Monads and their applications

Dr. Daniel Schäppi's course lecture notes

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Categorical preliminaries

Definition 1.0.1 (Categories)**.** A *category* $\mathbf{C}$ consists of:

1. a collection of objects $\mathrm{Ob}(\mathbf{C})$;

2. a collection of arrows $\mathrm{Ar}(\mathbf{C})$;

3. two maps $\mathrm{dom}, \mathrm{cod} \colon \mathrm{Ar}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$;

4. a map $\mathrm{id}_{-} \colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ar}(\mathbf{C})$ with $\mathrm{dom}(\mathrm{id}_c) = c = \mathrm{cod}(\mathrm{id}_c)$;

5. for every $f, g \in \mathrm{Ar}(\mathbf{C})$ such that $\mathrm{cod}(f) = \mathrm{dom}(g)$ a unique composite morphism gf such that $\mathrm{cod}(gf) = \mathrm{cod}(g)$, $\mathrm{dom}(gf) = f$.

This data has to satisfy the following axioms

1. given $f \in \mathrm{Ar}(\mathbf{C})$, $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$, $\mathrm{id}_{c'} f = f = \mathrm{id}_c$, that is the composition is unital;

2. given a composable triple $f, g, h \in \mathrm{Ar}(\mathbf{C})$, $h(gf) = (hg)f$, that is the composition is associative.

An arrow f such that $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$ is denoted $f \colon c \to c'$.

Definition 1.0.2 (Functors)**.**

Definition 1.0.3 (Full functors, faithful functor)**.**

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Proof. □

Chapter 2

Monads and algebras

2.1 Introduction

Throughout mathematics we encounter structures defined by some action morphisms. Here we give some examples.

Example 2.1.1. Given a group G, we may consider a G-set X described by an action map $G \times X \to X$.

Example 2.1.2. Given an abelian group M and a ring R, we can get an R-module M by fixing a group homomorphism $R \otimes_{\mathbb{Z}} M \to M$.

Example 2.1.3. Given a monoid M in **Set**, we get a map $\Pi_{k=1}^{n} M \to M$, $(m_1, \ldots, m_n) \mapsto ((\ldots((m_1 m_2) m_3) \ldots) m_{n-1}) m_n$. This induces an action map from $W(M) = \amalg_{n \in \mathbb{N}} \Pi_{k=1}^{n} M$, the set of words on M, to M.

Example 2.1.4. Given a set X, let $\mathcal{U}X$ be the set of ultrafilters on it. Any compact T2 topology on X allows us to see each ultrafilter as a system of neighborhoods of a unique point in X, hence it gives us a unique map $\mathcal{U}X \to X$ sending each ultrafilter to the respective point.

Example 2.1.5. Given a directed graph $D = (V, E, E \overset{s}{\underset{t}{\rightrightarrows}} V)$, we can create its free category FD, where the objects are the vertices and $FD(v, w) = \{\text{finite paths } v \to \ldots \to w\}$. We set id_v to be the path of length 0, while composition is just the concatenation of paths.

In particular, if D is the directed graph with $V = \{0, \ldots, n\}$ and an edge $j \to k$ if and only if $k = j + 1$, we have $FD \cong [n]$.

If $D = \{*\}$ and $E = \{* \to *\}$, then $FD(*, *) \cong \mathbb{N}$.

Given a small category $\mathbf{C}$, we may consider the underlying graph $U\,\mathbf{C} = D$ with $V = \mathrm{Ob}(\mathbf{C})$, $E = \mathrm{Ar}(\mathbf{C})$, $s = \mathrm{dom}$ and $t = \mathrm{cod}$. We get then an action map $UFU\,\mathbf{C} \to U\,\mathbf{C}$ sending a finite path to its composite. This map is a morphism of directed graphs.

Notice that we always have a category $\mathbf{C}$ and some functor $T\colon \mathbf{C} \to \mathbf{C}$ with an action map $T\,\mathbf{C} \to \mathbf{C}$. How can we see all of these examples as specific instances of a general phenomenon?

Definition 2.1.6. A *monad* on a category $\mathbf{C}$ is a triple (T, μ, η) where $T\colon \mathbf{C} \to \mathbf{C}$ is a functor, while $\mu\colon T^2 \Rightarrow T$ and $\eta\colon \mathrm{id}_{\mathbf{C}} \Rightarrow T$ are natural transformations such that the following diagrams commute:

$$
\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "\mathrm{id}_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "\mathrm{id}_T"] \\
 & T &
\end{tikzcd}
$$

μ is called the *multiplicative map*, while η is the *unit* of T.

The commutativity of the first diagram is equivalent to stating that the following two diagrams are equal:

$$
\begin{tikzcd}
 & \mathbf{C} \arrow[r, "T"] \arrow[drr, bend right, "T"'] \arrow[d, phantom, "\Downarrow\mu"] & \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathbf{C} \arrow[ur, "T"] \arrow[rrr, "T"'] & {} & {} & \mathbf{C}
\end{tikzcd}
=
\begin{tikzcd}
 & \mathbf{C} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathbf{C} \arrow[ur, "T"] \arrow[urr, bend right, "T"] \arrow[rrr, "T"'] & {} & {} & \mathbf{C}
\end{tikzcd}
$$

On the other hand, the second diagram can be rephrased as follows:

$$
\begin{tikzcd}
 & \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\eta\ \Downarrow\mu"] & \\
\mathbf{C} \arrow[ur, bend left, "\mathrm{id}_{\mathbf{C}}"] \arrow[ur, bend right, "T"'] \arrow[rr, "T"'] & {} & \mathbf{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{C} \arrow[d, bend right, "T"'] \arrow[d, bend left, "T"] \arrow[d, phantom, "="] \\
\mathbf{C}
\end{tikzcd}
=
\begin{tikzcd}
 & \mathbf{C} \arrow[dr, bend left, "\mathrm{id}_{\mathbf{C}}"] \arrow[dr, bend right, "T"'] \arrow[d, phantom, "\Downarrow\mu\ \Downarrow\eta"] & \\
\mathbf{C} \arrow[ur, "T"] \arrow[rr, "T"'] & {} & \mathbf{C}
\end{tikzcd}
$$

A monad naturally defines other algebraic structures, which we now introduce.

