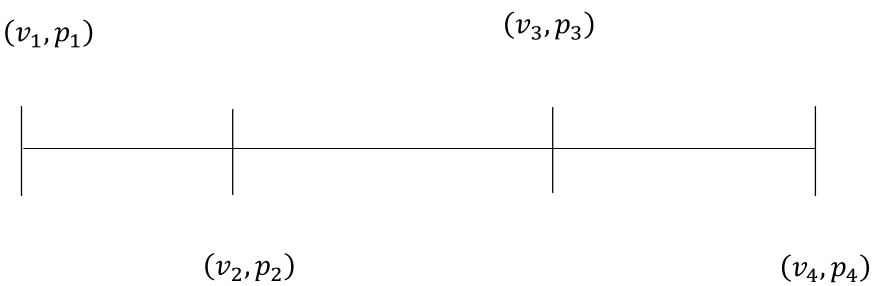
Método de los Elementos Finitos para el problema de dinámica de fluidos en una dimensión con funciones de forma lineales y con peso de Galerkin

$$Ec. 1\ f = \bar{v}\frac{d}{dx}v_x - \eta\frac{d}{dx}\left(\frac{d}{dx}v_x\right) + \frac{1}{\rho}\frac{d}{dx}p$$

$$Ec. 2\ \frac{d}{dx}\vec{v} = 0$$

Mallado:

Asumimos que las incógnitas están en los nodos



Interpolación:

Utilizaremos funciones de forma lineales para una dimensión.

$$N_i = \frac{x_{i+1} - x}{x_{i+1} - x_i}$$

$$N_{i+1} = \frac{x - x_i}{x_{i+1} - x_i}$$

Discretización:

$$v \approx [N_i \quad N_{i+1}] \begin{bmatrix} v_i \\ v_{i+1} \end{bmatrix} = \boldsymbol{Nv}, \qquad N(x)$$

$$p \approx [N_i \quad N_{i+1}] \begin{bmatrix} P_i \\ P_{i+1} \end{bmatrix} = \boldsymbol{Np}, \qquad N(x)$$

Combinando Ecuaciones:

$$\bar{v}\frac{d}{dx}\boldsymbol{Nv} - \eta\frac{d}{dx}\left(\frac{d}{dx}\boldsymbol{Nv}\right) + \frac{1}{\rho}\frac{d}{dx}\boldsymbol{Np} - f + \frac{d}{dx}\boldsymbol{Nv} \approx 0$$

Calculo Residual:

$$\bar{v}\frac{d}{dx}\boldsymbol{Nv} - \eta\frac{d}{dx}\left(\frac{d}{dx}\boldsymbol{Nv}\right) + \frac{1}{\rho}\frac{d}{dx}\boldsymbol{Np} - f + \frac{d}{dx}\boldsymbol{Nv} = \xi$$

Método de los residuos ponderados:

$$\int_{\Omega} \xi_i w_i \, d\Omega = 0$$

$$\mathbf{W} = \begin{bmatrix} w_{x_i} \\ w_{x_i+1} \end{bmatrix}$$

$$\int_{\Omega} \boldsymbol{W}\left[\bar{v}\frac{d}{dx}\boldsymbol{N}\boldsymbol{v} - \eta\frac{d}{dx}\left(\frac{d}{dx}\boldsymbol{N}\boldsymbol{v}\right) + \frac{1}{\rho}\frac{d}{dx}\boldsymbol{N}\boldsymbol{p} - f + \frac{d}{dx}\boldsymbol{N}\boldsymbol{v}\right] d\Omega = 0$$

Método de Galerkin:

$$W_i = N_i$$

Forma Fuerte:

$$\int_{x_i}^{x_{i+1}} \boldsymbol{W}\left[\bar{v}\frac{d}{dx}\boldsymbol{N}\boldsymbol{v} - \eta\frac{d}{dx}\left(\frac{d}{dx}\boldsymbol{N}\boldsymbol{v}\right) + \frac{1}{\rho}\frac{d}{dx}\boldsymbol{N}\boldsymbol{p} - f + \frac{d}{dx}\boldsymbol{N}\boldsymbol{v}\right] \mathrm{dx} = 0$$

$$\int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}}\bar{v}\frac{d}{dx}\boldsymbol{N}\boldsymbol{v}dx - \int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}}\eta\frac{d}{dx}\left(\frac{d}{dx}\boldsymbol{N}\boldsymbol{v}\right)dx + \int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}}\frac{1}{\rho}\frac{d}{dx}\boldsymbol{N}\boldsymbol{p}dx - \int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}} f dx + \int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}}\frac{d}{dx}\boldsymbol{N}\boldsymbol{v}dx = 0$$

