Pricing and hedging barrier options via reflection

- In the Black-Scholes model,

$$dS_t/S_t = r\,dt + \sigma\,dW_t^{\mathbb{Q}},$$
$$S_t = S_0\,\exp((r - \sigma^2/2)\,t + \sigma\,W_t^{\mathbb{Q}}),$$

- we can find the price of a knock-out (upper) barrier option,

$$X = H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t < B\}},$$

  explicitly in terms of the prices of European options.

- This pricing method is due to *Carr-Bowie 1994* and is based on the reflection principle for Brownian motion.

Reflection principle for Brownian motion

- Let W be a 1-dim. Brownian motion and F be a given function, vanishing in $[B, \infty)$.
- Consider

$$\mathbb{E}\left(F(W_T)\mathbf{1}_{\{\max_{t\in[0,T]} W_t < B\}}\right) = \mathbb{E}\left(F(W_T)\mathbf{1}_{\{\tau > T\}}\right)$$
$$= \mathbb{E}\, F(W_T) - \mathbb{E}\left(F(B + \tilde{W}_{T-\tau})\mathbf{1}_{\{\tau \leq T\}}\right),$$

where

$$\tau := \inf\{t \geq 0 : \; W_t \geq B\}, \quad \tilde{W}_t := W_{t+\tau} - B.$$

- Strong Markov property of Brownian motion implies that $\tilde{W}$ is a Brownian motion independent of τ.
- On the other hand $-\tilde{W}$ is also a Brownian motion independent of τ.
- Thus, we can replace $(\tilde{W}, \tau)$ by $(-\tilde{W}, \tau)$ in the above expectation...

Reflection principle for Brownian motion

$$\tau := \inf\{t \geq 0 : \; W_t \geq B\}, \quad \tilde{W}_t := W_{t+\tau} - B.$$

- ...obtaining

$$\mathbb{E}\left(F(W_T)\mathbf{1}_{\{\max_{t\in[0,T]} W_t < B\}}\right) = \mathbb{E}\, F(W_T) - \mathbb{E}\left(F(B + \tilde{W}_{T-\tau})\mathbf{1}_{\{\tau \leq T\}}\right)$$
$$= \mathbb{E}\, F(W_T) - \mathbb{E}\left(F(B - \tilde{W}_{T-\tau})\mathbf{1}_{\{\tau \leq T\}}\right)$$
$$= \mathbb{E}\, F(W_T) - \mathbb{E}\left(F(2B - W_T)\mathbf{1}_{\{\max_{t\in[0,T]} W_t \geq B\}}\right),$$

  where, to obtain the last equality, we used the definition of $\tilde{W}$.

- Notice that

$$\mathbb{E}\left(F(2B - W_T)\mathbf{1}_{\{\max_{t\in[0,T]} W_t \geq B\}}\right) = \mathbb{E}\, F(2B - W_T),$$

  since $F(2B - W_T)$ is non-zero only when $W_T \geq B$.

- Thus, $\mathbb{E}\left(F(W_T)\mathbf{1}_{\{\max_{t\in[0,T]} W_t < B\}}\right) = \mathbb{E}\, F(W_T) - \mathbb{E}\, F(2B - W_T)$.

Back to barrier options

$$dS_t/S_t = r\,dt + \sigma\,dW_t^{\mathbb{Q}},$$
$$S_t = S_0\,\exp((r-\sigma^2/2)\,t + \sigma\,W_t^{\mathbb{Q}}).$$

- Let $C := \frac{r}{\sigma} - \frac{\sigma}{2}$ and define the measure $\tilde{\mathbb{Q}}$ by

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \exp\left(-\int_0^T C\,dW_t^{\tilde{\mathbb{Q}}} - \frac{T\,C^2}{2}\right) = \exp\left(-C\,W_T^{\tilde{\mathbb{Q}}} - \frac{1}{2}T\,C^2\right),$$

- so that $S_t = S_0 \exp\left(\sigma W_t^{\tilde{\mathbb{Q}}}\right)$.
- Then, the price of a knock-out (upper) barrier option is

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t<B\}}\right]$$
$$= e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\exp\left(C\,W_T^{\tilde{\mathbb{Q}}} - \frac{T\,C^2}{2}\right) H\left(S_0\exp\left(\sigma W_T^{\tilde{\mathbb{Q}}}\right)\right)\mathbf{1}_{\{\max_{t\in[0,T]} W_t^{\tilde{\mathbb{Q}}}<\frac{1}{\sigma}\log\frac{B}{S_0}\}}\right]$$

Price of a barrier option in BS model

- Thus, we have

$$
\begin{aligned}
& e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t<B\}}\right] \\
& = e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\exp\left(C\,W_T^{\tilde{\mathbb{Q}}} - \frac{T\,C^2}{2}\right) H\left(S_0\exp\left(\sigma W_T^{\tilde{\mathbb{Q}}}\right)\right)\mathbf{1}_{\{\max_{t\in[0,T]} W_t^{\tilde{\mathbb{Q}}}<\frac{1}{\sigma}\log\frac{B}{S_0}\}}\right] \\
& =: e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[F(W_T^{\tilde{\mathbb{Q}}})\mathbf{1}_{\{\max_{t\in[0,T]} W_t^{\tilde{\mathbb{Q}}}<\tilde{B}\}}\right].
\end{aligned}
$$

