Monads and their applications

Dr. Daniel Schäppi's course lecture notes

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Introduction

Chapter 1

Categorical preliminaries

Definition 1.0.1 (Categories)**.** A category $\mathbf{C}$ consists of:

1. a collection of objects $\mathrm{Ob}(\mathbf{C})$;
2. a collection of arrows $\mathrm{Ar}(\mathbf{C})$;
3. two maps $\mathrm{dom}, \mathrm{cod} \colon \mathrm{Ar}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$;
4. a map $\mathrm{id}_- \colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ar}(\mathbf{C})$ with $\mathrm{dom}(\mathrm{id}_c) = c = \mathrm{cod}(\mathrm{id}_c)$;
5. for every $f, g \in \mathrm{Ar}(\mathbf{C})$ such that $\mathrm{cod}(f) = \mathrm{dom}(g)$ a unique composite morphism gf such that $\mathrm{cod}(gf) = \mathrm{cod}(g)$, $\mathrm{dom}(gf) = f$.

This data has to satisfy the following axioms

1. given $f \in \mathrm{Ar}(\mathbf{C})$, $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$, $\mathrm{id}_{c'} f = f = \mathrm{id}_c$, that is the composition is unital;
2. given a composable triple $f, g, h \in \mathrm{Ar}(\mathbf{C})$, $h(gf) = (hg)f$, that is the composition is associative.

An arrow f such that $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$ is denoted $f \colon c \to c'$.

Definition 1.0.2 (Functors)**.**

Definition 1.0.3 (Full functors, faithful functor)**.**

Definition 1.0.4 (Natural transformations)**.**

Definition 1.0.5 (Equivalent functors)**.**

Definition 1.0.6 (Representable Functors)**.**

Definition 1.0.7 (Whiskering)**.**

Definition 1.0.8 (Horizontal and vertical composition of nat.transf.)**.**

Definition 1.0.9 (adjunctions)**.**

Lemma 1.0.10 (Yoneda)**.**

Proof. □

Chapter 2

Monads and algebras

2.1 Introduction

Throughout mathematics we encounter structures defined by some action morphisms. Here we give some examples.

Example 2.1.1. Given a group G, we may consider a G-set X described by an action map $G \times X \to X$.

Example 2.1.2. Given an abelian group M and a ring R, we can get an R-module M by fixing a group homomorphism $R \otimes_{\mathbb{Z}} M \to M$.

Example 2.1.3. Given a monoid M in **Set**, we get a map $\Pi_{k=1}^{n} M \to M$, $(m_1, \ldots, m_n) \mapsto ((\ldots((m_1 m_2)m_3)\ldots)m_{n-1})m_n$. This induces an action map from $W(M) = \amalg_{n\in\mathbb{N}} \Pi_{k=1}^{n} M$, the set of words on M, to M.

Example 2.1.4. Given a set X, let $\mathcal{U}X$ be the set of ultrafilters on it. Any compact T2 topology on X allows us to see each ultrafilter as a system of neighborhoods of a unique point in X, hence it gives us a unique map $\mathcal{U}X \to X$ sending each ultrafilter to the respective point.

Example 2.1.5. Given a directed graph $D = (V, E, E \underset{t}{\overset{s}{\rightrightarrows}} V)$, we can create its free category FD, where the objects are the vertices and $FD(v, w) = \{\text{finite paths } v \to \ldots \to w\}$. We set id_v to be the path of length 0, while composition is just the concatenation of paths.

In particular, if D is the directed graph with $V = \{0, \ldots, n\}$ and an edge $j \to k$ if and only if $k = j + 1$, we have $FD \cong [n]$.

If $D = \{*\}$ and $E = \{* \to *\}$, then $FD(*, *) \cong \mathbb{N}$.

