Monads and their applications 1

Exercise 1.
Let (T, μ, η) be a monad on $\mathscr{C}$ and let (A, α) be a T-algebra. Show that for any isomorphism $g\colon B \to A$ there exists a unique T-algebra structure $\beta\colon TB \to B$ on B such that g is a morphism of T-algebras $(B, \beta) \to (A, \alpha)$.

Exercise 2.
A monad (T, μ, η) on $\mathscr{C}$ is called *idempotent* if the multiplication $\mu\colon T^2 \Rightarrow T$ is an isomorphism. Show that for any idempotent monad $T\colon \mathscr{C} \to \mathscr{C}$ and any object $A \in \mathscr{C}$, there exists at most one T-algebra structure on A. Moreover, show that for any pair of T-algebras (A, α) and (B, β), every morphism $f\colon A \to B$ in $\mathscr{C}$ is a morphism of T-algebras (in other words, the forgetful functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathscr{C}$ is full).

Exercise 3.
Let R be a commutative ring. Show that the functor $\mathbf{Mod}_R \to \mathbf{Mod}_R$, which sends M to $\oplus_{n\in\mathbb{N}} M^{\otimes n}$ can be endowed with the structure of a monad whose category of algebras is (isomorphic to) the category of R-algebras. Find a monad on $\mathbf{Mod}_R$ whose category of algebras is the category of *commutative* R-algebras.

Exercise 4. *??*
Let $F\colon \mathscr{C} \to \mathscr{D}$ and $U\colon \mathscr{D} \to \mathscr{C}$ be two functors.

(a) Show that there is a bijection between the set of pairs natural transformations $\eta\colon \mathrm{id}_{\mathscr{D}} \Rightarrow UF$ and $\varepsilon\colon FU \to \mathrm{id}_{\mathscr{C}}$ which satisfy the triangle identites ($\varepsilon F \cdot F\eta = 1_F$, $U\varepsilon \cdot \eta U = 1_U$) on the one hand and the set of natural isomorphisms

$$\varphi\colon \mathscr{D}(F-, -) \Rightarrow \mathscr{C}(-, U-)\colon \mathscr{D}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$$

on the other.

(b) Assume that for each $c \in \mathscr{C}$, the functor $\mathscr{C}(c, U-)$ is representable. Suppose that there is an explicit choice of representing object $d_c \in \mathscr{D}$, that is, a choice of a natural isomorphism $\mathscr{D}(d_c, -) \cong \mathscr{C}(c, U-)$. Show that there exists a functor $G\colon \mathscr{C} \to \mathscr{D}$ with $Gc = d_c$ such that G is left adjoint to U.

(c) Show that if both F and G are left adjoint to U, then there exists a natural isomorphism $F \cong G$.

Exercise 5. *??*

An *ultrafilter* on a set X is a set $\mathscr{F}$ of subsets of X such that the following axiom holds: for all subsets $A \subseteq X$, A belongs to F if and only if for all $B_1, \ldots, B_n \in \mathscr{F}$, the intersection $A \cap B_1 \cap \ldots \cap B_n$ is non-empty. The *principal ultrafilter* of $x \in X$ is $\mathscr{F}_x = \{A \subseteq X | x \in A\}$. Given a subset $A \subseteq X$, we write $[A]$ for the set of ultrafilters $\mathscr{F}$ which contain A. We write UX for the set of ultrafilters on X. Given a function $f\colon X \to Y$ and an ultrafilter $\mathscr{F}$ on X, we call

$$f_*\mathscr{F} := \{B \subseteq Y | f^{-1}(B) \in \mathscr{F}\}$$

the pushforward of $\mathscr{F}$.

(a) Show that UX defines an endofunctor of **Set** via the pushforward.

(b) Show that $\eta_X\colon X \to UX$, $x \mapsto \mathscr{F}_x$ and $\mu_X\colon UUX \to UX$ defined by

$$\mu(\mathscr{F}) = \{A \subseteq X | [A] \in \mathscr{F}\}$$

endow U with the structure of a monad.

(c) Let $\xi\colon UX \to X$ be an algebra for the ultrafilter monad. We call a subset $U \subseteq X$ *open* if

$$\forall x \in X \ \forall \mathscr{F} \in UX\colon (x \in U \text{ and } \xi(\mathscr{F}) = x) \Rightarrow U \in \mathscr{F}$$

holds. Show that these open sets form a topology on X.

Monads and their applications 2

Exercise 1.
Let $\mathscr{C}$ be a category with finite coproducts. For an object $c \in \mathscr{C}$, let $c/\mathscr{C}$ denote the slice category, whose objects are morphisms with domain c and whose morphisms are commutative triangles. Show that the forgetful functor $c/\mathscr{C} \to \mathscr{C}$ is monadic (using Beck's theorem) and describe the monad in question.

Exercise 2.
Let $\mathscr{C}$ be the category of torsion free abelian groups.

(a) Show that the inclusion $\mathscr{C} \to \mathbf{Ab}$ is monadic (using the monadicity theorem).

(b) Show that the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ is monadic.

(c) Show that the composite $\mathscr{C} \to \mathbf{Set}$ of the above two functors is *not* monadic. (Hint: what happens to the canonical presentation of a finite abelian group?)

Exercise 3.
An object $g \in \mathscr{C}$ is called a *strong generator* if $\mathscr{C}(g, -)\colon \mathscr{C} \to \mathbf{Set}$ is conservative.

(a) Assume that $\mathscr{C}$ has small colimits and that g is a strong generator such that $\mathscr{C}(g, -)$ preserves small colimits (an object satisfying the latter condition is sometimes called *small projective*). Show that there exists a monoid M such that $\mathscr{C}$ is equivalent to the category of M-sets. (Hint: is $\mathscr{C}(g, -)$ monadic?)

