Monads and their applications 1

Exercise 1.
Let (T, μ, η) be a monad on $\mathscr{C}$ and let (A, α) be a T-algebra. Show that for any isomorphism $g\colon B \to A$ there exists a unique T-algebra structure $\beta\colon TB \to B$ on B such that g is a morphism of T-algebras $(B, \beta) \to (A, \alpha)$.

Exercise 2.
A monad (T, μ, η) on $\mathscr{C}$ is called *idempotent* if the multiplication $\mu\colon T^2 \Rightarrow T$ is an isomorphism. Show that for any idempotent monad $T\colon \mathscr{C} \to \mathscr{C}$ and any object $A \in \mathscr{C}$, there exists at most one T-algebra structure on A. Moreover, show that for any pair of T-algebras (A, α) and (B, β), every morphism $f\colon A \to B$ in $\mathscr{C}$ is a morphism of T-algebras (in other words, the forgetful functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathscr{C}$ is full).

Exercise 3.
Let R be a commutative ring. Show that the functor $\mathbf{Mod}_R \to \mathbf{Mod}_R$, which sends M to $\oplus_{n\in\mathbb{N}} M^{\otimes n}$ can be endowed with the structure of a monad whose category of algebras is (isomorphic to) the category of R-algebras. Find a monad on $\mathbf{Mod}_R$ whose category of algebras is the category of *commutative* R-algebras.

Exercise 4. *??*
Let $F\colon \mathscr{C} \to \mathscr{D}$ and $U\colon \mathscr{D} \to \mathscr{C}$ be two functors.

(a) Show that there is a bijection between the set of pairs natural transformations $\eta\colon \mathrm{id}_{\mathscr{D}} \Rightarrow UF$ and $\varepsilon\colon FU \to \mathrm{id}_{\mathscr{C}}$ which satisfy the triangle identites ($\varepsilon F \cdot F\eta = 1_F$, $U\varepsilon \cdot \eta U = 1_U$) on the one hand and the set of natural isomorphisms

$$\varphi\colon \mathscr{D}(F-,-) \Rightarrow \mathscr{C}(-,U-)\colon \mathscr{D}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$$

on the other.

(b) Assume that for each $c \in \mathscr{C}$, the functor $\mathscr{C}(c, U-)$ is representable. Suppose that there is an explicit choice of representing object $d_c \in \mathscr{D}$, that is, a choice of a natural isomorphism $\mathscr{D}(d_c, -) \cong \mathscr{C}(c, U-)$. Show that there exists a functor $G\colon \mathscr{C} \to \mathscr{D}$ with $Gc = d_c$ such that G is left adjoint to U.

(c) Show that if both F and G are left adjoint to U, then there exists a natural isomorphism $F \cong G$.

Exercise 5. *??*

An *ultrafilter* on a set X is a set $\mathscr{F}$ of subsets of X such that the following axiom holds: for all subsets $A \subseteq X$, A belongs to F if and only if for all $B_1, \ldots, B_n \in \mathscr{F}$, the intersection $A \cap B_1 \cap \ldots \cap B_n$ is non-empty. The *principal ultrafilter* of $x \in X$ is $\mathscr{F}_x = \{A \subseteq X | x \in A\}$. Given a subset $A \subseteq X$, we write $[A]$ for the set of ultrafilters $\mathscr{F}$ which contain A. We write UX for the set of ultrafilters on X. Given a function $f\colon X \to Y$ and an ultrafilter $\mathscr{F}$ on X, we call

$$f_*\mathscr{F} := \{B \subseteq Y | f^{-1}(B) \in \mathscr{F}\}$$

the pushforward of $\mathscr{F}$.

(a) Show that UX defines an endofunctor of **Set** via the pushforward.

(b) Show that $\eta_X\colon X \to UX$, $x \mapsto \mathscr{F}_x$ and $\mu_X\colon UUX \to UX$ defined by

$$\mu(\mathscr{F}) = \{A \subseteq X | [A] \in \mathscr{F}\}$$

endow U with the structure of a monad.

(c) Let $\xi\colon UX \to X$ be an algebra for the ultrafilter monad. We call a subset $U \subseteq X$ *open* if

$$\forall x \in X \; \forall \mathscr{F} \in UX\colon (x \in U \text{ and } \xi(\mathscr{F}) = x) \Rightarrow U \in \mathscr{F}$$

holds. Show that these open sets form a topology on X.