Advanced Derivative Models (MSCF 46915). Exotic options

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Lecture outline

1. Barrier options
   - Basic facts about barrier options
   - Reflection-based pricing and hedging
   - Pricing and hedging via PDE
   - Pricing and hedging via Monte Carlo
2. American options
   - Basic facts about American options
   - Pricing and hedging via PDE
   - Pricing and hedging via DPP (and Monte Carlo)
3. Spread options
   - Basic facts about spread options
   - Margrabe's formula

Introduction

- There is a wide variety of traded option types.
- European-exercise calls and puts on a single underlying are referred to as vanilla options. Anything else is an exotic option.
- We have already encountered some exotic options, including the log contract, variance swaps, volatility swaps, VIX futures and VIX options.
- Here we will explore three additional types of exotic options.
    - Barrier options with knock-in/knock-out provisions. These are path-dependent where the payoff amount depends on not only the underlying value at expiration but also on its max. or min. value.
    - American-exercise options with the exercise time not predetermined but chosen by the holder. The payoff depends on the holder's exercise strategy, and the price is determined assuming that the holder acts optimally.
    - Spread options, whose payoff is a function of the difference between the prices of two underlying assets at the terminal time.
- These features are often used as ingredients in even more exotic options/structured products.

Barrier options

- Barrier options are path-dependent and similar to ordinary options except become activated or extinguished when the underlying breaches a predetermined barrier level.
- Knock-in options can be exercised only if barrier is breached before expiration.
    - Up-and-in underlying price starts below the barrier and must move higher to activate option.
    - Down-and-in underlying starts above the barrier and must move lower to activate option.
- Knock-out options can be exercised only if barrier is not breached before expiration.
    - Up-and-out underlying starts below the barrier and must move higher to extinguish option.
    - Down-and-out underlying starts above the barrier and must move lower to extinguish option.
- Barrier options are particularly popular in FX markets.

Example: up-and-out call

- Suppose the underlying price is $S_0 = 100$ and an up-and-out European call is written on it with strike price $K = 110$ and barrier $B = 120$.
- This option becomes worthless should the underlying reach 120 on or before the expiration time. It does not reactivate if the underlying falls below 120 again.
- On the other hand, if the underlying never reaches 120 then the option behaves like a vanilla call.
- The payoff os this option is

$$X = (S_T - K)^+ \mathbf{1}_{\{\max_{t\in[0,T]} S_t < B\}}$$

In-and-out parity

- The combination of one knock-in and one knock-out with the same underlying, expiration time T, strike K, and barrier B equals the price of the corresponding vanilla option:

$$(S_T - K)^+ \mathbf{1}_{\{\max_{t\in[0,T]} S_t < B\}} + (S_T - K)^+ \mathbf{1}_{\{\max_{t\in[0,T]} S_t \geq B\}} = (S_T - K)^+,$$
$$C(T,K) = C^{\mathrm{in}}(T,K,B) + C^{\mathrm{out}}(T,K,B).$$

- Before any knock-in/out event occurs, both barrier options have strictly positive prices and are both strictly less than the vanilla price.
- Once the barrier is reached, one of the barrier options becomes worthless and the other's price then coincides with the vanilla.
- If the vanilla does not expire worthless, then one barrier option's payoff will exactly match the vanilla, and the other will be zero (depending on whether the barrier was reached or not before the expiration).

Barrier events and variations

- Barrier events can be difficult to determine.
    - What if the underlying never trades beyond the barrier and trades at the barrier exactly once?
    - How big would this single trade need to be?
    - Must it have occurred on an exchange or can it be between private parties?
- In practice, it is important that the option contract clearly defines what does and does not constitute a barrier event.
- Barriers can be defined with several variations. Examples include:
    - a discrete barrier allows the barrier event to be triggered at only discrete times;
    - a Parisian option is a barrier option which is triggered in/out only when the underlying has spent a minimum amount of time beyond the barrier;
    - a double-no-touch option knocks out when either an upper or lower barrier is breached.

Pricing and hedging barrier options via reflection

- In the Black-Scholes model,

$$dS_t/S_t = r\,dt + \sigma\,dW_t^{\mathbb{Q}},$$
$$S_t = S_0\,\exp((r - \sigma^2/2)\,t + \sigma\,W_t^{\mathbb{Q}}),$$

- we can find the price of a knock-out (upper) barrier option,

$$X = H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t < B\}},$$

explicitly in terms of the prices of European options.

- This pricing method is due to *Carr-Bowie 1994* and is based on the reflection principle for Brownian motion.

