AMCA - Aplicación 6

1/ a) $f(t) = t^2$ en $[-\pi, \pi]$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos(nt)\,dt = \begin{bmatrix} u = t^2 & dv = \cos(nt)dt \\ du = 2t\,dt & v = \frac{\operatorname{sen}(nt)}{n} \end{bmatrix} =$$

$$= \frac{1}{\pi}\left(\cancelto{0}{\left[\frac{t^2 \operatorname{sen}(nt)}{n}\right]_{-\pi}^{\pi}} - \frac{2}{n}\int_{-\pi}^{\pi} t\operatorname{sen}(nt)\,dt \right) = -\frac{2}{n\pi}\int_{-\pi}^{\pi} t\operatorname{sen}(nt)\,dt =$$

$$= \begin{bmatrix} u = t & dv = \operatorname{sen}(nt)dt \\ du = dt & v = \frac{-\cos(nt)}{n} \end{bmatrix} = -\frac{2}{n\pi}\left(\left[\frac{-t\cos(nt)}{n}\right]_{-\pi}^{\pi} + \cancelto{0}{\frac{1}{n}\int_{-\pi}^{\pi} \cos(nt)\,dt} \right) =$$

cos es par

$$= -\frac{2}{n\pi}\left(\frac{-\pi\cos(n\pi)}{n} - \frac{\pi\cos(-n\pi)}{n} \right) = \frac{4\pi}{n\pi}\ \frac{\cos(n\pi)}{n} =$$

$$= \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \operatorname{sen}(nt)\,dt = \begin{bmatrix} u = t^2 & dv = \operatorname{sen}(nt)dt \\ du = 2t\,dt & v = \frac{-\cos(nt)}{n} \end{bmatrix} =$$

cos es par

$$= \frac{1}{\pi}\left[-\frac{t^2\cos(nt)}{n}\right]_{-\pi}^{\pi} + \frac{2}{\pi}\int_{-\pi}^{\pi} \frac{t\cos(nt)}{n}\,dt = \frac{1}{\pi}\left(\cancel{-\frac{\pi^2\cos(n\pi)}{n}} + \cancel{\frac{\pi^2\cos(-n\pi)}{n}} \right.$$

$$\left. + 2\int_{-\pi}^{\pi} \frac{t\cos(nt)}{n}\,dt \right) = \begin{bmatrix} u = t & dv = \cos(nt)dt \\ du = dt & v = \frac{\operatorname{sen}(nt)}{n} \end{bmatrix} =$$

$$= \frac{2}{\pi}\left(\cancelto{0}{\left[\frac{t\operatorname{sen}(nt)}{n}\right]_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} \frac{\operatorname{sen}(nt)}{n}\,dt \right) = \frac{-2}{\pi n}\cancelto{0}{\left[\frac{\cos(nt)}{n}\right]_{-\pi}^{\pi}} = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

Serie de Fourier de f

$$\frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left(a_n \cos(nt) + b_n \operatorname{sen}(nt)\right)$$

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nt)$$

b) $f(t) = \begin{cases} \dfrac{t}{\pi} & 0 \le t < 1/2 \\ -\dfrac{t}{\pi} & 1/2 \le t \le 1 \end{cases}$

$$a_n = \frac{1}{\pi}\left[\int_0^{1/2} \frac{t}{\pi}\cos(nt)\,dt - \int_{1/2}^{1} \frac{t}{\pi}\cos(nt)\,dt\right] = \frac{1}{\pi^2}\left[\int_0^{1/2} t\cos(nt)\,dt - \int_{1/2}^{1} t\cos(nt)\,dt\right] =$$

$$= \left[\begin{matrix} u = t & dv = \cos(nt)dt \\ du = dt & v = \dfrac{\operatorname{sen}(nt)}{n}\end{matrix}\right] = \frac{1}{\pi^2}\left[\left[\frac{t\operatorname{sen}(nt)}{n}\right]_0^{1/2} - \frac{1}{n}\int_0^{1/2}\operatorname{sen}(nt)\,dt - \left[\frac{t\operatorname{sen}(nt)}{n}\right]_{1/2}^{1}\right.$$

$$\left. + \frac{1}{n}\int_{1/2}^{1}\operatorname{sen}(nt)\,dt\right] = \frac{1}{\pi^2}\left[\frac{\operatorname{sen}(n/2)}{2n} + \frac{1}{n}\left[\frac{\cos(nt)}{n}\right]_0^{1/2} - \frac{\operatorname{sen}(n) - 1/2\operatorname{sen}(n/2)}{n}\right.$$

$$\left. - \frac{1}{n}\left[\frac{\cos(nt)}{n}\right]_{1/2}^{1}\right] = \frac{1}{\pi^2}\left[\frac{\operatorname{sen}(n/2)}{2n} + \frac{\cos(n/2) - 1}{n^2} - \frac{2\operatorname{sen}(n) - \operatorname{sen}(n/2)}{2n}\right.$$

