Find Maximum Independent Set in a Graph of Treewidth k by Dynamic Programming

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https://en.wikipedia.org/wiki/Tree_decomposition

Denotations

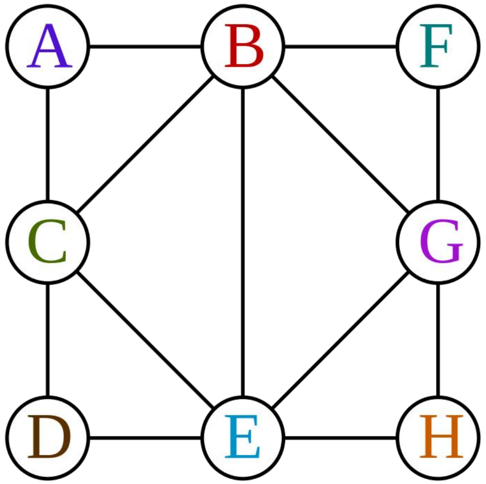
- X_i: a node of the tree decomposition
- D_i: the union of the sets X_j descending from X_i
- $A(S, i)$: the size of the largest independent subset I of D_i s.t.
  - $I \cap X_i = S$
- $B(S, i, j)$: the size of the largest independent subset I of D_i s.t.
  - X_i and X_j are an adjacent pair
  - X_i is farther from the root of the tree than X_j
  - $I \cap X_i \cap X_j = S$

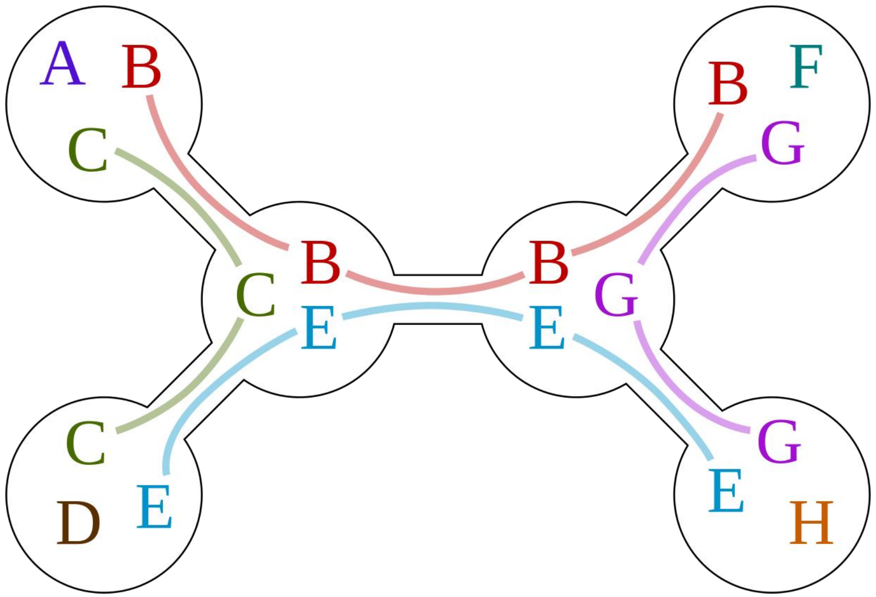
Bottom-up Dynamic Programming

- $A(S, i) = |S| + \sum_j (B(S \cap X_j, j, i) - |S \cap X_j|)$
- $B(S, i, j) = \max_{\substack{S' \subset X_i \\ S = S' \cap X_j}} A(S', i)$

