UNIVERSITÀ DEGLI STUDI DI GENOVA

DIBRIS

DEPARTMENT OF COMPUTER SCIENCE AND TECHNOLOGY,
BIOENGINEERING, ROBOTICS AND SYSTEM ENGINEERING

MODELING AND CONTROL OF MANIPULATORS

First Assignment

Equivalent representations of orientation matrices

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Contents

Mathematical expression	Definition	MATLAB expression
$< w >$	World Coordinate Frame	w
${}^a_bR$	Rotation matrix of frame $< b >$ with respect to frame $< a >$	aRb
${}^a_bT$	Transformation matrix of frame $< b >$ with respect to frame $< a >$	aTb

Table 1: Nomenclature Table

1 Introduction

The first assignment of Modeling and Control of Manipulators consists of four different exercises and focuses on the geometric fundamentals and algorithmic tools underlying any robotics application. The concepts of transformation matrix, orientation matrix and the equivalent representations of orientation matrices (Equivalent angle-axis representation, Euler Angles and Quaternions).

2 Exercise 1 - Equivalent Angle-Axis Representation (Exponential representation)

The case of study of this section is the representation of 3D rotation matrices with the so-called "*angle-axis representation*" or "*exponential representation*", which admits the following analytical expression, also known as Rodrigues formula:

$$\mathbf{R}(^*\mathbf{v},\theta) = e^{[^*\mathbf{v}\wedge]\theta} = e^{[\rho\wedge]} = \mathbf{I}_{3x3} + [^*\mathbf{v}\wedge]\sin(\theta) + [^*\mathbf{v}\wedge]^2(1-\cos(\theta))$$

Considering a geometric unit vector $\mathbf{v}$ and an angle θ, the formula above specifies the rotation around this axis. In the following sections, several examples will be shown with the final frame obtained, starting from a specified geometric unit vector and angle of rotation.

2.1 Q1.2

The given geometric unit vector and angle of rotation were: $\mathbf{v} = [1,0,0]$, $\theta = \pi/6$. Since this is an elementary rotation around the x axis of 30°, named p1 in Figure 1, the obtained rotation matrix has the first column, representing the x axis initial frame projections to the arrival frame x', y', z', respectively called t1, t2 and t3 in Figure 1, correctly equal to $[1,0,0]$.
This is because the x component is maintained, the axis does not change, whereas the arrival axes frame y' and z' still remain orthogonal to x so the projections are zero. It is relevant to observe that the zeros are repeated in the second and third column, this is due to the orthogonality between the initial frame axes y, z and the obtained final frame axis x' = x.
The resulting plots can be seen in Figure 1, whereas the rotation matrix of a frame $< v >$ with respect to a generic frame $< w >$ computed by using the Rodrigues formula is the following:

$$^w_vR = \begin{bmatrix} 1.000 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix}$$

(a) Q1.1 Initial frame. (b) Q1.2 Obtained frame.

Figure 1: Counterclockwise rotation of 30° around the x axis.

2.2 Q1.3

The given geometric unit vector and angle of rotation were: $\mathbf{v} = [0,1,0]$ and $\theta = \pi/4$. The same considerations made in section 2.1 hold for this elementary rotation around the y axis of 45°, but in this case it is necessary to observe the second column of the matrix. The y component is maintained, the axis does not change, whereas the arrival axes frame x' and z' still remain orthogonal to y, therefore the projections are zero.

The resulting plots can be seen in Figure 2, whereas the rotation matrix of a frame $< v >$ with respect to a generic frame $< w >$ computed by using the Rodrigues formula is the following:

$$ {}^{w}_{v}R = \begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix} $$

(a) Q1.3 Initial frame. (b) Q1.3 Obtained frame.

Figure 2: Counterclockwise rotation of 45° around the y axis.

2.3 Q1.4

The given geometric unit vector and angle of rotation were: **v** $= [0, 0, 1]$ and $\theta = \pi/2$ The same considerations made in the previous sections apply for this elementary rotation around the z axis of 90°, but in this case it is enough to observe the third column of the matrix.
The resulting plots can be seen in Figure 3, whereas the rotation matrix of a frame $< v >$ with respect to a generic frame $< w >$ computed by using the Rodrigues formula is the following:

$$ {}^{w}_{v}R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

(a) Q1.4 Initial frame. (b) Q1.4 Obtained frame.

Figure 3: Counterclockwise rotation of 90° around the z axis.

