

- b. (10 pts) Using the formula for $K_{0,T}^{\text{var}}$ in Heston model, given in the lecture, calibrate (v_0, \bar{v}, λ) to the market values of $\{K_{0,T_i}^{\text{var}}\}$ obtained in part (a) by minimizing the sum of squared errors.

To solve the minimization problem, use

'opt.differential_evolution(modelVarianceStrikeObjectiveFunction, bounds = [(0.0, 10.0), (0.0, 10.0), (0.0, 1000.0)], seed = 0, polish = True, maxiter = 10000)'.

Plot the model-implied and the market-implied break-even strikes on the same graph (as a function of expiry). Comment on the quality of fit.

3. Consider the GARCH model:

$$\begin{aligned} dS_t/S_t &= (r - q) dt + \sqrt{v_t} dW_t^{\mathbb{Q}}, \\ dv_t &= \lambda(\bar{v} - v_t) dt + \eta v_t dZ_t, \quad dW_t^{\mathbb{Q}} dZ_t = \rho dt. \end{aligned}$$

Assume that

$$\begin{aligned} v_0 &= 0.08364961, \quad \bar{v} = 0.05127939, \quad \lambda = 1.697994, \\ \eta &= 8.396695, \quad \rho = -0.6921993. \end{aligned}$$

- a. (10 pts) Estimate the price of a down-and-out barrier put paying

$$(K - S_T)^+ \mathbf{1}_{\{\min_{t \in [0, T]} S_t > B\}}$$

in the above model via Monte Carlo method with Milstein scheme, as of 8/7/2024. Use 5,000 as the size of the Monte Carlo sample and the number of time steps. Use the expiry T corresponding to 9/6/2024 and the strike $K = 5150$. Estimate the discount factor and the forward price from the options' prices observed on 8/7/2024.

Plot the resulting price of the barrier option as a function of the barrier level $B \in [0, 5200]$.

- b. (10 pts) Compute the price of the above barrier option in the Black-Scholes model via the Carr-Bowie formula, as shown in the lectures, using numerical integration of the associated European payoff with respect to the appropriate normal distribution. Use the same T , K , the discount factor and the forward price as in part (a), and the implied volatility computed from the put option with expiry T and with the closest available strike to K , as of 8/7/2024.

Plot the resulting price of the barrier option as a function of the barrier level $B \in [0, 5200]$ on the same graph as the plot from part (a). Comment on the similarities and the differences between the two plots.

4. (15 pts) In this exercise, you implement and test the Vega immunization strategy, as described in the lectures. Use the options' price data provided in the .csv file, but download the historical prices of the SP500 index from the Federal Reserve Economic Data, as in Homework 1.

You need to hedge a short position in one call option with expiry 6/18/2026 and strike 5,000, starting on 8/7/2024 and ending on the last day of Feb 2025, using the dynamic trading (with daily rebalancing) in the underlying, in the money-market account, and in the call option with expiry 3/31/2025 and strike 5,900.

Assume that the money market account pays a constant (continuously compounded) interest rate r and that the SP500 index pays a constant (continuously compounded)

dividend rate q . Estimate r and q from the discount factor and the forward price implied by the options' prices with expiry 3/31/2025 (as of 8/7/2024).

Implement the Vega immunization strategy and plot a histogram of the absolute daily returns (i.e., daily changes in the total wealth) of your overall portfolio (the short position and the hedge).

Repeat the same experiment without using the additional call option in your hedge (note that the formula for Δ_t changes in this case and turns into the usual delta-hedge) and comment on how the two strategies compare to each other.