Primer examen metodos I

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1)

a) *Linealidad: $Tr((\alpha a+\beta b)^{\dagger}C) = Tr((\alpha a)^{\dagger}C) + Tr((\beta b)^{\dagger}C)$

$= \alpha Tr(a^{\dagger}C) + \beta(b^{\dagger}C)$

*Positividad:

$Tr(A^{\dagger}A) = (z_1)^2 + (z_3 z_3^*) + (z_2 z_2^*) + (z_4)^2$

z_1 y $z_4 \subset \mathbb{R}$ y $z_2 z_2^* = a^2+b^2$, por lo que $Tr(A^{\dagger}A) \geq 0$

*Simetría

$Tr(A^{\dagger}B) = z_1 C_1 + z_2 C_3 + z_3 C_2 + z_4 C_4$

$= C_1 z_1 + C_3 z_2 + C_2 z_3 + C_4 z_4 = Tr(B^{\dagger}A)$

b) $\|A\| = \sqrt{\langle a|a \rangle}$ donde $\langle a|a \rangle = Tr(A^{\dagger}A)$

$Tr(A^{\dagger}A) = (z_1)^2 + (z_2 z_2^*) + (z_3 z_3^*) + (z_4)^2$

$= a_1^2 + (a_2)^2 + (b_2)^2 + (a_3)^2 + (b_3)^2 + a_4^2$

$\|A\| = \sqrt{(a_1)^2+(a_2)^2+(a_3)^2+(a_4)^2+(b_2)^2+(b_3)^2}$

c) $\|A-B\|_F = \sqrt{Tr((A-B)^{\dagger}(A-B))}$ = distancia

$d = \sqrt{Tr(A^{\dagger}A - A^{\dagger}B - B^{\dagger}A - B^{\dagger}B)}$

$d = \sqrt{Tr(A^{\dagger}A) - 2Tr(A^{\dagger}B) - Tr(B^{\dagger}B)}$

d) $Tr(\sigma_1^{\dagger}\sigma_2) = Tr\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i - i = \underline{0}$, $Tr(\sigma_1^{\dagger}\sigma_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \underline{0}$,

$Tr(\sigma_2^{\dagger}\sigma_3) = Tr\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \underline{0}$

Como la traza entre las matrices es cero, son ortogonales entre ellas

f) $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \sigma_1^{+} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \sigma_1 \in V$

$\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \Rightarrow \sigma_2^{+} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow \sigma_2 \in V$

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \sigma_3^{+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \sigma_3 \in V$

Por lo tanto pertenece a V

Independencia lineal:

$z_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + z_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + z_4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$(a_1 + b_1 i) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (a_2 + b_2 i) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + (a_3 + b_3 i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + (a_4 b_4 i) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$a_3 + ib_3 + a_4 + ib_4 = 0$
$a_1 + ib_1 - a_2 i + b_2 = 0$
$a_1 + ib_1 + a_2 i - b_2 = 0$
$-a_3 - ib_3 + a_4 + ib_4 = 0$

1
$a_3 + a_4 = 0$
$a_1 + b_2 = 0$
$a_1 - b_2 = 0$
$-a_3 + a_4 = 0$

y

2
$ib_3 + ib_4 = 0$
$ib_1 - ia_2 = 0$
$ib_1 + ia_2 = 0$
$-ib_3 + ib_4 = 0$

Para 1

$a_3 = -a_4 \Rightarrow -(-a_4) + a_4 = 0 \Rightarrow \underline{a_4 = 0} \Rightarrow \underline{a_3 = 0}$

$a_1 = -b_2 \Rightarrow -b_2 - b_2 = 0 \Rightarrow \underline{b_2 = 0} \Rightarrow \underline{a_1 = 0}$

Para 2

$b_3 = -b_4 \Rightarrow ib_4 + ib_4 = 0 \Rightarrow \underline{b_4 = 0} \Rightarrow \underline{b_3 = 0}$

$b_1 = a_2 \Rightarrow ia_2 + ia_2 = 0 \Rightarrow \underline{a_2 = 0} \Rightarrow \underline{b_1 = 0}$

$\therefore$ Dado que $z_1 = z_2 = z_3 = z_4 = 0$, el conjunto es L.I

Ahora: $A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = A^{+} = \begin{pmatrix} z_1^* & z_3^* \\ z_2^* & z_4^* \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{pmatrix} = \begin{pmatrix} a_1 - ib_1 & a_3 - ib_3 \\ a_2 - ib_2 & a_4 - ib_4 \end{pmatrix}$

