

Monads and their applications

Dr. Daniel Sch  ppi's course lecture notes

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Introduction

Chapter 1

Categorical preliminaries

Definition 1.0.1 (Categories). A category \mathbf{C} consists of:

1. a collection of objects $\text{Ob}(\mathbf{C})$;
2. a collection of arrows $\text{Ar}(\mathbf{C})$;
3. two maps $\text{dom}, \text{cod}: \text{Ar}(\mathbf{C}) \rightarrow \text{Ob}(\mathbf{C})$;
4. a map $\text{id}_-: \text{Ob}(\mathbf{C}) \rightarrow \text{Ar}(\mathbf{C})$ with $\text{dom}(\text{id}_c) = c = \text{cod}(\text{id}_c)$;
5. for every $f, g \in \text{Ar}(\mathbf{C})$ such that $\text{cod}(f) = \text{dom}(g)$ a unique composite morphism gf such that $\text{cod}(gf) = \text{cod}(g)$, $\text{dom}(gf) = \text{dom}(f)$.

This data has to satisfy the following axioms

1. given $f \in \text{Ar}(\mathbf{C})$, $c = \text{dom}(f)$ and $c' = \text{cod}(f)$, $\text{id}_{c'} f = f = f \text{id}_c$, that is the composition is unital;
2. given a composable triple $f, g, h \in \text{Ar}(\mathbf{C})$, $h(gf) = (hg)f$, that is the composition is associative.

An arrow f such that $c = \text{dom}(f)$ and $c' = \text{cod}(f)$ is denoted $f: c \rightarrow c'$.

Definition 1.0.2 (Functors).

Definition 1.0.3 (Full functors, faithful functor).

Definition 1.0.4 (Natural transformations).

Definition 1.0.5 (Equivalent functors).

Definition 1.0.6 (Representable Functors).

Definition 1.0.7 (Whiskering).

Definition 1.0.8 (Horizontal and vertical composition of nat.transf.).

Definition 1.0.9 (adjunctions).

Lemma 1.0.10 (Yoneda).

Proof.

□

Chapter 2

Monads and algebras

2.1 Introduction

Throughout mathematics we encounter structures defined by some action morphisms. Here we give some examples.

Example 2.1.1. Given a group G , we may consider a G -set X described by an action map $G \times X \rightarrow X$.

Example 2.1.2. Given an abelian group M and a ring R , we can get an R -module M by fixing a group homomorphism $R \otimes_{\mathbb{Z}} M \rightarrow M$.

Example 2.1.3. Given a monoid M in **Set**, we get a map $\prod_{k=1}^n M \rightarrow M$, $(m_1, \dots, m_n) \mapsto ((\dots((m_1 m_2) m_3) \dots) m_{n-1}) m_n$. This induces an action map from $W(M) = \prod_{n \in \mathbb{N}} \prod_{k=1}^n M$, the set of words on M , to M .

Example 2.1.4. Given a set X , let $\mathcal{U}X$ be the set of ultrafilters on it. Any compact T2 topology on X allows us to see each ultrafilter as a system of neighborhoods of a unique point in X , hence it gives us a unique map $\mathcal{U}X \rightarrow X$ sending each ultrafilter to the respective point.

Example 2.1.5. Given a directed graph $D = (V, E, E \stackrel{s}{\dashrightarrow}_t V)$, we can create its free category FD , where the objects are the vertices and $FD(v, w) = \{\text{finite paths } v \rightarrow \dots \rightarrow w\}$. We set id_v to be the path of length 0, while composition is just the concatenation of paths.

In particular, if D is the directed graph with $V = \{0, \dots, n\}$ and an edge $j \rightarrow k$ if and only if $k = j + 1$, we have $FD \cong [n]$.

If $D = \{*\}$ and $E = \{* \rightarrow *\}$, then $FD(*, *) \cong \mathbb{N}$.

Given a small category \mathbf{C} , we may consider the underlying graph $U \mathbf{C} = D$ with $V = \text{Ob}(\mathbf{C})$, $E = \text{Ar}(\mathbf{C})$, $s = \text{dom}$ and $t = \text{cod}$. We get then an action map $UFU \mathbf{C} \rightarrow U \mathbf{C}$ sending a finite path to its composite. This map is a morphism of directed graphs.

