Monads and their applications

Dr. Daniel Schäppi's course lecture notes

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Lemma 1.0.9 (Yoneda).

Proof. □

Chapter 2

Monads and algebras

Throughout mathematics we encounter structures defined by some action morphisms. Here we give some examples.

Example 2.0.1. Given a group G, we may consider a G-set X described by an action map $G \times X \to X$.

Example 2.0.2. Given an abelian group M and a ring R, we can get an R-module M by fixing a group homomorphism $R \otimes_{\mathbb{Z}} M \to M$.

Example 2.0.3. Given a monoid M in Set, we get a map $\Pi_{k=1}^{n} M \to M$, $(m_1, \ldots, m_n) \mapsto ((\ldots((m_1 m_2) m_3) \ldots) m_{n-1}) m_n$. This induces an action map from $W(M) = \amalg_{n \in \mathbb{N}} \Pi_{k=1}^{n} M$, the set of words on M, to M.

Example 2.0.4. Given a set X, let $\mathcal{U}X$ be the set of ultrafilters on it. Any compact T2 topology on X allows us to see each ultrafilter as a system of neighborhoods of a unique point in X, hence it gives us a unique map $\mathcal{U}X \to X$ sending each ultrafilter to the respective point.

Example 2.0.5. Given a directed graph $D = (V, E, E \overset{s}{\underset{t}{\rightrightarrows}} V)$, we can create its free category FD, where the objects are the vertices and $FD(v, w) = \{\text{finite paths } v \to \ldots \to w\}$. We set Id_v to be the path of length 0, while composition is just the concatenation of paths.

2.1 Monads

Definition 2.1.1 (monad)**.** [...]

$$
\begin{tikzcd}
& \mathsf{C} \arrow[r, "T"] \arrow[drr, bend right, "T"'] \arrow[d, phantom, "\Downarrow\mu"] & \mathsf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathsf{C} \arrow[ur, "T"] \arrow[rrr, "T"'] & \phantom{\mathsf{C}} & \phantom{\mathsf{C}} & \mathsf{C}
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{C} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \mathsf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathsf{C} \arrow[ur, "T"] \arrow[urr, bend right, "T"] \arrow[rrr, "T"'] & \phantom{\mathsf{C}} & \phantom{\mathsf{C}} & \mathsf{C}
\end{tikzcd}
$$

2.2 Algebras

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Chapter 3

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Chapter 4

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Chapter 5

Monads in ∞-category theory