Reflection principle for Brownian motion

- Let W be a 1-dim. Brownian motion and F be a given function, vanishing in $[B,\infty)$.
- Consider

$$\mathbb{E}\left(F(W_T)\mathbf{1}_{\left\{\max_{t\in[0,T]} W_t<B\right\}}\right)=\mathbb{E}\left(F(W_T)\mathbf{1}_{\{\tau>T\}}\right)$$
$$=\mathbb{E}\,F(W_T)-\mathbb{E}\left(F(B+\tilde{W}_{T-\tau})\mathbf{1}_{\{\tau\le T\}}\right),$$

  where

$$\tau:=\inf\{t\ge 0:\ W_t\ge B\},\quad \tilde{W}_t:=W_{t+\tau}-B.$$

- Strong Markov property of Brownian motion implies that $\tilde{W}$ is a Brownian motion independent of τ.
- On the other hand $-\tilde{W}$ is also a Brownian motion independent of τ.
- Thus, we can replace $(\tilde{W},\tau)$ by $(-\tilde{W},\tau)$ in the above expectation...

Reflection principle for Brownian motion

$$\tau := \inf\{t \geq 0 : \ W_t \geq B\}, \quad \tilde{W}_t := W_{t+\tau} - B.$$

- ...obtaining

$$\mathbb{E}\left(F(W_T)\mathbf{1}_{\{\max_{t\in[0,T]} W_t < B\}}\right) = \mathbb{E}\,F(W_T) - \mathbb{E}\left(F(B + \tilde{W}_{T-\tau})\mathbf{1}_{\{\tau \leq T\}}\right)$$
$$= \mathbb{E}\,F(W_T) - \mathbb{E}\left(F(B - \tilde{W}_{T-\tau})\mathbf{1}_{\{\tau \leq T\}}\right)$$
$$= \mathbb{E}\,F(W_T) - \mathbb{E}\left(F(2B - W_T)\mathbf{1}_{\{\max_{t\in[0,T]} W_t \geq B\}}\right),$$

  where, to obtain the last equality, we used the definition of $\tilde{W}$.

- Notice that

$$\mathbb{E}\left(F(2B - W_T)\mathbf{1}_{\{\max_{t\in[0,T]} W_t \geq B\}}\right) = \mathbb{E}\,F(2B - W_T),$$

  since $F(2B - W_T)$ is non-zero only when $W_T \geq B$.

- Thus, $\mathbb{E}\left(F(W_T)\mathbf{1}_{\{\max_{t\in[0,T]} W_t < B\}}\right) = \mathbb{E}\,F(W_T) - \mathbb{E}\,F(2B - W_T)$.

Back to barrier options

$$dS_t/S_t = r\,dt + \sigma\,dW_t^{\mathbb{Q}},$$
$$S_t = S_0\,\exp((r-\sigma^2/2)\,t + \sigma\,W_t^{\mathbb{Q}}).$$

- Let $C := \frac{r}{\sigma} - \frac{\sigma}{2}$ and define the measure $\tilde{\mathbb{Q}}$ by

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \exp\left(-\int_0^T C\,dW_t^{\tilde{\mathbb{Q}}} - \frac{T\,C^2}{2}\right) = \exp\left(-C\,W_T^{\tilde{\mathbb{Q}}} - \frac{1}{2}T\,C^2\right),$$

- so that $S_t = S_0 \exp\left(\sigma W_t^{\tilde{\mathbb{Q}}}\right)$.
- Then, the price of a knock-out (upper) barrier option is

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t < B\}}\right]$$
$$= e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\exp\left(C\,W_T^{\tilde{\mathbb{Q}}} - \frac{T\,C^2}{2}\right) H\left(S_0 \exp\left(\sigma W_T^{\tilde{\mathbb{Q}}}\right)\right)\mathbf{1}_{\{\max_{t\in[0,T]} W_t^{\tilde{\mathbb{Q}}} < \frac{1}{\sigma}\log\frac{B}{S_0}\}}\right]$$

Price of a barrier option in BS model

- Thus, we have

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t<B\}}\right]$$
$$=e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\exp\left(C\,W^{\tilde{\mathbb{Q}}}_T-\frac{T\,C^2}{2}\right)H\left(S_0\exp\left(\sigma W^{\tilde{\mathbb{Q}}}_T\right)\right)\mathbf{1}_{\{\max_{t\in[0,T]} W^{\tilde{\mathbb{Q}}}_t<\frac{1}{\sigma}\log\frac{B}{S_0}\}}\right]$$
$$=: e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[F(W^{\tilde{\mathbb{Q}}}_T)\mathbf{1}_{\{\max_{t\in[0,T]} W^{\tilde{\mathbb{Q}}}_t<\tilde{B}\}}\right].$$

