MSCF 46915 (Advanced Derivative Models), Fall 2025, Mini 2.

Homework 4

Due: Wed, Dec 3, 2025, NO LATER than 11:59pm.

SP500 options' price data needed for this assignment is provided in a separate .csv file.

1. A (continuously monitored) corridor variance swap pays

$$X = \frac{1}{T}\int_0^T \mathbf{1}_{(L,U)}(S_t)\, v_t\, dt - H$$

at the expiry time T, with H being the strike/rate of the variance swap. In other words, the payoff accumulates realized variance only when the spot price lies within the corridor (L, U).

Assume that the underlying price S follows a general SV model,

$$dS_t/S_t = \mu\, dt + \sqrt{v_t}\, dW_t,$$

and that the interest rate is zero, $r = 0$.

a. (5 pts) Use Itô's formula to show that

$$X = g(S_T) - g(S_0) - \int_0^T g'(S_t)\, S_t\, \mu\, dt - \int_0^T g'(S_t)\, S_t \sqrt{v_t}\, dW_t - H,$$

where

$$g(x) := \frac{2}{T}\left(\left[\frac{1}{U} - \frac{1}{L}\right] x - \log\frac{L}{U}\right)\mathbf{1}_{[0,L]}(x) + \frac{2}{T}\left(\frac{x}{U} - 1 - \log\frac{x}{U}\right)\mathbf{1}_{(L,U)}(x).$$

b. (10 pts) Using the results of part (a), construct a strategy that replicates the payoff X by dynamic trading in the underlying and in the money market account (the latter pays zero interest) and by taking static positions in European calls and puts with expiry T. (You may assume that vanilla options of all strikes are available and ignore the transaction costs.)

c. (5 pts) The break-even corridor variance strike is the value of H such that the initial price of X is zero. Use the results of part (b) to find a formula expressing the break-even corridor variance strike in terms of European put and call prices.

2. Consider the Heston model:

$$dS_t/S_t = (r - q)\, dt + \sqrt{v_t}\, dW_t^{\mathbb{Q}},$$
$$dv_t = \lambda(\bar{v} - v_t)\, dt + \eta\sqrt{v_t}\, dZ_t, \quad dW_t^{\mathbb{Q}}\, dZ_t = \rho\, dt.$$

a. (10 pts) Using the prices of call and put options of all strikes and expiries available in the .csv file, as of $8/7/2024$, compute the options-implied break-even strikes $\{K^{\text{var}}_{0,T_i}\}_i$ of the continuously compounded variance swaps on SP500, as shown in the lecture. Use the discount factor and the forward price implied by the options' prices observed on $8/7/2024$. Plot the resulting function $T_i \mapsto K^{\text{var}}_{0,T_i}$.

b. (10 pts) Using the formula for $K_{0,T}^{\text{var}}$ in Heston model, given in the lecture, calibrate $(v_0, \bar{v}, \lambda)$ to the market values of $\{K_{0,T_i}^{\text{var}}\}$ obtained in part (a) by minimizing the sum of squared errors.

To solve the minimization problem, use

'opt.differential_evolution(modelVarianceStrikeObjectiveFunction, bounds = [(0.0, 10.0), (0.0, 10.0), (0.0, 1000.0)], seed = 0, polish = True, maxiter = 10000)'.

Plot the model-implied and the market-implied break-even strikes on the same graph (as a function of expiry). Comment on the quality of fit.

3. Consider the GARCH model:

$$
\begin{aligned}
dS_t/S_t &= (r-q)\,dt + \sqrt{v_t}\,dW_t^{\mathbb{Q}}, \\
dv_t &= \lambda(\bar{v} - v_t)\,dt + \eta\, v_t\, dZ_t, \quad dW_t^{\mathbb{Q}}\, dZ_t = \rho\, dt.
\end{aligned}
$$

Assume that

$$
\begin{aligned}
&v_0 = 0.08364961, \quad \bar{v} = 0.05127939, \quad \lambda = 1.697994, \\
&\eta = 8.396695, \quad \rho = -0.6921993.
\end{aligned}
$$

a. (10 pts) Estimate the price of a down-and-out barrier put paying

$$
(K - S_T)^+\, \mathbf{1}_{\{\min_{t\in[0,T]} S_t > B\}}
$$

in the above model via Monte Carlo method with Milstein scheme, as of $8/7/2024$. Use $5,000$ as the size of the Monte Carlo sample and the number of time steps. Use the expiry T corresponding to $9/6/2024$ and the strike $K = 5150$. Estimate the discount factor and the forward price from the options' prices observed on $8/7/2024$.

Plot the resulting price of the barrier option as a function of the barrier level $B \in [0, 5200]$.

b. (10 pts) Compute the price of the above barrier option in the Black-Scholes model via the Carr-Bowie formula, as shown in the lectures, using numerical integration of the associated European payoff with respect to the appropriate normal distribution. Use the same T, K, the discount factor and the forward price as in part (a), and the implied volatility computed from the put option with expiry T and with the closest available strike to K, as of $8/7/2024$.

Plot the resulting price of the barrier option as a function of the barrier level $B \in [0, 5200]$ on the same graph as the plot from part (a). Comment on the similarities and the differences between the two plots.

4. (15 pts) In this exercise, you implement and test the Vega immunization strategy, as described in the lectures. Use the options' price data provided in the .csv file, but download the historical prices of the SP500 index from the Federal Reserve Economic Data, as in Homework 1.

You need to hedge a short position in one call option with expiry $6/18/2026$ and strike $5,000$, starting on $8/7/2024$ and ending on the last day of Feb 2025, using the dynamic trading (with daily rebalancing) in the underlying, in the money-market account, and in the call option with expiry $3/31/2025$ and strike $5,900$.

Assume that the money market account pays a constant (continuously compounded) interest rate r and that the SP500 index pays a constant (continuously compounded) dividend rate q. Estimate r and q from the discount factor and the forward price implied by the options' prices with expiry $3/31/2025$ (as of $8/7/2024$).

Implement the Vega immunization strategy and plot a histogram of the absolute daily returns (i.e., daily changes in the total wealth) of your overall portfolio (the short position and the hedge).

Repeat the same experiment without using the additional call option in your hedge (note that the formula for Δ_t changes in this case and turns into the usual delta-hedge) and comment on how the two strategies compare to each other.