Integrando:

$$\int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}} \bar{v} \frac{d}{dx} \boldsymbol{N} \boldsymbol{v} dx$$

$$= \bar{v} \int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}} \frac{d}{dx} \boldsymbol{N} dx \boldsymbol{v}$$

$$= \bar{v} \int_{x_i}^{x_{i+1}} \begin{bmatrix} \dfrac{x_{i+1} - x}{x_{i+1} - x_i} \\ \dfrac{x - x_i}{x_{i+1} - x_i} \end{bmatrix} \begin{bmatrix} \dfrac{-1}{x_{i+1} - x_i} & \dfrac{1}{x_{i+1} - x_i} \end{bmatrix} dx \boldsymbol{v}$$

$$= \frac{\bar{v}}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \boldsymbol{v}$$

$$= \boldsymbol{A}_{\boldsymbol{2x2}} \boldsymbol{v}$$

Integración por partes:

$$-\int_{x_i}^{x_{i+1}} \boldsymbol{N^T}\eta\frac{d}{dx}\left(\frac{d}{dx}\boldsymbol{Nv}\right)dx$$

$$u = \boldsymbol{N^T}$$

$$du = \frac{d}{dx}\boldsymbol{N^T}dx$$

$$dv = \eta\frac{d}{dx}\left(\frac{d}{dx}\boldsymbol{Nv}\right)\,dx$$

$$v = \eta\frac{d}{dx}\boldsymbol{Nv}$$

$$= -\boldsymbol{N^T}\eta\frac{d}{dx}\boldsymbol{Nv}\bigg|_{\Gamma} + \eta\int_{x_i}^{x_{i+1}} \frac{d}{dx}\boldsymbol{N^T}\frac{d}{dx}\boldsymbol{Nv}dx$$

$$= \frac{\eta}{x_{i+1}-x_i}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\boldsymbol{v}$$

$$= \boldsymbol{B_{2x2}v}$$

Integración:

$$\int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}} \frac{1}{\rho} \frac{d}{dx} \boldsymbol{N}\boldsymbol{p} dx$$

$$= \frac{1}{\rho} \int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}} \frac{d}{dx} \boldsymbol{N}\boldsymbol{p} dx$$

$$= \frac{1}{2\rho} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \boldsymbol{p}$$

$$= \boldsymbol{C}_{\boldsymbol{2x2}} \boldsymbol{p}$$

Integración:

$$\int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}} f dx = f \int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}} dx = \frac{-f(x_{i+1} - x_i)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\boldsymbol{b}_{\boldsymbol{2x1}}$$

Integración:

$$\int_{x_i}^{x_{i+1}} \boldsymbol{N}^{\boldsymbol{T}} \frac{d}{dx} \boldsymbol{N}\boldsymbol{v} dx = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \boldsymbol{v} = \boldsymbol{D}_{\boldsymbol{2x2}} \boldsymbol{v}$$

Sistema de Ecuaciones Lineales:

$$Ec.1\ \boldsymbol{A_{2x2}v} + \boldsymbol{B_{2x2}v} + \boldsymbol{C_{2x2}p} = \boldsymbol{b_{2x1}} + \boldsymbol{N^T}\eta\frac{d}{dx}\boldsymbol{Nv}\Big|_{\Gamma}$$

$$Ec.2\ \boldsymbol{D_{2x2}v} = \boldsymbol{0}$$

$$\begin{bmatrix} \boldsymbol{A+B} & \boldsymbol{C} \\ \boldsymbol{D} & \boldsymbol{0} \end{bmatrix}_{\boldsymbol{4x4}} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{p} \end{bmatrix}_{\boldsymbol{4x1}} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}_{\boldsymbol{4x1}} + \boldsymbol{N^T}\eta\frac{d}{dx}\boldsymbol{Nv}\Big|_{\Gamma}$$

$$\boldsymbol{KX} = \boldsymbol{W}$$

Para cada elemento i:

$$k = \begin{bmatrix} (A+B)_{i11} & (A+B)_{i12} & C_{i11} & C_{i12} \\ (A+B)_{i21} & (A+B)_{i22} & C_{i21} & C_{i22} \\ D_{i11} & D_{i12} & 0 & 0 \\ D_{i21} & D_{i22} & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} v_{i1} \\ v_{i2} \\ p_{i1} \\ p_{i2} \end{bmatrix}$$

$$b = \begin{bmatrix} b_{i1} \\ b_{i2} \\ 0 \\ 0 \end{bmatrix}$$

Así por ejemplo:

$$(A+B)_{i11} = \frac{-v}{2} + \frac{\eta}{x_{i+1} - x_i}$$

Ensamblaje:

Elemento	$\boldsymbol{i}$	$\boldsymbol{i+1}$
1	1	2
2	2	3
3	3	4

$$\begin{vmatrix} (A+B)_{111} & (A+B)_{112} & 0 & 0 & C_{111} & C_{112} & 0 & 0 \\ (A+B)_{121} & (A+B)_{122}+(A+B)_{211} & (A+B)_{212} & 0 & C_{121} & C_{122}+C_{211} & C_{212} & 0 \\ 0 & (A+B)_{221} & (A+B)_{222}+(A+B)_{311} & (A+B)_{312} & 0 & C_{221} & C_{222}+C_{311} & C_{312} \\ 0 & 0 & (A+B)_{321} & (A+B)_{322} & 0 & 0 & C_{321} & C_{322} \\ D_{111} & D_{112} & 0 & 0 & 0 & 0 & 0 & 0 \\ D_{121} & D_{122}+D_{211} & D_{212} & 0 & 0 & 0 & 0 & 0 \\ 0 & D_{221} & D_{222}+D_{311} & D_{312} & 0 & 0 & 0 & 0 \\ 0 & 0 & D_{321} & D_{322} & 0 & 0 & 0 & 0 \end{vmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12}+b_{21} \\ b_{22}+b_{31} \\ b_{32} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$