- Setting $H(x)=0$ for $x>B$, we have $F(x)=0$ for $x>B$ and use the reflection principle:

$$\mathbb{E}^{\tilde{\mathbb{Q}}}\left[F(W^{\tilde{\mathbb{Q}}}_T)\mathbf{1}_{\{\sup_{t\in[0,T]} W^{\tilde{\mathbb{Q}}}_t<\tilde{B}\}}\right]=\mathbb{E}^{\tilde{\mathbb{Q}}}F(W^{\tilde{\mathbb{Q}}}_T)-\mathbb{E}^{\tilde{\mathbb{Q}}}F(2\tilde{B}-W^{\tilde{\mathbb{Q}}}_T).$$

Price of a barrier option in BS model

$$C = \frac{r}{\sigma} - \frac{\sigma}{2}, \quad \tilde{B} = \frac{1}{\sigma}\log\frac{B}{S_0}, \quad F(x) = e^{C\,x - \frac{T\,C^2}{2}}\,H\left(S_0\,e^{\sigma x}\right), \quad S_t = S_0 \exp\left(\sigma W_t^{\tilde{\mathbb{Q}}}\right)$$

- Thus, we have

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t < B\}}\right] = \mathbb{E}^{\tilde{\mathbb{Q}}}F(W_T^{\tilde{\mathbb{Q}}}) - e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}F(2\tilde{B} - W_T^{\tilde{\mathbb{Q}}})$$

$$= e^{-rT}\mathbb{E}^{\mathbb{Q}}H(S_T) - e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[e^{2C\tilde{B} - C\,W_T^{\tilde{\mathbb{Q}}} - \frac{T\,C^2}{2}}\,H\left(S_0 e^{2\tilde{B}\sigma - \sigma W_T^{\tilde{\mathbb{Q}}}}\right)\right]$$

$$= e^{-rT}\mathbb{E}^{\mathbb{Q}}H(S_T) - e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[e^{C\,W_T^{\tilde{\mathbb{Q}}} - \frac{T\,C^2}{2}}\,e^{2C\tilde{B} - 2C\,W_T^{\tilde{\mathbb{Q}}}}\,H\left(S_0 e^{2\tilde{B}\sigma - \sigma W_T^{\tilde{\mathbb{Q}}}}\right)\right]$$

$$= e^{-rT}\mathbb{E}^{\mathbb{Q}}H(S_T) - e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[B^{\frac{2r}{\sigma^2} - 1}\,S_T^{1 - \frac{2r}{\sigma^2}}\,H\left(S_T^{-1}\,B^2\right)\right]$$

$$=: e^{-rT}\mathbb{E}^{\mathbb{Q}}H(S_T) - e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\tilde{H}(S_T),$$

- and the price of a knock-out (upper) barrier option, which has a terminal payoff function H, in the Black-Scholes model, coincides with the price of the European option that has payoff $H - \tilde{H}$, prior to the knock-out event.

Semi-static hedging of barrier options in BS model

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t<B\}}\right] = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[H(S_T)-\tilde{H}(S_T)],$$
$$\tilde{H}(x) = (x/B)^{1-\frac{2r}{\sigma^2}}\,H\left(B^2/x\right).$$

- The same conclusion applies to lower knock-out options, and to knock-in options (with different formulas for $\tilde{H}$).
- In fact, the derivation we have done to obtain the above can be repeated with conditional expectations, to conclude that, prior to knock-in or knock-out event, the price of a barrier option in the BS model coincides with the price of an associated European option.
- Thus, we obtain a semi-static hedging strategy: to hedge a short position in a knock-out barrier option,
    1. buy a European option with payoff $H-\tilde{H}$,
    2. when/if the barrier event occurs, sell it (at zero price).

Semi-static hedging with $r = 0$

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t<B\}}\right] = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[H(S_T) - \tilde{H}(S_T)],$$
$$\tilde{H}(x) = (x/B)^{1-\frac{2r}{\sigma^2}}\, H\left(B^2/x\right).$$

- If $r = 0$ (recall that we assumed $q = 0$ throughout) and $H(x) = (K - x)^+$, with $K < B$ (so that $H(x) = 0$ for $x > B$), then

$$\tilde{H}(x) = (x/B)\,\left(K - B^2/x\right)^+ = (Kx/B - B)^+ = (K/B)\,\left(x - B^2/K\right)^+.$$

- Thus, in Black's model (i.e., the BS model with $r = q = 0$), the price of up-and-out put with strike K coincides with

$$P(T, K) - (K/B)\,C(B^2/K).$$

- The semi-static hedge of a short position in the up-and-out put is given by a long position in a vanilla put with strike K and a short position in K/B shares of vanilla calls with strike B^2/K.

