Monads and their applications

Dr. Daniel Schäppi's course lecture notes

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Contents

Introduction **iii**

1 Categorical preliminaries **1**

2 Monads and algebras **2**

2.1 Introduction 2

2.2 Monadic functors 4

2.3 The category of T-actions 8

2.4 Limits and colimits 9

3 Beck's monadicity theorem **12**

4 Monads in 2-category theory **13**

5 Monads in ∞-category theory **14**

Introduction

Chapter 1

Categorical preliminaries

Definition 1.0.1 (Categories)**.** A category $\mathbf{C}$ consists of:

1. a collection of objects $\mathrm{Ob}(\mathbf{C})$;
2. a collection of arrows $\mathrm{Ar}(\mathbf{C})$;
3. two maps $\mathrm{dom}, \mathrm{cod} \colon \mathrm{Ar}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$;
4. a map $\mathrm{id}_{-} \colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ar}(\mathbf{C})$ with $\mathrm{dom}(\mathrm{id}_c) = c = \mathrm{cod}(\mathrm{id}_c)$;
5. for every $f, g \in \mathrm{Ar}(\mathbf{C})$ such that $\mathrm{cod}(f) = \mathrm{dom}(g)$ a unique composite morphism gf such that $\mathrm{cod}(gf) = \mathrm{cod}(g)$, $\mathrm{dom}(gf) = f$.

This data has to satisfy the following axioms

1. given $f \in \mathrm{Ar}(\mathbf{C})$, $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$, $\mathrm{id}_{c'} f = f = \mathrm{id}_c$, that is the composition is unital;
2. given a composable triple $f, g, h \in \mathrm{Ar}(\mathbf{C})$, $h(gf) = (hg)f$, that is the composition is associative.

An arrow f such that $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$ is denoted $f \colon c \to c'$.

Definition 1.0.2 (Functors)**.**

Definition 1.0.3 (Full functors, faithful functor)**.**

Definition 1.0.4 (Natural transformations)**.**

Definition 1.0.5 (Equivalent functors)**.**

Definition 1.0.6 (Representable Functors)**.**

Definition 1.0.7 (Whiskering)**.**

Definition 1.0.8 (Horizontal and vertical composition of nat.transf.)**.**

Definition 1.0.9 (adjunctions)**.**

Lemma 1.0.10 (Yoneda)**.**

Proof. □

Chapter 2

Monads and algebras

2.1 Introduction

Throughout mathematics we encounter structures defined by some action morphisms. Here we give some examples.

Example 2.1.1. Given a group G, we may consider a G-set X described by an action map $G \times X \to X$.

Example 2.1.2. Given an abelian group M and a ring R, we can get an R-module M by fixing a group homomorphism $R \otimes_{\mathbb{Z}} M \to M$.

Example 2.1.3. Given a monoid M in **Set**, we get a map $\Pi_{k=1}^{n} M \to M$, $(m_1, \ldots, m_n) \mapsto ((\ldots((m_1 m_2)m_3)\ldots)m_{n-1})m_n$. This induces an action map from $W(M) = \amalg_{n\in\mathbb{N}} \Pi_{k=1}^{n} M$, the set of words on M, to M.

Example 2.1.4. Given a set X, let $\mathcal{U}X$ be the set of ultrafilters on it. Any compact T2 topology on X allows us to see each ultrafilter as a system of neighborhoods of a unique point in X, hence it gives us a unique map $\mathcal{U}X \to X$ sending each ultrafilter to the respective point.

Example 2.1.5. Given a directed graph $D = (V, E, E \underset{t}{\overset{s}{\rightrightarrows}} V)$, we can create its free category FD, where the objects are the vertices and $FD(v, w) = \{\text{finite paths } v \to \ldots \to w\}$. We set id_v to be the path of length 0, while composition is just the concatenation of paths.

In particular, if D is the directed graph with $V = \{0, \ldots, n\}$ and an edge $j \to k$ if and only if $k = j + 1$, we have $FD \cong [n]$.

If $D = \{*\}$ and $E = \{* \to *\}$, then $FD(*, *) \cong \mathbb{N}$.

Given a small category $\mathbf{C}$, we may consider the underlying graph $U\,\mathbf{C} = D$ with $V = \mathrm{Ob}(\mathbf{C})$, $E = \mathrm{Ar}(\mathbf{C})$, $s = \mathrm{dom}$ and $t = \mathrm{cod}$. We get then an action map $UFU\,\mathbf{C} \to U\,\mathbf{C}$ sending a finite path to its composite. This map is a morphism of directed graphs.

