satisfies our sentences, Bess likes Cody, so the statement that Bess likes Cody is a logical conclusion from our set of sentences.

Unfortunately, determining logical entailment by checking all possible worlds is impractical in general. There are usually many, many possible worlds; and in some cases there can be infinitely many.

The alternative is *logical reasoning*, that is, the application of reasoning rules to derive logical conclusions and produce a *logical proofs*, i.e., sequences of reasoning steps that leads from premises to conclusions.

As an example, consider the following informal proof that starts with the premises shown above and proves that Bess likes Cody.

> *We know that Abby likes everyone that Bess likes, and we know that Abby does not like Dana. Therefore, Bess must not like Dana either. (If Bess did like Dana, then Abby would like her as well.) At the same time, we know that Bess likes Cody or Dana. Consequently, since Bess does not like Dana, she must like Cody.*

The concept of proof, in order to be meaningful, requires that we be able to recognize certain reasoning steps as immediately obvious. In other words, we need to be familiar with the reasoning "atoms" out of which complex proof "molecules" are built.

One of Aristotle's great contributions to philosophy was his recognition that what makes a step of a proof immediately obvious is its form rather than its content. It does not matter whether

we are talking about blocks or stocks or sorority girls. What matters is the structure of the facts with which you are working. Such patterns are called *rules of inference*.

As an example, consider the reasoning step shown below. We know that all Accords are Hondas, and we know that all Hondas are Japanese cars. Consequently, we can conclude that all Accords are Japanese cars.

All Accords are Hondas.
All Hondas are Japanese.
Therefore, all Accords are Japanese.

Now consider another example. We know that all borogoves are slithy toves, and we know that all slithy toves are mimsy. Consequently, we can conclude that all borogoves are mimsy. What's more, in order to reach this conclusion, we do not need to know anything about borogoves or slithy toves or what it means to be mimsy.

All borogoves are slithy toves.
All slithy toves are mimsy.
Therefore, all borogoves are mimsy.

What is interesting about these examples is that they share the same reasoning structure, that is, the pattern shown below.

All x are y.
All y are z.
Therefore, all x are z.

The existence of such reasoning patterns is fundamental in Logic but raises important questions. Which patterns are correct? Are there many such patterns or just a few?

Let us consider the first of these questions. Obviously, there are patterns that are just plain wrong in the sense that they can lead to incorrect conclusions. Consider, as an example, the (faulty) reasoning pattern shown below.

All x are y.
Some y are z.
Therefore, some x are z.

Now let us take a look at an instance of this pattern. If we replace *x* by *Toyotas* and *y* by *cars* and *z* by *made in America*, we get the following line of argument, leading to a conclusion that happens to be correct.

All Toyotas are cars.
Some cars are made in America.
Therefore, some Toyotas are made in America.

On the other hand, if we replace x by *Toyotas* and y by *cars* and z by *Porsches*, we get a line of argument leading to a conclusion that is questionable.

All Toyotas are cars.
Some cars are Porsches.
Therefore, some Toyotas are Porsches.

What distinguishes a correct pattern from one that is incorrect is that it must always lead to correct conclusions, i.e., conclusions that are logically entailed by the premises. As we will see, this is the defining criterion for what we call *deduction*.

Now, it is noteworthy that there are patterns of reasoning that are sometimes useful but do not satisfy this strict criterion. There is inductive reasoning, abductive reasoning, reasoning by analogy, and so forth.

Induction is reasoning from the particular to the general. The example shown below illustrates this. If we see enough cases in which something is true and we never see a case in which it is false, we tend to conclude that it is always true.

I have seen 1000 black ravens.
I have never seen a raven that is not black.
Therefore, every raven is black.
Now try red Hondas.

Abduction is reasoning from effects to possible causes. Many things can cause an observed result. We often tend to infer a cause even when our enumeration of possible causes is incomplete.

If there is no fuel, the car will not start.
If there is no spark, the car will not start.
There is spark.
The car will not start.
Therefore, there is no fuel.
What if the car is in a vacuum chamber?

Reasoning by *analogy* is reasoning in which we infer a conclusion based on similarity of two situations, as in the following example.

The flow in a pipe is proportional to its diameter.
Wires are like pipes.
Therefore, the current in a wire is proportional to diameter.
Now try price.

Of all types of reasoning, deductive reasoning is the only one that *guarantees* its conclusions in all cases. It has some very special properties and holds a unique place in Logic. In this book, we concentrate entirely on deduction and leave these other forms of reasoning to others.

1.3 FORMALIZATION

So far, we have illustrated everything with sentences in English. While natural language works well in many circumstances, it is not without its problems. Natural language sentences can be complex; they can be ambiguous; and failing to understand the meaning of a sentence can lead to errors in reasoning.

Even very simple sentences can be troublesome. Here we see two grammatically legal sentences. They are the same in all but the last word, but their structure is entirely different. In the first, the main verb is *blossoms*, while in the second *blossoms* is a noun and the main verb is *sank*.

The cherry blossoms in the Spring.
The cherry blossoms in the Spring sank.

As another example of grammatical complexity, consider the following excerpt taken from the University of Michigan lease agreement. The sentence in this case is sufficiently long and the grammatical structure sufficiently complex that people must often read it several times to understand precisely what it says.

> *The University may terminate this lease when the Lessee, having made application and executed this lease in advance of enrollment, is not eligible to enroll or fails to enroll in the University or leaves the University at any time prior to the expiration of this lease, or for violation of any provisions of this lease, or for violation of any University regulation relative to resident Halls, or for health reasons, by providing the student with written notice of this termination 30 days prior to the effective data of termination, unless life, limb, or property would be jeopardized, the Lessee engages in the sales of purchase of controlled substances in violation of federal, state or local law, or the Lessee is no longer enrolled as a student, or the Lessee engages in the use or possession of firearms, explosives, inflammable liquids, fireworks, or other dangerous weapons within the building, or turns in a false alarm, in which cases a maximum of 24 hours notice would be sufficient.*

As an example of ambiguity, suppose I were to write the sentence *There's a girl in the room with a telescope*. See Figure 1.4 for two possible meanings of this sentence. Am I saying that there is a girl in a room containing a telescope? Or am I saying that there is a girl in the room and she is holding a telescope?

Such complexities and ambiguities can sometimes be humorous if they lead to interpretations the author did not intend. See the examples in Figure 1.5 for some infamous newspaper headlines with multiple interpretations. Using a formal language eliminates such unintentional ambiguities (and, for better or worse, avoids any unintentional humor as well).

As an illustration of errors that arise in reasoning with sentences in natural language, consider the following examples. In the first, we use the transitivity of the *better* relation to derive a conclusion about the relative quality of champagne and soda from the relative quality of champagne and beer and the relative quality or beer and soda. So far so good.

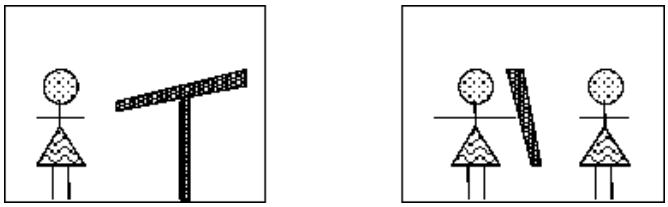


Figure 1.4: *There's a girl in the room with a telescope.*

Crowds Rushing to See Pope Trample 6 to Death
Journal Star, Peoria, 1980

Scientists Grow Frog Eyes and Ears
The Daily Camera, Boulder, 2000

British Left Waffles On Falkland Islands

Food Stamp Recipients Turn to Plastic
The Miami Herald, 1991

Indian Ocean Talks
The Plain Dealer, 1977

Fried Chicken Cooked in Microwave Wins Trip
The Oregonian, Portland, 1981

Figure 1.5: Various newspaper headlines.

Champagne is better than beer.
Beer is better than soda.
Therefore, champagne is better than soda.

Now, consider what happens when we apply the same transitivity rule in the case illustrated below. The form of the argument is the same as before, but the conclusion is somewhat less believable. The problem in this case is that the use of *nothing* here is syntactically similar to the use of *beer* in the preceding example, but in English it means something entirely different.

Bad sex is better than nothing.
Nothing is better than good sex.
Therefore, bad sex is better than good sex.

Logic eliminates these difficulties through the use of a formal language for encoding information. Given the syntax and semantics of this formal language, we can give a precise definition for the notion of logical entailment. Moreover, we can establish precise reasoning rules that produce all and only logically entailed conclusions.

In this regard, there is a strong analogy between the methods of Formal Logic and those of high school algebra. To illustrate this analogy, consider the following algebra problem.

Xavier is three times as old as Yolanda. Xavier's age and Yolanda's age add up to twelve. How old are Xavier and Yolanda?

Typically, the first step in solving such a problem is to express the information in the form of equations. If we let x represent the age of Xavier and y represent the age of Yolanda, we can capture the essential information of the problem as shown below.

$$\begin{aligned} x - 3y &= 0 \\ x + \ \ y &= 12 \end{aligned}$$

Using the methods of algebra, we can then manipulate these expressions to solve the problem. First we subtract the second equation from the first.

$$\begin{array}{r} x - 3y = 0 \\ x + \ \ y = 12 \\ \hline -4y = -12 \end{array}$$

Next, we divide each side of the resulting equation by -4 to get a value for y. Then substituting back into one of the preceding equations, we get a value for x.

$$\begin{aligned} x &= 9 \\ y &= 3 \end{aligned}$$

Now, consider the following logic problem.

If Mary loves Pat, then Mary loves Quincy. If it is Monday and raining, then Mary loves Pat or Quincy. If it is Monday and raining, does Mary love Quincy?

As with the algebra problem, the first step is formalization. Let p represent the possibility that Mary loves Pat; let q represent the possibility that Mary loves Quincy; let m represent the possibility that it is Monday; and let r represent the possibility that it is raining.

With these abbreviations, we can represent the essential information of this problem with the following logical sentences. The first says that p *implies* q, i.e., if Mary loves Pat, then Mary loves Quincy. The second says that m *and* r *implies* p *or* q, i.e., if it is Monday and raining, then Mary loves Pat or Mary loves Quincy.

$$\begin{array}{lcl} p & \Rightarrow & q \\ m \wedge r & \Rightarrow & p \vee q \end{array}$$

As with Algebra, Formal Logic defines certain operations that we can use to manipulate expressions. The operation shown below is a variant of what is called *Propositional Resolution*. The expressions above the line are the premises of the rule, and the expression below is the conclusion.

$$\begin{array}{lcl} p_1 \wedge ... \wedge p_k & \Rightarrow & q_1 \vee ... \vee q_l \\ r_1 \wedge ... \wedge r_m & \Rightarrow & s_1 \vee ... \vee s_n \\ \hline p_1 \wedge ... \wedge p_k \wedge r_1 \wedge ... \wedge r_m & \Rightarrow & q_1 \vee ... \vee q_l \vee s_1 \vee ... \vee s_n \end{array}$$

There are two elaborations of this operation. (1) If a proposition on the left-hand side of one sentence is the same as a proposition in the right-hand side of the other sentence, it is okay to drop the two symbols, with the proviso that *only one* such pair may be dropped. (2) If a constant is repeated on the same side of a single sentence, all but one of the occurrences can be deleted.

We can use this operation to solve the problem of Mary's love life. Looking at the two premises above, we notice that p occurs on the left-hand side of one sentence and the right-hand side of the other. Consequently, we can cancel the p and thereby derive the conclusion that, if is Monday and raining, then Mary loves Quincy or Mary loves Quincy.

$$\begin{array}{lcl} p & \Rightarrow & q \\ m \wedge r & \Rightarrow & p \vee q \\ \hline m \wedge r & \Rightarrow & q \vee q \end{array}$$

Dropping the repeated symbol on the right-hand side, we arrive at the conclusion that, if it is Monday and raining, then Mary loves Quincy.

$$\begin{array}{lcl} m \wedge r & \Rightarrow & q \vee q \\ \hline m \wedge r & \Rightarrow & q \end{array}$$

This example is interesting in that it showcases our formal language for encoding logical information. As with algebra, we use symbols to represent relevant aspects of the world in question, and we use operators to connect these symbols in order to express information about the things those symbols represent.

The example also introduces one of the most important operations in Formal Logic — Resolution (in this case a restricted form of Resolution). Resolution has the property of being *complete* for an important class of logic problems, i.e., it is the *only* operation necessary to solve any problem in the class.

1.4 AUTOMATION

The existence of a formal language for representing information and the existence of a corresponding set of mechanical manipulation rules together have an important consequence—the possibility of *automated reasoning* using digital computers.

The idea is simple. We use our formal representation to encode the premises of a problem as data structures in a computer, and we program the computer to apply our mechanical rules in a systematic way. The rules are applied until the desired conclusion is attained or until it is determined that the desired conclusion cannot be attained. (Unfortunately, in some cases, this determination cannot be made; and the procedure never halts. Nevertheless, as discussed in later chapters, the idea is basically sound.)

Although the prospect of automated reasoning has achieved practical realization only in the last few decades, it is interesting to note that the concept itself is not new. In fact, the idea of building machines capable of logical reasoning has a long tradition.

One of the first individuals to give voice to this idea was Leibniz. He conceived of "a universal algebra by which all knowledge, including moral and metaphysical truths, can some day be brought within a single deductive system." Having already perfected a mechanical calculator for arithmetic, he argued that, with this universal algebra, it would be possible to build a machine capable of rendering the consequences of such a system mechanically.

Boole gave substance to this dream in the 1800s with the invention of Boolean algebra and with the creation of a machine capable of computing accordingly.

The early twentieth century brought additional advances in Logic, notably the invention of the predicate calculus by Russell and Whitehead and the proof of the corresponding completeness and incompleteness theorems by Gödel in the 1930s.

The advent of the digital computer in the 1940s gave increased attention to the prospects for automated reasoning. Research in artificial intelligence led to the development of efficient algorithms for logical reasoning, highlighted by Robinson's invention of resolution theorem proving in the 1960s.

Today, the prospect of automated reasoning has moved from the realm of possibility to that of practicality, with the creation of *logic technology* in the form of automated reasoning systems, such as Vampire, Prover9, the Prolog Technology Theorem Prover, Epilog, and others.

The emergence of this technology has led to the application of logic technology in a wide variety of areas. The following paragraphs outline some of these uses.

Mathematics. Automated reasoning programs can be used to check proofs and, in some cases, to produce proofs or portions of proofs.

Engineering. Engineers can use the language of Logic to write specifications for their products and to encode their designs. Automated reasoning tools can be used to simulate designs and in some cases validate that these designs meet their specification. Such tools can also be used to diagnose failures and to develop testing programs.

Database Systems. By conceptualizing database tables as sets of simple sentences, it is possible to use Logic in support of database systems. For example, the language of Logic can be used to define virtual views of data in terms of explicitly stored tables, and it can be used to encode constraints on databases. Automated reasoning techniques can be used to compute new tables, to detect problems, and to optimize queries.

Data Integration. The language of Logic can be used to relate the vocabulary and structure of disparate data sources, and automated reasoning techniques can be used to integrate the data in these sources.

Logical Spreadsheets. Logical spreadsheets generalize traditional spreadsheets to include logical constraints as well as traditional arithmetic formulas. Examples of such constraints abound. For example, in scheduling applications, we might have timing constraints or restrictions on who can reserve which rooms. In the domain of travel reservations, we might have constraints on adults and infants. In academic program sheets, we might have constraints on how many courses of varying types that students must take.

Law and Business. The language of Logic can be used to encode regulations and business rules, and automated reasoning techniques can be used to analyze such regulations for inconsistency and overlap.

1.5 READING GUIDE

Although Logic is a single field of study, there is more than one logic in this field. In the three main units of this book, we look at two different types of logic, each more sophisticated than the last.

Propositional Logic is the logic of propositions. Symbols in the language represent "conditions" in the world, and complex sentences in the language express interrelationships among these conditions. The primary operators are Boolean connectives, such as *and*, *or*, and *not*.

Relational Logic expands upon Propositional Logic by providing a means for explicitly talking about individual objects and their interrelationships (not just monolithic conditions). In order to do so, we expand our language to include object constants, function constants, relation constants, variables, and quantifiers.

Each chapter brings new issues and capabilities to light. Despite these differences, there are many commonalities among these logics. In particular, in each case, there is a language with a formal syntax and a precise semantics; there is a notion of logical entailment; and there are legal rules for manipulating expressions in the language.

These similarities allow us to compare the two logics and to gain an appreciation of the tradeoff between expressiveness and computational complexity. On the one hand, the introduction of additional linguistic complexity makes it possible to say things that cannot be said in more restricted languages. On the other hand, the introduction of additional linguistic flexibility has adverse effects on computability. As we proceed though the material, our attention will range from the completely computable case of Propositional Logic to a variant of Logic that is not at all computable.

One final comment. In the hopes of preventing difficulties, it is worth pointing out a potential source of confusion. This book exists in the *meta* world. It contains sentences about sentences; it contains proofs about proofs. In some places, we use similar mathematical symbology both for sentences *in* Logic and sentences *about* Logic. Wherever possible, we try to be clear about this distinction, but the potential for confusion remains. Unfortunately, this comes with the territory. We are using Logic to study Logic. It is our most powerful intellectual tool.

RECAP

Logic is the study of information encoded in the form of logical sentences. Each logical sentence divides the set of all possible world into two subsets - the set of worlds in which the sentence is true and the set of worlds in which the set of sentences is false. A set of premises *logically entails* a conclusion if and only if the conclusion is true in every world in which all of the premises are true. *Deduction* is a form of symbolic reasoning that produces conclusions that are logically entailed by premises (distinguishing it from other forms of reasoning, such as *induction*, *abduction*, and *analogical reasoning*). A *proof* is a sequence of simple, more-or-less obvious deductive steps that justifies a conclusion that may not be immediately obvious from given premises. In Logic, we usually encode logical information as sentences in formal languages; and we use rules of inference appropriate to these languages. Such formal representations and methods are useful for us to use ourselves. Moreover, they allow us to automate the process of deduction, though the computability of such implementations varies with the complexity of the sentences involved.

EXERCISES

1.1 Consider the state of the Sorority World depicted below.

	Abby	Bess	Cody	Dana
Abby		✓	✓	
Bess			✓	
Cody	✓	✓		✓
Dana		✓	✓	

For each of the following sentences, say whether or not it is true in this state of the world.

(*a*) *Abby likes Dana.*
(*b*) *Dana does not like Abby.*
(*c*) *Abby likes Cody or Dana.*
(*d*) *Abby likes someone who likes her.*
(*e*) *Somebody likes everybody.*

1.2 Come up with a table of likes and dislikes for the Sorority World that makes *all* of the following sentences true. Note that there is more than one such table.

Dana likes Cody.
Abby does not like Dana.
Bess likes Cody or Dana.
Abby likes everyone whom Bess likes.
Cody likes everyone who likes her.
Nobody likes herself.

1.3 Consider a set of Sorority World premises that are true in the four states of Sorority World shown below and that are false in all other states.

	Abby	Bess	Cody	Dana
Abby			✓	
Bess			✓	
Cody	✓	✓		✓
Dana			✓	

	Abby	Bess	Cody	Dana
Abby		✓	✓	
Bess			✓	
Cody	✓	✓		✓
Dana			✓	

	Abby	Bess	Cody	Dana
Abby			✓	
Bess			✓	
Cody	✓	✓		✓
Dana		✓	✓	

	Abby	Bess	Cody	Dana
Abby		✓	✓	
Bess			✓	
Cody	✓	✓		✓
Dana		✓	✓	

For each of the following sentences, say whether or not it is logically entailed by these premises.

(*a*) *Abby likes Bess or Bess likes Abby.*
(*b*) *Somebody likes herself.*
(*c*) *Everybody likes somebody.*

1.4 Say whether or not the following reasoning patterns are logically correct.

(*a*) *All x are z. All y are z. Therefore, some x are y.*
(*b*) *Some x are y. All y are z. Therefore, some x are z.*
(*c*) *All x are y. Some y are z. Therefore, some x are z.*

CHAPTER 2

Propositional Logic

2.1 INTRODUCTION

Propositional Logic is concerned with propositions and their interrelationships. The notion of a proposition here cannot be defined precisely. Roughly speaking, a *proposition* is a possible condition of the world that is either true or false, e.g., the possibility that it is raining, the possibility that it is cloudy, and so forth. The condition need not be true in order for it to be a proposition. In fact, we might want to say that it is false or that it is true if some other proposition is true.

In this chapter, we first look at the syntactic rules that define the legal expressions in Propositional Logic. We then look at the semantics of the expressions specified by these rules. Given this semantics, we talk about evaluation, satisfaction, and the properties of sentences. We then define the concept of propositional entailment, which identifies for us, at least in principle, all of the logical conclusions we can draw from any set of propositional sentences.

2.2 SYNTAX

In Propositional Logic, there are two types of sentences – simple sentences and compound sentences. Simple sentences express simple facts about the world. Compound sentences express logical relationships between the simpler sentences of which they are composed.

Simple sentences in Propositional Logic are often called *proposition constants* or, sometimes, *logical constants*. In what follows, we write proposition constants as strings of letters, digits, and underscores ("_"), where the first character is a lower case letter. For example, *raining* is a proposition constant, as are *rAiNiNg*, *r32aining*, and *raining_or_snowing*. *Raining* is not a logical constant because it begins with an upper case character. 324567 fails because it begins with a number. *raining-or-snowing* fails because it contains hyphens.

Compound sentences are formed from simpler sentences and express relationships among the constituent sentences. There are five types of compound sentences: negations, conjunctions, disjunctions, implications, and biconditionals.

A *negation* consists of the negation operator ¬ and a simple or compound sentence, called the *target*. For example, given the sentence p, we can form the negation of p as shown below.

$$(\neg p)$$

A *conjunction* is a sequence of sentences separated by occurrences of the $\wedge$ operator and enclosed in parentheses, as shown below. The constituent sentences are called *conjuncts*. For example, we can form the conjunction of p and q as follows.

$$(p \wedge q)$$

A *disjunction* is a sequence of sentences separated by occurrences of the $\vee$ operator and enclosed in parentheses. The constituent sentences are called *disjuncts*. For example, we can form the disjunction of p and q as follows.

$$(p \vee q)$$

An *implication* consists of a pair of sentences separated by the $\Rightarrow$ operator and enclosed in parentheses. The sentence to the left of the operator is called the *antecedent*, and the sentence to the right is called the *consequent*. The implication of p and q is shown below.

$$(p \Rightarrow q)$$

An *equivalence*, or *biconditional*, is a combination of an implication and a reduction. For example, we can express the biconditional of p and q as shown below.

$$(p \Leftrightarrow q)$$

Note that the constituent sentences within any compound sentence can be either simple sentences or compound sentences or a mixture of the two. For example, the following is a legal compound sentence.

$$((p \vee q) \Rightarrow r)$$

One disadvantage of our notation, as written, is that the parentheses tend to build up and need to be matched correctly. It would be nice if we could dispense with parentheses, e.g., simplifying the preceding sentence to the one shown below.

$$p \vee q \Rightarrow r$$

Unfortunately, we cannot do without parentheses entirely, since then we would be unable to render certain sentences unambiguously. For example, the sentence shown above could have resulted from dropping parentheses from either of the following sentences.

$$((p \vee q) \Rightarrow r)$$
$$(p \vee (q \Rightarrow r))$$

The solution to this problem is the use of *operator precedence*. The following table gives a hierarchy of precedences for our operators. The $\neg$ operator has higher precedence than $\wedge$; $\wedge$ has higher precedence than $\vee$; and $\vee$ has higher precedence than $\Rightarrow$ and $\Leftrightarrow$.

$$\neg$$
$$\wedge$$
$$\vee$$
$$\Rightarrow \Leftrightarrow$$

In unparenthesized sentences, it is often the case that an expression is flanked by operators, one on either side. In interpreting such sentences, the question is whether the expression associates with the operator on its left or the one on its right. We can use precedence to make this determination. In particular, we agree that an operand in such a situation always associates with the operator of higher precedence. When an operand is surrounded by operators of equal precedence, the operand associates to the right. The following examples show how these rules work in various cases. The expressions on the right are the fully parenthesized versions of the expressions on the left.

$\neg p \wedge q$	$((\neg p) \wedge q)$
$p \wedge \neg q$	$(p \wedge (\neg q))$
$p \wedge q \vee r$	$((p \wedge q) \vee r)$
$p \vee q \wedge r$	$(p \vee (q \wedge r)$
$p \Rightarrow q \Rightarrow r$	$(p \Rightarrow (q \Rightarrow r))$
$p \Rightarrow q \Leftrightarrow r$	$(p \Rightarrow (q \Leftrightarrow r))$

Note that just because precedence allows us to delete parentheses in some cases does not mean that we can dispense with parentheses entirely. Consider the example shown earlier. Precedence eliminates the ambiguity by dictating that the unparenthesized sentence is an implication with a disjunction as antecedent. However, this makes for a problem for those cases when we want to express a disjunction with an implication as a disjunct. In such cases, we must retain at least one pair of parentheses.

We end the section with two simple definitions that are useful in discussing Propositional Logic. A *propositional vocabulary* is a set of proposition constants. A *propositional language* is the set of all propositional sentences that can be formed from a propositional vocabulary.

2.3 SEMANTICS

The treatment of semantics in Logic is similar to its treatment in Algebra. Algebra is unconcerned with the real-world significance of variables. What is interesting are the relationships among the

values of the variables expressed in the equations we write. Algebraic methods are designed to respect these relationships, independent of what the variables represent.

In a similar way, Logic is unconcerned with the real world significance of proposition constants. What is interesting is the relationship among the truth values of simple sentences and the truth values of compound sentences within which the simple sentences are contained. As with Algebra, logical reasoning methods are independent of the significance of proposition constants; all that matter is the form of sentences.

Although the values assigned to logical constants are not crucial in the sense just described, in talking about Logic, it is sometimes useful to make truth assignments explicit and to consider various assignments or all assignments and so forth. Such an assignment is called a truth assignment.

Formally, a *truth assignment* for Propositional Logic is a function assigning a truth value to each of the proposition constants of the language. In what follows, we use the digit 1 as a synonym for *true* and 0 as a synonym for *false*; and we refer to the value of a constant or expression under a truth assignment i by superscripting the constant or expression with i as the superscript.

The assignment shown below is an example for the case of a logical language with just three proposition constants: p, q, and r.

$$p^i = 1$$
$$q^i = 0$$
$$r^i = 1$$

The following assignment is another truth assignment for the same language.

$$p^i = 0$$
$$q^i = 0$$
$$r^i = 1$$

Note that the formulas above are not themselves sentences in Propositional Logic. Propositional Logic does not allow superscripts and does not use the = symbol. Rather, these are informal, metalevel statements *about* particular truth assignments. Although talking about propositional logic using a notation similar to that propositional logic can sometimes be confusing, it allows us to convey meta-information precisely and efficiently. To minimize problems, in this book we use such meta-notation infrequently and only when there is little chance of confusion.

Looking at the preceding truth assignments, it is important to bear in mind that, as far as logic is concerned, any truth assignment is as good as any other. It does not directly fix the truth assignment of individual proposition constants.

On the other hand, *given* a truth assignment for the logical constants of a language, logic *does* fix the truth assignment for all compound sentences in that language. In fact, it is possible to determine the truth value of a compound sentence by repeatedly applying the following rules.

If the truth value of a sentence is *true*, the truth value of its negation is *false*. If the truth value of a sentence is *false*, the truth value of its negation is *true*.

ϕ	$\neg\phi$
1	0
0	1

The truth value of a conjunction is *true* if and only if the truth value of its conjuncts are both *true*; otherwise, the truth value is *false.*

ϕ	ψ	$\phi \wedge \psi$
1	1	1
1	0	0
0	1	0
0	0	0

The truth value of a disjunction is *true* if and only if the truth value of at least one its disjuncts is *true*; otherwise, the truth value is *false*. Note that this is the *inclusive or* interpretation of the $\vee$ operator and is differentiated from the *exclusive or* interpretation in which a disjunction is true if and only if an odd number of its disjuncts are false.

ϕ	ψ	$\phi \vee \psi$
1	1	1
1	0	1
0	1	1
0	0	0

The truth value of an implication is *false* if and only if its antecedent is *true* and its consequent is *false*; otherwise, the truth value is *true*. This is called *material implication*.

ϕ	ψ	$\phi \Rightarrow \psi$
1	1	1
1	0	0
0	1	1
0	0	1

A biconditional is *true* if and only if the truth values of its constituents agree, i.e., they are either both *true* or both *false*.

ϕ	ψ	$\phi \Leftrightarrow \psi$
1	1	1
1	0	0
0	1	0
0	0	1

Given the semantic definitions in the last section, we can easily determine the truth value for any sentence, whether simple or compound, given a truth assignment for our proposition constants.

The technique is simple. We substitute true and false values for the proposition constants in our sentence, forming an expression with 1s and 0s and logical operators. We use our operator semantics to evaluate subexpressions with these truth values as arguments. We then repeat, working from the inside out, until we have a truth value for the sentence as a whole.

As an example, consider the truth assignment i shown below.

$$p^i = 1$$
$$q^i = 0$$
$$r^i = 1$$

Using our evaluation method, we can see that i satisfies $(p \vee q) \wedge (\neg q \vee r)$.

$$(p \vee q) \wedge (\neg q \vee r)$$
$$(1 \vee 0) \wedge (\neg 0 \vee 1)$$
$$1 \wedge (\neg 0 \vee 1)$$
$$1 \wedge (1 \vee 1)$$
$$1 \wedge 1$$
$$1$$

Now consider truth assignment j defined as follows.

$$p^j = 1$$
$$q^j = 1$$
$$r^j = 0$$

In this case, j does not satisfy $(p \vee q) \wedge (\neg q \vee r)$.

$$(p \vee q) \wedge (\neg q \vee r)$$
$$(1 \vee 1) \wedge (\neg 1 \vee 0)$$
$$1 \wedge (\neg 1 \vee 0)$$
$$1 \wedge (0 \vee 0)$$
$$1 \wedge 0$$
$$0$$

Using this technique, we can evaluate the truth of arbitrary sentences in our language. The cost is proportional to the size of the sentence.

We finish up this section with a few definitions for future use. We say that a truth assignment *satisfies* a sentence if and only if it is *true* under that truth assignment. We say that a truth assignment *falsifies* a sentence if and only if it is *false* under that truth assignment. A truth assignment satisfies a *set* of sentences if and only if it satisfies *every* sentence in the set. A truth assignment falsifies a *set* of sentences if and only if it falsifies *at least* one sentence in the set.

2.4 SATISFACTION

Satisfaction is the opposite of evaluation. We begin with one or more compound sentences and try to figure out which truth assignments satisfy those sentences.

One way to do this is using a truth table for the language. A *truth table* for a propositional language is a table showing all of the possible truth assignments for the propositional constants in the language.

The following figure shows a truth table for a propositional language with just three propositional constants (p, q, and r). Each column corresponds to one proposition constant, and each row corresponds to a single truth assignment. The truth assignments i and j defined in the preceding section correspond to the third and second rows of this table, respectively.

p	q	r
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

Note that, for a propositional language with n logical constants, there are n columns in the truth tables and 2^n rows.

Here is an intuitively simple method of finding truth assignments that satisfy given sentences. (1) We start with a truth table for the proposition constants of our language, and we add columns for each sentence in our set. (2) We then evaluate each sentence for each of the rows of the truth table. (3) Finally, any row that satisfies all sentences in the set is a solution to the problem as a whole. Note that there might be more than one.

2.5 LOGICAL PROPERTIES OF PROPOSITIONAL SENTENCES

Evaluation and satisfaction are processes that involve specific sentences and specific truth assignments. In Logic, we are sometimes more interested in the properties of sentences that hold across truth assignments. In particular, the notion of satisfaction imposes a classification of sentences in a language based on whether there are truth assignments that satisfy that sentence.

A sentence is *valid* if and only if it is satisfied by *every* truth assignment. For example, the sentence $p \vee \neg\, p$ is valid.

A sentence is *unsatisfiable* if and only if it is not satisfied by any truth assignment. For example, the sentence $p \Leftrightarrow \neg\, p$ is unsatisfiable. No matter what truth assignment we take, the sentence is always false.

A sentence is *contingent* if and only if there is some truth assignment that satisfies it and some truth assignment that falsifies it.

It is frequently useful to group these classifications into two groups. A sentence is *satisfiable* if and only if it is valid or contingent. A sentence is *falsifiable* if and only if it is unsatisfiable or contingent. We have already seen several examples of satisfiable and falsifiable sentences.

In one sense, valid sentences and unsatisfiable sentences are useless. Valid sentences do not rule out any possible truth assignments; unsatisfiable sentences rule out all truth assignments; thus they say nothing about the world. On the other hand, from a logical perspective, they are extremely useful in that, as we shall see, they serve as the basis for legal transformations that we can perform on other logical sentences.

2.6 PROPOSITIONAL ENTAILMENT

Validity, satisfiability, and unsatisfiability are properties of individual sentences. In logical reasoning, we are not so much concerned with individual sentences as we are with the relationships between sentences. In particular, we would like to know, given some sentences, whether other sentences are or are not logical conclusions. This relative property is known as *logical entailment*. When we are speaking about Propositional Logic, we use the phrase *propositional entailment*.

A set of sentences Δ *logically entails* a sentence ϕ (written $\Delta \models \phi$) if and only if every truth assignment that satisfies Δ also satisfies ϕ.

For example, the sentence p logically entails the sentence $(p \vee q)$. Since a disjunction is true whenever one of its disjuncts is true, then $(p \vee q)$ must be true whenever p is true. On the other hand, the sentence p does *not* logically entail $(p \wedge q)$. A conjunction is true if and only if *both* of its conjuncts are true, and q may be false. Of course, any set of sentences containing both p and q does logically entail $(p \wedge q)$.

Note that the relationship of logical entailment is a logical one. Even if the premises of a problem do not logically entail the conclusion, this does not mean that the conclusion is necessarily false, even if the premises are true. It just means that it is *possible* that the conclusion is false.

Once again, consider the case of $(p \wedge q)$. Although p does not logically entail this sentence, it is *possible* that both p and q are true and, therefore, $(p \wedge q)$ is true. However, the logical entailment does not hold because it is also possible that q is false and, therefore, $(p \wedge q)$ is false.

Note that logical entailment is not the same as logical equivalence. The sentence p logically entails $(p \vee q)$, but $(p \vee q)$ does not logically entail p. Logical entailment is not analogous to arithmetic equality; it is closer to arithmetic inequality.

One way of determining whether or not a set of premises logically entails a possible conclusion is to check the truth table for the proposition constants in the language. This is called the truth table method. (1) First, we form a truth table for the proposition constants and add a column for the premises and a column for the conclusion. (2) We then evaluate the premises. (3) We evaluate the conclusion. (4) Finally, we compare the results. If every row that satisfies the premises also satisfies the conclusion, then the premises logically entail the conclusion.

As an example, let's use this method to show that p logically entails $(p \vee q)$. We set up our truth table and add a column for our premise and a column for our conclusion. In this case, the premise is just p and so evaluation is straightforward; we just copy the column. The conclusion is true if and only if p is true or q is true. Finally, we notice that every row that satisfies the premise also satisfies the conclusion.

p	q	p	$p \vee q$
1	1	1	1
1	0	1	1
0	1	0	1
0	0	0	0

Now, let's do the same for the premise p and the conclusion $(p \wedge q)$. We set up our table as before and evaluate our premise. In this case, there is only one row that satisfies our conclusion. Finally, we notice that the assignment in the second row satisfies our premise but does not satisfy our conclusion; so logical entailment does not hold.

p	q	p	$p \wedge q$
1	1	1	1
1	0	1	0
0	1	0	0
0	0	0	0

Finally, let's look at the problem of determining whether the set of propositions $\{p, q\}$ logically entails $(p \wedge q)$. Here we set up our table as before, but this time we have two premises to satisfy. Only one truth assignment satisfies both premises, and this truth assignment also satisfies the conclusion; hence in this case logical entailment does hold.

p	q	p	q	$p \wedge q$
1	1	1	1	1
1	0	1	0	0
0	1	0	1	0
0	0	0	0	0

As a final example, let's return to the love life of the fickle Mary. Here is the problem from the course introduction. We know $(p \Rightarrow q)$, i.e., if Mary loves Pat, then Mary loves Quincy. We know $(m \Rightarrow p \vee q)$, i.e., if it is Monday, then Mary loves Pat or Quincy. Let's confirm that, if it is Monday, then Mary loves Quincy. We set up our table and evaluate our premises and our conclusion. Both premises are satisfied by the truth assignments on rows 1, 3, 5, 7, and 8; and we notice that those truth assignments make the conclusion true. Hence, the logical entailment holds.

m	p	q	$m \Rightarrow p \vee q$	$p \Rightarrow q$	$m \Rightarrow q$
1	1	1	1	1	1
1	1	0	1	0	0
1	0	1	1	1	1
1	0	0	0	1	0
0	1	1	1	1	1
0	1	0	1	0	1
0	0	1	1	1	1
0	0	0	1	1	1

The disadvantage of the truth table method is computational complexity. As mentioned above, the size of a truth table for a language grows exponentially with the number of proposition constants in the language. When the number of constants is small, the method works well. When the number is large, the method becomes impractical. Even for moderate sized problems, it can be tedious. Even for an application like Sorority World, where there are only 16 proposition constants, there are 65,536 truth assignments.

Over the years, researchers have proposed ways to improve the performance of truth table checking. We discuss some of these in Chapter 3. The other approach is to use symbolic manipulation (i.e. logical reasoning and proofs) in place of truth table checking. We discuss these methods in the next chapter.

Before we end this section, it is worth noting that logical entailment and satisfiability have an interesting relationship to each other. In particular, if a set Δ of sentences logically entails a sentence ϕ, then Δ together with the negation of ϕ must be unsatisfiable. The reverse is also true. If Δ and the negation of ϕ is unsatisfiable, then Δ must logically entail ϕ. This is guaranteed by a metatheorem called the unsatisfiability theorem.

Theorem 2.1 *Unsatisfiability Theorem:* $\Delta \models \phi$ if and only if $\Delta \cup \{\neg\phi\}$ is unsatisfiable.

Proof. Suppose that $\Delta \models \phi$. If a truth assignment satisfies Δ, then it must also satisfy ϕ. But then it cannot satisfy $\neg\phi$. Therefore, $\Delta \cup \{\neg\phi\}$ is unsatisfiable. Suppose that $\Delta \cup \{\neg\phi\}$ is unsatisfiable. Then every truth assignment that satisfies Δ must fail to satisfy $\neg\phi$, i.e., it must satisfy ϕ. Therefore, $\Delta \models \phi$. □

An interesting consequence of this result is that we can determine logical entailment by checking for unsatisfiability. This turns out to be useful in various logical proof methods.

