Monads and their applications

Dr. Daniel Schäppi's course lecture notes

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Introduction

Chapter 1

Categorical preliminaries

Definition 1.0.1 (Categories)**.** A category $\mathbf{C}$ consists of:

1. a collection of objects $\mathrm{Ob}(\mathbf{C})$;

2. a collection of arrows $\mathrm{Ar}(\mathbf{C})$;

3. two maps $\mathrm{dom}, \mathrm{cod}\colon \mathrm{Ar}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$;

4. a map $\mathrm{id}_-\colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ar}(\mathbf{C})$ with $\mathrm{dom}(\mathrm{id}_c) = c = \mathrm{cod}(\mathrm{id}_c)$;

5. for every $f, g \in \mathrm{Ar}(\mathbf{C})$ such that $\mathrm{cod}(f) = \mathrm{dom}(g)$ a unique composite morphism gf such that $\mathrm{cod}(gf) = \mathrm{cod}(g)$, $\mathrm{dom}(gf) = f$.

This data has to satisfy the following axioms

1. given $f \in \mathrm{Ar}(\mathbf{C})$, $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$, $\mathrm{id}_{c'} f = f = \mathrm{id}_c$, that is the composition is unital;

2. given a composable triple $f, g, h \in \mathrm{Ar}(\mathbf{C})$, $h(gf) = (hg)f$, that is the composition is associative.

An arrow f such that $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$ is denoted $f\colon c \to c'$.

Definition 1.0.2 (Functors)**.**

Definition 1.0.3 (Full functors, faithful functor)**.**

Definition 1.0.4 (Natural transformations)**.**

Definition 1.0.5 (Equivalent functors)**.**

Definition 1.0.6 (Representable Functors)**.**

Definition 1.0.7 (Whiskering)**.**

Definition 1.0.8 (Horizontal and vertical composition of nat.transf.)**.**

Definition 1.0.9 (adjunctions)**.**

Lemma 1.0.10 (Yoneda)**.**

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Chapter 2

Monads and algebras

2.1 Introduction

Throughout mathematics we encounter structures defined by some action morphisms. Here we give some examples.

Example 2.1.1. Given a group G, we may consider a G-set X described by an action map $G \times X \to X$.

Example 2.1.2. Given an abelian group M and a ring R, we can get an R-module M by fixing a group homomorphism $R \otimes_{\mathbb{Z}} M \to M$.

Example 2.1.3. Given a monoid M in **Set**, we get a map $\Pi_{k=1}^{n} M \to M$, $(m_1, \ldots, m_n) \mapsto ((\ldots((m_1 m_2)m_3)\ldots)m_{n-1})m_n$. This induces an action map from $W(M) = \amalg_{n \in \mathbb{N}} \Pi_{k=1}^{n} M$, the set of words on M, to M.

Example 2.1.4. Given a set X, let $\mathcal{U}X$ be the set of ultrafilters on it. Any compact T2 topology on X allows us to see each ultrafilter as a system of neighborhoods of a unique point in X, hence it gives us a unique map $\mathcal{U}X \to X$ sending each ultrafilter to the respective point.

Example 2.1.5. Given a directed graph $D = (V, E, E \underset{t}{\overset{s}{\rightrightarrows}} V)$, we can create its free category FD, where the objects are the vertices and $FD(v, w) = \{\text{finite paths } v \to \ldots \to w\}$. We set id_v to be the path of length 0, while composition is just the concatenation of paths.

In particular, if D is the directed graph with $V = \{0, \ldots, n\}$ and an edge $j \to k$ if and only if $k = j + 1$, we have $FD \cong [n]$.

If $D = \{*\}$ and $E = \{* \to *\}$, then $FD(*, *) \cong \mathbb{N}$.

Given a small category $\mathbf{C}$, we may consider the underlying graph $U\,\mathbf{C} = D$ with $V = \mathrm{Ob}(\mathbf{C})$, $E = \mathrm{Ar}(\mathbf{C})$, $s = \mathrm{dom}$ and $t = \mathrm{cod}$. We get then an action map $UFU\,\mathbf{C} \to U\,\mathbf{C}$ sending a finite path to its composite. This map is a morphism of directed graphs.

Notice that we always have a category $\mathbf{C}$ and some functor $T\colon \mathbf{C} \to \mathbf{C}$ with an action map $T\,\mathbf{C} \to \mathbf{C}$. How can we see all of these examples as specific instances of a general phenomenon?

