Monads and their applications 12

Exercise 1.
Let $\mathscr{V}$ be a cosmos, A and B monoids in $\mathscr{V}$. The category $\mathscr{V}_A$ of left A-modules is precisely the functor category $[A,\mathscr{V}]$. Show that the category of left $\mathscr{V}$-adjoints $\mathscr{V}_A \to \mathscr{V}_B$ is equivalent to the category of A-B-bimodules (Hint: use the theory of free cocompletions). Apply this to the case of equivalences to show that the module categories are equivalent as $\mathscr{V}$-categories if and only if there exists an A-B bimodule M and a B-A-bimodule N with $N \otimes_B M \cong A$ and $M \otimes_A N \cong B$ (this is called *Morita equivalence* of monoids).

Exercise 2.
Recall that an Eilenberg–Moore object (EM-object for short) represents actions of a monoad in a 2-category $\mathscr{K}$. A *Kleisli-object* is the dual notion, that is, an EM-object in $\mathscr{K}^{\mathrm{op}}$: given a monad $t\colon C \to C$, the Kleisli-object is the universal morphism $f\colon C \to C^t$ with an action $ft \Rightarrow f$.

Show that **CAT** has Kleisli-objects, described as follows: for a monad $T\colon \mathscr{C} \to \mathscr{C}$, the category $\mathscr{C}^T$ has the same objects as $\mathscr{C}$ and $\mathscr{C}^T(A,B) := \mathscr{C}(A,TB)$. The composition is defined using the monad multiplication. Show that $\mathscr{C}^T$ is equivalent to the full subcategory of T-**Alg** consisting of the free algebras.

Exercise 3.
Let $\mathscr{K}$ be a 2-category with EM-objects and let $C \in \mathscr{K}$. Write $\mathrm{Mnd}(C)$ for the category of monads on C. Call a 1-cell $g\colon A \to C$ *tractable* if the right Kan extension of g along itself exists. Write $\mathscr{K}_0'/C$ for the full subcategory of the slice (1-)category consisting of tractable 1-cells. Show that the functor

$$(-)\text{-}\mathbf{Alg}\colon \mathrm{Mnd}(C)^{\mathrm{op}} \to \mathscr{K}_0'/C$$

which sends t to its EM-object is right adjoint to the functor which sends g to the density monad $\mathrm{Ran}_g\, g$.

Exercise 4.
Let $\mathscr{C}$ be an unenriched cocomplete category. Let $\mathscr{G}$ be a full subcategory consisting of κ-presentable objects and suppose that $\mathscr{G}$ is a *strong* generator: the functor

$$\widetilde{K} = \mathrm{Hom}_{\mathscr{G}}(K,-)\colon \mathscr{C} \to [\mathscr{G}^{\mathrm{op}},\mathbf{Set}]$$

is conservative. The goal of this exercise is to show that $\mathscr{C}$ is locally κ-presentable, that is, there automatically exists a *dense* generator of κ-small objects. More precisely, let $\mathscr{A}$ be the closure of $\mathscr{G}$ under κ-small colimits. Show that $\mathscr{A}$ is dense using the following steps.

(a) Show that the category $\mathscr{A}/C$ is κ-filtered and that the canonical diagram $\mathscr{A}/C \to \mathscr{C}$ is sent to a colimit diagram by the functor $\widetilde{K}$.

(b) Show the following fact about conservative functors $F\colon \mathscr{A} \to \mathscr{B}$: if $\mathscr{A}$ has colimits of a given shape, F preserves colimits of that shape, and a specific cocone is sent to a colimit cocone by F, then the cocone in question is already a colimit cocone. In other words, a conservative functor *detects* all the colimits that exist in the domain and that it preserves.

(c) Conclude that $\mathscr{A}$ is dense and thus that $\mathscr{C}$ is locally κ-presentable.

Exercise 5. *??*
Let $\mathscr{V}$ be a cosmos. Given two *small* $\mathscr{V}$-categories, show that there exists a $\mathscr{V}$-category $[\mathscr{A}, \mathscr{B}]$ whose underlying category is $\mathscr{V}\text{-}\mathbf{Cat}(\mathscr{A}, \mathscr{B})$. The hom-object can be defined using the usual equalizer in $\mathscr{V}$ which would give natural transformations for $\mathscr{V} = \mathbf{Set}$. Show that this defines an internal hom-object in the monoidal category $\mathscr{V}\text{-}\mathbf{Cat}$, that is, $- \otimes \mathscr{A} \dashv [\mathscr{A}, -]$.