2.4 Q1.5

In this section we deal with a non unit geometric vector, that is necessarily normalized in order to compute the rotation matrix, and angle of rotation equal to: $\mathbf{v} = [0.408, 0.816, -0.408]$ and $\theta = 0.2449$
The resulting plots can be seen in Figure 4, whereas the rotation matrix of a frame $< v >$ with respect to a generic frame $< w >$ computed by using the Rodrigues formula is the following:

$$ {}^w_vR = \begin{bmatrix} 0.9751 & 0.1089 & 0.1930 \\ -0.0890 & 0.9901 & -0.1089 \\ -0.2029 & 0.0890 & 0.9751 \end{bmatrix} $$

(a) Q1.5 Initial frame. (b) Q1.5 Obtained frame.

Figure 4: Counterclockwise rotation of 14.0137° around a non unit axis.

2.5 Q1.6

Given a non unit geometric vector, the angle of rotation is computed, from the Rodrigues formula, as the norm of the vector: $\rho = [0, \pi/2, 0]$; $\theta = \pi/2$
The resulting plots can be seen in Figure 5, whereas the rotation matrix of a frame $< v >$ with respect to a generic frame $< w >$ computed by using the Rodrigues formula is the following:

$$ {}^w_vR = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} $$

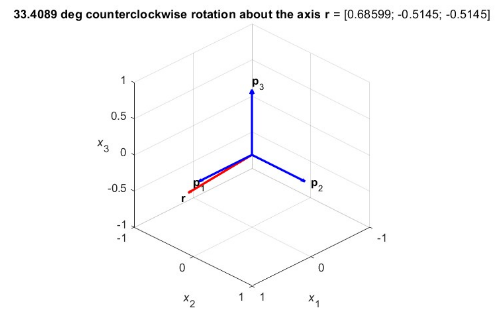
(a) Q1.6 Initial frame. (b) Q1.6 Obtained frame.

Figure 5: Counterclockwise rotation of 90° around the a non unit axis.

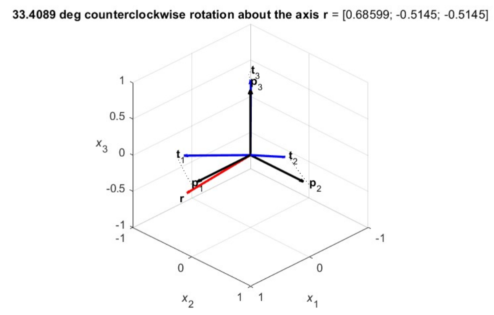
2.6 Q1.7

Given a non unit geometric vector, the angle of rotation is computed, from the Rodrigues formula, as the norm of the vector: $\rho = [0.4, -0.3, -0.3]$; $\theta = 33.4089°$
The resulting plots can be seen in Figure 6, whereas the rotation matrix of a frame $< v >$ with respect to a generic frame $< w >$ computed by using the Rodrigues formula is the following:

$$ {}^w_vR = \begin{bmatrix} 0.9125 & 0.2250 & -0.3416 \\ -0.3416 & 0.8785 & -0.3340 \\ 0.2250 & 0.4215 & 0.8785 \end{bmatrix} $$



(a) Q1.7 Initial frame.



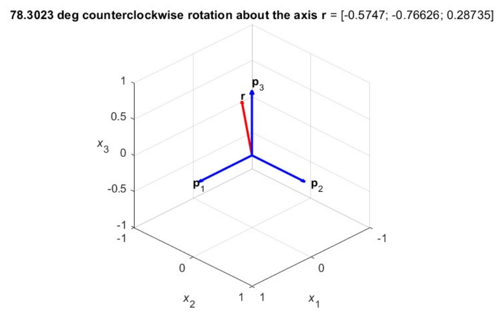
(b) Q1.7 Obtained frame.

Figure 6: Counterclockwise rotation of 33.4089° around a non unit axis.