Definition 2.1.6. A monad on a category $\mathbf{C}$ is a triple (T, μ, η) where $T\colon \mathbf{C} \to \mathbf{C}$ is a functor, while $\mu\colon T^2 \Rightarrow T$ and $\eta\colon \mathrm{id}_{\mathbf{C}} \Rightarrow T$ are natural transformations such that the following diagrams commute:
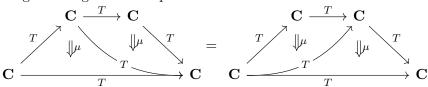
$$
\begin{array}{ccc}
T^3 \xRightarrow{\;T\mu\;} T^2 & \qquad & T \xRightarrow{\;\eta T\;} T^2 \xLeftarrow{\;T\eta\;} T \\[4pt]
\Big\Downarrow{\mu T} \qquad \Big\Downarrow{\mu} & & \quad \mathrm{id}_T \searrow \;\; \Big\Downarrow{\mu} \;\; \swarrow \mathrm{id}_T \\[4pt]
T^2 \xRightarrow{\;\mu\;} T & & T
\end{array}
$$

μ is called the multiplicative map, while η is the unit of T.

The commutativity of the first diagram is equivalent to stating that the following two diagrams are equal:

$$
\begin{array}{ccc}
\mathbf{C} \xrightarrow{\;T\;} \mathbf{C} & & \mathbf{C} \xrightarrow{\;T\;} \mathbf{C} \\
{}^{T}\nearrow \;\Downarrow\mu\; \Downarrow\mu\; \nwarrow{}^{T} & = & {}^{T}\nearrow \;\Downarrow\mu\; \Downarrow\mu\; \nwarrow{}^{T} \\
\mathbf{C} \xrightarrow{\;\;T\;\;} \mathbf{C} & & \mathbf{C} \xrightarrow{\;\;T\;\;} \mathbf{C}
\end{array}
$$

On the other hand, the second diagram can be rephrased as follows:

$$
\begin{array}{ccccc}
\mathbf{C} & & \mathbf{C} & & \mathbf{C} \\
\mathrm{id}_{\mathbf{C}} \nearrow \;\Downarrow\eta\; {}_T \Downarrow\mu\; {}^{T} & = & T \Big(=\Big) T & = & {}^{T}\nearrow \;\Downarrow\mu\; {}_T \Downarrow\eta\; \searrow \mathrm{id}_{\mathbf{C}} \\
\mathbf{C} \xrightarrow{\;T\;} \mathbf{C} & & \mathbf{C} & & \mathbf{C} \xrightarrow{\;T\;} \mathbf{C}
\end{array}
$$

A monad naturally defines other algebraic structures, which we now introduce.

Definition 2.1.7. Given a monad (T, μ, η), a T-algebra or T-module is a pair (a, α), where $a \in \mathrm{Ob}(\mathbf{C})$ and $\alpha\colon Ta \to a$ is such that the following diagrams commute:

$$
\begin{array}{ccc}
T^2 a \xrightarrow{\;T\alpha\;} Ta & \qquad & a \xrightarrow{\;\eta_a\;} Ta \\
\Big\downarrow{\mu_a} \qquad \Big\downarrow{\alpha} & & \mathrm{id}_a \searrow \;\; \Big\downarrow{\alpha} \\
Ta \xrightarrow{\;\alpha\;} a & & a
\end{array}
$$

Definition 2.1.8. A morphism of T-algebras $(a, \alpha) \to (b, \beta)$ is a morphism $f\colon a \to b$ such that the following diagram commutes:

$$
\begin{array}{ccc}
Ta \xrightarrow{\;Tf\;} Tb \\
\Big\downarrow{\alpha} \qquad \Big\downarrow{\beta} \\
a \xrightarrow{\;f\;} b
\end{array}
$$

T-algebras form a category $T\text{-}\mathbf{Alg}$, which has a natural forgetful functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$.

We now show how to recover the examples previously given with this language.

Example 2.1.9.

$$
\begin{aligned}
T = G \times - \colon\ & \mathbf{Set} \to \mathbf{Set} \\
\mu_A \colon\ & G \times (G \times A) \to G \times A \\
& (g, (h, a)) \mapsto (gh, a) \\
\eta_A \colon\ & A \to G \times A \\
& a \mapsto (e, a)
\end{aligned}
$$

is a monad and (A, α) is a T-algebra if and only if A is a G-set. It follows that $T\text{-}\mathbf{Alg} \cong G\text{-}\mathbf{Set}$.

