Monads and their applications

Dr. Daniel Schäppi's course lecture notes

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Introduction

Chapter 1

Categorical preliminaries

Definition 1.0.1 (Categories)**.** A category $\mathbf{C}$ consists of:

1. a collection of objects $\mathrm{Ob}(\mathbf{C})$;
2. a collection of arrows $\mathrm{Ar}(\mathbf{C})$;
3. two maps $\mathrm{dom}, \mathrm{cod} \colon \mathrm{Ar}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$;
4. a map $\mathrm{id}_- \colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ar}(\mathbf{C})$ with $\mathrm{dom}(\mathrm{id}_c) = c = \mathrm{cod}(\mathrm{id}_c)$;
5. for every $f, g \in \mathrm{Ar}(\mathbf{C})$ such that $\mathrm{cod}(f) = \mathrm{dom}(g)$ a unique composite morphism gf such that $\mathrm{cod}(gf) = \mathrm{cod}(g)$, $\mathrm{dom}(gf) = f$.

This data has to satisfy the following axioms

1. given $f \in \mathrm{Ar}(\mathbf{C})$, $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$, $\mathrm{id}_{c'} f = f = \mathrm{id}_c$, that is the composition is unital;
2. given a composable triple $f, g, h \in \mathrm{Ar}(\mathbf{C})$, $h(gf) = (hg)f$, that is the composition is associative.

An arrow f such that $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$ is denoted $f \colon c \to c'$.

Definition 1.0.2 (Functors)**.**

Definition 1.0.3 (Full functors, faithful functor)**.**

Definition 1.0.4 (Natural transformations)**.**

Definition 1.0.5 (Equivalent functors)**.**

Definition 1.0.6 (Representable Functors)**.**

Definition 1.0.7 (Whiskering)**.**

Definition 1.0.8 (Horizontal and vertical composition of nat.transf.)**.**

Definition 1.0.9 (adjunctions)**.**

Lemma 1.0.10 (Yoneda)**.**

Proof. □

Chapter 2

Monads and algebras

2.1 Introduction

Throughout mathematics we encounter structures defined by some action morphisms. Here we give some examples.

Example 2.1.1. Given a group G, we may consider a G-set X described by an action map $G \times X \to X$.

Example 2.1.2. Given an abelian group M and a ring R, we can get an R-module M by fixing a group homomorphism $R \otimes_{\mathbb{Z}} M \to M$.

Example 2.1.3. Given a monoid M in **Set**, we get a map $\Pi_{k=1}^{n} M \to M$, $(m_1, \ldots, m_n) \mapsto ((\ldots((m_1 m_2) m_3) \ldots) m_{n-1}) m_n$. This induces an action map from $W(M) = \amalg_{n \in \mathbb{N}} \Pi_{k=1}^{n} M$, the set of words on M, to M.

Example 2.1.4. Given a set X, let $\mathcal{U}X$ be the set of ultrafilters on it. Any compact T2 topology on X allows us to see each ultrafilter as a system of neighborhoods of a unique point in X, hence it gives us a unique map $\mathcal{U}X \to X$ sending each ultrafilter to the respective point.

Example 2.1.5. Given a directed graph $D = (V, E, E \overset{s}{\underset{t}{\rightrightarrows}} V)$, we can create its free category FD, where the objects are the vertices and $FD(v, w) = \{\text{finite paths } v \to \ldots \to w\}$. We set id_v to be the path of length 0, while composition is just the concatenation of paths.

In particular, if D is the directed graph with $V = \{0, \ldots, n\}$ and an edge $j \to k$ if and only if $k = j + 1$, we have $FD \cong [n]$.

If $D = \{*\}$ and $E = \{* \to *\}$, then $FD(*, *) \cong \mathbb{N}$.

Given a small category $\mathbf{C}$, we may consider the underlying graph $U\,\mathbf{C} = D$ with $V = \mathrm{Ob}(\mathbf{C})$, $E = \mathrm{Ar}(\mathbf{C})$, $s = \mathrm{dom}$ and $t = \mathrm{cod}$. We get then an action map $UFU\,\mathbf{C} \to U\,\mathbf{C}$ sending a finite path to its composite. This map is a morphism of directed graph.

