Advanced Derivative Models (MSCF 46915). Variance and Volatility Derivatives

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Fall 2025, Mini 2

Lecture outline

1. Variance swaps
    - Break-even strike/rate
    - Continuously compounded realized variance
    - Replication of the break-even strike
    - Variance swaps in Heston model
2. Nonlinear volatility derivatives
    - Volatility swaps
    - VIX index
    - Term structure of variance
    - VIX futures
    - VIX options

Variance swap: definition

- Variance swap is a derivative, whose payoff X at expiry time T is a function of the realized path of the underlying price (S_t).
- Namely, the (normalized) payoff of a variance swap (written on the underlying (S_t)) with expiry $T = t_N$ and strike K pays

$$X := \frac{252}{N} \sum_{i=1}^{N} \left(\log(S_{t_i}/S_{t_{i-1}})\right)^2 - K,$$

where $\frac{252}{N} \sum_{i=1}^{N} \left(\log(S_{t_i}/S_{t_{i-1}})\right)^2$ is the (close-to-close) realized variance for a sequence of consecutive business day closing times $\{t_i\}_{i=0}^{N}$.

- The swap is "spot starting" if $t_0 = 0$, and it is "forward starting" if $t_0 > 0$.
- This structure is similar to a forward contract: one party agrees to pay a fixed amount, and the other makes a payment dependent on the future values of the underlying.
- If $\{t_i\}$ are measured in days (typically true in practice), then K represents the average daily payment of the fixed leg of the contract. Hence, K is often referred to as the "variance swap rate".

Variance swap: break-even variance strike (variance swap rate)

$$X = \frac{252}{N} \sum_{i=1}^{N} \left(\log(S_{t_i}/S_{t_{i-1}})\right)^2 - K.$$

- The break-even variance strike,

$$K^{\mathrm{var}}_{t_0,t_N} := \mathbb{E}^{\mathbb{Q}} \left[\frac{252}{N} \sum_{i=1}^{N} \left(\log(S_{t_i}/S_{t_{i-1}})\right)^2 \right],$$

is chosen to produce a zero fair value of the contract at time zero.

Continuously compounded approximation of realized variance

- Consider a general SV model

$$dS_t/S_t = r\,dt + \sqrt{v_t}\,dW_t.$$

- If the monitoring frequency $t_i - t_{i-1}$ (one day) is small relative to the full period $t_N - t_0$, then

$$\frac{252}{N}\sum_{i=1}^{N}\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2 \approx \frac{1}{t_N - t_0}\int_{t_0}^{t_N} v_t\,dt.$$

- In such a case, the close-to-close and the continuously compounded realized variances are nearly equal, and the break-even strike can be approximated well via

$$K^{\mathrm{var}}_{t_0,t_N} \approx \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{t_N - t_0}\int_{t_0}^{t_N} v_t\,dt\right].$$

Replication of expected continuously compounded realized variance

$$K^{\text{var}}_{t_0,t_N} \approx \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{t_N - t_0}\int_{t_0}^{t_N} v_t\,dt\right],$$

$$dS_t/S_t = r\,dt + \sqrt{v_t}\,dW_t.$$

- How to approximate the above expectation using observed price data?
- Recall the static replication of log-contract: applying Itô's formula,

$$d\log S_t = (r - v_t/2)\,dt + \sqrt{v_t}\,dW_t.$$

- Rearranging terms and integrating gives the payoff of (continuously compounded) realized variance

$$\int_0^T v_t\,dt = -2\log\frac{S_T}{F_{0,T}} + 2\int_0^T \sqrt{v_t}\,dW_t$$

as a sum of the payoffs of the log contract and of a zero-cost self-financing trading strategy (in (B,S)).

