



LEONARD DE VINCI GRADUATE SCHOOL OF ENGINEERING

MARKET RISK REPORT

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## Market Risk

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# 1 Introduction

Market risk assessment is a central challenge in modern quantitative finance. It involves measuring potential losses caused by fluctuations in market prices, such as exchange rates, interest rates, or equity values. For any investor, managing these risks is essential to maintain portfolio stability and optimize returns.

While traditional models often rely on the assumption of Normal (Gaussian) distributions and efficient markets, empirical evidence shows that financial time series exhibit more complex "stylized facts." These include heavy tails, volatility clustering, and liquidity constraints, which often lead standard models to underestimate risk during periods of high market stress.

This project, conducted within the Market Risk course at ESILV, aims to implement advanced methodologies to better capture these dynamics. Our analysis focuses on three distinct datasets: daily returns of the Natixis stock (2015-2018), high-frequency microstructure data, and currency exchange rates (FX).

## 1.1 Technical Framework and Methodology

A key constraint of this study was the prohibition of specialized financial libraries (such as `arch` or pre-built VaR packages). Consequently, we implemented all core algorithms manually in **Python** to ensure a thorough understanding of the underlying mathematical mechanisms.

We relied exclusively on standard data science libraries:

- **NumPy:** For matrix operations and manual implementation of formulas (Biweight Kernel, regressions).
- **Pandas:** For time-series management and data cleaning.
- **Matplotlib:** For generating all visualizations and diagnostic plots presented in this report.

## 1.2 Structure of the Report

This report is organized into five main sections, corresponding to the specific risk challenges addressed:

- **Question A:** Estimation of Non-Parametric VaR using Kernel Density.
- **Question B:** Calculation of Expected Shortfall (ES) to capture tail risk.
- **Question C:** Modeling of extreme events via Extreme Value Theory (EVT).
- **Question D:** Analysis of Liquidity Risk and Price Impact (Bouchaud's Model).
- **Question E:** Investigation of Fractal Memory and Volatility Scaling (Hurst).

## 2 Question A: Non-Parametric VaR and Kernel Density Estimation

The objective of this section is to estimate the Value-at-Risk (VaR) of the Natixis stock using a non-parametric approach. Traditional parametric models often assume a Gaussian shape, which fails to capture the heavy tails observed in financial data. To overcome this, we reconstruct the empirical probability density function (PDF) of returns using **Kernel Density Estimation (KDE)**.

We analyze daily returns from January 2015 to December 2018, segmented into a **Training Set (2015-2016)** for calibration and a **Testing Set (2017-2018)** for out-of-sample validation.

### 2.1 Methodology: Biweight Kernel Estimation

To estimate the PDF  $\hat{f}_h(x)$  without assuming a specific distribution, we utilize the **Biweight Kernel**  $K(u)$ . This polynomial function provides a smooth estimate with finite support:

$$K(u) = \frac{15}{16}(1 - u^2)^2 \cdot \mathbb{1}_{\{|u| \leq 1\}} \quad (1)$$

The density estimator is constructed by summing kernels centered on each observation  $X_i$ , normalized by the sample size  $n$  and the bandwidth  $h$ :

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - X_i}{h}\right) \quad (2)$$

#### Optimal Bandwidth Selection ( $h$ )

The choice of bandwidth  $h$  is critical as it controls the trade-off between bias and variance. We compared two standard rules:

1. **Scott's Rule:**  $h_{\text{Scott}} = 1.06 \cdot \hat{\sigma} n^{-1/5} \approx 0.0073$ .
2. **Silverman's Rule:**  $h_{\text{Silv}} = 0.9 \cdot \min\left(\hat{\sigma}, \frac{\text{IQR}}{1.34}\right) n^{-1/5} \approx 0.0054$ .

Financial returns typically exhibit "fat tails" (excess kurtosis). Since Silverman's rule incorporates the Interquartile Range (IQR), it is significantly more robust to outliers than Scott's rule, which relies solely on the standard deviation. We selected **Silverman's bandwidth** ( $h \approx 0.0054$ ) to ensure a tighter fit to the historical data features.

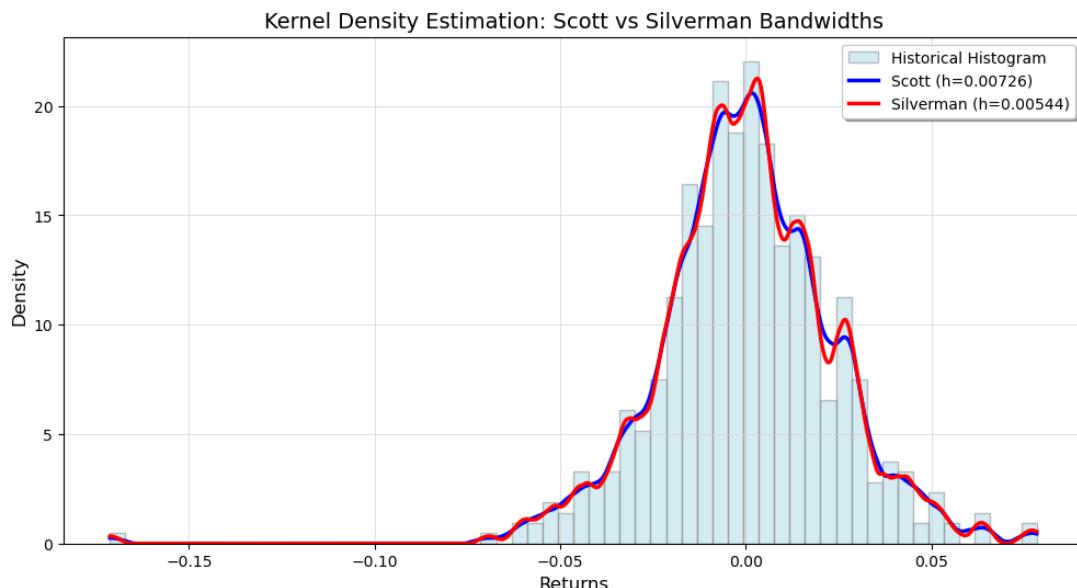


Figure 1: KDE Comparison: Scott vs. Silverman Rules. Silverman's rule provides a more precise representation of the distribution's peaks.