RECAP

The syntax of propositional logic begins with a set of *proposition constants*. Compound sentences are formed by combining simpler sentences with logical operators. In the version of Propositional Logic used here, there are five types of compound sentences—negations, conjunctions, disjunctions,

implications, and biconditionals. A *truth assignment* for Propositional Logic is a mapping that assigns a truth value to each of the propositional constants in the language. The truth or falsity of compound sentences can be determined from a truth assignment using rules based on the five logical operators of the language. A truth assignment *i satisfies* a sentence if and only if the sentence is *true* under that truth assignment. A sentence is *valid* if and only if it is satisfied by *every* truth assignment. A sentence is *unsatisfiable* if and only if it is not satisfied by any truth assignment. A sentence is *contingent* if and only if it is both satisfiable and falsifiable, i.e., it is neither valid nor unsatisfiable. A sentence is *satisfiable* if and only if it is either valid or contingent. A sentence is *falsifiable* if and only if it is unsatisfiable or contingent. A set of sentences Δ *logically entails* a sentence ϕ (written $\Delta \models \phi$) if and only if every truth assignment that satisfies Δ also satisfies ϕ.

EXERCISES

2.1 Say whether each of the following expressions is a syntactically legal sentence of Propositional Logic.

(a) $p \wedge \neg p$
(b) $\neg p \vee \neg p$
(c) $\neg(q \vee r)\ \neg q \Rightarrow \neg\neg p$
(d) $(p \wedge q) \vee (p \neg\wedge q)$
(e) $p \vee \neg q \wedge \neg p \vee \neg q \Rightarrow p \vee q$

2.2 How many distinct truth assignments are possible for a language with n proposition constants $-2n, n^2, 2^n, 2^{n^2}, 2^{2^n}$?

2.3 A small company makes widgets in a variety of constituent materials (aluminum, copper, iron), colors (red, green, blue, grey), and finishes (matte, textured, coated). Although there are more than one thousand possible combinations of widget features, the company markets only a subset of the possible combinations. The following sentences are constraints that characterize the possibilities. Suppose that a customer places an order for a copper widget that is both green and blue with a matte coating. Your job is to determine which constraints are satisfied and which are violated.

(a) $aluminum \vee copper \vee iron$
(b) $aluminum \Rightarrow grey$
(c) $copper \wedge \neg coated \Rightarrow red$
(d) $coated \wedge \neg copper \Rightarrow green$
(e) $green \vee blue \Leftrightarrow \neg textured \wedge \neg iron$

2.4 A small company makes widgets in a variety of constituent materials (aluminum, copper, iron), colors (red, green, blue, grey), and finishes (matte, textured, coated). Although there are more than one thousand possible combinations of widget features, the company markets only a subset of the possible combinations. The sentences below are some constraints that

characterize the possibilities. Your job here is to select materials, colors, and finishes in such a way that *all* of the product constraints are satisfied. Note that there are multiple ways this can be done.

$$
\begin{aligned}
&\mathit{aluminum} \vee \mathit{copper} \vee \mathit{iron}\\
&\mathit{red} \vee \mathit{green} \vee \mathit{blue} \vee \mathit{grey}\\
&\mathit{aluminum} \Rightarrow \mathit{grey}\\
&\mathit{copper} \wedge \neg \mathit{coated} \Rightarrow \mathit{red}\\
&\mathit{iron} \Rightarrow \mathit{coated}
\end{aligned}
$$

2.5 Say whether each of the following sentences is valid, contingent, or unsatisfiable.

(*a*) $(p \Rightarrow q) \vee (q \Rightarrow p)$
(*b*) $p \wedge (p \Rightarrow \neg q) \wedge q$
(*c*) $(p \Rightarrow (q \wedge r)) \Leftrightarrow (p \Rightarrow q) \wedge (p \Rightarrow r)$
(*d*) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \wedge q) \Rightarrow r)$
(*e*) $(p \Rightarrow q) \wedge (p \Rightarrow \neg q)$
(*f*) $(\neg p \vee \neg q) \Rightarrow \neg(p \wedge q)$
(*g*) $((\neg p \Rightarrow q) \Rightarrow (\neg q \Rightarrow p)) \wedge (p \vee q)$
(*h*) $(\neg p \vee q) \Rightarrow (q \wedge (p \Leftrightarrow q))$
(*i*) $((\neg r \Rightarrow \neg p \wedge \neg q) \vee s) \Leftrightarrow (p \vee q \Rightarrow r \vee s)$
(*j*) $(p \wedge (q \Rightarrow r)) \Leftrightarrow ((\neg p \vee q) \Rightarrow (p \wedge r))$

2.6 Use the Truth Table Method to answer the following questions about logical entailment.

(*a*) $\{p \Rightarrow q \vee r\} \models (p \Rightarrow r)$
(*b*) $\{p \Rightarrow r\} \models (p \Rightarrow q \vee r)$
(*c*) $\{q \Rightarrow r\} \models (p \Rightarrow q \vee r)$
(*d*) $\{p \Rightarrow q \vee r, p \Rightarrow r\} \models (q \Rightarrow r)$
(*e*) $\{p \Rightarrow q \vee r, q \Rightarrow r\} \models (p \Rightarrow r)$

CHAPTER 3

Satisfiability

3.1 INTRODUCTION

The propositional satisfiability problem (often called SAT) is the problem of determining whether a set of sentences in Propositional Logic is satisfiable. The problem is significant both because the question of satisfiability is important in its own right and because many other questions in Propositional Logic can be reduced to that of propositional satisfiability. In practice, many automated reasoning problems in Propositional Logic are first reduced to satisfiability problems and then solved by using a satisfiability solver. Today, SAT solvers are commonly used in hardware design, software analysis, planning, mathematics, computer security analysis, and many other areas.

In this chapter, we look at several basic methods for solving SAT problems. Many modern SAT solvers are optimized variants of these basic methods.

3.2 TRUTH TABLE METHOD

Let $\Delta = \{p \vee q, p \vee \neg q, \neg p \vee q, \neg p \vee \neg q \vee \neg r, \neg p \vee r\}$. To determine whether Δ is satisfiable, we build the truth table below.

p	q	r	$p \vee q$	$p \vee \neg q$	$\neg p \vee q$	$\neg p \vee \neg q \vee \neg r$	$\neg p \vee r$	Δ satisfied
0	0	0	0	1	1	1	1	no
0	0	1	0	1	1	1	1	no
0	1	0	1	0	1	1	1	no
0	1	1	1	0	1	1	1	no
1	0	0	1	1	0	1	0	no
1	0	1	1	1	0	1	1	no
1	1	0	1	1	1	1	0	no
1	1	1	1	1	1	0	1	no

There is one row for each possible truth assignment. For each truth assignment, each of the sentences in Δ is evaluated. If any sentence evaluates to 0, then Δ as a whole is not satisfied by the truth assignment. If a satisfying truth assignment is found, then Δ is determined to be satisfiable. If no satisfying truth assignment is found, then ϕ is unsatisfiable. In this example, every row ends with Δ not satisfied. So the truth table method concludes that Δ is unsatisfiable.

The truth table method is complete because every truth assignment is checked. However, the method is impractical for all but very small problem instances. In our example with 3 propositions, there are $2^3 = 8$ rows. For a problem instance with 10 propositions, there are $2^{10} = 1024$ rows—still

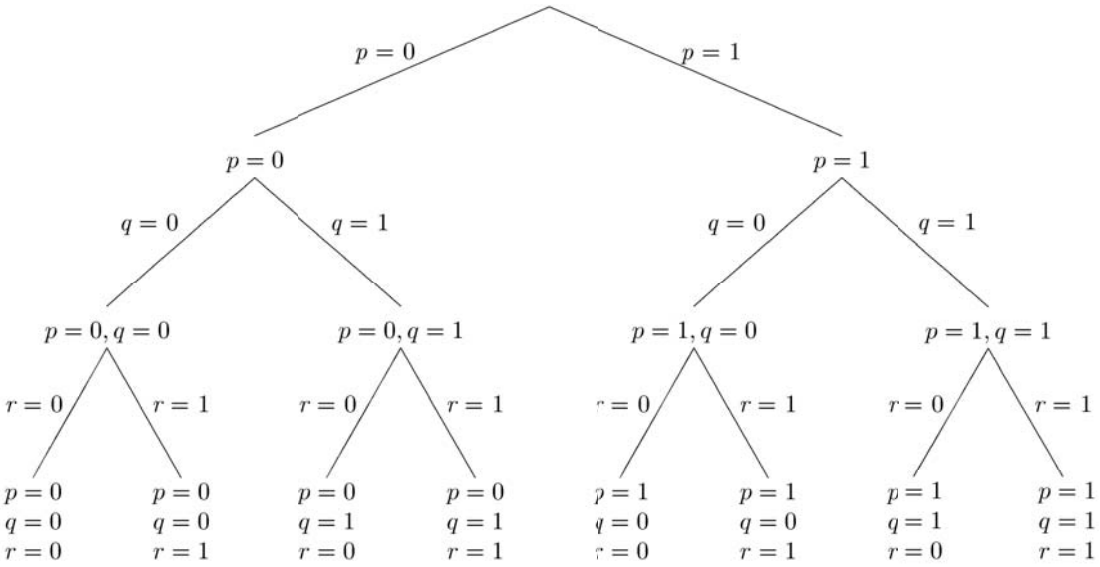
quite small for a modern computer. But as the number of propositions grow, the number of rows quickly overwhelm even the fastest computers. A more efficient method is needed.

3.3 BASIC BACKTRACKING SEARCH

Looking at the example in the preceding section, we can observe that, often times, a partial truth assignment is all that is required to determine whether the input Δ is satisfied.

For example, let's consider the partial assignment $\{p = 1, q = 0\}$. Even without the truth value for r, we can see that $\neg p \vee q$ evaluates to 0 and therefore Δ is not satisfied. Furthermore, we can conclude that no truth assignment that extends this partial assignment can satisfy Δ because the sentence $\neg p \vee q$ would always evaluate to 0 in every extension. So by determining whether the input in satisfied by a partial assignment, we can save the work of checking all extensions of the partial assignment. In this case, we can conclude that neither the truth assignment $\{p = 1, q = 0, r = 0\}$ nor the assignment $\{p = 1, q = 0, r = 1\}$ satisfies Δ. The saving is small in this case; but, as the number of propositions increases, the technique can eliminate many more truth assignments from consideration. Basic backtracking search is a method that allows us to realize these savings by systematically searching partial assignments.

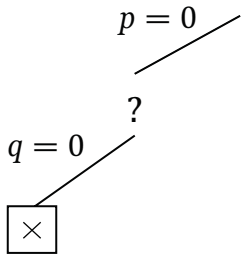
To search the space of truth assignments systematically, both partial and complete, we can set the propositions one at a time. The process can be visualized as a tree search where each branch sets the truth value of one proposition, each interior node is a partial truth assignment, and each leaf node is a complete truth assignment. Below is a tree whose fringe represents all complete truth assignments.



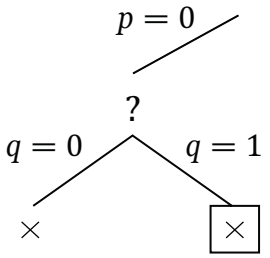
In basic backtracking search, we start at the root of the tree (the empty assignment where nothing is known) and work our way down a branch. At each node, we check whether the input Δ is satisfied. If Δ is satisfied, we move down the branch by choosing a truth value for a currently unassigned proposition. If Δ is falsified, then, knowing that all the (partial) truth assignments further down the branch also falsify Δ, we *backtrack* to the most recent decision point and proceed down a different branch. At some point, we will either find a truth assignment that satisfies all the sentences of Δ or determine that none exists.

Let's look at an example. At each step, we show the part of the tree explored so far. A boxed node is the current node. An $\times$ mark at a node indicates that the truth assignment at that node falsifies at least one sentence in Δ. A $\checkmark$ indicates that the truth assignment at that node satisfies all the sentences in Δ. A ? indicates that the partial truth assignment at the node neither satisfies nor falsifies Δ.

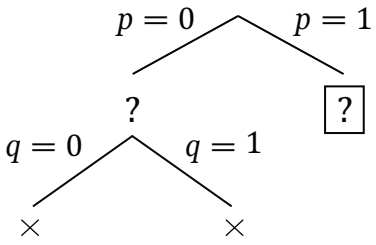
Let's start with the assumption that p is false. This leads to the partial tree shown below.



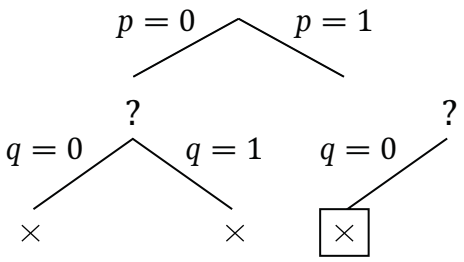
Now, let's assume that q is false. In this case, Δ is falsified, and the current branch is closed.



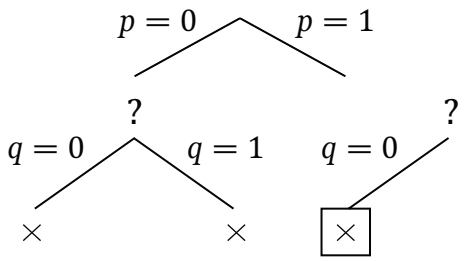
Since the current branch is closed, we backtrack to the most recent choice point where another branch can be taken. This time, let's assume that q is true.



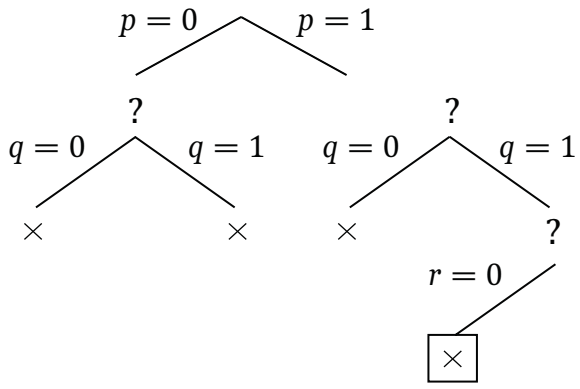
Again, Δ is falsified, and the current branch is closed. Again we backtrack to the most recent choice point where another branch can be taken. Let p be true.



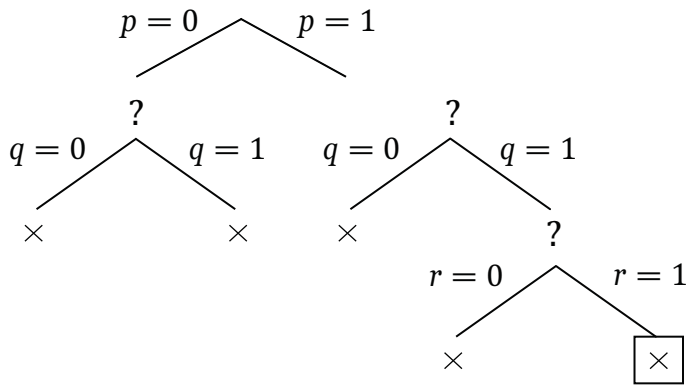
Let q be false.



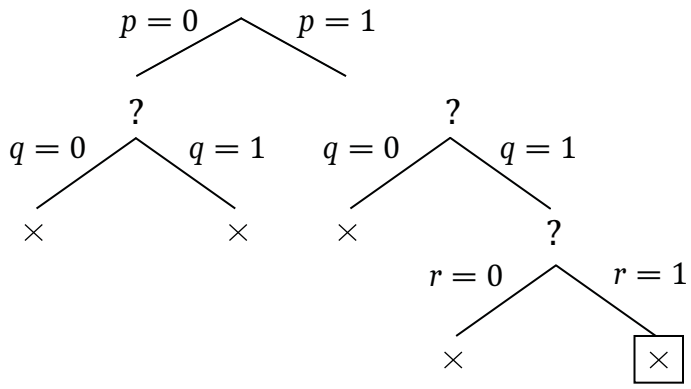
Again, our sentences are falsified, and we backtrack. Let q be true.



Let r be false.

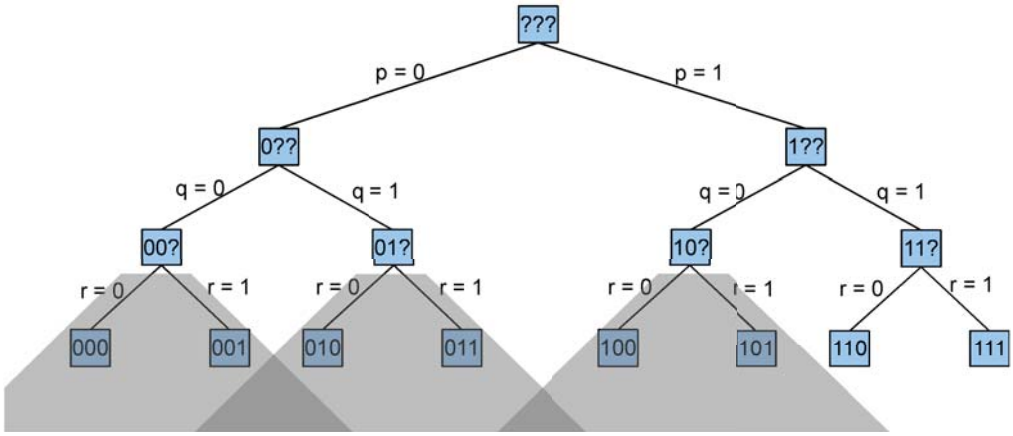


Our sentences are falsified by this assignment, and we backtrack one last time. Let r be true.



Once again, our sentences are falsified. Since all branches have been explored and closed, the method determines that Δ is unsatisfiable.

Looking at the full search tree compared to the portion explored by the basic backtracking search, we can see that the greyed out subtrees are all pruned from the search space. In this particular example, the savings are not spectacular. But in a bigger example with more propositions, the pruned subtrees can be much bigger.



3.4 SIMPLIFICATION AND UNIT PROPAGATION

In this section, we consider two optimizations of basic backtracking search—*simplification* and *unit propagation*. In order for these methods to work, we assume that our sentences have been transformed into logically equivalent disjunctions (using a method like the one described in Chapter 5). As we choose the truth values of some propositions in a partial truth assignment for these disjunctions, there are opportunities to simplify the set of sentences that need to be checked.

Suppose, for example, a proposition p has been assigned the truth value 1. (1) Each disjunction containing a disjunct p may be ignored because it must already be satisfied by the current partial assignment. (2) Each disjunction ϕ containing a disjunct $\neg p$ may be modified (call the result ϕ') by removing from it all occurrences of the disjunct $\neg p$ because under all truth assignments where p has truth value 1, ϕ holds if and only if ϕ' holds.

If a proposition p has been assigned the truth value 0, we can simplify our sentences analogously. (1) Each disjunction containing a disjunct $\neg p$ may be ignored because it must already be satisfied by the current partial assignment. (2) Each disjunction ϕ containing a disjunct p may be modified by removing from it all occurrences of the disjunct p.

Consider once again the example in the preceding section. Under the partial assignment $p = 1$, we can simplify our sentences as shown below. The first two sentences are dropped, and the literal $\neg p$ is dropped from the other three sentences.

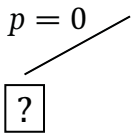
Original	Simplified
$p \vee q$	–
$p \vee \neg q$	–
$\neg p \vee q$	q
$\neg p \vee \neg q \vee \neg r$	$\neg q \vee \neg r$
$\neg p \vee r$	r

While simplifying sentences is helpful in and of itself, the real value of simplification is that it enables a further optimization that can drastically decrease the search space.

In the course of the backtracking search, if we see a sentence that consists of single atom, say p, we know that the only possible satisfying assignments further down the branch must set p to true. In this case, we can fix p to be true and ignore the subbranch that sets p to false. Similarly, when we encounter a sentence that consists of a single negated atom, say $\neg p$, we can fix p to be false and ignore the other subbranch. This optimization is called *unit propagation* because sentences of the form p or $\neg p$ (a simple sentence or the negation of a simple sentence) are called *units*.

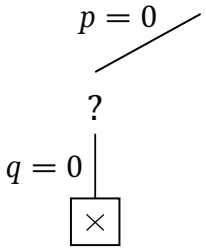
Let's redo example 1 with formula simplification and unit propagation. As we proceed through this example, we illustrate each step with the search tree on that step (on the left) and the simplified sentence set (in the table on the right).

To start, let p be false.



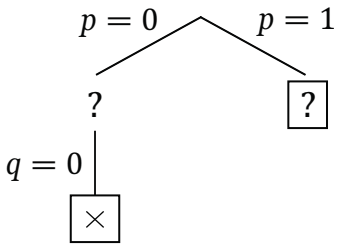
Original	Simplified
$p \vee q$	q
$p \vee \neg q$	$\neg q$
$\neg p \vee q$	–
$\neg p \vee \neg q \vee \neg r$	–
$\neg p \vee r$	–

In the simplified set of sentences, we have the unit $\neg q$, so we fix q to be false (unit propagation). (We also have the unit q, so we could have fixed q to true. The final result is the same in either case.)



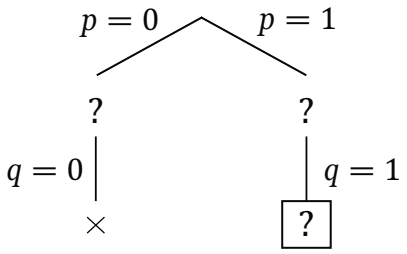
Original	Simplified
$p \vee q$	*false*
$p \vee \neg q$	–
$\neg p \vee q$	–
$\neg p \vee \neg q \vee \neg r$	–
$\neg p \vee r$	–

Δ is falsified, so we backtrack to the most recent decision point, all the way back at the root. Let p be true.



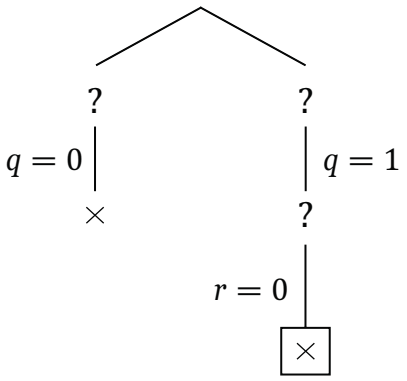
Original	Simplified
$p \vee q$	–
$p \vee \neg q$	–
$\neg p \vee q$	q
$\neg p \vee \neg q \vee \neg r$	$\neg q \vee \neg r$
$\neg p \vee r$	r

In the simplified set of sentences, we have the unit q so we do unit propagation, fixing q to be true. (We could also have performed unit propagation using the other unit r.)



Original	Simplified
$p \vee q$	–
$p \vee \neg q$	–
$\neg p \vee q$	–
$\neg p \vee \neg q \vee \neg r$	$\neg r$
$\neg p \vee r$	r

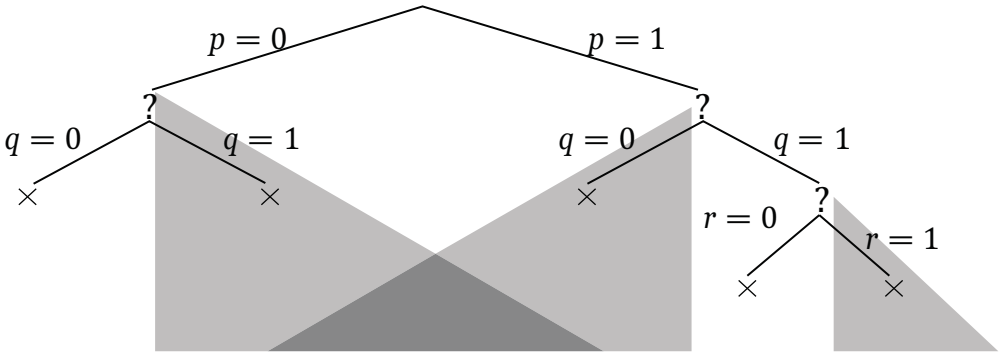
In the simplified set of sentences, we have the unit $\neg r$ so we do unit propagation, fixing r to be false.



Original	Simplified
$p \vee q$	–
$p \vee \neg q$	–
$\neg p \vee q$	–
$\neg p \vee \neg q \vee \neg r$	–
$\neg p \vee r$	*false*

Δ is falsified on this branch, so the branch is closed. All branches are closed, so the method determines that Δ is unsatisfiable.

Compared to the tree explored by the basic backtracking search, we see that the greyed out subtrees are pruned away from the search space.



3.5 DPLL

The *Davis-Putnam-Logemann-Loveland method* (DPLL) is a classic method for SAT solving. It is essentially backtracking search along with unit propagation and pure literal elimination (described in

Chapter 8). Most modern, complete SAT solvers are based on DPLL, with additional optimizations not discussed here. These SAT solvers are routinely used to solve SAT problems with large numbers of propositions.

3.6 GSAT

Many practical SAT solvers based on complete search for models are routinely used to solve satisfiability problems of significant size. However, some problem instances are still impractical to solve using complete search for models.

In these problem instances, it is impractical to exhaustively eliminate every truth assignment as a possible model. In such situations, one may consider incomplete search methods - methods that answer correctly when they can but sometimes fail to answer. The basic idea is to sample a subset of truth assignments. If a satisfying assignment is found, then we can conclude that the input is satisfiable. If no satisfying assignment is found, then we don't know whether the input is satisfiable. In practice, modern incomplete SAT solvers tend to be very good at finding the satisfying assignments in a reasonable amount of time when they exist, but they still lack the ability to answer definitively that an input is unsatisfiable.

The first idea may be to uniformly sample a number of truth assignments from the space of all truth assignments. However, in large problem instances with few satisfying assignments, one would expect to take a very long time before happening upon a satisfying assignment. What enables incomplete SAT solvers to work efficiently is the use of heuristics that frequently steer the search toward satisfying assignments. One family of incomplete SAT solvers uses local search heuristics to look for a truth assignment that maximizes the number of sentences it satisfies. Clearly, a set of sentences is satisfiable if and only if it is satisfied by an optimal truth assignment (optimal in terms of maximizing the number of sentences satisfied).

We look at one particular method called GSAT in more detail. The GSAT method starts with an arbitrary truth assignment. Then GSAT moves from one truth assignment to the next by flipping one of the propositions to achieve the greatest increase in the number of sentences satisfied. The search stops when it reaches a truth assignment such that the number of sentences satisfied cannot be increased by flipping the truth value of one proposition (in other words, a locally optimal truth assignment is reached).

Consider the set of sentences $\{p \vee q \vee r, \neg p, \neg q, \neg r\}$. This set is clearly unsatisfiable. What follows is one possible execution of GSAT on this case. For each step, we show the truth assignment for the proposition constants; we show the truth values for the sentences in our set; and we show how many sentences are satisfied. (In this case, we need this number to be 4 in order for the set as a whole to be satisfied.) We start with an arbitrary assignment in which all proposition constants are true.

p	q	r	$p \vee q \vee r$	$\neg p$	$\neg q$	$\neg r$	# satisfied
1	1	1	1	0	0	0	1

Only one sentence is satisfied by this assignment. However, we can improve matters by flipping the value of p.

p	q	r	$p \vee q \vee r$	$\neg p$	$\neg q$	$\neg r$	# satisfied
0	1	1	1	1	0	0	2

Still not satisfied, but we can improve matters by flipping the value of q.

p	q	r	$p \vee q \vee r$	$\neg p$	$\neg q$	$\neg r$	# satisfied
0	0	1	1	1	1	0	3

Unfortunately, at this point, we are stuck. No flipping of a single proposition value can increase the number of sentences satisfied. The search terminates without finding a satisfying assignment. Indeed, the input set of sentences is unsatisfiable.

Now, let's see what happens when we apply the method to a set or sentences that is satisfiable. Our set in this case is $\{p, \neg p \vee q, \neg p \vee r\}$. As before, we start with an arbitrary assignment.

p	q	r	p	$\neg p \vee q$	$\neg p \vee r$	# satisfied
1	0	0	1	0	0	1

Flipping the value of q improves matters.

p	q	r	p	$\neg p \vee q$	$\neg p \vee r$	# satisfied
1	1	0	1	1	0	2

Flipping the value of r improves matters further.

p	q	r	p	$\neg p \vee q$	$\neg p \vee r$	# satisfied
1	1	1	1	1	1	3

Since all three sentences are satisfied, the search terminates, concluding that the set is satisfiable.

Unfortunately, local search of this sort is not guaranteed to be complete—the search may fail to find a satisfying assignment even when one exists. Depending on the starting assignment and the arbitrary choices in breaking ties between flipping one proposition vs. another, the GSAT method may fail to find a satisfying assignment for a satisfiable set of input sentences.

To see an example of incompleteness, let's look a different execution of GSAT on the same set of sentences as above. We begin with the same initial assignment as before.

p	q	r	p	$\neg p \vee q$	$\neg p \vee r$	# satisfied
1	0	0	1	0	0	1

This time, instead of flipping the value of q, let's flip the value of p.

p	q	r	p	$\neg p \vee q$	$\neg p \vee r$	# satisfied
0	0	0	0	1	1	2

Unfortunately, at this point, we are stuck. No flipping of a proposition value can increase the number of sentences satisfied. Although a satisfying assignment exists, the search terminates without finding it.

To avoid becoming stuck without finding a satisfying assignment, one can modify the search procedure by allowing some combination of the following.

- Restarting at a random truth assignment (randomized restarts).
- Flipping a proposition to move to a truth assignment that satisfies the same number of sentences (plateau moves).
- Flipping a random proposition regardless of whether the move increases the number of satisfied sentences (noisy move).

Active research and engineering efforts continue in developing search methods that can find a satisfying assignment more quickly (when one exists).

RECAP

The *propositional satisfiability problem* (often called *SAT*) is the problem of determining whether a set of sentences in Propositional Logic is satisfiable. Many other questions in Propositional Logic (such as logical entailment) can be reduced to that of propositional satisfiability. The Truth Table Method for testing satisfiability checks every truth assignment one by one to see whether any truth assignment satisfies the input sentences. The method is straightforward but prohibitively expensive for all but the smallest problems. *Backtracking* explores the partial truth assignments using a systematic tree search. When a partial assignment is found to falsify at least one input sentence, all extensions of the partial assignment can be eliminated from the search space. *Simplification* is a technique that simplifies the input set of sentences based on the current partial truth assignment. *Unit propagation* eliminates from the search space truth assignments that disagree with *units*—a simple sentence or the negation of a simple sentence—in the simplified set of input sentences. Backtracking search with simplification and unit propagation is a practical method for testing satisfiability. The method is *complete* in the sense that it is guaranteed to terminate with the correct answer. Most modern, complete SAT solvers are highly optimized versions of this basic method. Another class of SAT solvers are the *incomplete* SAT solvers. Instead of exhausting the space of truth assignments, incomplete SAT solvers use heuristics to quickly identify satisfying assignments. In practice, incomplete SAT solvers tend to be very good at quickly finding satisfying assignments, but they lack the ability to answer definitely that an input is unsatisfiable.

EXERCISES

3.1 Using Backtracking to determine whether the set of formulas $\{\neg p \vee q, r \vee p\}$ is satisfiable, which of the following is a possible assignment to explore after the partial assignment $\{p{=}1, q{=}0\}$?

(*a*) $\{p{=}1, q{=}0, r{=}1\}$
(*b*) $\{p{=}1, q{=}0, r{=}0\}$
(*c*) $\{p{=}1, q{=}1\}$

3.2 Under the partial truth assignment $\{p{=}0, q{=}1\}$, the set of formulas $\{p \vee \neg q \vee r\}$ simplifies to which of the following?

(*a*) $\{r\}$
(*b*) $\{p \vee \neg q\}$
(*c*) $\{p \vee \neg q \vee r\}$
(*d*) $\{\}$

3.3 Under the partial truth assignment $\{p{=}0, q{=}1\}$, the set of formulas $\{p \vee q \vee r\}$ simplifies to which of the following?

(*a*) $\{r\}$
(*b*) $\{\neg p \vee q \vee r\}$
(*c*) $\{p \vee q \vee r\}$
(*d*) $\{q \vee r\}$
(*e*) $\{\}$

3.4 Using Backtracking with Simplification and Unit Propagation to determine whether the set of formulas $\{p \vee r, \neg q\}$ is satisfiable, which of the following are possible assignments that may be explored in the process?

(*a*) $\{p{=}1, q{=}1\}$
(*b*) $\{p{=}1, q{=}0\}$
(*c*) $\{p{=}0, q{=}1\}$
(*d*) $\{p{=}0, q{=}0\}$
(*e*) $\{p{=}0\}$

3.5 Using Backtracking to determine whether the set of formulas $\{\neg p \vee q, r \vee p\}$ is satisfiable, which of the following are possible assignments that may be explored after the partial assignment $\{p{=}1, q{=}0\}$?

(*a*) $\{p{=}1, q{=}1\}$
(*b*) $\{p{=}1, r{=}1\}$
(*c*) $\{p{=}0\}$
(*d*) $\{p{=}0, q{=}1\}$

CHAPTER 4

Propositional Proofs

4.1 INTRODUCTION

Checking logical entailment with truth tables has the merit of being conceptually simple. However, it is not always the most practical method. The number of truth assignments of a language grows exponentially with the number of logical constants. When the number of logical constants in a propositional language is large, it may be impossible to process its truth table.

Proof methods provide an alternative way of checking logical entailment that addresses this problem. In many cases, it is possible to create a proof of a conclusion from a set of premises that is much smaller than the truth table for the language; moreover, it is often possible to find such proofs with less work than is necessary to check the entire truth table.

We begin this lesson with a discussion of simple, linear proofs. We then move on to hierarchically structured proofs, where proofs can be nested inside of other proofs. Once we have seen both linear and structured proofs, we show how they are combined in the popular Fitch proof system. We finish with definitions for soundness and completeness—the standards by which proof systems are judged.

4.2 LINEAR PROOFS

As we saw in the introductory lesson, the essence of logical reasoning is symbolic manipulation. We start with premises, apply rules of inference to derive conclusions, stringing together such derivations to form logical proofs. The idea is simple. Getting the details right requires a little care. Let's start by defining schemas and rules of inference.

A *schema* is an expression satisfying the grammatical rules of our language except for the occurrence of *metavariables* (written here as Greek letters) in place of various subparts of the expression. For example, the following expression is a schema with metavariables ϕ and ψ.

$$\phi \Rightarrow \psi$$

A *rule of inference* is a pattern of reasoning consisting of some schemas, called *premises*, and one or more additional schemas, called *conclusions*. Rules of inference are often written as shown below. The schemas above the line are the premises, and the schemas below the line are the conclusions. The rule in this case is called *Implication Elimination* (or IE), because it eliminates the implication from the first premise.

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}$$

An instance of a rule of inference is the rule obtained by consistently substituting sentences for the metavariables in the rule. For example, the following is an instance of Implication Elimination.

$$\frac{p \Rightarrow q \quad p}{q}$$

If a metavariable occurs more than once, the same expression must be used for every occurrence. For example, in the case of Implication Elimination, it would not be acceptable to replace one occurrence of ϕ with one expression and the other occurrence of ϕ with a different expression.

Note that the replacement can be an arbitrary expression so long as the result is a legal expression. For example, in the following instance, we have replaced the variables by compound sentences.

$$\frac{(p \Rightarrow q) \Rightarrow (q \Rightarrow r) \quad (p \Rightarrow q)}{(q \Rightarrow r)}$$

Remember that there are infinitely many sentences in our language. Even though we start with finitely many propositional constants (in a propositional vocabulary) and finitely many operators, we can combine them in arbitrarily many ways. The upshot is that there are infinitely many instances of any rule of inference involving metavariables.

A rule *applies* to a set of sentences if and only if there is an instance of the rule in which all of the premises are in the set. In this case, the conclusions of the instance are the results of the rule application.

For example, if we had a set of sentences containing the sentence p and the sentence $(p \Rightarrow q)$, then we could apply Implication Elimination to derive q as a result. If we had a set of sentences containing the sentence $(p \Rightarrow q)$ and the sentence $(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$, then we could apply Implication Elimination to derive $(q \Rightarrow r)$ as a result.

Note that it is okay to have rules of inference with no premises. Rules with no premises are sometimes called *axiom schemas*. Axiom schemas are usually written without the horizontal line used in displaying ordinary rules of inference. Here are some examples.

The *Implication Creation schema* (IC), when used in combination with Implication Elimination, allows us to infer implications.

$$\phi \Rightarrow (\psi \Rightarrow \phi)$$

The *Implication Distribution schema* (ID) allows us to distribute one implication over another. If a sentence ϕ implies that ψ implies χ, then, if ϕ implies ψ, ϕ implies χ.

$$(\phi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi))$$

The *Contradiction Realization schema* (CR) permits us to infer a sentence if the negation of that sentence implies some sentence and its negation.

$$(\neg\phi \Rightarrow \psi) \Rightarrow ((\neg\phi \Rightarrow \neg\psi) \Rightarrow \phi)$$

These three axiom schemas together form the *Mendelson axiom schemas* for Propositional Logic. The interesting thing about the Mendelson axiom schemas is that, together with implication Elimination, they alone are sufficient to prove all logical consequences from any set of premises.

By writing down premises, writing instances of axiom schemas, and applying rules of inference, it is possible to derive conclusions that cannot be derived in a single step. This idea of stringing things together in this way leads to the notion of a linear proof.

A *linear proof* of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either (1) a premise, (2) an instance of an axiom schema, or (3) the result of applying a rule of inference to earlier items in sequence.

Here is an example. We start our proof by writing out our premises. We believe p; we believe $(p \Rightarrow q)$; and we believe that $(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$. Using Implication Elimination on the first premise and the second premise, we derive q. Applying Implication Elimination to the second premise and the third premise, we derive $(q \Rightarrow r)$. Finally, we use the derived premises on lines 4 and 5 to arrive at our desired conclusion.

1.	p	Premise
2.	$p \Rightarrow q$	Premise
3.	$(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$	Premise
4.	q	IE: 2, 1
5.	$q \Rightarrow r$	IE: 3, 2
6.	r	IE: 5, 4

Here is a somewhat more complicated example, which illustrates the use of axiom schemas. Whenever p is true, q is true. Whenever q is true, r is true. With these as premises, we can prove that, whenever p is true, r is true. On line 3, we write down an instance of Implication Creation. From this conclusion and premise 2, we use Implication Elimination to derive the sentence $(p \Rightarrow (q \Rightarrow r))$. On line 5, we write down an instance of Implication Distribution. We combine the conclusions from lines 4 and 5 to derive the sentence on line 6. Finally, we use Implication Elimination once again to derive the desired conclusion. Not an easy proof, but it works. Later in this chapter, we see how to produce a simpler proof of this result.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$	IC
4.	$p \Rightarrow (q \Rightarrow r)$	IE: 3, 2
5.	$p \Rightarrow (q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$	ID
6.	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	IE: 5, 4
7.	$p \Rightarrow r$	IE: 6, 1

Let R be a set of rules of inference. If there exists a proof of a sentence ϕ from a set Δ of premises using the rules of inference in R, we say that ϕ is *provable* from Δ using R. We usually write this as $\Delta \vdash_R \phi$, using the provability operator $\vdash$ (which is sometimes called *single turnstile*). If the set of rules is clear from context, we sometimes drop the subscript, writing just $\Delta \vdash \phi$.

The Mendelson System, mentioned earlier, is a well-known proof system for Propositional Logic. It consists of the Implication Elimination rule of inference and the three axiom schemas we saw earlier. The Mendelson System is interesting in that is sufficient to prove all logical consequences from any set of premises expressed using only $\neg$ and $\Rightarrow$. And, as we shall see later, this subset of sentences is interesting because, for every sentence in Propositional Logic, there is a logically equivalent sentence written in terms of these two operators. Unfortunately, proofs in the Mendelson system can be complicated. The good news is that we can eliminate much of this complexity by moving from linear proofs to structured proofs.