Replication of expected continuously compounded realized variance

$$\int_0^T v_t \, dt = -2 \log \frac{S_T}{F_{0,T}} + 2 \int_0^T \sqrt{v_t} \, dW_t.$$

- Recall that the log contract can replicated with a linear combination of vanilla options:

$$\log \frac{S_T}{F_{0,T}} = \frac{1}{F_{0,T}} (S_T - F_{0,T}) - \int_0^{F_{0,T}} \frac{1}{K^2} (K - S_T)^+ \, dK$$
$$- \int_{F_{0,T}}^{\infty} \frac{1}{K^2} (S_T - K)^+ \, dK.$$

Replication of expected continuously compounded realized variance

$$\int_0^T v_t\,dt = -\frac{2}{F_{0,T}}\left(S_T - F_{0,T}\right) + 2\int_0^{F_{0,T}} \frac{1}{K^2}\left(K - S_T\right)^+ dK + 2\int_{F_{0,T}}^{\infty} \frac{1}{K^2}\left(S_T - K\right)^+ dK + 2\int_0^T \sqrt{v_t}\,dW_t.$$

- Taking expectation in the above, we find the expected continuous realized variance (under RNM):

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T v_t\,dt\right] = -2\mathbb{E}^{\mathbb{Q}}\left[\log \frac{S_T}{F_{0,T}}\right] = 2e^{rT}\left(\int_0^{F_{0,T}} \frac{1}{K^2} P(T,K)\,dK + \int_{F_{0,T}}^{\infty} \frac{1}{K^2} C(T,K)\,dK\right),$$

where $C(T,K)$ and $P(T,K)$ are call and put prices, respectively.

Replication of the break-even strike

- Thus,

$$K^{\mathrm{var}}_{0,T} \approx \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T v_t\,dt\right]$$
$$= \frac{2e^{rT}}{T}\left(\int_0^{F_{0,T}} \frac{1}{K^2} P(T,K)\,dK + \int_{F_{0,T}}^{\infty} \frac{1}{K^2} C(T,K)\,dK\right).$$

- '$\approx$' in the above is not '$=$' only due to time-discretization.
- The above formula for $K^{\mathrm{var}}_{0,T}$ holds in any SV model (written under RNM)

$$dS_t/S_t = r\,dt + \sqrt{v_t}\,dW_t,$$

regardless of the distribution/dynamics of (v_t).

- However, the introduction of jumps in a model would change the formula for $K^{\mathrm{var}}_{0,T}$.

Break-even strikes/rates of forward starting variance swaps

- How to compute $K^{\text{var}}_{t_0,t_N}$?
- Recall

$$K^{\text{var}}_{t_0,t_N} = \mathbb{E}^{\mathbb{Q}}\left[\frac{252}{N}\sum_{i=1}^{N}\left(\log(S_{t_i}/S_{t_{i-1}})\right)^2\right].$$

- Then, assuming that $t_i - t_{i-1}$ is one day and introducing $\{u_j\}_{j=0}^{M+N}$ with $u_0 = 0$, $u_M = t_0$, and with $u_j - u_{j-1}$ being one day, we have

$$K^{\text{var}}_{t_0,t_N} = \frac{252}{N}\left(\mathbb{E}^{\mathbb{Q}}\sum_{j=1}^{M+N}\left(\log(S_{u_j}/S_{u_{j-1}})\right)^2 - \mathbb{E}^{\mathbb{Q}}\sum_{j=1}^{N}\left(\log(S_{u_j}/S_{u_{j-1}})\right)^2\right)$$

$$= \frac{252}{N}\left(\frac{N+M}{252}K^{\text{var}}_{0,t_N} - \frac{M}{252}K^{\text{var}}_{0,t_0}\right) \approx \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{t_N - t_0}\int_{t_0}^{t_N} v_t\, dt\right].$$

Break-even strike in Heston model

- In certain models, the continuous approximation of the break-even strike can be computed explicitly in terms of model parameters.
- In Heston model,

$$v_t = v_0 + \lambda \int_0^t (\bar{v} - v_s)\, ds + \eta \int_0^t \sqrt{v_s}\, dZ_s.$$