Given a small category $\mathbf{C}$, we may consider the underlying graph $U\,\mathbf{C} = D$ with $V = \mathrm{Ob}(\mathbf{C})$, $E = \mathrm{Ar}(\mathbf{C})$, $s = \mathrm{dom}$ and $t = \mathrm{cod}$. We get then an action map $UFU\,\mathbf{C} \to U\,\mathbf{C}$ sending a finite path to its composite. This map is a morphism of directed graph.

How can we see all of these examples as specific instances of a general phenomenon?

Notice that we always have a category $\mathbf{C}$ and some functor $T\colon \mathbf{C} \to \mathbf{C}$ with an action map $T\,\mathbf{C} \to \mathbf{C}$.

Definition 2.1.6. A monad on a category $\mathbf{C}$ is a triple (T, μ, η) where $T\colon \mathbf{C} \to \mathbf{C}$ is a functor, while $\mu\colon T^2 \Rightarrow T$ and $\eta\colon \mathrm{id}_{\mathbf{C}} \Rightarrow T$ are natural transformations such that the following diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "\mathrm{id}_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "\mathrm{id}_T"] \\ & T & \end{tikzcd}$$

μ is called the multiplicative map, while η is the unit of T.

The commutativity of the first diagram is equivalent to stating that the following two diagrams are equal:

$$\begin{tikzcd} & \mathbf{C} \arrow[r, "T"] \arrow[drr, bend right, "T"'] \arrow[d, phantom, "\Downarrow\mu"] & \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\ \mathbf{C} \arrow[ur, "T"] \arrow[rrr, "T"'] & {} & {} & \mathbf{C} \end{tikzcd} = \begin{tikzcd} & \mathbf{C} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\ \mathbf{C} \arrow[ur, "T"] \arrow[urr, bend right, "T"] \arrow[rrr, "T"'] & {} & {} & \mathbf{C} \end{tikzcd}$$

On the other hand, the second diagram can be rephrased as follows:

$$\begin{tikzcd} & \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\eta\ \Downarrow\mu"] & \\ \mathbf{C} \arrow[ur, bend left, "\mathrm{id}_{\mathbf{C}}"] \arrow[ur, bend right, "T"'] \arrow[rr, "T"'] & {} & \mathbf{C} \end{tikzcd} = \begin{tikzcd} \mathbf{C} \arrow[d, bend right, "T"'] \arrow[d, bend left, "T"] \arrow[d, phantom, "="] \\ \mathbf{C} \end{tikzcd} = \begin{tikzcd} & \mathbf{C} \arrow[dr, bend right, "T"'] \arrow[dr, bend left, "\mathrm{id}_{\mathbf{C}}"] \arrow[d, phantom, "\Downarrow\mu\ \Downarrow\eta"] & \\ \mathbf{C} \arrow[ur, "T"] \arrow[rr, "T"'] & {} & \mathbf{C} \end{tikzcd}$$

A monad naturally defines other algebraic structures, which we now introduce.

Definition 2.1.7. Given a monad (T, μ, η), a T-algebra or T-module is a pair (a, α), where $a \in \mathrm{Ob}(\mathbf{C})$ and $\alpha\colon Ta \to a$ is such that the following diagrams commute:

$$\begin{tikzcd} T^2a \arrow[r, "T\alpha"] \arrow[d, "\mu_a"] & Ta \arrow[d, "\alpha"] \\ Ta \arrow[r, "\alpha"] & a \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[dr, "\mathrm{id}_a"'] & Ta \arrow[d, "\alpha"] \\ & a \end{tikzcd}$$

Definition 2.1.8. A morphism of T-algebras $(a, \alpha) \to (b, \beta)$ is a morphism $f\colon a \to b$ such that the following diagram commutes:

$$\begin{tikzcd} Ta \arrow[r, "Tf"] \arrow[d, "\alpha"] & Tb \arrow[d, "\beta"] \\ a \arrow[r, "f"] & b \end{tikzcd}$$

T-algebras form a category $T - Alg$, which has a natural forgetful functor $U^T : T - Alg \to \mathbf{C}$.