4.3 STRUCTURED PROOFS

Structured proofs are similar to linear proofs in that they are sequences of reasoning steps. However, they differ from linear proofs in that they have more structure. In particular, sentences can be grouped into subproofs nested within outer superproofs.

As an example, consider the structured proof shown below. It resembles a linear proof except that we have grouped the sentences on lines 3 through 5 into a subproof within our overall proof.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$\quad p$	Assumption
4.	$\quad q$	Implication Elimination: 3, 1
5.	$\quad r$	Implication Elimination: 4, 2
6.	$p \Rightarrow r$	Implication Introduction: 3, 5

The main benefit of structured proofs is that they allow us to prove things more easily than with ordinary rules of inference. In structured proofs, we can make assumptions within subproofs; we can prove conclusions from those assumptions; and, from those derivations, we can derive implications outside of those subproofs, with our assumptions as antecedents and our conclusions as consequents.

The structured proof above illustrates this. On line 3, we begin a subproof with the assumption that p is true. Note that p is not a premise in the overall problem. In a subproof, we can make whatever assumptions that we like. From p, we derive q using the premise on line 1; and, from that q, we prove r using the premise on line 2. That terminates the subproof. Finally, from this subproof, we derive $(p \Rightarrow r)$ in the outer proof. Given p, we can prove r; and so we know $(p \Rightarrow r)$. The rule used in this case is called Implication Introduction, or II for short.

As this example illustrates, there are three basic operations involved in creating structured proofs—(1) making assumptions, (2) using ordinary rules of inference to derive conclusions, and (3) using structured rules of inference to derive conclusions outside of subproofs. Let's look at each of these operations in turn.

In a structured proof, it is permissible to make an arbitrary assumption in any nested proof. The assumptions need not be members of the initial premise set. Note that such assumptions cannot be used directly outside of the subproof, only as conditions in derived implications, so they do not contaminate the superproof or any unrelated subproofs.

For example, in the proof we just saw, we used this assumption operation in the nested subproof even though p was not among the given premises.

An ordinary rule of inference applies to a particular subproof of a structured proof if and only if there is an instance of the rule in which all of the premises occur earlier in the subproof or in some superproof of the subproof. Importantly, it is not permissible to use sentence in subproofs of that subproof or in other subproofs of its superproofs.

For example, in the structured proof we have been looking at, it is okay to apply Implication Elimination to 1 and 3. And it is okay to use Implication Elimination on lines 2 and 4.

However, it is *not* acceptable to use a sentence from a nested proof in applying an ordinary rule of inference to an outer proof, as in the two malformed proofs shown below.

The last line of the malformed proof shown below gives an example of this. It is *not* permissible to use Implication Elimination as shown here because it uses a conclusion from a subproof as a premise in an application of an ordinary rule of inference in its superproof.

	1.	$p \Rightarrow q$	Premise
	2.	$q \Rightarrow r$	Premise
	3.	$\quad p$	Assumption
	4.	$\quad q$	Implication Elimination: 1, 3
	5.	$\quad r$	Implication Elmination: 2, 4
	6.	$p \Rightarrow r$	Implication Introduction: 3, 5
Wrong!	7.	r	Implication Elimination: 2, 4 **Wrong!**

The malformed proof shown below is another example. Here, line 8 is illegal because line 4 is not in the current subproof or a superproof of the current subproof.

	1.	$p \Rightarrow q$	Premise	
	2.	$q \Rightarrow r$	Premise	
	3.	$\quad p$	Assumption	
	4.	$\quad q$	Implication Elmination: 1, 3	
	5.	$\quad r$	Implication Elmination: 2, 4	
	6.	$p \Rightarrow r$	Implication Introduction: 3, 5	
	7.	$\quad \neg r$	Assumption	
Wrong!	8.	$\quad r$	Implication Elmination: 2, 4	**Wrong!**
	9.	$\neg r \Rightarrow r$	Implication Introduction: 7, 8	

Currectly utilizing results derived in subproofs is the responsibility of a new type of rule of inference. Like an ordinary rule of inference, a structured rule of inference is a pattern of reasoning consisting of one or more premises and one or more conclusions. As before, the premises and conclusions can be schemas. However, the premises can also include conditions of the form $\phi \vdash \psi$, as in the following example. The rule in this case is called Implication Introduction, because it allows us to introduce new implications.

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

Once again, looking at the correct example, we see that there is an instance of Implication Introduction (shown here on the left) in deriving line 6 from the subproof on lines 3–5. The application of Implication Introduction in the malformed proof is also okay in deriving line 7 from the subproof in lines 4–6.

Finally, we define a *structured proof* of a conclusion from a set of premises to be a sequence of (possibly nested) sentences terminating in an occurrence of the conclusion at the *top level* of the proof. Each step in the proof must be either (1) a premise (at the top level) or an assumption (other than at the top level) or (2) the result of applying an ordinary or structured rule of inference to earlier items in the current subproof or a superproof of the current subproof.

4.4 FITCH

Fitch is a proof system that is particularly popular in the Logic community. It is as powerful as many other proof systems and is far simpler to use. Fitch achieves this simplicity through its support for structured proofs and its use of structured rules of inference in addition to ordinary rules of inference.

Fitch has ten rules of inference in all. Nine of these are ordinary rules of inference. The other rule (Implication Introduction) is a structured rule of inference.

And Introduction (shown below on the left) allows us to derive a conjunction from its conjuncts. If a proof contains sentences ϕ_1 through ϕ_n, then we can derive their conjunction. *And Elimination* (shown below on the right) allows us to derive conjuncts from a conjunction. If we have the conjunction of ϕ_1 through ϕ_n, then we can infer any of the conjuncts.

And Introduction

$$\frac{\begin{array}{l}\phi_1 \\ \ldots \\ \phi_n\end{array}}{\phi_1 \wedge \ldots \wedge \phi_n}$$

And Elimination

$$\frac{\phi_1 \wedge \ldots \wedge \phi_n}{\phi_i}$$

Or Introduction allows us to infer an arbitrary disjunction so long as at least one of the disjuncts is already in the proof. *Or Elimination* is a little more complicated than And Elimination. Since we do not know which of the disjuncts is true, we cannot just drop the $\vee$. However, if we know that every disjunct entails some sentence, than we can infer that sentence even if we do not know which disjunct is true.

Or Introduction

$$\frac{\phi_i}{\phi_1 \vee \ldots \vee \phi_n}$$

Or Elimination

$$\frac{\begin{array}{l}\phi_1 \vee \ldots \vee \phi_n \\ \phi_1 \Rightarrow \psi \\ \ldots \\ \phi_n \Rightarrow \psi\end{array}}{\psi}$$

Negation Introduction allows us to derive the negation of a sentence if it leads to a contradiction. If we believe $(\phi \Rightarrow \psi)$ and $(\phi \Rightarrow \neg\psi)$, then we can derive that phi is false. *Negation Elimination* allows us to delete double negatives.

Negation Introduction

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi \Rightarrow \neg\psi\end{array}}{\neg\phi}$$

Negation Elimination

$$\frac{\neg\neg\phi}{\phi}$$

Implication Introduction is the structured rule we saw in Section 4.3. If, by assuming ϕ, we can derive ψ, then we can derive $(\phi \Rightarrow \psi)$. *Implication Elimination* is the first rule we saw in Section 4.2.

Implication Introduction

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

Implication Elimination

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}$$

Biconditional Introduction allows us to infer a biconditional (also called an equivalence) from an implication and its converse. *Biconditional Elimination* goes the other way, allowing us to infer two implications from a single biconditional.

Biconditional Introduction

$$\frac{\phi \Rightarrow \psi \quad \psi \Rightarrow \phi}{\phi \Leftrightarrow \psi}$$

Biconditional Elimination

$$\frac{\phi \Leftrightarrow \psi}{\phi \Rightarrow \psi \quad \psi \Rightarrow \phi}$$

In addition to these rules of inference, it is common to include in Fitch proof editors several additional operations that are of use in constructing Fitch proofs. For example, the Premise operation allows one to add a new premise to a proof. The Reiteration operation allows one to reproduce an earlier conclusion for the sake of clarity.

The Fitch rules are all fairly simple to use; and, as we discuss in the next section, they are all that we need to prove any result that follows logically from any set of premises. Unfortunately, figuring out which rules to use in any given situation is not always that simple. Fortunately, there are a few tricks that help in many cases.

(1) To prove a conjunction, prove the conjuncts and then use And Introduction to produce the desired conjunction.

(2) To prove an implication, i.e. a sentence of the form $\phi \Rightarrow \psi$, assume ϕ, thereby starting a subproof; try to prove ψ; and, if successful, use Implication Introduction to discharge the subproof and prove the desired implication.

(3) To prove a negation, assume the target of the negated sentence to produce two implications with contradictory conclusions and then use negation introduction to produce the desired negation.

These particular tricks are very useful, but there are many more. The best way to become adept at producing proofs is to start by proving simple things (e.g. various valid sentences) and then build up incrementally to more complex conclusions.

4.5 SOUNDNESS AND COMPLETENESS

In talking about logic, we now have two notions—logical entailment and provability. A set of premises logically entails a conclusion if and only if every truth assignment that satisfies the premises also satisfies the conclusion. A sentence is provable from a set of premises if and only if there is a finite proof of the conclusion from the premises.

The concepts are quite different. One is based on truth assignments; the other is based on symbolic manipulation of expressions. Yet, for the proof systems we have been examining, they are closely related.

We say that a proof system is *sound* if and only if every provable conclusion is logically entailed. In other words, if $\Delta \vdash \phi$, then $\Delta \models \phi$. We say that a proof system is *complete* if and only if every logical conclusion is provable. In other words, if $\Delta \models \phi$, then $\Delta \vdash \phi$.

The Mendelson System is both sound and complete for all problems in which the premises and conclusions are expressed using only $\neg$ and $\Rightarrow$. In other words, for this system, logical entailment and provability are identical.

The Fitch system is sound and complete for the full language. In other words, for this system, logical entailment and provability are also identical. An arbitrary set of sentences Δ logically entails an arbitrary sentence ϕ if and only if ϕ is provable from Δ using Fitch.

The upshot of this result is significant. On large problems, the proof method often takes fewer steps than the truth table method. (Disclaimer: In the worst case, the proof method may take just as many or more steps to find an answer as the truth table method.) Moreover, proofs are usually much smaller than the corresponding truth tables. So writing an argument to convince others does not take as much space.

RECAP

A *pattern* is an expression satisfying the grammatical rules of our language except for the occurrence of *metavariables* in place of various subparts of the expression. An *instance* of a schema is the expression obtained by substituting expressions of the appropriate sort for the metavariables in the pattern so that the result is a legal expression. A *rule of inference* is a pattern of reasoning consisting of one set of patterns, called *premises*, and a second set of schemas, called *conclusions*. An *axiom schema* is effectively a rule of inference with no premises. A *linear proof* of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either (1) a premise, (2) an instance of an axiom schema, or (3) the result of applying a rule of inference to earlier items in sequence. If there exists a proof of a sentence ϕ from a set Δ of premises and the axiom schemas and rules of inference of a proof system, then ϕ is said to be *provable* from Δ (written as $\Delta \vdash \phi$) and is called a *theorem* of Δ. *Fitch* is a powerful yet simple proof system that supports structured proofs. A proof system is *sound* if and only if every provable conclusion is logically entailed. A proof system is *complete* if and only if every logical conclusion is provable. Mendelson is sound and complete for all sentences that can be written in terms of $\neg$ and $\Rightarrow$. Fitch is sound and complete for the full language.

EXERCISES

4.1 Given $\neg p \Rightarrow q$ and $\neg p \Rightarrow \neg q$, use the Mendelson System to prove p.

4.2 Given $\neg\neg p$, use the Mendelson System to prove p.

4.3 Use the Mendelson System to prove $p \Rightarrow p$.

4.4 Given $p \Rightarrow q$ and $m \Rightarrow p \vee q$, use the Fitch System to prove $m \Rightarrow q$.

4.5 Given $p \Rightarrow (q \Rightarrow r)$, use the Fitch System to prove $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$.

4.6 Use the Fitch System to prove $p \Rightarrow (q \Rightarrow p)$.

4.7 Use the Fitch System to prove $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.

4.8 Use the Fitch System to prove $(\neg p \Rightarrow q) \Rightarrow ((\neg p \Rightarrow \neg q) \Rightarrow p)$.

4.9 Given p, use the Fitch System to prove $\neg\neg p$.

4.10 Given $p \Rightarrow q$, use the Fitch System to prove $\neg q \Rightarrow \neg p$.

4.11 Given $p \Rightarrow q$, use the Fitch System to prove $\neg p \vee q$.

4.12 Use the Fitch System to prove $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$.

CHAPTER 5

Propositional Resolution

5.1 INTRODUCTION

Propositional Resolution is a powerful rule of inference for Propositional Logic. Using Propositional Resolution (without axiom schemas or other rules of inference), it is possible to build a theorem prover that is sound and complete for all of Propositional Logic. What's more, the search space using Propositional Resolution is much smaller than for standard Propositional Logic proof systems.

This chapter is devoted entirely to Propositional Resolution. We start with a look at clausal form, a variation of the language of Propositional Logic. We then examine Propositional Resolution. We close with some examples.

5.2 CLAUSAL FORM

Propositional Resolution works only on expressions in *clausal form*. Before the rule can be applied, the premises and conclusions must be converted to this form. Fortunately, as we shall see, there is a simple procedure for making this conversion.

A *literal* is either an atomic sentence or a negation of an atomic sentence. For example, if p is a logical constant, the sentences are both literals.

$$p$$
$$\neg p$$

A *clausal sentence* is either a literal or a disjunction of literals. If p and q are logical constants, then the following are clausal sentences.

$$p$$
$$\neg p$$
$$\neg p \vee q$$

A *clause* is the set of literals in a clausal sentence. For example, the following sets are the clauses corresponding to the clausal sentences above.

$$\{p\}$$
$$\{\neg p\}$$
$$\{\neg p, q\}$$

Note that the empty set {} is also a clause. It is equivalent to an "empty disjunction" and, therefore, is unsatisfiable. As we shall see, it is a particularly important special case.

As mentioned earlier, there is a simple procedure for converting an arbitrary set of Propositional Logic sentences to an equivalent set of clauses. The conversion rules are summarized below and should be applied in order.

1. Implications (I):

$$\begin{array}{lcl} \phi \Rightarrow \psi & \rightarrow & \neg\phi \vee \psi \\ \phi \Leftarrow \psi & \rightarrow & \phi \vee \neg\psi \\ \phi \Leftrightarrow \psi & \rightarrow & (\neg\phi \vee \psi) \wedge (\phi \vee \neg\psi) \end{array}$$

2. Negations (N):

$$\begin{array}{lcl} \neg\neg\phi & \rightarrow & \phi \\ \neg\,(\phi \wedge \psi) & \rightarrow & \neg\phi \vee \neg\psi \\ \neg\,(\phi \vee \psi) & \rightarrow & \neg\phi \wedge \neg\psi \end{array}$$

3. Distribution (D):

$$\begin{array}{lcl} \phi \vee (\psi \wedge \chi) & \rightarrow & (\phi \vee \psi) \wedge (\phi \vee \chi) \\ (\phi \wedge \psi) \vee \chi & \rightarrow & (\phi \vee \chi) \wedge (\psi \vee \chi) \\ \phi \vee (\phi_1 \vee \cdots \vee \phi_n) & \rightarrow & \phi \vee \phi_1 \vee \cdots \vee \phi_n \\ (\phi_1 \vee \cdots \vee \phi_n) \vee \phi & \rightarrow & \phi_1 \vee \cdots \vee \phi_n \vee \phi \\ \phi \wedge (\phi_1 \wedge \cdots \wedge \phi_n) & \rightarrow & \phi \wedge \phi_1 \wedge \cdots \wedge \phi_n \\ (\phi_1 \wedge \cdots \wedge \phi_n) \wedge \phi & \rightarrow & \phi_1 \wedge \cdots \wedge \phi_n \wedge \phi \end{array}$$

4. Operators (O):

$$\begin{array}{lcl} \phi_1 \vee \cdots \vee \phi & \rightarrow & \{\phi_1, \ldots, \phi_n\} \\ \phi_1 \wedge \cdots \wedge \phi_n & \rightarrow & \{\phi_1\}, \ldots, \{\phi_n\} \\ \phi_n & & \end{array}$$

As an example, consider the job of converting the sentence $(g \wedge (r \Rightarrow f))$ to clausal form. The conversion process is shown below.

$$\begin{array}{ll} & g \wedge (r \Rightarrow f) \\ \text{I} & g \wedge (\neg r \vee f) \\ \text{N} & g \wedge (\neg r \vee f) \\ \text{D} & g \wedge (\neg r \vee f) \\ \text{O} & \{g\} \\ & \{\neg r, f\} \end{array}$$

As a slightly more complicated case, consider the following conversion. We start with the same sentence except that, in this case, it is negated.

$$
\begin{array}{ll}
 & \neg\,(g \wedge (r \Rightarrow f)) \\
\mathrm{I} & \neg\,(g \wedge (\neg r \vee f)) \\
\mathrm{N} & \neg g \vee \neg\,(\neg r \vee f) \\
 & \neg g \vee (\neg\neg r \wedge \neg f) \\
 & \neg g \vee (r \wedge \neg f) \\
\mathrm{D} & (\neg g \vee r) \wedge (\neg g \vee \neg f) \\
\mathrm{O} & \{\neg g, r\} \\
 & \{\neg g, \neg f\}
\end{array}
$$

Note that, even though the sentences in these two examples are similar to start with (disagreeing on just one $\neg$ operator), the results are quite different.

5.3 RESOLUTION PRINCIPLE

The idea of Propositional Resolution is simple. Suppose we have the clause $\{p, q\}$. In other words, we know that p is true or q is true. Suppose we also have the clause $\{\neg q, r\}$. In other words, we know that q is false or r is true. One clause contains q, and the other contains $\neg q$. If q is false, then by the first clause p must be true. If q is true, then, by the second clause, r must be true. Since q must be either true or false, then it must be the case that either p is true or r is true. So we should be able to derive the clause $\{p, r\}$.

This intuition is the basis for the rule of inference shown below. Given a clause containing a literal χ and another clause containing the literal $\neg\chi$, we can infer the clause consisting of all the literals of both clauses minus the complementary pair. This rule of inference is called *Propositional Resolution* or the *Resolution Principle*.

$$
\frac{\{\phi_1, \ldots, \chi, \ldots, \phi_m\} \quad \{\psi_1, \ldots, \neg\chi, \ldots, \psi_n\}}{\{\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n\}}
$$

The case we just discussed is an example. If we have the clause $\{p, q\}$ and we also have the clause $\{\neg q, r\}$, then we can derive the clause $\{p, r\}$ in a single step.

$$
\frac{\{p, q\} \quad \{\neg q, r\}}{\{p, r\}}
$$

Note that, since clauses are sets, there cannot be two occurrences of any literal in a clause. Therefore, in drawing a conclusion from two clauses that share a literal, we merge the two occurrences into one, as in the following example.

$$
\frac{\{\neg p, q\} \quad \{p, q\}}{\{q\}}
$$

If either of the clauses is a singleton set, we see that the number of literals in the result is less than the number of literals in the other clause. For example, from the clause $\{p, q, r\}$ and the singleton clause $\{\neg p\}$, we can derive the shorter clause $\{q, r\}$.

$$\begin{array}{l} \{p, q, r\} \\ \{\neg p\} \\ \hline \{q, r\} \end{array}$$

Resolving two singleton clauses leads to the *empty clause*; i.e., the clause consisting of no literals at all, as shown below. The derivation of the empty clause means that the initial set of clauses contains a logical contradiction.

$$\begin{array}{l} \{p\} \\ \{\neg p\} \\ \hline \{\} \end{array}$$

If two clauses resolve, they may have more than one resolvent because there can be more than one way in which to choose the literals to resolve upon. Consider the following deductions.

$$\begin{array}{l} \{p, q\} \\ \{\neg p, \neg q\} \\ \hline \{p, \neg p\} \\ \{q, \neg q\} \end{array}$$

Note, however, when two clauses have multiple pairs of complementary literals, only *one pair* of literals may be resolved at a time. For example, the following is *not* a legal application of Propositional Resolution.

$$\begin{array}{l} \{p, q\} \\ \{\neg p, \neg q\} \quad \text{Wrong!} \\ \hline \{\} \end{array}$$

If we were to allow this to go through, we would be saying the these two clauses are inconsistent (that is, they form a logical contradiction). However, it is perfectly possible for $(p \vee q)$ to be true and $(\neg p \vee \neg q)$ to be true at the same time. For example, we just let p be true and q be false, and we have satisfied both clauses.

It is noteworthy that Resolution is related to many of our other rules of inference. Consider, for example, Implication Elimination, shown below on the left. If we have $(p \Rightarrow q)$ and we have p, then we can deduce q. The clausal form of the premises and conclusion are shown below on the right. The implication $(p \Rightarrow q)$ corresponds to the clause $\{\neg p, q\}$, and p corresponds to the singleton clause $\{p\}$. We have two clauses with a complementary literal, and so we cancel the complementary literals and derive the clause $\{q\}$, which is the clausal form of q.

$$\begin{array}{l} p \Rightarrow q \\ p \\ \hline q \end{array} \qquad\qquad \begin{array}{l} \{\neg p, q\} \\ \{p\} \\ \hline \{q\} \end{array}$$

As another example, consider an instance of the Negation Introduction rule shown below. The implication $(p \Rightarrow q)$ corresponds to the clause $\{\neg p, q\}$, and $(p \Rightarrow \neg q)$ corresponds to the clause $\{\neg p, \neg q\}$. We have two clauses with a complementary literal, and so we cancel the complementary literals and derive the clause $\{\neg p\}$, which is the clausal form of $\neg p$.

$$\frac{p \Rightarrow q \qquad p \Rightarrow \neg q}{\neg p} \qquad\qquad \frac{\{\neg p, q\} \qquad \{\neg p, \neg q\}}{\{\neg p\}}$$

For one more example, recall the example of formal reasoning introduced in Chapter 1. We said that, when we have two implications in which the left-hand side of one contains a proposition constant that occurs on the right-hand side of the other, then we can cancel those constants and deduce a new implication by combining the remaining constants on the left-hand sides of both implications and the remaining constants on the right-hand sides of both implications. As it turns out, this is just Propositional Resolution.

Recall that we illustrated this rule with the deduction shown below on the left. Given $(m \Rightarrow p \vee q)$ and $(p \Rightarrow q)$, we deduce $(m \Rightarrow q)$. On the right, we have the clausal form of the sentences on the left. In place of the first sentence, we have the clause $\{\neg m, p, q\}$; and, in place of the second sentence, we have $\{\neg p, q\}$. Using Propositional Logic, we can deduce $\{\neg m, q\}$, which is the clausal form of the sentence we derived on the left.

$$\frac{m \Rightarrow p \vee q \qquad p \Rightarrow q}{m \Rightarrow q} \qquad\qquad \frac{\{\neg m, p, q\} \qquad \{\neg p, q\}}{\{\neg m, q\}}$$

5.4 RESOLUTION REASONING

Reasoning with the Resolution Principle is analogous to reasoning with other rules of inference. We start with premises; we apply the Resolution Principle to those premises; we apply the rule to the results of those applications; and so forth until we get to our desired conclusion or we run out of things to do.

Formally, we define a *Resolution derivation* of a conclusion from a set of premises to be a finite sequence of clauses terminating in the conclusion in which each clause is either a premise or the result of applying the Resolution Principle to earlier members of the sequence.

Note that our definition of Resolution derivation is analogous to our definition of linear proof. However, in this case, we do not use the word *proof*, because we reserve that word for a slightly different concept, which is discussed below.

In many cases, it is possible to find resolution derivations of conclusions from premises. Suppose, for example, we are given the clauses $\{\neg p, r\}$ and $\{\neg q, r\}$ and $\{p, q\}$. Then we can derive the conclusion $\{r\}$ as shown below.

1.	$\{\neg p, r\}$	Premise
2.	$\{\neg q, r\}$	Premise
3.	$\{p, q\}$	Premise
4.	$\{q, r\}$	1, 3
5.	$\{r\}$	2, 4

It is noteworthy that the Resolution is not *generatively complete*, i.e., it is not possible to find Resolution derivations for all clauses that are logically entailed by a set of premise clauses.

For example, given the clause $\{p\}$ and the clause $\{q\}$, there is no resolution derivation of $\{p, q\}$, despite the fact that it is logically entailed by the premises in this case.

As another example, consider that valid clauses (such as $\{p, \neg p\}$) are always true, and so they are logically entailed by any set of premises, including the empty set. However, Propositional Resolution requires some premises to have any effect. Given an empty set of premises, we would not be able to derive any conclusions, including these valid clauses.

On the other hand, we can be sure of one thing. If a set Δ of clauses is unsatisfiable, then there is guaranteed to be a Resolution derivation of the empty clause from Δ. More generally, if a set Δ of Propositional Logic sentences is unsatisfiable, then there is guaranteed to be a Resolution derivation of the empty clause from the clausal form of Δ.

As an example, consider the clauses $\{p, q\}$, $\{p, \neg q\}$, $\{\neg p, q\}$, and $\{\neg p, \neg q\}$. There is no truth assignment that satisfies all four of these clauses. Consequently, starting with these clauses, we should be able to derive the empty clause; and we can. A resolution derivation is shown below.

1.	$\{p, q\}$	Premise
2.	$\{p, \neg q\}$	Premise
3.	$\{\neg p, q\}$	Premise
4.	$\{\neg p, \neg q\}$	Premise
5.	$\{p\}$	1, 2
6.	$\{\neg p\}$	3, 4
7.	$\{\}$	5, 6

The good news is that we can use the relationship between unsatisfiability and logical entailment to produce a method for determining logical entailment as well. Recall that the Refutation Theorem introduced in Chapter 2 tells us that a set Δ of sentences logically entails a sentence ϕ if and only if the set of sentences $\Delta \cup \{\neg\phi\}$ is inconsistent. So, to determine logical entailment, all we need to do is to negate our goal, add it to our premises, and use Resolution to determine whether the resulting set is unsatisfiable.

Let's capture this idea with some definitions. A *resolution proof* of a sentence ϕ from a set Δ of sentences is a resolution derivation of the empty clause from the clausal form of $\Delta \cup \{\neg\phi\}$. A sentence ϕ is *provable* from a set of sentences Δ by Propositional Resolution (written $\Delta \vdash_{\texttt{Res}} \phi$) if and only if there is a resolution proof of ϕ from Δ.

As an example of a resolution proof, consider one of the problems we saw in Chapter 3. We have three premises: p, $(p \Rightarrow q)$, and $(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$. Our job is to prove r. A resolution

proof is shown below. The first two clauses in the proof correspond to the first two premises of the problem. The third and fourth clauses in the proof correspond to the third premise. The fifth clause comes from the negation of the goal. Resolving the first clause with the second, we get the clause q, shown on line 6. Resolving this with the fourth clause gives us r. And resolving this with the clause on line 5 gives us the empty clause.

1.	$\{p\}$	Premise
2.	$\{\neg p, q\}$	Premise
3.	$\{p, \neg q, r\}$	Premise
4.	$\{\neg q, r\}$	Premise
5.	$\{\neg r\}$	Premise
6.	$\{q\}$	1, 2
7.	$\{r\}$	4, 6
8.	$\{\}$	5, 7

Here is another example, this time illustrating the way in which we can use Resolution to prove valid sentences. Let's say that we have no premises at all and we want to prove $(p \Rightarrow (q \Rightarrow p))$, an instance of the Implication Creation axiom schema.

The first step is to negate this sentence and convert to clausal form. A trace of the conversion process is shown below. Note that we end up with three clauses.

	$\neg (p \Rightarrow (q \Rightarrow p))$
I	$\neg (\neg p \vee (\neg q \vee p))$
N	$\neg\neg p \wedge \neg (\neg q \vee p)$
	$p \wedge (\neg\neg q \wedge \neg p)$
D	$p \wedge q \wedge \neg p$
O	$\{p\}$
	$\{q\}$
	$\{\neg p\}$

Finally, we take these clauses and produce a resolution derivation of the empty clause in one step.

1.	$\{p\}$	Premise
2.	$\{q\}$	Premise
3.	$\{\neg p\}$	Premise
4.	$\{\}$	1, 3

One of the best features of Propositional Resolution is that it much more focussed than the other proof methods we have seen. There is no need to choose instantiations carefully or to search through infinitely many possible instantiations for rules of inference.

Moreover, unlike the other methods we have seen, Propositional Resolution can be used in a proof procedure that always terminates without losing completeness. In fact, Propositional

Resolution is sound and complete for Propositional Logic: for a set of sentences Δ and a sentence ϕ, $\Delta \models \phi$ if and only if $\Delta \vdash_{\texttt{Res}} \phi$. Since there are only finitely many clauses that can be constructed from a finite set of proposition constants, the procedure eventually runs out of new conclusions to draw, and when this happens we can terminate our search for a proof.

RECAP

Propositional Resolution is a rule of inference for Propositional Logic. Propositional Resolution works only on expressions in *clausal form*. A *literal* is either an atomic sentence or a negation of an atomic sentence. For example, if p is a logical constant, the sentences p and $\neg p$ are both literals. A *clausal sentence* is either a literal or a disjunction of literals. A *clause* is the set of literals in a clausal sentence. The empty set {} is also a clause; it is equivalent to an "empty disjunction" and, therefore, is unsatisfiable. Given a clause containing a literal χ and another clause containing the literal $\neg\chi$, we can infer the clause consisting of all the literals of both clauses minus the complementary pair. This rule of inference is called *Propositional Resolution* or the *Resolution Principle*. A *resolution derivation* of a conclusion from a set of premises is a finite sequence of clauses terminating in the conclusion in which each clause is either a premise or the result of applying the Resolution Principle to earlier members of the sequence. A *resolution proof* of a sentence ϕ from a set Δ of sentences is a resolution derivation of the empty clause from the clausal form of $\Delta \cup \{\neg\phi\}$. A sentence ϕ is *provable* from a set of sentences Δ by Propositional Resolution if and only if there is a resolution proof of ϕ from Δ. Resolution is not *generatively complete*, i.e., it is not possible to find resolution derivations for all clauses that are logically entailed by a set of premise clauses. On the other hand, it is *complete* in another sense—if a set Δ of clauses is unsatisfiable, then there is guaranteed to be a resolution derivation of the empty clause from Δ. More generally, if a set Δ of Propositional Logic sentences is unsatisfiable, then there is guaranteed to be a resolution derivation of the empty clause from the clausal form of Δ. Propositional Resolution can be used in a proof procedure that always terminates without losing completeness.

EXERCISES

5.1 Convert the following sentences to clausal form.

(a) $p \wedge q \Rightarrow r \vee s$
(b) $p \vee q \Rightarrow r \vee s$
(c) $\neg(p \vee q \vee r)$
(d) $\neg(p \wedge q \wedge r)$
(e) $p \wedge q \Leftrightarrow r$

5.2 What are the results of applying Propositional Resolution to the following pairs of clauses.

(*a*) $\{p, q, \neg r\}$ and $\{r, s\}$
(*b*) $\{p, q, r\}$ and $\{r, \neg s, \neg t\}$
(*c*) $\{q, \neg q\}$ and $\{q, \neg q\}$
(*d*) $\{\neg p, q, r\}$ and $\{p, \neg q, \neg r\}$

5.3 Use Resolution to show that the clauses $\{p, q\}$, $\{\neg p, r\}$, $\{\neg p, \neg r\}$, $\{p, \neg q\}$ are not simultaneously satisfiable.

5.4 Given the premises $(p \Rightarrow q)$ and $(r \Rightarrow s)$, use Propositional Resolution to prove the conclusion $(p \vee r \Rightarrow q \vee s)$.

CHAPTER 6

Relational Logic

6.1 INTRODUCTION

Propositional Logic does a good job of allowing us to talk about relationships among individual propositions, and it gives us the machinery to derive logical conclusions based on these relationships. Suppose, for example, we believe that, if Jack knows Jill, then Jill knows Jack. Suppose we also believe that Jack knows Jill. From these two facts, we can conclude that Jill knows Jack using a simple application of Implication Elimination.

Unfortunately, when we want to say things more generally, we find that Propositional Logic is inadequate. Suppose, for example, that we wanted to say that, in general, if one person knows a second person, then the second person knows the first. Suppose, as before, that we believe that Jack knows Jill. How do we express the general fact in a way that allows us to conclude that Jill knows Jack? Here, Propositional Logic is inadequate; it gives us no way of succinctly encoding this more general belief in a form that captures its full meaning and allows us to derive such conclusions.

Relational Logic is an extension of Propositional Logic that solves this problem. The trick is to augment our language with two new linguistic features: *variables* and *quantifiers*. With these new features, we can express information about multiple objects without enumerating those objects; and we can express the existence of objects that satisfy specified conditions without saying which objects they are.

In this chapter, we proceed through the same stages as in the introduction to Propositional Logic. We start with syntax and semantics. We then discuss evaluation and satisfaction. Finally, we move on to logical entailment. As with Propositional Logic, we leave the discussion of proofs to later chapters.

6.2 SYNTAX

In Propositional Logic, sentences are constructed from a basic vocabulary of propositional constants. In Relational Logic, there are no propositional constants; instead we have *object constants*, *function constants*, *relation constants*, and *variables*.

In our examples here, we write both variables and constants as strings of letters, digits, and a few non-alphanumeric characters (e.g., "_"). By convention, variables begin with letters from the end of the alphabet (u, v, w, x, y, z). Examples include x, *ya*, and *z_2*. By convention, all constants begin with either alphabetic letters (other than u, v, w, x, y, z) or digits. Examples include a, b, 123, *comp*225, and *barack_obama*.

Note that there is no distinction in spelling between object constants, function constants, and relation constants. The type of each such word is determined by its usage or, in some cases, in an explicit specification.

As we shall see, function constants and relation constants are used in forming complex expressions by combining them with an appropriate number of arguments. Accordingly, each function constant and relation constant has an associated *arity*, i.e., the number of arguments with which that function constant or relation constant can be combined. A function constant or relation constant that can combined with a single argument is said to be *unary*; one that can be combined with two arguments is said to be *binary*; one that can be combined with three arguments is said to be *ternary*; more generally, a function or relation constant that can be combined with n arguments is said to be n-ary.

A *vocabulary* consists of a non-empty set of object constants, a set of function constants, a set of relation constants, and an assignment of arities for each of the function constants and relation constants in the vocabulary. (Note that this definition here is slightly non-traditional. In many textbooks, a vocabulary (sometimes called a *signature*) includes a specification of function constants and relation constants but not object constants, whereas our definition here includes all three types of constants.)

A *term* is defined to be a variable, an object constant, or a functional expression (as defined below). Terms typically denote objects presumed or hypothesized to exist in the world; and, as such, they are analogous to noun phrases in natural language, e.g., *Joe* or *the car's left front wheel*.

A *functional expression*, or *functional term*, is an expression formed from an n-ary function constant and n terms enclosed in parentheses and separated by commas. For example, if f is a binary function constant and if a and y are terms, then $f(a,y)$ is a functional expression, as are $f(a,a)$ and $f(y,y)$.

Note that functional expressions can be nested within other functional expressions. For example, if g is a unary function constant and if a is a term, $g(a)$ and $g(g(a))$ and $g(g(g(a)))$ are all functional expressions.

There are three types of *sentences* in Relational Logic: relational sentences (the analog of propositions in Propositional Logic), logical sentences (analogous to the logical sentences in Propositional Logic), and quantified sentences (which have no analog in Propositional Logic).

A *relational sentence* is an expression formed from an n-ary relation constant and n terms. For example, if q is a relation constant with arity 2 and if a and y are terms, then the expression shown below is a syntactically legal relational sentence.

$$q(a, y)$$

Logical sentences are defined as in Propositional Logic. There are negations, conjunctions, disjunctions, implications, and equivalences. The syntax is exactly the same, except that the elementary components are relational sentences and equations rather than proposition constants.

Quantified sentences are formed from a *quantifier*, a variable, and an embedded sentence. The embedded sentence is called the *scope* of the quantifier. There are two types of quantified sentences in Relational Logic: universally quantified sentences and existentially quantified sentences.

A *universally quantified sentence* is used to assert that all objects have a certain property. For example, the following expression is a universally quantified sentence asserting that, if p holds of an object, then q holds of that object and itself.

$$(\forall x.(p(x) \Rightarrow q(x,x)))$$

An *existentially quantified sentence* is used to assert that some object has a certain property. For example, the following expression is an existentially quantified sentence asserting that there is an object that satisfies p and, when paired with itself, satisfies q as well.

$$(\exists x.(p(x) \wedge q(x,x)))$$

Note that quantified sentences can be nested within other sentences. For example, in the first sentence below, we have quantified sentences inside of a disjunction. In the second sentence, we have a quantified sentence nested inside of another quantified sentence.

$$(\forall x.p(x)) \vee (\exists x.q(x,x))$$
$$(\forall x.(\exists y.q(x, y)))$$

As with Propositional Logic, we can drop unneeded parentheses in Relational Logic, relying on precedence to disambiguate the structure of unparenthesized sentences. In Relational Logic, the precedence relations of the logical operators are the same as in Propositional Logic, and quantifiers have higher precedence than logical operators.

The following examples show how to parenthesize sentences with both quantifiers and logical operators. The sentences on the right are partially parenthesized versions of the sentences on the left. (To be fully parenthesized, we would need to add parentheses around each of the sentences as a whole.)

$$\forall x.p(x) \Rightarrow q(x) \qquad (\forall x.p(x)) \Rightarrow q(x)$$
$$\exists x.p(x) \wedge q(x) \qquad (\exists x.p(x)) \wedge q(x)$$

Notice that, in each of these examples, the quantifier does *not* apply to the second relational sentence, even though, in each case, that sentence contains an occurrence of the variable being quantified. If we want to apply the quantifier to a logical sentence, we must enclose that sentence in parentheses, as in the following examples.

$$\forall x.(p(x) \Rightarrow q(x))$$
$$\exists x.(p(x) \wedge q(x))$$

An expression in Relational Logic is *ground* if and only if it contains no variables. For example, the sentence $p(a)$ is ground, whereas the sentence $\forall x.p(x)$ is not.

An occurrence of a variable is *free* if and only if it is not in the scope of a quantifier of that variable. Otherwise, it is *bound*. For example, y is free and x is bound in the following sentence.

$$\exists x.q(x,y)$$

A sentence is *open* if and only if it has free variables. Otherwise, it is *closed*. For example, the first sentence below is open and the second is closed.

$$p(y) \Rightarrow \exists x.q(x,y)$$
$$\forall y.(p(y) \Rightarrow \exists x.q(x,y))$$

6.3 SEMANTICS

The semantics of Relational Logic presented here is termed *Herbrand semantics*. It is named after the logician Herbrand, who developed some of its key concepts. As Herbrand is French, it should properly be pronounced "air-brahn". However, most people resort to the Anglicization of this, instead pronouncing it "her-brand". (One exception is Stanley Peters, who has been known at times to pronounce it "hare-brained".)