Notice that we always have a category \mathbf{C} and some functor $T: \mathbf{C} \rightarrow \mathbf{C}$ with an action map $T\mathbf{C} \rightarrow \mathbf{C}$. How can we see all of these examples as specific instances of a general phenomenon?

Definition 2.1.6. A monad on a category \mathbf{C} is a triple (T, μ, η) where $T: \mathbf{C} \rightarrow \mathbf{C}$ is a functor, while $\mu: T^2 \Rightarrow T$ and $\eta: \text{id}_{\mathbf{C}} \Rightarrow T$ are natural transformations such that the following diagrams commute:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \Downarrow \mu T & & \Downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array} \quad \begin{array}{ccccc} T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T \\ & \searrow \text{id}_T & \Downarrow \mu & \swarrow \text{id}_T & \\ & & T & & \end{array}$$

μ is called the multiplicative map, while η is the unit of T .

The commutativity of the first diagram is equivalent to stating that the following two diagrams are equal:

$$\begin{array}{ccc} & \mathbf{C} & \xrightarrow{T} \mathbf{C} \\ & \uparrow T & \searrow T \\ \mathbf{C} & \xrightarrow{T} & \mathbf{C} \end{array} \quad \begin{array}{ccc} & \mathbf{C} & \xrightarrow{T} \mathbf{C} \\ & \uparrow T & \searrow T \\ \mathbf{C} & \xrightarrow{T} & \mathbf{C} \end{array} = \begin{array}{ccc} & \mathbf{C} & \xrightarrow{T} \mathbf{C} \\ & \uparrow T & \searrow T \\ \mathbf{C} & \xrightarrow{T} & \mathbf{C} \end{array}$$

On the other hand, the second diagram can be rephrased as follows:

$$\begin{array}{ccc} & \mathbf{C} & \\ \text{id}_{\mathbf{C}} \nearrow & \uparrow \eta & \searrow T \\ \mathbf{C} & \xrightarrow{T} & \mathbf{C} \end{array} = T \left(\begin{array}{c} \mathbf{C} \\ \text{id}_{\mathbf{C}} \end{array} \right) T = \begin{array}{ccc} & \mathbf{C} & \\ T \nearrow & \uparrow \mu & \searrow T \\ \mathbf{C} & \xrightarrow{T} & \mathbf{C} \end{array}$$

A monad naturally defines other algebraic structures, which we now introduce.

Definition 2.1.7. Given a monad (T, μ, η) , a T -algebra or T -module is a pair (a, α) , where $a \in \text{Ob}(\mathbf{C})$ and $\alpha: Ta \rightarrow a$ is such that the following diagrams commute:

$$\begin{array}{ccc} T^2a & \xrightarrow{T\alpha} & Ta \\ \downarrow \mu_a & & \downarrow \alpha \\ Ta & \xrightarrow{\alpha} & a \end{array} \quad \begin{array}{ccc} a & \xrightarrow{\eta_a} & Ta \\ \downarrow \text{id}_a & & \downarrow \alpha \\ a & & a \end{array}$$

Definition 2.1.8. A morphism of T -algebras $(a, \alpha) \rightarrow (b, \beta)$ is a morphism $f: a \rightarrow b$ such that the following diagram commutes:

$$\begin{array}{ccc} Ta & \xrightarrow{Tf} & Tb \\ \downarrow \alpha & & \downarrow \beta \\ a & \xrightarrow{f} & b \end{array}$$

T -algebras form a category $T\text{-}\mathbf{Alg}$, which has a natural forgetful functor $U^T: T\text{-}\mathbf{Alg} \rightarrow \mathbf{C}$.

We now show how to recover the examples previously given with this language.

Example 2.1.9.

$$\begin{aligned} T &= G \times -: \mathbf{Set} \rightarrow \mathbf{Set} \\ \mu_A &: G \times (G \times A) \rightarrow G \times A \\ &\quad (g, (h, a)) \mapsto (gh, a) \\ \eta_A &: A \rightarrow G \times A \\ &\quad a \mapsto (e, a) \end{aligned}$$

is a monad and (A, α) is a T -algebra if and only if A is a G -set. It follows that $T\text{-}\mathbf{Alg} \cong G\text{-}\mathbf{Set}$.