- Taking expectations, we obtain

$$\mathbb{E}^{\mathbb{Q}} v_t = v_0 + \lambda \int_0^t (\bar{v} - \mathbb{E}^{\mathbb{Q}} v_s)\, ds,$$

- which leads to the ODE for $\mathbb{E}^{\mathbb{Q}} v_t$:

$$\frac{d}{dt}\mathbb{E}^{\mathbb{Q}} v_t = \lambda(\bar{v} - \mathbb{E}^{\mathbb{Q}} v_t).$$

- Its unique solution is

$$\mathbb{E}^{\mathbb{Q}} v_t = \bar{v} + e^{-\lambda\, t}(v_0 - \bar{v}).$$

Break-even strike in Heston model

- Plug the expected spot variance $\mathbb{E}^{\mathbb{Q}} v_t$ into (the continuous approximation of) the break-even variance strike formula

$$K^{\text{var}}_{0,T} \approx \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{T} \int_0^T v_t \, dt \right] = \frac{1}{T} \int_0^T \mathbb{E}^{\mathbb{Q}} v_t \, dt$$

$$= \frac{1}{T} \int_0^T \left(\bar{v} + e^{-\lambda t} (v_0 - \bar{v}) \right) \, dt = \bar{v} + \frac{1 - e^{-\lambda T}}{\lambda T} (v_0 - \bar{v}).$$

- This, of course, is consistent with the model independent formula (assuming Heston model is correct), though it is not so easy to verify directly.

Volatility swaps

- Namely, the (normalized) payoff of a (spot started) volatility swap (written on the underlying (S_t)) with expiry t_N and strike K pays

$$\sqrt{\frac{252}{N}\sum_{i=1}^{N}\left(\log(S_{t_i}/S_{t_{i-1}})\right)^2} - K,$$

where $\{t_i\}_{i=0}^{N}$ form a sequence of consecutive business day closing times, with $t_0 = 0$.

- The break-even volatility strike/rate,

$$K_{0,t_N}^{\mathrm{vol}} := \mathbb{E}^{\mathbb{Q}}\sqrt{\frac{252}{N}\sum_{i=1}^{N}\left(\log(S_{t_i}/S_{t_{i-1}})\right)^2},$$

is chosen to produce a zero fair value of the contract at time zero.

Continuously compounded approximation of break-even strike

- As before, in a general SV model

$$dS_t/S_t = r\,dt + \sqrt{v_t}\,dW_t,$$

- the break-even strike can be approximated by the expectation of a continuously compounded volatility:

$$K^{\mathrm{vol}}_{0,t_N} := \mathbb{E}^{\mathbb{Q}}\sqrt{\frac{252}{N}\sum_{i=1}^{N}\left(\log(S_{t_i}/S_{t_{i-1}})\right)^2}$$
$$\approx \mathbb{E}^{\mathbb{Q}}\sqrt{\frac{1}{t_N}\int_0^{t_N} v_t\,dt}.$$

- The right hand aside of the above in not linear in v, which, in particular, implies that we cannot use the Carr-Madan formula for replicating K^{vol}_{0,t_N}.

Volatility swaps in Heston model

- In Heston model

$$dS_t/S_t = r\,dt + \sqrt{v_t}\,dW_t,$$
$$dv_t = \lambda(\bar{v} - v_t)\,dt + \eta\sqrt{v_t}\,dZ_t,$$

- one can derive (*Cox-Ingersoll-Ross 1985*):

$$\mathbb{E}^{\mathbb{Q}} e^{-z\int_0^T v_t\,dt} = A\,e^{-z\,v_0\,B},$$

$$A := \left(\frac{2\phi\,e^{(\phi+\lambda)T/2}}{(\phi+\lambda)(e^{\phi\,T}-1)+2\phi}\right)^{2\lambda\bar{v}/\eta^2},$$

$$B := \frac{2(e^{\phi\,T}-1)}{(\phi+\lambda)(e^{\phi\,T}-1)+2\phi}, \quad \phi := \sqrt{\lambda^2 + 2z\,\eta^2},$$