Semi-static hedging of barrier options in BS model

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t<B\}}\right] = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[H(S_T)-\tilde{H}(S_T)].$$

- Beyond the case $r=0$, the above formula is not very useful for pricing: since $\tilde{H}$ is not a vanilla payoff, we still need to solve a PDE to find its price (and the price of a barrier option can be found via the PDE directly).
- If we replicate $\tilde{H}$ by the vanilla payoffs, the Carr-Bowie formula becomes more attractive (especially for small maturities).
- The semi-static hedge is a more important contribution of the analysis based on reflection principle, though, in practice, it also requires replication of $\tilde{H}$ via calls and puts. The latter can be done efficiently for short expiration times.

Semi-static hedging of barrier options beyond BS model

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t<B\}}\right] = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[H(S_T)-\tilde{H}(S_T)].$$

- Semi-static hedge is a bet on the shape of the implied vol surface at the time when the barrier is hit: the hedge succeeds if and only if the price of $H(S_T)$ equals the price of $\tilde{H}(S_T)$ at that (random) time.
- The presence (and persistency) of implied smile means that the BS semi-static hedge has no reason to succeed in practice.
    - In particular, the presence of implied skew means that the BS semi-static hedge will typically super-replicate the up-and-out put, assuming $r=0$.
- Semi-static hedge succeeds (via a very different argument)
    - in any time-homogeneous LV model (*Carr-Nadtochiy 2011*),
    - or simply under the assumption that the market implied vol surface is given by the one produced by a chosen LV model, at the time when the barrier is hit (*Nadtochiy-Obloj 2017*).

BS PDE for barrier options

- For the sake of illustration, we consider the Black-Scholes model,

$$dS_t/S_t = r\,dt + \sigma\,dW_t^{\mathbb{Q}},$$

  and the up-and-out put with payoff

$$X = (K - S_T)^+\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t < B\}}.$$

- The arbitrage-free price of an up-and-out put is

$$\pi_t(X) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(K - S_T)^+\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t < B\}}\,|\,\mathcal{F}_t].$$

- The payoff is a function of S_T and of $I_T := \mathbf{1}_{\{S_T^* < B\}}$, where we introduced the running maximum process $S_t^* := \max_{u\in[0,t]} S_u$.
- (S_t, I_t) form a Markov system.
- Thus, we can express $\pi_t(X)$ as a function of (t, S_t, I_t).
- Since I_t takes value 0 or 1, with the option's price vanishing for $I_t = 0$, we can focus on the price before the barrier is hit – i.e., for $I_t = 1$.

BS PDE for barrier options

- Thus, we consider the price before the first time τ when the barrier is hit:

$$V(t,S_t) := e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(K-S_T)^+ \mathbf{1}_{\{\max_{u\in[t,T]} S_u<B\}} \,|\, S_t].$$

- The actual option's price is

$$\pi_t(X) = V(t,S_t)\,\mathbf{1}_{\{\tau>t\}} = V(t,S_t)\,\mathbf{1}_{\{\max_{u\in[0,t]} S_u<B\}} = V(t,S_t)\,I_t.$$

- Recall that $(e^{-rt}\pi_t(X))_{t\in[0,T]}$ is a martingale.
- Then, $(e^{-rt}V(t,S_t))_{t\in[0,\tau]}$ is a martingale (on the stochastic interval $[0,\tau]$).
- Using Itô's formula, we deduce

$$\partial_t V(t,S_t) + \frac{1}{2}\sigma^2\, S_t^2\, \partial^2_{ss} V(t,S_t) + r\, S_t\, \partial_s V(t,S_t) - r\, V(t,S_t) = 0, \quad t<\tau.$$

- S_t can take any value in $(0,B)$ when $t<\tau$ (and no other values are possible). Thus, we obtain the usual BS PDE, but in a restricted space domain:

$$\partial_t V + \frac{1}{2}\sigma^2\, S^2\, \partial^2_{SS} V + r\, S\, \partial_S V - r\, V = 0, \quad t<T,\ S\in(0,B),$$
$$V(T,S) = (K-S)^+, \quad V(t,B) = 0.$$

PDE-based (dynamic) hedging

- As in the case of European options, the price of a barrier option, before the barrier is hit, satisfies

$$dV(t,S_t) = r\,V(t,S_t)\,dt + \sigma\,S_t\,\partial_s V(t,S_t)\,dW_t^{\mathbb{Q}}, \quad t<\tau.$$

- After τ, we continue by hedging the corresponding European option which the barrier option turns into (and which is zero for knock-out options).
- Thus, we can replicate the price of a barrier option perfectly, in the BS model, by following the delta-hedging strategy.
- PDE-based pricing works in exactly the same way (with non-constant coefficients in the PDE) in LV and SV models.
- The difference between LV and SV prices may be significant for barrier options, even if both LV and SV models are calibrated to the same vanilla prices.
- The dynamic hedging works in LV models, and partial hedging strategies can be designed in SV models.