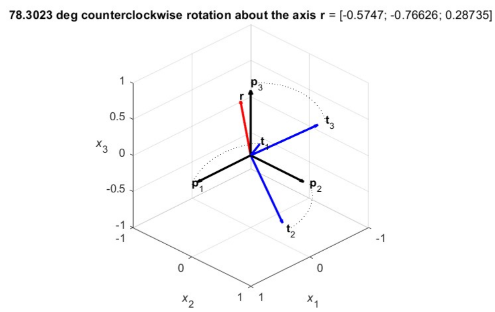
2.7 Q1.8

Given a non unit geometric vector, the angle of rotation is computed, from the Rodrigues formula, as the norm of the vector: $\rho = [-\pi/4, -\pi/3, \pi/8]$; $\theta = 78.803°$
The resulting plots can be seen in Figure 7, whereas the rotation matrix of a frame $< v >$ with respect to a generic frame $< w >$ computed by using the Rodrigues formula is the following:

$$ {}^{w}_{v}R = \begin{bmatrix} 0.4661 & 0.0697 & -0.8820 \\ 0.6325 & 0.6709 & 0.3872 \\ 0.6187 & -0.7383 & 0.2686 \end{bmatrix} $$



(a) Q1.8 Initial frame.



(b) Q1.8 Obtained frame.

Figure 7: Counterclockwise rotation of 78.803° around a non unit axis.

3 Exercise 2 - Inverse Equivalent Angle-Axis Problem

The aim of this section is to compute the equivalent angle-axis representation values, respectively the angle θ and the axis **v**, given a rotation matrix as input to a Matlab custom function.
This latter must satisfy the following conditions in order to compute the equivalent angle-axis representation values: being a square matrix of dimension three, being orthogonal and therefore with its inverse equal to the transpose, the determinant must be equal to 1.
Once all these conditions hold, we could compute θ and **v** with the following expressions:

$$\theta = acos((trace(R) - 1)/2)$$
$$\mathbf{v} = 1/sin(\theta)vex((R - R^T)/2)$$

but it is necessary to observe that the expression for the axis **v** poses an ill-defined problem; this is because the values of the angles are limited and occur in $[0, pi]$, therefore we have a not well behaved axis for the extreme values of the interval.
In order to compute the axis vector we exploit the features of the rotation matrix and its eigenvalues. It is known from the Rodrigues formula that the rotation matrix, for an angle θ equal to zero reduces to the identity matrix, therefore if we compute the product between the rotation matrix and the axis vector with the angle considered above, we obtain as a result the axis vector itself; this means that one of the eigenvalue of the rotation matrix is equal to 1, so the axis vector is computed as the eigenvector of the rotation matrix corresponding to this specific eigenvalue.

Considering two reference frames $< a >$ and $< b >$, referred to a common world coordinate system $< w >$, and their orientation with respect to the world frame $< w >$ expressed in Figure 8, we want to compute ${}^a_bR$.

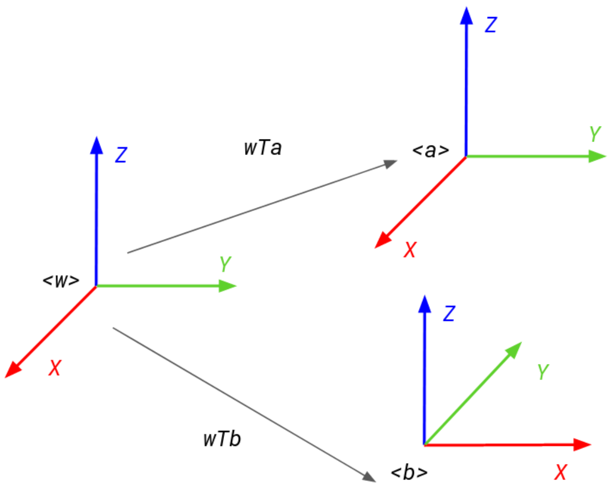


Figure 8: Exercise 2 frames

3.1 Q2.1 - Q2.2

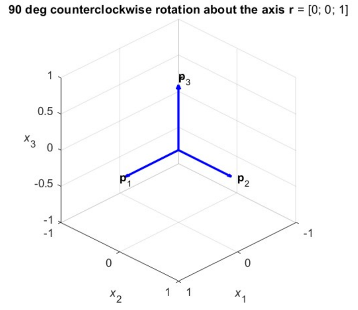
In order to compute the orientation matrix ${}^a_bR$, we can express it as a tree of frames:

$${}^a_bR = {}^a_wR\,{}^w_bR$$

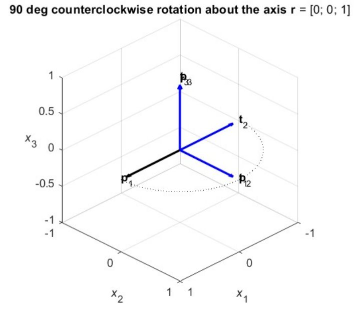
this expression obtained can be further developed:

$${}^a_bR = {}^w_aR^T\,{}^w_bR$$

Since the frames $< a >$ and $< w >$ have the same orientation, the corresponding rotation matrix is equal to the identity matrix because of what explained in the previous paragraph; whereas the rotation matrix R^w_b can be computed as a rotation of 90° around the z axis, this results in a rotation matrix R^a_b having the same form seen in section Q1.4.
The obtained plots are shown in Figure 9, while the values for the angle - axis representation are: **v** $= [0, 0, 1]$; $\theta = \pi/2$



(a) Q2.1 Initial frame.



