Homework 3

Course: Advanced Derivative Models

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ represent a probability space with filtration $\{\mathcal{F}_t \subset \mathcal{F} |\, t \in [0, +\infty)\}$. Furthermore, let $\{W_t\}_{t\geq 0}$ and $\{X_t\}_{t\geq 0}$ represent two independent $\mathcal{F}_t$-adapted Brownian motions under the measure $\mathbb{P}$. Consider a risky asset $\{S_t\}_{t\geq 0}$ paying no dividends and associated spot variance process $\{v_t\}_{t\geq 0}$ satisfying the SDE system

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu_t dt + \sqrt{v_t} dW_t \\ dv_t &= \omega_t dt + \eta\sqrt{v_t} dZ_t\end{aligned}$$

with money market account $\{B_t\}_{t\geq 0}$ satisfying

$$\frac{dB_t}{B_t} = r dt$$

where $\{\mu_t\}_{t\geq 0}$ and $\{\omega_t\}_{t\geq 0}$ are $\mathcal{F}_t$-predictable and square integrable drift processes, $r \in \mathbb{R}$, $\eta > 0$, $\rho \in (-1, 1)$, and $Z_t = \rho W_t + \sqrt{1-\rho^2} X_t$. Thus $\{Z_t\}_{t\geq 0}$ is another Brownian motion under $\mathbb{P}$ having a constant correlation ρ with $\{W_t\}_{t\geq 0}$.

Let θ_t and ϕ_t be $\mathcal{F}_t$-adapted processes satisfying the Novikov condition

$$\mathbb{E}^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^t \theta_s^2 ds}\right] < +\infty \qquad\qquad \mathbb{E}^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^t \phi_s^2 ds}\right] < +\infty$$

and define the $\mathbb{P}$-martingale process

$$\xi_t = \exp\left\{-\int_0^t \theta_s dW_s - \int_0^t \phi_s dX_s - \frac{1}{2}\int_0^t \left(\theta_s^2 + \phi_s^2\right) ds\right\}.$$

Then, by Girsanov's theorem

$$\widehat{W}_t = W_t + \int_0^t \theta_s ds \qquad\qquad \widehat{X}_t = X_t + \int_0^t \phi_s ds$$

are independent Brownian motions with respect to the measure $\widehat{\mathbb{P}}$ defined by the Radon-Nikodym derivative

$$\left.\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \xi_t.$$

1. Find the dynamics of S_t and v_t in terms of $\widehat{W}_t$ and $\widehat{Z}_t = \rho\widehat{W}_t + \sqrt{1-\rho^2}\widehat{X}_t$ (i.e. write down SDEs under $\widehat{\mathbb{P}}$). Under what condition is $\widehat{\mathbb{P}}$ a risk-neutral measure?

   Observe $d\widehat{W}_t d\widehat{Z}_t = dW_t dZ_t = \rho dt$ and substitute $dW_t = d\widehat{W}_t - \theta_t dt$ and $dX_t = d\widehat{X}_t - \phi_t dt$ into the SDEs

   $$\begin{aligned}\frac{dS_t}{S_t} &= \left(\mu_t - \sqrt{v_t}\theta_t\right) dt + \sqrt{v_t} d\widehat{W}_t \\ dv_t &= \left(\omega_t - \eta\sqrt{v_t}\left(\rho\theta_t + \sqrt{1-\rho^2}\phi_t\right)\right) dt + \eta\sqrt{v_t} d\widehat{Z}_t\end{aligned}$$

   to find the dynamics in terms of $\widehat{W}_t$ and $\widehat{Z}_t$.
   In order for $\widehat{\mathbb{P}}$ to be a risk-neutral measure, $e^{-rt}S_t$ (risky asset divided by the money market account $B_t = e^{rt}$) must be a $\widehat{\mathbb{P}}$-martingale. This happens when $\mathbb{E}^{\widehat{\mathbb{P}}}\left[dS_t/S_t |\, \mathcal{F}_t\right] = r dt$. This occurs if and only if $\mu_t - \sqrt{v_t}\theta_t = r$. It should be noted that it is necessary for either $v_t \neq 0$ or else $\mu_t = r$ at each time $t \geq 0$ for this condition to hold.

2. Under what additional condition does $\widehat{\mathbb{P}}$ result in Heston model dynamics with $v_t > 0$ for all $t \geq 0$, $\widehat{\mathbb{P}}$-a.s.? Is this sufficient to ensure $\widehat{\mathbb{P}}$ is unique? If not, then how should the pricing measure be determined?