PDE-based (dynamic) hedging

- Challenge: the numerical approximation of $\partial_S V$ is challenging when V is close to being discontinuous.
- This happens for knock-out options whose terminal payoff function does not vanish at the barrier, for S near B, and for small T.
- For example, up-and-out call with $K < B$:

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} V + r S \partial_S V - r V = 0, \quad t < T, \; S \in (0, B),$$
$$V(T, S) = (S - K)^+, \quad V(t, B) = 0.$$

- A popular fix to this problem is to use $\partial_S V$ corresponding to a different barrier B' that is further away from the current underlying value. This, of course, super-replicates the knock-out options and sub-replicates the knock-in options.
- Another approach is to use the semi-static hedge, which comes with a bet on future implied vol, but is expected to work well for short maturities (note the transaction costs and the need to replicate a general, potentially discontinuous, European payoff with vanilla options).

Monte Carlo methods for pricing and hedging barrier options

- Monte Carlo simulation is a common alternative to solving a PDE and is particularly useful for more complicated (high-dimensional) models or more exotic options.
- Simulation is composed of N independent trajectories on a time grid of M time steps using an Euler or Milstein scheme.
- As we discussed before, it makes sense to use $M = O(N) \approx C\,N$, as $N \to \infty$.
- However, the constant C may need to be large when the price is close to being discontinuous w.r.t. the model factors (e.g., when the spot is close to the barrier and the expiry is short).
- Additional tricks exist for estimating the derivatives/sensitivities of the price function via Monte Carlo (see the book by Glasserman on Monte Carlo methods in Finance).
- Hedging strategies designed by the principle of "canceling sensitivities" of the overall portfolio (hedge + a position in the barrier option) can be computed by combining the Monte Carlo and regression methods.

What is an American option

- American option pays $H(S_\tau)$ at the time $\tau \leq T$ chosen by the option's holder.
- H and T are chosen at the time the option is initiated/settled.
- τ is not fixed at the time the option is initiated/settled. Hence, we view it as a random time (i.e., it is not known from the beginning).
- Clearly, the holder will choose τ to maximize option's payoff, but this is not so easy, as the decision to stop or not to stop at time t can only depend on the information available at time t, while the maximum value of $H(S_u)$ over $u \in [0, T]$ depends on the future path $(S_u)_{u\in[t,T]}$.
- The random times that respect this principle of "not looking into the future" are known as stopping times:

$$\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \quad \forall\, t \in [0, T].$$

Optimal stopping problem

- Once the exercise time/strategy τ is chosen, the American option becomes a more standard contingent claim (though still not European, because τ is fixed as a function of the underlying path, not as a number). Then, NA is equivalent to the existence of RNM $\mathbb{Q}$ s.t. the time-zero price of this claim is

$$\mathbb{E}^{\mathbb{Q}}[e^{-r\tau}H(S_\tau)].$$

- Clearly, the option's holder wants to maximize the value of the claim, so she chooses τ that solves the following optimal stopping problem:

$$\sup_{\tau} \mathbb{E}^{\mathbb{Q}}[e^{-r\tau}\, H(S_\tau)],$$

  where the supremum is taken over all stopping times.

- The above can be made more rigorous: i.e., under some assumptions, one can show that NA is equivalent to the existence of RNM $\mathbb{Q}$ s.t. the price of the American option is given by the above supremum.

- In moderate-dimensional Markov models, there exist efficient PDE, Monte Carlo and mixed methods (BSDEs, Longstaff-Schwartz method) for solving the above optimal stopping problems (with theoretical guarantees).

Practical aspects

- American-exercise feature can be present along with more complex payoff structures: e.g., American-type lookback options, convertible bonds, etc.
- The set of times at which one is allowed to exercise an option may be restricted to a given (finite) set, giving rise to Bermuda options.
- All call and put options on individual stocks listed in the US exchanges are of American type.
- The price of American option is $\geq$ the price of its European counterpart and the immediate payoff.
- If the underlying pays no dividends, the price of American call coincides with the price of the associated European call.
- If the interest rate is zero, the price of American put coincides with the price of the associated European put.
- Typically, American call is exercised if the underlying rises sufficiently high, while American put is exercised if the underlying falls sufficiently low.
- Discrete-time payment of dividends makes pricing of American calls and puts more interesting/challenging.