Definition 2.1.7. Given a monad (T, μ, η), a T*-algebra* or T*-module* is a pair (a, α), where $a \in \mathrm{Ob}(\mathbf{C})$ and $\alpha\colon Ta \to a$ is such that the following diagrams commute:

$$
\begin{tikzcd}
T^2a \arrow[r, "T\alpha"] \arrow[d, "\mu_a"'] & Ta \arrow[d, "\alpha"] \\
Ta \arrow[r, "\alpha"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[dr, "\mathrm{id}_a"'] & Ta \arrow[d, "\alpha"] \\
 & a
\end{tikzcd}
$$

Definition 2.1.8. A *morphism of* T*-algebras* $(a, \alpha) \to (b, \beta)$ is a morphism $f\colon a \to b$ such that the following diagram commutes:

$$
\begin{tikzcd}
Ta \arrow[r, "Tf"] \arrow[d, "\alpha"'] & Tb \arrow[d, "\beta"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}
$$

T-algebras form a category $T\text{-}\mathbf{Alg}$, which has a natural forgetful functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$.

We now show how to recover the examples previously given with this language.

Example 2.1.9.

$$\begin{aligned} T = G \times -\colon &\ \mathbf{Set} \to \mathbf{Set} \\ \mu_A\colon &\ G \times (G \times A) \to G \times A \\ &\ (g,(h,a)) \mapsto (gh,a) \\ \eta_A\colon &\ A \to G \times A \\ &\ a \mapsto (e,a) \end{aligned}$$

is a monad and (A,α) is a T-algebra if and only if A is a G-set. It follows that $T\text{-}\mathbf{Alg} \cong G\text{-}\mathbf{Set}$.

Example 2.1.10. Given a ring R, $T = R \otimes_{\mathbb{Z}} -\colon \mathbf{Ab} \to \mathbf{Ab}$ is a monad when considered with the following natural transformations:

$$\begin{aligned} \mu_-\colon &\ R \otimes_{\mathbb{Z}} (R \otimes_{\mathbb{Z}} -) \cong (R \otimes_{\mathbb{Z}}) \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \\ \eta_-\colon &\ - \cong \mathbb{Z} \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \end{aligned}$$

We have that $(R \otimes_{\mathbb{Z}} -)\text{-}\mathbf{Alg} \cong \mathbf{Mod}_R$.

Example 2.1.11. Consider $W\colon \mathbf{Set} \to \mathbf{Set}$ given by $WX = \amalg_{n\in\mathbb{N}} \Pi_{k=1}^{n} X$. Multiplication $\mu_X\colon WWX \to WX$ is given by concatenation of words, while the unit $\eta_X\colon X \to WX$ is just $x \mapsto (x)$. With this, $W\text{-}\mathbf{Alg} \cong \mathrm{Mon}(\mathbf{Set})$.

Example 2.1.12. The functor $\mathcal{U}$ defined in Example 2.1.4, equipped with suitable natural transformations, is a monad on $\mathbf{Set}$ and $\mathcal{U}\text{-}\mathbf{Alg} \cong \mathbf{CHTop}$, the category of compact T2 spaces.

Example 2.1.13. The free-forgetful adjunction $F \dashv U$ between categories and directed graphs induces a monad on the latter, with $UF\text{-}\mathbf{Alg} \cong \mathbf{Cat}$.

2.2 Monadic functors

Now that we have introduced these structures, our aim is to characterize *monadic functors*, that is functors $U\colon \mathbf{A} \to \mathbf{C}$ which are equivalent to $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ for some monad (T,μ,η) on $\mathbf{C}$.

First of all, notice that U^T is faithful by construction, hence U must be faithful, but more is true.

Lemma 2.2.1. The functor U^T is conservative, that is if $U^T f$ is an isomorphism then f is an isomorphism of T-algebras.

Proof. Suppose that g is the inverse of $f\colon a \to b$ and f is a morphism $(a, \alpha) \to (b, \beta)$. We only need to prove that the square on the left commutes, that is $g\beta = \alpha Tg$:

$$\begin{tikzcd} Tb \arrow[r, "Tg"] \arrow[d, "\beta"'] & Ta \arrow[r, "Tf"] \arrow[d, "\alpha"'] & Tb \arrow[d, "\beta"] \\ b \arrow[r, "g"'] & a \arrow[r, "f"'] & b \end{tikzcd}$$

We see that $fg\beta = \beta$ and $f\alpha Tg = \beta TfTg = \beta T(fg) = \beta T\,\mathrm{id}_b = \beta$, hence the thesis. ◻

Remark 2.2.2. Notice that the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ can't be monadic since it does not reflect isomorphisms. However, if we restrict it to the full subcategory of **Top** spanned by compact Hausdorff spaces we indeed obtain a monadic functor.

Proposition 2.2.3. The functor $U^T : T\text{-}\mathbf{Alg} \to \mathbf{C}$ has a left adjoint $F^T : \mathbf{C} \to T\text{-}\mathbf{Alg}$ such that $F^T c = (Tc, \mu_c)$, $F^T f\colon (Tc, \mu_c) \xrightarrow{Tf} (Td, \mu_d)$ and $U^T F^T = T$. Furthermore, the unit of this adjunction is given by $\gamma_c = \eta_c\colon c \to U^T F^T c = Tc$ and the counit has components $\epsilon_{(a,\alpha)} = \alpha\colon (Ta, \mu_a) \to (a, \alpha)$.

Proof. (i) To show that (Tc, μ_c) is a T-algebra we need the following diagrams to be commutative.

$$\begin{tikzcd} T^3c \arrow[r, "T\mu_c"] \arrow[d, "\mu_{Tc}"'] & T^2c \arrow[d, "\mu_c"] \\ T^2c \arrow[r, "\mu_c"'] & Tc \end{tikzcd} \qquad \begin{tikzcd} Tc \arrow[r, "\eta_{Tc}"] \arrow[dr, equal] & T^2c \arrow[d, "\mu_c"] \\ & Tc \end{tikzcd}$$

These are exactly the associativity and one of the unit laws for (T, μ, η).