Example 2.1.10. Given a ring R, $T = R \otimes_{\mathbb{Z}} -\colon \mathbf{Ab} \to \mathbf{Ab}$ is a monad when considered with the following natural transformations:

$$
\begin{aligned}
\mu_- \colon\ & R \otimes_{\mathbb{Z}} (R \otimes_{\mathbb{Z}} -) \cong (R \otimes_{\mathbb{Z}}) \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \\
\eta_- \colon\ & - \cong \mathbb{Z} \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} -
\end{aligned}
$$

We have that $(R \otimes_{\mathbb{Z}} -)\text{-}\mathbf{Alg} \cong \mathbf{Mod}_R$.

Example 2.1.11. Consider $W\colon \mathbf{Set} \to \mathbf{Set}$ given by $WX = \amalg_{n \in \mathbb{N}} \Pi_{k=1}^{n} X$. Multiplication $\mu_X\colon WWX \to WX$ is given by concatenation of words, while the unit $\eta_X\colon X \to WX$ is just $x \mapsto (x)$. With this, $W\text{-}\mathbf{Alg} \cong \mathrm{Mon}(\mathbf{Set})$.

Example 2.1.12. The functor $\mathcal{U}$ defined in Example 2.1.4, equipped with suitable natural transformations, is a monad on $\mathbf{Set}$ and $\mathcal{U}\text{-}\mathbf{Alg} \cong \mathbf{CHTop}$, the category of compact T2 spaces.

Example 2.1.13. The free-forgetful adjunction $F \dashv U$ between categories and directed graphs induces a monad on the latter, with $UF\text{-}\mathbf{Alg} \cong \mathbf{Cat}$.

2.2   Monadic functors

Now that we have introduced these structures, our aim is to characterize *monadic functors*, that is functors $U\colon \mathbf{A} \to \mathbf{C}$ which are equivalent to $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ for some monad (T, μ, η) on $\mathbf{C}$.

First of all, notice that U^T is faithful by construction, hence U must be faithful, but more is true.

Lemma 2.2.1. The functor U^T is conservative, that is if $U^T f$ is an isomorphism then f is an isomorphism of T-algebras.

Proof. Suppose that g is the inverse of $f \colon a \to b$ and f is a morphism $(a, \alpha) \to (b, \beta)$. We only need to prove that the square on the left commutes, that is $g\beta = \alpha Tg$:

$$
\begin{array}{ccccc}
Tb & \xrightarrow{\ Tg\ } & Ta & \xrightarrow{\ Tf\ } & Tb \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
b & \xrightarrow[\ g\ ]{} & a & \xrightarrow[\ f\ ]{} & b
\end{array}
$$

We see that $fg\beta = \beta$ and $f\alpha Tg = \beta TfTg = \beta T(fg) = \beta T\operatorname{id}_b = \beta$, hence the thesis. $\square$

Remark 2.2.2. Notice that the forgetful functor $U \colon \mathbf{Top} \to \mathbf{Set}$ can't be monadic since it does not reflect isomorphisms. However, if we restrict it to the full subcategory of $\mathbf{Top}$ spanned by compact Hausdorff spaces we indeed obtain a monadic functor.

Proposition 2.2.3. The functor $U^T \colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ has a left adjoint $F^T \colon \mathbf{C} \to T\text{-}\mathbf{Alg}$ such that $F^T c = (Tc, \mu_c)$, $F^T f \colon (Tc, \mu_c) \xrightarrow{Tf} (Td, \mu_d)$ and $U^T F^T = T$. Furthermore, the unit of this adjunction is given by $\gamma_c = \eta_c \colon c \to U^T F^T c = Tc$ and the counit has components $\epsilon_{(a,\alpha)} = \alpha \colon (Ta, \mu_a) \to (a, \alpha)$.

Proof.   (i) To show that (Tc, μ_c) is a T-algebra we need the following diagrams to be commutative.

$$
\begin{array}{ccc}
T^3 c & \xrightarrow{\ T\mu_c\ } & T^2 c \\
{\scriptstyle \mu_{Tc}}\downarrow & & \downarrow{\scriptstyle \mu_c} \\
T^2 c & \xrightarrow[\ \mu_c\ ]{} & Tc
\end{array}
\qquad
\begin{array}{ccc}
Tc & \xrightarrow{\ \eta_{Tc}\ } & T^2 c \\
 & \diagdown\diagdown & \downarrow{\scriptstyle \mu_c} \\
 & & Tc
\end{array}
$$

These are exactly the associativity and one of the unit laws for (T, μ, η).