PDE approach: heuristics for Markov representation

- Assume the Black-Scholes model and consider the continuation value of an American option (which coincides with its time-t price if the option has not been exercised by then):

$$V_t := \sup_{\tau \geq t} \mathbb{E}^{\mathbb{Q}}[e^{-r(\tau - t)} H(S_\tau) \,|\, \mathcal{F}_t].$$

- At time t, we can either exercise the option or wait. Whichever decision we make, the set of possible (conditional) distributions of the future payoffs (from which we choose by prescribing when to exercise) depends only on S_t.
- Thus, the optimal decision on whether to exercise or not at time t is also a function of S_t only.
- Hence, the optimal exercise time τ^* must be the first time when the underlying S_t enters a moving set $\Gamma(t)$ (which is to be found):

$$\tau^* := \inf\{t \geq 0 : \; S_t \in \Gamma(t)\}.$$

- In particular, the continuation value has the usual Markov representation:

$$V_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(\tau^* - t)} H(S_{\tau^*}) \,|\, \mathcal{F}_t] =: V(t, S_t).$$

PDE approach

$$V(t, S_t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(\tau^*-t)} H(S_{\tau^*}) \,|\, \mathcal{F}_t].$$

- $(e^{-rt} V(t, S_t))_{t \in [0,\tau]}$ is a $\mathbb{Q}$-martingale, hence its drift must vanish for $t < \tau$.
- Applying Itô's rule, we deduce from the above the usual BS PDE:

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} V + r\, S\, \partial_S V - r\, V = 0.$$

- Challenge: the above PDE only holds until the exercise/stopping time τ. How do we reflect this in the equation?
- Main idea: the option's holder will wait (and not exercise the option) as long as $V > H$.
- Thus, the above PDE holds in the domain $\{(t, S) : V(t, S) > H(S)\}$.
- Since V is not known a priori, the above condition does not produce any fixed domain for the PDE. Instead it is used as an additional boundary condition (in addition to the usual $V(T, S) = H(S)$), to determine the so-called "free boundary" (a.k.a. early exercise boundary).

(Early) exercise boundary

$$
\begin{aligned}
&V(t,S_t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(\tau^*-t)}H(S_{\tau^*})\,|\,\mathcal{F}_t],\\
&\partial_t V + \frac{1}{2}\sigma^2\, S^2\, \partial^2_{SS} V + r\, S\, \partial_S V - r\, V = 0, \quad (t,S):\ V(t,S) > H(S),\\
&V(T,S) = H(S).
\end{aligned}
$$

- Heuristically speaking, as soon as (t,S_t) reach the boundary of the set $\{(t,S):\ V(t,S) \leq H(S)\}$, represented as the graph of a function $t \mapsto S_{\mathrm{exer}}(t)$, it is optimal to exercise the option:

$$\tau^* = \inf\{t \in [0,T]:\ S_t = S_{\mathrm{exer}}(t)\}.$$

- The above PDE belongs to the class of "obstacle problems", which is a sub-class of the "free-boundary problems".
- There exists rigorous mathematical theory for this type of PDEs: i.e., precise formulation, existence, uniqueness, numerical methods with theoretical guarantees, and a precise connection between solutions of this PDE and the optimal stopping problem.

Early exercise boundary

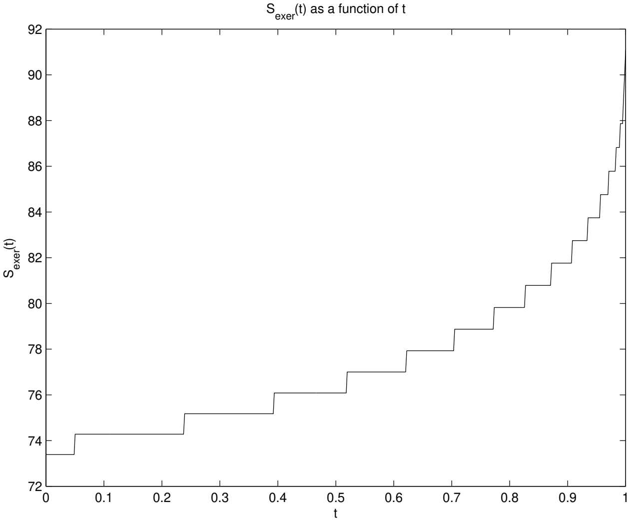

Figure: Early exercise boundary of an American put option.

Numerical solution to BS PDE for American options

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} V + r\,S\,\partial_S V - r\,V = 0, \quad (t,S):\ V(t,S) > H(S),$$
$$V(T,S) = H(S), \quad \tau^* = \inf\{t \in [0,T]:\ V(t,S_t) \le H(S_t)\}.$$

- Discretize time and space with the steps Δt, ΔS.
- Then, the discretized PDE becomes

$$\frac{1}{\Delta t}(V(t+\Delta t) - V(t)) + \frac{1}{2}\sigma^2 S^2 \bar{\partial}^2_{SS} V(t+\Delta t) + r\,S\,\bar{\partial}_S V(t+\Delta t)$$
$$- r\,V(t+\Delta t) = 0,$$

  where we suppressed the dependence of V on S and used the notation $\bar{\partial}$ to denote the finite-difference approximation of the space derivative.
- The above gives

$$V(t) = V(t+\Delta t) - \frac{\Delta t}{2}\sigma^2 S^2 \bar{\partial}^2_{SS} V(t+\Delta t) - \Delta t\,r\,S\,\bar{\partial}_S V(t+\Delta t)$$
$$+ \Delta t\,r\,V(t+\Delta t) = 0, \quad t = T - \Delta t,\ T - 2\Delta t, \ldots, 0.$$