How can we see all of these examples as specific instances of a general phenomenon?

Notice that we always have a category $\mathbf{C}$ and some functor $T\colon \mathbf{C} \to \mathbf{C}$ with an action map $T\,\mathbf{C} \to \mathbf{C}$.

Definition 2.1.6. A monad on a category $\mathbf{C}$ is a triple (T, μ, η) where $T\colon \mathbf{C} \to \mathbf{C}$ is a functor, while $\mu\colon T^2 \Rightarrow T$ and $\eta\colon \mathrm{id}_{\mathbf{C}} \Rightarrow T$ are natural transformations such that the following diagrams commute:

$$
\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "\mathrm{id}_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "\mathrm{id}_T"] \\
& T &
\end{tikzcd}
$$

μ is called the multiplicative map, while η is the unit of T.

The commutativity of the first diagram is equivalent to stating that the following two diagrams are equal:

$$
\begin{tikzcd}
& \mathbf{C} \arrow[r, "T"] \arrow[drr, bend right, "T"'] \arrow[d, phantom, "\Downarrow\mu"] & \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathbf{C} \arrow[ur, "T"] \arrow[rrr, "T"'] & {} & {} & \mathbf{C}
\end{tikzcd}
=
\begin{tikzcd}
& \mathbf{C} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathbf{C} \arrow[ur, "T"] \arrow[urr, bend right, "T"'] \arrow[rrr, "T"'] & {} & {} & \mathbf{C}
\end{tikzcd}
$$

On the other hand, the second diagram can be rephrased as follows:

$$
\begin{tikzcd}
& \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\eta\quad\Downarrow\mu"] & \\
\mathbf{C} \arrow[ur, bend left=50, "\mathrm{id}_{\mathbf{C}}"] \arrow[ur, bend right, "T"'] \arrow[rr, "T"'] & {} & \mathbf{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{C} \arrow[d, bend right, "T"'] \arrow[d, bend left, "T"] \arrow[d, phantom, "="] \\
\mathbf{C}
\end{tikzcd}
=
\begin{tikzcd}
& \mathbf{C} \arrow[dr, bend right, "T"'] \arrow[dr, bend left=50, "\mathrm{id}_{\mathbf{C}}"] \arrow[d, phantom, "\Downarrow\mu\quad\Downarrow\eta"] & \\
\mathbf{C} \arrow[ur, "T"] \arrow[rr, "T"'] & {} & \mathbf{C}
\end{tikzcd}
$$

A monad naturally defines other algebraic structures, which we now introduce.

Definition 2.1.7. Given a monad (T, μ, η), a T-algebra or T-module is a pair (a, α), where $a \in \mathrm{Ob}(\mathbf{C})$ and $\alpha\colon Ta \to a$ is such that the following diagrams commute:

$$
\begin{tikzcd}
T^2 a \arrow[r, "T\alpha"] \arrow[d, "\mu_a"] & Ta \arrow[d, "\alpha"] \\
Ta \arrow[r, "\alpha"] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[dr, "\mathrm{id}_a"'] & Ta \arrow[d, "\alpha"] \\
& a
\end{tikzcd}
$$

Definition 2.1.8. A morphism of T-algebras $(a, \alpha) \to (b, \beta)$ is a morphism $f\colon a \to b$ such that the following diagram commutes:

$$
\begin{tikzcd}
Ta \arrow[r, "Tf"] \arrow[d, "\alpha"] & Tb \arrow[d, "\beta"] \\
a \arrow[r, "f"] & b
\end{tikzcd}
$$

T-algebras form a category $T-Alg$, which has a natural forgetful functor $U^T\colon T-Alg \to \mathbf{C}$.

We now show how to recover the examples previously given with this language.

Example 2.1.9.

$$\begin{aligned} T = G \times -\colon\ & \mathbf{Set} \to \mathbf{Set} \\ \mu_A\colon\ & G \times (G \times A) \to G \times A \\ & (g,(h,a)) \mapsto (gh,a) \\ \eta_A\colon\ & A \to G \times A \\ & a \mapsto (e,a) \end{aligned}$$

is a monad and (A,α) is a T-algebra if and only if A is a G-set. It follows that $T - Alg \cong G - \mathbf{Set}$.

