

Find Maximum Independent Set in a Graph of Treewidth k by Dynamic Programming

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Denotations

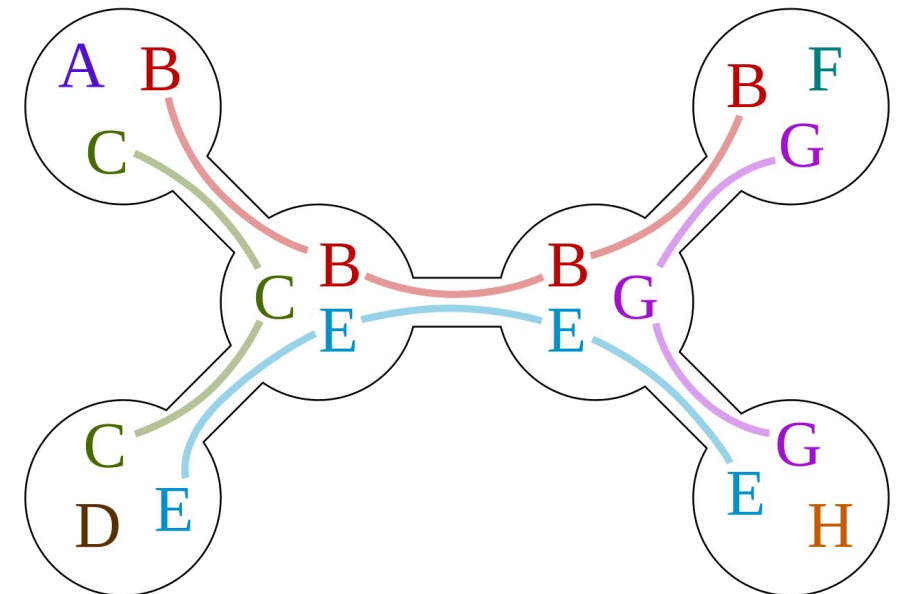
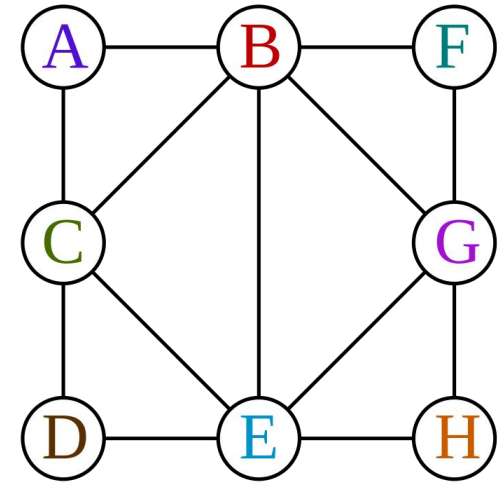
- X_i : a node of the tree decomposition
- D_i : the union of the sets X_j descending from X_i
- $A(S, i)$: the size of the largest independent subset I of D_i s.t.
 - $I \cap X_i = S$
- $B(S, i, j)$: the size of the largest independent subset I of D_i s.t.
 - X_i and X_j are an adjacent pair
 - X_i is farther from the root of the tree than X_j
 - $I \cap X_i \cap X_j = S$

Bottom-up Dynamic Programming

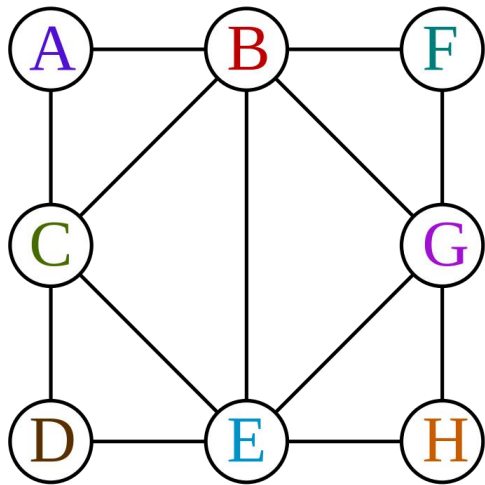
- $A(S, i) = |S| + \sum_j (B(S \cap X_j, j, i) - |S \cap X_j|)$
- $B(S, i, j) = \max_{\substack{S' \subset X_i \\ S = S' \cap X_j}} A(S', i)$