Example Graph

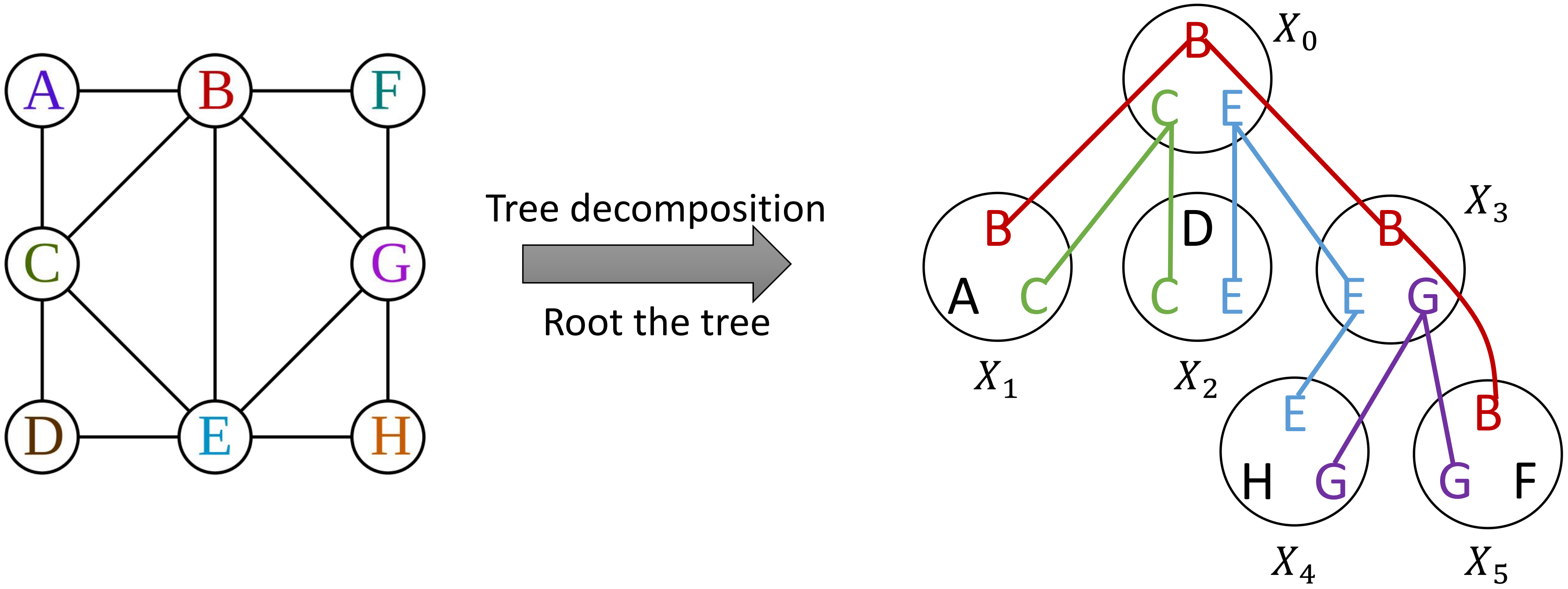
- A graph with eight vertices, and a tree decomposition of it onto a tree with six nodes. Each graph edge connects two vertices that are listed together at some tree node, and each graph vertex is listed at the nodes of a contiguous subtree of the tree. Each tree node lists at most three vertices, so the width of this decomposition is two.



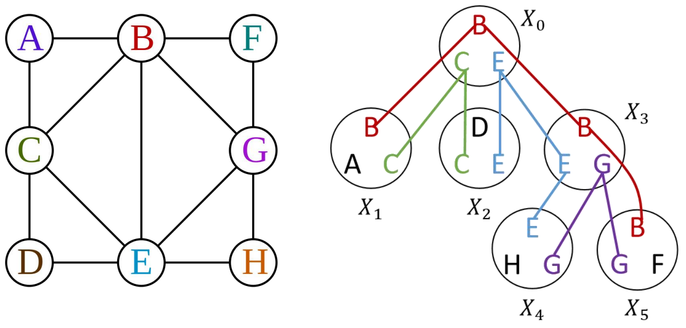


https://en.wikipedia.org/wiki/Tree_decomposition#/media/File:Tree_decomposition.svg

Root the Tree



Initialization



- $|X_i| = 3, S = I \cap X_i \Rightarrow |S| = 1$
- X_i is adjacent to but farther than X_j

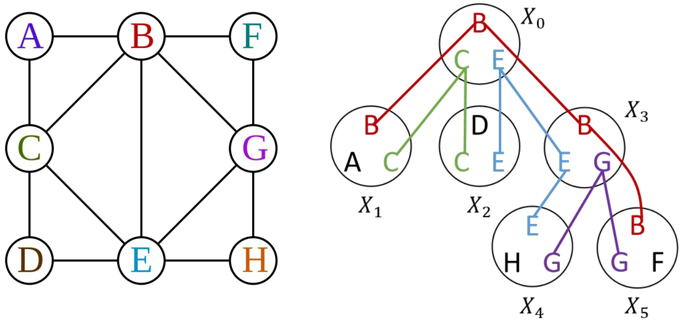
S \ i	5	4	3	2	1	0
A					1	
B	1				1	
C				1	1	
D				1		
E		1		1		
F	1					
G	1	1				
H		1				
Ø						

Table of $A(S, i)$

S \ (i, j)	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B					
C					
E					
G					
Ø					

Table of $B(S, i, j)$

Calculation of Bottom $B(\emptyset, i, j)$



- $B(\emptyset, 5,3) = \max\limits_{\substack{S' \subset \{B,F,G\} \\ \emptyset = S' \cap \{B,E,G\}}} A(S', 5) = A(F, 5)$