Example 2.1.10. Given a ring R, $T = R\otimes_{\mathbb{Z}}\colon Ab \to Ab$ is a monad when considered with the following natural transformations:

$$\begin{aligned} \mu_-\colon\ & R \otimes_{\mathbb{Z}} (R \otimes_{\mathbb{Z}} -) \cong (R\otimes_{\mathbb{Z}}) \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \\ \eta_-\colon\ & - \cong \mathbb{Z} \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \end{aligned}$$

We have that $(R \otimes_{\mathbb{Z}} -) - Alg \cong Mod_R$.

Example 2.1.11. COnsider $W\colon \mathbf{Set} \to \mathbf{Set}$ given by $WX = \amalg_{n\in\mathbb{N}}\Pi_{k=1}^{n} X$. Multiplication $\mu_X\colon WWX \to WX$ is given by concatenation of words, while the unit $\eta_X\colon X \to WX$ is just $x \mapsto (x)$.

With this, $W - Alg \cong Mon(\mathbf{Set})$.

Example 2.1.12. The functor $\mathcal{U}$ with the right natural transformations is a monad on **Set** and $\mathcal{U} - Alg \cong CHTop$, the category of compact T2 spaces.

Example 2.1.13. UF also induces a monad on the category of directed graphs and $UF - Alg \cong \mathbf{Cat}$.

2.2 Monadic functors

Now that we have introduced these structures, our aim is to characterize monadic functors, that is functors $U\colon \mathbf{A} \to \mathbf{C}$ which are equivalent to $U^T\colon T - Alg \to \mathbf{C}$ for some monad (T,μ,η) on $\mathbf{C}$.

First of all, notice that U^T is faithful by construction, hence U must be faithful, but more is true.

Lemma 2.2.1. The functor U^T is conservative, that is if $U^T f$ is an isomorphism then f is an isomorphism of T-algebras.

Proof. Suppose that g is the inverse of $f: a \to b$ and f is a morphism $(a, \alpha) \to (b, \beta)$. We only need to prove that the square on the left commutes, that is $g\beta = \alpha Tg$:

$$\begin{tikzcd} Tb \arrow[r, "Tg"] \arrow[d, "\beta"] & Ta \arrow[r, "Tf"] \arrow[d, "\alpha"] & Tb \arrow[d, "\beta"] \\ b \arrow[r, "g"] & a \arrow[r, "f"] & b \end{tikzcd}$$

We see that $fg\beta = \beta$ and $f\alpha Tg = \beta TfTg = \beta T(fg) = \beta T \,\mathrm{id}_b = \beta$, hence the thesis. □

Remark 2.2.2. Notice that the forgetful functor U: **Top** $\to$ **Set** can't be monadic since it does not reflect isomorphisms. However, if we restrict it to the full subcategory of **Top** spanned by compact Hausdorff spaces we indeed obtain a monadic functor.

Proposition 2.2.3. The functor $U^T: T - Alg \to \mathbf{C}$ has a left adjoint $F^T: \mathbf{C} \to T - Alg$ such that $F^T c = (Tc, \mu_c)$, $F^T f: (Tc, \mu_c) \xrightarrow{Tf} (Td, \mu_d)$ and $U^T F^T = T$. Furthermore, the unit of this adjunction is given by $\gamma_c = \eta_c: c \to U^T F^T c = Tc$ and the counit has components $\epsilon_{(a,\alpha)} = \alpha: (Ta, \mu_a) \to (a, \alpha)$.

Proof. (i) To show that (Tc, μ_c) is a T-algebra we need the following diagrams to be commutative.

$$\begin{tikzcd} T^3c \arrow[r, "T\mu_c"] \arrow[d, "\mu_{Tc}"'] & T^2c \arrow[d, "\mu_c"] \\ T^2c \arrow[r, "\mu_c"'] & Tc \end{tikzcd} \qquad \begin{tikzcd} Tc \arrow[r, "\eta_{Tc}"] \arrow[dr, equal] & T^2c \arrow[d, "\mu_c"] \\ & Tc \end{tikzcd}$$

These are exactly the associativity and one of the unit laws for (T, μ, η).

