Milstein scheme

- In the discretization error,

$$|\mathbb{E}H(S_T) - \mathbb{E}H(\hat{S}_T)| = O(\Delta t^{1/2}),$$

- the error coming from the Euler approximation of the drift term $\int a(v_t)\,dt$ is of the order $O(\Delta t)$,
- while the error coming from the Euler approximation of the diffusion term $\int b(v_t)\,dZ_t$ is of the order $O(\Delta t^{1/2})$.
- Main idea is to find a higher-order approximation of the diffusion term:

$$\begin{aligned} b(v_s) &\approx b(v_t) + \partial_v b(v_t)[v_s - v_t] \\ &\approx b(v_t) + b(v_t)\,\partial_v b(v_t)\,[Z_s - Z_t]. \end{aligned}$$

- Using the above, we obtain

$$\begin{aligned} v_{t+\Delta t} - v_t \;&=\; \int_t^{t+\Delta t} dv_s \;\approx\; a(v_t)\Delta t + \int_t^{t+\Delta t} b(v_s)dZ_s \\ &\approx a(v_t)\Delta t + b(v_t)[Z_{t+\Delta t} - Z_t] + b(v_t)\,\partial_v b(v_t) \int_t^{t+\Delta t} [Z_s - Z_t]dZ_s. \end{aligned}$$

Milstein scheme

$$\begin{aligned} v_{t+\Delta t} - v_t &= \int_t^{t+\Delta t} dv_s \approx a(v_t)\Delta t + \int_t^{t+\Delta t} b(v_s) dZ_s \\ &\approx a(v_t)\Delta t + b(v_t)[Z_{t+\Delta t} - Z_t] + b(v_t)\, \partial_v b(v_t) \int_t^{t+\Delta t} [Z_s - Z_t] dZ_s. \end{aligned}$$

- Using Itô's formula, we have

$$\int_t^{t+\Delta t} [Z_s - Z_t] dZ_s = \frac{1}{2}[Z_{t+\Delta t} - Z_t]^2 - \frac{\Delta t}{2}.$$

- Denote $Z_{t+\Delta t} - Z_t = \sqrt{\Delta t}\, \epsilon^2_{t+\Delta t}$, where $\{\epsilon^2_t\}$ are i.i.d. standard normals.
- Then the above approximation becomes:

$$v_{t+\Delta t} = v_t + a(v_t)\Delta t + b(v_t)\sqrt{\Delta t}\, \epsilon^2_{t+\Delta t} + b(v_t)\, \partial_v b(v_t)\, \frac{\Delta t}{2}[(\epsilon^2_{t+\Delta t})^2 - 1].$$

- Discretization error of Milstein's scheme is $O(\Delta t)$. So, $\Delta t \sim N^{-1/2}$.

Milstein scheme: implementation details

- Simulated paths of (v_t) can still become negative, so we apply the same boundary conditions to address that.
- To simulate (S_t, v_t) jointly, we need to generate i.i.d. normal noise vectors $\{\epsilon_t = (\epsilon^1_t, \epsilon^2_t)\}$, with $\mathrm{Var}(\epsilon^i_t) = 1$ and $\mathrm{Cov}(\epsilon^1_t, \epsilon^2_t) = \rho$.
- To generate the desired $\epsilon_t = (\epsilon^1_t, \epsilon^2_t)$, we generate independent standard normals ζ^1_t and ζ^2_t and define

$$\epsilon^1_t := \zeta^1_t, \quad \epsilon^2_t := \rho\,\zeta^1_t + \sqrt{1-\rho^2}\,\zeta^2_t.$$

- Instead of approximating (S_t) from its SDE directly, we approximate $(X_t := \log S_t)$:

$$X_{t+\Delta t} = X_t + (r - v_t/2)\,\Delta t + \sqrt{v_t}\,\sqrt{\Delta t}\,\epsilon^1_{t+\Delta t}.$$

- Discretization error of this scheme is still $O(\Delta t)$. So, $\Delta t \sim N^{-1/2}$.