Notice that we always have a category $\mathbf{C}$ and some functor $T\colon \mathbf{C} \to \mathbf{C}$ with an action map $T\,\mathbf{C} \to \mathbf{C}$. How can we see all of these examples as specific instances of a general phenomenon?

Definition 2.1.6. A monad on a category $\mathbf{C}$ is a triple (T, μ, η) where $T\colon \mathbf{C} \to \mathbf{C}$ is a functor, while $\mu\colon T^2 \Rightarrow T$ and $\eta\colon \mathrm{id}_{\mathbf{C}} \Rightarrow T$ are natural transformations such that the following diagrams commute:

$$\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "\mathrm{id}_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "\mathrm{id}_T"] \\
& T &
\end{tikzcd}$$

μ is called the multiplicative map, while η is the unit of T.

The commutativity of the first diagram is equivalent to stating that the following two diagrams are equal:

$$\begin{tikzcd}
& \mathbf{C} \arrow[r, "T"] \arrow[drr, bend right, "T"'] \arrow[d, phantom, "\Downarrow\mu"] & \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathbf{C} \arrow[ur, "T"] \arrow[rrr, "T"'] & & & \mathbf{C}
\end{tikzcd}
=
\begin{tikzcd}
& \mathbf{C} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathbf{C} \arrow[ur, "T"] \arrow[urr, bend right, "T"] \arrow[rrr, "T"'] & & & \mathbf{C}
\end{tikzcd}$$

On the other hand, the second diagram can be rephrased as follows:

$$\begin{tikzcd}
& \mathbf{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\eta \;\; \Downarrow\mu"] & \\
\mathbf{C} \arrow[ur, bend left, "\mathrm{id}_{\mathbf{C}}"] \arrow[ur, bend right, "T"'] \arrow[rr, "T"'] & & \mathbf{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{C} \arrow[d, bend right, "T"'] \arrow[d, bend left, "T"] \arrow[d, phantom, "="] \\
\mathbf{C}
\end{tikzcd}
=
\begin{tikzcd}
& \mathbf{C} \arrow[dr, bend left, "\mathrm{id}_{\mathbf{C}}"] \arrow[dr, bend right, "T"'] \arrow[d, phantom, "\Downarrow\mu \;\; \Downarrow\eta"] & \\
\mathbf{C} \arrow[ur, "T"] \arrow[rr, "T"'] & & \mathbf{C}
\end{tikzcd}$$

A monad naturally defines other algebraic structures, which we now introduce.

Definition 2.1.7. Given a monad (T, μ, η), a T-algebra or T-module is a pair (a, α), where $a \in \mathrm{Ob}(\mathbf{C})$ and $\alpha\colon Ta \to a$ is such that the following diagrams commute:

$$\begin{tikzcd}
T^2 a \arrow[r, "T\alpha"] \arrow[d, "\mu_a"] & Ta \arrow[d, "\alpha"] \\
Ta \arrow[r, "\alpha"] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[dr, "\mathrm{id}_a"'] & Ta \arrow[d, "\alpha"] \\
& a
\end{tikzcd}$$

Definition 2.1.8. A morphism of T-algebras $(a, \alpha) \to (b, \beta)$ is a morphism $f\colon a \to b$ such that the following diagram commutes:

$$\begin{tikzcd}
Ta \arrow[r, "Tf"] \arrow[d, "\alpha"] & Tb \arrow[d, "\beta"] \\
a \arrow[r, "f"] & b
\end{tikzcd}$$

T-algebras form a category T- **Alg**, which has a natural forgetful functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$.

We now show how to recover the examples previously given with this language.

Example 2.1.9.

$$\begin{aligned}
T = G \times -\colon\ & \mathbf{Set} \to \mathbf{Set} \\
\mu_A\colon\ & G \times (G \times A) \to G \times A \\
& (g,(h,a)) \mapsto (gh, a) \\
\eta_A\colon\ & A \to G \times A \\
& a \mapsto (e, a)
\end{aligned}$$

is a monad and (A, α) is a T-algebra if and only if A is a G-set. It follows that $T\text{-}\mathbf{Alg} \cong G\text{-}\mathbf{Set}$.

