

# Homework 4

**Course: Advanced Derivative Models**

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For all problems in this assignment, let  $W_t$  be a Brownian motion under risk-neutral measure  $\mathbb{Q}$ ,  $r \in \mathbb{R}$ , and assume the risky asset follows a stochastic volatility model

$$\frac{dS_t}{S_t} = rdt + \sqrt{v_t} dW_t$$

with spot variance process  $v_t$ .

For any subset of the number line  $A \subset \mathbb{R}$ , an associated indicator function  $\mathbf{1}_A : \mathbb{R} \mapsto \mathbb{R}$  is defined by

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \in \mathbb{R} \setminus A. \end{cases}$$

1. Define the function  $g : [0, +\infty) \rightarrow \mathbb{R}$  by

$$g(x) = \begin{cases} \frac{2}{T} \left( \left( \frac{1}{U} - \frac{1}{L} \right) x - \log \frac{L}{U} \right), & 0 \leq x \leq L \\ \frac{2}{T} \left( \frac{x}{U} - 1 - \log \frac{x}{U} \right), & L < x < U \\ 0, & x \geq U \end{cases}$$

where  $0 \leq L \leq U < +\infty$ .

Find  $w : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$\int_0^T w(S_t) v_t dt = g(S_T) - g(S_0) - \int_0^T g'(S_t) S_t r dt - \int_0^T g'(S_t) S_t \sqrt{v_t} dW_t$$

holds.

First differentiate  $g$  twice

$$g'(x) = \begin{cases} \frac{2}{T} \left( \frac{1}{U} - \frac{1}{L} \right), & 0 \leq x \leq L \\ \frac{2}{T} \left( \frac{1}{U} - \frac{1}{x} \right), & L < x < U \\ 0, & x \geq U \end{cases}$$

$$g''(x) = \begin{cases} \frac{2}{Tx^2}, & L < x < U \\ 0, & \text{otherwise} \end{cases}$$

and observe that  $g'$  is continuous and  $g''$  is continuous almost everywhere. These may be further simplified

$$g'(x) = \frac{2}{Tx} \left( \frac{1}{L} (L-x)^+ - \frac{1}{U} (U-x)^+ \right)$$

$$g''(x) = \frac{2}{Tx^2} \mathbf{1}_{(L,U)}(x).$$

Thus, by Itô's Lemma,

$$\begin{aligned} d(g(S_t)) &= g'(S_t) dS_t + \frac{1}{2} g''(S_t) (dS_t)^2 \\ &= g'(S_t) S_t r dt + g'(S_t) S_t \sqrt{v_t} dW_t + \frac{1}{2} g''(S_t) S_t^2 v_t dt. \end{aligned}$$

Rearranging terms and integrating reveals

$$\int_0^T \frac{1}{2} g''(S_t) S_t^2 v_t dt = g(S_T) - g(S_0) - \int_0^T g'(S_t) S_t r dt - \int_0^T g'(S_t) S_t \sqrt{v_t} dW_t.$$

Thus the desired representation is achieved with

$$w(x) = \frac{1}{2} g''(x) x^2 = \frac{1}{T} \mathbf{1}_{(L,U)}(x).$$

## 2. A (continuously monitored) corridor variance swap pays

$$X = \frac{1}{T} \int_0^T \mathbf{1}_{(L,U)}(S_t) v_t dt - H$$

at time  $T$ , where  $H \geq 0$  is the variance strike and  $0 \leq L \leq U < +\infty$ . In other words, the payment accumulates realized variance only when the spot price lies within the corridor  $S_t \in (L, U)$ .

Construct a strategy that replicates the contingent claim with time  $T$  payoff  $X$  by taking static and/or dynamic positions in the risk-free asset, risky asset, and European calls and puts on the risky asset. (*Hint: replicate each term found in the previous problem.*)

Using the representation from the previous problem, the payoff may be expressed as

$$\begin{aligned} X &= \frac{1}{T} \int_0^T \mathbf{1}_{(L,U)}(S_t) v_t dt - H \\ &= g(S_T) - g(S_0) - \int_0^T g'(S_t) S_t r dt - \int_0^T g'(S_t) S_t \sqrt{v_t} dW_t - H. \end{aligned} \quad (1)$$

Each of these terms may be replicated with hedging strategies.