(b) Show that the monoid M above is isomorphic to the endomorphism monoid $\mathscr{C}(g, g)$ of g.

Exercise 4.
Let $F\colon \mathscr{C} \to \mathscr{C}$ be an endofunctor. An F*-algebra* is a pair (c, γ) of an object $c \in \mathscr{C}$ and a morphism $\gamma\colon Fc \to c$ (not subject to any axioms). A morphism of algebras $(c, \gamma) \to (d, \delta)$ is a morphism $f\colon c \to d$ making the evident square commutative. We denote the category of F-algebras by $F\text{-}\mathbf{Alg}$. Show that the forgetful functor $F\text{-}\mathbf{Alg} \to \mathscr{C}$ is monadic if and only if it has a left adjoint.

Exercise 5.

Complete the proof of the monadicity theorem by showing that the natural transformations $\overline{\eta}$ and $\overline{\varepsilon}$ described in the lecture satisfy the triangle identities.

Monads and their applications 3

Exercise 1.
Let $F\colon \mathscr{A} \to \mathscr{C}$, $K\colon \mathscr{A} \to \mathscr{B}$ and $L\colon \mathscr{B} \to \mathscr{C}$ be functors. A natural transformation $\eta\colon F \Rightarrow L \circ K$ is said to exhibit L as left Kan extension of F along K if the composite

$$[\mathscr{B},\mathscr{C}](L,G) \xrightarrow{-\circ K} [\mathscr{A},\mathscr{C}](LK,G) \xrightarrow{\eta^*} [\mathscr{A},\mathscr{C}](F,G)$$

is a bijection for all functors $G\colon \mathscr{B} \to \mathscr{C}$. If a left Kan extension of F along K exists, then it is unique up to unique natural isomorphism and it is denoted by $\operatorname{Lan}_K F$.

(a) Show that left adjoints preserve left Kan extensions in the following sense: if $\eta\colon F \Rightarrow LK$ exhibits L as left Kan extension of F along K and $H\colon \mathscr{C} \to \mathscr{D}$ is a left adjoint, then $H\eta$ exhibits HL as left Kan extension of HF along K.

(b) Show that left Kan extensions compose: if $\operatorname{Lan}_K F$ exists and

$$K'\colon \mathscr{B} \to \mathscr{B}'$$

is any functor, then $\operatorname{Lan}_{K'} \operatorname{Lan}_K F$ exists if and only if $\operatorname{Lan}_{K'K} F$ exists. Moreover, show that in this case there is a natural isomorphism $\operatorname{Lan}_{K'} \operatorname{Lan}_K F \cong \operatorname{Lan}_{K'K} F$.

Exercise 2.
The notion of *right* Kan extension is dual to left Kan extension: it is given by a universal natural transformation $\gamma\colon RK \Rightarrow F$ and denoted by $\operatorname{Ran}_K F$.

(a) Let $F\colon \mathscr{A} \to \mathscr{C}$ be a functor such that the right Kan extension

$$\operatorname{Ran}_F F\colon \mathscr{C} \to \mathscr{C}$$

of F along itself exists. Show that $\operatorname{Ran}_F F$ has the structure of a monad in a natural way. This monad is called the *codensity monad* of F.

(b) If $\mathscr{A} = *$ is the terminal category, then giving a functor $F\colon \mathscr{A} \to \mathscr{C}$ amounts to picking an object $c \in \mathscr{C}$, $c = F(*)$. Show that, in this case, $\operatorname{Ran}_F F$ exists if $\mathscr{C}$ has products. The resulting codensity monad is called the *endomorphism monad* of c and denoted by $\langle c, c\rangle$.

Exercise 3.
Let k be a field and $\mathbf{Vect}_k$ the category of k-vector spaces. Let $\mathscr{A} = \{k\}$ be the full subcategory on the one-dimensional vector space k. Note that every object of $\mathbf{Vect}_k$ is a colimit of some diagram that factors through $\mathscr{A}$ (since all vector spaces are free).

(a) Show that, nevertheless, the inclusion $\mathscr{A} \to \mathbf{Vect}_k$ is *not* dense.

(b) Let $\mathscr{B} = \{k \oplus k\}$ be the full subcategory one the two-dimensional vector space. Show that the inclusion $\mathscr{B} \to \mathbf{Vect}_k$ is dense.

Exercise 4.
Let $\mathscr{A}$ be a small category and let $Y\colon \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ be the Yoneda embedding.

(a) Use the Yoneda lemma to show that the canonical cocone on Y/F exhibits F as colimit of the domain functor $\mathrm{dom}\colon Y/F \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$, $(\varphi\colon \mathscr{A}(-, a) \Rightarrow F) \mapsto \mathscr{A}(-, a)$.

(b) The category $\mathrm{el}(F)$ of elements of F has objects the pairs (a, x) where $a \in \mathscr{A}$ and $x \in Fa$ and morphisms $(a, x) \to (b, y)$ the morphisms $f\colon a \to b$ in $\mathscr{A}$ which satisfy $Ff(y) = x$. Show that there is an isomorphism $Y/F \cong \mathrm{el}(F)^{\mathrm{op}}$.

Exercise 5. *(bonus)*
An object $c \in \mathscr{C}$ is called *strongly finitely presentable* if the representable functor $\mathscr{C}(c, -)\colon \mathscr{C} \to \mathbf{Set}$ preserves sifted colimits. A cocomplete category $\mathscr{C}$ is called *locally strongly finitely presentable* if there exists a small dense subcategory $\mathscr{A}$ of $\mathscr{C}$ which consists of strongly finitely presentable objects.