Example 2.1.10. Given a ring R, $T = R \otimes_{\mathbb{Z}} -\colon \mathbf{Ab} \to \mathbf{Ab}$ is a monad when considered with the following natural transformations:

$$\begin{aligned}
\mu_-\colon\ & R \otimes_{\mathbb{Z}} (R \otimes_{\mathbb{Z}} -) \cong (R \otimes_{\mathbb{Z}}) \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \\
\eta_-\colon\ & - \cong \mathbb{Z} \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} -
\end{aligned}$$

We have that $(R \otimes_{\mathbb{Z}} -)\text{-}\mathbf{Alg} \cong \mathbf{Mod}_R$.

Example 2.1.11. Consider $W\colon \mathbf{Set} \to \mathbf{Set}$ given by $WX = \amalg_{n \in \mathbb{N}} \Pi_{k=1}^{n} X$. Multiplication $\mu_X\colon WWX \to WX$ is given by concatenation of words, while the unit $\eta_X\colon X \to WX$ is just $x \mapsto (x)$. With this, $W\text{-}\mathbf{Alg} \cong \mathrm{Mon}(\mathbf{Set})$.

Example 2.1.12. The functor $\mathcal{U}$ defined in Example 2.1.4, equipped with suitable natural transformations, is a monad on **Set** and $\mathcal{U}\text{-}\mathbf{Alg} \cong \mathbf{CHTop}$, the category of compact T2 spaces.

Example 2.1.13. The free-forgetful adjunction $F \dashv U$ between categories and directed graphs induces a monad on the latter, with $UF\text{-}\mathbf{Alg} \cong \mathbf{Cat}$.

2.2 Monadic functors

Now that we have introduced these structures, our aim is to characterize *monadic functors*, that is functors $U\colon \mathbf{A} \to \mathbf{C}$ which are equivalent to $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ for some monad (T, μ, η) on $\mathbf{C}$.

First of all, notice that U^T is faithful by construction, hence U must be faithful, but more is true.

Lemma 2.2.1. The functor U^T is conservative, that is if $U^T f$ is an isomorphism then f is an isomorphism of T-algebras.

Proof. Suppose that g is the inverse of $f\colon a \to b$ and f is a morphism $(a,\alpha) \to (b,\beta)$. We only need to prove that the square on the left commutes, that is $g\beta = \alpha Tg$:

$$\begin{tikzcd} Tb \arrow[r, "Tg"] \arrow[d, "\beta"] & Ta \arrow[r, "Tf"] \arrow[d, "\alpha"] & Tb \arrow[d, "\beta"] \\ b \arrow[r, "g"] & a \arrow[r, "f"] & b \end{tikzcd}$$

We see that $fg\beta = \beta$ and $f\alpha Tg = \beta TfTg = \beta T(fg) = \beta T\,\mathrm{id}_b = \beta$, hence the thesis. □

Remark 2.2.2. Notice that the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ can't be monadic since it does not reflect isomorphisms. However, if we restrict it to the full subcategory of **Top** spanned by compact Hausdorff spaces we indeed obtain a monadic functor.

Proposition 2.2.3. The functor $U^T : T\text{-}\mathbf{Alg} \to \mathbf{C}$ has a left adjoint $F^T\colon \mathbf{C} \to T\text{-}\mathbf{Alg}$ such that $F^Tc = (Tc, \mu_c)$, $F^Tf\colon (Tc,\mu_c) \xrightarrow{Tf} (Td,\mu_d)$ and $U^TF^T = T$. Furthermore, the unit of this adjunction is given by $\gamma_c = \eta_c\colon c \to U^TF^Tc = Tc$ and the counit has components $\epsilon_{(a,\alpha)} = \alpha\colon (Ta,\mu_a) \to (a,\alpha)$.

Proof. (i) To show that (Tc,μ_c) is a T-algebra we need the following diagrams to be commutative.

$$\begin{tikzcd} T^3c \arrow[r, "T\mu_c"] \arrow[d, "\mu_{Tc}"'] & T^2c \arrow[d, "\mu_c"] \\ T^2c \arrow[r, "\mu_c"'] & Tc \end{tikzcd} \qquad \begin{tikzcd} Tc \arrow[r, "\eta_{Tc}"] \arrow[dr, equal] & T^2c \arrow[d, "\mu_c"] \\ & Tc \end{tikzcd}$$

These are exactly the associativity and one of the unit laws for (T,μ,η).