Numerical solution to BS PDE for American options

$$\tilde{V}(t,S) := V(t+\Delta t, S) - \frac{\Delta t}{2}\sigma^2 S^2 \,\bar{\partial}^2_{SS} V(t+\Delta t, S) - \Delta t\, r\, S\, \bar{\partial}_S V(t+\Delta t, S)$$
$$+ \Delta t\, r\, V(t+\Delta t, S) = 0, \quad t = T-\Delta t,\ T-2\Delta t, \dots, 0.$$

- The above iterative scheme does not take into account the free-boundary condition $V(t,S) > H(S)$.
- To take it into account, we introduce an additional step

$$V(t,S) := \max\left(H(S),\ \tilde{V}(t,S)\right).$$

- And the decision on whether to exercise/stop or not is made for each (t,S), depending on whether $V(t,S) = H(S)$ or $V(t,S) > H(S)$.

American vs European put

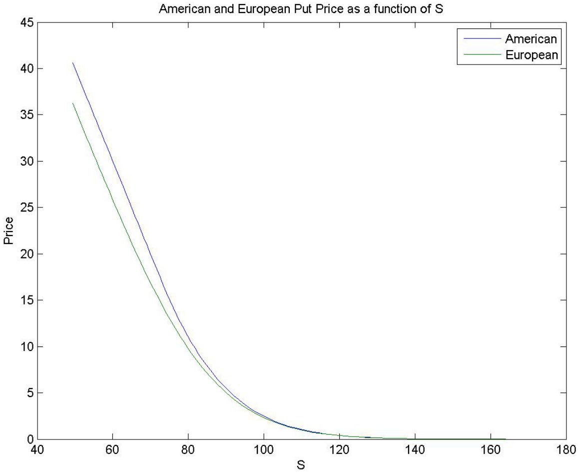

Figure: Prices of American and European options.

PDE methods for American options

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} V + r\, S\, \partial_S V - r\, V = 0, \quad (t,S):\ V(t,S) > H(S),$$
$$V(T,S) = H(S), \quad \tau^* = \inf\{t \in [0,T]:\ V(t,S_t) \leq H(S_t)\}.$$

- Hedging in the Black-Scholes model: once the price function V is found, we can estimate its sensitivity w.r.t. S and use it to implement the usual Delta-hedging strategy.
- Pricing and hedging can be done in LV models using the same PDE approach.
- In SV models, if the number of state processes that make the model Markov is small, then, the PDE methods can be used for pricing and for partial hedging.
- Otherwise, one has to resort to Monte Carlo methods.

Dynamic Programming Principle (DPP) for American options

$$V(t, S_t) = \max_{\tau \geq t} \mathbb{E}^{\mathbb{Q}}[e^{-r(\tau - t)} H(S_\tau) \,|\, \mathcal{F}_t].$$

- Discretize time creating $\mathbb{T} := \{0, \Delta t, 2\Delta t, \dots, T\}$.
- DPP reduces the global optimization problem (over the sequences of decisions "stop/continue" for all time $t \in \mathbb{T}$) to a sequence of local optimization problems (with the decision "stop/continue" at each individual time $t \in \mathbb{T}$):

$$V(t, x) = \max\left(H(x),\, \mathbb{E}^{\mathbb{Q}}[e^{-r\,\Delta t} V(t + \Delta t, S_{t+\Delta t}) \,|\, S_t = x]\right),$$
$$t = T - \Delta, T - 2\Delta t, \dots, 0.$$

- The above scheme is easy to implement once we figure out how to efficiently compute the conditional expectation

$$\mathbb{E}^{\mathbb{Q}}[e^{-r\,\Delta t} V(t + \Delta t, S_{t+\Delta t}) \,|\, S_t = x].$$

Longstaff-Schwartz 2001

$$\mathbb{E}^{\mathbb{Q}}[e^{-r\,\Delta t} V(t+\Delta t, S_{t+\Delta t}) \,|\, S_t = x].$$

- Simulate i.i.d. paths $\{S^{(i)}\}_{i=1}^{N}$.
- For $t = T = \Delta t$, the next-step continuation value $V(t+\Delta t, S_{t+\Delta t}) = H(S_{t+\Delta t})$ is known.
- Thus, we solve a (nonlinear) regression problem

  $$V(t+\Delta t, S^{(i)}_{t+\Delta t}) \approx F_{t+\Delta t}(\theta; S^{(i)}_{t+\Delta t}),$$

  where $\{F_{t+\Delta t}(\theta; \cdot)\}$ is a chosen parametric class of functions to be fitted to the sample (e.g., linear functions, polynomials, neural networks, etc.).
- We repeat the above for $t = T - 2\Delta t, \dots, 0$.
- The main advantage of this method is that it allows for more general (moderate-dimensional) state processes than simply (S_t), without creating prohibitively high computational complexity.