Example Graph

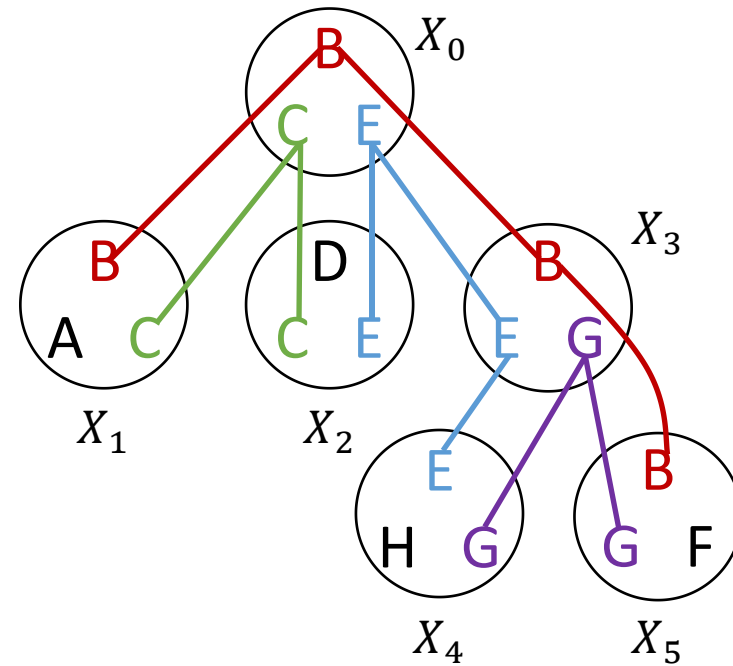
- A graph with eight vertices, and a tree decomposition of it onto a tree with six nodes. Each graph edge connects two vertices that are listed together at some tree node, and each graph vertex is listed at the nodes of a contiguous subtree of the tree. Each tree node lists at most three vertices, so the width of this decomposition is two.



Root the Tree

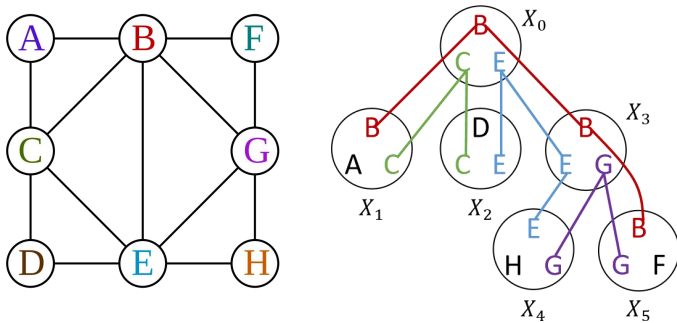


Tree decomposition
Root the tree



Initialization

- $|X_i| = 3, S = I \cap X_i \Rightarrow |S| = 1$
- X_i is adjacent to but farther than X_j



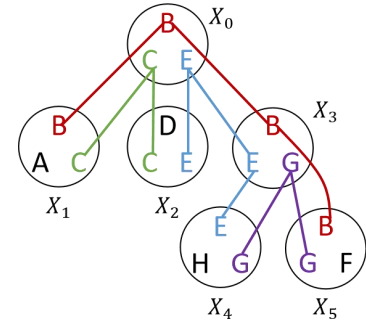
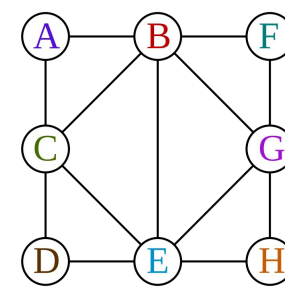
$S \setminus i$	5	4	3	2	1	0
A					1	
B	1				1	
C				1	1	
D				1		
E		1		1		
F	1					
G	1	1				
H		1				
\emptyset						

Table of $A(S, i)$

$S \setminus (i, j)$	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B					
C					
E					
G					
\emptyset					

Table of $B(S, i, j)$

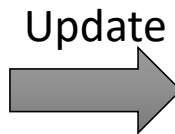
Calculation of Bottom $B(\emptyset, i, j)$



$$\bullet B(\emptyset, 5, 3) = \max_{\substack{S' \subset \{B, F, G\} \\ \emptyset = S' \cap \{B, E, G\}}} A(S', 5) = A(F, 5)$$

$S \setminus i$	5	4	3	2	1	0
A					1	
B	1				1	
C				1	1	
D				1		
E		1		1		
F	1					
G	1	1				
H		1				
\emptyset						

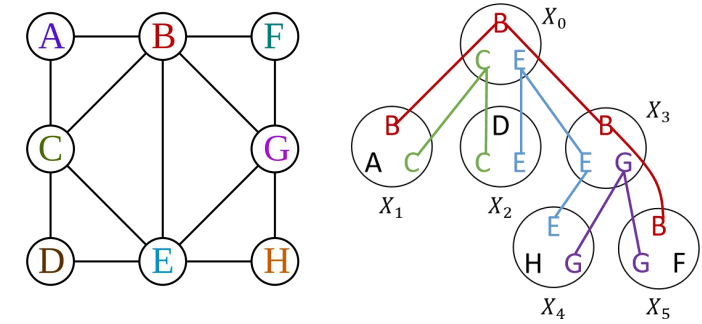
Table of $A(S, i)$



$S \setminus (i, j)$	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B					
C					
E					
G					
\emptyset	$ \{F\} $	$ \{H\} $	$ \{D\} $	$ \{A\} $	