(a) Show that finite coproducts of strongly finitely presentable objects are strongly finitely presentable.

(b) Let $U\colon \mathscr{D} \to \mathscr{C}$ have a left adjoint $F\colon \mathscr{C} \to \mathscr{D}$. Show the following claim: if U preserves sifted colimits, then F preserves strongly finitely presentable objects.

(c) Let $\mathscr{C}$ be a strongly finitely presentable category and let $T\colon \mathscr{C} \to \mathscr{C}$ be a monad which commutes with sifted colimits. Show that T-$\mathbf{Alg}$ is locally strongly finitely presentable. (Hint: let $\mathscr{A}$ be a dense subcategory of $\mathscr{C}$ consisting of strongly finitely presentable objects. Show that the objects (Ta, μ_a) form a dense subcategory of T-$\mathbf{Alg}$).

Monads and their applications 4

Exercise 1.
Let $F\colon \mathscr{A} \to \mathscr{C}$, $K\colon \mathscr{A} \to \mathscr{B}$ be two functors where $\mathscr{A}$ is essentially small. Note that K is not assumed to be full and faithful.

(a) Show that $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](\widetilde{K}c, \widetilde{F}-)\colon \mathscr{D} \to \mathbf{Set}$ is representable if and only if the colimit of the diagram $K/c \to \mathscr{D}$ which sends $(a, \varphi\colon Ka \to c)$ to $Fa \in \mathscr{D}$ has a colimit.

(a) Assume that the colimit of part (a) always exist. By Yoneda, there exists a functor $L\colon \mathscr{C} \to \mathscr{D}$ with bijections

$$\mathscr{D}(Lc, d) \cong [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](\widetilde{K}c, \widetilde{F}d)$$

which are natural in c and d. Show that, in this case, there exists a natural transformation $\eta\colon F \Rightarrow LK$ which exhibits L as left Kan extension of F along K.

Exercise 2.

(a) Let $(\mathscr{C}_i)_{i\in I}$ be a family of locally finitely presentable categories. Show that the product $\prod_{i\in I} \mathscr{C}_i$ is locally finitely presentable.

(b) Let $\mathscr{A}$ be a small category and $\mathscr{C}$ locally finitely presentable. Show that $[\mathscr{A}, \mathscr{C}]$ is locally finitely presentable.

Exercise 3.
Let $\mathscr{C}$ be a cocomplete category. An object $a \in \mathscr{C}$ is called *small projective* if $\mathscr{C}(a, -)$ preserves *all* small colimits. Suppose there exists a small subcategory $\mathscr{A} \subseteq \mathscr{C}$ such that the closure of $\mathscr{A}$ under colimits is all of $\mathscr{C}$. Show that $\mathscr{C} \simeq [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$. (Hint: start by showing that the inclusion $K\colon \mathscr{A} \to \mathscr{C}$) is dense.

Exercise 4.
Let $\mathscr{C}$ be locally finitely presentable, $\mathscr{A}$ a small dense subcategory consisting of finitely presentable objects. Let $\mathscr{A}'$ be the closure of $\mathscr{A}$ under finite colimits. Let $\mathscr{C}_{\mathrm{fp}}$ denote the full subcategory consisting of finitely presentable objects. From the lecture, we know that $\mathscr{A}' \subseteq \mathscr{C}_{\mathrm{fp}}$. Show that this in fact an equality: every finitely presentable object lies in the closure of $\mathscr{A}$ under finite colimits.

Exercise 5. *??*

Show that the finitely presentable objects in the category of topological spaces are precisely the finite discrete spaces, and that they do not form a dense generator. Thus **Top** is not finitely presentable.

Monads and their applications 5

Exercise 1.
Let (S,σ) be a well-pointed endofunctor of $\mathscr{D}$. Let $\mathscr{C}$ be a finitely cocomplete category and let $F\colon \mathscr{D} \to \mathscr{C}$ be left adjoint to $U\colon \mathscr{C} \to \mathscr{D}$. Show that the endofunctor (S',σ') of $\mathscr{C}$ defined by the pushout diagram

$$\begin{tikzcd} FU \arrow[r, "F\sigma U"] \arrow[d, "\varepsilon"'] & FSU \arrow[d] \\ \mathrm{id} \arrow[r, "\sigma'"'] & S' \end{tikzcd}$$

in $[\mathscr{C},\mathscr{C}]$ is a well-pointed endofunctor and that there is an induced diagram

$$\begin{tikzcd} (S',\sigma')\text{-}\mathbf{Alg} \arrow[r, "\overline{U}"] \arrow[d] & (S,\sigma)\text{-}\mathbf{Alg} \arrow[d] \\ \mathscr{C} \arrow[r, "U"'] & \mathscr{D} \end{tikzcd}$$

which is a (strict) pullback diagram of categories.

Exercise 2.
Let $\big((S_i,\sigma_i)\big)_{i\in I}$ be a family of accessible well-pointed endofunctors of a locally presentable category $\mathscr{C}$. Let S be the colimit in $[\mathscr{C},\mathscr{C}]$ of the "star-shaped" diagram $\sigma_i\colon \mathrm{id}_{\mathscr{C}} \Rightarrow S_i$ (that is, a star with center $\mathrm{id}_{\mathscr{C}}$). Write σ for the induced composite $\mathrm{id} \Rightarrow S_i \Rightarrow S$ (which is by construction independent of i). Show that (S,σ) is an accessible well-pointed endofunctor and that (S,σ)- $\mathbf{Alg}$ is the intersection $\bigcap_{i\in I}(S_i,\sigma_i)$- $\mathbf{Alg}$.

