Vega immunization

- Recall the Sigma-hedge:

$$\Sigma_t = \frac{\partial_v V^X}{\partial_v C}, \quad \Delta_t = \partial_s V^X - \Sigma_t \partial_s C.$$

- If the hedged option is European, i.e., $X = H(S_T)$, then it can be written as a (possibly infinite) linear combination of calls or puts:

$$H(S_T) = \sum_j h_j \, (S_T - K_j)^+$$

- Then, a popular approach is to represent the prices of associated options via Black-Scholes formulas,

$$C_t = \mathrm{C}^{\mathrm{BS}}(S_t, \sigma_t^{\mathrm{imp}}(M, K), M, K), \;\; \pi(X)_t = \sum_j h_j \, \mathrm{C}^{\mathrm{BS}}(S_t, \sigma_t^{\mathrm{imp}}(T, K_j), T, K_j).$$

- and to replace $\partial_v C$ and $\partial_v V^X$ by

$$\partial_{\sigma^{\mathrm{imp}}} C = \partial_{\sigma^{\mathrm{imp}}} \mathrm{C}^{\mathrm{BS}}(S_t, \sigma_t^{\mathrm{imp}}(M, K), M, K),$$
$$\partial_{\sigma^{\mathrm{imp}}} V^X = \sum_j h_j \, \partial_{\sigma^{\mathrm{imp}}} \mathrm{C}^{\mathrm{BS}}(S_t, \sigma_t^{\mathrm{imp}}(T, K_j), T, K_j).$$

Vega immunization

- Recall the Sigma-hedge

$$\Sigma_t = \frac{\partial_v V^X}{\partial_v C}, \quad \Delta_t = \partial_s V^X - \Sigma_t \partial_s C.$$

- Replacing $\partial_v C$ and $\partial_v V^X$ by $\partial_{\sigma^{\mathrm{imp}}} C$ and $\partial_{\sigma^{\mathrm{imp}}} V^X$, we obtain the Vega immunization strategy (Δ_t, Σ_t):

$$\Sigma_t = \frac{\partial_{\sigma^{\mathrm{imp}}} V^X}{\partial_{\sigma^{\mathrm{imp}}} C}, \quad \Delta_t = \partial_s V^X - \Sigma_t \partial_s C.$$

- Note that the above hedge can be computed without prescribing any particular model for (S, v)! It is very popular among practitioners.
- In a two-factor SV model, where Sigma-hedge replicates the claim X perfectly, Vega immunization gives exactly the same strategy.
- Strictly speaking, Vega immunization is limited to European claims, but one can imagine less rigorous extensions of this approach to other claims, e.g., via regression.

Static hedging of European options with calls and puts

- *Carr-Madan 1988*: for any twice differentiable $H : [0, \infty) \to \mathbb{R}$ and any $x, a \geq 0$,

$$H(x) = H(a) + H'(a)(x - a) + \int_0^a H''(K)(K - x)^+ \, dK + \int_a^\infty H''(K)(x - K)^+ \, dK.$$

- As a consequence, the payoff $H(S_T)$ is exactly replicated by a static (buy and hold) position consisting of
  - a bond position, worth initially $e^{-rT}(H(a) - H'(a)a)$,
  - $H'(a)$ shares of the risky asset,
  - and an infinite combination of vanilla puts and calls, with the same expiration T and with all possible strikes K, taken with weights $H''(K)$.
- In practice, integrals must be truncated and discretized. Thus, real-world static hedging is approximate.

Valuation

- Apply the Carr-Madan formula with $a = e^{rT} S_0 = F_{0,T}$, multiply by the discount factor e^{-rT}, and take expectation, to obtain:

$$\mathbb{E}^{\mathbb{Q}}[e^{-rT} H(S_T)] = e^{-rT} H(F_{0,T})$$
$$+ \int_0^{F_{0,T}} H''(K)\, P(T,K)\, dK + \int_{F_{0,T}}^{\infty} H''(K)\, C(T,K)\, dK.$$

- Thus, we can compute the price of any European-type claim via vanilla puts and calls.
- This representation is model-independent in the sense that it does not depend on a specific choice of the distribution/dynamics of (S, v) (though it still depends on our assumption of linear pricing and ignores the transaction costs).
- If the underlying pays dividends, the formula $e^{rT} S_0 = F_{0,T}$ fails, but the Carr-Madan formula still holds.

Log contract

- A very important application of the Carr-Madan formula is the pricing of a log-contract:

$$
\begin{aligned}
&H(S_T) = \log S_T, \\
&H''(K) = -\frac{1}{K^2}, \\
&\mathbb{E}^{\mathbb{Q}}[e^{-rT} \log S_T] = e^{-rT} \log F_{0,T} \\
&\quad - \int_0^{F_{0,T}} \frac{1}{K^2} P(T,K)\, dK - \int_{F_{0,T}}^{\infty} \frac{1}{K^2} C(T,K)\, dK.
\end{aligned}
$$

- This formula is used for variance swap replication/valuation.
- A particularly important special case of the above is the construction of VIX index.