$$\left. - \frac{\cos(n) - 1/2\cos(n/2)}{n^2}\right] = \frac{\operatorname{sen}(n/2) - \operatorname{sen}(n)}{\pi^2 n} + \frac{\cos(n/2) - 1 - \cos(n) + \frac{1}{2}\cos(n/2)}{\pi^2 n^2}$$

$$a_0 = \int_0^{1/2} \frac{t}{\pi}\,dt - \int_{1/2}^{1} \frac{t}{\pi}\,dt = \left[\frac{t^2}{2\pi}\right]_0^{1/2} - \left(\frac{t^2}{2\pi}\right)_{1/2}^{1} = \frac{1/4}{2\pi} - \frac{1}{2\pi} + \frac{1/4}{2\pi} =$$

$$= \frac{1}{4\pi} - \frac{1}{2\pi} = -\frac{1}{4\pi}$$

$$b_n = \frac{1}{\pi^2}\left[\int_0^{1/2} t\,sen(nt)\,dt - \int_{1/2}^{1} t\,sen(nt)\,dt\right] = \begin{bmatrix} u=t & dv = sen(nt)dt \\ du = dt & v = \frac{-cos(nt)}{n} \end{bmatrix} =$$

$$= \frac{1}{\pi^2}\left[\left[\frac{-t\,cos(nt)}{n}\right]_0^{1/2} + \int_0^{1/2} \frac{cos(nt)}{n}\,dt + \left[\frac{t\,cos(nt)}{n}\right]_{1/2}^{1} - \int_{1/2}^{1}\frac{cos(nt)}{n}\,dt\right] =$$

$$= \frac{1}{\pi^2}\left[\frac{-cos(n/2)}{2n} + \left[\frac{sen(nt)}{n^2}\right]_0^{1/2} + \frac{cos(n) - 1/2\,cos(n/2)}{n} - \left[\frac{sen(nt)}{n^2}\right]_{1/2}^{1}\right] =$$

$$= \frac{1}{\pi^2}\left[\frac{-cos(n/2)}{2n} + \frac{sen(n/2)}{n^2} + \frac{2cos(n) - cos(n/2)}{n} - \frac{sen(n) - sen(n/2)}{n^2}\right] =$$

$$= \frac{1}{\pi^2}\left[\frac{cos(n) - cos(n/2)}{n} - \frac{sen(n)}{n^2}\right]$$

Serie de Fourier de f

$$\frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left(a_n cos(nt) + b_n sen(nt)\right)$$

$$-\frac{1}{8\pi} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\left(\frac{sen(n/2) - sen(n)}{n} + \frac{3/2\,cos(n/2) - 1 - cos(n)}{n^2}\right)cos(nt)\right.$$

$$\left. + \left(\frac{cos(n) - cos(n/2)}{n} - \frac{sen(n)}{n^2}\right)\right]$$

2/ $f(t) = \begin{cases} 2t-1 & 0 \leq t < \frac{1}{2} \\ -2t+1 & \frac{1}{2} \leq t < 1 \end{cases}$

$$a = \int_0^1 \phi(t) f(t)\,dt = \int_0^{1/2} (2t-1)\,dt + \int_{1/2}^1 (-2t+1)\,dt = \left[t^2-t\right]_0^{1/2} + \left[-t^2+t\right]_{1/2}^1 =$$

$$= \left(\frac{1}{4} - \frac{1}{2}\right) - \left(\cancel{-1}+\cancel{1}+\frac{1}{4} - \frac{1}{2}\right) = 0$$

$$a_{0,0} = \int_0^1 \psi_{0,0}(t) f(t)\,dt = \int_0^{1/2} (2t-1)\,dt + \int_{1/4}^{1/2} (2t-1)\,dt = \left[t^2-t\right]_0^1 = 0$$

$$a_{1,0} = \int_0^1 \psi_{1,0}(t) f(t)\,dt = \int_0^{1/4} (2t-1)\sqrt{2}\,dt - \int_{1/4}^{1/2} (2t-1)\sqrt{2}\,dt = \sqrt{2}\left(\left[t^2-t\right]_0^{1/4} - \left[t^2-t\right]_{1/4}^{1/2}\right) =$$

$$= \sqrt{2}\left(\left(\frac{1}{16} - \frac{1}{4}\right) - \left(\frac{1}{4} - \frac{1}{2} - \frac{1}{16} + \frac{1}{4}\right)\right) = \sqrt{2}\left(\frac{1}{8} - \frac{1}{4}\right) = \frac{-\sqrt{2}}{8}$$

$$a_{1,1} = \int_0^1 \psi_{1,1}(t) f(t)\,dt = \sqrt{2}\left(\int_{1/2}^{3/4} (-2t+1)\,dt - \int_{3/4}^1 (-2t+1)\,dt\right) = \sqrt{2}\left(\left[-t^2+t\right]_{1/2}^{3/4} + \left[t^2-t\right]_{3/4}^1\right) =$$