Exercise 3.
Let $k\colon a \to b$ be a morphism of a category $\mathscr{C}$. An object $c \in \mathscr{C}$ is called *orthogonal* to k if for any $f\colon a \to c$, there exists a unique dashed arrow making the triangle

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "k"'] & c \\ b \arrow[ur, dashed] & \end{tikzcd}$$

commutative. The full subcategory of objects orthogonal to k is denoted by $\{k\}^{\perp} \subseteq \mathscr{C}$. Similarly, given a set Σ of morphisms in $\mathscr{C}$, we write $\Sigma^{\perp} \subseteq \mathscr{C}$ for the class of objects which are orthogonal to all the morphisms in Σ.

(a) Let $\mathscr{C}$ be locally presentable and let $k\colon a \to b$ be a morphism in $\mathscr{C}$. This defines a functor $k\colon [1] \to \mathscr{C}$, where $[1] = \{0 \to 1\}$ denotes the category consisting of a single non-trivial morphism. Show that the right adjoint of the induced adjunction $\mathrm{Lan}_Y k\colon \mathbf{Set}^{[1]^{\mathrm{op}}} \rightleftarrows \mathscr{C} \colon \widetilde{k}$ is accessible.

(b) Show that there exists an accessible well-pointed endofunctor (S_k, σ_k) of $\mathscr{C}$ such that (S_k, σ_k)- $\mathbf{Alg}$ is equal to the full subcategory $\{k\}^{\perp} \subseteq \mathscr{C}$ of objects orthogonal to k. (Hint: apply Exercise 1 to the adjunction of (a).)

(c) Let Σ be a set of morphisms in $\mathscr{C}$. Show that there exists an accessible well-pointed endofunctor of $\mathscr{C}$ whose category of algebras is $\Sigma^{\perp} \subseteq \mathscr{C}$.

Exercise 4.
Let $\mathscr{A}$ be a small category and consider a set $\left(D^k\colon \mathscr{I}_k \to \mathscr{A}\right)_{k\in K}$ of diagrams in $\mathscr{A}$. For each diagram D^k, fix a cocone $\kappa_i\colon D^k_i \to a_k$ in $\mathscr{A}$. Let $\mathscr{C} \subseteq [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ be the full subcategory of presheaves $F\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ with the property that $F\kappa_i\colon Fa_k \to FD^k_i$ is a limit cone for each $k \in K$. Show that there exists a set Σ of morphisms in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ such that $\mathscr{C} = \Sigma^{\perp}$.

Exercise 5. *??*
From the lecture, we know that the category $\mathscr{C}$ in Exercise 4 is reflective. Show that the composite

$$\mathscr{A} \xrightarrow{Y} [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathscr{C}$$

of the Yoneda embedding and the left adjoint of the inclusion is the universal functor to a cocomplete category which sends the given cones to colimit cones.

Monads and their applications 6

Exercise 1.
Let (S,σ) be a well-pointed endofunctor of $\mathscr{C}$. Let $L\colon \mathscr{C} \to \mathscr{C}$ be an endofunctor of $\mathscr{C}$ and $\pi\colon S \Rightarrow L$ a natural transformation such that $\pi_c\colon Sc \to Lc$ is an epimorphism for all $c \in \mathscr{C}$. Show that $(L,\pi\sigma)$ is a well-pointed endofunctor and that $(L,\pi\sigma)$- **Alg** is equivalent to the full subcategory of (S,σ)- **Alg** consisting of objects (a,α) such that $\pi_a\colon Sa \to La$ is an isomorphism.

Exercise 2.
Let $\mathscr{C}$ be a cocomplete category and let (T,τ) be a pointed endofunctor of $\mathscr{C}$. Let (S,σ) be the well-pointed endofunctor of the arrow category $\mathscr{C}^{[1]}$ whose algebras are the isomorphisms: $S(c \to c') = \mathrm{id}_{c'}$. Recall that we defined a well-pointed endofunctor (S',σ') on the slice category $T/\mathscr{C}$ by the pushout

$$\begin{tikzcd}
\tau_! \tau^* \arrow[r, "\tau_! \sigma \tau^*"] \arrow[d, "\varepsilon"'] & \tau_! S \tau^* \arrow[d] \\
\mathrm{id} \arrow[r, "\sigma'"'] & S'
\end{tikzcd}$$

in $[T/\mathscr{C}, T/\mathscr{C}]$. Show that S' sends $(a,b,\alpha\colon Ta \to b)$ to $(b,c,\beta\colon Tb \to c)$, where β denotes the coequalizer

$$\begin{tikzcd}
T^2a \arrow[r, shift left, "T\tau_a"] \arrow[r, shift right, "\tau_{Ta}"'] & T^2a \arrow[r, "T\alpha"] & Tb \arrow[r, "\beta"] & c
\end{tikzcd}$$

in $\mathscr{C}$. (Hint: if you get stuck, this is discussed in Kelly's "transfinite constructions," §17).

Exercise 3.
Let $\mathscr{C}$ be a locally λ-presentable category, $F\colon \mathscr{C} \to \mathscr{D}$ λ-accessible, and let $G\colon \mathscr{C} \to \mathscr{D}$ be a right adjoint. Recall that the slice category F/G has objects the triples $(a,b,\gamma\colon Fa \to Gb)$, with morphisms given by pairs of morphisms in $\mathscr{C}$ making the evident square commutative.

(a) Show that the slice category F/G is locally λ-presentable.

(b) Suppose that G is also λ-accessible. Show that, in this case, both the domain functor and the codomain functor $F/G \to \mathscr{C}$ which send (a,b,γ) to a respectively b are λ-accessible.