The first term in (1) may be replicated with a static hedge found by applying the Carr-Madan Theorem

$$g(S_T) = g(a) + g'(a)(S_T - a) + \int_0^a g''(K)(K - x)^+ dK + \int_a^{+\infty} g''(K)(x - K)^+ dK$$

for any fixed  $a \geq 0$ . This results in

- a risk-free asset position worth  $e^{-rT}(g(a) - g'(a)a)$ ,
- $g'(a)$  shares of the risky asset, and
- an infinite strip of European puts and calls expiring at time  $T$  with amount  $g''(K)$  held in options with strike price  $K$ .

The second and fifth terms in (1) are constant and thus replicated by a static position in the risk-free asset worth

$$-e^{-rT}(g(S_0) + H)$$

when combined.

Consider the strategy which initially invests in a put expiring at time  $t \leq T$  with strike price  $K$  and then reinvests the put proceeds in the risk-free asset. This strategy's value at time  $T$  is  $e^{r(T-t)}(K - S_t)^+$ . Observe that

$$-g'(S_t) S_t r = \frac{2r}{T} \left( \frac{1}{U} (U - S_t)^+ - \frac{1}{L} (L - S_t)^+ \right)$$

and thus the third term in (1) is replicated by holding two infinite strips of options across expiration times  $t \in [0, T]$ :

- long put with strike price  $U$  in the amount  $\frac{2r}{TU}e^{-r(T-t)}$ ; and
- short put with strike price  $L$  in the amount  $\frac{2r}{TL}e^{-r(T-t)}$ .

As each of these puts expires, the net time  $t$  proceeds  $\frac{2r}{T}e^{-r(T-t)} \left( \frac{1}{U}(U - S_t)^+ - \frac{1}{L}(L - S_t)^+ \right)$  are reinvested in the risk-free asset and thus worth  $\frac{2r}{T} \left( \frac{1}{U}(U - S_t)^+ - \frac{1}{L}(L - S_t)^+ \right)$  at time  $T$ . Taken together, the time  $T$  value equals

$$\int_0^T \frac{2r}{T} \left( \frac{1}{U}(U - S_t)^+ - \frac{1}{L}(L - S_t)^+ \right) dt = - \int_0^T g'(S_t) S_t r dt,$$

replicating the third term in (1).

For the forth term in (1), define

$$V_t = -e^{-r(T-t)} \int_0^t g'(S_s) S_s \sqrt{v_s} dW_s,$$

apply Itô's lemma, and substitute  $S_t \sqrt{v_t} dW_t = S_t r dt - dS_t$  to find

$$\begin{aligned} dV_t &= rV_t dt - e^{-r(T-t)} g'(S_t) S_t \sqrt{v_t} dW_t \\ &= \left( V_t + e^{-r(T-t)} g'(S_t) S_t \right) r dt - e^{-r(T-t)} g'(S_t) dS_t. \end{aligned}$$

This shows  $V_t$  is a dynamic self-financing strategy holding

- a total of  $-e^{-r(T-t)} g'(S_t)$  shares of the risky asset,
- and the remainder worth  $V_t + e^{-r(T-t)} g'(S_t) S_t$  held in the risk-free asset.

Finally check that it replicates the forth term in (1)

$$V_T = -e^{-r(T-T)} \int_0^T g'(S_s) S_s \sqrt{v_s} dW_s = - \int_0^T g'(S_t) S_t \sqrt{v_t} dW_t.$$

Combining each of the above hedges replicates all of the terms in (1) and hence the corridor variance swap itself.

3. Find a formula expressing the break-even corridor variance strike in terms of put and/or call prices, i.e. the value of  $H$  which results in  $\mathbb{E}^{\mathbb{Q}}[X] = 0$ .