Table of $B(S, i, j)$

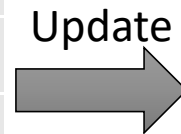
Calculation of Bottom $B(S, i, j)$



$$\bullet B(\{B\}, 5, 3) = \max_{\substack{S' \subset \{B, F, G\} \\ \{B\} = S' \cap \{B, E, G\}}} A(S', 5) = A(\{B\}, 5)$$

$S \setminus i$	5	4	3	2	1	0
A					1	
B	1				1	
C				1	1	
D				1		
E		1		1		
F	1					
G	1	1				
H		1				
\emptyset						

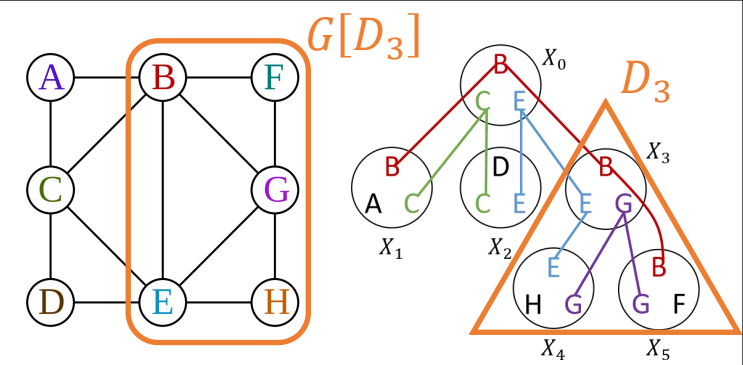
Table of $A(S, i)$



$S \setminus (i, j)$	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B	1			1	
C			1	1	
E		1	1		
G	1	1			
\emptyset	$ \{F\} $	$ \{H\} $	$ \{D\} $	$ \{A\} $	

Table of $B(S, i, j)$

Calculation of $A(S, 3)$



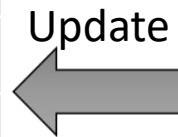
$$\begin{aligned}
 A(\{B\}, 3) &= |\{B\}| + \sum_{j \in \{4,5\}} (B(\{B\} \cap X_j, j, 3) - |\{B\} \cap X_j|) \\
 &= |\{B\}| + (B(\emptyset, 4, 3) - |\emptyset|) + (B(\{B\}, 5, 3) - |\{B\}|)
 \end{aligned}$$

$$= |\{B\}| + (|\{H\}| - 0) + (1 - 1)$$

$$= 2 \text{ (MIS of } G[D_3] \text{ including } \{B\} \text{ is } \{B, H\})}$$

$S \setminus i$	5	4	3	2	1	0
A					1	
B	1		$ \{B, H\} $		1	
C				1	1	
D				1		
E		1	$ \{E, F\} $	1		
F	1					
G	1	1	$ \{G\} $			
H		1				
\emptyset			$ \{F, H\} $			

Table of $A(S, i)$

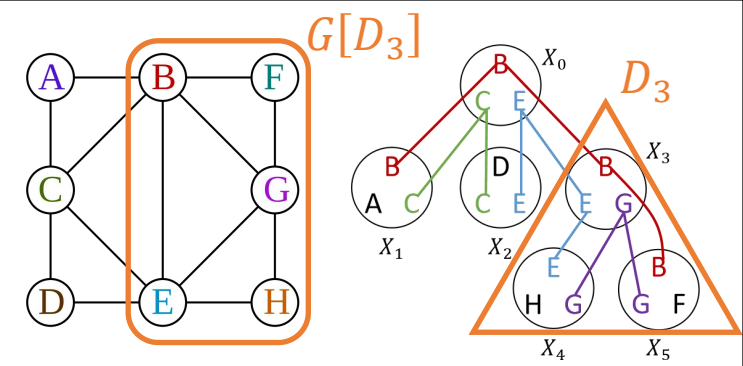


$S \setminus (i, j)$	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B	1			1	
C			1	1	
E		1	1		
G	1	1			
\emptyset	$ \{F\} $	$ \{H\} $	$ \{D\} $	$ \{A\} $	