S \ i	5	4	3	2	1	0
A					1	
B	1				1	
C				1	1	
D				1		
E		1		1		
F	1					
G	1	1				
H		1				
∅						

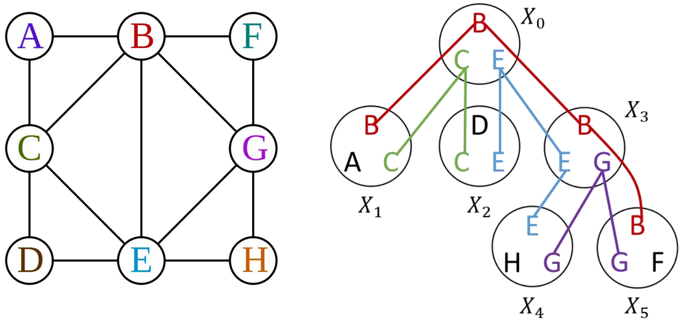
Table of $A(S, i)$

Update

S \ (i, j)	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B					
C					
E					
G					
∅	$\lvert\{F\}\rvert$	$\lvert\{H\}\rvert$	$\lvert\{D\}\rvert$	$\lvert\{A\}\rvert$	

Table of $B(S, i, j)$

Calculation of Bottom $B(S, i, j)$



- $B(\{B\}, 5{,}3) = \max_{\substack{S' \subset \{B, F, G\} \\ \{B\} = S' \cap \{B, E, G\}}} A(S', 5) = A(\{B\}, 5)$

S \ i	5	4	3	2	1	0
A					1	
B	1				1	
C				1	1	
D				1		
E		1		1		
F	1					
G	1	1				
H		1				
Ø						

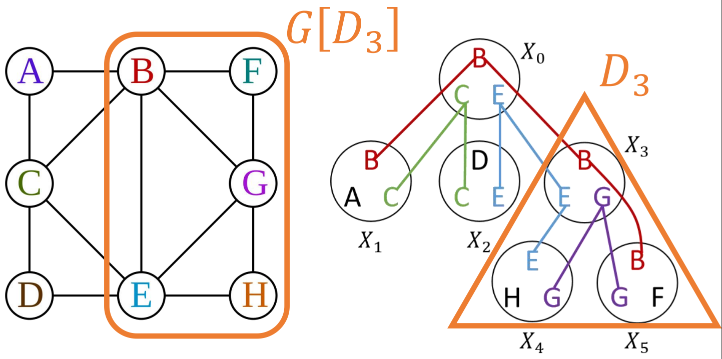
Table of $A(S, i)$

Update

S \ (i, j)	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B	1			1	
C			1	1	
E		1	1		
G	1	1			
Ø	$\lvert\{F\}\rvert$	$\lvert\{H\}\rvert$	$\lvert\{D\}\rvert$	$\lvert\{A\}\rvert$	

Table of $B(S, i, j)$

Calculation of $A(S, 3)$



- $A(\{B\}, 3) = |\{B\}| + \sum_{j \in \{4,5\}} (B(\{B\} \cap X_j, j, 3) - |\{B\} \cap X_j|)$

$$= |\{B\}| + (B(\emptyset, 4{,}3) - |\emptyset|) + (B(\{B\}, 5{,}3) - |\{B\}|)$$

$$= |\{B\}| + (|\{H\}| - 0) + (1 - 1)$$

$= 2$ (MIS of $G[D_3]$ including $\{B\}$ is $\{B, H\}$)

S \ i	5	4	3	2	1	0
A					1	
B	1		$\|\{B, H\}\|$		1	
C				1	1	
D				1		
E		1	$\|\{E, F\}\|$	1		
F	1					
G	1	1	$\|\{G\}\|$			
H		1				
Ø			$\|\{F, H\}\|$			

Table of $A(S, i)$

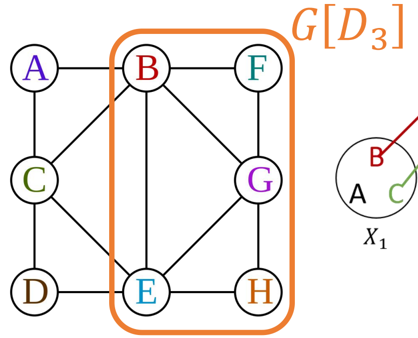
Update

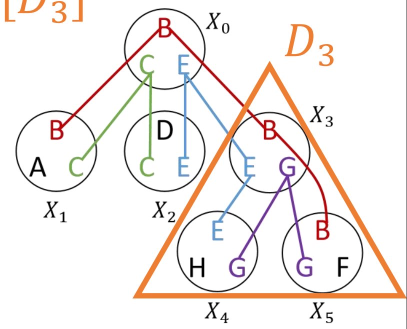
S \ (i, j)	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B	1			1	
C			1	1	
E		1	1		
G	1	1			
Ø	$\|\{F\}\|$	$\|\{H\}\|$	$\|\{D\}\|$	$\|\{A\}\|$	