Example 2.1.10. Given a ring R , $T = R \otimes_{\mathbb{Z}} -: \mathbf{Ab} \rightarrow \mathbf{Ab}$ is a monad when considered with the following natural transformations:

$$\begin{aligned} \mu_- &: R \otimes_{\mathbb{Z}} (R \otimes_{\mathbb{Z}} -) \cong (R \otimes_{\mathbb{Z}}) \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \\ \eta_- &: - \cong \mathbb{Z} \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \end{aligned}$$

We have that $(R \otimes_{\mathbb{Z}} -)\text{-}\mathbf{Alg} \cong \mathbf{Mod}_R$.

Example 2.1.11. Consider $W: \mathbf{Set} \rightarrow \mathbf{Set}$ given by $WX = \coprod_{n \in \mathbb{N}} \prod_{k=1}^n X$. Multiplication $\mu_X: WWX \rightarrow WX$ is given by concatenation of words, while the unit $\eta_X: X \rightarrow WX$ is just $x \mapsto (x)$. With this, $W\text{-}\mathbf{Alg} \cong \mathbf{Mon}(\mathbf{Set})$.

Example 2.1.12. The functor \mathcal{U} defined in Example 2.1.4, equipped with suitable natural transformations, is a monad on \mathbf{Set} and $\mathcal{U}\text{-}\mathbf{Alg} \cong \mathbf{CHTop}$, the category of compact T2 spaces.

Example 2.1.13. The free-forgetful adjunction $F \dashv U$ between categories and directed graphs induces a monad on the latter, with $UF\text{-}\mathbf{Alg} \cong \mathbf{Cat}$.

2.2 Monadic functors

Now that we have introduced these structures, our aim is to characterize *monadic functors*, that is functors $U: \mathbf{A} \rightarrow \mathbf{C}$ which are equivalent to $U^T: T\text{-}\mathbf{Alg} \rightarrow \mathbf{C}$ for some monad (T, μ, η) on \mathbf{C} .

First of all, notice that U^T is faithful by construction, hence U must be faithful, but more is true.

Lemma 2.2.1. The functor U^T is conservative, that is if $U^T f$ is an isomorphism then f is an isomorphism of T -algebras.

Proof. Suppose that g is the inverse of $f: a \rightarrow b$ and f is a morphism $(a, \alpha) \rightarrow (b, \beta)$. We only need to prove that the square on the left commutes, that is $g\beta = \alpha Tg$:

$$\begin{array}{ccccc} Tb & \xrightarrow{Tg} & Ta & \xrightarrow{Tf} & Tb \\ \downarrow \beta & & \downarrow \alpha & & \downarrow \beta \\ b & \xrightarrow{g} & a & \xrightarrow{f} & b \end{array}$$

We see that $fg\beta = \beta$ and $f\alpha Tg = \beta T f Tg = \beta T(fg) = \beta T \text{id}_b = \beta$, hence the thesis. \square

Remark 2.2.2. Notice that the forgetful functor $U: \mathbf{Top} \rightarrow \mathbf{Set}$ can't be monadic since it does not reflect isomorphisms. However, if we restrict it to the full subcategory of \mathbf{Top} spanned by compact Hausdorff spaces we indeed obtain a monadic functor.

Proposition 2.2.3. The functor $U^T: T\text{-}\mathbf{Alg} \rightarrow \mathbf{C}$ has a left adjoint $F^T: \mathbf{C} \rightarrow T\text{-}\mathbf{Alg}$ such that $F^T c = (Tc, \mu_c)$, $F^T f: (Tc, \mu_c) \xrightarrow{Tf} (Td, \mu_d)$ and $U^T F^T = T$. Furthermore, the unit of this adjunction is given by $\gamma_c = \eta_c: c \rightarrow U^T F^T c = Tc$ and the counit has components $\epsilon_{(a, \alpha)} = \alpha: (Ta, \mu_a) \rightarrow (a, \alpha)$.

Proof. (i) To show that (Tc, μ_c) is a T -algebra we need the following diagrams to be commutative.

$$\begin{array}{ccc} T^3 c & \xrightarrow{T\mu_c} & T^2 c \\ \mu_{Tc} \downarrow & & \downarrow \mu_c \\ T^2 c & \xrightarrow{\mu_c} & Tc \end{array} \quad \begin{array}{ccc} Tc & \xrightarrow{\eta_{Tc}} & T^2 c \\ & \searrow & \downarrow \mu_c \\ & & Tc \end{array}$$

These are exactly the associativity and one of the unit laws for (T, μ, η) .