(ii) For every $f\colon c \to c'$, Tf is a morphism of algebras $(Tc, \mu_c) \to (Tc', \mu_{c'})$. The diagram

$$\begin{tikzcd} T^2c \arrow[r, "T^2f"] \arrow[d, "\mu_c"'] & T^2c' \arrow[d, "\mu_{c'}"] \\ Tc \arrow[r, "Tf"'] & Tc' \end{tikzcd}$$

is commutative because of the naturality of μ. Hence F^T is defined on morphisms. It is a functor by the functoriality of T.

(iii) The unit is natural by assumption. We claim that $\epsilon_{(a,\alpha)} = \alpha$ is a morphism of algebras

$$F^T U^T(a, \alpha) = F^T a = (Ta, \mu_a) \to \mathrm{id}_{T\text{-}\mathbf{Alg}}(a, \alpha) = (a, \alpha)$$

and ϵ is a natural transformation $F^T U^T \Rightarrow \mathrm{id}_{T\text{-}\mathbf{Alg}}$. Let's check it. We know that α is a morphism of algebras if and only if

$$\begin{tikzcd} T^2a \arrow[r, "T\alpha"] \arrow[d, "\mu_a"'] & Ta \arrow[d, "\alpha"] \\ Ta \arrow[r, "\alpha"'] & a \end{tikzcd}$$

is commutative. But this is one of the two T-algebra axioms! Moreover, to prove that ϵ is natural, we need to show that

$$\begin{tikzcd} (Ta, \mu_a) \arrow[r, "\alpha=\epsilon_{(a,\alpha)}"] \arrow[d, "Tf"'] & (a, \alpha) \arrow[d, "f"] \\ (Tb, \mu_b) \arrow[r, "\beta=\epsilon_{(b,\beta)}"'] & (b, \beta) \end{tikzcd}$$

is commutative, but this is the axiom for f to be a morphism of T-algebras!

(iv) It remains to check the two triangular identities $\epsilon F^T \cdot F^T \eta = \mathrm{id}_{F^T}$ and $U^T \epsilon \cdot \eta U^T = \mathrm{id}_{U^T}$. These are to be checked on the components at c and (a, α), respectively.

$$\begin{tikzcd} (Tc, \mu_c) \arrow[r, "T\eta_c"] \arrow[dr, equal] & (T^2c, \mu_{Tc}) \arrow[d, "\mu_{Tc}"] \\ & (Tc, \mu_c) \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[dr, equal] & Ta \arrow[d, "\alpha"] \\ & a \end{tikzcd}$$

The commutativity of these diagrams is ensured by the second unit law for a monad and the unit law for the T-algebra (a, α), respectively. □

Definition 2.2.4. Algebras of the form (Tc, μ_c) are called *free T-algebras*.

Thanks to the proposition above we can prove that, given a monad T we can always find an adjunction that generates it. Actually, the converse holds too.

Proposition 2.2.5. If $U\colon \mathbf{D} \to \mathbf{C}$ has a left adjoint F with unit η and counit ϵ, then $(UF, U\epsilon F, \eta)$ is a monad on $\mathbf{C}$. Also, if (T, μ, η) is a monad on $\mathbf{C}$, then $(U^T F^T, U^T \epsilon F^T, \eta) = (T, \mu, \eta)$.

Proof. Let us check the axioms. First of all, the associativity holds due to the following equations.

$$\begin{tikzcd} & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & & & & & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \\ \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] & {} & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] & & {} & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
 & & \mathbf{C} \arrow[dr, "F"] & & \mathbf{C} \arrow[dr, "F"] & & \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] \arrow[rr, phantom, bend left=40, "\epsilon\Downarrow"] & & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] \arrow[rr, phantom, bend left=40, "\epsilon\Downarrow"] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C}
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
 & & & & \mathbf{C} \arrow[dr, "F"] & & & & \mathbf{C} \arrow[dr, "F"] & & \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] \arrow[rr, phantom, bend left=40, "\epsilon\Downarrow"] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] \arrow[rr, phantom, bend left=40, "\epsilon\Downarrow"] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C}
\end{tikzcd}$$

Unit laws:

$$\begin{tikzcd}
 & & \mathbf{C} \arrow[dr, "F"] & & \\
\mathbf{C} \arrow[r, "F"] \arrow[urr, equal, bend left] \arrow[urr, phantom, "\eta\Downarrow"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] \arrow[rr, phantom, bend left=40, "\epsilon\Downarrow"] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C}
\end{tikzcd}$$

is equal to 1_{UF}, since $\epsilon F \cdot F\eta = 1_F$ by one of the triangular identities of the adjunction $F \dashv U$. Furthermore,

$$\begin{tikzcd}
 & & \mathbf{C} \arrow[dr, "F"] \arrow[drr, equal, bend left] \arrow[drr, phantom, "\eta\Downarrow"] & & \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] \arrow[rr, phantom, bend left=40, "\epsilon\Downarrow"] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C}
\end{tikzcd}$$

is equal to 1_{UF}. This follows from the explicit description of the unit and the counit of the adjunction $F^T \dashv U^T$, in fact

$$U^T \epsilon F^T c = U^T \epsilon_{(Tc, \eta_c)} = \mu_c \qquad \square$$

Example 2.2.6 (Interesting adjunction, boring monad)**.** Let us consider the adjunction $U \colon \mathbf{Top} \rightleftarrows \mathbf{Set} \colon \mathsf{Disc} =: F$, whose left adjoint assigns to every set X the discrete topological space $FX = (X, 2^X)$. It's immediate to see that $UFX = X$, hence $UF = \mathrm{id}_{\mathbf{Set}}$. How many natural transformations $\mathrm{id}_{\mathbf{Set}} = UF \overset{\alpha}{\Rightarrow} UF = \mathrm{id}_{\mathbf{Set}}$ are there? We know that $\mathrm{id}_{\mathbf{Set}} \cong \mathrm{Hom}(*, -)$, so $\mathrm{Nat}(\mathrm{id}_{\mathbf{Set}}, \mathrm{id}_{\mathbf{Set}}) \cong \mathrm{Nat}(\mathrm{Hom}(*, -), \mathrm{Hom}(*, -)) \cong \mathrm{Hom}(*, *) = \{\mathrm{id}_*\}$ by Yoneda, hence $\alpha = \mathrm{id}$. Therefore $(UF, U\epsilon F, \eta) = (\mathrm{id}_{\mathbf{Set}}, \mathrm{id}, \mathrm{id})$

Example 2.2.7. If S is a set, $\mathbf{Set}(S, -) \colon \mathbf{Set} \to \mathbf{Set}$ is right adjoint to $S \times - \colon \mathbf{Set} \to \mathbf{Set}$, so we get a monad $X \mapsto \mathbf{Set}(S, S \times X)$. This is called *the state monad* and is important in Computer Science.