Table of $B(S, i, j)$

MIS of $G[D_3]$ excluding X_3 is $\{F, H\}$

Calculation of $B(S, 3, 0)$



$$\bullet B(\{B\}, 3, 0) = \max_{\substack{S' \subset \{B, E, G\} \\ \{B\} = S' \cap \{B, C, E\}}} A(S', 3) = A(\{B\}, 3) = |\{B, H\}|$$

$S \setminus i$	5	4	3	2	1	0
A					1	
B	1		$ \{B, H\} $		1	
C				1	1	
D				1		
E		1	$ \{E, F\} $	1		
F	1					
G	1	1	$ \{G\} $			
H		1				
\emptyset			$ \{F, H\} $			

Table of $A(S, i)$

(MIS of $G[D_3]$ including $X_3 \cap X_0 = \{B\}$ is $\{B, H\}$)

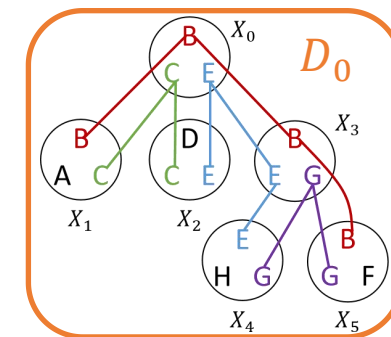
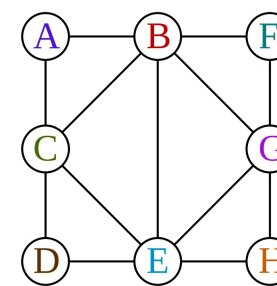
Update
→

$S \setminus (i, j)$	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B	1			1	$ \{B, H\} $
C			1	1	
E		1	1		$ \{E, F\} $
G	1	1			
\emptyset	$ \{F\} $	$ \{H\} $	$ \{D\} $	$ \{A\} $	$ \{F, H\} $

Table of $B(S, i, j)$

MIS of $G[D_3]$ excluding $X_3 \cap X_0$ is $\{F, H\}$

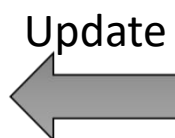
Calculation of $A(S, 0)$



$$\begin{aligned}
 \bullet A(\{B\}, 0) &= |\{B\}| + \sum_{j \in \{1,2,3\}} (B(\{B\} \cap X_j, j, 0) - |\{B\} \cap X_j|) \\
 &= |\{B\}| + (B(\{B\}, 1, 0) - |\{B\}|) + (B(\emptyset, 2, 0) - |\emptyset|) + (B(\{B\}, 3, 0) - |\{B\}|) \\
 &= \{B\} + (1 - 1) + (\{D\} - 0) + (\{B, H\} - \{B\}) \\
 &= |\{B, D, H\}| \text{ (MIS of } G \text{ including } \{B\} \text{ is } \{B, D, H\})
 \end{aligned}$$

$S \setminus i$	5	4	3	2	1	0
A					1	
B	1		$ \{B, H\} $		1	$ \{B, D, H\} $
C				1	1	$ \{C, F, H\} $
D				1		
E		1	$ \{E, F\} $	1		$ \{A, E, F\} $
F	1					
G	1	1	$ \{G\} $			
H		1				
\emptyset			$ \{F, H\} $			$ \{A, D, F, H\} $

Table of $A(S, i)$



$S \setminus (i, j)$	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B	1			1	$ \{B, H\} $
C			1	1	
E		1	1		$ \{E, F\} $
G	1	1			
\emptyset	$ \{F\} $	$ \{H\} $	$ \{D\} $	$ \{A\} $	$ \{F, H\} $

Table of $B(S, i, j)$

MIS of G excluding X_0 is $\{A, D, F, H\}$

MIS of G

- $|\text{MIS of } G| = \max_S A(S, 0) = 4$
- MIS of G is $\{A, D, F, H\}$

$S \setminus i$	5	4	3	2	1	0
A					1	
B	1		$ \{B, H\} $		1	$ \{B, D, H\} $
C				1	1	$ \{C, F, H\} $
D				1		
E		1	$ \{E, F\} $	1		$ \{A, E, F\} $
F	1					
G	1	1	$ \{G\} $			
H		1				
\emptyset			$ \{F, H\} $			$ \{A, D, F, H\} $

Table of $A(S, i)$

$S \setminus (i, j)$	(5, 3)	(4, 3)	(2, 0)	(1, 0)	(3, 0)
B	1			1	$ \{B, H\} $
C			1	1	
E		1	1		$ \{E, F\} $
G	1	1			
\emptyset	$ \{F\} $	$ \{H\} $	$ \{D\} $	$ \{A\} $	$ \{F, H\} $

Table of $B(S, i, j)$