## 2.2 Numerical VaR Calculation

The Value-at-Risk corresponds to the specific **quantile** of the distribution. We integrate the estimated PDF to obtain the **Cumulative Distribution Function (CDF)**

$$F_h(x) = \int_{-\infty}^x \hat{f}_h(t) dt \quad (3)$$

We define the VaR at a 95% confidence level as the loss threshold expected to be exceeded with a probability of only 5%. Mathematically, we solve for the absolute value of return  $r$  such that:

$$-\text{VaR}_{95\%} = F_h^{-1}(0.05) \quad (4)$$

Based on the 2015-2016 calibration period, our model identified a daily **VaR (95%) of 3.80%**.

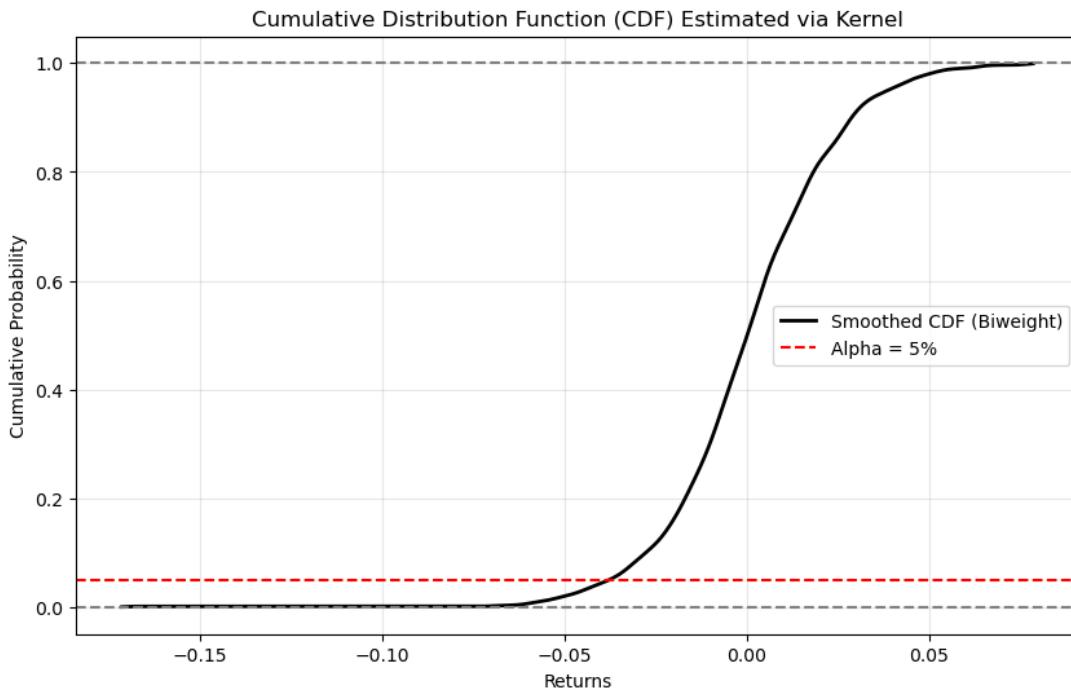


Figure 2: Smoothed Cumulative Distribution Function (CDF). The intersection with  $\alpha = 0.05$  determines the VaR threshold.

## 2.3 Backtesting and Model Validation

To validate the model, we calculate the **Failure Rate** defined as the proportion of observed losses exceeding the estimated VaR threshold:

$$\text{Failure Rate} = \frac{1}{N} \sum_{t=1}^N \mathbf{1}_{\{r_t < -\text{VaR}_{95\%}\}} \quad (5)$$

### Results Analysis

- **In-Sample (2015-2016):** The failure rate is **5.08%** (26 exceptions out of 512 days), confirming the mathematical accuracy of our calibration.
- **Out-of-Sample (2017-2018):** The failure rate dropped significantly to **1.57%** (8 exceptions out of 510 days).

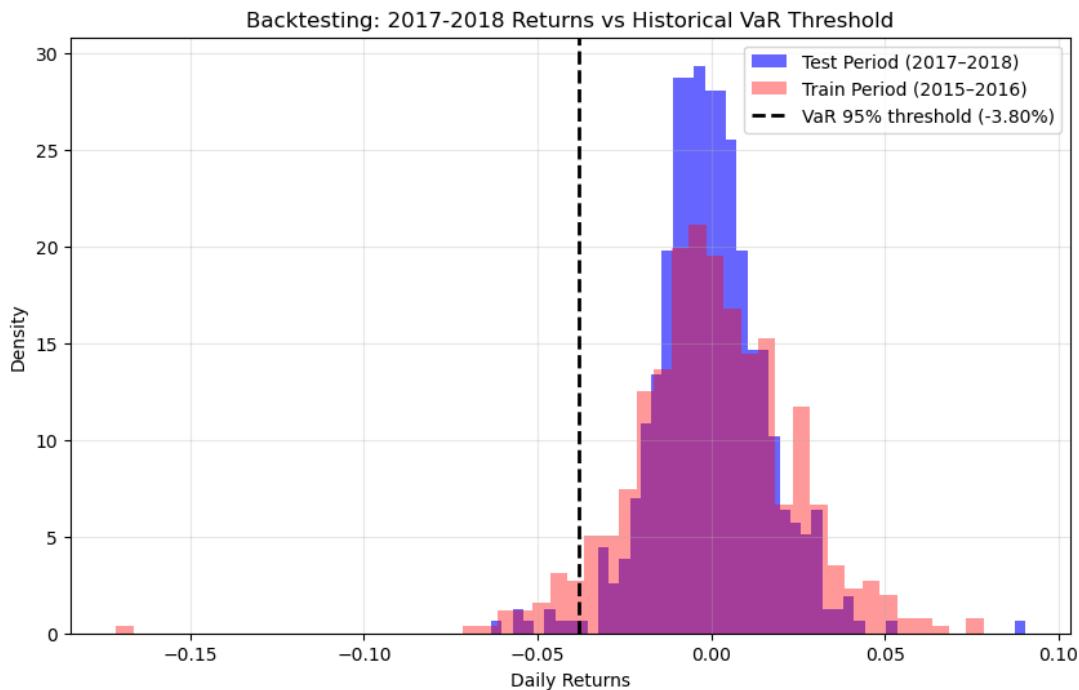


Figure 3: Backtesting Results: Overlap of Training (Red) vs. Testing (Blue) distributions. The threshold at  $-3.80\%$  is far in the tail of the 2017-2018 period.