The *Herbrand base* for a vocabulary (with at least one object constant) is the set of all ground relational sentences that can be formed from the constants of the language. Said another way, it is the set of all sentences of the form $r(t_1,...,t_n)$, where r is an n-ary relation constant and t_1, ... , t_n are ground terms.

For a vocabulary with object constants a and b, no function constants, and relation constants p and q where p has arity 1 and q has arity 2, the Herbrand base is shown below.

$$\{p(a),\ p(b),\ q(a,a),\ q(a,b),\ q(b,a),\ q(b,b)\}$$

It is worthwhile to note that, for a given relation constant and a finite set of terms, there is an upper bound on the number of ground relational sentences that can be formed using that relation constant. In particular, for a set of terms of size b, there are b^n distinct n-tuples of object constants; and hence there are b^n ground relational sentences for each n-ary relation constant. Since the number of relation constants in a vocabulary is finite, this means that the Herbrand base is also finite.

Of course, not all Herbrand bases are finite. In the presence of function constants, the number of ground terms is infinite; and so the Herbrand base is infinite. For example, in a language with a single object constant a and a single unary function constant f and a single unary relation constant p, the Herbrand base consists of the sentences shown below.

$$\{p(a),\ p(f(a)),\ p(f(f(a))),\ ...\}$$

A *truth assignment* for a Relational Logic language is a function that maps each ground relational sentence in the Herbrand base to a truth value. As in Propositional Logic, we use the digit 1 as a synonym for true and 0 as a synonym for false; and we refer to the value assigned to a ground relational sentence by writing the relational sentence with the name of the truth assignment as a superscript. For example, the truth assignment i defined below is an example for the case of the language mentioned a few paragraphs above.

$$
\begin{aligned}
p(a)^i &= 1 \\
p(b)^i &= 0 \\
q(a,a)^i &= 1 \\
q(a,b)^i &= 0 \\
q(b,a)^i &= 1 \\
q(b,b)^i &= 0
\end{aligned}
$$

As with Propositional Logic, once we have a truth assignment for the ground relational sentences of a language, the semantics of our operators prescribes a unique extension of that assignment to the complex sentences of the language.

The rules for logical sentences in Relational Logic are the same as those for logical sentences in Propositional Logic. A truth assignment i satisfies a negation $\neg\phi$ if and only if i does not satisfy ϕ. Truth assignment i satisfies a conjunction $(\phi_1 \wedge ... \wedge \phi_n)$ if and only if i satisfies every ϕ_i. Truth assignment i satisfies a disjunction $(\phi_1 \vee ... \vee \phi_n)$ if and only if i satisfies at least one ϕ_i. Truth assignment i satisfies an implication $(\phi \Rightarrow \psi)$ if and only if i does not satisfy ϕ or does satisfy ψ. Truth assignment i satisfies an equivalence $(\phi \Leftrightarrow \psi)$ if and only if i satisfies both ϕ and ψ or it satisfies neither ϕ nor ψ.

In order to define satisfaction of quantified sentences, we need the notion of instances. An *instance* of an expression is an expression in which all variables have been consistently replaced by ground terms. Consistent replacement here means that, if one occurrence of a variable is replaced by a ground term, then all occurrences of that variable are replaced by the same ground term.

A universally quantified sentence is true for a truth assignment if and only if *every* instance of the scope of the quantified sentence is true for that assignment. An existentially quantified sentence is true for a truth assignment if and only if *some* instance of the scope of the quantified sentence is true for that assignment.

As an example of these definitions, consider the sentence $\forall x.(p(x) \Rightarrow q(x,x))$. What is the truth value under the truth assignment shown above? According to our definition, a universally quantified sentence is true if and only if every instance of its scope is true. For this language, , with object constants a and b and no function constants, there are just two instances. See below.

$$
\begin{aligned}
p(a) &\Rightarrow q(a,a) \\
p(b) &\Rightarrow q(b,b)
\end{aligned}
$$

We know that $p(a)$ is true and $q(a,a)$ is true, so the first instance is true. $q(b,b)$ is false, but so is $p(b)$ so the second instance is true as well. Since both instances are true, the original quantified sentence is true.

Now let's consider a case with nested quantifiers. Is $\forall x.\exists y.q(x, y)$ true or false for the truth assignment shown above? As before, we know that this sentence is true if every instance of its scope is true. The two possible instances are shown below.

$$
\begin{aligned}
&\exists y.q(a, y) \\
&\exists y.q(b, y)
\end{aligned}
$$

To determine the truth of the first of these existential sentences, we must find at least one instance of the scope that is true. The possibilities are shown below. Of these, the first is true; and so the first existential sentence is true.

$$q(a,a)$$
$$q(a,b)$$

Now, we do the same for the second existentially quantified. The possible instances follow. Of these, again the first is true; and so the second existential sentence is true.

$$q(b,a)$$
$$q(b,b)$$

Since both existential sentences are true, the original universally quantified sentence must be true as well.

A truth assignment *i* *satisfies* a sentence with free variables if and only if it satisfies every instance of that sentence. A truth assignment *i* *satisfies* a set of sentences if and only if *i* satisfies every sentence in the set.

6.4 EXAMPLE: SORORITY WORLD

Consider once again the Sorority World example introduced in Chapter 1. Recall that this world focusses on the interpersonal relations of a small sorority. There are just four members: Abby, Bess, Cody, and Dana. Our goal is to represent information about who likes whom.

In order to encode this information in Relational Logic, we adopt a vocabulary with four object constants (*abby*, *bess*, *cody*, *dana*) and one binary relation constant (*likes*).

If we had complete information about the likes and dislikes of the girls, we could completely characterize the state of affairs as a set of ground relational sentences or negations of ground relational sentences, like the ones shown below, with one sentence for each member of the Herbrand base. (In our example here, we have written the positive literals in black and the negative literals in grey in order to distinguish the two more easily.)

¬*likes*(*abby*,*abby*)	¬*likes*(*abby*,*bess*)	*likes*(*abby*,*cody*)	¬*likes*(*abby*,*dana*)
¬*likes*(*bess*,*abby*)	¬*likes*(*bess*,*bess*)	*likes*(*bess*,*cody*)	¬*likes*(*bess*,*dana*)
likes(*cody*,*abby*)	*likes*(*cody*,*bess*)	¬*likes*(*cody*,*cody*)	*likes*(*cody*,*dana*)
¬*likes*(*dana*,*abby*)	¬*likes*(*dana*,*bess*)	*likes*(*dana*,*cody*)	¬*likes*(*dana*,*dana*)

To make things more interesting, let's assume that we do *not* have complete information, only fragments of information about the girls likes and dislikes. Let's see how we can encode such fragments in Relational Logic.

Let's start with a simple disjunction. *Bess likes Cody or Dana.* Encoding a sentence with a disjunctive noun phrase (such as *Cody or Dana*) is facilitated by first rewriting the sentence as a disjunction of simple sentences. *Bess likes Cody or Bess likes Dana.* In Relational Logic, we can express this fact as a simple disjunction with the two possibilities as disjuncts.

$$likes(bess,cody) \vee likes(bess,dana)$$

Abby likes everyone Bess likes. Again, paraphrasing helps translate. *If Bess likes a girl, then Abby also likes her.* Since this is a fact about everyone, we use a universal quantifier.

$$\forall y.(likes(bess,y) \Rightarrow likes(abby,y))$$

Cody likes everyone who likes her. In other words, *if some girl likes Cody, then Cody likes that girl.* Again, we use a universal quantifier.

$$\forall y.(likes(y,cody) \Rightarrow likes(cody,y))$$

Bess likes somebody who likes her. The word *somebody* here is a tip-off that we need to use an existential quantifier.

$$\exists y.(likes(bess,y) \wedge likes(y,bess))$$

Nobody likes herself. The use of the word *nobody* here suggests a negation. A good technique in such cases is to rewrite the English sentence as the negation of a positive version of the sentence before translating to Relational Logic.

$$\neg\exists x.likes(x,x)$$

Everybody likes somebody. Here we have a case requiring two quantifiers, one universal and one existential. The key to this case is getting the quantifiers in the right order. Reversing them leads to a very different statement.

$$\forall x.\exists y.likes(x,y)$$

There is someone everyone likes. The preceding sentence tells us that everyone likes someone, but that someone can be different for different people. This sentence tells us that everybody likes the same person.

$$\exists x.\forall y.likes(x,y)$$

6.5 EXAMPLE: BLOCKS WORLD

The Blocks World is a popular application area for illustrating ideas in the field of Artificial Intelligence. A typical Blocks World scene is shown in Figure 6.1.

Most people looking at this figure interpret it as a configuration of five toy blocks. Some people conceptualize the table on which the blocks are resting as an object as well; but, for simplicity, we ignore it here.

In order to describe this scene, we adopt a vocabulary with five object constants, as shown below, with one object constant for each of the five blocks in the scene. The intent here is for each of these object constants to represent the block marked with the corresponding capital letter in the scene.

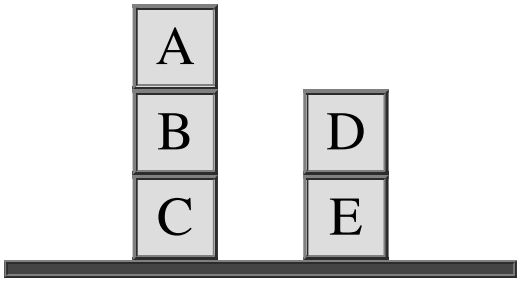


Figure 6.1: One state of Blocks World.

$$\{a, b, c, d, e\}$$

In a spatial conceptualization of the Blocks World, there are numerous meaningful relations. For example, it makes sense to think about the relation that holds between two blocks if and only if one is resting on the other. In what follows, we use the relation constant *on* to refer to this relation. We might also think about the relation that holds between two blocks if and only if one is anywhere above the other, i.e., the first is resting on the second or is resting on a block that is resting on the second, and so forth. In what follows, we use the relation constant *above* to talk about this relation. There is the relation that holds of three blocks that are stacked one on top of the other. We use the relation constant *stack* as a name for this relation. We use the relation constant *clear* to denote the relation that holds of a block if and only if there is no block on top of it. We use the relation constant *table* to denote the relation that holds of a block if and only if that block is resting on the table. The relational vocabulary corresponding to this conceptualization is shown below.

$$\{on, above, stack, clear, table\}$$

The arities of these relation constants are determined by their intended use. Since *on* is intended to denote a relation between two blocks, it has arity 2. Similarly, *above* has arity 2. The *stack* relation constant has arity 3. Relation constants *clear* and *table* each have arity 1.

Given this vocabulary, we can describe the scene in Figure 6.1 by writing ground literals that state which relations hold of which objects or groups of objects. Let's start with *on*. The following sentences tell us directly for each ground relational sentence whether it is true or false. (Once again, we have written the positive literals in black and the negative literals in grey in order to distinguish the two more easily.)

$\neg on(a,a)$	$on(a,b)$	$\neg on(a,c)$	$\neg on(a,d)$	$\neg on(a,e)$
$\neg on(b,a)$	$\neg on(b,b)$	$on(b,c)$	$\neg on(b,d)$	$\neg on(b,e)$
$\neg on(c,a)$	$\neg on(c,b)$	$\neg on(c,c)$	$\neg on(c,d)$	$\neg on(c,e)$
$\neg on(d,a)$	$\neg on(d,b)$	$\neg on(d,c)$	$\neg on(d,d)$	$on(d,e)$
$\neg on(e,a)$	$\neg on(e,b)$	$\neg on(e,c)$	$\neg on(e,d)$	$\neg on(e,e)$

We can do the same for the other relations. However, there is an easier way. Each of the remaining relations can be defined in terms of *on*. These definitions together with our facts about the *on* relation logically entail every other ground relational sentence or its negation. Hence, given these definitions, we do not need to write out any additional data.

A block satisfies the *clear* relation if and only if there is nothing on it.

$$\forall y.(clear(y) \Leftrightarrow \neg\exists x.on(x,y))$$

A block satisfies the *table* relation if and only if it is not on some block.

$$\forall x.(table(x) \Leftrightarrow \neg\exists y.on(x,y))$$

Three blocks satisfy the *stack* relation if and only if the first is on the second and the second is on the third.

$$\forall x.\forall y.\forall z.(\ stack(x,y,z) \Leftrightarrow on(x,y) \wedge on(y,z))$$

The *above* relation is a bit tricky to define correctly. One block is above another block if and only if the first block is on the second block or it is on another block that is above the second block. Also, no block can be above itself. Given a complete definition for the *on* relation, these two axioms determine a unique *above* relation.

$$\forall x.\forall z.(above(x,\ z) \Leftrightarrow on(x,z) \vee \exists y.(on(x,y) \wedge above(y,z)))$$
$$\neg\exists x.above(x,x)$$

One advantage to defining relations in terms of other relations is economy. If we record *on* information for every object and encode the relationship between the *on* relation and the these other relations, there is no need to record any ground relational sentences for those relations.

Another advantage is that these general sentences apply to Blocks World scenes other than the one pictured here. It is possible to create a Blocks World scene in which none of the atomic sentences we have listed is true, but these general definitions are still correct.

6.6 EXAMPLE: MODULAR ARITHMETIC

In this example, we show how to characterize Modular Arithmetic in Relational Logic. In Modular Arithmetic, there are only finitely many objects. For example, in Modular Arithmetic with modulus 4, we would have just four integers—0, 1, 2, 3—and that's all. Our goal here to define the addition relation. Admittedly, this is a modest goal; but, once we see how to do this; we can use the same approach to define other arithmetic relations.

Let's start with the *same* relation, which is true of every number and itself and is false for numbers that are different. We can completely characterize the *same* relation by writing ground relational sentences, one positive sentence for each number and itself and negative sentences for all of the other cases.

$same(0,0)$	$\neg same(0,1)$	$\neg same(0,2)$	$\neg same(0,3)$
$\neg same(1,0)$	$same(1,1)$	$\neg same(1,2)$	$\neg same(1,3)$
$\neg same(2,0)$	$\neg same(2,1)$	$same(2,2)$	$\neg same(2,3)$
$\neg same(3,0)$	$\neg same(3,1)$	$\neg same(3,2)$	$same(3,3)$

Now, let's axiomatize the *next* relation, which, for each number, gives the next larger number, wrapping back to 0 after we reach 3.

$$next(0,1)$$
$$next(1,2)$$
$$next(2,3)$$
$$next(3,0)$$

Properly, we should write out the negative literals as well. However, we can save that work by writing a single axiom asserting that *next* is a functional relation, i.e., for each member of the Herbrand base, there is just one successor.

$$\forall x.\forall y.\forall z.(next(x,y) \wedge next(x,z) \Rightarrow same(y,z))$$

In order to see why this saves us the work of writing out the negative literals, we can write this axiom in the equivalent form shown below.

$$\forall x.\forall y.\forall z.(next(x,y) \wedge \neg same(y,z) \Rightarrow \neg next(x,z))$$

The addition table for Modular Arithmetic is the usual addition table for arbitrary numbers except that we wrap around whenever we get past 3. For such a small arithmetic, it is easy to write out the ground relational sentences for addition, as shown below.

$plus(0,0,0)$	$plus(1,0,1)$	$plus(2,0,2)$	$plus(3,0,3)$
$plus(0,1,1)$	$plus(1,1,2)$	$plus(2,1,3)$	$plus(3,1,0)$
$plus(0,2,2)$	$plus(1,2,3)$	$plus(2,2,0)$	$plus(3,2,1)$
$plus(0,3,3)$	$plus(1,3,0)$	$plus(2,3,1)$	$plus(3,3,2)$

As with *next*, we avoid writing out the negative literals by writing a suitable functionality axiom for *plus*.

$$\forall x.\forall y.\forall z.\forall w.(plus(x,y,z) \wedge \neg same(z,w) \Rightarrow \neg plus(x,y,w))$$

That's one way to do things, but we can do better. Rather than writing out all of those relational sentences, we can use Relational Logic to define *plus* in terms of *same* and *next* and use that axiomatization to deduce the ground relational sentences. The definition is shown below. First, we have an identity axiom. Adding 0 to any number results in the same number. Second we have a successor axiom. If z is the sum of x and y, then the sum of the successor of x and y is the successor of z. Finally, we have our functionality axiom once again.

$$\forall.plus(0,y,y)$$
$$\forall x.\forall y.\forall z.\forall x2.\forall z2.(plus(x,y,z) \wedge next(x,x2) \wedge next(z,z2) \Rightarrow plus(x2,y,z2))$$
$$\forall x.\forall y.\forall z.\forall w.(plus(x,y,z) \wedge \neg same(z,w) \Rightarrow \neg plus(x,y,w))$$

One advantage of doing things this way is economy. With these sentences, we do not need the ground relational sentences about *plus* given above. They are all logically entailed by our sentences about *next* and the definitional sentences. A second advantage is versatility. Our sentences define *plus* in terms of *next* for arithmetic with any modulus, not just modulus 4.

6.7 EXAMPLE: PEANO ARITHMETIC

Now, let's look at the problem of encoding arithmetic for all of the natural numbers. Since there are infinitely many natural numbers, we need infinitely many terms.

One way to get infinitely many terms is to have a vocabulary with infinitely many object constants. While there is nothing wrong with this in principle, it makes the job of axiomatizing arithmetic effectively impossible, as we would have to write out infinitely many literals to capture our successor relation.

An alternative approach is to represent numbers using a single object constant (e.g., 0) and a single unary function constant (e.g., s). We can then represent every number n by applying the function constant to 0 exactly n times. In this encoding, $s(0)$ represents 1; $s(s(0))$ represents 2; and so forth. With this encoding, we automatically get an infinite universe of terms, and we can write axioms defining addition and multiplication as simple variations on the axioms of Modular Arithmetic.

Unfortunately, even with this representation, axiomatizing Peano Arithmetic is more challenging than axiomatizing Modular Arithmetic. We cannot just write out ground relational sentences to characterize our relations, because there are infinitely many cases to consider. For Peano Arithmetic, we must rely on logical sentences and quantified sentences, not just because they are more economical but because they are the only way we can characterize our relations in finite space.

Let's look at the same relation first. The axioms shown here define the *same* relation in terms of 0 and s.

$$\forall x.same(x,x)$$
$$\forall x.(\neg same(0,s(x)) \wedge \neg same(s(x),0))$$
$$\forall x.\forall y.(\neg same(x,y) \Rightarrow \neg same(s(x),s(y)))$$

It is easy to see that these axioms completely characterize the *same* relation. By the first axiom, the same relation holds of every term and itself.

$$same(0,0)$$
$$same(s(0),s(0))$$
$$same(s(s(0)),s(s(0)))$$
$$\ldots$$

The other two axioms tell us what is not true. The second axiom tells us that 0 is not the same as any composite term. The same hold true with the arguments reversed.

$$\begin{array}{ll} \neg same(0,s(0)) & \neg same(s(0),0) \\ \neg same(0,s(s(0))) & \neg same(s(s(0)),0) \\ \neg same(0,s(s(s(0)))) & \neg same(s(s(s(0))),0) \\ \ldots & \ldots \end{array}$$

The third axiom builds on these results to show that non-identical composite terms of arbitrary complexity do not satisfy the same relation. Viewed the other way around, to see that two non-identical terms are not the same, we just strip away occurrences of s from each term till one of the two terms becomes 0 and the other one is not 0. By the second axiom, these are not the same, and so the original terms are not the same.

$$\begin{array}{ll} \neg same(s(0),s(s(0))) & \neg same(s(s(0)),s(0)) \\ \neg same(s(0),s(s(s(0)))) & \neg same(s(s(s(0))),s(0)) \\ \neg same(s(0),s(s(s(\ s(0))))) & \neg same(s(s(s(s(0)))),\ s(0)) \\ \ldots & \ldots \end{array}$$

Once we have the *same* relation, we can define the other relations in our arithmetic. The following axioms define the plus relation in terms of 0, s, and *same*. Adding 0 to any number results in that number. If adding a number x to a number y produces a number z, then adding the successor of x to y produces the successor of z. Finally, we have a functionality axiom for *plus*.

$$\begin{array}{l} \forall y.plus(0,y,y) \\ \forall x.\forall y.\forall z.(plus(x,y,z) \Rightarrow plus(s(x),y,s(z))) \\ \forall x.\forall y.\forall z.\forall w.(plus(x,y,z) \wedge \neg same(z,w) \Rightarrow \neg plus(x,y,w)) \end{array}$$

The axiomatization of multiplication is analogous. Multiplying any number by 0 produces 0. If a number z is the product of x and y and w is the sum of y and z, then w is the product of the successor of x and y. As before, we have a functionality axiom.

$$\begin{array}{l} \forall y.times(0,y,0) \\ \forall x.\forall y.\forall z.\forall w.(times(x,y,z) \wedge plus(y,z,w) \Rightarrow times(s(x),y,w)) \\ \forall x.\forall y.\forall z.\forall w.(times(x,y,z) \wedge \neg same(z,w) \Rightarrow \neg times(x,y,w)) \end{array}$$

That's all we need—just three axioms for *same* and three axioms for each arithmetic function.

Before we leave our discussion of Peano arithmetic, it is worthwhile to look at the concept of Diophantine equations. A *polynomial equation* is a sentence composed using only addition, multiplication, and exponentiation with fixed exponents (that is numbers not variables). For example, the expression shown below in traditional math notation is a polynomial equation.

$$x^2 + 2y = 4z$$

A *natural Diophantine equation* is a polynomial equation in which the variables are restricted to the natural numbers. For example, the polynomial equation here is also a Diophantine equation and happens to have a solution in the natural numbers: x=4 and y=8 and z=8.

Diophantine equations can be readily expressed as sentences in Peano Arithmetic. For example, we can represent the Diophantine equation above with the sentence shown below.

$$\forall x.\forall y.\forall z.\forall u.\forall v.\forall w.(\mathit{times}(x,x,u) \wedge \mathit{times}(2,y,v) \wedge \mathit{plus}(u,v,w) \Rightarrow \mathit{times}(4,z,w))$$

This is a little messy, but it is doable. And we can always clean things up by adding a little syntactic sugar to our notation to make it look like traditional math notation.

Once this mapping is done, we can use the tools of logic to work with these sentences. In some cases, we can find solutions; and, in some cases, we can prove that solutions do not exist. This has practical value in some situations, but it also has significant theoretical value in establishing important properties of Relational Logic, a topic that we discuss in a later section.

6.8 EXAMPLE: LINKED LISTS

A list is finite sequence of objects. Lists can be flat, e.g., $[a, b, c]$. Lists can also be nested within other lists, e.g., $[a, [b, c], d]$.

A linked list is a way of representing nested lists of variable length and depth. Each element is represented by a cell containing a value and a pointer to the remainder of the list. Our goal in this example is to formalize linked lists and define some useful relations.

To talk about lists of arbitrary length and depth, we use the binary function constant *cons*, and we use the object constant *nil* to refer to the empty list. In particular, a term of the form $\mathit{cons}(\tau_1, \tau_2)$ designates a sequence in which τ_1 denotes the first element and τ_2 denotes the rest of the list. With this function constant, we can encode the list $[a, b, c]$ as follows.

$$\mathit{cons}(a, \mathit{cons}(b, \mathit{cons}(c, \mathit{nil})))$$

The advantage of this representation is that it allows us to describe functions and relations on lists without regard to length or depth.

As an example, consider the definition of the binary relation *member*, which holds of an object and a list if the object is a top-level member of the list. Using the function constant *cons*, we can characterize the *member* relation as shown below. Obviously, an object is a member of a list if it is the first element; however, it is also a member if it is member of the rest of the list.

$$\forall x.\forall y.\mathit{member}(x, \mathit{cons}(x, y))$$
$$\forall x.\forall y.\forall z.(\mathit{member}(x, z) \Rightarrow \mathit{member}(x, \mathit{cons}(y, z)))$$

We also can define functions to manipulate lists in different ways. For example, the following axioms define a relation called *append*. The value of *append* (its last argument) is a list consisting of the elements in the list supplied as its first argument followed by the elements in the list supplied as its second. For example, we would have $\mathit{append}(\mathit{cons}(a,\mathit{nil}), \mathit{cons}(b, \mathit{cons}(c, \mathit{nil})), \mathit{cons}(a, \mathit{cons}(b, \mathit{cons}(c, \mathit{nil}))))$.

$$\forall z.append(nil, z, z)$$
$$\forall x.\forall y.\forall z.(append(y, z, w) \Rightarrow append(cons(x, y), z, cons(x,w)))$$

We can also define relations that depend on the structure of the elements of a list. For example, the *among* relation is true of an object and a list if the object is a member of the list, if it is a member of a list that is itself a member of the list, and so on.

$$\forall x.among(x, x)$$
$$\forall x.\forall y.\forall z.(among(x, y) \vee among(x, z) \Rightarrow among(x, cons(y, z)))$$

Lists are an extremely versatile representational device, and the reader is encouraged to become as familiar as possible with the techniques of writing definitions for functions and relations on lists. As is true of many tasks, practice is the best approach to gaining skill.

6.9 EXAMPLE: PSEUDO ENGLISH

Pseudo English is a formal language that is intended to approximate the syntax of the English language. One way to define the syntax of Pseudo English is to write grammatical rules in Backus Naur Form (BNF). The rules shown below illustrate this approach for a small subset of Pseudo English. A sentence is a noun phrase followed by a verb phrase. A noun phrase is either a noun or two nouns separated by the word *and*. A verb phrase is a verb followed by a noun phrase. A noun is either the word *Mary* or the word *Pat* or the word *Quincy*. A verb is either *like* or *likes*.

```
<sentence> ::= <np> <vp>
<np> ::= <noun>
<np> ::= <noun> "and" <noun>
<vp> ::= <verb> <np>
<noun> ::= "mary" | "pat" | "quincy"
<verb> ::= "like" | "likes"
```

Alternatively, we can use Relational Logic to formalize the syntax of Pseudo English. The sentences shown below express the grammar described in the BNF rules above. (We have dropped the universal quantifiers here to make the rules a little more readable.) Here, we are using the *append* relation defined in the section of lists.

$$np(x) \wedge vp(y) \wedge append(x,y,z) \Rightarrow sentence(z)$$
$$noun(x) \Rightarrow np(x)$$
$$noun(x) \wedge noun(y) \wedge append(x,\text{and},z) \wedge append(z,y,w) \Rightarrow np(w)$$
$$verb(x) \wedge np(y) \wedge append(x,y,z) \Rightarrow vp(z)$$
$$noun(mary)$$
$$noun(pat)$$
$$noun(quincy)$$
$$verb(like)$$
$$verb(likes)$$

Using these sentences, we can test whether a given sequence of words is a syntactically legal sentence in Pseudo English and we can use our logical entailment procedures to enumerate syntactically legal sentences, like those shown below.

mary likes pat
pat and quincy like mary
mary likes pat and quincy

One weakness of our BNF and the corresponding axiomatization is that there is no concern for agreement in number between subjects and verbs. Hence, with these rules, we can get the following expressions, which in Natural English are ungrammatical.

× *mary like pat*
× *pat and quincy likes mary*

Fortunately, we can fix this problem by elaborating our rules just a bit. In particular, we add an argument to some of our relations to indicate whether the expression is singular or plural. Here, 0 means singular, and 1 means plural. We then use this to block sequences of words where the number do not agree.

$$np(x,w) \wedge vp(y,w) \wedge append(x,y,z) \Rightarrow sentence(z)$$
$$noun(x) \Rightarrow np(x,0)$$
$$noun(x) \wedge noun(y) \wedge append(x,\text{and},z) \wedge append(z,y,w) \Rightarrow np(w,1)$$
$$verb(x,w) \wedge np(y,v) \wedge append(x,y,z) \Rightarrow vp(z,w)$$
$$noun(mary)$$
$$noun(pat)$$
$$noun(quincy)$$
$$verb(like,1)$$
$$verb(likes,0)$$

With these rules, the syntactically correct sentences shown above are still guaranteed to be sentences, but the ungrammatical sequences are blocked. Other grammatical features can be formalized in similar fashion, e.g., gender agreement in pronouns (*he* versus *she*), possessive adjectives (*his* versus *her*), reflexives (like *himself* and *herself*), and so forth.

6.10 EXAMPLE: METALEVEL LOGIC

Throughout this book, we have been writing sentences in English about sentences in Logic, and we have been writing informal proofs in English about formal proofs in Logic. A natural question to ask is whether it is possible formalize Logic within Logic. The answer is yes. The limits of what can be done are very interesting. In this section, we look at a small subset of this problem—using Relational Logic to formalize information about Propositional Logic.

The first step in formalizing Propositional Logic in Relational Logic is to represent the syntactic components of Propositional Logic.

In what follows, we make each proposition constant in our Propositional Logic language an object constant in our Relational Logic formalization. For example, if our Propositional Logic language has relation constants p, q, and r, then p, q, and r are object constants in our formalization.

Next, we introduce function constants to represent constructors of complex sentences. There is one function constant for each logical operator: *not* for $\neg$, *and* for $\wedge$, *or* for $\vee$, *if* for $\Rightarrow$, and *iff* for $\Leftrightarrow$. Using these function constants, we represent Propositional Logic sentences as terms in our language. For example, we use the term $and(p,q)$ to represent the Propositional Logic sentence $(p \wedge q)$; and we use the term $if(and(p,q),r)$ to represent the Propositional Logic sentence $(p \wedge q \Rightarrow r)$.

Finally, we introduce a selection of relation constants to express the types of various expressions in our Propositional Logic language. We use the unary relation constant *proposition* to assert that an expression is a proposition. We use the unary relation constant *negation* to assert that an expression is a negation. We use the unary relation constant *conjunction* to assert that an expression is a conjunction. We use the unary relation constant *disjunction* to assert that an expression is a disjunction. We use the unary relation constant *implication* to assert that an expression is an implication. We use the unary relation constant *biconditional* to assert that an expression is a biconditional. And we use the unary relation constant *sentence* to assert that an expression is a proposition.

With this vocabulary, we can characterize the syntax of our language as follows. We start with declarations of our proposition constants.

$$\begin{array}{l} proposition(p) \\ proposition(q) \\ proposition(r) \end{array}$$

Next, we define the types of expressions involving our various logical operators.

$$\begin{array}{l} \forall x.(sentence(x) \Rightarrow negation(not(x))) \\ \forall x.(sentence(x) \wedge sentence(y) \Rightarrow conjunction(and(x,y))) \\ \forall x.(sentence(x) \wedge sentence(y) \Rightarrow disjunction(or(x,y))) \\ \forall x.(sentence(x) \wedge sentence(y) \Rightarrow implication(if(x,y))) \\ \forall x.(sentence(x) \wedge sentence(y) \Rightarrow biconditional(iff(x,y))) \end{array}$$

Finally, we define sentences as expressions of these types.

$$\begin{array}{l} \forall x.(proposition(x) \Rightarrow sentence(x)) \\ \forall x.(negation(x) \Rightarrow sentence(x)) \\ \forall x.(conjunction(x) \Rightarrow sentence(x)) \\ \forall x.(disjunction(x) \Rightarrow sentence(x)) \\ \forall x.(implication(x) \Rightarrow sentence(x)) \\ \forall x.(biconditional(x) \Rightarrow sentence(x)) \end{array}$$

Note that these sentences constrain the types of various expressions but do not define them completely. For example, we have not said that $not(p)$ is *not* a conjunction. It is possible to make

our definitions more complete by writing negative sentences. However, they are a little messy, and we do not need them for the purposes of this section.

With a solid characterization of syntax, we can formalize our rules of inference. We start by representing each rule of inference as a relation constant. For example, we use the ternary relation constant *ai* to represent And Introduction, and we use the binary relation constant *ae* to represent And Elimination. With this vocabulary, we can define these relations as shown below.

$$\forall x.(sentence(x) \wedge sentence(y) \Rightarrow ai(x,y,and(x,y)))$$
$$\forall x.(sentence(x) \wedge sentence(y) \Rightarrow ae(and(x,y),x))$$
$$\forall x.(sentence(x) \wedge sentence(y) \Rightarrow ae(and(x,y),y))$$

In similar fashion, we can define proofs—both linear and structured. We can even define truth assignments, satisfaction, and the properties of validity, satisfiability, and so forth. Having done all of this, we can use the proof methods discussed in the next chapters to prove our metatheorems about Propositional Logic.

We can use a similar approach to formalizing Relational Logic within Relational Logic. However, in that case, we need to be very careful. If done incorrectly, we can write paradoxical sentences, i.e., sentences that are neither true or false. For example, a careless formalization leads to formal versions of sentences like *This sentence is false*, which is self-contradictory, i.e., it cannot be true and cannot be false. Fortunately, with care it is possible to avoid such paradoxes and thereby get useful work done.

6.11 PROPERTIES OF SENTENCES IN RELATIONAL LOGIC

Although the languages of Propositional Logic and Relational Logic are different, many of the key concepts of the two logics are the same. Notably, the concepts of validity, contingency, unsatisfiability, and so forth for Relational Logic have the same definitions in Relational Logic as in Propositional Logic.

A sentence is *satisfiable* if and only if it is satisfied by at least one truth assignment. A sentence is *falsifiable* if and only if there is at least one truth assignment that makes it false. We have already seen several examples of satisfiable and falsifiable sentences. A sentence is *unsatisfiable* if and only if it is not satisfied by any truth assignment, i.e., no matter what truth assignment we take, the sentence is always false. A sentence is *contingent* if and only if it is both satisfiable and falsifiable, i.e., it is neither valid nor unsatisfiable.

Not only are the definitions of these concepts the same; some of the results are the same as well. If we think of ground relational sentences and ground equations as propositions, we get similar results for the two logics—a ground sentence in Relational Logic is valid / contingent / unsatisfiable if and only if the corresponding sentence in Propositional Logic is valid / contingent / unsatisfiable.

Here, for example, are Relational Logic versions of common Propositional Logic validities—the Law of the Excluded Middle, Double Negation, and deMorgan's laws for distributing negation over conjunction and disjunction.

$$p(a) \vee \neg p(a)$$
$$p(a) \Leftrightarrow \neg\neg p(a)$$
$$\neg(p(a) \wedge q(a,b)) \Leftrightarrow (\neg p(a) \vee \neg q(a,b))$$
$$\neg(p(a) \vee q(a,b)) \Leftrightarrow (\neg p(a) \wedge \neg q(a,b))$$

Of course, not all sentences in Relational Logic are ground. There are valid sentences of Relational Logic for which there are no corresponding sentences in Propositional Logic.

The *Common Quantifier Reversal* tells us that reversing quantifiers of the same type has no effect on truth assignment.

$$\forall x.\forall y.q(x,y) \Leftrightarrow \forall y.\forall x.q(x,y)$$
$$\exists x.\exists y.q(x,y) \Leftrightarrow \exists y.\exists x.q(x,y))$$

Existential Distribution tells us that it is okay to move an existential quantifier inside of a universal quantifier. (Note that the reverse is not valid, as we shall see later.)

$$\exists y.\forall x.q(x,y) \Rightarrow \forall x.\exists y.q(x,y)$$

Finally, *Negation Distribution* tells us that it is okay to distribute negation over quantifiers of either type by flipping the quantifier and negating the scope of the quantified sentence.

$$\neg\forall x.p(x) \Leftrightarrow \exists x.\neg p(x)$$

6.12 LOGICAL ENTAILMENT

A set of Relational Logic sentences Δ *logically entails* a sentence ϕ (written $\Delta \models \phi$) if and only if every truth assignment that satisfies Δ also satisfies ϕ.

As with validity and contingency and satisfiability, this definition is the same for Relational Logic as for Propositional Logic. As before, if we treat ground relational sentences and ground equations as propositions, we get similar results. In particular, a set of ground premises in Relational Logic logically entails a ground conclusion in Relational Logic if and only if the corresponding set of Propositional Logic premises logically entails the corresponding Propositional Logic conclusion.

For example, we have the following results. The sentence $p(a)$ logically entails $(p(a) \vee p(b))$. The sentence $p(a)$ does *not* logically entail $(p(a) \wedge p(b))$. However, any set of sentences containing both $p(a)$ and $p(b)$ does logically entail $(p(a) \wedge p(b))$.

The presence of variables allows for additional logical entailments. For example, the premise $\exists y.\forall x.q(x,y)$ logically entails the conclusion $\forall x.\exists y.q(x, y)$. If there is *some* object y that is paired with every x, then every x has some object that it pairs with, namely y.

Here is another example. The premise $\forall x.\forall y.q(x,y)$ logically entails the conclusion $\forall x.\forall y.q(y, x)$. The first sentence says that q is true for all pairs of objects, and the second sentence says the exact same thing. In cases like this, we can interchange variables.

Understanding logical entailment for Relational Logic is complicated by the fact that it is possible to have free variables in Relational Logic sentences. Consider, for example, the premise $q(x,y)$ and the conclusion $q(y,x)$. Does $q(x,y)$ logically entail $q(y,x)$ or not?

Our definition for logical entailment and the semantics of Relational Logic give a clear answer to this question. Logical entailment holds if and only if every truth assignment that satisfies the premise satisfies the conclusion. A truth assignment satisfies a sentence with free variables if and only if it satisfies every instance. In other words, a sentence with free variables is equivalent to the sentence in which all of the free variables are universally quantified. In other words, $q(x,y)$ is satisfied if and only if $\forall x.\forall y.q(x,y)$ is satisfied, and similarly for $q(y,x)$. So, the first sentence here logically entails the second if and only if $\forall x.\forall y.q(x,y)$ logically entails $\forall x.\forall y.q(y,x)$; and, as we just saw, this is, in fact, the case.

6.13 FINITE RELATIONAL LOGIC

Finite Relational Logic (FRL) is that subset of Relational Logic in which the Herbrand base is finite. In order to guarantee a finite Herbrand base, we must restrict ourselves to vocabularies in which there are at most finitely many object constants and no function constants at all.

One interesting feature of FRL is that it is expressively equivalent to Propositional Logic (PL). For any FRL language, we can produce a pairing between the ground relational sentences that language and the proposition constants in a Propositional Logic language. Given this correspondence, for any set of *arbitrary* sentences in our FRL language, there is a corresponding set of sentences in the language of PL such that any FRL truth assignment that satisfies our FRL sentences agrees with the corresponding Propositional Logic truth assignment applied to the Propositional Logic sentences.

The procedure for transforming our FRL sentences to PL has multiple steps, but each step is easy. We first convert our sentences to prenex form, then we ground the result, and we rewrite in Propositional Logic. Let's look at these steps in turn.

A sentence is in *prenex form* if and only if it is closed and all of the quantifiers are on the outside of all logical operators. Converting a set of FRL sentences to a logically equivalent set in prenex form is simple. First, we rename variables in different quantified sentences to eliminate any duplicates. We then apply quantifier distribution rules in reverse to move quantifiers outside of logical operators. Finally, we universally quantify any free variables in our sentences.

For example, to convert the closed sentence $\forall y.p(x,y) \vee \exists y.q(y)$ to prenex form, we first rename one of the variables. In this case, let's rename the y in the second disjunct to z. This results in the sentence $\forall y.p(x,y) \vee \exists z.q(z)$. We then apply the distribution rules in reverse to produce $\forall y.\exists z.(p(x, y) \vee q(z))$. Finally, we universally quantify the free variable x to produce the prenex form of our original sentence: $\forall x.\forall y.\exists z.(p(x,y) \vee q(z))$

Once we have a set of sentences in prenex form, we compute the grounding. We start with our initial set Δ of sentences and we incrementally build up our grounding Γ. On each step we process a sentence in Δ, using the procedure described below. The procedure terminates when Δ becomes empty. The set Γ at that point is the grounding of the input.