As in the lecture slides, let  $P(T, K) = \mathbb{E}^{\mathbb{Q}}[e^{-rT}(K - S_T)^+]$  and  $C(T, K) = \mathbb{E}^{\mathbb{Q}}[e^{-rT}(S_T - K)^+]$  denote put and call prices expiring at time  $T$  and with strike price  $K$ , respectively.

First take expectations of the relation found in Problem #1

$$\mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{T} \int_0^T \mathbf{1}_{(L,U)}(S_t) v_t dt \right] = \mathbb{E}^{\mathbb{Q}}[g(S_T)] - g(S_0) - \mathbb{E}^{\mathbb{Q}} \left[ \int_0^T g'(S_t) S_t r dt \right].$$

Second, as in Problem #2, apply the Carr-Madan Theorem to  $g(S_T)$  and again take expectations

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[g(S_T)] &= g(U) + g'(U)(\mathbb{E}^{\mathbb{Q}}[S_T] - U) \\ &\quad + e^{rT} \left( \int_0^U g''(K) P(T, K) dK + \int_U^{+\infty} g''(K) C(T, K) dK \right) \\ &= \frac{2e^{rT}}{T} \int_L^U \frac{P(T, K)}{K^2} dK \end{aligned}$$

where  $a = U$  is chosen to simplify the expression (since  $g(x) = g'(x) = g''(x) = 0$  for all  $x \geq U$ ). Third, recalling  $\int_0^T \frac{2r}{T} \left( \frac{1}{U} (U - S_t)^+ - \frac{1}{L} (L - S_t)^+ \right) dt = - \int_0^T g'(S_t) S_t r dt$ , observe

$$\begin{aligned} -\mathbb{E}^{\mathbb{Q}} \left[ \int_0^T g'(S_t) S_t r dt \right] &= \mathbb{E}^{\mathbb{Q}} \left[ \int_0^T \frac{2r}{T} \left( \frac{1}{U} (U - S_t)^+ - \frac{1}{L} (L - S_t)^+ \right) dt \right] \\ &= \frac{2re^{rt}}{T} \int_0^T \left( \frac{1}{U} P(t, U) - \frac{1}{L} P(t, L) \right) dt. \end{aligned}$$

Combining the above findings yields

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{T} \int_0^T \mathbf{1}_{(L,U)}(S_t) v_t dt \right] &= \mathbb{E}^{\mathbb{Q}} [g(S_T)] - g(S_0) - \mathbb{E}^{\mathbb{Q}} \left[ \int_0^T g'(S_t) S_t r dt \right] \\ &= \frac{2e^{rT}}{T} \int_L^U \frac{P(T, K)}{K^2} dK \\ &\quad + \frac{2re^{rt}}{T} \int_0^T \left( \frac{1}{U} P(t, U) - \frac{1}{L} P(t, L) \right) dt - g(S_0). \end{aligned}$$

Finally, setting  $\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{T} \int_0^T \mathbf{1}_{(L,U)}(S_t) v_t dt - H \right] = 0$  and solving for  $H$  gives

$$\begin{aligned} H &= \mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{T} \int_0^T \mathbf{1}_{(L,U)}(S_t) v_t dt \right] \\ &= \frac{2}{T} \left( e^{rT} \int_L^U \frac{1}{K^2} P(T, K) dK + re^{rt} \int_0^T \left( \frac{1}{U} P(t, U) - \frac{1}{L} P(t, L) \right) dt \right) - g(S_0). \end{aligned}$$

#### 4. Consider the log-normal forward variance model

$$\frac{d\xi_t^T}{\xi_t^T} = \phi e^{-\kappa(T-t)} dZ_t, \quad t \leq T$$

where  $Z_t$  is another Brownian motion under risk-neutral measure  $\mathbb{Q}$  with  $dW_t dZ_t = \rho dt$ , for some  $\phi > 0$ ,  $\kappa > 0$ , and  $\rho \in (-1, 1)$ . Compute the SDE for spot variance  $v_t = \xi_t^t$  and either prove or disprove whether  $(S_t, v_t)$  is a Markovian system.

