Monads and their applications

Dr. Daniel Schäppi's course lecture notes

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Definition 1.0.8 (adjunctions).

Lemma 1.0.9 (Yoneda).

Proof. □

Chapter 2

Monads and algebras

Throughout mathematics we encounter structures defined by some action morphisms. Here we give some examples.

Example 2.0.1. Given a group G, we may consider a G-set X described by an action map $G \times X \to X$.

Example 2.0.2. Given an abelian group M and a ring R, we can get an R-module M by fixing a group homomorphism $R \otimes_{\mathbb{Z}} M \to M$.

Example 2.0.3. Given a monoid M in *Set*, we get a map $\Pi_{k=1}^{n} M \to M$, $(m_1, \ldots, m_n) \mapsto ((\ldots((m_1 m_2)m_3)\ldots)m_{n-1})m_n$. This induces an action map from $W(M) = \amalg_{n\in\mathbb{N}} \Pi_{k=1}^{n} M$, the set of words on M, to M.

Example 2.0.4. Given a set X, let $\mathcal{U}X$ be the set of ultrafilters on it. Any compact T2 topology on X allows us to see each ultrafilter as a system of neighborhoods of a unique point in X, hence it gives us a unique map $\mathcal{U}X \to X$ sending each ultrafilter to the respective point.

Example 2.0.5. Given a directed graph $D = (V, E, E \xrightarrow{s} V, E \xrightarrow{t} E)$, we can create its free category FD, where the objects are the vertices and $FD(v, w) = \{\text{finite paths } v \to \ldots \to w\}$. We set Id_v to be the path of length 0, while composition is just the concatenation of paths.

2.1 Monads

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Chapter 3

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