What is a spread option

- The payoff of a spread option is a function of the difference (spread) between the prices of two assets.
- For example, a spread call option with strike K and expiry T pays

$$(S_T^2 - S_T^1 - K)^+$$

- Spread options are popular in energy markets, where several alternative sources can be used to produce energy (e.g., electricity).
- Another application is to hedge the oscillations in the price difference between raw input (oil) and the output (gasoline).
- More applications exist in other markets.

Pricing via reduction to European options

- If a spread option with payoff $H(S_T^1 - S_T^2)$ needs to be priced "in isolation" from other products/tasks, we can simply model the evolution of $S_t := S_t^1 - S_t^2$ under RNM $\mathbb{Q}$ directly.
- For example, if S_t^1 and S_t^2 are futures prices, they are martingales under $\mathbb{Q}$, and hence their spread $S_t = S_t^1 - S_t^2$ is a martingale as well.
- Since S_t does not have to stay positive, a reasonably good model for its short- and medium-term evolution is the Bachelier model:

$$dS_t = \sigma\, dW_t^{\mathbb{Q}}.$$

- Then, the spread option, effectively, becomes a European option on S.

Pricing spread options in 2-dim BS model

- In some cases, it is desirable to have models for S^1 and S^2 individually.
- Then, the PDE or Monte Carlo methods can be used for pricing and hedging.
- On the other hand, clever tricks may lead to much more efficient methods.
- For example, consider the 2-dim BS model

$$\begin{cases} dB_t = rB_t dt, & B_0 = 1, \\ dS^1_t = rS_t dt + \sigma^1 S_t dW^{\mathbb{Q}}_t, & S^1_0 > 0, \\ dS^2_t = rS_t dt + \sigma^2 S_t \left(\rho dW^{\mathbb{Q}}_t + \sqrt{1-\rho^2} dZ_t\right), & S^2_0 > 0. \end{cases}$$

- Let us find the price of a zero-strike spread call that pays

$$(S^2_T - S^1_T)^+.$$

- This can be accomplished by the change of numeraire, and it is known as the "Margrabe's formula".

Change of numeraire

- In the market (B, S^1, S^2), let us use S^1 as the numeraire.
- Define $\tilde{S}^2_t := S^2_t / S^1_t$ and $\tilde{B}_t := B_t / S^1_t$.
- A tedious exercise in stochastic calculus shows that there exists a measure $\tilde{\mathbb{Q}} \sim \mathbb{Q}$ and independent Brownian motions $\tilde{W}$, $\tilde{Z}$ under this measure, s.t.

$$d\tilde{S}^2_t = (\sigma^2\rho - \sigma^1)\tilde{S}^2_t d\tilde{W}_t + \sigma^2\sqrt{1-\rho^2}\tilde{S}^2_t d\tilde{Z}_t,$$
$$d\tilde{B}_t = -\sigma^1 \tilde{B}_t d\tilde{W}_t.$$

- Notice that

$$d\tilde{S}^2_t = \sigma \tilde{S}^2_t d\hat{W}_t,$$

where

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2},$$
$$\hat{W}_t = \frac{\sigma^2\rho - \sigma^1}{\sigma}\tilde{W}_t + \frac{\sigma^2\sqrt{1-\rho^2}}{\sigma}\tilde{Z}_t,$$

and $\hat{W}$ is a BM under $\tilde{\mathbb{Q}}$.

Margrabe's formula

$$d\tilde{S}_t^2 = \sigma \tilde{S}_t^2 d\hat{W}_t.$$

- The price of the spread option is

$$V_t = S_t^1 \, \mathbb{E}^{\tilde{\mathbb{Q}}}((S_T^2 - S_T^1)/S_T^1 \,|\, \mathcal{F}_t) = \mathbb{E}^{\tilde{\mathbb{Q}}}((\tilde{S}_T^2 - 1)^+ \,|\, \mathcal{F}_t),$$

where $\tilde{S}^2$ follows a GBM with zero drift under $\tilde{\mathbb{Q}}$.

- Thus, we obtain the Margrabe's formula:

$$V_t = S_t^1 C^{\mathrm{BS}} \left(\frac{S_t^2}{S_t^1}, 0, \sigma, T - t, 1 \right).$$