(ii) For every $f: c \to c'$, Tf is a morphism of algebras $(Tc, \mu_c) \to (Tc', \mu_{c'})$. The diagram

$$\begin{tikzcd} T^2c \arrow[r, "T^2f"] \arrow[d, "\mu_c"'] & T^2c' \arrow[d, "\mu_{c'}"] \\ Tc \arrow[r, "Tf"'] & Tc' \end{tikzcd}$$

is commutative because of the naturality of μ. Hence F^T is defined on morphisms. It is a functor by the functoriality of T.

(iii) The unit is natural by assumption. We claim that $\epsilon_{(a,\alpha)} = \alpha$ is a morphism of algebras

$$F^T U^T(a, \alpha) = F^T a = (Ta, \mu_a) \to \mathrm{id}_{T-Alg}(a, \alpha) = (a, \alpha)$$

and ϵ is a natural transformation $F^T U^T \Rightarrow \mathrm{id}_{T-Alg}$. Let's check it. We know that α is a morphism of algebras if and only if

$$\begin{tikzcd} T^2 a \arrow[r, "T\alpha"] \arrow[d, "\mu_a"'] & Ta \arrow[d, "\alpha"] \\ Ta \arrow[r, "\alpha"'] & a \end{tikzcd}$$

is commutative. But this is one of the two T-algebra axioms! Moreover, to prove that ϵ is natural, we need to show that

$$\begin{tikzcd} (Ta, \mu_a) \arrow[r, "\alpha=\epsilon_{(a,\alpha)}"] \arrow[d, "Tf"'] & (a, \alpha) \arrow[d, "f"] \\ (Tb, \mu_b) \arrow[r, "\beta=\epsilon_{(b,\beta)}"'] & (b, \beta) \end{tikzcd}$$

is commutative, but this is the axiom for f to be a morphism of T-algebras!

(iv) It remains to check the two triangular identities $\epsilon F^T \circ F^T \eta = id_{F^T}$ and $U^T \epsilon \circ \eta U^T = id_{U^T}$. These are to be checked on the components at c and (a, α), respectively.

$$\begin{tikzcd} (Tc, \mu_c) \arrow[r, "T\eta_c"] \arrow[dr, equal] & (T^2 c, \mu_{Tc}) \arrow[d, "\mu_{Tc}"] \\ & (Tc, \mu_c) \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[dr, equal] & Ta \arrow[d, "\alpha"] \\ & a \end{tikzcd}$$

The commutativity of these two diagrams is ensured by the second unit law for a monad and the unit law for the T-algebra (a, α), respectively. □

Definition 2.2.4. Algebras of the form (Tc, μ_c) are called free T-algebras.

Thanks to the proposition above we can prove that, given a monad T we can always find an adjunction that generates it. Actually, the converse holds too.

Proposition 2.2.5. If $U\colon \mathbf{D} \to \mathbf{C}$ has a left adjoint F with unit η and counit ϵ, then $(UF, U\epsilon F, \eta)$ is a monad on $\mathbf{C}$. Also, if (T, μ, η) is a monad on $\mathbf{C}$, then $(U^T F^T, U^T \epsilon F^T, \eta) = (T, \mu, \eta)$.

Proof. Let us check the axioms. First of all, the associativity holds due to the following equations.

$$
\begin{tikzcd}
 & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow\epsilon"] & & & & & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow\epsilon"] & & \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, bend right, equal] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, bend right, equal] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C}
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
 & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow\epsilon"] & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow\epsilon"] & & \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, bend right, equal] & & \mathbf{D} \arrow[ur, "U"] \arrow[rr, bend right, equal] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C}
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
 & & & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow\epsilon"] & & & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow\epsilon"] & & \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, bend right, equal] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, bend right, equal] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C}
\end{tikzcd}
$$

$\square$

Chapter 3

Beck's monadicity theorem

Chapter 4

Monads in 2-category theory

Chapter 5

Monads in ∞-category theory