   First $\widehat{\mathbb{P}}$ must be a risk-neutral measure, meaning $\mu_t - \sqrt{v_t}\theta_t = r$. Solving for θ_t gives

$$\theta_t = \frac{\mu_t - r}{\sqrt{v_t}}$$

   with $v_t > 0$ satisfied by assumption. Substituting this into the SDEs found in the first problem yields

$$\begin{aligned} \frac{dS_t}{S_t} &= r dt + \sqrt{v_t} d\widehat{W}_t \\ dv_t &= \left(\omega_t - \eta\sqrt{v_t}\left(\rho\frac{\mu_t - r}{\sqrt{v_t}} + \sqrt{1-\rho^2}\phi_t\right)\right) dt + \eta\sqrt{v_t} d\widehat{Z}_t. \end{aligned}$$

   This is of Heston form provided the stochastic volatility drift term satisfies

$$\omega_t - \eta\sqrt{v_t}\left(\rho\frac{\mu_t - r}{\sqrt{v_t}} + \sqrt{1-\rho^2}\phi_t\right) = \lambda\left(\overline{v} - v_t\right)$$

   for some $\lambda > 0$ and $\overline{v} > 0$. Solving for ϕ_t gives

$$\phi_t = \frac{\omega_t - \eta\rho\left(\mu_t - r\right) - \lambda\left(\overline{v} - v_t\right)}{\eta\sqrt{\left(1-\rho^2\right) v_t}}.$$

   The Feller condition

$$2\lambda\overline{v} \geq \eta^2$$

   and $v_0 \geq 0$ must furthermore be satisfied to ensure $v_t > 0$ for all $t \geq 0$, $\widehat{\mathbb{P}}$-a.s.
   The parameters λ and $\overline{v}$ may be otherwise chosen arbitrarily. Therefore $\widehat{\mathbb{P}}$ is not unique, and there are infinitely many risk-neutral measures equivalent to $\mathbb{P}$ which result in Heston model dynamics satisfying Feller's condition.
   Model parameters should be calibrated to the market implied volatility surface (or smile). This provides specific values for each parameter and thus indicates a unique risk-neutral measure consistent with market prices.

3. By the Feynman-Kac formula, for any arbitrage-free stochastic volatility model that is Markovian in the variables (S_t, v_t), the risk-neutral pricing formula

$$V(t,x,y) = \mathbb{E}\left[e^{-r(T-t)} H\left(S_T\right) \middle| S_t = x, v_t = y\right]$$

   of a European payoff $H(S_T)$ is the solution to some PDE. Derive this PDE for a GARCH diffusion model by making use of the fact that $e^{-rt}V(t, S_t, v_t)$ is a martingale under risk-neutral measure—do not simply apply the Feynman-Kac formula. Be sure to include the terminal condition.

   Under a GARCH diffusion model, S_t and v_t satisfy the system of SDEs

$$\begin{aligned} \frac{dS_t}{S_t} &= r dt + \sqrt{v_t} dW_t \\ dv_t &= \lambda\left(\overline{v} - v_t\right) dt + \eta v_t dZ_t \\ dW_t dZ_t &= \rho dt \end{aligned}$$

   Markovian in the state variables (S_t, v_t), hence we will derive a PDE for the pricing function

$$V(t,x,y) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} H\left(S_T\right) \middle| S_t = x, v_t = y\right].$$

Applying Itô's Lemma to the discounted contingent claim $e^{-rt}V(t, S_t, v_t)$ results in

$$\begin{aligned}
d\left(e^{-rt}V(t,S_t,v_t)\right) &= \left(e^{-rt}\frac{\partial V}{\partial t} - re^{-rt}V\right)dt + e^{-rt}\frac{\partial V}{\partial x}dS_t + e^{-rt}\frac{\partial V}{\partial y}dv_t \\
&\quad + \frac{1}{2}e^{-rt}\frac{\partial^2 V}{\partial x^2}dS_t^2 + e^{-rt}\frac{\partial^2 V}{\partial x \partial y}dS_t dv_t + \frac{1}{2}e^{-rt}\frac{\partial^2 V}{\partial y^2}dv_t^2 \\
&= e^{-rt}\left(\frac{\partial V}{\partial t} - rV + rS_t\frac{\partial V}{\partial x} + \lambda\left(\overline{v} - v_t\right)\frac{\partial V}{\partial y}\right)dt \\
&\quad + e^{-rt}\left(\frac{1}{2}S_t^2 v_t\frac{\partial^2 V}{\partial x^2} + \rho\eta S_t v_t^{3/2}\frac{\partial^2 V}{\partial x \partial y} + \frac{1}{2}\eta^2 v_t^2\frac{\partial^2 V}{\partial y^2}\right)dt \\
&\quad + e^{-rt}S_t\sqrt{v_t}\frac{\partial V}{\partial x}dW_t + e^{-rt}\eta v_t\frac{\partial V}{\partial y}dZ_t
\end{aligned}$$

where function arguments have been omitted for brevity. This is a martingale, and hence the "dt" term must be zero. This yields the pricing PDE

$$\frac{\partial V}{\partial t} + rx\frac{\partial V}{\partial x} + \lambda\left(\overline{v} - y\right)\frac{\partial V}{\partial y} + \frac{1}{2}x^2 y\frac{\partial^2 V}{\partial x^2} + \rho\eta x y^{3/2}\frac{\partial^2 V}{\partial x \partial y} + \frac{1}{2}\eta^2 y^2\frac{\partial^2 V}{\partial y^2} - rV = 0$$

(where the substitutions $x = S_t$ and $y = v_t$ have been made). This holds for $(t, x, y) \in [0, T) \times (0, +\infty) \times (0, +\infty)$ with terminal condition $V(T, x, y) = H(x)$.