$\cancel{a_1} + ib_1 = \cancel{a_1} - ib_1 \wedge a_2 + ib_2 = a_3 - ib_3 \wedge a_3 + ib_3 = a_2 - ib_2 \wedge \cancel{a_4} + ib_4 = \cancel{a_4} - ib_4$

$ib_1 + ib_1 = 0$ $\qquad a_2 = a_3 - ib_3 \longrightarrow \cancel{a_3} + ib_3 = \cancel{a_3} - ib_3 - ib_2$

$2ib_1 = 0 \Rightarrow b_1 = 0$ $\qquad b_3 + b_3 = b_2$

$b_3 = \frac{b_2}{2}$

$ib_4 + ib_4 = 0 \Rightarrow b_4 = 0$ $\quad a_2 = a_3 - \frac{1}{2} i b_2$

$\begin{pmatrix} a_1 & (a_3 - \frac{1}{2} ib_2) + ib_2 \\ a_3 - \frac{1}{2} ib_2 & a_4 \end{pmatrix}$ El vector general tiene cuatro variables libres, $\mathbb{V}$ es de dim(4)

Por lo tanto el conjunto es base de $\mathbb{V}$

h)

$$A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + C\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + D\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} A+C & -iB+D \\ iB+D & A-C \end{pmatrix} = H$$

Si $A, B, C, D \in \mathbb{C}$, no se podrá crear un subespacio puro dado que tendrán entradas complejas

2)

a) $f(x) = x^2 + x + 3 = a_0(1) + a_1(x) + a_2\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$

$$= a_0 - \frac{1}{2}a_2 + a_1 x + \frac{3}{2}$$

$a_0 - \frac{1}{2}a_2 = 3 \rightarrow a_0 = 3 + \frac{1}{3} \rightarrow a_0 = \frac{10}{3}$

$a_1 = 1$

$\frac{3}{2}a_2 = 1 \rightarrow a_2 = \frac{2}{3}$

El polinomio $f(x)$ expandido por los Polinomios de legendre es:

$$x^2 = \frac{10}{3}P_0(x) + P_1(x) + \frac{2}{3}P_2(x)$$

b) $P^p_{(x)} \otimes P^q_{(y)} = (x^2 + x + 3) \otimes (y + 1)$

$= x^2(y+1) + x(y+1) + 3(y+1)$

$= x^2 y + x^2 + xy + x + 3y + 3$

c) $P^{p\otimes q} = P(x) \otimes q(x) = x^2y + x^2 + xy + x + 3y + 3$

$= (y+1)x^2 + (y+1)x + 3y + 3$

Expansión de la base $\{x^2, x, 1\}$ $\longrightarrow$ $= (y+1)P_2(x) + (y+1)P_1(x) + (3y+3)P_0(x)$

$P^{p\otimes q} = P(x) \otimes q(x)$

$= x^2y + x^2 + xy + x + 3y + 3$ Expansión del tensor en base $\{y^2, y, 1\}$

$= (x^2 + x + 3)y + (x^2 + x + 3)$

$= 0\,P_2(y) + (x^2 + x + 3)P_1(y) + (x^2 + x + 3)P_0(y)$

d) $x^2y + x^2 + xy + x + 3y + 3 = \tilde{C}^{11}(1) + \tilde{C}^{12}(x) + \tilde{C}^{13}\frac{1}{2}(3x^2 - 1)$

$(y+1)x^2 + (y+1)x + (3y+3) = \tilde{C}^{11}(1) + \tilde{C}^{12}(x) + \frac{3}{2}x^2\tilde{C}^{13} - \tilde{C}^{13}(1)\frac{1}{2}$

$\tilde{C}^{12} = y + 1$

$(3y+3) = \tilde{C}^{11} - \frac{1}{2}\tilde{C}^{13} \rightarrow (3y+3) + \frac{y+1}{3} = \tilde{C}^{11} \rightarrow \tilde{C}^{11} = \frac{10y+10}{3}$

$\tilde{C}^{13} = \frac{2y+2}{3}$

$(x^2 + x + 3)y + (x^2 + 3) = \tilde{C}^{21}(1) + \tilde{C}^{22}(y) + \tilde{C}^{23}(\frac{1}{2}(3x^2 - 1))$

$\tilde{C}^{23} = 0$ $\quad \tilde{C}^{22} = x^2 + x + 3$ $\quad \tilde{C}^{21} = x^2 + 3$