(b) Q2.1 Obtained frame.

Figure 9: Frames related to the rotation ${}^a_bR$

3.2 Q2.3

It is possible to compute the orientation matrix ${}^c_bR$ given the following transformation matrix:

$$ {}^w_cT = \begin{bmatrix} 0.835959 & -0.283542 & -0.46986 & 0 \\ 0.271321 & 0.957764 & -0.0952472 & -1.23 \\ 0.47703 & -0.0478627 & 0.877583 & 14 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

This is because we can express the rotation matrix ${}^c_bR$ as:

$$ {}^c_bR = {}^c_wR\,{}^w_bR $$

which can be further developed into:

$$ {}^c_bR = {}^w_cR^T\,{}^w_bR $$

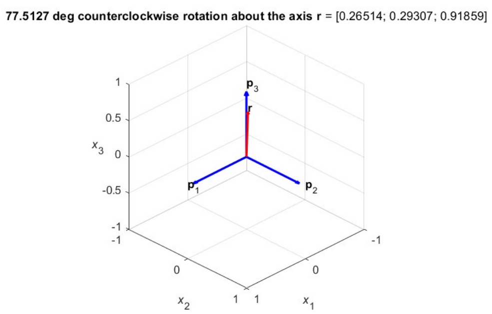
Knowing the structure of a generic transformation matrix, we can find ${}^w_cR$ as the 3x3 elements of the transformation matrix ${}^w_cT$, therefore:

$$ {}^w_cR = \begin{bmatrix} 0.835959 & -0.283542 & -0.46986 \\ 0.271321 & 0.957764 & -0.0952472 \\ 0.47703 & -0.0478627 & 0.877583 \end{bmatrix} $$

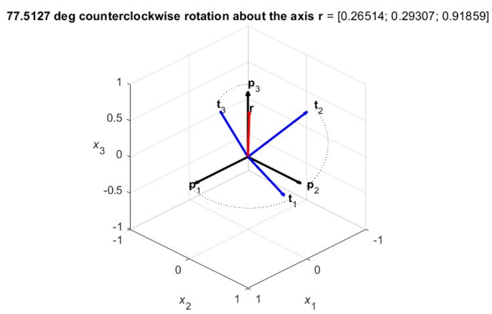
The computed orientation matrix ${}^c_bR$ is the following:

$$ {}^c_bR = \begin{bmatrix} -0.271 & 0.835 & 0.477 \\ -0.957 & -0.283 & -0.047 \\ 0.0952 & -0.469 & 0.877 \end{bmatrix} $$

The obtained plots are shown in Figure 10, while the values for the angle - axis representation are: $\mathbf{v} = [0.2651, 0.2931, 1.9186]$; $\theta = 77.5127°$.



(a) Q2.3 Initial frame.



(b) Q2.3 Obtained frame.

Figure 10: Frames related to the rotation ${}^c_bR$

4 Exercise 3 - Euler angles (Z-X-Z) vs Tait-Bryan angles (Yaw-Pitch-Roll)

Any orientation matrix can be expressed in terms of three elementary rotations in sequence. These can occur either about the axes of a fixed coordinate system (extrinsic rotations), or about the axes of a rotating coordinate system (intrinsic rotations) initially aligned with the fixed one. Then we can distinguish:

- Proper Euler angles: X-Z-X, Y-Z-Y, ...
- Tait-Bryan angles: Z-Y-X, X-Y-Z, ...