(ii) For every $f\colon c \to c'$, Tf is a morphism of algebras $(Tc,\mu_c) \to (Tc',\mu_{c'})$. The diagram

$$\begin{tikzcd} T^2c \arrow[r, "T^2f"] \arrow[d, "\mu_c"'] & T^2c' \arrow[d, "\mu_{c'}"] \\ Tc \arrow[r, "Tf"'] & Tc' \end{tikzcd}$$

is commutative because of the naturality of μ. Hence F^T is defined on morphisms. It is a functor by the functoriality of T.

(iii) The unit is natural by assumption. We claim that $\epsilon_{(a,\alpha)} = \alpha$ is a morphism of algebras

$$F^TU^T(a,\alpha) = F^Ta = (Ta,\mu_a) \to \mathrm{id}_{T\text{-}\mathbf{Alg}}(a,\alpha) = (a,\alpha)$$

and ϵ is a natural transformation $F^T U^T \Rightarrow \mathrm{id}_{T\text{-}\mathbf{Alg}}$. Let's check it. We know that α is a morphism of algebras if and only if

$$\begin{tikzcd}
T^2a \arrow[r, "T\alpha"] \arrow[d, "\mu_a"'] & Ta \arrow[d, "\alpha"] \\
Ta \arrow[r, "\alpha"'] & a
\end{tikzcd}$$

is commutative. But this is one of the two T-algebra axioms! Moreover, to prove that ϵ is natural, we need to show that

$$\begin{tikzcd}
(Ta, \mu_a) \arrow[r, "\alpha=\epsilon_{(a,\alpha)}"] \arrow[d, "Tf"'] & (a, \alpha) \arrow[d, "f"] \\
(Tb, \mu_b) \arrow[r, "\beta=\epsilon_{(b,\beta)}"'] & (b, \beta)
\end{tikzcd}$$

is commutative, but this is the axiom for f to be a morphism of T-algebras!

(iv) It remains to check the two triangular identities $\epsilon F^T \circ F^T \eta = \mathrm{id}_{F^T}$ and $U^T \epsilon \circ \eta U^T = \mathrm{id}_{U^T}$. These are to be checked on the components at c and (a, α), respectively.

$$\begin{tikzcd}
(Tc, \mu_c) \arrow[r, "T\eta_c"] \arrow[dr, equal] & (T^2c, \mu_{Tc}) \arrow[d, "\mu_{Tc}"] \\
& (Tc, \mu_c)
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[dr, equal] & Ta \arrow[d, "\alpha"] \\
& a
\end{tikzcd}$$

The commutativity of these two diagrams is ensured by the second unit law for a monad and the unit law for the T-algebra (a, α), respectively. □

Definition 2.2.4. Algebras of the form (Tc, μ_c) are called free T-algebras.

Thanks to the proposition above we can prove that, given a monad T we can always find an adjunction that generates it. Actually, the converse holds too.

Proposition 2.2.5. If $U\colon \mathbf{D} \to \mathbf{C}$ has a left adjoint F with unit η and counit ϵ, then $(UF, U\epsilon F, \eta)$ is a monad on $\mathbf{C}$. Also, if (T, μ, η) is a monad on $\mathbf{C}$, then $(U^T F^T, U^T \epsilon F^T, \eta) = (T, \mu, \eta)$.

Proof. Let us check the axioms. First of all, the associativity holds due to the following equations.

$$\begin{tikzcd}
& & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & & & & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C}
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
& & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] & & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C}
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
& & & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & & & & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C}
\end{tikzcd}$$

Unit laws:

$$\begin{tikzcd}
& & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \\
\mathbf{C} \arrow[urr, equal, bend left] \arrow[r, "F"] \arrow[ur, phantom, "\eta\Downarrow"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] & & \mathbf{D} \arrow[r, "U"] & \mathbf{C}
\end{tikzcd}$$

is equal to 1_{UF}, since $\epsilon F \circ F\eta = 1_F$ by one of the triangular identities of the adjunction $F \dashv U$. Furthermore,

$$\begin{tikzcd}
& & \mathbf{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] \arrow[drr, equal, bend left] & & \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[ur, "U"] \arrow[rr, equal] & & \mathbf{D} \arrow[r, "U"] \arrow[u, phantom, "\eta\Downarrow"] & \mathbf{C}
\end{tikzcd}$$

is equal to 1_{UF}. This follows from the explicit description of the unit and the counit of the adjunction $F^T \dashv U^T$, in fact