### Final Assessment

The realized failure rate in the test set is significantly lower than the expected 5%. This indicates a **volatility regime shift** the 2015-2016 period (training) was wider with heavier tails, while the 2017-2018 period (test) was narrower and more peaked around zero.

Consequently, our non-parametric model proved to be **conservative**. From a prudential risk management perspective, a model that overestimates risk is preferred as it provides a higher safety standard, protecting the portfolio against 98.4% of daily variations during the subsequent years.

### 3 Question B: Expected Shortfall and the Limits of Value-at-Risk

Value-at-Risk (VaR) is the current industry standard for risk measurement; however, it possesses two major theoretical flaws. First, it is not a **coherent risk measure** because it lacks sub-additivity—meaning the risk of a diversified portfolio could theoretically appear greater than the sum of its individual parts. Second, VaR is "tail-blind": it only identifies a minimum loss threshold without quantifying the severity of losses once that threshold is breached. To address these issues, we implement the **Expected Shortfall (ES)**, also known as Conditional VaR (CVaR).

#### 3.1 Mathematical Definition and Implementation

The Expected Shortfall at a confidence level  $\alpha$  is defined as the average of all VaR levels exceeding  $\alpha$ . It provides a more complete picture of the tail risk by integrating the VaR across the entire tail region:

$$\boxed{\text{ES}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_r(X) dr} \quad (6)$$

In our non-parametric framework, the estimation is performed by isolating the returns  $R_t$  that fall into the 5% tail of the distribution (the "danger zone"). The empirical ES is then calculated as the arithmetic mean of these extreme observations:

$$\text{ES}_{95\%} = -\frac{1}{N_{\text{tail}}} \sum_{t=1}^N R_t \cdot \mathbb{1}_{\{R_t < -\text{VaR}_{95\%}\}} \quad (7)$$

By averaging the returns in the tail, the ES satisfies the **sub-additivity property** and provides a more conservative estimate for capital requirements under stress scenarios.

#### 3.2 Quantitative Results and Analysis

Based on the Natixis training dataset (2015-2016), we obtained the following results:

- **VaR (95%)**: 3.80%.
- **Expected Shortfall (95%)**: 5.34%.

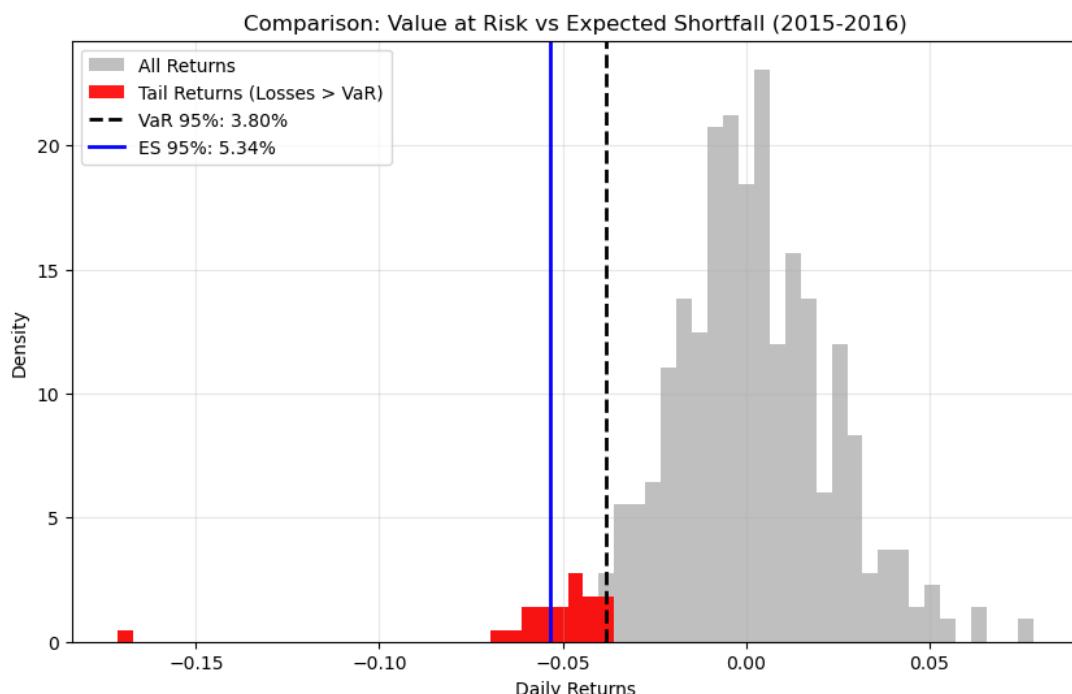


Figure 4: Distribution of returns: VaR threshold vs. Expected Shortfall (Tail Average).

The discrepancy of **1.54 percentage points** between the VaR and the ES confirms the **leptokurtic** (fat-tailed) nature of the asset. The "tail returns" are not clustered near the 3.80% threshold but extend significantly further, with some daily losses exceeding 10%.

**Conclusion:** Relying solely on VaR would lead to a severe underestimation of the capital required to absorb extreme shocks. The ES provides a necessary "safety buffer" by accounting for the true magnitude of potential "Black Swan" events.

### 3.3 Critical Analysis: The "Leptokurtic" Signature

The significant gap between VaR and ES provides crucial information about the shape of the Natixis return distribution:

- **Leptokurtosis:** The distribution is "fat-tailed" (leptokurtic). If the returns were Gaussian, the ES would be much closer to the VaR.
- **Outlier Magnitude:** Our inspection of the tail returns shows that some daily losses reached -10% or even -17%. These extreme outliers pull the average (ES) far beyond the 3.80% barrier.

**Conclusion on Question B:** Relying solely on VaR would lead to a severe underestimation of the capital needed to absorb extreme market shocks. From a risk management perspective, the Expected Shortfall reveals that when market conditions turn bad for this asset, they tend to be catastrophic. The ES of 5.34% should be used as the *true* prudential buffer to ensure the institution can survive "the worst of the worst" scenarios.