(1) The first rule covers the case when the sentence ϕ being processed is ground. In this case, we remove the sentence from Delta and add it to Gamma.

$$\Delta_{i+1} = \Delta_i - \{\phi\}$$
$$\Gamma_{i+1} = \Gamma_i \cup \{\phi\}$$

(2) If our sentence is of the form $\forall v.\phi[v]$, we eliminate the sentence from Δ_i and replace it with copies of the scope, one copy for each object constant τ in our language.

$$\Delta_{i+1} = \Delta_i - \{\forall v.\phi[v]\} \cup \{\phi[\tau] \mid \tau \text{ an object constant}\}$$
$$\Gamma_{i+1} = \Gamma_i$$

(3) If our sentence of the form $\exists v.\phi[v]$, we eliminate the sentence from Δ_i and replace it with a disjunction, where each disjunct is a copy of the scope in which the quantified variable is replaced by an object constant in our language.

$$\Delta_{i+1} = \Delta_i \text{ - } \{\exists v.\phi[v]\} \cup \{\phi[\tau_1] \vee ... \vee \phi[\tau_n]\}$$
$$\Gamma_{i+1} = \Gamma_i$$

The procedure halts when Δ_i becomes empty. The set Γ_i is the grounding of the input. It is easy to see that Γ_i is logically equivalent to the input set.

Here is an example. Suppose we have a language with just two object constants a and b. And suppose we have the set of sentences shown below. We have one ground sentence, one universally quantified sentence, and one existentially quantified sentence. All are in prenex form.

$$\{p(a), \forall x.(p(x) \Rightarrow q(x)), \exists x.\neg q(x)\}$$

A trace of the procedure is shown below. The first sentence is ground, so we remove it from Δ add it to Γ. The second sentence is universally quantified, so we replace it with a copy for each of our two object constants. The resulting sentences are ground, and so we move them one by one from Δ to Γ. Finally, we ground the existential sentence and add the result to Δ and then move the ground sentence to Γ. At this point, since Δ is empty, Γ is our grounding.

$$\Delta_0 = \{\{p(a), \forall x.(p(x) \Rightarrow q(x)), \exists x.\neg q(x)\}$$
$$\Gamma_0 = \{\}$$

$$\Delta_1 = \{\forall x.(p(x) \Rightarrow q(x)), \exists x.\neg q(x)\}$$
$$\Gamma_1 = \{p(a)\}$$

$$\Delta_2 = \{p(a) \Rightarrow q(a),\ p(b) \Rightarrow q(b), \exists x.\neg q(x)\}$$
$$\Gamma_2 = \{p(a)\}$$

$$\Delta_3 = \{p(b) \Rightarrow q(b), \exists x.\neg q(x)\}$$
$$\Gamma_3 = \{p(a),\ p(a) \Rightarrow q(a)\}$$

$$\Delta_4 = \{\exists x.\neg q(x)\}$$
$$\Gamma_4 = \{p(a),\ p(a) \Rightarrow q(a),\ p(b) \Rightarrow q(b)\}$$

$$\Delta_5 = \{\neg q(a) \vee \neg q(b)\}$$
$$\Gamma_5 = \{p(a),\ p(a) \Rightarrow q(a),\ p(b) \Rightarrow q(b)\}$$

$$\Delta_6 = \{\}$$
$$\Gamma_6 = \{p(a),\ p(a) \Rightarrow q(a),\ p(b) \Rightarrow q(b),\ \neg q(a) \vee \neg q(b)\}$$

Once we have a grounding Γ, we replace each ground relational sentence in Γ by a proposition constant. The resulting sentences are all in Propositional Logic; and the set is equivalent to the sentences in Δ in that any FRL truth assignment that satisfies our FRL sentences agrees with the corresponding Propositional Logic truth assignment applied to the Propositional Logic sentences.

For example, let's represent the FRL sentence $p(a)$ with the proposition pa; let's represent $p(a)$ with pa; let's represent $q(a)$ with qa; and let's represent $q(b)$ with qb. With this correspondence, we can represent the sentences in our grounding with the Propositional Logic sentences shown below.

$$\{pa,\ pa \Rightarrow qa,\ pb \Rightarrow qb,\ \neg qa \vee \neg qb\}$$

Since the question of unsatisfiability for PL is decidable, then the question of unsatisfiability for FRL is also decidable. Since logical entailment and unsatisfiability are correlated, we also know that the question of logical entailment for FRL is decidable.

Another consequence of this correspondence between FRL and PL is that, like PL, FRL is *compact*—every unsatisfiable set of sentences contains a finite subset that is unsatisfiable. This is important as it assures us that we can demonstrate the unsatisfiability by analyzing just a finite set of sentences; and, as we shall see in the next chapter, logical entailment can be demonstrated with finite proofs.

6.14 OMEGA RELATIONAL LOGIC

Omega Relational Logic (ORL) is function-free Relational Logic. Like FRL, there are no function constants; but, unlike FRL, we can have infinitely many object constants.

Recall that a logic is *compact* if and only if every unsatisfiable set of sentences has a finite unsatisfiable subset. Suppose that the vocabulary for a language consists of the object constants 1, 2, 3, 4, ... and the unary relation constant p. Take the set of sentences $p(1)$, $p(2)$, $p(3)$, $p(4)$, ... and add in the sentence $\exists x.\neg p(x)$. This set is unsatisfiable. (Our ground atoms say that p is true of everything, but our existentially quantified sentence says that there is something for which p is not true.) However, if we were to remove any one of the sentences, it would be satisfiable; and so every finite subset is satisfiable.

Although it is not compact, ORL is well-behaved in one important way. In particular, the question of unsatisfiability for *finite* sets of sentences is *semidecidable*, i.e., there is a procedure that takes a finite set of sentences in ORL as argument and is guaranteed to halt in finite time with a positive answer if the set is unsatisfiable. Given a satisfiable set of sentences, the procedure *may* halt with a negative answer. However, this is not guaranteed. If the input set is satisfiable, the procedure may run forever.

The first step of the procedure is to convert our sentences to prenex form using the technique described in the section of Finite Relational Logic. We then ground the resulting set. Finally, we examine finite subsets of the grounding for unsatisfiability.

Unfortunately, the grounding technique for FRL does not work for ORL. Since we can have infinitely many object constants in ORL, in dealing with an existentially quantified sentence, we would have to generate an infinitely large disjunction, and sentences of infinite size are not permitted in our language. Fortunately, there is a an alternative form of grounding, called *omega grounding*, which serves our purpose without this defect.

Given a set of sentences in prenex form, we define its omega grounding as follows. Let Δ_0 be our initial set of sentences in prenex form. Let Λ_0 be the empty set. And let Γ_0 be the empty set. On each step of the definition, we consider one sentence in Δ, taking the sentences in the order in which they were added to Δ; we apply the following rules to define successive versions of Δ and Γ and Lambda; according to the following rules.

(1) If our sentence ϕ is ground, we drop the sentence from Δ and add it to Γ.

$$\Delta_{i+1} = \Delta_i - \{\phi\}$$
$$\Lambda_{i+1} = \Lambda_i$$
$$\Gamma_{i+1} = \Gamma_i \cup \{\phi\}$$

(2) If our sentence has the form $\forall \nu.\phi[\nu]$, we remove it from Δ, add it to Λ, and add instances of the scope for each constant τ_i that appears in any of our sets of sentences. Note that we do not add copies for every constant in the language, only those used in our sets of sentences.

$$\Delta_{i+1} = \Delta_i - \{\forall \nu.\phi[\nu]\} \cup \{\phi[\tau_1], \ldots, \phi[\tau_k]\}$$
$$\Lambda_{i+1} = \Lambda_i \cup \{\forall \nu.\phi[\nu]\}$$
$$\Gamma_{i+1} = \Gamma_i$$

(3) If our sentence has the from $\exists \nu.\phi[\nu]$, we eliminate the sentence from Δ and add in its place an instance of the scope for some unused constant τ. There must always be an unused constant, since at each point we have only finitely many sentences while the language has infinitely many constants. We also add to Δ instances of the scopes of all universally quantified sentences in Λ using the new constant τ.

$$\Delta_{i+1} = \Delta_i - \{\exists \nu.\phi[\nu]\} \cup \{\phi[\tau]\} \cup \{\psi[\tau] \mid \forall \mu.\psi[\mu] \in \Lambda_i\}$$
$$\Lambda_{i+1} = \Lambda_i$$
$$\Gamma_{i+1} = \Gamma_i$$

If Δ ever becomes empty, the process terminates. In any case, the omega grounding of the initial Δ is the union of all of the Γi sets. It is possible to show that this Γ is satisfiable if and only if the original Δ is satisfiable.

Let's look at an example. Suppose we have a language with object constants 1, 2, 3, ... And suppose we have the set of sentences shown below. We have one ground sentence, one universally quantified sentence, and one existentially quantified sentence. All are in prenex form.

$$\{p(1), \forall x.(p(x) \Rightarrow q(x)), \exists x.\neg q(x)\}$$

The definition goes as follows. The first sentence is ground, so we remove it from Δ add it to Γ. The second sentence is universally quantified, so we add it to Λ; and we replace it on Δ with a copy with 1 substituted for the universally quantified variable. Next, we focus on the existentially quantified sentence. We drop the sentence from Δ and add an instance in its place. In this case, we replace the existential variable by the previously unused object constant 2. We also use 2 in another instance of the sentence in Λ. At this point, everything in Δ is ground, so we just move the sentences, one by one, to Γ. At this point, since we have run out of sentences in Δ, we are done and the current value of Γ is our omega grounding.

$$\Delta_0 = \{p(1), \forall x.(p(x) \Rightarrow q(x)), \exists x.\neg q(x)\}$$
$$\Lambda_0 = \{\}$$
$$\Gamma_0 = \{\}$$

$$\Delta_1 = \{\forall x.(p(x) \Rightarrow q(x)), \exists x.\neg q(x)\}$$
$$\Lambda_1 = \{\}$$
$$\Gamma_1 = \{p(1)\}$$

$$\Delta_2 = \{\exists x.\neg q(x), p(1) \Rightarrow q(1)\}$$
$$\Lambda_2 = \{\forall x.(p(x) \Rightarrow q(x))\}$$
$$\Gamma_2 = \{p(1)\}$$

$$\Delta_3 = \{p(1) \Rightarrow q(1), \neg q(2), p(2) \Rightarrow q(2)\}$$
$$\Lambda_3 = \{\forall x.(p(x) \Rightarrow q(x))\}$$
$$\Gamma_3 = \{p(1)\}$$

$$\Delta_4 = \{\neg q(2), p(2) \Rightarrow q(2)\}$$
$$\Lambda_4 = \{\forall x.(p(x) \Rightarrow q(x))\}$$
$$\Gamma_4 = \{p(1), p(1) \Rightarrow q(1)\}$$

$$\Delta_5 = \{p(2) \Rightarrow q(2)\}$$
$$\Lambda_5 = \{\forall x.(p(x) \Rightarrow q(x))\}$$
$$\Gamma_5 = \{p(1), p(1) \Rightarrow q(1), \neg q(2)\}$$

$$\Delta_6 = \{\}$$
$$\Lambda_6 = \{\forall x.(p(x) \Rightarrow q(x))\}$$
$$\Gamma_6 = \{p(1), p(1) \Rightarrow q(1), \neg q(2), p(2) \Rightarrow q(2)\}$$

Note that, when there are existentially quantified sentences nested within universally quantified sentences, the result of this definition may be infinite in size. Consider, as an example, the set of sentences shown below.

$$\{p(1), \forall x.\exists y.q(x,y)\}$$

In this case, the procedure goes as follows. The first sentence is ground, so we remove it from our input set add it to Γ. The next sentence is universally quantified, so we add it to Λ and replace it in Δ with a copy for each constant mentioned in our sentences. In this case, there is just one such constant: 1. Next, we ground the existential sentence using a new constant, in this case 2, and add the result to Δ We also add a corresponding instance of the sentence in Λ. Our ground instance is then moved to Γ. Again, we ground our existential sentence, in this case using the object constant 3; and, again, we instantiate the sentence in Λ. As before, we move the ground instance to Γ. The construction goes on repeatedly in this fashion.

$\Delta_0 = \{p(1), \forall x.\exists y.q(x,y)\ \}$
$\Lambda_0 = \{\}$
$\Gamma_0 = \{\}$

$\Delta_1 = \{\forall x.\exists y.q(x, y)\}$
$\Lambda_1 = \{\}$
$\Gamma_1 = \{p(1)\}$

$\Delta_2 = \{\exists y.q(1,y)\}$
$\Lambda_2 = \{\forall x.\exists y.q(x, y)\}$
$\Gamma_2 = \{p(1)\}$

$\Delta_3 = \{q(1,2), \exists y.q(2,y)\}$
$\Lambda_3 = \{\forall x.\exists y.q(x, y)\}$
$\Gamma_3 = \{p(1)\}$

$\Delta_4 = \{\exists y.q(2,y)\}$
$\Lambda_4 = \{\forall x.\exists y.q(x, y)\}$
$\Gamma_4 = \{p(1), q(1,2)\}$

$\Delta_5 = \{q(2,3), \exists y.q(3,y)\}$
$\Lambda_5 = \{\forall x.\exists y.q(x, y)\}$
$\Gamma_5 = \{p(1), q(1,2)\}$

$\Delta_6 = \{\exists y.q(3,y)\}$
$\Lambda_6 = \{\forall x.\exists y.q(x, y)\}$
$\Gamma_6 = \{p(1), q(1,2), q(2,3)\}$

...

The union of this infinite sequence of Γ sets is the omega grounding for Δ. Although the answer in this case in infinitely large, it is okay, because we do not need to compute the entire thing, only finite subsets of increasing size.

Note that the omega grounding for a finite set of ORL sentences is all ground. We know that, if a set of ground sentences is unsatisfiable, then there must be a finite subset that is unsatisfiable. Hence, to determine unsatisfiability, we can just check finite subsets of our omega grounding. If the set is unsatisfiable, we will eventually encounter an unsatisfiable set. Otherwise, the procedure may run forever. In any case, this demonstrates the semidecidability of for finite sets of sentences in ORL.

6.15 GENERAL RELATIONAL LOGIC

As we have seen Finite Relational Logic is very well-behaved. The questions of unsatisfiability and logical entailment are decidable. In ORL we can express more; and, although we lose decidability, we retain semidecidability. What happens when we relax our restrictions and look at General Relational Logic (GRL), i.e., Relational Logic without any restrictions whatsoever?

The good news about GRL is that it is highly expressive. We can formalize things in GRL that cannot be formalized (at least in finite form) in the other subsets we have examined thus far. For example, in an earlier section, we showed how to define addition and multiplication in finite form. This is not possible with restricted forms of Relational Logic (such as FRL and ORL) and in other logics (e.g., First Order Logic).

The bad news is that the questions of unsatisfiability and logical entailment for GRL are not semidecidable. Explaining this in detail is beyond the scope of this course. However, we can give a line of argument that suggests why it is true. The argument reduces a problem that is generally accepted to be non-semideciable to the question of unsatisfiability / logical entailment for General Relational Logic. If our logic were semidecidable, then this other question would be semidecidable as well; and, since it is known not to be semidecidable, the GRL must not be semidecidable either.

As we know, Diophantine equations can be readily expressed as sentences in GRL. For example, we can represent the solvability of Diophantine equation $3x^2=1$ with the sentence shown below.

$$\exists x.\exists y.(\mathit{times}(x, x, y) \wedge \mathit{times}(s(s(s(0))), y, s(0)))$$

We can represent every Diophantine in an analogous way. We can express the unsolvability of a Diophantine equation by negating the corresponding sentence. We can then ask the question of whether the axioms of arithmetic logically entail this negation or, equivalently, whether the axioms of Arithmetic together with the unnegated sentence are unsatisfiable.

The problem is that it is well known that determining whether Diophantine equations are unsolvable is not semidecidable. If we could determine the unsatisfiability of our GRL encoding of a Diophantine equation, we could decide whether it is unsolvable, contradicting the non-semidecidability of that problem.

Note that this does not mean GRL is useless. In fact, it is great for expressing such information; and we can prove many results, even though, in general, we cannot prove everything that follows from arbitrary sets of sentences in Relational Logic. We discuss this issue further in later chapters.

RECAP

Relational Logic is an extension of Propositional Logic that includes some additional linguistic features: constants, variables and quantifiers. In Relational Logic, simple sentences have more structure than in Propositional Logic. Furthermore, using variables and quantifiers, we can express information about multiple objects without enumerating those objects; and we can express the existence of

objects that satisfy specified conditions without saying which objects they are. The syntax of Relational Logic begins with object constants, function constants, and relation constants. *Relational sentences* are the atomic elements from which more complex sentences are built. *Logical sentences* are formed by combining simpler sentences with logical operators. In the version of Relational Logic used here, there are five types of logical sentences: negations, conjunctions, disjunctions, implications, and equivalences. There are two types of *quantified sentences*: *universal sentences* and *existential sentences*. The *Herbrand base* for a Relational Logic language is the set of all ground relational sentences in the language. A *truth assignment* for a Relational Logic language is a mapping that assigns a truth value to each element of it Herbrand base. The truth or falsity of compound sentences is determined from an truth assignment using rules based on the five logical operators of the language. A truth assignment *i satisfies* a sentence if and only if the sentences is *true* under that truth assignment. A sentence is *valid* if and only if it is satisfied by *every* truth assignment. A sentence is *satisfiable* if and only if it is satisfied by at least one truth assignment. A sentence is *falsifiable* if and only if there is at least one truth assignment that makes it false. A sentence is *unsatisfiable* if and only if it is not satisfied by any truth assignment. A sentence is *contingent* if and only if it is both satisfiable and falsifiable, i.e., it is neither valid nor unsatisfiable. A set of sentences Δ *logically entails* a sentence ϕ (written $\Delta \models \phi$) if and only if every truth assignment that satisfies Δ also satisfies ϕ. A class of questions is *decidable* if and only if there is an effective procedure that is guaranteed to halt on any question in the class and give the correct answer. A class of questions is *semidecidable* if and only if there is an effective procedure that is guaranteed to halt on any question in the class in case the answer is true but is not guaranteed to halt if the answers is false. The questions of unsatisfiability and logical entailment for Finite Relational Logic is decidable. The questions of unsatisfiability and logical entailment for Omega Relational Logic is semidecidable for finite sets of sentences. The questions of unsatisfiability and logical entailment for General Relational Logic is not even semidecidable. A logic is *compact* if and only if every unsatisfiable set of sentences contains a finite subset that is unsatisfiable. Finite Relational Logic is compact, but Omega Relational Logic and General Relational Logic are not.

EXERCISES

6.1 Say whether each of the following expressions is a syntactically legal sentence of Relational Logic. Assume that *red*, *green*, *jim*, and *molly* are object constants; assume that *color* is a binary function constant; and assume that *parent* is a binary relation constant.

(*a*) *parent*(*red*, *green*)
(*b*) ¬*color*(*jim*, *green*)
(*c*) *color*(*blue*, *molly*)
(*d*) *parent*(*molly*, *molly*)
(*e*) *parent*(*molly*, *z*)
(*f*) ∃*x*.*parent*(*molly*, *x*)

(g) $\exists y.parent(molly, jim)$
(h) $\forall z.(z(jim, molly) \Rightarrow z(molly, jim))$

6.2 Consider a language with n object constants, no function constants, and a single binary relation constant.

(a) How many ground terms are there in this language $-n$, n^2, 2^n, 2^{n^2}, 2^{2^n}?
(b) How many ground atomic sentences are there in this language $-n$, n^2, 2^n, 2^{n^2}, 2^{2^n}?
(c) How many distinct truth assignments are possible for this language $-n$, n^2, 2^n, 2^{n^2}, 2^{2^n}?

6.3 Consider a language with object constants a and b, no function constants, and relation constants p and q where p has arity 1 and q has arity 2. Imagine a truth assignment that makes $p(a)$, $q(a,b)$, $q(b,a)$ true and all other ground atoms false. Say whether each of the following sentences is true or false under this truth assignment.

(a) $\forall x.(p(x) \Rightarrow q(x,x))$
(b) $\forall x.\exists y.q(x,y)$
(c) $\exists y.\forall x.q(x,y)$
(d) $\forall x.(p(x) \Rightarrow \exists y.q(x,y))$
(e) $\forall x.p(x) \Rightarrow \exists y.q(y,y)$

6.4 Consider a state of the Sorority World that satisfies the following sentences.

$\neg likes(abby,abby)$	$likes(abby,bess)$	$\neg likes(abby,cody)$	$likes(abby,dana)$
$likes(bess,abby)$	$\neg likes(bess,bess)$	$likes(bess,cody)$	$\neg likes(bess,dana)$
$\neg likes(cody,abby)$	$likes(cody,bess)$	$\neg likes(cody,cody)$	$likes(cody,dana)$
$likes(dana,abby)$	$\neg likes(dana,bess)$	$likes(dana,cody)$	$\neg likes(dana,dana)$

Say which of the following sentences is satisfied by this state of the world.

(a) $likes(dana,cody)$
(b) $\neg likes(abby,dana)$
(c) $likes(bess,cody) \vee likes(bess,dana)$
(d) $\forall y.(likes(bess,y) \Rightarrow likes(abby,y))$
(e) $\forall y.(likes(y,cody) \Rightarrow likes(cody,y))$
(f) $\forall x.\neg likes(x,x)$

6.5 Consider a version of the Blocks World with just three blocks - a, b, and c. The on relation is axiomatized below.

$\neg on(a,a)$	$on(a,b)$	$\neg on(a,c)$
$\neg on(b,a)$	$\neg on(b,b)$	$on(b,c)$
$\neg on(c,a)$	$\neg on(c,b)$	$\neg on(c,c)$

Let's suppose that the *above* relation is defined as follows.

$$\forall x.\forall z.(above(x,z) \Leftrightarrow on(x,z) \vee \exists y.(above(x,y) \wedge above(y,z)))$$

A sentence ϕ is consistent with a set Δ of sentences if and only if there is a truth assignment that satisfies all of the sentences in $\Delta \cup \{\phi\}$. Say whether each of the following sentences is consistent with the sentences about *on* and *above* shown above.

(*a*) $above(a,c)$
(*b*) $above(a,a)$
(*c*) $above(c,a)$

6.6 Say whether each of the following sentences is valid, contingent, or unsatisfiable.

(*a*) $\forall x.p(x) \Rightarrow \exists x.p(x)$
(*b*) $\exists x.p(x) \Rightarrow \forall x.p(x)$
(*c*) $\forall x.p(x) \Rightarrow p(x)$
(*d*) $\exists x.p(x) \Rightarrow p(x)$
(*e*) $p(x) \Rightarrow \forall x.p(x)$
(*f*) $p(x) \Rightarrow \exists x.p(x)$
(*g*) $\forall x.\exists y.p(x,y) \Rightarrow \exists y.\forall x.p(x,y)$
(*h*) $\forall x.(p(x) \Rightarrow q(x)) \Rightarrow \exists x.(p(x) \wedge q(x))$
(*i*) $\forall x.(p(x) \Rightarrow q(x)) \wedge \exists x.(p(x) \wedge \neg q(x))$
(*j*) $(\exists x.p(x) \Rightarrow \forall x.q(x)) \vee (\forall x.q(x) \Rightarrow \exists x.r(x))$

6.7 Let Γ be a set of Relational Logic sentences, and let ϕ and ψ be individual Relational Logic sentences. For each of the following claims, state whether it is true or false.

(*a*) $\forall x.\phi \models \phi$
(*b*) $\phi \models \forall x.\phi$
(*c*) If $\Gamma \models \neg\phi[\tau]$ for some ground term τ, then $\Gamma \not\models \forall x.\phi[x]$
(*d*) If $\Gamma \models \phi[\tau]$ for some ground term τ, then $\Gamma \models \exists x.\phi[x]$
(*e*) If $\Gamma \models \phi[\tau]$ for every ground term τ, then $\Gamma \models \forall x.\phi[x]$

CHAPTER 7

Relational Logic Proofs

7.1 INTRODUCTION

Logical entailment for Relational Logic is defined the same as for Propositional Logic. A set of premises logically entails a conclusion if and only if every truth assignment that satisfies the premises also satisfies the conclusions. In the case of Propositional Logic, we can check logical entailment by examining a truth table for the language. With finitely many proposition constants, the truth table is large but finite. For Relational Logic, things are not so easy. It is possible to have Herbrand bases of infinite size; and, in such cases, truth assignments are infinitely large and there are infinitely many of them, making it impossible to check logical entailment using truth tables.

All is not lost. As with Propositional Logic, we can establish logical entailment in Relational Logic by writing proofs. In fact, it is possible to show that, with a few simple restrictions, a set of premises logically entails a conclusions if and only if there is a finite proof of the conclusion from the premises, even when the Herbrand base is infinite. Moreover, it is possible to find such proofs in a finite time. That said, things are not perfect. If a set of sentences does *not* logically entail a conclusion, then the process of searching for a proof might go on forever. Moreover, if we remove the restrictions mentioned above, we lose the guarantee of finite proofs. Still, the relationship between logical entailment and finite provability, given those restrictions, is a very powerful result and has enormous practical benefits.

In this chapter, we start by extending the Fitch system from Propositional Logic to Relational Logic. We then illustrate the system with a few examples. Finally, we talk about soundness and completeness.

7.2 PROOFS

Formal proofs in Relational Logic are analogous to formal proofs in Propositional Logic. The major difference is that there are four additional mechanisms to deal quantified sentences.

The Fitch system for Relational Logic is an extension of the Fitch system for Propositional Logic. In addition to the ten logical rules of inference, there are four ordinary rules of inference for quantified sentences. Let's look at each of these in turn. (If you're like me, the prospect of going through a discussion of so many rules of inference sounds a little repetitive and boring. However, it is not so bad. Each of the rules has its own quirks and idiosyncrasies, its own personality. In fact, a couple of the rules suffer from a distinct excess of personality. If we are to use the rules correctly, we need to understand these idiosyncrasies.)

Universal Introduction (UI) allows us to reason from arbitrary sentences to universally quantified versions of those sentences.

Universal Introduction

$$\frac{\phi}{\forall v.\phi}$$

where v does not occur free in both ϕ and an active assumption

Typically, UI is used on sentences with free variables to make their quantification explicit. For example, if we have the sentence *hates*(*jane*,*y*), then, we can infer $\forall y.$*hates*(*jane*,*y*).

Note that we can also apply the rule to sentences that do not contain the variable that is quantified in the conclusion. For example, from the sentence *hates*(*jane*,*jill*), we can infer $\forall x.$*hates*(*jane*,*jill*). And, from the sentence *hates*(*jane*,*y*), we can infer $\forall x.$*hates*(*jane*,*y*). These are not particularly sensible conclusions. However, the results are correct, and the deduction of such results is necessary to ensure that our proof system is complete.

There is one important restriction on the use of Universal Introduction. If the variable being quantified appears in the sentence being quantified, it must not appear free in any *active assumption*, i.e., an assumption in the current subproof or any superproof of that subproof. For example, if there is a subproof with assumption $p(x)$ and from that we have managed to derive $q(x)i$, then we cannot just write $\forall x.q(x)$.

If we want to quantify a sentence in this situation, we must first use Implication Introduction to discharge the assumption and then we can apply Universal Introduction. For example, in the case just described, we can first apply Implication Introduction to derive the result $(p(x) \Rightarrow q(x))$ in the parent of the subproof containing our assumption, and we can then apply Universal Introduction to derive $\forall x.(p(x) \Rightarrow q(x))$.

Universal Elimination (UE) allows us to reason from the general to the particular. It states that, whenever we believe a universally quantified sentence, we can infer a version of the target of that sentence in which the universally quantified variable is replaced by an appropriate term.

Universal Elimination

$$\frac{\forall v.\phi[v]}{\phi[\tau]}$$

where τ is substitutable for v in ϕ

For example, consider the sentence $\forall y.$*hates*(*jane*,*y*). From this premise, we can infer that Jane hates Jill, i.e., *hates*(*jane*,*jill*). We also can infer that Jane hates her mother, i.e., *hates*(*jane*,*mother*(*jane*)). We can even infer than Jane hates herself, i.e., *hates*(*jane*,*jane*).

In addition, we can use Universal Elimination to create conclusions with free variables. For example, from $\forall x.$*hates*(*jane*,*x*), we can infer *hates*(*jane*,*x*) or, equivalently, *hates*(*jane*,*y*).

In using Universal Elimination, we have to be careful to avoid conflicts with other variables and quantifiers in the quantified sentence. This is the reason for the constraint on the replacement term. As an example of what can go wrong without this constraint, consider the sentence $\forall x.\exists y.$*hates*(x, y),

i.e., everybody hates somebody. From this sentence, it makes sense to infer $\exists y.\mathit{hates}(\mathit{jane},y)$, i.e., Jane hates somebody. However, we do not want to infer $\exists y.\mathit{hates}(y, y)$; i.e., there is someone who hates herself.

We can avoid this problem by obeying the restriction on the Universal Elimination rule. We say that a term τ is *substitutable* for a variable ν in a sentence ϕ if and only if no free occurrence of ν occurs within the scope of a quantifier of some variable in τ. For example, the term x is substitutable for y in $\exists z.\mathit{hates}(y, z)$. However, the term z is not substitutable for y, since y is being replaced by z and y occurs within the scope of a quantifier of z. Thus, we cannot substitute z for y in this sentence, and we avoid the problem we have just described.

Existential Introduction (EI). If we believe a sentence $\phi[\tau]$ that is the result of replacing every free occurrence of a variable ν by a term τ, then we can infer the existentially quantified sentence $\exists\nu.\phi[\nu]$.

Existential Introduction

$$\frac{\phi[\tau]}{\exists\nu.\phi[\nu]}$$

where τ is substitutable for ν in ϕ

The rule is stated in a way that requires thinking "backwards". For example, from the sentence $\mathit{hates}(\mathit{jill},\mathit{jill})$, we can infer that there is someone who hates herself, i.e., $\exists x.\mathit{hates}(x,x)$, because $\mathit{hates}(\mathit{jill},\mathit{jill})$ can be obtained from $\mathit{hates}(x,x)$ by replacing every free occurrence of the variable x by the term *jill*. We can also infer that there is someone Jill hates, i.e., $\exists x.\mathit{hates}(\mathit{jill},x)$, because $\mathit{hates}(\mathit{jill},\mathit{jill})$ can be obtained from $\mathit{hates}(\mathit{jill},x)$ by replacing every free occurrence of the variable x by the term *jill*. Finally, we can also infer that there is someone who hates Jill, i.e., $\exists x.\mathit{hates}(x,\mathit{jill})$, because $\mathit{hates}(\mathit{jill},\mathit{jill})$ can be obtained from $\mathit{hates}(x,\mathit{jill})$ by replacing every free occurrence of the variable x by the term *jill*.

For a further example, by two applications of Existential Introduction, we can infer that someone hates someone, i.e., $\exists x.\exists y.\mathit{hates}(x,y)$.

Notice that the rule avoids certain unwarranted inferences. For example, starting from $\exists x.\mathit{hates}(\mathit{jane},x)$, one might be tempted to infer $\exists x.\exists x.\mathit{hates}(x,x)$. It is an odd sentence since it contains nested quantifiers of the same variable. Nevertheless, it is a legal sentence, and it states that there is someone who hates himself, which does not follow from the premise in this case.

The inference is disallowed under Existential Introduction because the premise $\exists x.\mathit{hates}(\mathit{jane},x)$, cannot be obtained from the sentence $\exists x.\mathit{hates}(x,x)$ by replacing every *free* occurrence of x by *jane*.

Consider another example of a disallowed inference under Existential Introduction. If we disregarded the condition that τ be substitutable for ν in ϕ, we could start from the premise $\forall x.\exists y.\mathit{hates}(x, y)$ (everyone hates someone) and infer the conclusion $\exists z.\forall x.\exists y.\mathit{hates}(x, z)$ (there is a somebody whom everyone hates) because $\forall x.\exists y.\mathit{hates}(x, y)$ can be obtained from $\forall x.\exists y.\mathit{hates}(x, z)$ by replacing every free occurrence of z by y. The inference is disallowed under Existential Introduction because y is not substitutable for z in $\forall x.\exists y.\mathit{hates}(x, z)$.

Existential Elimination (EE). Suppose we have an existentially quantified sentence with target ϕ; and suppose we have a universally quantified implication in which the antecedent is the same as the scope of our existentially quantified sentence and the conclusion does not contain any free occurrences of the quantified variable. Then, we can use Existential Elimination to infer the consequent.

Existential Elimination

$$\frac{\exists v_1.\phi[v_1] \quad \forall v_2.(\phi[v_2] \Rightarrow \psi)}{\psi}$$

where v_2 does not occur free in ψ

For example, if we have the sentence $\forall x.(\mathit{hates}(\mathit{jane},x) \Rightarrow \neg \mathit{nice}(\mathit{jane}))$ and we have the sentence $\exists x.\mathit{hates}(\mathit{jane},x)$, then we can conclude $\neg \mathit{nice}(\mathit{jane})$.

It is interesting to note that Existential Elimination is analogous to Or Elimination. This parallel is expected because an existential sentence is effectively a disjunction. Recall that, in Or Elimination, we start with a disjunction with n disjuncts and n implications, one for each of the disjuncts and produce as conclusion the consequent shared by all of the implications. An existential sentence (like the first premise in any instance of Existential Elimination) is effectively a disjunction over the set of all ground terms; and a universal implication (like the second premise in any instance of Existential Elimination) is effectively a set of implications, one for each ground term in the language. The conclusion of Existential Elimination is the common consequent of these implications, just as in Or Elimination.

As in Chapter 3, we define a *structured proof* of a conclusion from a set of premises to be a sequence of (possibly nested) sentences terminating in an occurrence of the conclusion at the *top level* of the proof. Each step in the proof must be either (1) a premise (at the top level) or an assumption (other than at the top level) or (2) the result of applying an ordinary or structured rule of inference to earlier items in the sequence (subject to the constraints given above and in Chapter 3).

7.3 EXAMPLE

As an illustration of these concepts, consider the following problem. Suppose we believe that everybody loves somebody. And suppose we believe that everyone loves a lover. Our job is to prove that Jack loves Jill.

First, we need to formalize our premises. Everybody loves somebody. For all y, there exists a z such that $\mathit{loves}(y,z)$.

$$\forall y.\exists z.\mathit{loves}(y,z)$$

Everybody loves a lover. If a person is a lover, then everyone loves him. If a person loves another person, then everyone loves him. For all x and for all y and for all z, $\mathit{loves}(y,z)$ implies $\mathit{loves}(x,y)$.

$$\forall x.\forall y.\forall z.(\mathit{loves}(y,z) \Rightarrow \mathit{loves}(x,y))$$

Our goal is to show that Jack loves Jill. In other words, starting with the preceding sentences, we want to derive the following sentence.

$$loves(jack,jill)$$

A proof of this result is shown below. Our premises appear on lines 1 and 2. The sentence on line 3 is the result of applying Universal Elimination to the sentence on line 1, substituting *jill* for y. The sentence on line 4 is the result of applying Universal Elimination to the sentence on line 2, substituting *jack* for x. The sentence on line 5 is the result of applying Universal Elimination to the sentence on line 4, substituting *jill* for y. Finally, we apply Existential Elimination to produce our conclusion on line 6.

1.	$\forall y.\exists z.loves(y,z)$	Premise
2.	$\forall x.\forall y.\forall z.(loves(y,z) \Rightarrow loves(x,y))$	Premise
3.	$\exists z.loves(jill,z)$	UE: 1
4.	$\forall y.\forall z.(loves(y,z) \Rightarrow loves(jack,y))$	UE: 2
5.	$\forall z.(loves(jill,z) \Rightarrow loves(jack,jill))$	UE: 4
6.	$loves(jack,jill)$	EE: 3, 5

Now, let's consider a slightly more interesting version of this problem. We start with the same premises. However, our goal now is to prove the somewhat stronger result that everyone loves everyone. For all x and for all y, x loves y.

$$\forall x.\forall y.loves(x,y)$$

The proof shown below is analogous to the proof above. The only difference is that we have free variables in place of object constants at various points in the proof. Once again, our premises appear on lines 1 and 2. Once again, we use Universal Elimination to produce the result on line 3; but this time the result contains a free variable. Note that we have replaced the We get the results on lines 4 and 5 by successive application of Universal Elimination to the sentence on line 2. We deduce the result on line 6 using Existential Elimination. Finally, we use two applications of Universal Introduction to generalize our result and produce the desired conclusion.

1.	$\forall y.\exists z.loves(y,z)$	Premise
2.	$\forall x.\forall y.\forall z.(loves(y,z) \Rightarrow loves(x,y))$	Premise
3.	$\exists z.loves(y,z)$	UE: 1
4.	$\forall y.\forall z.(loves(y,z) \Rightarrow loves(x,y))$	UE: 2
5.	$\forall z.(loves(y,z) \Rightarrow loves(x,y))$	UE: 4
6.	$loves(x,y)$	EE: 3, 5
7.	$\forall y.loves(x,y)$	UI: 6
8.	$\forall x.\forall y.loves(x,y)$	UI: 7

This second example illustrates the power of free variables. We can manipulate them as though we are talking about specific individuals (though each one could be any object); and, when we are done, we can universalize them to derive universally quantified conclusions.

7.4 EXAMPLE

As another illustration of Relational Logic proofs, consider the following problem. We know that horses are faster than dogs and that there is a greyhound that is faster than every rabbit. We know that Harry is a horse and that Ralph is a rabbit. Our job is to derive the fact that Harry is faster than Ralph.

First, we need to formalize our premises. The relevant sentences follow. Note that we have added two facts about the world not stated explicitly in the problem: that greyhounds are dogs and that our *faster than* relationship is transitive.

$$\forall x.\forall y.(h(x) \wedge d(y) \Rightarrow f(x,y))$$
$$\exists y.(g(y) \wedge \forall z.(r(z) \Rightarrow f(y,z)))$$
$$\forall y.(g(y) \Rightarrow d(y))$$
$$\forall x.\forall y.\forall z.(f(x,y) \wedge f(y,z) \Rightarrow f(x,z))$$
$$h(harry)$$
$$r(ralph)$$

Our goal is to show that Harry is faster than Ralph. In other words, starting with the preceding sentences, we want to derive the following sentence.

$$f(harry,ralph)$$

The derivation of this conclusion goes as shown below. The first six lines correspond to the premises just formalized. On line 7, we start a subproof with an assumption corresponding to the scope of the existential on line 2, with the idea of using Existential Elimination later on in the proof. Lines 8 and 9 come from And Elimination. Line 10 is the result of applying Universal Elimination to the sentence on line 9. On line 11, we use Implication Elimination to infer that y is faster than Ralph. Next, we instantiate the sentence about greyhounds and dogs and infer that y is a dog. Then, we instantiate the sentence about horses and dogs; we use And Introduction to form a conjunction matching the antecedent of this instantiated sentence; and we infer that Harry is faster than y. We then instantiate the transitivity sentence, again form the necessary conjunction, and infer the desired conclusion. Finally, we use Implication Introduction to discharge our subproof; we use Universal Introduction to universalize the results; and we use Existential Elimination to produce our desired conclusion.