- and use

$$\mathbb{E}^{\mathbb{Q}}\sqrt{\int_0^T v_t\,dt} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \left(1 - \mathbb{E}^{\mathbb{Q}}\left[e^{-z\int_0^T v_t\,dt}\right]\right) z^{-3/2}\,dz.$$

VIX index

- Published daily by Chicago Board Options Exchange (CBOE).
- Not a traded asset itself, but used as the underlying for some cash settled derivatives (VIX futures and VIX options).
- VIX is meant to approximate the square root of the break-even strike $K^{\text{var}}_{0,T}$, with T being one month, of a variance swap on the SP500 index.
- It is computed from the vanilla options' prices via the replication formula (approximated with a discrete set of strikes):

$$VIX \approx \sqrt{\frac{2e^{rT}}{T}\left(\int_0^{F_{0,T}} \frac{P(T,K)}{K^2}\,dK + \int_{F_{0,T}}^{\infty} \frac{C(T,K)}{K^2}\,dK\right)}.$$

- Recall that $VIX^2 \approx \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T v_t\,dt\right] \approx K^{\text{var}}_{0,T}$, though all these equalities are indeed approximate.
- The (more-or-less) precise formula for VIX can be found here: `https://cdn.cboe.com/resources/vix/vixwhite.pdf`

Term structure of (forward) variance

- For a given pair $0 \leq t \leq T < \infty$, the instantaneous T-forward variance observed at time t is defined as

$$\xi_t^T := \mathbb{E}^{\mathbb{Q}}[v_T \,|\, \mathcal{F}_t].$$

- The (forward) variance term structure (a.k.a. forward variance curve) is

$$(\xi_t^T)_{T \geq t}.$$

- Note that the break-even strike of a variance swap with expiry T can be written in terms of the forward variance curve

$$K_{0,T}^{\text{var}} \approx \mathrm{VS}_0^T := \frac{1}{T} \int_0^T \xi_0^s \, ds.$$

- The value of the break-even strike of a (spot started) variance swap with expiry $T > 0$ that will be settled/initiated at a future time $t \leq T$ is approximately equal to

$$\mathrm{VS}_t^T := \frac{1}{T-t} \int_t^T \xi_t^s \, ds = \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{T-t} \int_t^T v_s \, ds \,\Big|\, \mathcal{F}_t \right].$$

VIX via the term structure of variance

- Recall that

$$VIX_0^2 \approx \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T v_t\,dt\right] = \frac{1}{T}\int_0^T \xi_0^s\,ds = \mathrm{VS}_0^T,$$

  where T is one month.

- Similarly, the future value of VIX^2 is given by

$$VIX_t^2 \approx \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T-t}\int_t^T v_s\,ds \,|\, \mathcal{F}_t\right] = \frac{1}{T-t}\int_t^T \xi_t^s\,ds = \mathrm{VS}_t^T,$$

  where T is one month after t.

- Thus, we can price (and partially hedge) derivatives on VIX via modeling the future values of $(\mathrm{VS}_t^T)_{t,T}$ or, equivalently $(\xi_t^T)_{t,T}$.

VIX futures

- The time-zero price of a futures contract on VIX index, with expiry at time t, is given by K^{VIX} such that the payoff
$$\mathrm{VIX}_t - K^{\mathrm{VIX}}$$
has zero price at the initial time.
- By risk-neutral valuation and continuous approximation
$$K^{\mathrm{VIX}} \approx \mathbb{E}^{\mathbb{Q}}\sqrt{\mathrm{VS}_t^T} = \mathbb{E}^{\mathbb{Q}}\sqrt{\frac{1}{T-t}\int_t^T \xi_t^s\, ds}$$
$$= \mathbb{E}^{\mathbb{Q}}\sqrt{\frac{1}{T-t}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T v_s\, ds \,\middle|\, \mathcal{F}_t\right]},$$
where T is one month after t.
- Recall that the expectation cannot be interchanged with the square-root. Hence, VIX futures price cannot be replicated with vanilla options.
- This is why, just like in the case of volatility swaps, we need a model to find the value of K^{VIX}.