There is always a comparison morphism $\mathbf{D} \xrightarrow{\overline{U}} UF\text{-}\mathbf{Alg}$ s.t.

$$\begin{tikzcd}
\mathbf{D} \arrow[rr, "\overline{U}"] \arrow[dr, "U"'] & & UF\text{-}\mathbf{Alg} \arrow[dl, "U^{UF}"] \\
 & \mathbf{C} &
\end{tikzcd}$$

commutes. We set $\overline{U}f = (Ud, UFUd \xrightarrow{U\epsilon_d} Ud) = (Ud, U\epsilon_d)$. More specifically, for a given functor $G\colon \mathbf{D} \to \mathbf{C}$ we can ask what do we need to get an equivalence $\overline{G}\colon \mathbf{D} \to T\text{-}\mathbf{Alg}$. To get there, we will need a few more definitions and lemmas.

2.3 The category of T-actions

Just like a monad (T, μ, η) defines a category $T\text{-}\mathbf{Alg}$, it also allows us to construct another category from functors $\mathbf{D} \to \mathbf{C}$.

Definition 2.3.1. Given a monad (T, μ, η) on a category $\mathbf{C}$ and fixed another category $\mathbf{D}$, a *T-action* on a functor $G\colon \mathbf{D} \to \mathbf{C}$ is a natural transformation $\gamma\colon TG \Rightarrow G$ such that the diagrams

$$\begin{tikzcd} T^2G \arrow[r, Rightarrow, "T\gamma"] \arrow[d, Rightarrow, "\mu G"'] & TG \arrow[d, Rightarrow, "\gamma"] \\ TG \arrow[r, Rightarrow, "\gamma"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, equal] & TG \arrow[d, Rightarrow, "\gamma"] \\ & G \end{tikzcd}$$

commute.

A *morphism of T-actions* $(G, \gamma) \overset{\varphi}{\Rightarrow} (K, \kappa)$ is a natural transformation $\varphi\colon G \Rightarrow K$ such that

$$\begin{tikzcd} TG \arrow[r, Rightarrow, "T\varphi"] \arrow[d, Rightarrow, "\gamma"'] & TK \arrow[d, Rightarrow, "\kappa"] \\ G \arrow[r, Rightarrow, "\varphi"'] & K \end{tikzcd}$$

commutes.

Up to size, T-actions and their morphisms assemble into a category $T\text{-}\mathsf{Act}(\mathbf{D})$.

Example 2.3.2. The functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ has a T-action given by $(U^T, \alpha\colon TU^T \Rightarrow U^T)$, where $\alpha_{(b,\beta)} := \beta\colon Tb \to b$.

Example 2.3.3. Given an adjunction $F \dashv U\colon \mathbf{C} \rightleftarrows \mathbf{D}$ with unit $\eta\colon \mathrm{id}_{\mathbf{C}} \Rightarrow UF$ and counit $\epsilon\colon FU \Rightarrow \mathrm{id}_{\mathbf{D}}$, we get a monad on $(UF, U\epsilon F, \eta)$ on $\mathbf{C}$. We have then a UF-action $U\epsilon\colon UFU \Rightarrow U$, where the axioms follow from the triangular identities and the naturality of $U\epsilon$.

Proposition 2.3.4. (U^T, α) is the universal T-action, that is for any category $\mathbf{D}$ the functor $\mathbf{Cat}(\mathbf{D}, T\text{-}\mathbf{Alg}) \to T\text{-}\mathsf{Act}(\mathbf{D})$ sending G to $(U^TG, \alpha G)$ and $\beta\colon G \Rightarrow H$ to $U^T\beta\colon (U^TG, \alpha G) \Rightarrow (U^TH, \alpha H)$ is an isomorphism of categories.

Proof. In other words, for every T-action (G, γ) there exists a unique lift $\overline{G}\colon \mathbf{D} \to T\text{-}\mathbf{Alg}$ such that $(U^T\overline{G}, \alpha\overline{G}) = (G, \gamma)$ and for every $\phi\colon (G, \gamma) \Rightarrow (K, \kappa)$ there is a unique $\overline{\phi}\colon \overline{G} \Rightarrow \overline{K}$ with $U^T\overline{\phi} = \phi$.

It is enough to set $\overline{G}d \coloneqq (Gd, \gamma_d)$ on objects, $\overline{G}f \coloneqq Gf$ on morphisms, $\overline{\phi}_d \coloneqq \phi_d$ and check the axioms.

$$\begin{tikzcd}
\mathbf{D} \arrow[r, dashed, "\exists! \overline{G}"] \arrow[dr, "G"'] & T\text{-}\mathbf{Alg} \arrow[d, "U^T"] \\
& \mathbf{C}
\end{tikzcd}$$

□

Following the construction in this proof, from the last example we get the comparison functor for the adjunction $F \dashv U$. In particular, $\overline{U}d = (Ud, U\epsilon_d)$. Furthermore, this means that $U\colon \mathbf{Top} \to \mathbf{Set}$ factors through identities.

2.4 Limits and colimits in the category of algebras

We have shown that the forgetful functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ is a right adjoint, and as such it preserves limits. However, more is true.

Proposition 2.4.1. For any monad (T, μ, η) on $\mathbf{C}$, the forgetful functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ strictly creates limits.

Proof. This statement means that, for any diagram $D\colon I \to T\text{-}\mathbf{Alg}$ such that $U^T D\colon I \to \mathbf{C}$ has a limit (l, κ_i) in $\mathbf{C}$, there is a unique T-algebra structure $\lambda\colon Tl \to l$ such that κ_i is a morphism of T-algebras for all $i \in I$ and this makes $((l, \lambda), \kappa_i)$ into a limit of D.

Now we begin the proof.