  (ii) For every $f \colon c \to c'$, Tf is a morphism of algebras $(Tc, \mu_c) \to (Tc', \mu_{c'})$. The diagram

$$
\begin{array}{ccc}
T^2 c & \xrightarrow{\ T^2 f\ } & T^2 c' \\
{\scriptstyle \mu_c}\downarrow & & \downarrow{\scriptstyle \mu_{c'}} \\
Tc & \xrightarrow[\ Tf\ ]{} & Tc'
\end{array}
$$

is commutative because of the naturality of μ. Hence F^T is defined on morphisms. It is a functor by the functoriality of T.

  (iii) The unit is natural by assumption. We claim that $\epsilon_{(a,\alpha)} = \alpha$ is a morphism of algebras

$$
F^T U^T (a, \alpha) = F^T a = (Ta, \mu_a) \to \operatorname{id}_{T\text{-}\mathbf{Alg}}(a, \alpha) = (a, \alpha)
$$

and ϵ is a natural transformation $F^T U^T \Rightarrow \mathrm{id}_{T\text{-}\mathbf{Alg}}$. Let's check it. We know that α is a morphism of algebras if and only if

$$
\begin{array}{ccc}
T^2 a & \xrightarrow{T\alpha} & Ta \\
{\scriptstyle \mu_a}\downarrow & & \downarrow{\scriptstyle \alpha} \\
Ta & \xrightarrow{\alpha} & a
\end{array}
$$

is commutative. But this is one of the two T-algebra axioms! Moreover, to prove that ϵ is natural, we need to show that
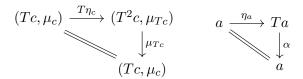
$$
\begin{array}{ccc}
(Ta, \mu_a) & \xrightarrow{\alpha = \epsilon_{(a,\alpha)}} & (a, \alpha) \\
{\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle f} \\
(Tb, \mu_b) & \xrightarrow[\beta = \epsilon_{(b,\beta)}]{} & (b, \beta)
\end{array}
$$

is commutative, but this is the axiom for f to be a morphism of T-algebras!

(iv) It remains to check the two triangular identities $\epsilon F^T \circ F^T \eta = id_{F^T}$ and $U^T \epsilon \circ \eta U^T = id_{U^T}$. These are to be checked on the components at c and (a, α), respectively.

$$
\begin{array}{ccc}
(Tc, \mu_c) & \xrightarrow{T\eta_c} & (T^2 c, \mu_{Tc}) \\
& \searrow & \downarrow{\scriptstyle \mu_{Tc}} \\
& & (Tc, \mu_c)
\end{array}
\qquad
\begin{array}{ccc}
a & \xrightarrow{\eta_a} & Ta \\
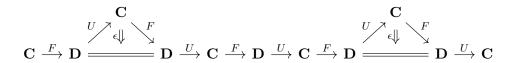& \searrow & \downarrow{\scriptstyle \alpha} \\
& & a
\end{array}
$$

The commutativity of these two diagrams is ensured by the second unit law for a monad and the unit law for the T-algebra (a, α), respectively.
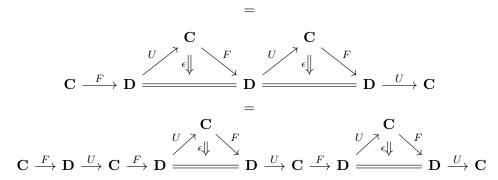□

Definition 2.2.4. Algebras of the form (Tc, μ_c) are called free T-algebras.

Thanks to the proposition above we can prove that, given a monad T we can always find an adjunction that generates it. Actually, the converse holds too.

Proposition 2.2.5. If $U \colon \mathbf{D} \to \mathbf{C}$ has a left adjoint F with unit η and counit ϵ, then $(UF, U\epsilon F, \eta)$ is a monad on $\mathbf{C}$. Also, if (T, μ, η) is a monad on $\mathbf{C}$, then $(U^T F^T, U^T \epsilon F^T, \eta) = (T, \mu, \eta)$.

