

Monads and their applications 5

Exercise 1.

Let (S, σ) be a well-pointed endofunctor of \mathcal{D} . Let \mathcal{C} be a finitely cocomplete category and let $F: \mathcal{D} \rightarrow \mathcal{C}$ be left adjoint to $U: \mathcal{C} \rightarrow \mathcal{D}$. Show that the endofunctor (S', σ') of \mathcal{C} defined by the pushout diagram

$$\begin{array}{ccc} FU & \xrightarrow{F\sigma U} & FSU \\ \varepsilon \downarrow & & \downarrow \\ \text{id} & \xrightarrow{\sigma'} & S' \end{array}$$

in $[\mathcal{C}, \mathcal{C}]$ is a well-pointed endofunctor and that there is an induced diagram

$$\begin{array}{ccc} (S', \sigma')\text{-}\mathbf{Alg} & \xrightarrow{\bar{U}} & (S, \sigma)\text{-}\mathbf{Alg} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{U} & \mathcal{D} \end{array}$$

which is a (strict) pullback diagram of categories.

Exercise 2.

Let $((S_i, \sigma_i))_{i \in I}$ be a family of accessible well-pointed endofunctors of a locally presentable category \mathcal{C} . Let S be the colimit in $[\mathcal{C}, \mathcal{C}]$ of the “star-shaped” diagram $\sigma_i: \text{id}_{\mathcal{C}} \Rightarrow S_i$ (that is, a star with center $\text{id}_{\mathcal{C}}$). Write σ for the induced composite $\text{id} \Rightarrow S_i \Rightarrow S$ (which is by construction independent of i). Show that (S, σ) is an accessible well-pointed endofunctor and that $(S, \sigma)\text{-}\mathbf{Alg}$ is the intersection $\bigcap_{i \in I} (S_i, \sigma_i)\text{-}\mathbf{Alg}$.

Exercise 3.

Let $k: a \rightarrow b$ be a morphism of a category \mathcal{C} . An object $c \in \mathcal{C}$ is called *orthogonal* to k if for any $f: a \rightarrow c$, there exists a unique dashed arrow making the triangle

$$\begin{array}{ccc} a & \xrightarrow{f} & c \\ k \downarrow & \nearrow \text{dashed} & \\ b & & \end{array}$$

commutative. The full subcategory of objects orthogonal to k is denoted by $\{k\}^\perp \subseteq \mathcal{C}$. Similarly, given a set Σ of morphisms in \mathcal{C} , we write $\Sigma^\perp \subseteq \mathcal{C}$ for the class of objects which are orthogonal to all the morphisms in Σ .

- (a) Let \mathcal{C} be locally presentable and let $k: a \rightarrow b$ be a morphism in \mathcal{C} . This defines a functor $k: [1] \rightarrow \mathcal{C}$, where $[1] = \{0 \rightarrow 1\}$ denotes the category consisting of a single non-trivial morphism. Show that the right adjoint of the induced adjunction $\text{Lan}_Y k: \mathbf{Set}^{[1]^{\text{op}}} \rightleftarrows \mathcal{C}: \tilde{k}$ is accessible.
- (b) Show that there exists an accessible well-pointed endofunctor (S_k, σ_k) of \mathcal{C} such that $(S_k, \sigma_k)\text{-Alg}$ is equal to the full subcategory $\{k\}^\perp \subseteq \mathcal{C}$ of objects orthogonal to k . (Hint: apply Exercise 1 to the adjunction of (a).)
- (c) Let Σ be a set of morphisms in \mathcal{C} . Show that there exists an accessible well-pointed endofunctor of \mathcal{C} whose category of algebras is $\Sigma^\perp \subseteq \mathcal{C}$.

Exercise 4.

Let \mathcal{A} be a small category and consider a set $(D^k: \mathcal{I}_k \rightarrow \mathcal{A})_{k \in K}$ of diagrams in \mathcal{A} . For each diagram D^k , fix a cocone $\kappa_i: D_i^k \rightarrow a_k$ in \mathcal{A} . Let $\mathcal{C} \subseteq [\mathcal{A}^{\text{op}}, \mathbf{Set}]$ be the full subcategory of presheaves $F: \mathcal{A}^{\text{op}} \rightarrow \mathbf{Set}$ with the property that $F\kappa_i: Fa_k \rightarrow FD_i^k$ is a limit cone for each $k \in K$. Show that there exists a set Σ of morphisms in $[\mathcal{A}^{\text{op}}, \mathbf{Set}]$ such that $\mathcal{C} = \Sigma^\perp$.

Exercise 5. ??

From the lecture, we know that the category \mathcal{C} in Exercise 4 is reflective. Show that the composite

$$\mathcal{A} \xrightarrow{Y} [\mathcal{A}^{\text{op}}, \mathbf{Set}] \longrightarrow \mathcal{C}$$

of the Yoneda embedding and the left adjoint of the inclusion is the universal functor to a cocomplete category which sends the given cones to colimit cones.