Exercise 4.
Let $\mathscr{C}$, $\mathscr{D}$ be locally λ-presentable, $F\colon \mathscr{C} \to \mathscr{D}$ left adjoint to $U\colon \mathscr{D} \to \mathscr{C}$. Let $\kappa \geq \lambda$ be a regular cardinal.

(a) Show that if U is κ-accessible, then $F(\mathscr{C}_\kappa) \subseteq \mathscr{D}_\kappa$. (Recall that $\mathscr{C}_\kappa$ denotes the full subcategory of κ-presentable objects in $\mathscr{C}$.)

(b) Show that there exists a regular cardinal $\mu \geq \lambda$ such that U is μ-accessible. (Hint: consider the composite of U with the full and faithful $\widetilde{K}\colon \mathscr{C} \to [\mathscr{C}_\lambda^{\mathrm{op}}, \mathbf{Set}]$, where $K\colon \mathscr{C}_\lambda \to \mathscr{C}$ denotes the inclusion.)

(c) Conclude that for any left adjoint F between locally λ-presentable categories, there exists a regular cardinal $\mu \geq \lambda$ such that $F(\mathscr{C}_\mu) \subseteq \mathscr{D}_\mu$.

Exercise 5. *??*
Show that left adjoint functors between locally presentable categories are precisely the cocontinuous functors and that right adjoints between locally presentable categories are precisely the continuous functors which are also accessible.

Monads and their applications 7

Exercise 1.
Complete the proof that $(-)\text{-}\mathbf{Alg}\colon \mathbf{Mnd}(\mathscr{C})^{\mathrm{op}} \to \mathbf{CAT}/\mathscr{C}$ is a functor and that it is full and faithful.

Exercise 2.
Recall that the *codensity monad* $\mathrm{Ran}_F F$ of $F\colon \mathscr{A} \to \mathscr{C}$ is the right Kan extension of F along itself.

(a) Show that the right Kan extension of any functor G along a right adjoint U always exists and is given by GL ("by adjunction").

(b) Show that the codensity monad of a right adjoint is precisely the monad $(UL, \eta, U\varepsilon L)$ associated to the adjunction.

Exercise 3.
We call a functor $F\colon \mathscr{A} \to \mathscr{C}$ *admissible* if $\mathrm{Ran}_F F$ exists and we write $\mathbf{CAT}'/\mathscr{C}$ for the full subcategory of admissible functors. From Exercise 2 it follows that $(-)\text{-}\mathbf{Alg}$ factors through the admissible functors. We denote the codensity monad of F by $S(F)$.

(a) Show that

$$(-)\text{-}\mathbf{Alg}\colon \mathbf{Mnd}(\mathscr{C})^{\mathrm{op}} \to \mathbf{CAT}'/\mathscr{C}$$

is right adjoint to

$$S\colon\ \mathbf{CAT}'/\mathscr{C} \to \mathbf{Mnd}(\mathscr{C})^{\mathrm{op}}$$

(this is called the *semantics-structure adjunction*).

(b) Use this adjunction to give a rigorous argument that the algebra functor

$$(-)\text{-}\mathbf{Alg}\colon \mathbf{Mnd}(\mathscr{C})^{\mathrm{op}} \to \mathbf{CAT}/\mathscr{C}$$

with target the full slice category sends colimits to limits if $\mathscr{C}$ is complete.

(c) Use the adjunction to give an alternative proof that $(-)\text{-}\mathbf{Alg}$ is full and faithful.

Exercise 4.
Directed graphs are presheaves on $G = \{\, 0 \rightrightarrows 1 \,\}$, that is, pairs of sets E and V with source and target maps $s\colon E \to V$ and $t\colon E \to V$. Use finitary endofunctors of the presheaf category such as $(E, V, s, t) \mapsto (V, V, \mathrm{id}, \mathrm{id})$ or the functor which sends a graph to the graph consisting of paths of length two in the original (that is, the edges in the new graph are given by the pullback of s along t) to construct a finitary monad on $[G^{\mathrm{op}}, \mathbf{Set}]$ whose category of algebras is isomorphic to $\mathbf{Cat}$.

Exercise 5. *??*
Let $\mathscr{C}$ be the category whose objects are small categories with a choice of colimit for each finite diagram and whose morphisms are the functors which preserve these colimits strictly. Use free finitary monads and colimits of such to show that $\mathscr{C}$ is finitarily monadic over $\mathbf{Cat}$.

Monads and their applications 8

Exercise 1.
Let $\mathscr{V}$, $\mathscr{W}$, $\mathscr{U}$ be monoidal categories and let $(F, \varphi_0, \varphi)\colon \mathscr{V} \to \mathscr{W}$ and $(G, \psi_0, \psi)\colon \mathscr{W} \to \mathscr{U}$ is lax monoidal with structure morphisms $\gamma_0 = G(\varphi_0)\cdot\psi_0$ and $\gamma_{X,Y} = G(\varphi_{X,Y}) \cdot \psi_{FX,FY}$. Show that monoidal natural transformations can be whiskered on either side with lax monoidal functors.

Exercise 2.
Let $(F, \varphi_0, \varphi)\colon \mathscr{V} \to \mathscr{W}$ be strong monoidal and suppose that the underlying functor F has a right adjoint U. Show that the composites

$$I_{\mathscr{V}} \xrightarrow{\eta_{I_{\mathscr{V}}}} UF(I) \xrightarrow{U(\varphi_0^{-1})} U(I_{\mathscr{W}})$$

and

$$\begin{tikzcd}
UX \otimes_{\mathscr{V}} UY \arrow[r, "\eta_{UX \otimes_{\mathscr{V}} UY}"] \arrow[d, dashed] & UF(UX \otimes_{\mathscr{V}} UY) \arrow[d, "U\varphi^{-1}_{UX,UY}"] \\
U(X \otimes_{\mathscr{W}} Y) & U(FUX \otimes_{\mathscr{W}} FUY) \arrow[l, "U(\varepsilon_X \otimes_{\mathscr{W}} \varepsilon_Y)"]
\end{tikzcd}$$

endow U with the structure of a lax monoidal functor, and that η, ε are monoidal natural transformations for this structure if the composites UF and FU are given the lax monoidal structure of Exercise 1.