First of all, remember that $D\phi\colon D_i \to D_j$ is a morphism of T-algebras for all $\phi\colon i \to j$ by assumption, hence the morphisms $\delta_i T\kappa_i\colon Tl \to D_i$ define a cone over D, where δ_i is the T-algebra structure on D_i. By the universal property of the limit, there is a unique morphism $\lambda\colon Tl \to l$ making the following diagram commute for all i.

$$\begin{tikzcd}
Tl \arrow[r, "T\kappa_i"] \arrow[d, dashed, "\lambda"'] & TD_i \arrow[d, "\delta_i"] \\
l \arrow[r, "\kappa_i"'] & D_i
\end{tikzcd}$$

This tells us that, if the limit $((l, \lambda), \kappa_i)$ of D exists, it is unique. We have to check that (l, λ) is a T-algebra.

Notice that for all i all of the faces of the following diagrams, except for possibly the top ones, commute:

$$
\begin{tikzcd}
 & Tl \arrow[rrd, "\lambda"] \arrow[ddd, "T\kappa_i"] & & & & Tl \arrow[dr, "\lambda"] \arrow[dd, "T\kappa_i"] & \\
T^2l \arrow[ur, "T\lambda"] \arrow[drr, "\mu_l"'] \arrow[ddd, "T^2\kappa_i"] & & & l \arrow[ddd, "\kappa_i"] & l \arrow[ur, "\eta_l"] \arrow[rr, equal] \arrow[dd, "\kappa_i"'] & & l \arrow[dd, "\kappa_i"] \\
 & & Tl \arrow[ur, "\lambda"] \arrow[ddd, "T\kappa_i"] & & & TD_i \arrow[dr, "\kappa_i"] & \\
 & TD_i \arrow[rrd, "\delta_i"] & & & D_i \arrow[ur, "\eta_{D_i}"] \arrow[rr, equal] & & D_i \\
T^2D_i \arrow[ur, "T\delta_i"] \arrow[drr, "\mu_{D_i}"] & & & D_i & & & \\
 & & TD_i \arrow[ur, "\delta_i"] & & & &
\end{tikzcd}
$$

Since the κ_i are jointly monic, the upper face commutes and therefore (l, λ) is a T-algebra. It remains to check that $((l, \lambda), \kappa_i)$ factors every other cone over D.

Let $\gamma_i \colon (x, \zeta) \to (D_i, \delta_i)$ be a cone over D. Then, there is a unique $f \colon x \to l$ in $\mathbf{C}$ such that $\kappa_i f = \gamma_i$. We only have to show that f is a morphism of T-algebras $(x, \zeta) \to (l, \lambda)$.

Consider the following diagram and notice that the outer square, the one on the right and the two triangles commute, hence the square on the left commutes as well since the κ_i are jointly monic.

$$
\begin{tikzcd}
Tx \arrow[r, "Tf"'] \arrow[rr, bend left, "T\gamma_i"] \arrow[d, "\zeta"] & Tl \arrow[r, "T\kappa_i"'] \arrow[d, "\lambda"] & TD_i \arrow[d, "\delta_i"] \\
x \arrow[r, "f"] \arrow[rr, bend right, "\gamma_i"'] & l \arrow[r, "\kappa_i"] & D_i
\end{tikzcd}
$$

□

A similar statement holds for colimits.

Proposition 2.4.2. Given a monad (T, μ, η) on $\mathbf{C}$, the forgetful functor $U^T \colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ strictly creates any colimit preserved by both T and T^2.

Proof. Similarly to the dual situation, this means that for any diagram $D \colon I \to T\text{-}\mathbf{Alg}$ such that $U^T D \colon I \to \mathbf{C}$ has a colimit (c, κ_i) preserved by both T and T^2, there is a unique T-algebra structure $\lambda \colon Tc \to c$ such that κ_i is a morphism of T-algebras for all $i \in I$. This makes $((c, \lambda), \kappa_i)$ into a colimit of D.

The proof is essentially dual to the one given earlier, in the sense that we find again a unique $\lambda\colon Tc \to c$ using the universal property of the colimit $(Tc, T\kappa_i)$ of TD.

$$\begin{tikzcd} TD_i \arrow[r, "T\kappa_i"] \arrow[d, "\delta_i"'] & Tc \arrow[d, dashed, "\lambda"] \\ D_i \arrow[r, "\kappa_i"'] & c \end{tikzcd}$$

To check that (c, λ) is an algebra we use the universal property of $(T^2c, T^2\kappa_i)$ (for μ) and the one of (c, κ_i) (for η). □

Remark 2.4.3. The same statement holds for monadic functors, except for the fact that they may not create limits and colimits strictly since they are just equivalent to a U^T.

Remark 2.4.4. If T is a monad on a complete category $\mathbf{C}$, then $T\text{-}\mathbf{Alg}$ is complete. If $\mathbf{C}$ is cocomplete and T is cocontinuous, then $T\text{-}\mathbf{Alg}$ is cocomplete.

Example 2.4.5. Let $\mathbf{C}$ be a small category. There is a cocontinuous monad on the category of $\mathrm{Ob}(\mathbf{C})$-indexed collections of sets whose category of algebras is the functor category $[\mathbf{C}, \mathbf{Set}]$. The underlying endofunctor of this monad is defined as

$$\begin{aligned} T\colon [\mathrm{Ob}(\mathbf{C}), \mathbf{Set}] &\to [\mathrm{Ob}(\mathbf{C}), \mathbf{Set}] \\ (X_c)_{c\in\mathbf{C}} &\mapsto \left(\coprod_{d\in\mathbf{C}} \mathbf{C}(d, c) \times X_d \right)_{c\in\mathbf{C}} \end{aligned}$$

Since $[\mathrm{Ob}(\mathbf{C}), \mathbf{Set}]$ is complete and cocomplete, so is $[\mathbf{C}, \mathbf{Set}]$ (with limits and colimits computed pointwise).

Chapter 3

Beck's monadicity theorem

The final ingredient we need is the observation that T-algebras admit canonical presentations using free algebras.

Example 3.0.1. Pick an epi $F \twoheadrightarrow G$ in the category of groups **Grp**, where F is a free group. The kernel of this homomorphism defines a (normal) subgroup K of F, giving rise to the sequence $K \rightarrowtail F \twoheadrightarrow G$. We can take another epi $F' \twoheadrightarrow K$, with F' again a free group. Therefore G is the cokernel of some morphism $F' \to F$. This argument applies to rings, algebras etc.