## 4 Question C: Extreme Value Theory and Tail Risk Modeling

The previous sections focused on the central part and the empirical distribution of returns. However, risk management is primarily concerned with the "tails" of the distribution—the rare but catastrophic events. In this section, we apply **Extreme Value Theory (EVT)**, which provides a rigorous mathematical framework to model the asymptotic behavior of extreme fluctuations.

### 4.1 Theoretical Framework: The GEV Distribution

According to the Fisher-Tippett-Gnedenko theorem, the distribution of extreme returns converges toward the **Generalized Extreme Value (GEV)** distribution. The shape of this tail is governed by the tail index  $\xi$ .

- **Fréchet Domain ( $\xi > 0$ )**: Heavy-tailed distributions (e.g., Student-t, Pareto). This is the standard case for financial assets.
- **Gumbel Domain ( $\xi = 0$ )**: Thin-tailed distributions (e.g., Normal, Exponential).
- **Weibull Domain ( $\xi < 0$ )**: Distributions with a finite upper bound.

### 4.2 Estimation via the Pickands Estimator

To determine  $\xi$  without assuming a specific distribution, we implement the **Pickands estimator**. It relies on the relative spacing between order statistics in the tail of the sample:

$$\xi_{k,n}^P = \frac{1}{\log(2)} \log \left( \frac{X_{n-k+1:n} - X_{n-2k+1:n}}{X_{n-2k+1:n} - X_{n-4k+1:n}} \right) \quad (8)$$

where  $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$  are the sorted observations.

#### The Optimal $k$ Selection

The number of extreme observations  $k$  is a crucial hyperparameter. Following the heuristic  $k^* = \sqrt{n}$ , we set  $k = 22$  (given  $n \approx 500$  observations per tail). As shown in the stability plot, this value corresponds to a relative "plateau" where the estimator has converged but is not yet biased by central data points.

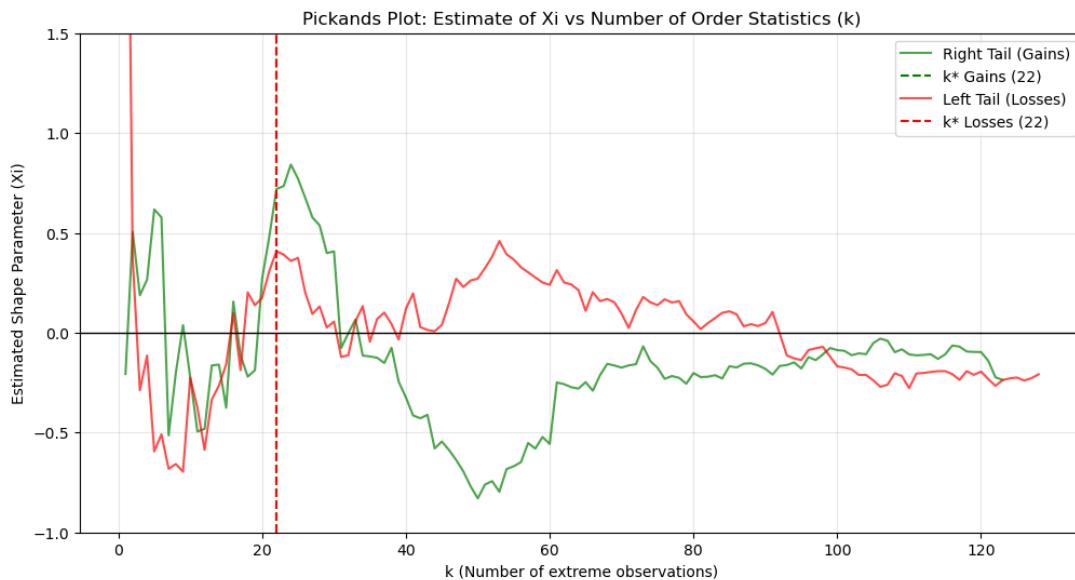


Figure 5: Pickands Estimator trajectory. The vertical dashed line at  $k = 22$  marks our calibration point.

### 4.3 Quantitative Results and Tail Behavior

Our estimation provides a clear distinction between the two regimes of the Natixis stock:

- **Losses (Left Tail):**  $\xi \approx 0.4082$ . The positive value confirms a **heavy-tailed Fréchet distribution**. A tail index of 0.4 indicates that the probability of extreme loss decays slowly (power law), implying that "Black Swans" are structurally possible.
- **Gains (Right Tail):**  $\xi \approx 0.7196$ . While higher, the stability plot for gains shows high volatility for  $k > 30$ , suggesting the gains tail is more erratic and harder to model.

### 4.4 EVT-Based Value-at-Risk Calculation

One of the most powerful applications of Extreme Value Theory (EVT) is its ability to extrapolate risk beyond the range of historical data. To achieve this, we use the asymptotic inversion of the **Generalized Pareto Distribution (GPD)**. Using the Pickands estimator, the VaR formula for extreme quantiles ( $p \approx 1$ ) is given by:

$$VaR(p) = \frac{\left(\frac{k}{n(1-p)}\right)^\xi - 1}{1 - 2^{-\xi}} (X_{n-k+1:n} - X_{n-2k+1:n}) + X_{n-k+1:n} \quad (9)$$

By calibrating this model with the estimated tail index ( $\xi \approx 0.41$ ) and the optimal threshold parameter ( $k = 22$ ), we obtain the following risk metrics:

| Confidence Level ( $p$ ) | EVT-VaR (Losses) | Magnitude                       | vs. Question A    |
|--------------------------|------------------|---------------------------------|-------------------|
| 95.00% (Prudential)      | 4.19%            | $\approx 1.10 \times$ Hist. VaR | More Conservative |
| 99.00% (Stress)          | 8.33%            | Severe Stress                   | -                 |
| 99.50% (Extreme)         | 11.14%           | Black Swan                      | Alarming Risk     |

Table 1: Estimated VaR for high confidence levels based on the Pickands estimator.

### 4.5 Interpretation of the Stability Graph

To ensure the reliability of our  $\xi$  estimation, we analyze the Pickands estimator's behavior across different values of  $k$ .