$$U^T \epsilon F^T c = U^T \epsilon_{(Tc, \eta_c)} = \mu_c$$

$\square$

Example 2.2.6 (Interesting adjunction, boring monad)**.** Let us consider the adjunction $U\colon \mathbf{Top} \rightleftarrows \mathbf{Set}\colon \mathsf{Disc} =: F$, whose left adjoint assigns to every set X the discrete topological space $FX = (X, 2^X)$. It's immediate to see that $UFX = X$, hence $UF = \mathrm{id}_{\mathbf{Set}}$. How many natural transformations $\mathrm{id}_{\mathbf{Set}} = UF \overset{\alpha}{\Rightarrow} UF = \mathrm{id}_{\mathbf{Set}}$ are there? We know that $\mathrm{id}_{\mathbf{Set}} \cong \mathrm{Hom}(*, -)$, so $\mathrm{Nat}(\mathrm{id}_{\mathbf{Set}}, \mathrm{id}_{\mathbf{Set}}) \cong \mathrm{Nat}(\mathrm{Hom}(*, -), \mathrm{Hom}(*, -)) \cong \mathrm{Hom}(*, *) = \{\mathrm{id}_*\}$ by Yoneda, hence $\alpha = \mathrm{id}$. Therefore $(UF, U\epsilon F, \eta) = (\mathrm{id}_{\mathbf{Set}}, \mathrm{id}, \mathrm{id})$

Example 2.2.7. If S is a set, $\mathbf{Set}(S, -)\colon \mathbf{Set} \to \mathbf{Set}$ is right adjoint to $S \times -\colon \mathbf{Set} \to \mathbf{Set}$, so we get a monad $X \mapsto \mathbf{Set}(S, S \times X)$. This is called *the state monad* and is important in Computer Science.

There is always a comparison morphism $\mathbf{D} \xrightarrow{\overline{U}} UF\text{-}\mathbf{Alg}$ s.t.

$$\begin{tikzcd}
\mathbf{D} \arrow[rr, "\overline{U}"] \arrow[dr, "U"'] & & UF\text{-}\mathbf{Alg} \arrow[dl, "U^{UF}"] \\
& \mathbf{C} &
\end{tikzcd}$$

commutes. We set $\overline{U}f = (Ud, UFUd \xrightarrow{U\epsilon_d} Ud) = (Ud, U\epsilon_d)$. More specifically, for a given functor $\mathbf{D} \xrightarrow{G} \mathbf{C}$ we can ask what do we need to get an equivalence $\overline{G}\colon \mathbf{D} \to T\text{-}\mathbf{Alg}$. To get there, we will need a few more definitions and lemmas.

2.3 The category of T-actions

Just like a monad (T, μ, η) defines a category $T\text{-}\mathbf{Alg}$, it also allows us to construct another category from functors $\mathbf{D} \to \mathbf{C}$.

Definition 2.3.1. Given a monad (T, μ, η) and fixed a category $\mathbf{D}$, a T-action on a functor $\mathbf{D} \xrightarrow{G} \mathbf{C}$ is a natural transformation $TG \overset{\gamma}{\Rightarrow} G$ such that the diagrams

$$\begin{tikzcd} T^2G \arrow[r, Rightarrow, "T\gamma"] \arrow[d, Rightarrow, "\mu G"'] & TG \arrow[d, Rightarrow, "\gamma"] \\ TG \arrow[r, Rightarrow, "\gamma"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, equal] & TG \arrow[d, Rightarrow, "\gamma"] \\ & G \end{tikzcd}$$

commute.

A morphism of T-actions $(G, \gamma) \Rightarrow (K, \kappa)$ is a natural transformation $\varphi\colon G \Rightarrow K$ such that

$$\begin{tikzcd} TG \arrow[r, Rightarrow, "T\varphi"] \arrow[d, Rightarrow, "\gamma"'] & TK \arrow[d, Rightarrow, "\kappa"] \\ G \arrow[r, Rightarrow, "\varphi"'] & K \end{tikzcd}$$

commutes.

Up to size, T-actions and their morphisms assemble into a category T-$\mathsf{Act}(\mathbf{D})$.

Example 2.3.2. $U^T : T\text{-}\mathbf{Alg} \to \mathbf{C}$ has a T-action given by $(U^T, \alpha\colon TU^T \Rightarrow U^T)$, where $\alpha_{(b,\beta)} \coloneqq \beta\colon Tb \to b$.