- Setting $H(x)=0$ for $x>B$, we have $F(x)=0$ for $x>B$ and use the reflection principle:

$$
\mathbb{E}^{\tilde{\mathbb{Q}}}\left[F(W_T^{\tilde{\mathbb{Q}}})\mathbf{1}_{\{\sup_{t\in[0,T]} W_t^{\tilde{\mathbb{Q}}}<\tilde{B}\}}\right] = \mathbb{E}^{\tilde{\mathbb{Q}}}F(W_T^{\tilde{\mathbb{Q}}}) - \mathbb{E}^{\tilde{\mathbb{Q}}}F(2\tilde{B}-W_T^{\tilde{\mathbb{Q}}}).
$$

Price of a barrier option in BS model

$$C=\frac{r}{\sigma}-\frac{\sigma}{2},\quad \tilde{B}=\frac{1}{\sigma}\log\frac{B}{S_0},\quad F(x)=e^{Cx-\frac{TC^2}{2}}H\left(S_0e^{\sigma x}\right),\quad S_t=S_0\exp\left(\sigma W_t^{\tilde{\mathbb{Q}}}\right)$$

- Thus, we have

$$\begin{aligned}
e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\mathbf{1}_{\{\max_{t\in[0,T]}S_t<B\}}\right] &= \mathbb{E}^{\tilde{\mathbb{Q}}}F(W_T^{\tilde{\mathbb{Q}}})-e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}F(2\tilde{B}-W_T^{\tilde{\mathbb{Q}}})\\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}H(S_T)-e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[e^{2C\tilde{B}-C\,W_T^{\tilde{\mathbb{Q}}}-\frac{TC^2}{2}}H\left(S_0e^{2\tilde{B}\sigma-\sigma W_T^{\tilde{\mathbb{Q}}}}\right)\right]\\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}H(S_T)-e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[e^{C\,W_T^{\tilde{\mathbb{Q}}}-\frac{TC^2}{2}}e^{2C\tilde{B}-2C\,W_T^{\tilde{\mathbb{Q}}}}H\left(S_0e^{2\tilde{B}\sigma-\sigma W_T^{\tilde{\mathbb{Q}}}}\right)\right]\\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}H(S_T)-e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[B^{\frac{2r}{\sigma^2}-1}S_T^{1-\frac{2r}{\sigma^2}}H\left(S_T^{-1}B^2\right)\right]\\
&=: e^{-rT}\mathbb{E}^{\mathbb{Q}}H(S_T)-e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\tilde{H}(S_T),
\end{aligned}$$

- and the price of a knock-out (upper) barrier option, which has a terminal payoff function H, in the Black-Scholes model, coincides with the price of the European option that has payoff $H-\tilde{H}$, prior to the knock-out event.

Semi-static hedging of barrier options in BS model

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t<B\}}\right] = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[H(S_T)-\tilde{H}(S_T)],$$

$$\tilde{H}(x) = (x/B)^{1-\frac{2r}{\sigma^2}}\,H\left(B^2/x\right).$$

- The same conclusion applies to lower knock-out options, and to knock-in options (with different formulas for $\tilde{H}$).
- In fact, the derivation we have done to obtain the above can be repeated with conditional expectations, to conclude that, prior to knock-in or knock-out event, the price of a barrier option in the BS model coincides with the price of an associated European option.
- Thus, we obtain a semi-static hedging strategy: to hedge a short position in a knock-out barrier option,
  1. buy a European option with payoff $H-\tilde{H}$,
  2. when/if the barrier event occurs, sell it (at zero price).

Semi-static hedging with $r = 0$

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t<B\}}\right] = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[H(S_T)-\tilde{H}(S_T)],$$
$$\tilde{H}(x) = (x/B)^{1-\frac{2r}{\sigma^2}}\,H\left(B^2/x\right).$$

- If $r = 0$ (recall that we assumed $q = 0$ throughout) and $H(x) = (K-x)^+$, with $K < B$ (so that $H(x) = 0$ for $x > B$), then

$$\tilde{H}(x) = (x/B)\,\left(K - B^2/x\right)^+ = (Kx/B - B)^+ = (K/B)\,\left(x - B^2/K\right)^+.$$

- Thus, in Black's model (i.e., the BS model with $r = q = 0$), the price of up-and-out put with strike K coincides with

$$P(T,K) - (K/B)\,C(B^2/K).$$

- The semi-static hedge of a short position in the up-and-out put is given by a long position in a vanilla put with strike K and a short position in K/B shares of vanilla calls with strike B^2/K.

Semi-static hedging of barrier options in BS model

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(S_T)\,\mathbf{1}_{\{\max_{t\in[0,T]} S_t<B\}}\right] = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[H(S_T)-\tilde{H}(S_T)].$$

- Beyond the case $r=0$, the above formula is not very useful for pricing: since $\tilde{H}$ is not a vanilla payoff, we still need to solve a PDE to find its price (and the price of a barrier option can be found via the PDE directly).
- If we replicate $\tilde{H}$ by the vanilla payoffs, the Carr-Bowie formula becomes more attractive (especially for small maturities).
- The semi-static hedge is a more important contribution of the analysis based on reflection principle, though, in practice, it also requires replication of $\tilde{H}$ via calls and puts. The latter can be done efficiently for short expiration times.