Proof. Let us check the axioms. First of all, the associativity holds due to the following equations.

$$
\mathbf{C} \xrightarrow{F} \mathbf{D}
\begin{array}{c}
\mathbf{C} \\
{\scriptstyle U}\nearrow \ {\scriptstyle \epsilon\Downarrow} \ \searrow{\scriptstyle F} \\
\Longrightarrow
\end{array}
\mathbf{D} \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D}
\begin{array}{c}
\mathbf{C} \\
{\scriptstyle U}\nearrow \ {\scriptstyle \epsilon\Downarrow} \ \searrow{\scriptstyle F} \\
\Longrightarrow
\end{array}
\mathbf{D} \xrightarrow{U} \mathbf{C}
$$

$$=$$

$$=$$

Unit laws:

is equal to 1_{UF}, since $\epsilon F \circ F\eta = 1_F$ by one of the triangular identities of the adjunction $F \dashv U$. Furthermore,
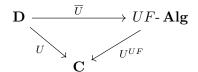

is equal to 1_{UF}. This follows from the explicit description of the unit and the counit of the adjunction $F^T \dashv U^T$, in fact

$$U^T \epsilon F^T c = U^T \epsilon_{(Tc, \eta_c)} = \mu_c$$

$\square$

Example 2.2.6 (Interesting adjunction, boring monad)**.** Let us consider the adjunction $U \colon \mathbf{Top} \rightleftarrows \mathbf{Set} \colon \mathsf{Disc} =: F$, whose left adjoint assigns to every set X the discrete topological space $FX = (X, 2^X)$. It's immediate to see that $UFX = X$, hence $UF = \mathrm{id}_{\mathbf{Set}}$. How many natural transformations $\mathrm{id}_{\mathbf{Set}} = UF \overset{\alpha}{\Rightarrow} UF = \mathrm{id}_{\mathbf{Set}}$ are there? We know that $\mathrm{id}_{\mathbf{Set}} \cong \hom(*, -)$, so $\mathrm{Nat}(\mathrm{id}_{\mathbf{Set}}, \mathrm{id}_{\mathbf{Set}}) \cong \mathrm{Nat}(\hom(*, -), \hom(*, -)) \cong \hom(*, *) = \{\mathrm{id}_*\}$ by Yoneda, hence $\alpha = \mathrm{id}$. Therefore $(UF, U\epsilon F, \eta) = (\mathrm{id}_{\mathbf{Set}}, \mathrm{id}, \mathrm{id})$

Example 2.2.7. If S is a set, $\mathbf{Set}(S, -) \colon \mathbf{Set} \to \mathbf{Set}$ is right adjoint to $S \times - \colon \mathbf{Set} \to \mathbf{Set}$, so we get a monad $X \mapsto \mathbf{Set}(S, S \times X)$. This is called *the state monad* and is important in Computer Science.

There is always a comparison morphism $\mathbf{D} \overset{\overline{U}}{\to} UF\text{-}\mathbf{Alg}$ s.t.

commutes. We set $\overline{U}f = (Ud, UFUd \xrightarrow{U\epsilon_d} Ud) = (Ud, U\epsilon_d)$. More generally, for a given functor $\mathbf{D} \xrightarrow{G} \mathbf{C}$ we can ask what do we need to get a lift $\overline{G}\colon \mathbf{D} \to T\text{-}\mathbf{Alg}$.

Definition 2.2.8. A T-action on a functor $\mathbf{D} \xrightarrow{G} \mathbf{C}$ is a natural transformation $TG \overset{\gamma}{\Rightarrow} G$ s.t.

$$
\begin{array}{ccc}
T^2G & \xrightarrow{T\gamma} & TG \\
\mu G \Downarrow & & \Downarrow \gamma \\
TG & \underset{\gamma}{\Longrightarrow} & G
\end{array}
\qquad
\begin{array}{ccc}
G & \xrightarrow{\eta G} & TG \\
& & \Downarrow \gamma \\
& & G
\end{array}
$$

commute.

Definition 2.2.9. A morphism of T-actions is a natural transformation $\varphi\colon G \Rightarrow K$ s.t.

$$
\begin{array}{ccc}
TG & \xrightarrow{T\varphi} & TK \\
\gamma \Downarrow & & \Downarrow \alpha \\
G & \underset{\varphi}{\Longrightarrow} & K
\end{array}
$$

commutes.

Up to size, we get a category $T\text{-}\mathsf{Act}(\mathbf{D})$, defined as follows.

Definition 2.2.10. Given a monad (T, μ, η)

Chapter 3

Beck's monadicity theorem

Theorem 3.0.1 (Beck)**.**

Chapter 4

Monads in 2-category theory

Chapter 5

Monads in ∞-category theory