

# Monads and their applications 11

## Exercise 1.

Let  $\mathcal{V}$  be the category  $\mathbf{Ch}(\mathbf{Ab})$  of chain complexes over abelian groups (“differential graded  $\mathbb{Z}$ -modules”). The tensor product of  $X_\bullet$  and  $Y_\bullet$  is given in degree  $n$  by  $\oplus_{i+j=n} X_i \otimes Y_j$ , with differential  $d(x \otimes y) = dx \otimes y + (-1)^i x \otimes dy$  for  $x \otimes y \in X_i \otimes Y_j$ . Show that this is a symmetric monoidal closed category and explicitly describe the self-enrichment of  $\mathcal{V}$  (specifically, the composition morphism).

## Exercise 2.

Let  $\mathcal{C}$  be a  $\mathcal{V}$ -category with powers,  $T: \mathcal{C} \rightarrow \mathcal{C}$  a  $\mathcal{V}$ -monad. Show that the forgetful  $\mathcal{V}$ -functor  $T\text{-}\mathbf{Alg} \rightarrow \mathcal{C}$  creates powers (that is, for every  $T$ -algebra  $(A, a)$  and  $V \in \mathcal{V}$  one can put a unique  $T$ -algebra structure on  $A^V$  making it a power of  $(A, a)$  in  $T\text{-}\mathbf{Alg}$ ).

## Exercise 3.

Let  $t: C \rightarrow C$  be a monad in the 2-category  $\mathcal{K}$ . A  $t$ -action on a 1-cell  $g: A \rightarrow C$  is a 2-cell  $\alpha: tg \Rightarrow g$  satisfying the laws for a  $t$ -algebra. This defines a 2-functor

$$t\text{-act}(-): \mathcal{K}^{\text{op}} \rightarrow \mathbf{Cat}$$

and we say that the *Eilenberg–Moore* object of  $t$  exists if this 2-functor is representable, that is, there exists a *universal*  $t$ -action  $(u, \alpha): C_t \rightarrow C$ .

- (a) Show that  $U: T\text{-}\mathbf{Alg} \rightarrow \mathcal{C}$  is an Eilenberg–Moore object in  $\mathcal{V}\text{-}\mathbf{Cat}$ .
- (b) Given any Eilenberg–More object  $u: C_t \rightarrow C$ , show that there exists a left adjoint  $f$  of  $u$  such that the monad  $uf$  is  $t$ . (Hint: define a suitable  $t$ -action on  $t$  itself).

## Exercise 4.

- (a) Let  $\mathcal{V} = \mathbf{Set}$  and consider the discrete category  $2 = \{0, 1\}$  on two objects. A weight  $W$  on this category amounts to a choice of two sets. What is the  $W$ -weighted colimit on a diagram  $2 \rightarrow \mathcal{C}$ ?
- (b) Let  $\mathcal{V} = \mathbf{Ab}$  and let  $R$  be a ring, considered as a 1-object  $\mathcal{V}$ -category. Let  $\mathcal{C} = \mathbf{Ab}$  as well. Show that weighted colimits over  $R$  correspond to the usual tensor product of a right and a left  $R$ -module.
- (c) Let  $\mathcal{V} = \mathbf{Cat}$  and consider the category  $\mathcal{J} = 0 \longrightarrow 2 \longleftarrow 1$ . Let  $W$  be the weight with  $W(0) = *$ ,  $W(1) = *$ , and  $W(2) = [1]$ , with

the morphisms  $W(i) \rightarrow [1]$  picking out  $i \in [1] = \{0 \rightarrow 1\}$ . Show that  $W$ -weighted limits in a 2-category  $\mathcal{K}$  are precisely comma-objects.

**Exercise 5.** ??

Show that  $\prod_{j \in J} \mathcal{V}$  is the free cocomplete  $\mathcal{V}$ -category on the discrete category  $j$ . More precisely, given a cocomplete  $\mathcal{V}$ -categories  $\mathcal{C}, \mathcal{D}$ , write  $\mathbf{Cocts}_0[\mathcal{C}, \mathcal{D}]$  for the category of cocontinuous  $\mathcal{V}$ -functors and  $\mathcal{V}$ -natural transformations. Show that the functor  $\mathbf{Cocts}_0[\prod_{j \in J} \mathcal{V}, \mathcal{C}] \rightarrow \prod_{j \in J} \mathcal{C}_0$  given by  $F \mapsto (FI_j)_{j \in J}$  is an equivalence of categories. Here  $I_j$  stands for the object which is given by  $I$  in degree  $j$  and by the initial object everywhere else. You need to be careful when showing that the functor is full!

If one checks all the requirements by only referring to copowers,  $\mathcal{V}$ -coproducts, and  $\mathcal{V}$ -coequalizers, (the latter is not even necessary), then one can use this to show that cocontinuous endofunctors of the product are equivalent, as a monoidal category, to  $\mathcal{V}$ -matrices on  $J$ . This can then be used to give a rigorous proof that any  $\mathcal{V}$ -category with object set  $J$  gives rise to the cocontinuous  $\mathcal{V}$ -monad  $T$  defined in the lecture.