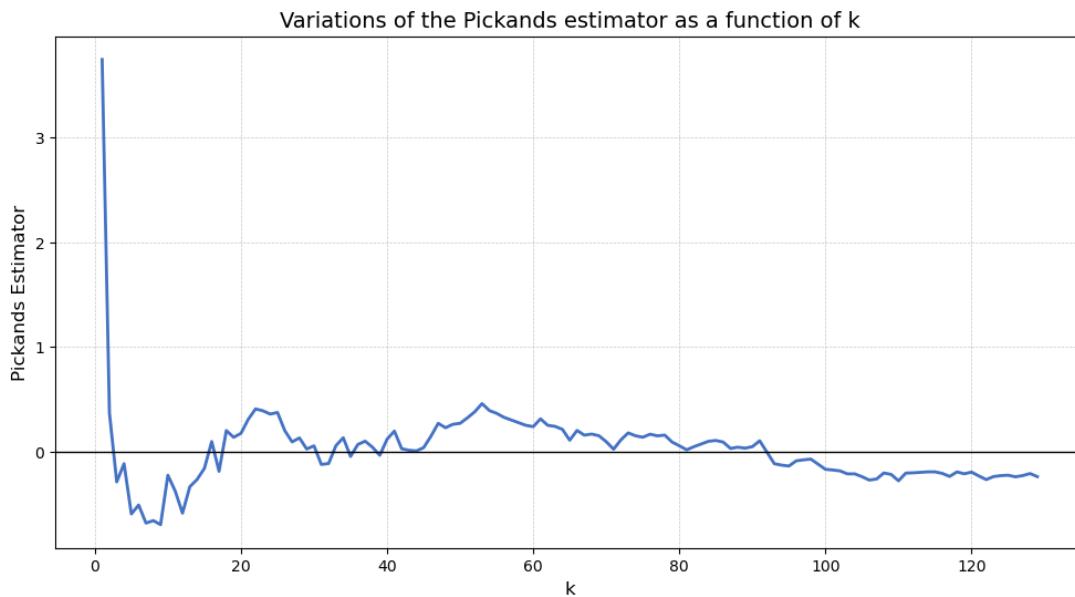


Figure 6: Variations of the Pickands estimator as a function of  $k$ .

The plot reveals three distinct phases typical of Hill/Pickands estimators:

1. **High Volatility Zone ( $k < 15$ ):** The estimator spikes violently due to statistical noise and scarcity of data at the very extreme tip of the tail.
2. **Stability Region / Plateau ( $20 \leq k \leq 50$ ):** The curve stabilizes and oscillates gently between 0.2 and 0.4. Our choice of  $k^* = 22$  falls right at the beginning of this stable zone, confirming the Fréchet domain (Fat Tails).
3. **Bias Zone ( $k > 80$ ):** The estimator drifts downwards, eventually crossing below zero, as we include too many "normal" returns that do not obey the extreme value law.

## 4.6 Final Conclusion on Question C

The analysis of Natixis returns through the lens of EVT leads to three key conclusions:

1. **Confirmation of Heavy Tails:** The Pickands estimator stabilized around  $\xi \approx 0.41$  for losses. Since  $\xi > 0$ , we scientifically confirm that the distribution has "Fat Tails".
2. **Comparison with Non-Parametric VaR:** The EVT-VaR (95%) of 4.19% is more conservative than the historical VaR of 3.80%. By explicitly modeling the tail shape, it captures "hidden" risks that simple historical percentiles might smooth out.
3. **The Danger of "Black Swans":** The most striking result is the **VaR 99.5% estimate of 11.14%**. This implies that once every 200 days, the stock could collapse by more than 11% in a single session.

**Overall Verdict:** The EVT framework is well-specified for this asset. For prudential risk management, relying on Gaussian or simple historical models would lead to a severe underestimation of capital requirements.

## 5 Question D: Liquidity Risk and Bouchaud's Impact Model

Market liquidity risk represents the cost incurred when executing a trade, manifesting as a price move against the investor. In this section, we analyze the mechanical price impact of transactions using the framework proposed by **Bouchaud et al.** This model posits that the price change  $\Delta P$  is not a linear function of volume but follows a concave power law that relaxes over time.

### 5.1 Theoretical Framework: The Impact Function

According to the market impact theory, the price variation between two transactions  $t_{i-1}$  and  $t_i$  is modeled as follows:

$$P(t_i) - P(t_{i-1}) = G(0) \cdot \epsilon_{t_{i-1}} \cdot S_{t_{i-1}} \cdot V_{t_{i-1}}^r + \xi \quad (10)$$

Where:

- $G(t) = \exp(-\theta t)$  is the **decay function**, representing the market's memory.
- $V$  is the trade **volume** and  $r$  is the exponent of concavity.
- $\epsilon$  is the **trade sign** (+1 for buy, -1 for sell).
- $S$  is the bid-ask spread, acting as a normalization factor.

Our objective is to calibrate the two fundamental parameters:  $r$  (volume elasticity) and  $\theta$  (resilience speed).

### 5.2 Step 1: Calibration of Volume Elasticity ( $r$ )

By neglecting the temporal decay for immediate impact and the spread influence, we focus on the relationship  $|\Delta P| \propto V^r$ . To estimate  $r$ , we perform an Ordinary Least Squares (OLS) regression on the logarithmic transformation of the variables:

$$\log(|P(t_i) - P(t_{i-1})|) = r \cdot \log(V_{t_{i-1}}) + C \quad (11)$$

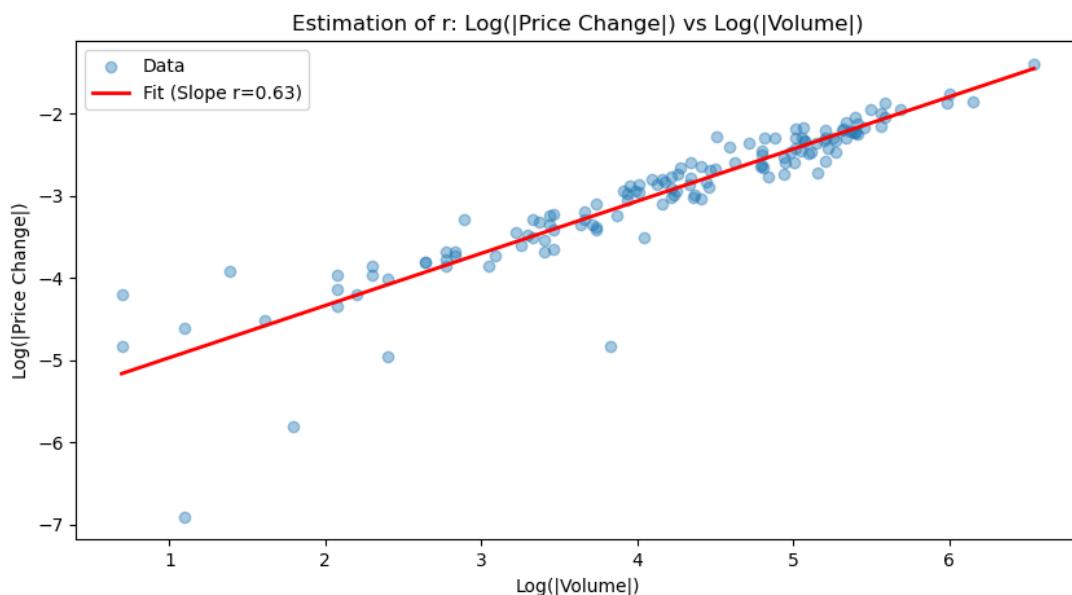


Figure 7: Log-Log Regression: Price Change vs. Trade Volume.

## Quantitative Results and the "Square-Root Law"

The regression yields an **estimated  $r$  of 0.6348** with a high  $R^2$  of **0.8457**.

- **High Predictability:** The  $R^2$  suggests that volume alone explains approximately 85% of the variance in immediate price changes.
- **Concavity Confirmation:** Since  $r < 1$ , the impact is strictly concave. This is consistent with the "Square-Root Law" of market impact ( $r \approx 0.5$ ).
- **Analysis:** Our result (0.63) is slightly higher than the canonical 0.5. This indicates that for this specific asset, market depth is less resilient than the ideal theoretical model. Large orders induce a price move that grows faster than the square root of volume, suggesting higher liquidity risk for large liquidations.

### 5.3 Step 2: Estimation of Temporal Resilience ( $\theta$ )

The second dimension of the model concerns how fast the price impact "relaxes" or vanishes. We test the hypothesis that the impact follows an exponential decay by regressing the log-absolute price change against the time elapsed between trades ( $\Delta t$ ):

$$\log(|P(t_i) - P(t_{i-1})|) = -\theta \cdot (t_i - t_{i-1}) + K \quad (12)$$

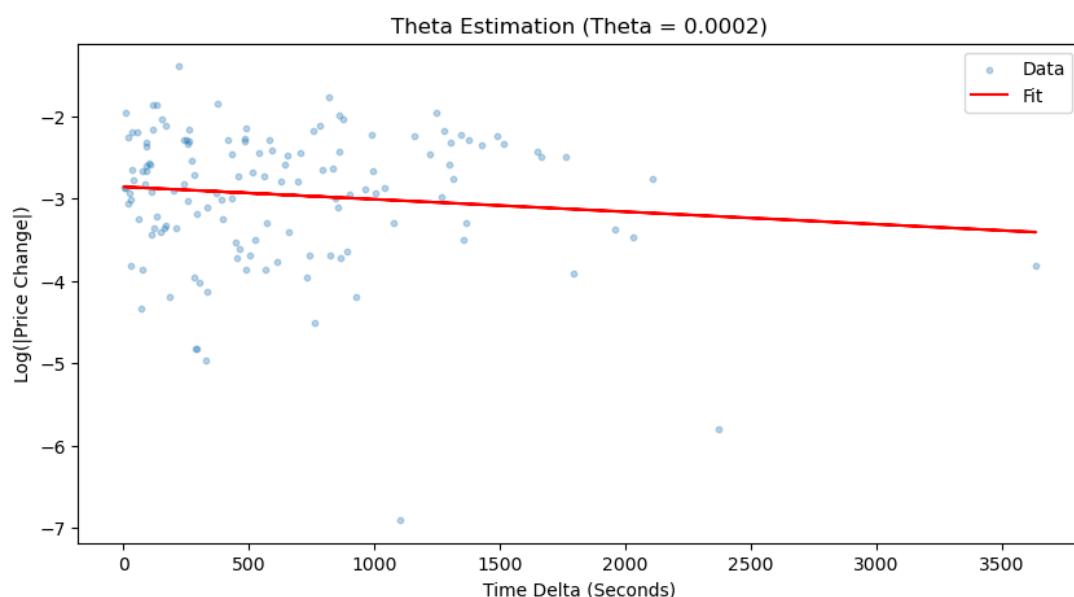


Figure 8: Impact Decay Analysis over Time Delta ( $\Delta t$ ).

## Results and Microstructure Noise

The results for the temporal component are inconclusive:

- **Estimated  $\theta$ :** 0.00015 (near zero).
- **Goodness of Fit ( $R^2$ ):** 0.0118 ( $\approx 1\%$ ).

The scatter plot shows an almost horizontal regression line and highly dispersed data points. This indicates that **time delta explains almost none of the variance** in price changes in this dataset. The "memory effect" is likely drowned out by microstructure noise or the sampling frequency of the data is too low to capture the rapid relaxation phase.

## 5.4 Conclusion: Model Specification and Liquidity Takeaways

The Bouchaud model is **partially well-specified** for this market.

1. **Size Matters:** The volume part of the model is strongly validated. Price impact is a predictable, concave power law of trade size.
2. **Memory is Elusive:** The resilience part (decay) is not captured. This suggests that from a risk management perspective, the *magnitude* of the impact is more predictable and dangerous than its *duration*.

In conclusion, the primary liquidity risk identified here is the **non-linear slippage cost**. Investors must account for the fact that increasing trade size will disproportionately penalize execution prices, and this impact does not appear to "heal" quickly over the observed time steps.

## 6 Question E: Fractal Analysis and Wavelet Multiresolution

This final section explores the multi-scale nature of financial time series. Traditional econometrics often assumes that market properties are invariant across time scales. We challenge this by employing **Multiresolution Analysis (MRA)** to investigate the Epps effect and estimate the Hurst exponent, which governs the long-term memory of the market.