1.	$\forall x.\forall y.(h(x) \wedge d(y) \Rightarrow f(x,y))$	Premise
2.	$\exists y.(g(y) \wedge \forall z.(r(z) \Rightarrow f(y,z)))$	Premise
3.	$\forall y.(g(y) \Rightarrow d(y))$	Premise
4.	$\forall x.\forall y.\forall z.(f(x,y) \wedge f(y,z) \Rightarrow f(x,z))$	Premise
5.	$h(harry)$	Premise
6.	$r(ralph)$	Premise

7.	$g(y) \wedge \forall z.(r(z) \Rightarrow f(y,z))$	Assumption
8.	$g(y)$	AE: 7
9.	$\forall z.(r(z) \Rightarrow f(y,z))$	AE: 7
10.	$r(ralph) \Rightarrow f(y,ralph)$	UE: 9
11.	$f(y,ralph)$	IE: 10, 6
12.	$g(y) \Rightarrow d(y)$	UE: 3
13.	$d(y)$	IE: 12, 8
14.	$\forall y.(h(harry) \wedge d(y) \Rightarrow f(harry,y))$	UE: 1
15.	$h(harry) \wedge d(y) \Rightarrow f(harry,y)$	UE: 14
16.	$h(harry) \wedge d(y)$	AI: 5, 13
17.	$f(harry,y)$	IE: 15, 16
18.	$\forall y.\forall z.(f(harry,y) \wedge f(y,z) \Rightarrow f(harry,z))$	UE: 4
19.	$\forall z.(f(harry,y) \wedge f(y,z) \Rightarrow f(harry,z))$	UE: 18
20.	$f(harry,y) \wedge f(y,ralph) \Rightarrow f(harry,ralph)$	UE: 19
21.	$f(harry,y) \wedge f(y,ralph)$	AI: 17, 11
22.	$f(harry,ralph)$	IE: 20, 21
23.	$g(y) \wedge \forall z.(r(z) \Rightarrow f(y,z)) \Rightarrow f(harry,ralph)$	II: 7, 22
24.	$\forall y.(g(y) \wedge \forall z.(r(z) \Rightarrow f(y,z)) \Rightarrow f(harry,ralph))$	UI: 23
25.	$f(harry,ralph)$	EE: 2, 24

This derivation is somewhat lengthy, but it is completely mechanical. Each conclusion follows from previous conclusions by a mechanical application of a rule of inference. On the other hand, in producing this derivation, we rejected numerous alternative inferences. Making these choices intelligently is one of the key problems in the process of inference.

7.5 EXAMPLE

In this section, we use our proof system to prove some basic results involving quantifiers.

Given $\forall x.\forall y.(p(x,y) \Rightarrow q(x))$, we know that $\forall x.(\exists y.p(x,y) \Rightarrow q(x))$. In general, given a universally quantified implication, it is okay to drop a universal quantifier of a variable outside the implication and apply an existential quantifier of that variable to the antecedent of the implication, provided that the variable does not occur in the consequent of the implication.

Our proof is shown below. As usual, we start with our premise. We start a subproof with an existential sentence as assumption. Then, we use Universal Elimination to strip away the outer quantifier from the premise. This allows us to derive $q(x)$ using Existential Elimination. Finally, we create an implication with Implication Introduction, and we generalize using Universal Introduction.

1.	$\forall x.\forall y.(p(x,y) \Rightarrow q(x))$	Premise
2.	$\quad \exists y.p(x,y)$	Assumption
3.	$\quad \forall y.(p(x,y) \Rightarrow q(x))$	UE: 1
4.	$\quad q(x)$	EE: 2, 3
5.	$\exists y.p(x,y) \Rightarrow q(x)$	II: 4
6.	$\forall x.(\exists y.p(x,y) \Rightarrow q(x))$	UI: 5

The relationship holds the other way around as well. Given $\forall x.(\exists y.p(x,y) \Rightarrow q(x))$, we know that $\forall x.\forall y.(p(x,y) \Rightarrow q(x))$. We can convert an existential quantifier in the antecedent of an implication into a universal quantifier outside the implication.

Our proof is shown below. As usual, we start with our premise. We start a subproof by making an assumption. Then we turn the assumption into an existential sentence to match the antecedent of the premise. We use Universal Implication to strip away the quantifier in the premise to expose the implication. Then, we apply Implication Elimination to derive $q(x)$. Finally, we create an implication with Implication Introduction, and we generalize using two applications of Universal Introduction.

1.	$\forall x.(\exists y.p(x,y) \Rightarrow q(x))$	Premise
2.	$\quad p(x,y)$	Assumption
3.	$\quad \exists y.p(x,y)$	EI: 2
4.	$\quad \exists y.p(x,y) \Rightarrow q(x)$	UE: 1
5.	$\quad q(x)$	IE: 4, 3
6.	$p(x,y) \Rightarrow q(x)$	II: 5
7.	$\forall x.\forall y.(p(x,y) \Rightarrow q(x))$	2 x UI: 6

RECAP

A Fitch system for Relational Logic can be obtained by extending the Fitch system for Propositional Logic with four additional rules to deal with quantifiers. The *Universal Introduction* rule allows us to reason from arbitrary sentences to universally quantified versions of those sentences. The *Universal Elimination* rule allows us to reason from a universally quantified sentence to a version of the target of that sentence in which the universally quantified variable is replaced by an appropriate term. The *Existential Introduction* rule allows us to reason from a sentence involving a term τ to an existentially quantified sentence in which one, some, or all occurrences of τ have been replaced by the existentially quantified variable. Finally, the *Existential Elimination* rule allows us to reason from an existentially quantified sentence $\exists \nu.\varphi[\nu]$ and a universally quantified implication $\forall \nu.(\varphi[\nu] \Rightarrow \psi)$ to the consequent ψ, under the condition that ν does not occur in ψ.

EXERCISES

7.1 Given $\forall x.\forall y.p(x,y)$, use the Fitch System to prove $\forall y.\forall x.p(x,y)$.

7.2 Given $\forall x.\forall y.p(x,y)$, use the Fitch System to prove $\forall x.\forall y.p(y,x)$.

7.3 Given $\exists x.\neg p(x)$, use the Fitch System to prove $\neg\forall x.p(x)$.

7.4 Given $\forall x.p(x)$, use the Fitch System to prove $\neg\exists x.\neg p(x)$.

7.5 Given $\exists y.\forall x.p(x,y)$, use the Fitch system to prove $\forall x.\exists y.p(x,y)$.

7.6 Given the premises $\forall x.(p(x) \Rightarrow q(x))$ and $\forall x.(q(x) \Rightarrow r(x))$, use the Fitch system to prove the conclusion $\forall x.(p(x) \Rightarrow r(x))$.

7.7 Given $\forall x.(p(x) \wedge q(x))$, use the Fitch System to prove $\forall x.p(x) \wedge \forall x.q(x)$.

7.8 Given $\forall x.(p(x) \Rightarrow q(x))$, use the Fitch System to prove $\forall x.p(x) \Rightarrow \forall x.q(x)$.

CHAPTER 8

Resolution

8.1 INTRODUCTION

The (general) *Resolution Principle* is a rule of inference for Relational Logic analogous to the Propositional Resolution Principle for Propositional Logic. Using the Resolution Principle alone (without axiom schemata or other rules of inference), it is possible to arrive at a proof method that proves the same consequences as the Fitch system given in Chapter 7. The search space using the Resolution Principle is smaller than the search space for generating Fitch proofs.

In our tour of Resolution, we look first at unification, which allows us to *unify* expressions by substituting terms for variables. We then move on to a definition of clausal form extended to handle variables. The Resolution Principle follows. We then look at some applications. Finally, we examine strategies for making the procedure more efficient.

8.2 CLAUSAL FORM

As with Propositional Resolution, (general) Resolution works only on expressions in *clausal form*. The definitions here are analogous. A *literal* is either a relational sentence or a negation of a relational sentence. A *clause* is a set of literals and, as in Propositional Logic, represents a disjunction of the literals in the set. A clause set is a set of clauses and represents a conjunction of the clauses in the set.

The procedure for converting Relational Logic sentences to clausal form is similar to that for Propositional Logic. Some of the rules are the same. However, there are a few additional rules to deal with the presence of variables and quantifiers. The conversion rules are summarized below and should be applied in order. The conversion process assumes that the input sentences have no free variables.

In the first step (Implications out), we eliminate all occurrences of the $\Rightarrow$, $\Leftarrow$, and $\Leftrightarrow$ operators by substituting equivalent sentences involving only the $\wedge$, $\vee$, and $\neg$ operators.

$$\begin{array}{lcl} \phi \Rightarrow \psi & \rightarrow & \neg\phi \vee \psi \\ \phi \Leftarrow \psi & \rightarrow & \phi \vee \neg\psi \\ \phi \Leftrightarrow \psi & \rightarrow & (\neg\phi \vee \psi) \wedge (\phi \vee \neg\psi) \end{array}$$

In the second step (Negations in), negations are distributed over other logical operators and quantifiers until each such operator applies to a single atomic sentence. The following replacement rules do the job.

$$\begin{array}{lcl} \neg\neg\phi & \rightarrow & \phi \\ \neg(\phi \wedge \psi) & \rightarrow & \neg\phi \vee \neg\psi \\ \neg(\phi \vee \psi) & \rightarrow & \neg\phi \wedge \neg\psi \\ \neg\forall v.\phi & \rightarrow & \exists v.\neg\phi \\ \neg\exists v.\phi & \rightarrow & \forall v.\neg\phi \end{array}$$

In the third step (Standardize variables), we rename variables so that each quantifier has a unique variable, i.e., the same variable is not quantified more than once within the same sentence. The following transformation is an example.

$$\forall x.(p(x) \Rightarrow \exists x.q(x)) \quad \rightarrow \quad \forall x.(p(x) \Rightarrow \exists y.q(y))$$

In the fourth step (Existentials out), we eliminate all existential quantifiers. The method for doing this is a little complicated, and we describe it in two stages.

If an existential quantifier does not occur within the scope of a universal quantifier, we simply drop the quantifier and replace all occurrences of the quantified variable by a new object constant; i.e., one that is not in our current language and hence is not used anywhere in our set of sentences. The object constant used to replace the existential variable in this case is called a *Skolem constant*. The following example assumes that a is not in the language. The conversion produces clauses in an expanded language with new object constants.

$$\exists x.p(x) \quad \rightarrow \quad p(a)$$

If an existential quantifier is within the scope of any universal quantifiers, there is the possibility that the value of the existential variable depends on the values of the associated universal variables. Consequently, we cannot replace the existential variable with an object constant. Instead, the general rule is to drop the existential quantifier and to replace the associated variable by a term formed from a new function symbol applied to the variables associated with the enclosing universal quantifiers. Any function defined in this way is called a *Skolem function*. The following example illustrates this transformation. It assumes that f is not in the current language. The conversion produces clauses in an expanded language with new function constants.

$$\forall x.(p(x) \wedge \exists z.q(x, y, z)) \rightarrow \forall x.(p(x) \wedge q(x, y, f(x, y)))$$

In the fifth step (Alls out), we drop all universal quantifiers. Because the remaining variables at this point are universally quantified, this does not introduce any ambiguities.

$$\forall x.(p(x) \wedge q(x, y, f(x, y))) \quad \rightarrow \quad p(x) \wedge q(x, y, f(x, y))$$

In the sixth step (Disjunctions in), we put the expression into *conjunctive normal form*, i.e., a conjunction of disjunctions of literals. This can be accomplished by repeated use of the following rules.

$$
\begin{array}{lcl}
\phi \vee (\psi \wedge \chi) & \rightarrow & (\phi \vee \psi) \wedge (\phi \vee \chi) \\
(\phi \wedge \psi) \vee \chi & \rightarrow & (\phi \vee \chi) \wedge (\psi \vee \chi) \\
\phi \vee (\phi_1 \vee ... \vee \phi_n) & \rightarrow & \phi \vee \phi_1 \vee ... \vee \phi_n \\
(\phi_1 \vee ... \vee \phi_n) \vee \phi & \rightarrow & \phi_1 \vee ... \vee \phi_n \vee \phi \\
\phi \wedge (\phi_1 \wedge ... \wedge \phi_n) & \rightarrow & \phi \wedge \phi_1 \wedge ... \wedge \phi_n \\
(\phi_1 \wedge ... \wedge \phi_n) \wedge \phi & \rightarrow & \phi_1 \wedge ... \wedge \phi_n \wedge \phi
\end{array}
$$

In the seventh step (Operators out), we eliminate operators by separating any conjunctions into its conjuncts and writing each disjunction as a separate clause.

$$
\begin{array}{lcl}
\phi_1 \wedge ... \wedge \phi_n & \rightarrow & \phi_1 \\
 & \rightarrow & ... \\
 & \rightarrow & \phi_n \\
\phi_1 \vee ... \vee \phi_n & \rightarrow & \{\phi_1, ..., \phi_n\}
\end{array}
$$

As an example of this conversion process, consider the problem of transforming the following expression to clausal form. The initial expression appears on the top line, and the expressions on the labeled lines are the results of the corresponding steps of the conversion procedure.

$$
\begin{array}{ll}
 & \exists y.(g(y) \wedge \forall z.(r(z) \Rightarrow f(y, z))) \\
\text{I} & \exists y.(g(y) \wedge \forall z.(\neg r(z) \vee f(y, z))) \\
\text{N} & \exists y.(g(y) \wedge \forall z.(\neg r(z) \vee f(y, z))) \\
\text{S} & \exists y.(g(y) \wedge \forall z.(\neg r(z) \vee f(y, z))) \\
\text{E} & g(gary) \wedge \forall z.(\neg r(z) \vee f(gary, z)) \\
\text{A} & g(gary) \wedge (\neg r(z) \vee f(gary, z)) \\
\text{D} & g(gary) \wedge (\neg r(z) \vee f(gary, z)) \\
\text{O} & \{g(gary)\} \\
 & \{\neg r(z), f(gary, z)\}
\end{array}
$$

Here is another example. In this case, the starting sentence is almost the same. The only difference is the leading ¬, but the result looks quite different.

$$\begin{array}{ll} & \neg\exists y.(g(y) \wedge \forall z.(r(z) \Rightarrow f(y,z))) \\ \text{I} & \neg\exists y.(g(y) \wedge \forall z.(\neg r(z) \vee f(y,z))) \\ \text{N} & \forall y.(\neg(g(y) \wedge \forall z.(\neg r(z) \vee f(y,z))) \\ & \forall y.(\neg g(y) \vee \neg\forall z.(\neg r(z) \vee f(y,z))) \\ & \forall y.(\neg g(y) \vee \exists z.\neg(\neg r(z) \vee f(y,z))) \\ & \forall y.(\neg g(y) \vee \exists z.(\neg\neg r(z) \wedge \neg f(y,z))) \\ & \forall y.(\neg g(y) \vee \exists z.(r(z) \wedge \neg f(y,z))) \\ \text{S} & \forall y.(\neg g(y) \vee \exists z.(r(z) \wedge \neg f(y,z))) \\ \text{E} & \forall y.(\neg g(y) \vee (r(k(y)) \wedge \neg f(y,k(y)))) \\ \text{A} & \neg g(y) \vee (r(k(y)) \wedge \neg f(y,k(y))) \\ \text{D} & (\neg g(y) \vee r(k(y))) \wedge (\neg g(y) \vee \neg f(y,k(y))) \\ \text{O} & \{\neg g(y) \vee r(k(y))\} \\ & \{\neg g(y) \vee \neg f(y,k(y))\} \end{array}$$

In Propositional Logic, the clause set corresponding to any sentence is logically equivalent to that sentence. In Relational Logic, this is not necessarily the case. For example, the clausal form of the sentence $\exists x.p(x)$ is $\{p(a)\}$. These are not logically equivalent to each other. They are not even in the same language.

On the other hand, the converted clause set has a special relationship to the original set of sentences—over the expanded language, the clause set satisfiable if and only if the original sentence is satisfiable (also over the expanded language). As we shall see, in resolution, this equivalence of satisfiability is all we need to obtain a proof method as powerful as the Fitch system presented in Chapter 7.

8.3 UNIFICATION

Unification is the process of determining whether two expressions can be *unified*, i.e., made identical by appropriate substitutions for their variables. As we shall see, making this determination is an essential part of Resolution.

A *substitution* is a finite mapping of variables to terms. In what follows, we write substitutions as sets of replacement rules, like the one shown below. In each rule, the variable to which the arrow is pointing is to be replaced by the term from which the arrow is pointing. In this case, x is to be replaced by a, y is to be replaced by $f(b)$, and z is to be replaced by v.

$$\{x \leftarrow a,\ y \leftarrow f(b),\ z \leftarrow v\}$$

The variables being replaced together constitute the *domain* of the substitution, and the terms replacing them constitute the *range*. For example, in the preceding substitution, the domain is $\{x, y, z\}$, and the range is $\{a, b, v\}$.

A substitution is *pure* if and only if all replacement terms in the range are free of the variables in the domain of the substitution. Otherwise, the substitution is *impure*. The substitution shown above is pure whereas the one shown below is impure.

$$\{x \leftarrow a,\ y \leftarrow f(b),\ z \leftarrow x\}$$

The result of applying a substitution σ to an expression ϕ is the expression $\phi\sigma$ obtained from the original expression by replacing every occurrence of every variable in the domain of the substitution by the term with which it is associated.

$$\begin{aligned} q(x, y)\{x \leftarrow a,\ y \leftarrow f(b),\ z \leftarrow v\} &= q(a,\ f(b)) \\ q(x, x)\{x \leftarrow a,\ y \leftarrow f(b),\ z \leftarrow v\} &= q(a, a) \\ q(x, w)\{x \leftarrow a,\ y \leftarrow f(b),\ z \leftarrow v\} &= q(a, w) \\ q(z, v)\{x \leftarrow a,\ y \leftarrow f(b),\ z \leftarrow v\} &= q(v, v) \end{aligned}$$

Note that, if a substitution is pure, application is idempotent, i.e., applying a substitution a second time has no effect.

$$\begin{aligned} q(x, x, y, w, z)\{x \leftarrow a,\ y \leftarrow f(b),\ z \leftarrow v\} &= q(a, a,\ f(b), w, v) \\ q(a, a,\ f(b), w, v)\{x \leftarrow a,\ y \leftarrow f(b),\ z \leftarrow v\} &= q(a, a,\ f(b), w, v) \end{aligned}$$

However, this is not the case for impure substitutions, as illustrated by the following example. Applying the substitution once leads to an expression with an x, allowing for a different answer when the substitution is applied a second time.

$$\begin{aligned} q(x, x, y, w, z)\{x \leftarrow a,\ y \leftarrow f(b),\ z \leftarrow x\} &= q(a, a,\ f(b), w, x) \\ q(a, a,\ f(b), w, x)\{x \leftarrow a,\ y \leftarrow f(b),\ z \leftarrow x\} &= q(a, a,\ f(b), w, a) \end{aligned}$$

Given two or more substitutions, it is possible to define a single substitution that has the same effect as applying those substitutions in sequence. For example, the substitutions $\{x \leftarrow a,\ y \leftarrow f(u),\ z \leftarrow v\}$ and $\{u \leftarrow d,\ v \leftarrow e\}$ can be combined to form the single substitution $\{x \leftarrow a,\ y \leftarrow f(d),\ z \leftarrow e,\ u \leftarrow d,\ v \leftarrow e\}$, which has the same effect as the first two substitutions when applied to any expression whatsoever.

Computing the *composition* of a substitution σ and a substitution τ is easy. There are two steps. (1) First, we apply τ to the range of σ. (2) Then we adjoin to σ all pairs from τ with different domain variables.

As an example, consider the composition shown below. In the right-hand side of the first equation, we have applied the second substitution to the replacements in the first substitution. In the second equation, we have combined the rules from this new substitution with the non-conflicting rules from the second substitution.

$$\begin{aligned} &\{x \leftarrow a,\ y \leftarrow f(u),\ z \leftarrow v\}\{u \leftarrow d,\ v \leftarrow e,\ z \leftarrow g\} \\ &\quad = \{x \leftarrow a,\ y \leftarrow f(d),\ z \leftarrow g\}\{u \leftarrow d,\ v \leftarrow e,\ z \leftarrow g\} \\ &\quad\quad = \{x \leftarrow a,\ y \leftarrow f(d),\ z \leftarrow g\}\{u \leftarrow d,\ v \leftarrow e\} \end{aligned}$$

It is noteworthy that composition does not necessarily preserve substitutional purity. The composition of two impure substitutions may be pure, and the composition of two pure substitutions may be impure.

This problem does not occur if the substitutions are *composable*. A substitution σ and a substitution τ are *composable* if and only if the domain of σ and the range of τ are disjoint. Otherwise, they are *noncomposable*.

$$\{x \leftarrow a, y \leftarrow b, z \leftarrow v\}\{x \leftarrow u, v \leftarrow b\}$$

By contrast, the following substitutions are noncomposable. Here, x occurs in both the domain of the first substitution and the range of the second substitution, violating the definition of composability.

$$\{x \leftarrow a, y \leftarrow b, z \leftarrow v\}\{x \leftarrow u, v \leftarrow x\}$$

The importance of composability is that it ensures preservation of purity. The composition of composable pure substitutions must be pure. In the sequel, we look only at compositions of composable pure substitutions.

A substitution σ is a *unifier* for an expression ϕ and an expression ψ if and only if $\phi\sigma = \psi\sigma$, i.e., the result of applying σ to ϕ is the same as the result of applying σ to ψ. If two expressions have a unifier, they are said to be *unifiable*. Otherwise, they are *nonunifiable*.

The expressions $p(x, y)$ and $p(a,v)$ have a unifier, e.g., $\{x \leftarrow a, y \leftarrow b, v \leftarrow b\}$ and are, therefore, unifiable. The results of applying this substitution to the two expressions are shown below.

$$p(x, y)\{x \leftarrow a, y \leftarrow b, v \leftarrow b\} = p(a, b)$$
$$p(a, v)\{x \leftarrow a, y \leftarrow b, v \leftarrow b\} = p(a, b)$$

Note that, although this substitution unifies the two expressions, it is not the only unifier. We do not have to substitute b for y and v to unify the two expressions. We can equally well substitute c or d or $f(c)$ or $f(w)$. In fact, we can unify the expressions without changing v at all by simply replacing y by v.

In considering these alternatives, it should be clear that some substitutions are more general than others are. We say that a substitution σ is *as general as or more general than* a substitution τ if and only if there is another substitution δ such that $\sigma\delta = \tau$. For example, the substitution $\{x \leftarrow a, y \leftarrow v\}$ is more general than $\{x \leftarrow a, y \leftarrow f(c), v \leftarrow f(c)\}$ since there is a substitution $\{v \leftarrow f(c)\}$ that, when applied to the former, gives the latter.

$$\{x \leftarrow a, y \leftarrow v\}\{v \leftarrow f(c)\} = \{x \leftarrow a, y \leftarrow f(c), v \leftarrow f(c)\}$$

In Resolution, we are interested only in unifiers with maximum generality. A *most general unifier*, or *mgu*, σ of two expressions has the property that it as general as or more general than any other unifier.

Although it is possible for two expressions to have more than one most general unifier, all of these most general unifiers are structurally the same, i.e., they are unique up to variable renaming. For example, $p(x)$ and $p(y)$ can be unified by either the substitution $\{x \leftarrow y\}$ or the substitution $\{y \leftarrow x\}$; and either of these substitutions can be obtained from the other by applying a third substitution. This is not true of the unifiers mentioned earlier.

One good thing about our language is that there is a simple and inexpensive procedure for computing a most general unifier of any two expressions if it exists.

The procedure assumes a representation of expressions as sequences of subexpressions. For example, the expression $p(a, f(b),z)$ can be thought of as a sequence with four elements, viz. the relation constant p, the object constant a, the term $f(b)$, and the variable z. The term $f(b)$ can in turn be thought of as a sequence of two elements, viz. the function constant f and the object constant b.

We start the procedure with two expression and a substitution, which is initially the empty substitution. We then recursively process the two expressions, comparing the subexpressions at each point. Along the way, we expand the substitution with variable assignments as described below. If, we fail to unify any pair of subexpression at any point in this process, the procedure as a whole fails. If we finish this recursive comparison of the expressions, the procedure as a whole succeeds, and the accumulated substitution at that point is the most general unifier.

The processing of subexpressions goes as follows.

1. If two expressions being compared are identical, then nothing more needs to be done.
2. If two expressions are not identical and both expressions are constants, then we fail, since there is no way to make them look alike.
3. If one of the expressions is a variable, we first check whether the variable has a binding in the current substitution. If so, we try to unify the binding with the second expression. If there is no binding, we check whether the second expression contains the variable. If the variable occurs within the expression, we fail; otherwise, we set the substitution to the composition of the old substitution and a new substitution in which we bind the variable to the second expression.
4. The only remaining possibility is that the two expressions are both sequences. In this case, we simply iterate across the expressions, comparing as described above.

As an example, consider the computation of the most general unifier for the expressions $p(x,b)$ and $p(a,y)$ with the initial substitution $\{\ \}$. A trace of the execution of the procedure for this case is shown below. We show the beginning of a comparison with a line labelled Compare together with the expressions being compared and the input substitution. We show the result of each comparison with a line labelled Result. The indentation shows the depth of recursion of the procedure.

Compare: $p(x,b)$, $p(a,y)$, $\{\ \}$
  Compare: p, p, $\{\ \}$
  Result: $\{\ \}$
  Compare: x, a, $\{\ \}$
  Result: $\{x \leftarrow a\}$
  Compare: y, b, $\{x \leftarrow a\}$
  Result: $\{x \leftarrow a, y \leftarrow b\}$
Result: $\{x \leftarrow a, y \leftarrow b\}$

As another example, consider the process of unifying the expression $p(x,x)$ and the expression $p(a,y)$. A trace is shown below. The main interest in this example comes in the step involving x and y. At this point, x has a binding, so we recursively compare the replacement a to y, which results in binding of y to a.

Compare: $p(x,x)$, $p(a,y)$, { }
  Compare: p, p, { }
  Result: { }
  Compare: x, a, { }
  Result: $\{x \leftarrow a\}$
  Compare: x, y, $\{x \leftarrow a\}$
    Compare: a, y, $\{x \leftarrow a\}$
    Result: $\{x \leftarrow a, y \leftarrow a\}$
  Result: $\{x \leftarrow a, y \leftarrow a\}$
Result: $\{x \leftarrow a, y \leftarrow a\}$

One important part of the unification procedure is the test for whether a variable occurs within an expression before the variable is bound to that expression. This test is called an *occur check* since it is used to check whether or not the variable occurs within the term with which it is being unified. For example, $p(x)$ is not unifiable with $p(f(x))$. When the algorithm tries to unify the two expressions using the substitution $x \leftarrow f(x)$, it performs the occur check. Upon finding that the variable x occurs in the term $f(x)$, the algorithm fails, concluding that no unifier can be found. Without this check, the algorithm would incorrectly find that expressions such as $p(x)$ and $p(f(x))$ are unifiable, even though there is no substitution for x that, when applied to both, makes them look alike.

8.4 RESOLUTION PRINCIPLE

The Resolution Principle is analogous to that of Propositional Resolution. The only difference is the use of unification to unify literals before applying the rule.

A simple version of the Resolution Principle is shown below. Given a clause with a literal ϕ and a second clause with a literal $\neg\psi$ such that ϕ and ψ have a most general unifier σ, we can derive a conclusion by applying σ to the clause consisting of the remaining literals from the two original clauses.

$$\frac{\{\phi_1, \ldots, \phi, \ldots, \phi_m\} \quad \{\psi_1, \ldots, \neg\psi, \ldots, \psi_n\}}{\{\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n\}\sigma}$$

where $\sigma = mgu(\phi, \psi)$

Consider the example shown below. The first clause contains the positive literal $p(a,y)$ and the second clause contains a negative occurrence of $p(x, f(x))$. The substitution $\{x \leftarrow a, y \leftarrow f(a)\}$ is a most general unifier of these two expressions. Consequently, we can collect the remaining literals $r(y)$ and $q(g(x))$ into a clause and apply the substitution to produce a conclusion.

$$\frac{\{p(a, y), r(y)\} \quad \{\neg p(x, f(x)), q(g(x))\}}{\{r(f(a)), q(g(a))\}}$$

Unfortunately, this simple version of the Resolution Principle is not quite good enough. Consider the two clauses shown below. Given the meaning of these two clauses, it should be possible to resolve them to produce the empty clause. However, the two atomic sentences do not unify. The variable x must be bound to a and b at the same time.

$$\{p(a, x)\}$$
$$\{\neg p(x, b)\}$$

Fortunately, this problem can easily be fixed by extending the Resolution Principle slightly as shown below. Before trying to resolve two clauses, we select one of the clauses and rename any variables the clause has in common with the other clause.

$$\frac{\{\phi_1, \ldots, \phi, \ldots, \phi_m\} \quad \{\psi_1, \ldots, \neg\psi, \ldots, \psi_n\}}{\{\phi_1\tau, \ldots, \phi_m\tau, \psi_1, \ldots, \psi_n\}\sigma}$$

where τ is a variable renaming on $\{\phi_1, \ldots, \phi, \ldots, \phi_m\}$

where $\sigma = mgu(\phi\tau, \psi)$

Renaming solves this problem. Unfortunately, we are still not quite done. There is one more technicality that must be addressed to finish the story. As stated, even with the extension mentioned above, the rule is not quite good enough. Given the clauses shown below, we should be able to infer the empty clause { }; however, this is not possible with the preceding definition. The clauses can be resolved in various ways, but the result is never the empty clause.

$$\{p(x), p(y)\}$$
$$\{\neg p(u), \neg p(v)\}$$

The good news is that we can solve this additional problem with one last modification to our definition of the Resolution Principle. If a subset of the literals in a clause Φ has a most general unifier γ, then the clause Φ' obtained by applying γ to Φ is called a *factor* of Φ. For example, the literals $p(x)$ and $p(f(y))$ have a most general unifier $\{x \leftarrow f(y)\}$, so the clause $\{p(f(y)), r(f(y), y)\}$ is a factor of $\{p(x), p(f(y)), r(x, y)\}$. Obviously, any clause is a trivial factor of itself.

Using the notion of factors, we can give a complete definition for the *Resolution Principle*. Suppose that Φ and Ψ are two clauses. If there is a literal ϕ in some factor of Φ and a literal $\neg\psi$ in some factor of Ψ, then we say that the two clauses Φ and Ψ *resolve* and that the new clause $((\Phi' - \{\phi\}) \cup (\Psi' - \{\neg\psi\}))\sigma$ is a *resolvent* of the two clauses.

$$\frac{\begin{array}{l}\Phi \\ \Psi\end{array}}{\begin{array}{l}((\Phi' - \{\phi\}) \cup (\Psi' - \{\neg\psi\}))\sigma \\ \text{where } \tau \text{ is a variable renaming on } \Phi \\ \text{where } \Phi\text{'} \text{ is a factor of } \Phi\tau \text{ and } \phi \in \Phi\text{'} \\ \text{where } \Psi\text{'} \text{ is a factor of } \Psi \text{ and } \neg\psi \in \Psi\text{'} \\ \text{where } \sigma = mgu(\phi, \psi)\end{array}}$$

Using this enhanced definition of Resolution, we can solve the problem mentioned above. Once again, consider the premises $\{p(x), p(y)\}$ and $\{\neg p(u), \neg p(v)\}$. The first premise has the factor $\{p(x)\}$, and the second has the factor $\{\neg p(u)\}$, and these two factors resolve to the empty clause in a single step.

8.5 RESOLUTION REASONING

Reasoning with the (general) Resolution Principle is analogous to reasoning with the Propositional Resolution Principle. We start with premises; we apply the Resolution Principle to those premises; we apply the rule to the results of those applications; and so forth until we get to our desired conclusion or until we run out of things to do.

As with Propositional Resolution, we define a *Resolution derivation* of a conclusion from a set of premises to be a finite sequence of clauses terminating in the conclusion in which each clause is either a premise or the result of applying the Resolution Principle to earlier members of the sequence. And, as with Propositional Resolution, we do not use the word *proof*, because we reserve that word for a slightly different concept, which is discussed in the next section.

As an example, consider a problem in the area kinship relations. Suppose we know that Art is the parent of Bob and Bud; suppose that Bob is the parent of Cal; and suppose that Bud is the parent of Coe. Suppose we also know that grandparents are parents of parents. Starting with these premises, we can use Resolution to conclude that Art is the grandparent of Coe. The derivation is shown below. We start with our five premises—four simple clauses for the four facts about the parent relation p and one more complex clause capturing the definition of the grandparent relation g. We start by resolving the clause on line 1 with the clause on line 5 to produce the clause on line 6. We then resolve the clause on line 3 with this result to derive that conclusion that Art is the grandparent of Cal. Interesting but not what we set out to prove; so we continue the process. We next resolve the clause on line 2 with the clause on line 5 to produce the clause on line 8. Then we resolve the clause on line 5 with this result to produce the clause on line 9, which is exactly what we set out to prove.

1.	$\{p(art, bob)\}$	Premise
2.	$\{p(art, bud)\}$	Premise
3.	$\{p(bob, cal)\}$	Premise
4.	$\{p(bud, coe)\}$	Premise
5.	$\{\neg p(x, y), \neg p(y, z), g(x, z)\}$	Premise
6.	$\{\neg p(bob, z), g(art, z)\}$	1, 5
7.	$\{g(art, cal)\}$	3, 6
8.	$\{\neg p(bud, z), g(art, z)\}$	2, 5
9.	$\{g(art, coe)\}$	4, 8

One thing to notice about this derivation is that there are some dead-ends. We first tried resolving the fact about Art and Bob before getting around to trying the fact about Art and Bud. Resolution does not eliminate all search. However, at no time did we ever have to make an arbitrary assumption or an arbitrary choice of a binding for a variable. The absence of such arbitrary choices is why Resolution is so much more focussed than natural deduction systems like Fitch.

Another worthwhile observation about Resolution is that, unlike Fitch, Resolution frequently terminates even when there is no derivation of the desired result. Suppose, for example, we were interested in deriving the clause $\{g(cal,art)\}$ from the premises in this case. This sentence, of course, does *not* follow from the premises. And Resolution is sound, so we would never generate this result. The interesting thing is that, in this case, the attempt to derive this result would eventually terminate. With the premises given, there are a few more Resolutions we could do, e.g., resolving the clause on line 1 with the second literal in the clause on line 5. However, having done these additional Resolutions, we would find ourselves with nothing left to do; and, unlike Fitch, the process would terminate.

Unfortunately, like Propositional Resolution, general Resolution is not *generatively complete*, i.e., it is not possible to find Resolution derivations for all clauses that are logically entailed by a set of premise clauses. For example, the clause $\{p(a), \neg p(a)\}$ is always true, and so it is logically entailed by any set of premises, including the empty set of premises. Resolution requires some premises to have any effect. Given an empty set of premises, we would not be able to derive any conclusions, including this valid clause.

Although Resolution is not *generatively* complete, problems like this one are solved by negating the goal and demonstrating that the resulting set of sentences is unsatisfiable.

8.6 UNSATISFIABILITY

One common use of Resolution is in demonstrating unsatisfiability. In clausal form, a contradiction takes the form of the empty clause, which is equivalent to a disjunction of no literals. Thus, to automate the determination of unsatisfiability, all we need do is to use Resolution to derive consequences from the set to be tested, terminating whenever the empty clause is generated.

Let's start with a simple example. See the derivation below. We have four premises. The derivation in this case is particularly easy. We resolve the first clause with the second to get the clause

shown on line 5. Next, we resolve the result with the third clause to get the unit clause on line 6. Note that $r(a)$ is the remaining literal from clause 3 after the Resolution, and $r(a)$ is also the remaining literal from clause 5 after the Resolution. Since these two literals are identical, they appear only once in the result. Finally, we resolve this result with the clause on 4 to produce the empty clause.

1.	$\{p(a,b), q(a,c)\}$	Premise
2.	$\{\neg p(x,y), r(x)\}$	Premise
3.	$\{\neg q(x,y), r(x)\}$	Premise
4.	$\{\neg r(z)\}$	Premise
5.	$\{q(a,c), r(a)\}$	1, 2
6.	$\{r(a)\}$	5, 3
7.	$\{\ \}$	6, 4

Here is a more complicated derivation, one that illustrates renaming and factoring. Again, we have four premises. Line 5 results from Resolution between the clauses on lines 1 and 3. This one is easy. Line 6 results from Resolution between the clauses on lines 2 and 4. In this case, renaming is necessary in order for the unification to take place. Line 7 results from renaming and factoring the clause on line 5 and resolving with the clause on line 6. Finally, line 8 results from factoring line 5 again and resolving with the clause on line 7. Note that we cannot just factor 5 and factor 6 and resolve the results in one step. Try it and see what happens.

1.	$\{\neg p(x,y), q(x,y,f(x,y))\}$	Premise
2.	$\{r(y,z), \neg q(a,y,z)\}$	Premise
3.	$\{p(x, g(x)), q(x,g(x),z)\}$	Premise
4.	$\{\neg r(x,y), \neg q(x,w,z)\}$	Premise
5.	$\{q(x,g(x),f(x, g(x))), q(x,g(x),z)\}$	1, 3
6.	$\{\neg q(a,x,y), \neg q(x,w,z)\}$	2, 4
7.	$\{\neg q(g(a),w,z)\}$	5, 6 (factoring 5)
8.	$\{\ \}$	5, 7 (factoring 5)

In demonstrating unsatisfiability, Resolution and Fitch (as given in Chapter 7) are equally powerful. Given a set of sentences, Resolution can derive the empty clause from the clausal form of the sentences if and only if Fitch can find a proof of a sentence of the form $\psi \wedge \neg\psi$. The benefit of using Resolution is that the search space is smaller.

8.7 LOGICAL ENTAILMENT

As with Propositional Logic, we can use a test for unsatisfiability to test logical entailment as well. Suppose we wish to show that the set of sentences Δ logically entails the formula ϕ. We can do this by finding a proof of ϕ from Δ, i.e., by establishing $\Delta \vdash \phi$. One version of the refutation theorem states that $\Delta \vdash \phi$ if and only if there is a sentence α of the form $\psi \wedge \neg\psi$ such that $\Delta \cup \{\neg\phi\} \vdash \alpha$. In addition, recall that Resolution and Fitch are equally powerful in demonstrating unsatisfiability.

Thus, if we show using Resolution that the set of formulas $\Delta \cup \{\neg\phi\}$ is unsatisfiable, we have demonstrated that $\Delta \vdash \phi$ and hence Δ logically entails ϕ.

To apply this technique of establishing logical entailment by establishing unsatisfiability using Resolution, we first negate ϕ and add it to Δ to yield Δ'. We then convert Δ' to clausal form and apply Resolution. If the empty clause is produced, the original Δ was unsatisfiable, and we have demonstrated that Δ logically entails ϕ. This process is called a *Resolution refutation*; it is illustrated by examples in the following sections.