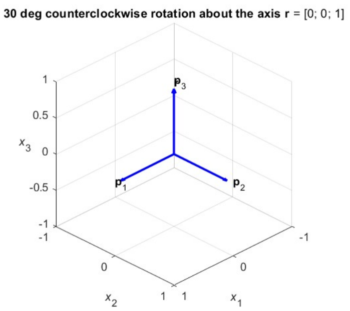
4.1 Q3.1 - Q3.2

Given two generic frames $< w >$ and $< b >$, the elementary orientation matrices for frame $< b >$ with respect to frame $< w >$, considered a generic angle θ, are defined as:

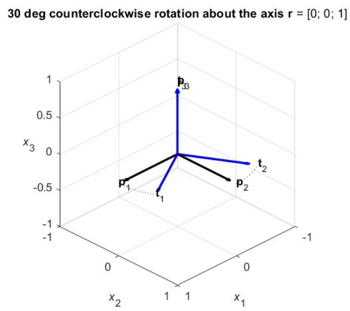
$$ {}^w_bR_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}, {}^w_bR_y = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix}, {}^w_bR_z = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Therefore, the elementary rotation of $30°$ around the z-axis is shown in Figure 11 and has the following orientation matrix:

$$ {}^w_bR_z = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



(a) Q3.1.a Initial frame.

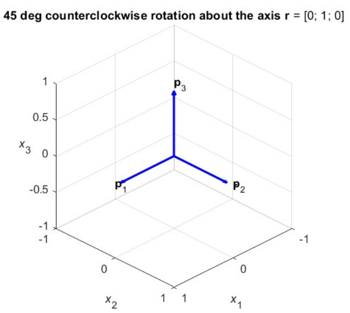


(b) Q3.1.a Obtained frame.

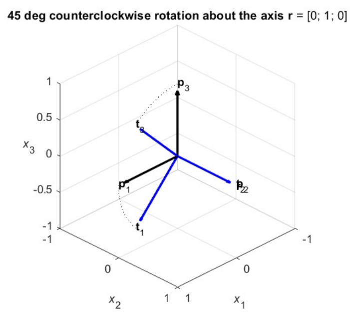
Figure 11: Rotation of 30° around the z axis

The elementary rotation of $45°$ around the y-axis is shown in Figure 12 and has the following orientation matrix:

$$ {}^w_bR_y = \begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix} $$



(a) Q3.1.b Initial frame.



(b) Q3.1.b Obtained frame.

Figure 12: Rotation of 45° around the y-axis

The elementary rotation of $15°$ around the x-axis is shown in Figure 13 and has the following orientation matrix:

$$
{}^{w}_{b}R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9659 & -0.2588 \\ 0 & 0.2588 & 0.9659 \end{bmatrix}
$$

(a) Q3.1.c Initial frame.

(b) Q3.1.c Obtained frame.

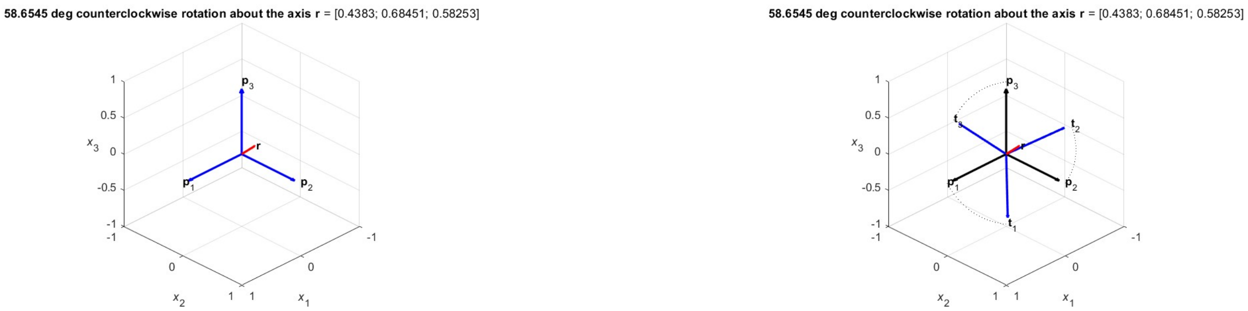
Figure 13: Rotation of 15° around the x-axis

4.2 Q3.3

To compute the z-y-x (yaw,pitch,roll) representation, it is enough to multiply the elementary rotations previously found, $R_{xyz} = R_x R_y R_z$, which correspond to the following values:

$$
R_{xyz} = \begin{bmatrix} 0.6124 & -0.3536 & 0.7071 \\ 0.6415 & 0.7450 & -0.1830 \\ -0.4621 & 0.5657 & 0.6830 \end{bmatrix}
$$

The plots obtained are shown in Figure 14, whereas the values found for the angle - axis representation are: **v** $= [0.4383, 0.6845, 0.5825]$; $\theta = 58.6545°$



(a) Q3.3 Initial frame.