Exercise 3.
Let $\mathscr{V}$, $\mathscr{W}$, be monoidal categories, $(F, \varphi_0, \varphi)\colon \mathscr{V} \to \mathscr{W}$ a strong monoidal left adjoint, and $f\colon S \to T$ a function of sets.

(a) Show that f induces a strong monoidal $f_*\colon \mathbf{Mat}(\mathscr{V}, S) \to \mathbf{Mat}(\mathscr{W}, T)$.

(b) Show that F induces a strong monoidal $F\colon \mathbf{Mat}(\mathscr{V}, S) \to \mathbf{Mat}(\mathscr{V}, S)$.

(c) Use Exercise 2 to show that these are both monoidal adjunctions.

Exercise 4.
Let $\mathscr{C}$ be a complete category, a, b objects of $\mathscr{C}$. Recall that $\langle a, b\rangle$ is defined to be the right Kan extension of $b\colon * \to \mathscr{C}$ along $a\colon * \to \mathscr{C}$. Show that

$$\big(\mathrm{Ob}(\mathscr{C}), (\langle a, b\rangle)_{(a,b)\in \mathrm{Ob}\,\mathscr{C}\times\mathrm{Ob}(\mathscr{C})}\big)$$

defines a category enriched in the monoidal category $[\mathscr{C}, \mathscr{C}]$ of endofunctors.

Exercise 5. *??*
There are two natural monoidal functors $\mathscr{V} \to [\mathscr{V}, \mathscr{V}]$ given by tensoring in either side. Under what conditions do these have a right adjoint? (For example, is locally presentable enough?) Applying Exercise 3(b) to the enriched category of Exercise 4, we get two $\mathscr{V}$-category structures on $\mathscr{V}$. Describe them explicitly. (Hint: you only need to know what the right adjoint does to functors of the form $\langle V, W\rangle$).

Monads and their applications 9

Exercise 1.
Show that the interchange law holds in $\mathscr{V}\text{-}\mathbf{CAT}$.

Exercise 2.
Let $F\colon \mathscr{V} \to \mathscr{W}$ be a lax monoidal functor. Show that F induces a 2-functor

$$F_*\colon \mathscr{V}\text{-}\mathbf{CAT} \to \mathscr{W}\text{-}\mathbf{CAT}$$

which sends a $\mathscr{V}$-category $\mathscr{A}$ to the $\mathscr{W}$-category $F_*\mathscr{A}$ with the same objects as $\mathscr{A}$, and hom-objects between a and b given by $F\big(\mathscr{A}(a,b)\big)$.

Exercise 3.
Let $\mathbf{Mon}(\mathbf{CAT})$ denote the 2-category of monoidal categories, lax monoidal functors, and monoidal natural transformations. Show that the assignment

$$(-)\text{-}\,\mathbf{Cat}\colon\ \mathbf{Mon}(\mathbf{CAT}) \to 2\text{-}\,\mathbf{CAT}$$

which sends $\mathscr{V}$ to the 2-category $\mathscr{V}\text{-}\,\mathbf{Cat}$ extends to a 2-functor, with action on 1-cells given by the 2-functor of Exercise 2.

Exercise 4.
Let $T\colon \mathscr{C} \to \mathscr{C}$ be a $\mathscr{V}$-monad. Complete the proof the $T\text{-}\,\mathbf{Alg}$ has the structure of a $\mathscr{V}$-category. Hint: at some point in the proof that algebra morphisms compose in the unenriched case, we use associativity, namely when we consider the composite

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, dashed] & TB \arrow[r, dashed] \arrow[d, "\beta"] & \cdot \arrow[d, dashed] \\
\cdot \arrow[r, dashed] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

in $\mathscr{C}$. This switches the operations "precompose with β" to "postcompose with β" and when translating the proof to $\mathscr{V}$-categories, one has to use the associator at this point.

Exercise 5.
Show that $T\text{-}\,\mathbf{Alg}$ has the same universal property in $\mathscr{V}\text{-}\,\mathbf{CAT}$ as the ordinary category of algebras for an unenriched monad has in $\mathbf{CAT}$. Namely, for each $\mathscr{V}$-category $\mathscr{A}$, define the category $T\text{-}\mathbf{Act}(\mathscr{A},\mathscr{C})$ of T-actions (now given by $\mathscr{V}$-functors with a $\mathscr{V}$-natural transformation $\rho\colon TG \Rightarrow G$) and show that $U^T\colon T\text{-}\,\mathbf{Alg} \to \mathscr{C}$ is the universal T-action.

Monads and their applications 10

Exercise 1.
Let $\mathscr{V}$ be a braided monoidal category. Show that the monoidal structure of $\mathscr{V}$ lifts to a monoidal structure of $\mathbf{Mon}(\mathscr{V})$ (that is, put a monoid structure on $M \otimes M'$ so that α, ρ, λ become monoid morphisms). If $\mathscr{V}$ is symmetric, show that γ lifts to a symmetry on $\mathbf{Mon}(\mathscr{V})$.

