Monads and their applications 9

Exercise 1.
Show that the interchange law holds in $\mathscr{V}\text{-}\mathbf{CAT}$.

Exercise 2.
Let $F\colon \mathscr{V} \to \mathscr{W}$ be a lax monoidal functor. Show that F induces a 2-functor

$$F_*\colon \mathscr{V}\text{-}\mathbf{CAT} \to \mathscr{W}\text{-}\mathbf{CAT}$$

which sends a $\mathscr{V}$-category $\mathscr{A}$ to the $\mathscr{W}$-category $F_*\mathscr{A}$ with the same objects as $\mathscr{A}$, and hom-objects between a and b given by $F\big(\mathscr{A}(a,b)\big)$.

Exercise 3.
Let $\mathbf{Mon}(\mathbf{CAT})$ denote the 2-category of monoidal categories, lax monoidal functors, and monoidal natural transformations. Show that the assignment

$$(-)\text{-}\mathbf{Cat}\colon\ \mathbf{Mon}(\mathbf{CAT}) \to 2\text{-}\mathbf{CAT}$$

which sends $\mathscr{V}$ to the 2-category $\mathscr{V}\text{-}\mathbf{Cat}$ extends to a 2-functor, with action on 1-cells given by the 2-functor of Exercise 2.

Exercise 4.
Let $T\colon \mathscr{C} \to \mathscr{C}$ be a $\mathscr{V}$-monad. Complete the proof the $T\text{-}\mathbf{Alg}$ has the structure of a $\mathscr{V}$-category. Hint: at some point in the proof that algebra morphisms compose in the unenriched case, we use associativity, namely when we consider the composite

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, dashed] & TB \arrow[r, dashed] \arrow[d, "\beta"] & \cdot \arrow[d, dashed] \\
\cdot \arrow[r, dashed] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

in $\mathscr{C}$. This switches the operations "precompose with β" to "postcompose with β" and when translating the proof to $\mathscr{V}$-categories, one has to use the associator at this point.

Exercise 5.
Show that $T\text{-}\mathbf{Alg}$ has the same universal property in $\mathscr{V}\text{-}\mathbf{CAT}$ as the ordinary category of algebras for an unenriched monad has in $\mathbf{CAT}$. Namely, for each $\mathscr{V}$-category $\mathscr{A}$, define the category $T\text{-}\mathbf{Act}(\mathscr{A},\mathscr{C})$ of T-actions (now given by $\mathscr{V}$-functors with a $\mathscr{V}$-natural transformation $\rho\colon TG \Rightarrow G$) and show that $U^T\colon T\text{-}\mathbf{Alg} \to \mathscr{C}$ is the universal T-action.