(ii) For every $f: c \rightarrow c'$, Tf is a morphism of algebras $(Tc, \mu_c) \rightarrow (Tc', \mu_{c'})$. The diagram

$$\begin{array}{ccc} T^2 c & \xrightarrow{T^2 f} & T^2 c' \\ \mu_c \downarrow & & \downarrow \mu_{c'} \\ Tc & \xrightarrow{Tf} & Tc' \end{array}$$

is commutative because of the naturality of μ . Hence F^T is defined on morphisms. It is a functor by the functoriality of T .

(iii) The unit is natural by assumption. We claim that $\epsilon_{(a, \alpha)} = \alpha$ is a morphism of algebras

$$F^T U^T(a, \alpha) = F^T a = (Ta, \mu_a) \rightarrow \text{id}_{T\text{-}\mathbf{Alg}}(a, \alpha) = (a, \alpha)$$

and ϵ is a natural transformation $F^T U^T \Rightarrow \text{id}_{T\text{-}\mathbf{Alg}}$. Let's check it. We know that α is a morphism of algebras if and only if

$$\begin{array}{ccc} T^2 a & \xrightarrow{T\alpha} & T a \\ \mu_a \downarrow & & \downarrow \alpha \\ T a & \xrightarrow{\alpha} & a \end{array}$$

is commutative. But this is one of the two T -algebra axioms! Moreover, to prove that ϵ is natural, we need to show that

$$\begin{array}{ccc} (T a, \mu_a) & \xrightarrow{\alpha = \epsilon_{(a, \alpha)}} & (a, \alpha) \\ T f \downarrow & & \downarrow f \\ (T b, \mu_b) & \xrightarrow{\beta = \epsilon_{(b, \beta)}} & (b, \beta) \end{array}$$

is commutative, but this is the axiom for f to be a morphism of T -algebras!

- (iv) It remains to check the two triangular identities $\epsilon F^T \circ F^T \eta = \text{id}_{F^T}$ and $U^T \epsilon \circ \eta U^T = \text{id}_{U^T}$. These are to be checked on the components at c and (a, α) , respectively.

$$\begin{array}{ccc} (T c, \mu_c) & \xrightarrow{T \eta_c} & (T^2 c, \mu_{T c}) \\ & \searrow & \downarrow \mu_{T c} \\ & & (T c, \mu_c) \end{array} \quad \begin{array}{ccc} a & \xrightarrow{\eta_a} & T a \\ & \searrow & \downarrow \alpha \\ & & a \end{array}$$

The commutativity of these two diagrams is ensured by the second unit law for a monad and the unit law for the T -algebra (a, α) , respectively. \square

Definition 2.2.4. Algebras of the form $(T c, \mu_c)$ are called free T -algebras.

Thanks to the proposition above we can prove that, given a monad T we can always find an adjunction that generates it. Actually, the converse holds too.

Proposition 2.2.5. If $U: \mathbf{D} \rightarrow \mathbf{C}$ has a left adjoint F with unit η and counit ϵ , then $(U F, U \epsilon F, \eta)$ is a monad on \mathbf{C} . Also, if (T, μ, η) is a monad on \mathbf{C} , then $(U^T F^T, U^T \epsilon F^T, \eta) = (T, \mu, \eta)$.

Proof. Let us check the axioms. First of all, the associativity holds due to the following equations.

$$\begin{array}{c} \mathbf{C} \\ \begin{array}{ccc} & \nearrow U & \searrow F \\ & \epsilon \Downarrow & \\ & \nearrow U & \searrow F \end{array} \\ \mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{\quad \quad \quad} \mathbf{D} \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{C} \end{array}$$

$$\begin{array}{c}
= \\
\begin{array}{c}
\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{C} \\
\begin{array}{ccc}
& \mathbf{C} & \\
U \nearrow & \Downarrow \epsilon & \searrow F \\
\mathbf{D} & & \mathbf{D}
\end{array}
\end{array}
\end{array}
=
\begin{array}{c}
\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{C} \\
\begin{array}{ccc}
& \mathbf{C} & \\
U \nearrow & \Downarrow \epsilon & \searrow F \\
\mathbf{D} & & \mathbf{D}
\end{array}
\end{array}$$