Exercise 2.
Show that a monoid in $\mathbf{Mon}(\mathscr{V})$ is a commutative monoid (this is known as the "Eckmann-Hilton argument"). More precisely, if $\big((M, \mu, \eta), \mu', \eta'\big)$ is a monoid in $\mathbf{Mon}(\mathscr{V})$, show that $\eta' = \eta$, $\mu' = \mu$ and $\mu\gamma_{M,M} = \mu$. (Hint: it is best to first prove this for $\mathscr{V} = \mathbf{Set}$ and then translate to general monoidal categories. First show that the two units $\eta = \eta'$ are the same and then turn the following pictorial "proof" into a precise argument, where we write one multiplication vertically, one horizontally:

$$\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & \eta \\ \eta & b \end{bmatrix} = \begin{bmatrix} \eta & a \\ b & \eta \end{bmatrix} = \begin{bmatrix} b & a \end{bmatrix}$$

for all $a, b \in M$.

Exercise 3.
Let $\mathscr{V}$ be a symmetric monoidal closed category, $\mathscr{C}$ a $\mathscr{V}$-category, and $C \in \mathscr{C}$. Show that the functor $\mathscr{C}(C, -)\colon \mathscr{C} \to \mathscr{V}$ defined in class is indeed a $\mathscr{V}$-functor.

Exercise 4.
Let $\mathscr{V}$ be a symmetric monoidal category. If $A \in \mathscr{V}$ is a monoid, then $A \otimes -$ is a monad. The $(A \otimes -)$-aglebras are called *A-modules* and we write $\mathscr{V}_A$ for $(A \otimes -)$- $\mathbf{Alg}$. Now suppose that A is commutative, that $\mathscr{V}$ has reflexive coequalizers, and that $V \otimes -$ preserves these for all $V \in \mathscr{V}$. Given A-modules M and N, define $M \otimes_A N$ by the reflexive coequalizer

$$M \otimes A \otimes N \rightleftarrows M \otimes N \longrightarrow M \otimes_A N$$

in $\mathscr{V}$. Show that this lifts to a functor $- \otimes_A -\colon \mathscr{V}_A \otimes \mathscr{V}_A \to \mathscr{V}_A$ making $\mathscr{V}_A$ into a symmetric monoidal category with unit A (Hint: use the fact that reflexive coequalizers are sifted and split coequalizers are absolute).

Exercise 5. *??*
Give an example of a braided monoidal category $\mathscr{V}$ such that the braiding does not lift to $\mathbf{Mon}(\mathscr{V})$.

Monads and their applications 11

Exercise 1.
Let $\mathscr{V}$ be the category $\mathrm{Ch}(\mathbf{Ab})$ of chain complexes over abelian groups ("differential graded $\mathbb{Z}$-modules"). The tensor product of $X_\bullet$ and $Y_\bullet$ is given in degree n by $\oplus_{i+j=n} X_i \otimes Y_j$, with differential $d(x \otimes y) = dx \otimes y + (-1)^i x \otimes dy$ for $x \otimes y \in X_i \otimes Y_j$. Show that this is a symmetric monoidal closed category and explicitly describe the self-enrichment of $\mathscr{V}$ (specifically, the composition morphism).

Exercise 2.
Let $\mathscr{C}$ be a $\mathscr{V}$-category with powers, $T\colon \mathscr{C} \to \mathscr{C}$ a $\mathscr{V}$-monad. Show that the forgetful $\mathscr{V}$-functor $T\text{-}\mathbf{Alg} \to \mathscr{C}$ creates powers (that is, for every T-algebra (A, a) and $V \in \mathscr{V}$ one can put a unique T-algebra structure on A^V making it a power of (A, a) in $T\text{-}\mathbf{Alg}$).

Exercise 3.
Let $t\colon C \to C$ be a monad in the 2-category $\mathscr{K}$. A t-action on a 1-cell $g\colon A \to C$ is a 2-cell $\alpha\colon tg \Rightarrow g$ satisfiying the laws for a t-algebra. This defines a 2-functor

$$t\text{-act}(-)\colon \mathscr{K}^{\mathrm{op}} \to \mathbf{Cat}$$

and we say that the *Eilenberg–Moore* object of t exists if this 2-functor is representable, that is, there exists a *universal* t-action $(u, \alpha)\colon C_t \to C$.

(a) Show that $U\colon T\text{-}\mathbf{Alg} \to \mathscr{C}$ is an Eilenberg–Moore object in $\mathscr{V}\text{-}\mathbf{Cat}$.

(b) Given any Eilenberg–More object $u\colon C_t \to C$, show that there exists a left adjoint f of u such that the monad uf is t. (Hint: define a suitable t-action on t itself).

Exercise 4.

(a) Let $\mathscr{V} = \mathbf{Set}$ and consider the discrete category $2 = \{0, 1\}$ on two objects. A weight W on this category amounts to a choice of two sets. What is the W-weighted colimit on a diagram $2 \to \mathscr{C}$?

(b) Let $\mathscr{V} = \mathbf{Ab}$ and let R be a ring, considered as a 1-object $\mathscr{V}$-category. Let $\mathscr{C} = \mathbf{Ab}$ as well. Show that weighted colimits over R correspond to the usual tensor product of a right and a left R-module.

(c) Let $\mathscr{V} = \mathbf{Cat}$ and consider the category $\mathscr{I} = 0 \longrightarrow 2 \longleftarrow 1$. Let W be the weight with $W(0) = *$, $W(1) = *$, and $W(2) = [1]$, with the morphisms $W(i) \to [1]$ picking out $i \in [1] = \{0 \to 1\}$. Show that W-weighted limits in a 2-category $\mathscr{K}$ are precisely comma-objects.