Modeling variance term structure

- VIX futures price is

$$K^{\mathrm{VIX}} \approx \mathbb{E}^{\mathbb{Q}} \sqrt{\frac{1}{T-t} \int_t^T \xi_t^s \, ds}.$$

- A popular approach is to model directly the evolution of the entire variance term structure

$$d\xi_t^{t+\tau} = \mu_t(\tau)\, dt + \sigma_t(\tau)\, dZ_t, \quad \tau \geq 0, \quad t \geq 0,$$

  and ensure that $(\mu_t(\tau), \sigma_t(\tau))$ are chosen so that, for any $T > 0$, the process $(\xi_t^T)_{t\in[0,T]}$ is a martingale (under an RNM $\mathbb{Q}$). This is known as the Heath-Jarrow-Morton (HJM) approach to modeling term structure.

- Another approach is to model the instantaneous variance process (v_t), e.g., via a SV model, and to compute

$$\mathbb{E}^{\mathbb{Q}}[v_T \mid \mathcal{F}_t] = \xi_t^T.$$

Variance term structure in Heston model

- In Heston model, the instantaneous variance v_t follows

$$dv_t = \lambda(\bar{v} - v_t)\,dt + \eta\,\sqrt{v_t}\,dZ_t.$$

- We have seen that

$$\xi_0^T = \mathbb{E}^{\mathbb{Q}} v_T = \bar{v} + e^{-\lambda\,T}(v_0 - \bar{v}).$$

- Using the Markov property of (v_t), it is easy to deduce from the above that

$$\xi_t^T = \mathbb{E}^{\mathbb{Q}}[v_T \,|\, \mathcal{F}_t] = \bar{v} + e^{-\lambda\,(T-t)}(v_t - \bar{v}).$$

- Then, the continuously compounded approximation of the break-even strike of a variance swap with expiry T started at a future time t is

$$\mathrm{VS}_t^T = \frac{1}{T-t}\int_t^T \xi_t^s\,ds = \bar{v} + \frac{1 - e^{-\lambda(T-t)}}{\lambda(T-t)}(v_t - \bar{v}).$$

Prices of VIX futures in Heston model

- VIX futures price is

$$K^{\mathrm{VIX}} \approx \mathbb{E}^{\mathbb{Q}} \sqrt{\frac{1}{T-t} \int_t^T \xi_t^s \, ds} = \mathbb{E}^{\mathbb{Q}} \sqrt{\mathrm{VS}_t^T}$$
$$= \mathbb{E}^{\mathbb{Q}} \sqrt{\bar{v} + \frac{1 - e^{-\lambda(T-t)}}{\lambda(T-t)} (v_t - \bar{v})} =: \mathbb{E}^{\mathbb{Q}} H(v_t),$$

  where T is one month after t.

- The price VIX futures in Heston model reduces to the price of a European option on v_t.
- We can either approximate the density q_t of v_t and compute (numerically)

$$\mathbb{E}^{\mathbb{Q}} H(v_t) = \int_0^\infty H(x) \, q_t(x) \, dx,$$

- or use Monte Carlo methods.

VIX options

- VIX options are European-type and cash settled, with their payoffs being functions of the future values of VIX index.
- For example, the price of a VIX call with expiry t and strike K is approximated by

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(\sqrt{\mathrm{VS}_t^T}-K\right)^+\right]=\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(\sqrt{\frac{1}{T-t}\int_t^T \xi_t^s\,ds}-K\right)^+\right],$$

where T is one month after t.

- Pricing VIX options is similar, in complexity, to pricing VIX futures (note that this is not the case for futures and options on equity).
- In Heston model,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(\sqrt{\mathrm{VS}_t^T}-K\right)^+\right]$$
$$=\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(\sqrt{\bar{v}+\frac{1-e^{-\lambda(T-t)}}{\lambda(T-t)}(v_t-\bar{v})}-K\right)^+\right].$$