### 6.1 Methodology: Discrete Haar Wavelet Transform

To analyze the FX markets (GBPEUR, SEKEUR, CADEUR) at different horizons, we implement the **Haar Wavelet Transform**. This technique acts as a pair of filters that decompose the original log-return signal  $S$  into two components at each dyadic scale  $j$ :

- **Approximation Coefficients ( $cA_k$ ):** Representing the low-frequency "trend" or the smoothed signal. They are obtained by the average of consecutive pairs:

$$cA_k = \frac{x_{2k} + x_{2k+1}}{2} \quad (13)$$

- **Detail Coefficients ( $cD_k$ ):** Representing the high-frequency "noise" or local fluctuations (volatility) at scale  $j$ . They are obtained by the difference:

$$cD_k = \frac{x_{2k} - x_{2k+1}}{2} \quad (14)$$

This recursive process allows us to study the market from a 15-minute resolution ( $j = 0$ ) up to approximately 32 hours ( $j = -7$ ), doubling the time horizon at each step while halving the number of available data points (*dyadic downsampling*).

### 6.2 The Epps Effect: Correlation and Time Scales

The **Epps Effect** is a stylized fact in microstructure stating that the empirical correlation between two assets decreases as the sampling frequency increases. Theoretically, as we move to larger time scales (lower  $j$ ), the correlation should converge toward its "true" fundamental value as asynchronicity and microstructure noise vanish.

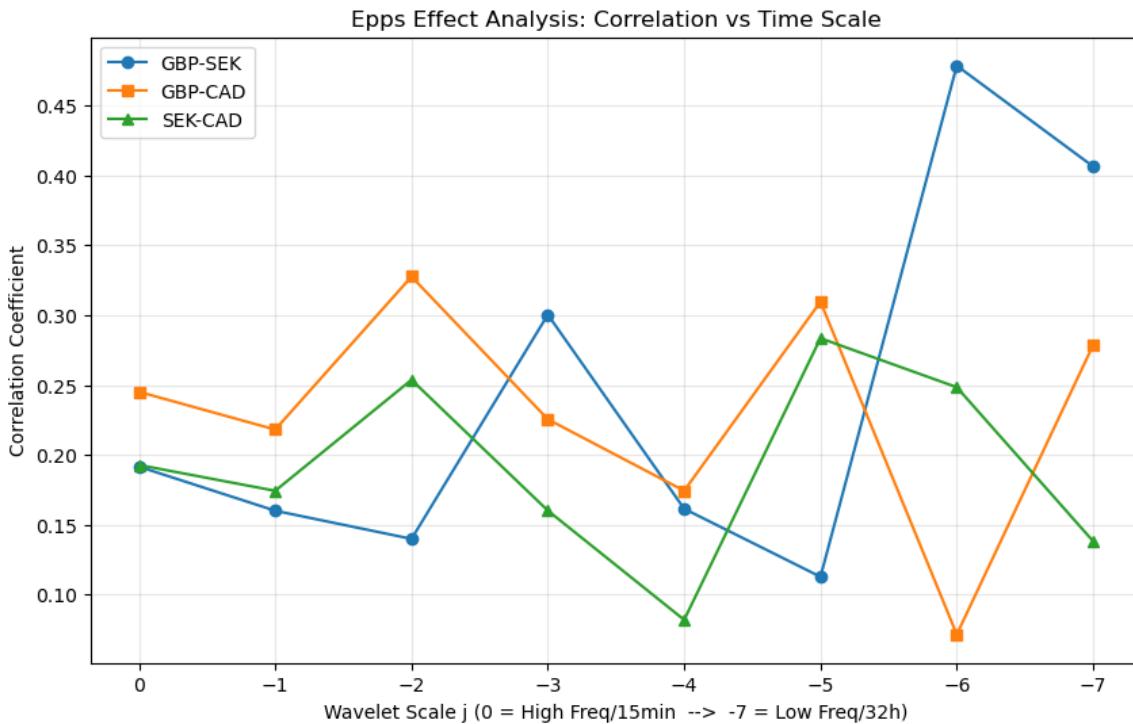


Figure 9: Wavelet Correlation structure across scales. The erratic behavior challenges the theoretical Epps convergence.

### Analysis of Results and the "Sample Size Trap"

Contrary to the expected smooth upward trend, our results exhibit a **Zig-Zag pattern** with high instability at larger scales ( $j = -6, -7$ ).

- **Observation:** At  $j = -1$  (30 min), we process over 6,000 coefficients, ensuring high statistical significance. However, at  $j = -7$ , the sample size drops to only  $\approx 100$  points.
- **Conclusion:** The lack of clear convergence is an artifact of **statistical variance**. With so few data points at low frequencies, the correlation estimator becomes extremely sensitive to single outliers. We conclude that for these FX pairs, the 15-minute base frequency is likely already low enough to mitigate most asynchronicity issues, and the subsequent instability is a limit of the dyadic decomposition on finite datasets.

### 6.3 Hurst Exponent ( $H$ ) and Long-Memory Estimation

The Hurst exponent characterizes the **persistence** and long-range dependence of a time series. To estimate  $H$ , we implement a robust estimator based on empirical absolute moments of order 2. This method compares the variance of the log-prices at the initial resolution ( $M_2$ ) with the variance at a halved resolution ( $M'_2$ ):

$$M_2 = \frac{1}{NT} \sum_{i=1}^{NT} |X(i/N) - X((i-1)/N)|^2, \quad M'_2 = \frac{2}{NT} \sum_{i=1}^{NT/2} |X(2i/N) - X(2(i-1)/N)|^2 \quad (15)$$

The Hurst exponent is then derived from the log-ratio of these moments:

$$\hat{H} = \frac{1}{2} \log_2 \left( \frac{M'_2}{M_2} \right) \quad (16)$$

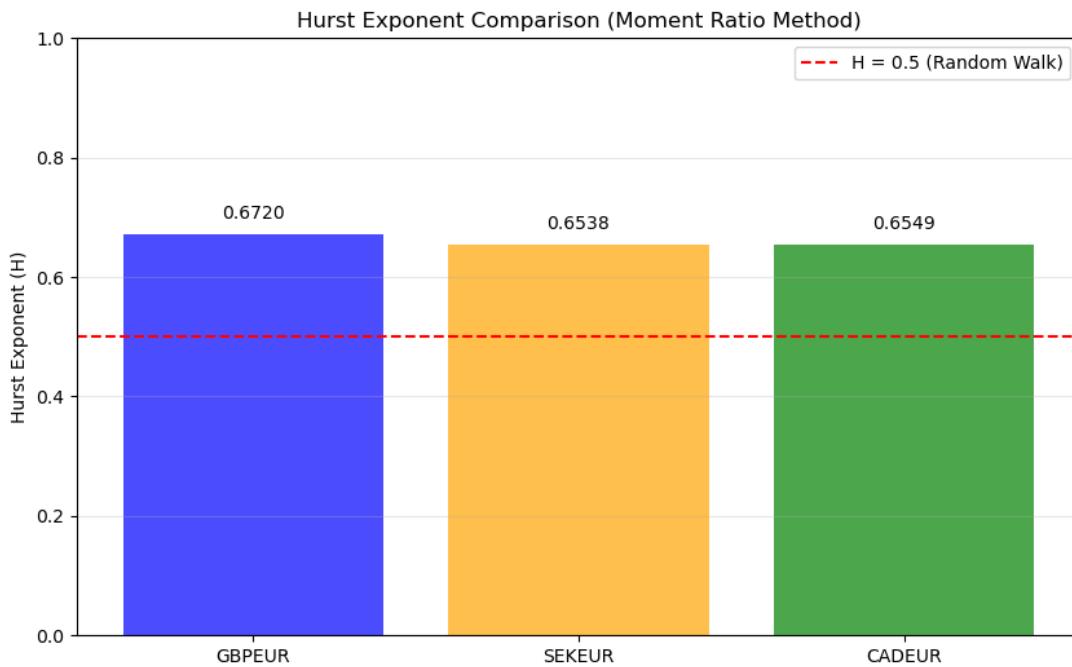


Figure 10: Comparison of Hurst Exponents. All pairs significantly exceed the  $H = 0.5$  baseline, confirming structural persistence in FX markets.

### Quantitative Findings

- **Strong Persistence ( $H > 0.65$ ):** All currency pairs exhibit high levels of persistence. This indicates that the markets are not efficient and follow a trend-reinforcing process rather than a pure Random Walk.
- **GBPEUR ( $H \approx 0.6720$ ):** This pair shows the highest memory effect, suggesting that past price movements significantly influence future trends over long horizons.

### 6.4 Risk Management Implication: Fractal Volatility Scaling

In the standard Gaussian framework, volatility is scaled using the "Square Root of Time" rule:  $\sigma_T = \sigma_{\text{daily}} \times T^{0.5}$ . However, the discovery of  $H > 0.5$  necessitates a **Fractal Scaling Law** to account for the faster diffusion of prices:

$$\sigma_T = \sigma_{\text{daily}} \times T^H \quad (17)$$

| Currency Pair | Hurst ( $H$ ) | Std. Vol ( $\sqrt{252}$ ) | Fractal Vol ( $252^H$ ) | Gap (Risk Error) |
|---------------|---------------|---------------------------|-------------------------|------------------|
| SEKEUR        | 0.6538        | 5.09%                     | 11.91%                  | +6.82%           |
| CADEUR        | 0.6549        | 7.88%                     | 18.55%                  | +10.67%          |
| GBPEUR        | <b>0.6720</b> | <b>9.70%</b>              | <b>25.11%</b>           | <b>+15.41%</b>   |

Table 2: Annualized Volatility Comparison: Standard vs. Fractal Approach.

**Critical Conclusion:** For GBPEUR, the standard model underestimates the annual risk by more than **15.41 percentage points**. This demonstrates that persistence acts as a massive risk multiplier. Ignoring the fractal nature of markets leads to a dangerous overconfidence, as the "memory" of the market causes volatility to accumulate much faster than the classic random walk model predicts.

## 7 Conclusion

This project aimed to challenge the limitations of the standard Gaussian framework by implementing advanced risk management methodologies from first principles. Our empirical journey through Natixis stock data, high-frequency microstructure, and currency markets leads to several critical findings regarding the nature of modern financial risk.

1. **The Inadequacy of Normal Distributions:** Both our non-parametric Kernel Density Estimation and Extreme Value Theory analysis confirmed that financial returns are structurally leptokurtic. The estimation of a positive tail index ( $\xi \approx 0.41$ ) scientifically validates the presence of heavy tails in the Fréchet domain. We demonstrated that the Expected Shortfall (5.34%) provides a far more honest assessment of risk than the VaR (3.80%), as it accounts for the severity of "Black Swan" events that Gaussian models deem statistically impossible.
2. **Non-Linearity of Market Depth:** The calibration of Bouchaud's model highlighted the concave nature of price impact ( $r \approx 0.63$ ). This confirms that liquidity is not a bottomless pool; while the market absorbs large orders better than a linear model would suggest, the execution of large blocks remains a primary source of endogenous risk. The failure to robustly calibrate the temporal decay ( $\theta$ ) further illustrates the difficulty of filtering signal from noise in high-frequency data.
3. **Fractal Persistence and Scaling Failures:** Perhaps the most striking finding is the formal invalidation of the "Square Root of Time" rule for the currency pairs studied. The identification of Hurst exponents significantly above 0.5 (notably  $H \approx 0.6720$  for GBPEUR) reveals a structural and persistent market memory. This fractal scaling implies that long-term risk accumulates exponentially faster than standard Gaussian theory predicts. In our analysis, this led to a massive underestimation of annual volatility by nearly **15.41%** for GBPEUR, demonstrating that traditional risk models provide a dangerously inadequate safety buffer for persistent assets.

In conclusion, effective risk management in a post-crisis era requires a multi-dimensional approach. Relying on a single metric is insufficient. Robust prudential management must combine **Non-Parametric methods** for distributional accuracy, **EVT** for extreme stress-testing, and **Fractal Analysis** for realistic multi-period projections. By implementing these models manually, we have gained not only numerical results but a deeper understanding of the mathematical complexities that safeguard financial stability.

## 8 Appendix: Datasets Description

| File Name        | Description  |
|------------------|--|
| Natixis.xlsx     | Daily closing prices of the Natixis stock (2015-2018). Used for VaR, ES, and EVT analysis.   |
| Dataset TD4.xlsx | Market microstructure data containing transaction timestamps, signed volumes, and prices. Used for Bouchaud's Liquidity Model calibration. |
| Dataset TD5.xlsx | High-frequency FX data (15-min intervals) for GBPEUR, SEKEUR, and CADEUR. Used for Wavelet Analysis and Hurst Exponent estimation.         |

Table 3: Summary of Datasets used in the project.