Apply Itô's Lemma to  $\log \xi_t^T$

$$\begin{aligned} d(\log \xi_t^T) &= \frac{d\xi_t^T}{\xi_t^T} - \frac{1}{2} \left( \frac{d\xi_t^T}{\xi_t^T} \right)^2 \\ &= \phi e^{-\kappa(T-t)} dZ_t - \frac{1}{2} \phi^2 e^{-2\kappa(T-t)} dt \end{aligned}$$

and rewrite in integral form

$$\begin{aligned} \log \xi_t^T &= \log \xi_0^T + \phi \int_0^t e^{-\kappa(T-s)} dZ_s - \frac{1}{2} \phi^2 \int_0^t e^{-2\kappa(T-s)} ds \\ &= \log \xi_0^T + \phi \int_0^t e^{-\kappa(T-s)} dZ_s - \frac{\phi^2}{4\kappa} e^{-2\kappa T} (e^{2\kappa t} - 1) \end{aligned}$$

to solve

$$\xi_t^T = \xi_0^T \exp \left\{ \phi \int_0^t e^{-\kappa(T-s)} dZ_s - \frac{\phi^2}{4\kappa} e^{-2\kappa T} (e^{2\kappa t} - 1) \right\}.$$

Substitute  $T = t$  to find

$$\begin{aligned} v_t &= \xi_t^t \\ &= \xi_0^t \exp \left\{ \phi \int_0^t e^{-\kappa(t-s)} dZ_s - \frac{\phi^2}{4\kappa} (1 - e^{-2\kappa t}) \right\} \\ &= \xi_0^t \exp \left\{ -\frac{\phi^2}{4\kappa} (1 - e^{-2\kappa t}) \right\} e^{X_t} \end{aligned} \quad (2)$$

where

$$X_t = \phi e^{-\kappa t} \int_0^t e^{\kappa s} dZ_s$$

is defined to help organize calculations. Observe

$$\begin{aligned} dX_t &= -\kappa \phi e^{-\kappa t} \int_0^t e^{\kappa s} dZ_s dt + \phi e^{-\kappa t} e^{\kappa t} dZ_t \\ &= -\kappa X_t dt + \phi dZ_t. \end{aligned}$$

Apply Itô's Lemma to (2)

$$\begin{aligned} dv_t &= \left( \frac{\frac{d}{dt} \xi_0^t}{\xi_0^t} - \frac{\phi^2}{2} e^{-2\kappa t} \right) v_t dt + v_t dX_t + \frac{1}{2} v_t (dX_t)^2 \\ &= \left( \frac{\frac{d}{dt} \xi_0^t}{\xi_0^t} - \frac{\phi^2}{2} e^{-2\kappa t} \right) v_t dt + v_t (-\kappa X_t dt + \phi dZ_t) + \frac{1}{2} v_t \phi^2 dt. \end{aligned}$$

Collect terms and simplify

$$dv_t = \left( \frac{\frac{d}{dt} \xi_0^t}{\xi_0^t} + \frac{\phi^2}{2} (1 - e^{-2\kappa t}) - \kappa X_t \right) v_t dt + \phi v_t dZ_t. \quad (3)$$

Finally, use (2) to express of  $X_t$  in terms of  $v_t$

$$X_t = \frac{\phi^2}{4\kappa} (1 - e^{-2\kappa t}) - \log \xi_0^t + \log v_t$$

and substitute into (3), finding the spot variance SDE

$$dv_t = \kappa \left( \frac{\frac{d}{dt} \xi_0^t}{\kappa \xi_0^t} + \frac{\phi^2}{4\kappa} (1 - e^{-2\kappa t}) + \log \xi_0^t - \log v_t \right) v_t dt + \phi v_t dZ_t.$$

This clearly results in a Markovian system in  $(S_t, v_t)$  since both SDEs involve only deterministic expressions of the variables  $t$ ,  $S_t$ , and  $v_t$ .