MSCF 46915 (Advanced Derivative Models), Fall 2025, Mini 2.

Homework 3

Due: Fri, Nov 21, 2025, NO LATER than 11:59pm.

SP500 options' price data needed for this assignment is provided in a separate .csv file. A Python template that demonstrates how to use the Fourier-based pricing method in Heston model is included with the homework assignment.

1. For any positive integer n, the payoff of a European "power put" is

$$G_n(x) = H_n(e^x) = \left[(K - e^x)^+\right]^n.$$

   a. (7 pts) Find its "shifted/modulated" Fourier transform:

$$\hat{G}_n(\omega - \mathrm{i}\,\beta) = \int_{-\infty}^{\infty} G_n(x)\, e^{-\mathrm{i}\,(\omega - \mathrm{i}\,\beta)\,x}\, dx,$$

   for the values of $\beta \in \mathbb{R}$ for which this transform is well defined (describe the set of all such $\beta \in \mathbb{R}$).

   b. (8 pts) Use the results of the previous part to find an integral representation for the (arbitrage-free) price of the European power put in terms of the prices of European options with payoffs $H(x) = x^z$, for various $z \in \mathbb{C}$.

2. (15 pts) Consider the Heston model:

$$dS_t/S_t = r\,dt + \sqrt{v_t}\,dW_t^{\mathbb{Q}},$$
$$dv_t = \lambda(\bar{v} - v_t)\,dt + \eta\,\sqrt{v_t}\,dZ_t, \quad dW_t^{\mathbb{Q}}\,dZ_t = \rho\,dt.$$

   Assume that

$$v_0 = 0.08364961, \quad \bar{v} = 0.05127939, \quad \lambda = 1.697994.$$

   Calibrate the remaining parameters (ρ, η) of Heston model to the market prices of European options expiring on $9/6/2024$ and with strikes $\{5105, 5155, 5205, 5255, 5305\}$, as of $8/7/2024$. To do this, you need to solve the minimization problem

$$\min_{\eta,\rho>0} \sum_{i=1}^{5} \left(\mathrm{C}^{\mathrm{mrkt}}(K_i) - \mathrm{C}^{\eta,\rho}(K_i)\right)^2,$$

   where $\mathrm{C}^{\eta,\rho}$ is the Heston price of the associated call option, computed via the Fourier transform method, as shown in the lectures and in the Python template (note that, in order to use this method, you need to compute the options-implied discount factor and forward price, as it was done in Homework 1). To solve the above optimization problem, use

   scipy.optimize.differential_evolution(objectiveFunction,
   bounds $= [(0.0001, 20),\ (-1, 1)]$, seed = 0, polish = True, maxiter = 10000)

   Print out the calibrated (ρ, η).

   Plot the implied volatility smile produced by the calibrated Heston model and the market implied smile for the options expiring on 9/6/2024. Restrict the strike range to within 500 points of at-the-money (ATM).

   Describe the quality of fit.

   Is Feller's condition satisfied for the calibrated parameters' values?

3. (15 pts) Use the parameters of Heston model that you calibrated in exercise 2 to compute the prices of power put options

$$\pi_0\left(\left[(K - S_T)^+\right]^n\right),$$

for $K \in [100, 12000]$, using the expiration date 9/6/2024 and powers $n \in \{1, 2, 3\}$. Use the method developed in exercise 1.

4. (20 pts) Consider the GARCH model:

$$\begin{aligned} dS_t/S_t &= r\,dt + \sqrt{v_t}\,dW_t^{\mathbb{Q}}, \\ dv_t &= \lambda(\bar{v} - v_t)\,dt + \eta\,v_t\,dZ_t, \quad dW_t^{\mathbb{Q}}\,dZ_t = \rho\,dt. \end{aligned}$$

Assume that

$$v_0 = 0.08364961, \quad \bar{v} = 0.05127939, \quad \lambda = 1.697994.$$

Calibrate the remaining parameters (ρ, η) to the European call prices with expiration on 9/6/2024 and with strikes $\{5105, 5155, 5205, 5255, 5305\}$, as of $8/7/2024$. Use the same optimization objective/method as in exercise 2, and compute the model-implied prices via the Monte-Carlo method and Milstein scheme, with the size of the Monte Carlo sample and the number of time steps being $5,000$.

Print out the calibrated (ρ, η).

Plot both the resulting calibrated GARCH diffusion model and market implied volatility smiles for the 9/6/2024 options expiration. Restrict the strike price range shown to within 500 points of ATM.

Describe the quality of fit.