Unit laws:

$$\begin{array}{c}
\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{C} \\
\begin{array}{ccc}
& \mathbf{C} & \\
\eta \Downarrow & U \nearrow & \searrow F \\
\mathbf{D} & & \mathbf{D}
\end{array}
\end{array}$$

is equal to 1_{UF} , since $\epsilon F \circ F \eta = 1_F$ by one of the triangular identities of the adjunction $F \dashv U$. Furthermore,

$$\begin{array}{c}
\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{C} \\
\begin{array}{ccc}
& \mathbf{C} & \\
U \nearrow & \Downarrow \epsilon & \searrow F \\
\mathbf{D} & & \mathbf{D}
\end{array}
\end{array}$$

is equal to 1_{UF} . This follows from the explicit description of the unit and the counit of the adjunction $F^T \dashv U^T$, in fact

$$U^T \epsilon F^T c = U^T \epsilon_{(Tc, \eta c)} = \mu_c$$

□

Example 2.2.6 (Interesting adjunction, boring monad). Let us consider the adjunction $U: \mathbf{Top} \rightleftarrows \mathbf{Set}: \text{Disc} =: F$, whose left adjoint assigns to every set X the discrete topological space $FX = (X, 2^X)$. It's immediate to see that $UF X = X$, hence $UF = \text{id}_{\mathbf{Set}}$. How many natural transformations $\text{id}_{\mathbf{Set}} = UF \xrightarrow{\alpha} UF = \text{id}_{\mathbf{Set}}$ are there? We know that $\text{id}_{\mathbf{Set}} \cong \text{hom}(*, -)$, so $\text{Nat}(\text{id}_{\mathbf{Set}}, \text{id}_{\mathbf{Set}}) \cong \text{Nat}(\text{hom}(*, -), \text{hom}(*, -)) \cong \text{hom}(*, *) = \{\text{id}_*\}$ by Yoneda, hence $\alpha = \text{id}$. Therefore $(UF, U\epsilon F, \eta) = (\text{id}_{\mathbf{Set}}, \text{id}, \text{id})$

Example 2.2.7. If S is a set, $\mathbf{Set}(S, -): \mathbf{Set} \rightarrow \mathbf{Set}$ is right adjoint to $S \times -: \mathbf{Set} \rightarrow \mathbf{Set}$, so we get a monad $X \mapsto \mathbf{Set}(S, S \times X)$. This is called *the state monad* and is important in Computer Science.

There is always a comparison morphism $\mathbf{D} \xrightarrow{\bar{U}} UF\text{-Alg}$ s.t.

$$\begin{array}{ccc}
\mathbf{D} & \xrightarrow{\bar{U}} & UF\text{-Alg} \\
U \searrow & & \swarrow U^{UF} \\
& \mathbf{C} &
\end{array}$$

commutes. We set $\bar{U}f = (Ud, UFUd \xrightarrow{U\epsilon_d} Ud) = (Ud, U\epsilon_d)$. More generally, for a given functor $\mathbf{D} \xrightarrow{G} \mathbf{C}$ we can ask what do we need to get a lift $\bar{G}: \mathbf{D} \rightarrow T\text{-}\mathbf{Alg}$.

Definition 2.2.8. A T -action on a functor $\mathbf{D} \xrightarrow{G} \mathbf{C}$ is a natural transformation $TG \xrightarrow{\gamma} G$ s.t.

$$\begin{array}{ccc} T^2G & \xrightarrow{T\gamma} & TG \\ \mu G \downarrow & & \downarrow \gamma \\ TG & \xrightarrow{\gamma} & G \end{array} \quad \begin{array}{ccc} G & \xrightarrow{\eta G} & TG \\ & \searrow & \downarrow \gamma \\ & & G \end{array}$$

commute.

Definition 2.2.9. A morphism of T -actions is a natural transformation $\varphi: G \Rightarrow K$ s.t.

$$\begin{array}{ccc} TG & \xrightarrow{T\varphi} & TK \\ \gamma \downarrow & & \downarrow \alpha \\ G & \xrightarrow{\varphi} & K \end{array}$$

commutes.

Up to size, we get a category $T\text{-Act}(\mathbf{D})$, defined as follows.

Definition 2.2.10. Given a monad (T, μ, η)

Chapter 3

Beck's monadicity theorem

Theorem 3.0.1. contenuto...

Chapter 4

Monads in 2-category theory

Chapter 5

Monads in ∞ -category theory