4. For any positive integer n, define the European "power put" payoff as

$$G_n(x) = H_n(e^x) = \left[(K - e^x)^+\right]^n$$

and find its Fourier transform

$$\widehat{G}_n(\omega) = \int_{-\infty}^{+\infty} G_n(x)\, e^{-\mathrm{i}\omega x} dx.$$

For which arguments $\omega \in \mathbb{C}$ is the Fourier transform $\widehat{G}_n(\omega)$ well defined?

Take the Fourier transform and apply the binomial theorem

$$\begin{aligned}
\widehat{G}_n(\omega) &= \int_{-\infty}^{+\infty} \left[(K - e^x)^+\right]^n e^{-\mathrm{i}\omega x} dx \\
&= \int_{-\infty}^{\log K} (K - e^x)^n e^{-\mathrm{i}\omega x} dx \\
&= \int_{-\infty}^{\log K} \left(\sum_{j=0}^{n} \binom{n}{j} K^{n-j} (-1)^j e^{jx}\right) e^{-\mathrm{i}\omega x} dx \\
&= \sum_{j=0}^{n} \binom{n}{j} (-1)^j K^{n-j} \int_{-\infty}^{\log K} e^{(j - \mathrm{i}\omega)x} dx.
\end{aligned}$$

Each integral in the resulting summation is easily computed

$$\int_{-\infty}^{\log K} e^{(j-\mathrm{i}\omega)x} dx = \left[\frac{e^{(j-\mathrm{i}\omega)x}}{j - \mathrm{i}\omega}\right]\Bigg|_{x=-\infty}^{\log K} = \frac{K^{j-\mathrm{i}\omega}}{j - \mathrm{i}\omega}$$

and valid when $\lim_{x \to -\infty} e^{(j-\mathrm{i}\omega)x} = 0$. This occurs if and only if

$$\mathrm{Re}\left(j - \mathrm{i}\omega\right) > 0,$$

or equivalently when $\mathrm{Im}(\omega) > -j$. Thus the Fourier transform is well defined when $\mathrm{Im}(\omega) > 0$ and given by

$$\widehat{G}_n(\omega) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j K^{n-j} \frac{K^{j-\mathrm{i}\omega}}{j-\mathrm{i}\omega} = K^{n-\mathrm{i}\omega} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j-\mathrm{i}\omega}.$$

It should be noted that this can be further simplified to the alternative form

$$\widehat{G}_n(\omega) = K^{n-\mathrm{i}\omega} n! \prod_{j=0}^{n} (j-\mathrm{i}\omega)^{-1},$$

though it is not necessary. This alternative form can furthermore be computed directly with repeated integration by parts.

5. Use the modulation/frequency shift Fourier transform property to find a valid integral representation for the European power put payoff $H_n(S_T)$ in terms of $\widehat{G}_n(\cdot)$.

   By the modulation/frequency shift property of the Fourier transform

$$G_n(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{G}_n(\omega - \mathrm{i}\beta)\, e^{(\beta+\mathrm{i}\omega)x} d\omega$$

   for any $\beta \in \mathbb{R}$. Taking $\beta < 0$ ensures $\widehat{G}_n(\omega - \mathrm{i}\beta)$ is defined for all $\omega \in \mathbb{R}$ and thus

$$\begin{aligned}
H_n(S_T) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{G}_n(\omega - \mathrm{i}\beta)\, e^{(\beta+\mathrm{i}\omega)\log S_T} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} K^{n-\mathrm{i}(\omega-\mathrm{i}\beta)} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j-\mathrm{i}(\omega-\mathrm{i}\beta)} e^{(\beta+\mathrm{i}\omega)\log S_T} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} K^{n-(\beta+\mathrm{i}\omega)} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j-(\beta+\mathrm{i}\omega)} e^{(\beta+\mathrm{i}\omega)\log S_T} d\omega \\
&= \frac{K^n}{2\pi} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \int_{-\infty}^{+\infty} (j-(\beta+\mathrm{i}\omega))^{-1} \left(\frac{S_T}{K}\right)^{(\beta+\mathrm{i}\omega)} d\omega.
\end{aligned}$$