As an example of using Resolution to determine logical entailment, let's consider a case we saw earlier. The premises are shown below. We know that everybody loves somebody and everybody loves a lover.

$$\forall x.\exists y.loves(x,y)$$
$$\forall u.\forall v.\forall w.(loves(v,w) \Rightarrow loves(u,v))$$

Our goal is to show that everybody loves everybody.

$$\forall x.\forall y.loves(x,y)$$

In order to solve this problem, we add the negation of our desired conclusion to the premises and convert to clausal form, leading to the clauses shown below. Note the use of a Skolem function in the first clause and the use of Skolem constants in the clause derived from the negated goal.

$$\{loves(x,f(x))\}$$
$$\{\neg loves(v,w), loves(u,v)\}$$
$$\{\neg loves(a,b)\}$$

Starting from these initial clauses, we can use Resolution to derive the empty clause and thus prove the result.

1.	$\{loves(x,f(x))\}$	Premise
2.	$\{\neg loves(v,w), loves(u,v)\}$	Premise
3.	$\{\neg loves(a,b)\}$	Premise
4.	$\{loves(u,x)\}$	1, 2
5.	$\{\ \}$	4, 3

As another example of Resolution, once again consider the problem of Harry and Ralph introduced in the preceding chapter. We know that every horse can outrun every dog. Some greyhounds can outrun every rabbit. Greyhounds are dogs. The relationship of being faster is transitive. Harry is a horse. Ralph is a rabbit.

$$\forall x.\forall y.(h(x) \wedge d(y) \Rightarrow f(x,y))$$
$$\exists y.(g(y) \wedge \forall z.(r(z) \Rightarrow f(y,z)))$$
$$\forall y.(g(y) \Rightarrow d(y))$$
$$\forall x.\forall y.\forall z.(f(x,y) \wedge f(y,z) \Rightarrow f(x,z))$$
$$h(harry)$$
$$r(ralph)$$

We desire to prove that Harry is faster than Ralph. In order to do this, we negate the desired conclusion.

$$\neg f(harry, ralph)$$

To do the proof, we take the premises and the negated conclusion and convert to clausal form. The resulting clauses are shown below. Note that the second premise has turned into two clauses. Note also that the Skolem constant $gary$ is introduced.

1.	$\{\neg h(x), \neg d(y), f(x,y)\}$	Premise
2.	$\{g(gary)\}$	Premise
3.	$\{\neg r(z), f(gary,z)\}$	Premise
4.	$\{\neg g(y), d(y)\}$	Premise
5.	$\{\neg f(x,y), \neg f(y,z), f(x,z)\}$	Premise
6.	$\{h(harry)\}$	Premise
7.	$\{r(ralph)\}$	Premise
8.	$\{\neg f(harry,ralph)\}$	Negated Goal

From these clauses, we can derive the empty clause, as shown in the following derivation.

9.	$\{d(gary)\}$	2, 4
10.	$\{\neg d(y), f(harry,y)\}$	6, 1
11.	$\{f(harry,gary)\}$	9, 10
12.	$\{f(gary,ralph)\}$	7, 3
13.	$\{\neg f(gary,z), f(harry,z)\}$	11, 1
14.	$\{f(harry,ralph\}$	12, 13
15.	$\{\ \}$	14, 8

Don't be misled by the simplicity of these examples. Resolution can and has been used in proving complex mathematical theorems, in proving the correctness of programs, and in many other applications.

8.8 ANSWER EXTRACTION

In a previous section, we saw how to use Resolution in answering true-or-false questions (e.g., *Is Art a grandparent of Coe?*). In this section, we show how Resolution can be used to answer fill-in-the-blank questions as well (e.g., *Who is a grandparent of Coe?*).

A fill-in-the-blank question is a sentence with free variables specifying the blanks to be filled in. The goal is to find bindings for the free variables such that the set of premises logically entails the sentence obtained by substituting the bindings into the original question. For example, consider the following premises:

1.	$\{f(art, jon)\}$	*art* is the father of *jon*
2.	$\{f(bob,kim)\}$	*bob* if the father of *kim*
3.	$\{\neg f(x,y), p(x,y)\}$	if x is the father of y then x is also a parent of y

To ask about Jon's parent, we would write the question $p(x,jon)$. Using the premises above, we see that *art* is an answer to this question, since the sentence $p(art,jon)$ is logically entailed by the premises.

An *answer literal* for a fill-in-the-blank question ϕ is a sentence $goal(v_1, \ldots, v_n)$, where $v_1, \ldots, v_n$ are the free variables in ϕ. To answer ϕ, we form an implication from ϕ and its answer literal and convert to clausal form. For example, the literal $p(x, jon)$is combined with its answer literal $goal(x)$ to form the rule $(p(x, jon) \Rightarrow goal(x))$, which leads to the clause $\{\neg p(x, jon), goal(x)\}$.

We then use Resolution as described above, except that we change the termination test. Rather than waiting for the empty clause to be produced, the procedure halts as soon as it derives a clause consisting of only answer literals. The following Resolution derivation shows how we compute the answer to *Who is Jon's parent?*

1.	$\{f(art,jon)\}$	Premise
2.	$\{f(bob,kim)\}$	Premise
3.	$\{\neg f(x,y),p(x,y)\}$	Premise
4.	$\{\neg p(x,jon),goal(x)\}$	Goal
5.	$\{\neg f(x,jon),goal(x)\}$	3, 4
6.	$\{goal(art)\}$	1, 5

If this procedure produces only one answer literal, the terms it contains constitute the only answer to the question. In some cases, the result of a fill-in-the-blank Resolution depends on the refutation by which it is produced. In general, several different refutations can result from the same query, leading to multiple answers.

Suppose, for example, that we knew the identities of both the father and mother of Jon and that we asked *Who is a parent of Jon?* The following Resolution trace shows that we can derive two answers to this question.

1.	$\{f(art,jon)\}$	Premise
2.	$\{m(ann,jon)\}$	Premise
3.	$\{\neg f(x,y),p(x,y)\}$	Premise
4.	$\{\neg m(x,y),p(x,y)\}$	Premise
5.	$\{\neg p(x,jon),goal(x)\}$	Goal
6.	$\{\neg f(x,jon),goal(x)\}$	3, 5
7.	$\{goal(art)\}$	1, 6
8.	$\{\neg m(x,jon),goal(x)\}$	4, 5
9.	$\{goal(ann)\}$	1, 8

Unfortunately, we have no way of knowing whether or not the answer statement from a given refutation exhausts the possibilities. We can continue to search for answers until we find enough of them. However, due to the undecidability of logical entailment, we can never know in general whether we have found all the possible answers.

Another interesting aspect of fill-in-the-blank Resolution is that in some cases the procedure can result in a clause containing more than one answer literal. The significance of this is that no one answer is guaranteed to work, but one of the answers must be correct.

The following Resolution trace illustrates this fact. Suppose we have a disjunction asserting that Art or Bob is the father of Jon, but we do not know which man is. The goal is to find a parent of John. After resolving the goal clause with the sentence about fathers and parents, we resolve the result with the premise disjunction, obtaining a clause that can be resolved a second time yielding a clause with two answer literals. This answers indicates not two answers but rather uncertainty as to which is the correct answer.

1.	$\{f(art,jon), f(bob,jon)\}$	Premise
2.	$\{\neg f(x,y), p(x,y)\}$	Premise
3.	$\{\neg p(x,jon), goal(x)\}$	Goal
4.	$\{\neg f(x,jon), goal(x)\}$	2, 3
5.	$\{f(art,jon), goal(bob)\}$	1, 4
6.	$\{goal(art), goal(bob)\}$	5, 4

In such situations, we can continue searching in hope of finding a more specific answer. However, given the undecidability of logical entailment, we can never know in general whether we can stop and say that no more specific answer exists.

8.9 STRATEGIES

One of the disadvantages of using the Resolution rule in an unconstrained manner is that it leads to many unnecessary inferences. Some inferences are redundant in that their conclusions can be derived in other ways. Some inferences are irrelevant in that they do not lead to derivations of the desired result.

This section presents a number of strategies for eliminating unnecessary work. In reading this material, it is important to bear in mind that we are concerned here not with the order in which inferences are done, but only with the size of a Resolution graph and with ways of decreasing that size by eliminating unnecessary deductions.

PURE LITERAL ELIMINATION

A literal occurring in a clause set is *pure* if and only if it has no instance that is complementary to an instance of another literal in the clause set. A clause that contains a pure literal is redundant for the purposes of refutation, since the literal can never be resolved away. Consequently, we can safely remove such a clause. Removing clauses with pure literals defines a deletion strategy known as *pure-literal elimination*.

The clause set that follows is unsatisfiable. However, in proving this we can ignore the second and third clauses, since they both contain the pure literal s. The example in this case involves clauses in Propositional Logic, but it applies equally well to Relational Logic.

$\{\neg p, \neg q, r\}$
$\{\neg p, s\}$
$\{\neg q, s\}$
$\{p\}$
$\{q\}$
$\{\neg r\}$

Note that, if a set of clauses contains no pure literals, there is no way we can derive any clauses with pure literals using Resolution. The upshot is that we need to apply pure-literal elimination only once to the starting set of premise clauses; we do not have to check each clause as it is generated.

TAUTOLOGY ELIMINATION

A *tautology* is a clause containing a pair of complementary literals. For example, the clause $\{p(f(a)), \neg p(f(a))\}$ is a tautology. The clause $\{p(x), q(y), \neg q(y), r(z)\}$ also is a tautology, even though it contains additional literals.

As it turns out, the presence of tautologies in a set of clauses has no effect on that set's satisfiability. A satisfiable set of clauses remains satisfiable, no matter what tautologies we add. An unsatisfiable set of clauses remains unsatisfiable, even if we remove all tautologies. Therefore, we can remove tautologies from a set of clauses, because we need never use them in subsequent inferences. The corresponding deletion strategy is called *tautology elimination*.

Note that the literals in a clause must be exact complements for tautology elimination to apply. We cannot remove non-identical literals, just because they are complements under unification. For example, the set of clauses $\{\neg p(a), p(x)\}$, $\{p(a)\}$, and $\{\neg p(b)\}$ is unsatisfiable. However, if we were to remove the first clause (by an incorrect application of tautology elimination), the remaining clauses would be satisfiable.

SUBSUMPTION ELIMINATION

In *subsumption elimination*, the deletion criterion depends on a relationship between two clauses in a database. A clause Φ *subsumes* a clause Ψ if and only if there exists a substitution σ such that $\Phi\sigma \subseteq \Psi$. For example, $\{p(x), q(y)\}$ subsumes $\{p(a), q(v), r(w)\}$, since there is a substitution $\{x \leftarrow a, y \leftarrow v\}$ that makes the former clause a subset of the latter.

If one member in a set of clauses subsumes another member, then the set remaining after eliminating the subsumed clause is satisfiable if and only if the original set is satisfiable. Therefore, subsumed clauses can be eliminated. Since the Resolution process itself can produce tautologies and subsuming clauses, we need to check for tautologies and subsumptions as we perform Resolutions.

UNIT RESOLUTION

A *unit resolvent* is one in which at least one of the parent clauses is a *unit clause*, i.e., one containing a single literal. A *unit derivation* is one in which all derived clauses are unit resolvents. A *unit refutation* is a unit derivation of the empty clause.

As an example of a unit refutation, consider the following proof. In the first two inferences, unit clauses from the initial set are resolved with binary clauses to produce two new unit clauses. These are resolved with the first clause to produce two additional unit clauses. The elements in these two sets of results are then resolved with each other to produce the contradiction.

1.	$\{p, q\}$	Premise
2.	$\{\neg p, r\}$	Premise
3.	$\{\neg q, r\}$	Premise
4.	$\{\neg r\}$	Premise
5.	$\{\neg p\}$	2, 4
6.	$\{\neg q\}$	3, 4
7.	$\{q\}$	1, 5
8.	$\{p\}$	1, 6
9.	$\{r\}$	3, 7
10.	$\{\ \}$	6, 7

Note that the proof contains only a subset of the possible uses of the Resolution rule. For example, clauses 1 and 2 can be resolved to derive the conclusion $\{q, r\}$. However, this conclusion and its descendants are never generated, since neither of its parents is a unit clause.

Inference procedures based on Unit Resolution are easy to implement and are usually quite efficient. It is worth noting that, whenever a clause is resolved with a unit clause, the conclusion has fewer literals than the parent does. This helps to focus the search toward producing the empty clause and thereby improves efficiency.

Unfortunately, inference procedures based on Unit Resolution generally are not complete. For example, the clauses $\{p, q\}$, $\{\neg p, q\}$, $\{p\ , \neg q\}$, and $\{\neg p, \neg q\}$ are inconsistent. Using unconstrained Resolution, it is easy to derive the empty clause. However, Unit Resolution fails in this case, since none of the initial clauses contains just one literal.

On the other hand, if we restrict our attention to Horn clauses (i.e., clauses with at most one positive literal), the situation is much better. In fact, it can be shown that there is a unit refutation of a set of Horn clauses if and only if there is a (unconstrained) Resolution refutation.

INPUT RESOLUTION

An *input resolvent* is one in which at least one of the two parent clauses is a member of the initial (i.e., input) set of clauses. An *input deduction* is one in which all derived clauses are input resolvents. An *input refutation* is an input deduction of the empty clause.

It can be shown that Unit Resolution and Input Resolution are equivalent in inferential power in that there is a unit refutation from a set of clauses if and only if there is an input refutation from the same set of clauses.

One consequence of this fact is that Input Resolution is complete for Horn clauses but incomplete in general. The unsatisfiable set of clauses $\{p, q\}$, $\{\neg p, q\}$, $\{p , \neg q\}$, and $\{\neg p, \neg q\}$ provides an example of a deduction on which Input Resolution fails. An input refutation must (in particular) have one of the parents of $\{\ \}$ be a member of the initial set of clauses. However, to produce the empty clause in this case, we must resolve either two single literal clauses or two clauses having single-literal factors. None of the members of the base set meet either of these criteria, so there cannot be an input refutation for this set.

LINEAR RESOLUTION

Linear Resolution (also called *Ancestry-filtered Resolution*) is a slight generalization of Input Resolution. A *linear resolvent* is one in which at least one of the parents is either in the initial set of clauses or is an ancestor of the other parent. A *linear deduction* is one in which each derived clause is a linear resolvent. A *linear refutation* is a linear deduction of the empty clause.

Linear Resolution takes its name from the linear shape of the proofs it generates. A linear deduction starts with a clause in the initial set of clauses (called the *top clause*) and produces a linear chain of Resolution. Each resolvent after the first one is obtained from the last resolvent (called the *near parent*) and some other clause (called the *far parent*). In Linear Resolution, the far parent must either be in the initial set of clauses or be an ancestor of the near parent.

Much of the redundancy in unconstrained Resolution derives from the Resolution of intermediate conclusions with other intermediate conclusions. The advantage of Linear Resolution is that it avoids many useless inferences by focusing deduction at each point on the ancestors of each clause and on the elements of the initial set of clauses.

Linear Resolution is known to be refutation complete. Furthermore, it is not necessary to try every clause in the initial database as top clause. It can be shown that, if a set of clauses Δ is satisfiable and $\Delta \cup \{\Phi\}$ is unsatisfiable, then there is a linear refutation with Φ as top clause. So, if we know that a particular subset of our input clauses is consistent, we need not attempt refutations with the elements of that subset as top clauses.

A *merge* is a resolvent that inherits a literal from each parent such that this literal is collapsed to a singleton by the most general unifier. The completeness of Linear Resolution is preserved even if the ancestors that are used are limited to merges. resolvent (i.e., clause $\{q\}$) is a merge.

SET OF SUPPORT RESOLUTION

If we examine Resolution traces, we notice that many conclusions come from Resolutions between clauses contained in a portion of the clauses that we know to be satisfiable. For example, in many cases, the set of premises is satisfiable, yet many of the resolvents are obtained by resolving premises

with other premises rather than the negated conclusion. As it turns out, we can eliminate these Resolutions without affecting the refutation completeness of Resolution.

A subset Γ of a set Δ is called a *set of support* for Δ if and only if $\Delta - \Gamma$ is satisfiable. Given a set of clauses Δ with set of support Γ, a *set of support resolvent* is one in which at least one parent is selected from Γ or is a descendant of Γ. A *set of support deduction* is one in which each derived clause is a set of support resolvent. A *set of support refutation* is a set of support deduction of the empty clause.

The following derivation is a set of support refutation, using as set of support the singleton set consisting of the clause $\{\neg r\}$. The clause $\{\neg r\}$ resolves with $\{\neg p, r\}$ and $\{\neg q, r\}$ to produce $\{\neg p\}$ and $\{\neg q\}$. These then resolve with clause 1 to produce $\{q\}$ and $\{p\}$. Finally, $\{q\}$ resolves with $\{\neg q\}$ to produce the empty clause.

1.	$\{p, q\}$	Premise
2.	$\{\neg p, r\}$	Premise
3.	$\{\neg q, r\}$	Premise
4.	$\{\neg r\}$	Set of Support
5.	$\{\neg p\}$	2, 4
6.	$\{\neg q\}$	3, 4
7.	$\{q\}$	1, 5
8.	$\{\ \}$	7, 6

Obviously, this strategy would be of little use if there were no easy way of selecting the set of support. Fortunately, there are several ways this can be done at negligible expense. For example, in situations where we are trying to prove conclusions from a consistent set of premises, the natural choice is to use the clauses derived from the negated goal as the set of support. This set satisfies the definition as long as the set of premises itself is satisfiable. With this choice of set of support, each Resolution must have a connection to the overall goal, so the procedure can be viewed as working backward from the goal. This is especially useful for premises in which the number of conclusions possible by working forward is larger. Furthermore, the goal-oriented character of such refutations often makes them more understandable than refutations using other strategies.

RECAP

The (general) *Resolution Principle* is a rule of inference for Relational Logic analogous to the Propositional Resolution Principle for Propositional Logic. As with Propositional Resolution, (general) Resolution works only on expressions in *clausal form*.

Unification is the process of determining whether two expressions can be *unified*, i.e., made identical by appropriate substitutions for their variables. A *substitution* is a finite mapping of variables to terms. The variables being replaced together constitute the *domain* of the substitution, and the terms replacing them constitute the *range*. A substitution is *pure* if and only if all replacement terms in the range are free of the variables in the domain of the substitution. Otherwise, the substitution

is *impure*. The result of applying a substitution σ to an expression ϕ is the expression $\phi\sigma$ obtained from the original expression by replacing every occurrence of every variable in the domain of the substitution by the term with which it is associated. The *composition* of two substitutions is a single substitution that has the same effect as applying those substitutions in sequence. A substitution σ is a *unifier* for an expression ϕ and an expression ψ if and only if $\phi\sigma = \psi\sigma$, i.e., the result of applying σ to ϕ is the same as the result of applying σ to ψ. If two expressions have a unifier, they are said to be *unifiable*. Otherwise, they are *nonunifiable*. A *most general unifier*, or *mgu*, σ of two expressions has the property that it as general as or more general than any other unifier. Although it is possible for two expressions to have more than one most general unifier, all of these most general unifiers are structurally the same, i.e., they are unique up to variable renaming.

The (general) Resolution Principle is a generalization of Propositional Resolution. The main difference is the use of unification to unify literals before applying the rule. A *Resolution derivation* of a conclusion from a set of premises is a finite sequence of clauses terminating in the conclusion in which each clause is either a premise or the result of applying the Resolution Principle to earlier members of the sequence. Resolution and Fitch are equally powerful. Given a set of sentences, Resolution can derive the empty clause from the clausal form of the sentences if and only if Fitch can find a proof of a contradiction. The benefit of using Resolution is that the search space is smaller.

EXERCISES

8.1 Consider a language with two object constants a and b and one function constant f. Give the clausal form for each of the following sentences in this language.

(a) $\exists y.\forall x.p(x,y)$
(b) $\forall x.\exists y.p(x,y)$
(c) $\exists x.\exists y.(p(x,y) \wedge q(x,y))$
(d) $\forall x.\forall y.(p(x,y) \Rightarrow q(x))$
(e) $\forall x.(\exists y.p(x,y) \Rightarrow q(x))$

8.2 For each of the following pairs of sentences, say whether the sentences are unifiable and give a most general unifier for those that are unifiable.

(a) $p(x,x)$ and $p(a,y)$
(b) $p(x,x)$ and $p(f(y),z)$
(c) $p(x,x)$ and $p(f(y),y)$
(d) $p(f(x,y),g(z,z))$ and $p(f(f(w,z),v),w)$

8.3 Give all resolvents, if any, for each of the following pairs of clauses.

(*a*) $\{p(x,f(x)), q(x)\}$ and $\{\neg p(a,y), r(y)\}$
(*b*) $\{p(x,b), q(x)\}$ and $\{\neg p(a,x), r(x)\}$
(*c*) $\{p(x), p(a), q(x)\}$ and $\{\neg p(y), r(y)\}$
(*d*) $\{p(x), p(a), q(x)\}$ and $\{\neg p(y), r(y)\}$
(*e*) $\{p(a), q(y)\}$ and $\{\neg p(x), \neg q(b)\}$
(*f*) $\{p(x), q(x,x)\}$ and $\{\neg q(a, f(a))\}$

8.4 Given the clauses $\{p(a), q(a)\}$, $\{\neg p(x), r(x)\}$, $\{\neg q(a)\}$, use Relational Resolution to derive the clause $\{r(a)\}$.

8.5 Given the premises $\forall x.(p(x) \Rightarrow q(x))$ and $\forall x.(q(x) \Rightarrow r(x))$, use Resolution to prove the conclusion $\forall x.(p(x) \Rightarrow r(x))$.

8.6 Given $\forall x.(p(x) \Rightarrow q(x))$, use Resolution to prove $\forall x.p(x) \Rightarrow \forall x.q(x)$.

8.7 Use Resolution to prove $\forall x.(((p(x) \Rightarrow q(x)) \Rightarrow p(x)) \Rightarrow p(x))$.

CHAPTER 9

Induction

9.1 INTRODUCTION

Induction is reasoning from the specific to the general. If various instances of a schema are true and there are no counterexamples, we are tempted to conclude a universally quantified version of the schema.

$$\begin{array}{l} p(a) \Rightarrow q(a) \\ p(b) \Rightarrow q(b) \\ p(c) \Rightarrow q(c) \end{array} \qquad \longrightarrow \qquad \forall x.\, p(x) \Rightarrow q(x)$$

Incomplete induction is induction where the set of instances is not exhaustive. From a reasonable collection of instances, we sometimes leap to the conclusion that a schema is always true even though we have not seen all instances. Consider, for example, the function f where $f(1) = 1$, and $f(n+1) = f(n) + 2n + 1$. If we look at some values of this function, we notice a certain regularity—the value of f always seems to be the square of its input. From this sample, we are tempted to leap to the conclusion that $f(n) = n^2$. Lucky guess. In this case, the conclusion happens to be true; and we can prove it.

n	$f(n)$	n^2
1	1	1
2	4	2^2
3	9	3^2
4	16	4^2
5	25	5^2

Here is another example. This one is due to the mathematician Fermat (1601–1665). He looked at values of the expression $2^{2^n} + 1$ for various values of n and noticed that they were all prime. So, he concluded the value of the expression was prime number. Unfortunately, this was not a lucky guess. His conjecture was ultimately disproved, in fact with the very next number in the sequence. (Mercifully, the counterexample was found after his death.)

n	$2^{2^n} + 1$	Prime?
1	5	Yes
17	4	Yes
257	9	Yes
65537	16	Yes

For us, this is not so good. In Logic, we are concerned with logical entailment. We want to derive only conclusions that are guaranteed to be true when the premises are true. Guesses like these are useful in suggesting possible conclusions, but they are not themselves proofs. In order to be sure of universally quantified conclusions, we must be sure that *all* instances are true. This is called *complete induction*. When applied to numbers, it is usually called *mathematical induction*.

The techniques for complete induction vary with the structure of the language to which it is applied. We begin this chapter with a discussion of domain closure, a rule that applies when the Herbrand base of our language is finite. We then move on to Linear Induction, i.e., the special case in which the ground terms in the language form a linear sequence. After that, we look at tree induction, i.e., the special case in which the ground terms of the language form a tree. Finally, we look at Structural Induction, which applies to all languages.

9.2 DOMAIN CLOSURE

Induction for finite languages is trivial. We simply use the Domain Closure rule of inference. For a language with object constants $\sigma_1, \ldots, \sigma_n$, the rule is written as shown below. If we believe a schema is true for every instance, then we can infer a universally quantified version of that schema.

$$\frac{\begin{array}{c}\phi[\sigma_1] \\ \ldots \\ \phi[\sigma_n]\end{array}}{\forall v.\phi[v]}$$

Recall that, in our formalization of the Sorority World, we have just four constants - *abby*, *bess*, *cody*, and *dana*. For this language, we would have the following Domain Closure rule.

Domain Closure (DC)

$$\frac{\begin{array}{l}\phi[abby] \\ \phi[bess] \\ \phi[cody] \\ \phi[dana]\end{array}}{\forall v.\phi[v]}$$

The following proof shows how we can use this rule to derive an inductive conclusion. Given the premises we considered earlier in this book, it is possible to infer that Abby likes someone, Bess likes someone, Cody likes someone, and Dana likes someone. Taking these conclusions as premises and using our Domain Closure rule, we can then derive the inductive conclusion $\forall x.\exists y.likes(x, y)$, i.e., everybody likes somebody.

1.	$\exists y.likes(abby, y)$	Premise
2.	$\exists y.likes(bess, y)$	Premise
3.	$\exists y.likes(cody, y)$	Premise
4.	$\exists y.likes(dana, y)$	Premise
5.	$\forall x.\exists y.likes(x, y)$	Domain Closure: 1, 2, 3, 4

Unfortunately, this technique does not work when there are infinitely many ground terms. Suppose, for example, we have a language with ground terms $\sigma_1, \sigma_2, \ldots$ A direct generalization of the Domain Closure rule is shown below.

$$\frac{\begin{array}{c}\phi[\sigma_1] \\ \phi[\sigma_2] \\ \ldots\end{array}}{\forall v.\phi[v]}$$

This rule is sound in the sense that the conclusion of the rule is logically entailed by the premises of the rule. However, it does not help us produce a proof of this conclusion. To use the rule, we would need to prove all of the rule's premises. Unfortunately, there are infinitely many premises. So, the rule cannot be used in generating a finite proof.

All is not lost. It is sometimes possible to write rules that cover all instances without enumerating them individually. The method depends on the structure of the language. The next sections describe how this can be done for languages with different structures.

9.3 LINEAR INDUCTION

Imagine an infinite set of dominoes placed in a line so that, when one falls, the next domino in the line also falls. If the first domino falls, then the second domino falls. If the second domino falls, then the third domino falls. And so forth. By continuing this chain of reasoning, it is easy for us to convince ourselves that every domino eventually falls. This is the intuition behind a technique called Linear Induction.

Consider a language with a single object constant a and a single unary function constant s. There are infinitely many ground terms in this language, arranged in what resembles a straight line. See below. Starting from the object constant, we move to a term in which we apply the function constant to that constant, then to a term in which we apply the function constant to that term, and so forth.

$$a \rightarrow s(a) \rightarrow s(s(a)) \rightarrow s(s(s(a))) \rightarrow \ldots$$

In this section, we concentrate on languages that have linear structure of this sort. Hereafter, we we call these *linear languages*. In all cases, there is a single object constant and a single unary function constant. In talking about a linear language, we call the object constant the *base element* of the language, and we call the unary function constant the *successor function*.

Although there are infinitely many ground terms in any linear language, we can still generate finite proofs that are guaranteed to be correct. The trick is to use the structure of the terms in the language in expressing the premises of an inductive rule of inference known as *Linear Induction*. See below for a statement of the induction rule for the language introduced above. In general, if we know that a schema holds of our base element and if we know that, whenever the schema holds of an element, it also holds of the successor of that element, then we can conclude that the schema holds of all elements.

Linear Induction

$$\frac{\phi[a] \qquad \forall \mu.(\phi[\mu] \Rightarrow \phi[s(\mu)])}{\forall \nu.\phi[\nu]}$$

A bit of terminology before we go on. The first premise in this rule is called the *base case* of the induction, because it refers to the base element of the language. The second premise is called the *inductive case*. The antecedent of the inductive case is called the *inductive hypothesis*, and the consequent is called, not surprisingly, the *inductive conclusion*. The conclusion of the rule is sometimes called the *overall conclusion* to distinguish it from the inductive conclusion.

For the language introduced above, our rule of inference is sound. Suppose we know that a schema is true of a and suppose that we know that, whenever the schema is true of an arbitrary ground term τ, it is also true of the term $s(\tau)$. Then, the schema must be true of everything, since there are no other terms in the language.

The requirement that the signature consists of no other object constants or function constants is crucial. If this were not the case, say there were another object constant b, then we would have trouble. It would still be true that ϕ holds for every element in the set $\{a, s(a), s(s(a)), \ldots\}$. However, because there are other elements in the Herbrand universe, e.g., b and $s(b)$, $\forall x.\phi(x)$ would no longer be guaranteed.

Here is an example of induction in action. Recall the formalization of Arithmetic introduced in Chapter 6. Using the object constant 0 and the unary function constant s, we represent each number n by applying the function constant to 0 exactly n times. For the purposes of this example, let's assume we have just one ternary relation constant, viz. *plus*, which we use to represent the addition table.

The following axioms describe *plus* in terms of 0 and s. The first sentence here says that adding 0 to any element produces that element. The second sentence states that adding the successor of a number to another number yields the successor of their sum. The third sentence is a functionality axiom for *plus*.

$$\forall y.plus(0, y, y)$$
$$\forall x.\forall y.\forall z.(plus(x, y, z) \Rightarrow plus(s(x), y, s(z)))$$
$$\forall x.\forall y.\forall z.\forall w.(plus(x, y, z) \wedge \neg same(z, w) \Rightarrow \neg plus(x, y, w))$$

It is easy to see that any table that satisfies these axioms includes all of the usual addition facts. The first axiom ensures that all cases with 0 as first arg are included. From this fact and the second axiom, we can see that all cases with $s(0)$ as first argument are included. And so forth.

The first axiom above tells us that 0 is a *left identity* for addition - 0 added to any number produces that number as result. As it turns out, given these definitions, 0 must also be a *right identity*, i.e., it must be the case that $\forall x.plus(x, 0, x)$.

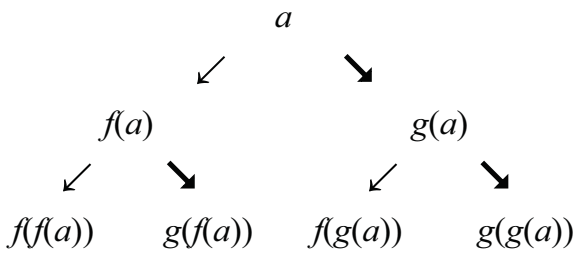
We can use induction to prove this result as shown below. We start with our premises. We use Universal Elimination on the first premise to derive the sentence on line 3. This takes care of the base case of our induction. We then start a subproof and assume the antecedent of the inductive case. We then use three applications of Universal Elimination on the second premise to get the sentence on line 5. We use Implication Elimination on this sentence and our assumption to derive the conclusion on line 6. We then discharge our assumption and form the implication shown on line 7 and then universalize this to get the result on line 8. Finally, we use Linear Induction to derive our overall conclusion.

1.	$\forall y.plus(0, y, y)$	Premise
2.	$\forall x.\forall y.\forall z.(plus(x, y, z) \Rightarrow plus(s(x), y, s(z)))$	Premise
3.	$plus(0, 0, 0)$	UE: 1
4.	$\quad plus(x, 0, x)$	Assumption
5.	$\quad plus(x, 0, x) \Rightarrow plus(s(x), 0, s(x))$	3 x UE: 2
6.	$\quad plus(s(x), 0, s(x))$	IE: 5, 4
7.	$plus(x, 0, x) \Rightarrow plus(s(x), 0, s(x))$	II: 4, 6
8.	$\forall x.(plus(x, 0, x) \Rightarrow plus(s(x), 0, s(x)))$	UI: 7
9.	$\forall x.plus(x, 0, x)$	Ind: 3, 8

Most inductive proofs have this simple structure. We prove the base case. We assume the inductive hypothesis; we prove the inductive conclusion; and, based on this proof, we have the inductive case. From the base case and the inductive case, we infer the overall conclusion.

9.4 TREE INDUCTION

Tree languages are a superset of linear languages. While in linear languages the terms in the language form a linear sequence, in tree languages the structure is more tree-like. Consider a language with an object constant a and two unary function constants f and g. Some of the terms in this language are shown below.



As with linear languages, we can write an inductive rule of inference for tree languages. The tree induction rule of inference for the language just described is shown below. Suppose we know that a schema ϕ holds of a. Suppose we know that, whenever the schema holds of any element, it holds of the term formed by applying f to that element. And suppose we know that, whenever the schema holds of any element, it holds of the term formed by applying g to that element. Then, we can conclude that the schema holds of every element.

Tree Induction

$$\frac{\phi[a] \quad \forall\mu.(\phi[\mu] \Rightarrow \phi[f(\mu)]) \quad \forall\mu.(\phi[\mu] \Rightarrow \phi[g(\mu)])}{\forall\nu.\phi[\nu]}$$

In order to see an example of tree induction in action, consider the ancestry tree for a particular dog. We use the object constant *rex* to refer to the dog; we use the unary function constant f to map an arbitrary dog to its father; and we use g map a dog to its mother. Finally, we have a single unary relation constant *purebred* that is true of a dog if and only if it is purebred.

Now, we write down the fundamental rule of dog breeding - we say that a dog is purebred if and only if both its father and its mother are purebred. See below. (This is a bit oversimplified on several grounds. Properly, the father and mother should be of the same breed. Also, this formalization suggests that every dog has an ancestry tree that stretches back without end. However, let's ignore these imperfections for the purposes of our example.)

$$\forall x.(\mathit{purebred}(x) \Leftrightarrow \mathit{purebred}(f(x)) \wedge \mathit{purebred}(g(x)))$$

Suppose now that we discover the fact that our dog *rex* is purebred. Then, we know that every dog in his ancestry tree must be purebred. We can prove this by a simple application of tree induction.

A proof of our conclusion is shown below. We start with the premise that Rex is purebred. We also have our constraint on purebred animals as a premise. We use Universal Elimination to instantiate the second premise, and then we use Biconditional Elimination on the biconditional in line 3 to produce the implication on line 4. On line 5, we start a subproof with the assumption the x is purebred. We use Implication Elimination to derive the conjunction on line 6, and then we use And Elimination to pick out the first conjunct. We then use Implication Introduction to discharge

our assumption, and we Universal Introduction to produce the inductive case for f. We then repeat this process to produce an analogous result for g on line 14. Finally, we use the tree induction rule on the sentences on lines 1, 9, and 14 and thereby derive the desired overall conclusion.

1.	$purebred(rex)$	Premise
2.	$\forall x.(purebred(x) \Leftrightarrow purebred(f(x)) \wedge purebred(g(x)))$	Premise
3.	$purebred(x) \Leftrightarrow purebred(f(x)) \wedge purebred(g(x))$	UE: 2
4.	$purebred(x) \Rightarrow purebred(f(x)) \wedge purebred(g(x))$	BE: 3
5.	$\quad purebred(x)$	Assumption
6.	$\quad purebred(f(x)) \wedge purebred(g(x))$	IE: 4, 6
7.	$\quad purebred(f(x))$	AE: 6
8.	$purebred(x) \Rightarrow purebred(f(x))$	II: 5, 7
9.	$\forall x.purebred(x) \Rightarrow purebred(f(x))$	UI: 8
10.	$\quad purebred(x)$	Assumption
11.	$\quad purebred(f(x)) \wedge purebred(g(x))$	IE: 4, 6
12.	$\quad purebred(g(x))$	AE: 7
13.	$purebred(x) \Rightarrow purebred(g(x))$	II: 6, 8
14.	$\forall x.purebred(x) \Rightarrow purebred(g(x))$	UI: 9
15.	$\forall x.purebred(x)$	Ind: 1, 9, 14

9.5 STRUCTURAL INDUCTION

Structural Induction is the most general form of induction. In Structural Induction, we can have multiple object constants, multiple function constants, and, unlike our other forms of induction, we can have function constants with multiple arguments. Consider a language with two object constants a and b and a single binary function constant c. See below for a list of some of the terms in the language. We do not provide a graphical rendering in this case, as the structure is more complicated than a line or a tree.

$$a, b, c(a,a), c(a,b), c(b,a), c(b,b), c(a,c(a,a)), c(c(a,a),a), c(c(a,a),c(a,a)), \ldots$$

The Structural Induction rule for this language is shown below. If we know that ϕ holds of our base elements a and b and if we know $\forall\mu.\forall\nu.((\phi[\mu] \wedge \phi[\nu]) \Rightarrow \phi[c(\mu,\nu)])$, then we can conclude $\forall\nu.\phi[\nu]$ in a single step.

Structural Induction

$$\frac{\begin{array}{l}\phi[a] \\ \phi[b] \\ \forall\lambda.\forall\mu.((\phi[\lambda] \wedge \phi[\mu]) \Rightarrow \phi[c(\lambda,\mu)])\end{array}}{\forall\nu.\phi[\nu]}$$

As an example of a domain where Structural Induction is appropriate, recall the world of lists and trees introduced in Chapter 6. Let's assume we have two object constants a and b, a binary

function constant c, and two unary relation constants p and q. Relation p is true of a structured object if and only if every fringe node is an a. Relation q is true of a structured object if and only if at least one fringe node is an a. The positive and negative axioms defining the relations are shown below.

$p(a)$	$q(a)$
$\forall u.\forall v.(p(u) \wedge p(v) \Rightarrow p(c(u,v)))$	$\forall u.\forall v.(q(u) \Rightarrow q(c(u,v)))$
$\neg p(b)$	$\forall u.\forall v.(q(v) \Rightarrow q(c(u,v)))$
$\forall u.\forall v.(p(c(u,v)) \Rightarrow p(u))$	$\neg q(b)$
$\forall u.\forall v.(p(c(u,v)) \Rightarrow p(v))$	$\forall u.\forall v.(q(c(u,v))) \Rightarrow q(u) \vee q(v))$

Now, as an example of Structural Induction in action, let's prove that every object that satisfies p also satisfies q. In other words, we want to prove the conclusion $\forall x.(p(x) \Rightarrow q(x))$. As usual, we start with our premises.

1.	$p(a)$	Premise
2.	$\forall u.\forall v.(p(u) \wedge p(v) \Rightarrow p(c(u,v)))$	Premise
3.	$\neg p(b)$	Premise
4.	$\forall u.\forall v.(p(c(u,v)) \Rightarrow p(u))$	Premise
5.	$\forall u.\forall v.(p(c(u,v)) \Rightarrow p(v))$	Premise
6.	$q(a)$	Premise
7.	$\forall u.\forall v.(q(u) \Rightarrow q(c(u,v)))$	Premise
8.	$\forall u.\forall v.(q(v) \Rightarrow q(c(u,v)))$	Premise
9.	$\neg q(b)$	Premise
10.	$\forall u.\forall v.(q(c(u,v)) \Rightarrow q(u) \vee q(v))$	Premise

To start the induction, we first prove the base cases for the conclusion. In this world, with two object constants, we need to show the result twice – once for each object constant in the language.