It is natural to ask if we can do this systematically for general T-algebras. Given (a, α) in T-**Alg**, we have $F^T U^T(a, \alpha) \to (a, \alpha)$ i.e. $(Ta, \mu_a) \xrightarrow{\alpha} (a, \alpha)$. A candidate[1] for F' would be $F^T U^T(Ta, \mu_a) = (T^2 a, \mu_{Ta})$. What are the "elements" of Ta? Notice that

$$(T^2 a, \mu_{Ta}) \underset{\mu_a}{\overset{T\alpha}{\rightrightarrows}} (Ta, \mu_a) \xrightarrow{\alpha} (a, \alpha)$$

is a well defined diagram in T-**Alg**, with $\alpha\mu_a = \alpha T\alpha$. Moreover, this is a coequalizer. We can use Proposition 2.4.2 to prove it, so that we need to check whether U^T sends the diagram above into a coequalizer preserved by T and T^2. In **C**, we get the diagram

$$\begin{tikzcd} T^2 a \arrow[r, shift left, "T\alpha"] \arrow[r, shift right, "\mu_a"'] & Ta \arrow[r, "\alpha"] \arrow[l, bend left=60, "\eta_{Ta}"] & a \arrow[l, bend left=60, "\eta_a"] \end{tikzcd}$$

in which the following equations hold true: $\alpha\eta_a = \mathrm{id}_a,\ \mu_a\eta_{Ta} = \mathrm{id}_{Ta}$ and $\eta_a\alpha = T\alpha\ \eta_{Ta}$ by naturality. It is a particular case of a more general concept.

Definition 3.0.2. A *split coequalizer* is a diagram of the form

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "h"] \arrow[l, bend left=60, "t"] & c \arrow[l, bend left=60, "s"] \end{tikzcd}$$

so that $hf = hg,\ hs = \mathrm{id}_c,\ gt = \mathrm{id}_b$, and $ft = sh$.

[1] Think about free groups: in that case we take words on Ta.

Proposition 3.0.3. In the above situation,

$$a \underset{g}{\overset{f}{\rightrightarrows}} b \xrightarrow{h} c$$

is a coequalizer. In particular, any functor preserves this coequalizer.

Proof. Take $k\colon b \to d$ such that $kf = kg$ and define $\overline{k} := ks$. Then we have

$$\overline{k}h = ksh = kft = kgt = k.$$

Uniqueness is clear since h is a (split) epi. $\square$

T and T^2 preserve split coequalizers, so they preserve our coequalizer.

Corollary 3.0.4. Let T be a monad on $\mathbf{C}$ and (a, α) a T-algebra. Then

$$(T^2a, \mu_{Ta}) \underset{\mu_a}{\overset{T\alpha}{\rightrightarrows}} (Ta, \mu_a) \xrightarrow{\alpha} (a, \alpha)$$

is a coequalizer in $T\text{-}\mathbf{Alg}$, which $U^T : T\text{-}\mathbf{Alg} \to \mathbf{C}$ sends to a split coequalizer in $\mathbf{C}$.

Proof. We have already observed that the second statement holds, so that coeq $(U^T(T\alpha), U^T(\mu_a))$ is preserved by T and T^2. Hence there exists a unique lift of the (split) coequalizer in $\mathbf{C}$ to a coequalizer in $T\text{-}\mathbf{Alg}$. $\square$

Results like the previous one inspire us to look at the parallel pairs of morphisms in a category which are sent to split coequalizers or, to say it better, to a parallel pair of morphisms that can be extended to a split coequalizer diagram. Those kinds of pairs will be of crucial importance in the following.

Definition 3.0.5. Let $U\colon \mathbf{D} \to \mathbf{C}$ be a functor. A pair of morphisms $f, g\colon d \rightrightarrows d'$ in $\mathbf{D}$ is *U-split* if $Uf, Ug\colon Ud \rightrightarrows Ud'$ is part of a split coequalizer in $\mathbf{C}$.

Remark 3.0.6. $T\alpha, \mu_a\colon (T^2a, \mu_{Ta}) \rightrightarrows (Ta, \mu_a)$ is a U^T-split pair. Moreover, $T\text{-}\mathbf{Alg}$ has coequalizers of U^T-split pairs and U^T preserves them. Hence, functors equivalent to U^T satisfy three conditions:

1. they have a left adjoint;
2. they are conservative;
3. U-split pairs have coequalizers which are preserved by U.

Theorem 1 (Beck). Let $U\colon \mathbf{D} \to \mathbf{C}$ be a right adjoint to $F\colon \mathbf{C} \to \mathbf{D}$. Let $(T = UF, U\epsilon F, \eta)$ be the induced monad and $\overline{U}\colon \mathbf{D} \to T\text{-}\mathbf{Alg}$ be the comparison functor.

1. If $\mathbf{D}$ has coequalizers of U-split pairs, then $\overline{U}$ has a left adjoint $\overline{F}\colon T\text{-}\mathbf{Alg} \to \mathbf{D}$;
2. if, in addition, U preserves coequalizers of U-split pairs, the unit $\overline{\eta}\colon \mathrm{id} \Rightarrow \overline{U}\,\overline{F}$ is an isomorphism;
3. if U is also conservative, then $\overline{U}$ is an equivalence of categories.