Table of $B(S, i, j)$

MIS of $G[D_3]$ excluding X_3 is $\{F, H\})$

Calculation of $B(S,3,0)$





- $B(\{B\},3,0) = \max\limits_{\substack{S' \subset \{B,E,G\} \\ \{B\} = S' \cap \{B,C,E\}}} A(S',3) = A(\{B\},3) = |\{B,H\}|$

(MIS of $G[D_3]$ including $X_3 \cap X_0 = \{B\}$ is $\{B,H\}$)

S \ i	5	4	3	2	1	0
A					1	
B	1		$\|\{B,H\}\|$		1	
C				1	1	
D				1		
E		1	$\|\{E,F\}\|$	1		
F	1					
G	1	1	$\|\{G\}\|$			
H		1				
Ø			$\|\{F,H\}\|$			

Table of $A(S,i)$

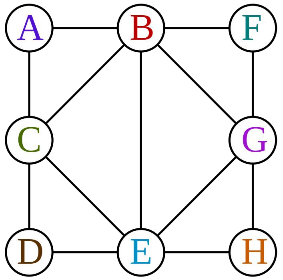
Update

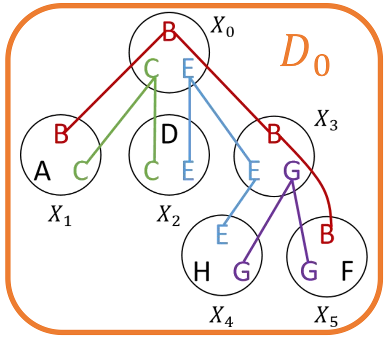
S \ (i, j)	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B	1			1	$\|\{B,H\}\|$
C			1	1	
E		1	1		$\|\{E,F\}\|$
G	1	1			
Ø	$\|\{F\}\|$	$\|\{H\}\|$	$\|\{D\}\|$	$\|\{A\}\|$	$\|\{F,H\}\|$

Table of $B(S,i,j)$

MIS of $G[D_3]$ excluding $X_3 \cap X_0$ is $\{F,H\}$)

Calculation of $A(S, 0)$





- $A(\{B\}, 0) = |\{B\}| + \sum_{j\in\{1,2,3\}} (B(\{B\} \cap X_j, j, 0) - |\{B\} \cap X_j|)$

$= |\{B\}| + (B(\{B\}, 1, 0) - |\{B\}|) + (B(\emptyset, 2, 0) - |\emptyset|) + (B(\{B\}, 3, 0) - |\{B\}|)$

$= \{B\} + (1 - 1) + (\{D\} - 0) + (\{B, H\} - \{B\})$

$= |\{B, D, H\}|$ (MIS of G including $\{B\}$ is $\{B, D, H\}$

S \ i	5	4	3	2	1	0
A					1	
B	1		$\|\{B,H\}\|$		1	$\|\{B,D,H\}\|$
C				1	1	$\|\{C,F,H\}\|$
D				1		
E		1	$\|\{E,F\}\|$	1		$\|\{A,E,F\}\|$
F	1					
G	1	1	$\|\{G\}\|$			
H		1				
Ø			$\|\{F,H\}\|$			$\|\{A,D,F,H\}\|$

Table of $A(S, i)$

Update

S \ (i, j)	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B	1			1	$\|\{B,H\}\|$
C			1	1	
E		1	1		$\|\{E,F\}\|$
G	1	1			
Ø	$\|\{F\}\|$	$\|\{H\}\|$	$\|\{D\}\|$	$\|\{A\}\|$	$\|\{F,H\}\|$

Table of $B(S, i, j)$

MIS of G excluding X_0 is $\{A, D, F, H\}$)

MIS of G

- $|\text{MIS of } G| = \max_{S} A(S, 0) = 4$
- MIS of G is $\{A, D, F, H\}$

S \ i	5	4	3	2	1	0
A					1	
B	1		$\lvert\{B, H\}\rvert$		1	$\lvert\{B, D, H\}\rvert$
C				1	1	$\lvert\{C, F, H\}\rvert$
D				1		
E		1	$\lvert\{E, F\}\rvert$	1		$\lvert\{A, E, F\}\rvert$
F	1					
G	1	1	$\lvert\{G\}\rvert$			
H		1				
Ø			$\lvert\{F, H\}\rvert$			$\lvert\{A, D, F, H\}\rvert$

Table of $A(S, i)$

S \ (i, j)	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B	1			1	$\lvert\{B, H\}\rvert$
C			1	1	
E		1	1		$\lvert\{E, F\}\rvert$
G	1	1			
Ø	$\lvert\{F\}\rvert$	$\lvert\{H\}\rvert$	$\lvert\{D\}\rvert$	$\lvert\{A\}\rvert$	$\lvert\{F, H\}\rvert$

Table of $B(S, i, j)$