Example 2.3.3. Given an adjunction $U \dashv F\colon \mathbf{C} \to \mathbf{D}$ with unit $\eta\colon \mathrm{id}_{\mathbf{C}} \Rightarrow UG$ and counit $\epsilon\colon GU \Rightarrow \mathrm{id}_{\mathbf{D}}$, we get a monad on $(UF, U\epsilon F, \eta)$ on $\mathbf{C}$. We have then a UF-action $U\epsilon\colon UFU \Rightarrow U$, where the axioms follow from the triangular identities.

Proposition 2.3.4. (U^T, α) is the universal T-action, that is for any category $\mathbf{D}$ the functor $\mathbf{Cat}(\mathbf{D}, T\text{-}\mathbf{Alg}) \to T - \mathsf{Act}(\mathbf{D})$ sending G to $(U^TG, \alpha G)$ and $\beta\colon G \Rightarrow H$ to $U^T\beta\colon (U^TG, \alpha G) \Rightarrow (U^TH, \alpha H)$ is an isomorphism of categories.

Proof. In other words, for every T-action (G, γ) there exists a unique lift $\overline{G}\colon \mathbf{D} \to T\text{-}\mathbf{Alg}$ such that $(U^T\overline{G}, \alpha\overline{G}) = (G, \gamma)$ and for every $\phi\colon (G, \gamma) \Rightarrow (K, \kappa)$ there is a unique $\overline{\phi}\colon \overline{G} \Rightarrow \overline{K}$ with $U^T\overline{\phi} = \phi$.

It is enough to set $\overline{G}d := (Gd, \gamma_d)$ on objects, $\overline{G}f := Gf$ on morphisms, $\overline{\phi}_d := \phi_d$ and check the axioms.

$$\begin{tikzcd} D \arrow[r, dotted, "\exists! \overline{G}"] \arrow[dr, "G"'] & T\text{-}\mathbf{Alg} \arrow[d, "U^T"] \\ & \mathbf{C} \end{tikzcd}$$

□

Following the construction in this proof, from the last example we get the comparison functor for the adjunction $F \dashv U$. In particular, $\overline{U}d = (Ud, U\epsilon_d)$. In particular, $U\colon \mathbf{Top} \to \mathbf{Set}$ factors through identities.

2.4 Limits and colimits

We have shown that the forgetful functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ is a right adjoint, hence as such it preserves limits. However, more is true.

Proposition 2.4.1. For any monad (T, μ, η) on $\mathbf{C}$, the forgetful functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ creates limits.

Proof. This statement means that, for any diagram $D\colon I \to T\text{-}\mathbf{Alg}$ such that $U^T D\colon I \to \mathbf{C}$ has a limit (l, κ_i), there is a unique T-algebra structure $\lambda\colon Tl \to l$ such that κ_i is a morphism of T-algebras for all $i \in I$. This makes $((l, \lambda), \kappa_i)$ into a limit of D.

Now we begin the proof.

First of all, remember that $D\phi\colon D_i \to D_j$ is a morphism of T-algebras for all $\phi\colon i \to j$ by assumption, hence the morphisms $\delta_i T\kappa_i\colon Tl \to D_i$ define a cone over D, where δ_i is the T-algebra structure on D_i. By the universal property of the limits, there is a unique morphism $\lambda\colon Tl \to l$ making the diagram commute for all i:

$$\begin{tikzcd} Tl \arrow[r, dotted, "\lambda"] \arrow[d, "T\kappa_i"] & l \arrow[d, "\kappa_i"] \\ TD_i \arrow[r, "\delta_i"] & D_i \end{tikzcd}$$

This tells us that, if the limit $((l, \lambda), \kappa_i)$ of D exists, it is unique. We have to check that (l, λ) is a T-algebra.