Proof. 1. For each free T-algebra (Ta, μ_a) we have

$$\begin{aligned} T\text{-}\mathbf{Alg}((Ta,\mu_a), \overline{U}-) &= T\text{-}\mathbf{Alg}(F^T a, \overline{U}-) \\ &\cong \mathbf{C}(a, U^T\overline{U}-) \\ &= \mathbf{C}(a, U-) \\ &\cong \mathbf{D}(Fa, -) \end{aligned}$$

therefore the value of $\overline{F}$ at (Ta, μ_a) has to be Fa. Since every T-algebra is a coequalizer of free algebras which is preserved by U^T, we may define $\overline{F}(a, \alpha)$ as the coequalizer of a pair of morphisms $FTa \rightrightarrows Fa$. We write this as $FUFU^T(a, \alpha) \rightrightarrows FU^T(a, \alpha)$. Consider the following pair of morphisms of functors

$$\begin{tikzcd} FUFU^T \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\epsilon FU^T"'] & FU^T \end{tikzcd}$$

in the functor category $[T\text{-}\mathbf{Alg}, \mathbf{D}]$. We claim that this pair has a coequalizer and $\overline{F}\colon T\text{-}\mathbf{Alg} \to \mathbf{D}$ is left adjoint to $\overline{U}$. Note that the pair of morphisms just above becomes split after the composition with $U\colon \mathbf{D} \to \mathbf{C}$. In fact

$$\begin{tikzcd} UFUFU^T \arrow[r, shift left, "UF\alpha"] \arrow[r, shift right, "U\epsilon FU^T"'] & UFU^T \arrow[r, "\alpha"] \arrow[l, bend left=60, "\eta UFU^T"] & U^T \arrow[l, bend left=60, "\eta U^T"] \end{tikzcd}$$

is a split coequalizer in $[T\text{-}\mathbf{Alg}, \mathbf{C}]$, given that it holds pointwise since $UF = T$. Let us denote by $\beta\colon FU^T \to \overline{F}$ the colimit (computed pointwise) of the pair $F\alpha, \epsilon FU^T\colon FUFU^T \rightrightarrows FU^T$. Precomposing this pair with $\overline{U}$ and recalling that $\alpha\overline{U} = U\epsilon$, $U^T\overline{U} = U$, we get the pair

$$\begin{tikzcd} FUFU \arrow[r, shift left, "FU\epsilon"] \arrow[r, shift right, "\epsilon FU"'] & FU, \end{tikzcd}$$

which is coequalized by $\epsilon\colon FU \Rightarrow \mathrm{id}_{\mathbf{D}}$.

$$\begin{tikzcd} FUFU \arrow[r, shift left, "FU\epsilon"] \arrow[r, shift right, "\epsilon FU"'] & FU \arrow[r, "\beta\overline{U}"] \arrow[dr, "\epsilon"'] & \overline{F}\,\overline{U} \arrow[d, dashed, "\exists!\overline{\epsilon}"] \\ & & \mathrm{id}_{\mathbf{D}} \end{tikzcd}$$

Since $\overline{F}\,\overline{U}$ is the coequalizer of the diagram above, there exists a unique $\overline{\epsilon}\colon \overline{F}\,\overline{U} \Rightarrow \mathrm{id}_{\mathbf{D}}$ such that $\overline{\epsilon}\cdot\beta\overline{U} = \epsilon$. To get the unit $\overline{\eta}\colon \mathrm{id}_{\mathbf{C}} \Rightarrow \overline{U}\,\overline{F}$ we need to describe a morphism of actions $(UT,\alpha) \to (UT\overline{U}\,\overline{F}, \alpha\overline{U}\,\overline{F})$. We claim that the comparison functor

$$\begin{tikzcd} U^T\overline{U}FUFU^T \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & U^T\overline{U}FU^T \arrow[r, "\alpha"] \arrow[d, equal] & U^T\overline{U}\,\overline{F} \arrow[d, dashed, "\exists!\overline{\epsilon}"] \\ UFUFU^T \arrow[r, shift left] \arrow[r, shift right] & UFU^T \arrow[r, "U\beta"'] & U^T \end{tikzcd}$$

is a morphism of T-actions.

Unraveling [...]

The only thing left to do is checking the triangular identities, which is left to the reader.

2. If U preserves coequalizers of U-split pairs, both $U\overline{F}$ and U^T are coequalizers of the above diagram, hence $\overline{\eta}$ is an isomorphism.

3. From the triangular identities, $\overline{\eta}\overline{U}\cdot\overline{U}\epsilon = \mathrm{id}_{FU}$, hence $\overline{U}\overline{\epsilon}$ is an isomorphism. Being $U^T\overline{U} = U$ conservative, $\overline{\epsilon}$ is an isomorphism as well. □

Definition 3.0.7. A pair $f, g\colon c \rightrightarrows d$ in a category $\mathbf{C}$ is *reflexive* if there exists a common section $i\colon d \to c$, that is $fi = gi = \mathrm{id}_d$.

A coequalizer of a reflexive pair is a *reflexive coequalizer.*

Remark 3.0.8. To give a cone of a reflexive pair it is enough to give a map $h\colon d \to x$ such that $hf = hg$, hence $\mathrm{colim}(c \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}} d) \cong \mathrm{colim}(c \rightrightarrows d)$.

Proposition 3.0.9. In Beck's monadicity theorem it suffices for (1) that coequalizers of reflexive U-split pairs exist, while in (2) and (3) we only need for them to be preserved.

Proof. The pair

$$FUFU^T \underset{\epsilon FU^T}{\overset{F\alpha}{\rightrightarrows}} FU^T$$

has $F\eta U^T$ as common section. Moreover, $\alpha\cdot\eta U^T = \mathrm{id}_{U^T}$ by the unit law of the T-action $\alpha\colon TU^T \Rightarrow U^T$ and $\epsilon F\cdot F\eta = \mathrm{id}_F$ by the triangular identities. □

Example 3.0.10. Let $\mathbf{A}$ and $\mathbf{B}$ be small categories, $\mathbf{C}$ a category which is both complete and cocomplete, and $G\colon \mathbf{A} \to \mathbf{B}$ a functor. The restriction along G, G^*, has both adjoints, given by left and right Kan extensions. Notice that the induced monad is cocontinuous since G^* is a left adjoint. Moreover, G^* is conservative if G is essentially surjective, thus any essentially surjective functor G induces a monadic adjunction as follows:

$$\begin{tikzcd}
{[\mathbf{B},\mathbf{C}]} \arrow[rr, "\overline{G^*}"] \arrow[dr, "G^*"'] & \arrow[d, phantom, "\overset{\sim}{\Longrightarrow}"] & G^*\,\mathrm{Lan}_G\text{-}\mathbf{Alg} \arrow[dl] \\
& {[\mathbf{A},\mathbf{C}]} &
\end{tikzcd}$$

We are going to show why reflexive coequalizers are useful, but first we need some preliminary definitions and results.