We now show how to recover the examples previously given with this language.

Example 2.1.9.

$$\begin{aligned} T = G \times - \colon &\ \mathbf{Set} \to \mathbf{Set} \\ \mu_A \colon &\ G \times (G \times A) \to G \times A \\ &\ (g, (h, a)) \mapsto (gh, a) \\ \eta_A \colon &\ A \to G \times A \\ &\ a \mapsto (e, a) \end{aligned}$$

is a monad and (A, α) is a T-algebra if and only if A is a G-set. It follows that $T - Alg \cong G - \mathbf{Set}$.

Example 2.1.10. Given a ring R, $T = R \otimes_{\mathbb{Z}} \colon Ab \to Ab$ is a monad when considered with the following natural transformations:

$$\begin{aligned} \mu_- \colon &\ R \otimes_{\mathbb{Z}} (R \otimes_{\mathbb{Z}} -) \cong (R \otimes_{\mathbb{Z}}) \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \\ \eta_- \colon &\ - \cong \mathbb{Z} \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \end{aligned}$$

We have that $(R \otimes_{\mathbb{Z}} -) - Alg \cong Mod_R$.

Example 2.1.11. COnsider $W \colon \mathbf{Set} \to \mathbf{Set}$ given by $WX = \amalg_{n \in \mathbb{N}} \Pi_{k=1}^{n} X$. Multiplication $\mu_X \colon WWX \to WX$ is given by concatenation of words, while the unit $\eta_X \colon X \to WX$ is just $x \mapsto (x)$.

With this, $W - Alg \cong Mon(\mathbf{Set})$.

Example 2.1.12. The functor $\mathcal{U}$ with the right natural transformations is a monad on **Set** and $\mathcal{U} - Alg \cong CHTop$, the category of compact T2 spaces.

Example 2.1.13. UF also induces a monad on the category of directed graphs and $UF - Alg \cong \mathbf{Cat}$.

2.2 Monadic functors

Now that we have introduced these structures, our aim is to characterize monadic functors, that is functors $U \colon \mathbf{A} \to \mathbf{C}$ which are equivalent to $U^T \colon T - Alg \to \mathbf{C}$ for some monad (T, μ, η) on $\mathbf{C}$.

First of all, notice that U^T is faithful by construction, hence U must be faithful, but more is true.

Lemma 2.2.1. The functor U^T is conservative, that is if $U^T f$ is an isomorphism then f is an isomorphism of T-algebras.

Proof. Suppose that g is the inverse of $f\colon a \to b$ and f is a morphism $(a,\alpha) \to (b,\beta)$. We only need to prove that the square on the left commutes, that is $g\beta = \alpha Tg$:

$$\begin{tikzcd} Tb \arrow[r, "Tg"] \arrow[d, "\beta"] & Ta \arrow[r, "Tf"] \arrow[d, "\alpha"] & Tb \arrow[d, "\beta"] \\ b \arrow[r, "g"] & a \arrow[r, "f"] & b \end{tikzcd}$$

We see that $fg\beta = \beta$ and $f\alpha Tg = \beta TfTg = \beta T(fg) = \beta T\,\mathrm{id}_b = \beta$, hence the thesis. □

Remark 2.2.2. Notice that the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ can't be monadic since it does not reflect isomorphisms. However, if we restrict it to the full subcategory of **Top** spanned by compact Hausdorff spaces we indeed obtain a monadic functor.

Proposition 2.2.3. The functor $U^T\colon T - Alg \to \mathbf{C}$ has a left adjoint $F^T\colon\ \mathbf{C} \to T - Alg$ such that $F^T c = (Tc, \mu_c)$ and $F^T f\colon (Tc,\mu_c) \to (Td,\mu_d)$.

Chapter 3

Beck's monadicity theorem

Chapter 4

Monads in 2-category theory

Chapter 5

Monads in ∞-category theory