Notice that for all i all of the faces of the following diagrams, except for possibly the top ones, commute:

$$
\begin{tikzcd}
 & Tl \arrow[drr, "\lambda"] \arrow[ddd, "T\kappa_i"] & & \\
T^2l \arrow[ur, "T\lambda"] \arrow[drr, "\mu_l"] \arrow[ddd, "T^2\kappa_i"'] & & & l \arrow[ddd, "\kappa_i"] \\
 & & Tl \arrow[ur, "\lambda"] \arrow[ddd, "T\kappa_i"] & \\
 & TD_i \arrow[drr, "\delta_i"] & & \\
T^2D_i \arrow[ur, "T\delta_i"] \arrow[drr, "\mu_{D_i}"] & & & D_i \\
 & & TD_i \arrow[ur, "\delta_i"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Tl \arrow[dr, "\lambda"] \arrow[dd, "T\kappa_i"] & \\
l \arrow[ur, "\eta_l"] \arrow[rr, equal] \arrow[dd, "\kappa_i"'] & & l \arrow[dd, "\kappa_i"] \\
 & TD_i \arrow[dr, "\kappa_i"] & \\
D_i \arrow[ur, "\eta_{D_i}"] \arrow[rr, equal] & & D_i
\end{tikzcd}
$$

Since the κ_i are jointly monic, the upper face commutes and therefore (l, λ) is a T-algebra. It remains to check that $((l, \lambda), \kappa_i)$ factors every other cone over D.

Let $\gamma_i \colon (x, \zeta) \to (D_i, \delta_i)$ be a cone over D. Then, there is a unique $f \colon x \to l$ in $\mathbf{C}$ such that $\kappa_i f = \gamma_i$. We only have to show that f is a morphism of T-algebras $(x, \zeta) \to (l, \lambda)$.

Consider the following diagram and notice that the outer square, the one on the right and the two triangles commute, hence the square on the left commutes as well since the κ_i are jointly monic.

$$
\begin{tikzcd}
Tx \arrow[r, "Tf"'] \arrow[rr, bend left, "T\gamma_i"] \arrow[d, "\zeta"] & Tl \arrow[r, "T\kappa_i"'] \arrow[d, "\lambda"] & TD_i \arrow[d, "\delta_i"] \\
x \arrow[r, "f"] \arrow[rr, bend right, "\gamma_i"'] & l \arrow[r, "\kappa_i"] & D_i
\end{tikzcd}
$$

□

A similar statement holds for colimits.

Proposition 2.4.2. For any monad (T, μ, η) on $\mathbf{C}$, the forgetful functor $U^T \colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ creates any limit preserved by both T and T^2.

Proof. Similarly to the dual situation, this means that for any diagram $D \colon I \to T\text{-}\mathbf{Alg}$ such that $U^T D \colon I \to \mathbf{C}$ has a colimit (c, κ) preserved by both T and T^2, there is a unique T-algebra structure $\lambda \colon Tc \to c$ such that

l_i is a morphism of T-algebras for all $i \in I$. This makes $((c, \lambda), \kappa_i)$ into a colimit of D.

The proof is essentially dual to the one given earlier, in the sense that again we find a unique $\lambda\colon Tc \to c$ using the universal property of the colimit (Tc, Tl_i) of TD.

$$\begin{tikzcd} TD_i \arrow[r, "Tl_i"] \arrow[d, "\delta_i"] & Tc \arrow[d, dotted, "\lambda"] \\ D_i \arrow[r, "l_i"] & c \end{tikzcd}$$

To check that (c, λ) is an algebra we use the universal property of (T^2c, T^2l_i) (for μ) and of (c, l_i) (for η). $\square$

Example 2.4.3. If T is a monad on a complete category $\mathbf{C}$, then T-$\mathbf{Alg}$ is complete. If $\mathbf{C}$ is cocomplete and T is cocontinuous, then T-$\mathbf{Alg}$ is cocontinuous.

In particular, let $\mathbf{C}$ be a small category. There is a cocontinuous monad on the category of $\mathrm{Ob}(\mathbf{C})$-indexed collections of sets whose category of algebras is the functor category $[\mathbf{C}, \mathbf{Set}]$. Its endofunctor $T\colon [\mathrm{Ob}(\mathbf{C}), \mathbf{Set}] \to [\mathrm{Ob}(\mathbf{C}), \mathbf{Set}]$ maps $(X_c)_{c \in \mathbf{C}}$ into $(\coprod_{d \in \mathbf{C}} \mathbf{C}(d, c) \times X_d)_{c \in \mathbf{C}}$. Since $[\mathrm{Ob}(\mathbf{C}), \mathbf{Set}]$ is complete and cocomplete, $[\mathbf{C}, \mathbf{Set}]$ is as well.

Chapter 3

Beck's monadicity theorem

Theorem 3.0.1 (Beck)**.**

Chapter 4

Monads in 2-category theory

Chapter 5

Monads in ∞-category theory