Definition 3.0.11. A functor $F\colon \mathbf{I} \to \mathbf{J}$ between small categories is called *final* if for any diagram $D\colon \mathbf{J} \to \mathbf{C}$ the comparison morphism $\mathrm{colim}_{\mathbf{I}}\, DF \to \mathrm{colim}_{\mathbf{J}}\, D$ is an isomorphism whenever both colimits exist.

Proposition 3.0.12. Let $F\colon \mathbf{I} \to \mathbf{J}$ be a functor between small categories. The following are equivalent:

1. F is final;
2. the unique isomorphism

$$\begin{tikzcd}
\mathbf{I}^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}"] \arrow[dr, "*"'] & \arrow[d, phantom, "\overset{\sim}{\Longrightarrow}"] & \mathbf{J}^{\mathrm{op}} \arrow[dl, "*"] \\
& \mathbf{Set} &
\end{tikzcd}$$

   exibits $*$ as $\mathrm{Lan}_{F^{\mathrm{op}}} *$;

3. for each $j \in \mathbf{J}$, the category $(j \downarrow F)$ is connected.

Proof. $(ii) \iff (iii)$ We have $\mathrm{Lan}_{F^{\mathrm{op}}} *(j) \cong \mathrm{colim}_{(j\downarrow F)} *$ by the formula for Kan extensions. A colimit of $(j \downarrow F) \to \mathbf{Set}$, $(\phi, j') \mapsto *$ is terminal if and only if $(j \downarrow F)$ is connected, hence the thesis.

$(ii) \implies (i)$ Let $D\colon \mathbf{J} \to \mathbf{C}$ be a diagram. We can then write $\mathrm{Cocone}(D,-)$ as follows:

$$\mathrm{Cocone}(D,X) \cong \mathrm{Nat}(*, \mathbf{C}(D-,X)) \cong [\mathbf{J}^{\mathrm{op}}, \mathbf{Set}](*, \mathbf{C}(D-,X))$$

By definition of left Kan extension, we also have

$$[I^{\mathrm{op}}, \mathbf{Set}(*, \mathbf{C}(DF-,X))] \cong [\mathbf{J}^{\mathrm{op}}, \mathbf{Set}](\mathrm{Lan}_{F^{\mathrm{op}}} *, \mathbf{C}(D-,X))$$

If $\mathrm{Lan}_{F^{\mathrm{op}}} * \cong *$, this shows that $\mathrm{colim}_{\mathbf{I}}\, DF \cong \mathrm{colim}_{\mathbf{J}}\, D$.

$(i) \implies (iii)$ Left as an exercise. □

Definition 3.0.13. A small category $\mathbf{I}$ is *sifted* if the diagonal $\Delta\colon \mathbf{I} \to \mathbf{I} \times \mathbf{I}$ is final. A colimit is sifted if the domain category is.

Example 3.0.14. For any filtered category $\mathbf{I}$, the category $((i, i') \downarrow \Delta)$ is again filtered and hence connected, thus filtered colimits are sifted.

Example 3.0.15. Reflexive coequalizers are sifted.

Proposition 3.0.16. If $F\colon \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ is a functor preserving reflexive coequalizers in each variable, that is for any $a \in \mathbf{A}$, $b \in \mathbf{B}$ the functors $F(a, -)\colon \mathbf{B} \to \mathbf{C}$ and $F(-, b)\colon \mathbf{A} \to \mathbf{C}$ preserve reflexive coequalizers, then F preserves reflexive coequalizers as well.

Proof. We need to check that, given a reflexive coequalizer $a_0 \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}} a_1 \longrightarrow a_2$ in $\mathbf{A}$ and $b_0 \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}} b_1 \longrightarrow b_2$ in $\mathbf{B}$, the diagonal of the following diagram is a coequalizer diagram in $\mathbf{C}$.

$$\begin{tikzcd}
F(a_0, b_0) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d, shift right=2] \arrow[dr, shift left=2] \arrow[dr, shift right=2] & F(a_1, b_0) \arrow[l] \arrow[r] \arrow[d, shift left=2] \arrow[d, shift right=2] & F(a_2, b_0) \arrow[d, shift left=2] \arrow[d, shift right=2] \\
F(a_0, b_1) \arrow[u] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d] & F(a_1, b_1) \arrow[l] \arrow[u] \arrow[ul] \arrow[r] \arrow[d] \arrow[dr] & F(a_2, b_1) \arrow[u] \arrow[d] \\
F(a_0, b_2) \arrow[r, shift left=2] \arrow[r, shift right=2] & F(a_1, b_2) \arrow[l] \arrow[r] & F(a_2, b_2)
\end{tikzcd}$$

From general facts, $F(a_2, b_2)$ is the colimit of the square in the top left. We may prove this using the sections, however in this case we can use the fact that, under certain hypothesis, given a diagram $D\colon \mathbf{I} \times \mathbf{I} \to \mathbf{C}$ we have $\operatorname{colim}_{\mathbf{I} \times \mathbf{I}} D \cong \operatorname{colim}_{\mathbf{I}} D\Delta$. Specifically, we want this to hold when $\mathbf{I} = \{a_0 \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}} a_1\}$. But the category $(i \downarrow \Delta)$ is clearly connected for any $i \in \mathbf{I}$, hence Δ is final and we have the thesis. $\square$

Example 3.0.17. The functor $\mathbf{Set} \times \mathbf{Set} \xrightarrow{- \times -} \mathbf{Set}$ satisfies the hypothesis of the theorem since $\mathbf{Set}$ is cartesian closed, hence $X \mapsto X \times X$ preserves reflexive coequalizers by the proposition. This shows that $\mathbf{Set}(X, -) \cong \Pi_{x \in X} \mathbf{Set}(*, -)$ preserves reflexive coequalizers if X is finite, hence the functor $\mathbf{Set}(X, -)\colon \mathbf{Set} \to \mathbf{Set}$ is monadic with $T = \mathbf{Set}(X, X \times -)$ for X finite.

$$\begin{tikzcd}
\mathbf{Set} \arrow[rr, "\cong"] \arrow[dr, "\mathbf{Set}(X,-)"'] & & T\text{-}\mathbf{Alg} \arrow[dl, "U^T"] \\
& \mathbf{Set} &
\end{tikzcd}$$

Chapter 4

Monads in 2-category theory

Chapter 5

Monads in ∞-category theory