Exercise 5. *??*
Show that $\prod_{j \in J} \mathscr{V}$ is the free cocomplete $\mathscr{V}$-cateogry on the discrete category j. More precisely, given a cocomplete $\mathscr{V}$-categories $\mathscr{C}$, $\mathscr{D}$, write $\mathbf{Cocts}_0[\mathscr{C}, \mathscr{D}]$ for the category of cocontinuous $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations. Show that the functor $\mathbf{Cocts}_0[\prod_{j \in J} \mathscr{V}, \mathscr{C}] \to \prod_{j \in J} \mathscr{C}_0$ given by $F \mapsto (FI_j)_{j \in J}$ is an equivalence of categories. Here I_j stands for the object which is given by I in degree j and by the initial object everywhere else. You need to be careful when showing that the functor is full!

If one checks all the requirements by only referring to copowers, $\mathscr{V}$-coproducts, and $\mathscr{V}$-coequalizers, (the latter is not even necessary), then one can use this to show that cocontinuous endofunctors of the product are equivalent, as a monoidal category, to $\mathscr{V}$-matrices on J. This can then be used to give a rigorous proof that any $\mathscr{V}$-category with object set J gives rise to the cocontinous $\mathscr{V}$-monad T defined in the lecture.

Monads and their applications 12

Exercise 1.
Let $\mathscr{V}$ be a cosmos, A and B monoids in $\mathscr{V}$. The category $\mathscr{V}_A$ of left A-modules is precisely the functor category $[A, \mathscr{V}]$. Show that the category of left $\mathscr{V}$-adjoints $\mathscr{V}_A \to \mathscr{V}_B$ is equivalent to the category of A-B-bimodules (Hint: use the theory of free cocompletions). Apply this to the case of equivalences to show that the module categories are equivalent as $\mathscr{V}$-categories if and only if there exists an A-B bimodule M and a B-A-bimodule N with $N \otimes_B M \cong A$ and $M \otimes_A N \cong B$ (this is called *Morita equivalence* of monoids).

Exercise 2.
Recall that an Eilenberg–Moore object (EM-object for short) represents actions of a monoad in a 2-category $\mathscr{K}$. A *Kleisli-object* is the dual notion, that is, an EM-object in $\mathscr{K}^{\mathrm{op}}$: given a monad $t\colon C \to C$, the Kleisli-object is the universal morphism $f\colon C \to C^t$ with an action $ft \Rightarrow f$.

Show that **CAT** has Kleisli-objects, described as follows: for a monad $T\colon \mathscr{C} \to \mathscr{C}$, the category $\mathscr{C}^T$ has the same objects as $\mathscr{C}$ and $\mathscr{C}^T(A,B) := \mathscr{C}(A,TB)$. The composition is defined using the monad multiplication. Show that $\mathscr{C}^T$ is equivalent to the full subcategory of T-**Alg** consisting of the free algebras.

Exercise 3.
Let $\mathscr{K}$ be a 2-category with EM-objects and let $C \in \mathscr{K}$. Write $\mathrm{Mnd}(C)$ for the category of monads on C. Call a 1-cell $g\colon A \to C$ *tractable* if the right Kan extension of g along itself exists. Write $\mathscr{K}_0'/C$ for the full subcategory of the slice (1-)category consisting of tractable 1-cells. Show that the functor

$$(-)\text{-}\mathbf{Alg}\colon \mathrm{Mnd}(C)^{\mathrm{op}} \to \mathscr{K}_0'/C$$

which sends t to its EM-object is right adjoint to the functor which sends g to the density monad $\mathrm{Ran}_g\, g$.

Exercise 4.
Let $\mathscr{C}$ be an unenriched cocomplete category. Let $\mathscr{G}$ be a full subcategory consisting of κ-presentable objects and suppose that $\mathscr{G}$ is a *strong* generator: the functor

$$\widetilde{K} = \mathrm{Hom}_{\mathscr{G}}(K,-)\colon \mathscr{C} \to [\mathscr{G}^{\mathrm{op}}, \mathbf{Set}]$$

is conservative. The goal of this exercise is to show that $\mathscr{C}$ is locally κ-presentable, that is, there automatically exists a *dense* generator of κ-small objects. More precisely, let $\mathscr{A}$ be the closure of $\mathscr{G}$ under κ-small colimits. Show that $\mathscr{A}$ is dense using the following steps.

(a) Show that the category $\mathscr{A}/C$ is κ-filtered and that the canonical diagram $\mathscr{A}/C \to \mathscr{C}$ is sent to a colimit diagram by the functor $\widetilde{K}$.

(b) Show the following fact about conservative functors $F\colon \mathscr{A} \to \mathscr{B}$: if $\mathscr{A}$ has colimits of a given shape, F preserves colimits of that shape, and a specific cocone is sent to a colimit cocone by F, then the cocone in question is already a colimit cocone. In other words, a conservative functor *detects* all the colimits that exist in the domain and that it preserves.

(c) Conclude that $\mathscr{A}$ is dense and thus that $\mathscr{C}$ is locally κ-presentable.

Exercise 5. *??*

Let $\mathscr{V}$ be a cosmos. Given two *small* $\mathscr{V}$-categories, show that there exists a $\mathscr{V}$-category $[\mathscr{A}, \mathscr{B}]$ whose underlying category is $\mathscr{V}\text{-}\mathbf{Cat}(\mathscr{A}, \mathscr{B})$. The hom-object can be defined using the usual equalizer in $\mathscr{V}$ which would give natural transformations for $\mathscr{V} = \mathbf{Set}$. Show that this defines an internal hom-object in the monoidal category $\mathscr{V}\text{-}\mathbf{Cat}$, that is, $- \otimes \mathscr{A} \dashv [\mathscr{A}, -]$.