(b) Q3.3 Obtained frame.

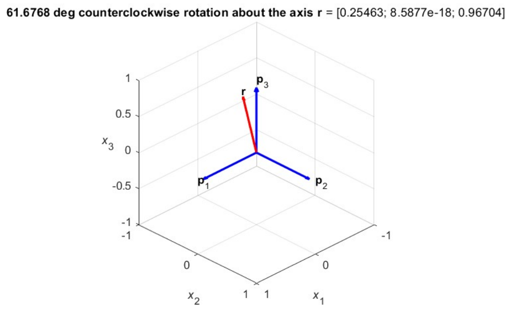
Figure 14: Rotation z-y-x (yaw,pitch,roll)

4.3 Q3.4

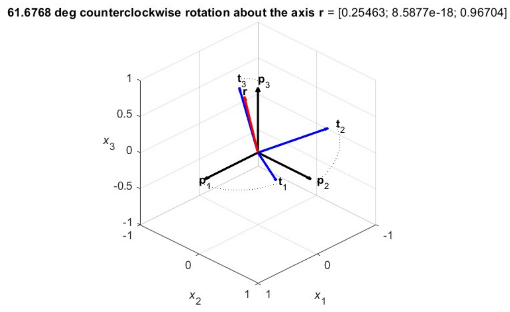
To compute the z-x-z representation, it is enough to multiply the elementary rotations previously found, $R_{zxz} = R_z R_x R_z$, which correspond to the following values:

$$
R_{zxz} = \begin{bmatrix} 0.5085 & -0.8513 & 0.1294 \\ 0.8513 & 0.4744 & -0.2241 \\ 0.1294 & 0.2241 & 0.9659 \end{bmatrix}
$$

The plots obtained are shown in Figure 15, whereas the values found for the angle - axis representation are: $\mathbf{v} = [0.2546, 0, 0.9670]$; $\theta = 61.6768°$



(a) Q3.4 Initial frame.



(b) Q3.4 Obtained frame.

Figure 15: Rotation z-x-z

5 Exercise 4 - Quaternions

The aim of this section is to express rotation matrices by using quaternions, which are a not ill-defined parametrization obtained by remapping the angles in the Rodrigues formula.
The parameters are the following:
$\mu = cos(\theta/2)$
$\epsilon = sin(\theta/2)$**h**
where θ and **h** are the angle - axis representation parameters.

5.1 Q4.1 - Q4.2

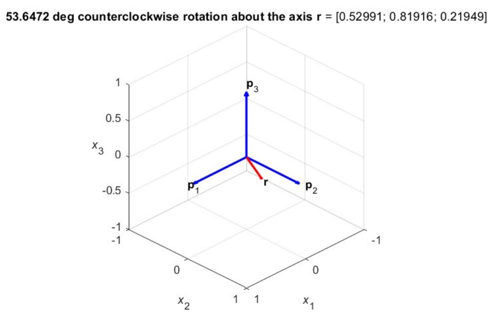
Considering the quaternion $q = 0.8924 + 0.23912i + 0.36964j + 0.099046k$ expressing how a reference frame $< b >$ is rotated with respect to $< a >$, we can compute the rotation matrix ${}^a_bR$ with the following formula:

$${}^a_bR = I_{3x3} + 2\mu[\epsilon\wedge] + 2[\epsilon\wedge]^2$$

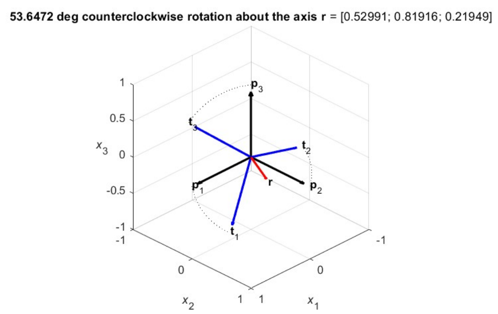
The plots obtained are shown in Figure 16 and the rotation matrix values are:

$${}^a_bR = \begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0.3536 & 0.866 & -0.3536 \\ -0.6124 & 0.5 & 0.6124 \end{bmatrix}$$

Whereas the angle-axis representation for the obtained orientation matrix is: **v** $= [0.5299, 0.8192, 0.2195]$; $\theta = 53.6472°$.



(a) Q4.1 Initial frame.



(b) Q4.1 Obtained frame.

Figure 16: Frames related to the rotation associated to the quaternion q