$$= \sqrt{2}\left(\left(-\frac{9}{16} + \frac{3}{4} + \frac{1}{4} - \frac{1}{2}\right) + \left(\cancel{1} - \cancel{1} - \frac{9}{16} + \frac{3}{4}\right)\right) = \sqrt{2}\left(-\frac{9}{8} + \frac{3}{2} - \frac{1}{4}\right) =$$

$$= \sqrt{2}\left(\frac{-9+12-2}{8}\right) = \frac{\sqrt{2}}{8}$$

Nuestra función varía de $(2t-1)$ a $(-2t+1)$ en $1/2$. Para calcular $a_{j,k}$ hay que distinguir cuando $k < 2^{j-1} - \frac{1}{2}$ y $k \geq 2^{j-1} - \frac{1}{2}$.

$$\underset{(k<2^{j-1}-\frac{1}{2})}{a_{j,k}} = \int_{\frac{k}{2^j}}^{\frac{k+\frac{1}{2}}{2^j}} 2^{j/2}(2t-1)\,dt - \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} 2^{j/2}(2t-1)\,dt =$$

$$= 2^{j/2}\left(\left[t^2-t\right]_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} - \left[t^2-t\right]_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}}\right) =$$

$$= 2^{j/2}\left(\frac{k^2+k+1/4}{2^{2j}} - \frac{k+1}{2^j} - \frac{k^2}{2^{2j}} + \frac{k}{2^{2j}}\right)$$

$$- 2^{j/2}\left(\frac{k^2+2k+1}{2^{2j}} - \frac{k+1}{2^j} - \frac{k^2+k+1/4}{2^{2j}} + \frac{k+1/2}{2^j}\right) =$$

$$= 2^{j/2}\left(\frac{k^2+k+1/4}{2^{2j-1}} - \frac{k+1/2}{2^{j-1}} - \frac{2k^2+2k+1}{2^{2j}} + \frac{2k+1}{2^j}\right)$$

$$\underset{(k\geq 2^{j-1}-\frac{1}{2})}{a_{j,k}} = \int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} 2^{j/2}(-2t+1)\,dt - \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} 2^{j/2}(-2t+1)\,dt =$$

$$= -\underset{(k<2^{j-1}-1/2)}{a_{j,k}}$$

$$S_n = \cancel{a\,\phi(t)}^{\,0} + \sum_{j=0}^{n}\sum_{k=0}^{2^j-1} a_{j,k}\,\psi_{j,k}(t)$$

b) $f(t) = t^2$ en $[0,1]$

$$a = \int_0^1 t^2 dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$$

$$a_{0,0} = \int_0^{1/2} t^2 dt - \int_{1/2}^1 t^2 dt = \left[\frac{t^3}{3}\right]_0^{1/2} - \left[\frac{t^3}{3}\right]_{1/2}^1 = \frac{1}{24} - \frac{1}{3} + \frac{1}{24} = \frac{-3}{12}$$

$$a_{j,k} = \int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} 2^{j/2} t^2 dt - \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} 2^{j/2} t^2 dt = 2^{j/2}\left(\left[\frac{t^3}{3}\right]_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} - \left[\frac{t^3}{3}\right]_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}}\right) =$$

$$= 2^{j/2}\left(\frac{(k+1/2)^3}{3\cdot 2^{3j}} - \frac{k^3}{3\cdot 2^{3j}} - \frac{(k+1)^3}{3\cdot 2^{3j}} + \frac{(k+1/2)^3}{3\cdot 2^{3j}}\right)$$

$$S_n = a\,\phi(t) + \sum_{j=0}^{n}\sum_{k=0}^{2^j-1} a_{j,k}\,\psi_{j,k}(t)$$

4/ Wavelets Daubechies

Introducidas en 1990 por Ingrid Daubechies. Son una familia de wavelets ortogonales que definen una transformación discreta de los wavelets.

La aproximación de f (una señal) es:

$$f = A^m + D^m + \dots + D^1$$

La m-ésima señal del promedio A^m es también la proyección ortogonal de f sobre $V^m = \text{lin}\{v_1^m, \dots, v_{N/2^m}^m\}$ y D^m es la proyección sobre de f sobre $W^m = \text{lin}\{w_1^m, \dots, w_{N/2^m}^m\}$.

Es decir $\mathbb{R}^N = V^m \oplus^{\perp} W^m \oplus^{\perp} \dots \oplus^{\perp} W^1$ con $m \leq K-1$ $(N=2^K)$.

Aplicación:

- Análisis de multirresolución. La posibilidad de descomponer una señal en subseñales (de promedio y detalle) por ser las Daubechies ortogonales.

Bibliografía: Wavelets de Haar y Daubechies y sus aplicaciones, Nahuel Oliveira Rodríguez (2018).