Let's start with a. The derivation is simple in this case. We assume $p(a)$, reiterate $q(a)$ from line 6, then use Implication Introduction to prove $(p(a) \Rightarrow q(a))$.

11.	$p(a)$	Assumption
12.	$q(a)$	Reiteration: 6
13.	$p(a) \Rightarrow q(a)$	Implication Introduction

Now, let's do the case for b. As before, we assume $p(b)$, and our goal is to derive $q(b)$. This is a little strange. We know that $q(b)$ is false. Still, we should be able to derive it since we have assumed $p(b)$, which is also false. The trick here is to generate contradictory conclusions from the assumption $\neg q(b)$. To this end, we assume $\neg q(b)$ and first prove $p(b)$. Having done so, we use Implication Introduction to get one implication. Then, we assume $\neg q(b)$ again and this time derive $\neg p(b)$ and get an implication. At this point, we can use Negation Introduction to derive $\neg\neg q(b)$ and Negation Elimination to get $q(b)$. Finally, we use Implication Introduction to prove $(p(b) \Rightarrow q(b))$.

14.	$p(b)$	Assumption
15.	$\quad \neg q(b)$	Assumption
16.	$\quad p(b)$	Reiteration: 14
17.	$\neg q(b) \Rightarrow p(b)$	Implication Introduction: 14, 16
18.	$\quad \neg q(b)$	Assumption
19.	$\quad \neg p(b)$	Reiteration: 3
20.	$\neg q(b) \Rightarrow \neg p(b)$	Implication Introduction: 18, 19
21.	$\neg\neg q(b)$	Negation Introduction: 17, 20
22.	$q(b)$	Negation Elimination: 21
23.	$p(b) \Rightarrow q(b)$	Implication Introduction: 17, 19

Having dealt with the base cases, the next step is to prove the inductive case. We need to show that, if our conclusion holds of u and v, then it also holds of $c(u, v)$. To this end, we assume the conjunction of our assumptions and then use And Elimination to split that conjunction into its two conjuncts. Our inductive conclusion is also an implication; and, to prove it, we assume its antecedent $p(c(u, v))$. From this, we derive $p(u)$; from that, we derive $q(u)$; and, from that, we derive $q(c(u, v))$. We then use Implication Introduction to get the desired implication. A final use of Implication Introduction and a couple of applications of Universal Introduction gives us the inductive case for the induction.

24.	$(p(u) \Rightarrow q(u)) \wedge (p(v) \Rightarrow q(v))$	Assumption
25.	$p(u) \Rightarrow q(u)$	And Elimination: 24
26.	$p(v) \Rightarrow q(v)$	And Elimination: 24
27.	$\quad p(c(u, v))$	Assumption
28.	$\quad \forall v.(p(c(u, v)) \Rightarrow p(u))$	Universal Elimination: 4
29.	$\quad p(c(u, v)) \Rightarrow p(u)$	Universal Elimination: 28
30.	$\quad p(u)$	Implication Elimination: 29, 27
31.	$\quad q(u)$	Implication Elimination: 25, 30
32.	$\quad \forall v.(q(u) \Rightarrow q(c(u, v)))$	Universal Elimination: 7
33.	$\quad q(u) \Rightarrow q(c(u, v))$	Universal Elimination: 32
34.	$\quad q(c(u, v))$	Implication Elimination: 33, 31
35.	$p(c(u, v)) \Rightarrow q(c(u, v))$	Implication Introduction: 27, 34
36.	$(p(u) \Rightarrow q(u)) \wedge (p(v) \Rightarrow q(v)) \Rightarrow (p(c(u, v)) \Rightarrow q(c(u, v)))$	Implication Introduction: 24, 35
37.	$\forall v.((p(u) \Rightarrow q(u)) \wedge (p(v) \Rightarrow q(v)) \Rightarrow (p(c(u, v))) \Rightarrow q(c(u, v)))$	Universal Introduction: 36
38.	$\forall u.\forall v.((p(u) \Rightarrow q(u)) \wedge (p(v) \Rightarrow q(v)) \Rightarrow (p(c(u, v))) \Rightarrow q(c(u, v)))$	Universal Introduction: 37

Finally, using the base case on lines 13 and 23 and the inductive case on line 38, we use Structural Induction to give us the conclusion we set out to prove.

39. $\forall x.(p(x) \Rightarrow q(x))$ Induction: 13, 23, 38

Although the proof in this case is longer than in the previous examples, the basic inductive structure is the same. Importantly, using induction, we can prove this result where otherwise it would not be possible.

9.6 MULTIDIMENSIONAL INDUCTION

In our look at induction thus far, we have been concentrating on examples in which the conclusion is a universally quantified sentence with just one variable. In many situations, we want to use induction to prove a result with more than one universally quantified variable. This is called *multidimensional induction* or, sometimes, *multivariate induction*.

In principle, multidimensional induction is straightforward. We simply use ordinary induction to prove the outermost universally quantified sentence. Of course, in the case of multidimensional induction the base case and the inductive conclusion are themselves universally quantified sentences; and, if necessary we use induction to prove these subsidiary results.

As an example, consider a language with a single object constant a, a unary function constant s, and a binary relation constant e. The axioms shown below define e.

$$e(a, a)$$
$$\forall x.\neg e(a, s(x))$$
$$\forall x.\neg e(s(x), a)$$
$$\forall x.\forall y.(e(x, y) \Rightarrow e(s(x), s(y)))$$
$$\forall x.\forall y.(e(s(x), s(y)) \Rightarrow e(x, y))$$

The relation e is an equivalence relation – it is reflexive, symmetric, and transitive. Proving reflexivity is easy and does not even require induction. Proving transitivity is left to an exercise. Our goal here is to prove symmetry.

Goal: $\forall x.\forall y.(e(x, y) \Rightarrow e(y, x))$.

In what follows, we use induction to prove the outer quantified formula and then use induction on each of the inner conclusions as well. This means we have two immediate subgoals - the base case for the outer induction and the inductive case for the outer induction.

Goal: $\forall y.(e(a, y) \Rightarrow e(y, a))$.
Goal: $\forall x.(\forall y.(e(x, y) \Rightarrow e(y, x)) \Rightarrow \forall y.(e(s(x), y) \Rightarrow e(y, s(x))))$.

As usual, we start with our premises. We then prove the base case of the inner induction. This is easy. We assume $e(a, a)$ and the use Implication Introduction to prove the base case in one step.

1.	$e(a, a)$	Premise
2.	$\forall x.\neg e(a, s(x))$	Premise
3.	$\forall x.\neg e(s(x), a)$	Premise
4.	$\forall x.\forall y.(e(x, y) \Rightarrow e(s(x), s(y)))$	Premise
5.	$\forall x.\forall y.(e(s(x), s(y)) \Rightarrow e(x, y))$	Premise
6.	$\quad e(a, a)$	Assumption
7.	$e(a, a) \Rightarrow e(a, a))$	Implication Introduction: 6, 6

The next step is to prove the inductive case for the inner induction. To this end, we assume inductive hypothesis and try to prove the inductive conclusion. Since the conclusion is itself an implication, we assume its antecedent and prove the consequent. As shown below, we do this by assuming the consequent is false and proving a sentence and its negation. We then use Negation Introduction and Negation Elimination to derives the consequent. We finish with two applications of Implication Introduction and an application of Universal Introduction.

8.	$e(a, y) \Rightarrow e(y, a))$	Assumption
9.	$\quad e(a, s(y))$	Assumption
10.	$\quad\quad \neg e(s(y), a)$	Assumption
11.	$\quad\quad e(a, s(y))$	Reiteration: 9
12.	$\quad \neg e(s(y), a,) \Rightarrow e(a, s(y))$	Implication Introduction: 10, 11
13.	$\quad\quad \neg e(s(y), a)$	Assumption
14.	$\quad\quad \neg e(a, s(y))$	Universal Instantiation: 2
15.	$\quad \neg e(s(y), a) \Rightarrow \neg e(a, s(y))$	Implication Introduction: 13, 14
16.	$\quad \neg\neg e(s(y), a)$	Negation Introduction: 12, 15
17.	$\quad e(s(y), a)$	Negation Elimination: 16
18.	$(e(a, y) \Rightarrow e(y, a)) \Rightarrow (e(a, s(y)) \Rightarrow e(s(y), a))$	Implication Introduction: 9, 17
19.	$\forall y.((e(a, y) \Rightarrow e(y, a)) \Rightarrow (e(a, s(y)) \Rightarrow e(s(y), a)))$	Universal Introduction: 18
20.	$\forall y.(e(a, y) \Rightarrow e(y, a))$	Induction: 7, 19

That's a lot of work just to prove the base case of the outer induction. The inductive case of the outer induction is even more complex, and it is easy to make mistakes. The trick to avoiding these mistakes is to be methodical.

In order to prove the inductive case for the outer induction, we assume the inductive hypothesis $\forall y.(e(x, y) \Rightarrow e(y, x))$; and we then prove the inductive conclusion $\forall y.(e(s(x), y) \Rightarrow e(y, s(x))))$. We prove this by induction on the variable y.

We start by proving the base case for this inner induction. We start with the inductive hypothesis. We then assume the antecedent of the base case.

21.	$\forall y.(e(x, y) \Rightarrow e(y, x))$	Assumption
22.	$\quad e(s(x), a)$	Assumption
23.	$\quad\quad \neg e(a, s(x))$	Assumption
24.	$\quad\quad e(s(x), a)$	Reiteration: 22
25.	$\quad \neg e(a, s(x))) \Rightarrow e(s(x), a)$	Implication Introduction: 23, 24
26.	$\quad\quad \neg e(a, s(x))$	Assumption
27.	$\quad\quad \neg e(s(x), a)$	Universal Elimination: 3
28.	$\quad \neg e(a, s(x))) \Rightarrow \neg e(s(x), a)$	Implication Introduction: 26, 27
29.	$\quad \neg\neg e(a, s(x)))$	Negation Introduction: 25, 28
30.	$\quad e(a, s(x)))$	Negation Elimination: 29
31.	$e(s(x), a) \Rightarrow e(a, s(x)))$	Implication Introduction: 22, 30

Next, we work on the inductive case for the second inner induction. We start by assuming the inductive hypothesis. We then assume the antecedent of the inductive conclusion.

32.	$\quad e(s(x), y) \Rightarrow e(y, s(x))$	Assumption
33.	$\quad\quad e(s(x), s(y))$	Assumption
34.	$\quad\quad \forall y.(e(s(x), s(y)) \Rightarrow e(x, y))$	Universal Elimination: 5
35.	$\quad\quad e(s(x), s(y)) \Rightarrow e(x, y)$	Universal Elimination: 34
36.	$\quad\quad e(x, y)$	Implication Elimination: 35, 33
37.	$\quad\quad e(x, y) \Rightarrow e(y, x)$	Universal Elimination: 21
38.	$\quad\quad e(y, x)$	Implication Elimination: 37, 36
39.	$\quad\quad \forall y.(e(y, x) \Rightarrow e(s(y), s(x)))$	Universal Elimination: 4
40.	$\quad\quad e(y, x) \Rightarrow e(s(y), s(x))$	Universal Elimination: 39
41.	$\quad\quad e(s(y), s(x))$	Implication Elimination: 40, 38
42.	$\quad e(s(x), s(y)) \Rightarrow e(s(y), s(x))$	Implication Introduction: 33, 41
43.	$(e(s(x), y) \Rightarrow e(y, s(x))) \Rightarrow$ $\quad (e(s(x), s(y)) \Rightarrow e(s(y), s(x)))$	Implication Introduction: 32, 42
44.	$\forall y.((e(s(x), y) \Rightarrow e(y, s(x))) \Rightarrow$ $\quad (e(s(x), s(y)) \Rightarrow e(s(y), s(x))))$	Universal Introduction: 43

From the results on lines 31 and 44, we can conclude the inductive case for the second inner induction. This gives us the inductive case for the outer induction.

45.	$\quad\forall y.(e(s(x), y) \Rightarrow e(y, s(x)))$	Induction: 31, 44
46.	$\forall y.(e(x, y) \Rightarrow e(y, x)) \Rightarrow$ $\quad \forall y.(e(s(x), y) \Rightarrow e(y, s(x)))$	Implication Introduction: 21, 45
47.	$\forall x.(\forall y.(e(x, y) \Rightarrow e(y, x)) \Rightarrow$ $\quad \forall y.(e(s(x), y) \Rightarrow e(y, s(x))))$	Universal Introduction: 46

Finally, from the base case for the outer induction and this inductive case, we can conclude our overall result.

48. $\forall x.\forall y.(e(x, y) \Rightarrow e(y, x))$ Induction: 7, 47

As this proof illustrates, the technique of using induction within induction works just fine. Unfortunately, it is tedious and error-prone, for this reason, many people prefer to use specialized forms of multidimensional induction.

9.7 EMBEDDED INDUCTION

In all of the examples in this chapter thus far, induction is used to prove the overall result. While this approach works nicely in many cases, it is not always successful. In some cases, it is easier to use induction on parts of a problem or to prove alternative conclusions and then use these intermediate results to derive the overall conclusions (using inductive or non-inductive methods).

As an example, consider a world characterized by a single object constant a, a single unary function constant s, and a single unary relation constant p. Assume we have the set of axioms shown below.

$$\forall x.(p(x) \Rightarrow p(s(s(x))))$$
$$p(a)$$
$$p(s(a))$$

A little thought reveals that these axioms logically entail the universal conclusion $\forall x.p(x)$. Unfortunately, we cannot derive this conclusion directly using Linear Induction. The base case is easy enough. And, from $p(x)$ we can easily derive $p(s(s(x)))$. However, it is not so easy to derive $p(s(x))$, which is what we need for the inductive case of Linear Induction.

The good news is that we can succeed in cases like this by proving a slightly more complicated intermediate conclusion and then using that conclusion to prove the result. One way to do this is shown below. In this case, we start by using Linear Induction to prove $\forall x.(p(x) \wedge p(s(x))$. he base case $p(a) \wedge p(s(a))$ is easy, since we are given the two conjuncts as axioms. The inductive case is straightforward. We assume $p(x) \wedge p(s(x))$. From this hypothesis, we use And Elimination to get $p(x)$ and $p(s(x))$. We then use Universal Elimination and Implication Elimination to derive $p(s(s(x)))$. We then conjoin these results, use Implication Introduction and Universal Introduction to get the inductive case for our induction. From the base case and the inductive case, we get our intermediate conclusion. Finally, starting with this conclusion, we use Universal Elimination, And Elimination, and Universal Introduction to get the overall result.

1.	$\forall x.(p(x) \Rightarrow p(s(s(x))))$	Premise
2.	$p(a)$	Premise
3.	$p(s(a)))$	Premise
4.	$p(a) \wedge p(s(a)))$	And Introduction
5.	$\quad p(x) \wedge p(s(x)))$	Assumption
6.	$\quad p(x)$	And Elimination
7.	$\quad p(s(x)))$	And Elimination
8.	$\quad p(s(x)) \Rightarrow p(s(s(x)))$	Universal Elimination: 1
9.	$\quad p(s(s(x)))$	Implication Elimination: 8, 7
10.	$\quad p(s(x))) \wedge p(s(s(x)))$	And Introduction: 7, 9
11.	$p(x) \wedge p(s(x))) \Rightarrow p(s(x))) \wedge p(s(s(x)))$	And Introduction: 7, 9
12.	$\forall x.(p(x) \wedge p(s(x))) \Rightarrow p(s(x))) \wedge p(s(s(x))))$	Universal Introduction: 11
13.	$\forall x.(p(x) \wedge p(s(x)))$	Induction: 4, 12
14.	$p(x) \wedge p(s(x))$	Universal Elimination: 13
15.	$p(x)$	And Elimination: 14
16.	$p(s(x))$	And Elimination: 14
17.	$\forall x.p(x)$	Universal Introduction: 16

In this case, we are lucky that there is a useful conclusion that we can prove with standard Linear Induction. Things are not always so simple; and in some cases we need more complex forms of induction. Unfortunately, there is no finite collection of approaches to induction that covers all cases. If there were, we could build an algorithm for determining logical entailment for Relational Logic in all cases; and, as we discussed in Chapter 6, there is no such algorithm.

9.8 RECAP

Induction is reasoning from the specific to the general. *Complete induction* is induction where the set of instances is exhaustive. *Incomplete induction* is induction where the set of instances is not exhaustive. *Linear Induction* is a type of complete induction for languages with a single object constants and a single unary function constant. *Tree Induction* is a type of complete induction for languages with a single object constants and multiple unary function constants. *Structural Induction* is a generalization of both Linear Induction and Tree Induction that works even in the presence of multiple object constants and multiple n-ary function constants.

EXERCISES

9.1 Assume a language with the object constants a and b and no function constants. Given $q(a)$ and $q(b)$, use the Fitch system with domain closure to prove $\forall x.(p(x) \Rightarrow q(x))$.

9.2 Assume a language with the object constant a and the function constant s. Given $r(a)$, $\forall x.(p(x) \Rightarrow r(s(x)))$, $\forall x.(q(x) \Rightarrow r(s(x)))$, and $\forall x.(r(x) \Rightarrow p(x) \vee q(x))$, use the Fitch system with Linear Induction to prove $\forall x.r(x)$.

9.3 Assume a language with object constant a and unary function constants f and g. Given $p(a)$, $\forall x.(p(x) \Rightarrow p(f(x)))$, and $\forall x.(p(f(x)) \Rightarrow p(g(x)))$, use the Fitch system with Tree Induction to prove $\forall x.p(x)$.

9.4 Consider a language with object constants a and b, binary function constant c, and unary relation constants m and p and q. The definitions for the relations are shown below. Relation m is true of a and only a. Relation p is true of a structured object if and only if it is a linear list (as defined in Chapter 6) with a top-level element that satisfies m. Relation q is true of a structured object if and only if there is an element anywhere in the structure that satisfies m.

$$
\begin{array}{lll}
m(a) & \forall u.\forall v.(m(u) \Rightarrow p(c(u,v))) & \forall u.(m(u) \Rightarrow q(u)) \\
\neg m(b) & \forall u.\forall v.(p(v) \Rightarrow p(c(u,v))) & \forall u.\forall v.(q(u) \Rightarrow q(c(u,v))) \\
\forall u.\forall v.\neg m(c(u,v)) & \neg p(a) & \forall u.\forall v.(q(v) \Rightarrow q(c(u,v))) \\
 & \neg p(b) & \neg m(a) \Rightarrow \neg q(a) \\
 & \forall u.\forall v.(p(c(u,v)) \Rightarrow m(u) \vee p(v)) & \neg m(b) \Rightarrow \neg q(b) \\
 & & \forall u.\forall v.(q(c(u,v)) \Rightarrow \\
 & & \quad q(u) \vee q(v))
\end{array}
$$

Your job is to show that any object that satisfies p also satisfies q. Starting with the preceding axioms, use Fitch with Structural Induction to prove $\forall x.(p(x) \Rightarrow q(x))$. Beware: The proof requires more than 50 steps (including the premises). The good news is that it is very similar to the proof in Section 9.5.

9.5 Starting with the axioms for e given in Section 9.6, it is possible to prove that e is transitive, i.e., $\forall x.\forall y.\forall z.(e(x,y) \wedge e(y,z) \Rightarrow e(x,z))$. Doing this requires a three variable induction, and it is quite messy. Your job in this exercise is to prove just the base case for the outermost induction, i.e., prove $\forall y.\forall z.(e(a,y) \wedge e(y,z) \Rightarrow e(a,z))$. Hint: Use the strategy illustrated in Section 9.6. Extra credit: Do the full proof of transitivity.

9.6 Consider a language with a single object constant a, a single unary function constant s, and two unary relation constants p and q. We start with the premises shown below. We know that p is true of $s(a)$ and only $s(a)$. We know that q is also true of $s(a)$, but we do not know whether it is true of anything else.

$$
\begin{array}{l}
\neg p(a) \\
p(s(a)) \\
\forall x.\neg p(s(s(x))) \\
q(s(a))
\end{array}
$$

Prove $\forall x.(p(x) \Rightarrow q(x))$. Hint: Break the problem into two parts - first prove the result for $s(x)$, and then use that intermediate conclusion to prove the overall result.

CHAPTER 10

Equality

10.1 INTRODUCTION

In our discussion of Relational Logic thus far, we have assumed that there is a one-to-one relationship between ground terms in our language and objects in the application area we are trying to describe. For example, in talking about people, we have been assuming a unique name for each person. In arithmetic, we have been assuming a unique term for each number. This makes things conceptually simple, and it is a reasonable way to go in many circumstances.

But not always. In natural language, we often find it convenient to use more than one term to refer to the same real world object. For example, we sometimes have multiple names for the same person – *Michael*, *Mike*, and so forth. And, in Arithmetic, we frequently use different terms to refer to the same number – $2 + 2$, $2 * 2$, and 4.

In Relational Logic, we can axiomatize the co-referentiality of terms in the form of equations. For example, to express the co-referentiality of $f(a)$ and $f(b)$, we write $equal(f(a), f(b))$. We can also distinguish terms by writing negated equations. To say that the terms $f(a)$ and $f(b)$ refer to different objects, we write $\neg equal(f(a), f(b))$.

Since equality is such a common relation, in what follows we write equations with the infix operator $=$, for example writing $(f(a) = f(b))$ in place of $equal(f(a), f(b))$. However, this is just syntactic sugar. We must remember that, as far our logic is concerned, syntactically and semantically, an equation is a relational sentence involving a relation constant like any other.

We start this chapter by axiomatizing the equality relation as an ordinary binary relation. We then discuss the substitution of equals for equals in other expressions. Finally, we look at how to expand our proof system to reason with equality.

10.2 PROPERTIES OF EQUALITY

The semantics of Relational Logic does not, by itself, tell us which terms are equal and which are not. In fact, it is possible for each and every term to refer to a different object, and it is possible for all terms to refer to the same object.

Although the semantics of Relational Logic by itself does not constrain the equality relation, the idea of co-referentiality does impose some constraints. For example, we cannot believe $a = b$ and $b = c$ and at the same time believe that $a \neq c$.

First of all, the equality relation must be *reflexive*. This means that the relation holds of every term in the language and itself.

$$\forall x.x = x$$

The relation must also be *symmetric*. If two terms refer to the same thing, it does not matter which one we write first in an equation.

$$\forall x.\forall y.(x = y \Rightarrow y = x)$$

Finally, the relation must be *transitive*. If we believe that a and b refer to the same object and we believe that b and c refer to the same object, then a and c must refer to the same object as well.

$$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$$

The three properties of reflexivity, symmetry, and transitivity are the defining conditions for what is know as an *equivalence relation*, and equality is the quintessential example.

Let's see how we can use these properties to solve some problems of equality. Suppose we know that $b = a$ and we know that $b = c$. The following is a proof that $a = c$. As usual, we start with our premises; and, since we are working with equations, we also include the three axioms of equality. We use two applications of Universal Elimination on our symmetry axiom to derive the sentence on line 6, substituting b for x and a for y. We then use Implication Elimination to conclude $a = b$. We then use Universal Elimination to instantiate the transitivity axiom, with x replaced by a, y replaced by b, and z replaced by c. We conjoin the result on line 7 with the premise on line 2 to derive the conjunction on line 9. Finally, we use Implication Elimination to derive our overall conclusion.

1.	$b = a$	Premise
2.	$b = c$	Premise
3.	$\forall x.x = x$	Reflexivity
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Symmetry
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Transitivity
6.	$b = a \Rightarrow a = b$	2×UE: 4
7.	$a = b$	IE: 6, 1
8.	$a = b \wedge b = c \Rightarrow a = c$	3×UE: 5
9.	$a = b \wedge b = c$	And Introduction: 7, 2
10.	$a = c$	IE: 8, 9

This proof is a little lengthy, but the method works. Later in the chapter, we show how to generate shorter proofs for simple results like this one.

10.3 SUBSTITUTION

The properties of equality seen thus far are only part of the story. There is also substitution. If two terms are equal (i.e., they refer to the same object), then every sentence that is true of the first term must be true of the second; and every sentence that is false of the first term must be false of the second.

In other words, it should be possible to substitute one term for an equal term without changing the truth value of any sentence. We can express this notion of substitutability by writing substitution axioms, one for each relation constant and one for each function constant in our vocabulary.

The sentence shown below is a substitution axiom for the unary relation constant p. If $x = y$ and $p(x)$ is true, then $p(y)$ must also be true.

$$\forall x.\forall y.(p(x) \wedge x = y \Rightarrow p(y))$$

The following sentence is a substitution axiom for the binary relation constant q. Note that we need to allow for equations for all of the arguments of relational sentences.

$$\forall u.\forall v.\forall x.\forall y.(q(u, v) \wedge u = x \wedge v = y \Rightarrow q(x, y))$$

The following is a substitution axiom for the unary function constant f.

$$\forall x.\forall y.\forall z.(f(x) = z \wedge x = y \Rightarrow f(y) = z)$$

Using the equality and substitution axioms, we can do proofs with equality. We simply add these additional axioms to our premises and prove our conclusions as before.

As an example, suppose that we believe $f(a) = b$ and $f(b) = a$. Let's prove that $f(f(a)) = a$. A proof of this conclusion using Fitch is shown below. The first two lines are our premises. The next four are our equality axioms and the substitution axiom for f.

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$\forall x.x = x$	Reflexivity
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Symmetry
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Transitivity
6.	$\forall x.\forall y.\forall z.(f(x) = z \wedge x = y \Rightarrow f(y) = z)$	Substitution
7.	$\forall y.(f(a) = y \Rightarrow y = f(a))$	Universal Elimination: 4
8.	$f(a) = b \Rightarrow b = f(a)$	Universal Elimination: 7
9.	$b = f(a)$	Implication Elimination: 8, 1
10.	$f(b) = a \wedge b = f(a)$	And Introduction: 2, 9
11.	$\forall x.\forall y.\forall z.(f(x) = z \wedge x = y \Rightarrow f(y) = z)$	Reiteration: 6
12.	$\forall y.\forall z.(f(b) = z \wedge b = y \Rightarrow f(y) = z)$	Universal Elimination: 11
13.	$\forall z.(f(b) = z \wedge b = f(a) \Rightarrow f(f(a)) = z)$	Universal Elimination: 12
14.	$f(b) = a \wedge b = f(a) \Rightarrow f(f(a)) = a$	Universal Elimination: 13
15.	$f(f(a)) = a$	Implication Elimination: 14, 10

A bit long, but it works. The simpler approach is to use built-in rules of inference for equality, as described in the next section.

10.4 FITCH WITH EQUALITY

As we have seen, we can use Fitch to create proofs for theories with equality by adding the equality and substitution axioms to our premise set. However, this is not the only way to go. Since equality is such a pervasive and important relation, it makes sense to treat it specially in our proof system. In this section we show how to extend Fitch to handle equations. As it turns out, all we need is one axiom schema and one rule of inference.

The Equality Introduction rule allows the insertion of an arbitrary instance of reflexivity.

Equality Introduction

$$\sigma = \sigma$$

where σ is any term

For example, without any premises whatsoever, we can write down equations like $(a = a)$ and $(f(a) = f(a))$ and $(f(x) = f(x))$.

Our rule of inference is called *Equality Elimination* (QE). Equality Elimination tells us that, when we have an equation and a sentence containing one or more occurrences of one of the terms in the equation, then we can deduce a version of the sentence in which the that term is replaced by the other term in the equation.

Equality Elimination

$$\frac{\phi[\tau_1] \qquad \tau_1 = \tau_2}{\phi[\tau_2]}$$

where τ_2 is substitutable for τ_1 in ϕ

In order to avoid the unintended capture of variables, QE requires that the replacement must be substitutable for the term being replaced in the sentence. This is the same substitutability condition that adorns the Universal Elimination rule of inference.

Note that the equation in the Equality Elimination rule can be used in either direction, i.e., an occurrence of τ_1 can be replaced by τ_2 or an occurrence of τ_2 can be replaced by τ_1.

The following is a proof of the result in Section 10.2 using these extensions in lieu of the equality axioms. Note that with these rules we can do the proof in a single step.

1.	$b = a$	Premise
2.	$b = c$	Premise
3.	$a = c$	Equality Elimination 2, 1

The following is a proof of the result in Section 10.3 using these two extensions in lieu of the equality and substitution axioms. Again, we can do the proof in a single step. Clearly, building things into our proof system makes for much shorter proofs.

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$f(f(a)) = a$	Equality Elimination 2, 1

Now, let's look at a couple of examples that illustrate how equational reasoning interacts with relational reasoning. We know that Quincy is the father of Pat, and we know that fathers are older than their children. Our job is to prove that Quincy is older than Pat.

Our proof is shown below. As usual, we start with our premises. The father of Pat is Quincy, and fathers are older than their children. Next, we use Universal Elimination to instantiate our quantified sentence. The father of Pat is older than Pat. Next we use Equality Elimination to replace *father(pat)* with *quincy* and thereby derive the conclusion that Quincy is older than Pat.

1.	$father(pat) = quincy$	Premise
2.	$\forall x.older(father(x), x)$	Premise
3.	$older(father(pat), pat)$	Universal Elimination 2
4.	$older(quincy, pat)$	Equality Elimination 3, 1

Finally, let's work through an example involving a disjunction of equations. We know that $p(a)$ is true and $p(b)$ is true, and we know that $(a = c \vee b = c)$ is true. Our job is to prove $p(c)$.

The proof follows. As always we start with our premises. On line 4 , we start a new subproof with the assumption that $a = c$ is true. From this assumption and our first premise, we conclude that $p(c)$ must be true. Of course, we are not done yet, since we have proved the result only under this assumption that $a = c$. We make this clear with a use of Implication Introduction to derive the sentence $(a = c \Rightarrow p(c))$. Now, we start another subproof, this time with the assumption $b = c$. As before, we derive $p(c)$; and, from this, we derive the implication $(b = c \Rightarrow p(c))$. Finally, we use Or Elimination to combine our two partial results with our disjunction of equations.

1.	$p(a)$	Premise
2.	$p(b)$	Premise
3.	$a = c \vee b = c$	Premise
4.	$\quad \vert\ a = c$	Assumption
5.	$\quad \vert\ p(c)$	Equality Elimination: 1, 4
6.	$a = c \Rightarrow p(c)$	Implication Introduction: 5
7.	$\quad \vert\ b = c$	Assumption
8.	$\quad \vert\ p(c)$	Equality Elimination: 2, 7
9.	$b = c \Rightarrow p(c)$	Implication Introduction: 8
10.	$p(c)$	Or Elimination: 3, 6, 9

10.5 EXAMPLE – GROUP THEORY

A group is an algebraic structure consisting of an arbitrary set of elements and a binary function on these elements such that the binary function is associative, there is a left and right identity for the function, and there is a left and right inverse for any element in the set.

We can express the group axioms in Relational Logic as shown below. Here, $*$ is the binary function, e is the identity for $*$, and *inv* gives the inverse of any element.

$$\forall x.\forall y.\forall z.(x * y) * z = x * (y * z)$$
$$\forall x.x * e = x$$
$$\forall x.e * x = x$$
$$\forall x.x * inv(x) = e$$
$$\forall x.inv(x) * x = e$$

Given these axioms, let's use Fitch with equality to prove that the inverse of the inverse of the element is equal to the original element, i.e, $\forall x.inv(inv(x)) = x$. As usual, we start with our axioms.

1.	$\forall x.\forall y.\forall z.(x * y) * z = x * (y * z)$	Premise
2.	$\forall x.x * e = x$	Premise
3.	$\forall x.e * x = x$	Premise
4.	$\forall x.x * inv(x) = e$	Premise
5.	$\forall x.inv(x) * x = e$	Premise

Next we instantiate the left inverse rule, substituting $inv(x)$ for x. We then use Equality Introduction to write down an equation saying that $(inv(inv(x)) * inv(x)) * x$ is equal to itself. Then, we use our instantiated left inverse rule to substitute e for $inv(inv(x))$ on the right hand side of our equation. We instantiate our left identity rule, and then we use that to simplify the right hand side of our equation to x.

6.	$inv(inv(x)) * inv(x) = e$	Universal Elimination: 5
7.	$(inv(inv(x)) * inv(x)) * x = (inv(inv(x)) * inv(x)) * x$	Equality Introduction
8.	$(inv(inv(x)) * inv(x)) * x = e * x$	Equality Elimination: 7, 6
9.	$e * x = x$	Universal Elimination: 3
10.	$(inv(inv(x)) * inv(x)) * x = x$	Equality Elimination: 9, 8

We now instantiate the associativity rule three times, replacing x by $inv(inv(x))$, y by $inv(x)$, and z by x. We then use Equality Elimination to re-associate the left hand side of the equation on line 10.

11.	$\forall y.\forall z.(inv(inv(x)) * y) * x = inv(inv(x)) * (y * x)$	Universal Elimination: 5
12.	$\forall z.(inv(inv(x)) * inv(x)) * x = inv(inv(x)) * (inv(x) * x)$	Universal Elimination: 11
13.	$(inv(inv(x)) * inv(x) * x = inv(inv(x)) * (inv(x) * x)$	Universal Elimination: 12
14.	$inv(inv(x)) * (inv(x) * x) = x$	Equality Elimination: 10, 13

We instantiate the left inverse rule again. We apply this to our equation to simplify the left hand side. We instantiate our right identity rule and apply the result to the simplified equation to produce $inv(inv(x)) = x$. Finally, we use Universal Introduction to produce the desired result.

15.	$inv(x) * x = e$	Universal Elimination: 5
16.	$inv(inv(x)) * e = x$	Equality Elimination: 14, 15
17.	$inv(inv(x)) * e = inv(inv(x))$	Universal Elimination: 2
18.	$inv(inv(x)) = x$	Equality Elimination: 16, 17
19.	$\forall x.inv(inv(x)) = x$	Universal Introduction: 18

Groups have many interesting properties. They are interesting in and of themselves. However, Group Theory is also an interesting domain for Logic, especially for theorem proving with equality.

10.6 RECAP

The *equality* relation is a way of stating that two terms refer to the same real world object. This idea of co-referentiality imposes some constraints on the equality relation. These include *reflexivity*, *symmetry*, *transitivity*, and *substitution* of equals for equals. These properties can be assured in reasoning by adding a finite number of axioms to every premise set. Alternatively, the same results can be obtained in Fitch by using *Equality Introduction* and *Equality Elimination*.

EXERCISES

10.1 Given $a = b$ and $b = c$ and the axioms of equality, use the Fitch system (without Equality Introduction or Equality Elimination) to prove $a = c$.

10.2 Given $a = b$ and $p(a)$ and the axioms of equality and the substitution axiom, use the Fitch system (without Equality Introduction or Equality Elimination) to prove $p(b)$.

10.3 Given $a = b$ and $b = c$, use the Fitch system with equality to prove $a = c$.

10.4 Given $a = b$ and $p(a)$, use the Fitch system with equality to prove $p(b)$.

10.5 Starting with the group axioms from Section 10.5, prove the left cancellation law for groups, i.e., prove $\forall x.\forall y.\forall z.(x * y = x * z \Rightarrow y = z)$. (Note that there is an analogous right cancellation law.)

APPENDIX A

Summary of Fitch Rules

Fitch for Propositional Logic (Fitch_{PL}) is the Fitch proof system with the propositional rules.

Fitch for Herbrand Logic (Fitch_{HL}) is the Fitch proof system with the propositional rules, the quantifier rules, and the induction rule.

Fitch for First Order Logic ($\text{Fitch}_{\text{FOL}}$) is the Fitch proof system with the propositional rules, the quantifier rules, and the equality rules.

Propositional Rules (PROP)

And Introduction (AI)

$$\frac{\varphi_1 \quad \cdots \quad \varphi_n}{\varphi_1 \wedge \cdots \wedge \varphi_n}$$

And Elimination (AE)

$$\frac{\varphi_1 \wedge \cdots \wedge \varphi_n}{\varphi_i}$$

Or Introduction (OI)

$$\frac{\varphi_i}{\varphi_1 \vee \cdots \vee \varphi_n}$$

Or Elimination (OE)

$$\frac{\varphi_1 \vee \cdots \vee \varphi_n \quad \varphi_1 \Rightarrow \psi \quad \cdots \quad \varphi_n \Rightarrow \psi}{\psi}$$

Negation Introduction (NI)

$$\frac{\varphi \Rightarrow \psi \qquad \varphi \Rightarrow \neg\psi}{\neg\varphi}$$

Negation Elimination (NE)

$$\frac{\neg\neg\varphi}{\varphi}$$

Implication Introduction (II)

$$\frac{\varphi \vdash \psi}{\varphi \Rightarrow \psi}$$

Implication Elimination (IE)

$$\frac{\varphi \Rightarrow \psi \qquad \varphi}{\psi}$$

Biconditional Introduction (BI)

$$\frac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\varphi \Leftrightarrow \psi}$$

Biconditional Elimination (BE)

$$\frac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\varphi \Leftrightarrow \psi}$$

Quantifier Rules (QU)

Universal Introduction (UI)

$$\frac{\varphi}{\forall \nu.\varphi}$$

[where ν does not occur free in both φ and an active assumption]

Universal Elimination (UE)

$$\frac{\forall \nu.\varphi[\nu]}{\varphi[\tau]}$$

[where τ is substitutable for ν in φ]

Existential Introduction (EI)

$$\frac{\varphi[\tau]}{\exists \nu.\varphi[\nu]}$$

[where τ is substitubale for ν in φ]

Existential Elimination (EE)

$$\frac{\exists \nu_1.\varphi[\nu_1] \qquad \forall \nu_2.(\varphi[\nu_2] \Rightarrow \psi)}{\psi}$$

[where ν_2 does not occur free in ψ]

Equality Rules (EQ)

Equality Introduction (QI)

$$\tau = \tau$$

[τ is a term]

Equality Elimination (QE)

$$\frac{\varphi[\tau_1] \qquad \tau_1 = \tau_2}{\varphi[\tau_2]}$$

[where τ_2 is substitutable for τ_1 in φ]

Induction Rules (IND)

General Induction (IND)

$$\frac{\{\varphi[\sigma] : \sigma \text{ is a object contant}\} \quad \{\forall \nu_1 \cdots \forall \nu_m.(\varphi[\nu_1] \wedge \cdots \wedge \varphi[\nu_m] \Rightarrow \varphi[f(\nu_1, \ldots, \nu_m)]) : f \text{ is a function of arity } m\}}{\forall \nu.\varphi[\nu]}$$

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Authors' Biographies

MICHAEL GENESERETH

Michael Genesereth is an associate professor in the Computer Science Department at Stanford University. He received his Sc.B. in Physics from M.I.T. and his Ph.D. in Applied Mathematics from Harvard University. He is best known for his research on Computational Logic and its applications. He has been teaching Logic to Stanford students and others for more than 20 years. He is the current director of the Logic Group at Stanford and founder and research director of CodeX (The Stanford Center for Legal Informatics).

ERIC KAO

Eric Kao is a Ph.D. candidate in the Computer Science Department at Stanford University. He holds a B.Math in Computer Science and Pure Mathematics from the University of Waterloo. His research centers on inconsistency-tolerant logic and its applications in data management. He has helped design and teach several logic courses.