Monads and their applications

Dr. Daniel Schäppi's course lecture notes

by

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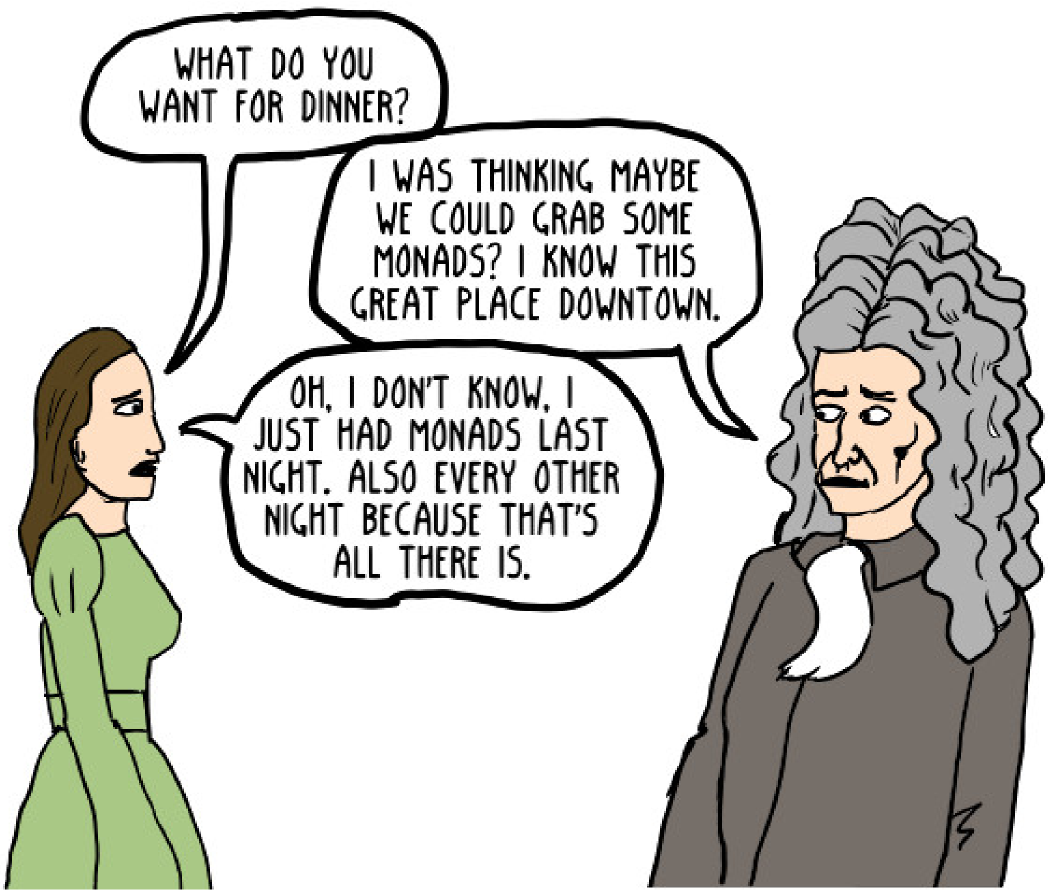

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Categorical preliminaries

Definition 0.0.1 (Categories)**.** A *category* $\mathcal{C}$ consists of:

1. a collection of objects $\mathrm{Ob}(\mathcal{C})$;
2. a collection of arrows $\mathrm{Ar}(\mathcal{C})$;
3. two maps $\mathrm{dom}, \mathrm{cod}\colon \mathrm{Ar}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$;
4. a map $\mathrm{id}_{-}\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ar}(\mathcal{C})$ with $\mathrm{dom}(\mathrm{id}_c) = c = \mathrm{cod}(\mathrm{id}_c)$;
5. for every $f, g \in \mathrm{Ar}(\mathcal{C})$ such that $\mathrm{cod}(f) = \mathrm{dom}(g)$ a unique composite morphism gf such that $\mathrm{cod}(gf) = \mathrm{cod}(g)$, $\mathrm{dom}(gf) = \mathrm{dom}(f)$.

This data has to satisfy the following axioms

1. given $f \in \mathrm{Ar}(\mathcal{C})$, $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$, $\mathrm{id}_{c'} f = f = \mathrm{id}_c$, that is the composition is unital;
2. given a composable triple $f, g, h \in \mathrm{Ar}(\mathcal{C})$, $h(gf) = (hg)f$, that is the composition is associative.

An arrow f such that $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$ is denoted $f\colon c \to c'$.

Definition 0.0.2 (Functors)**.**

Definition 0.0.3 (Full functors, faithful functor)**.**

Definition 0.0.4 (Natural transformations)**.**

Definition 0.0.5 (Equivalent functors)**.**

Definition 0.0.6 (Representable Functors)**.**

Definition 0.0.7 (Whiskering)**.**

Definition 0.0.8 (Horizontal and vertical composition of nat.transf.)**.**

Definition 0.0.9 (adjunctions)**.**

Lemma 0.0.10 (Yoneda)**.**

Proof. $\square$

We will denote by よ (the hiragana kata for “Yo”) the Yoneda embedding $\mathcal{C} \hookrightarrow \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

1 Monads and algebras

1.1 Introduction

Throughout mathematics we encounter structures defined by some action morphisms. Here we give some examples.

Example 1.1.1. Given a group G, we may consider a G-set X described by an action map $G \times X \to X$.

Example 1.1.2. Given an abelian group M and a ring R, we can get an R-module M by fixing a group homomorphism $R \otimes_{\mathbb{Z}} M \to M$.

Example 1.1.3. Given a monoid M in **Set**, we get a map $\Pi_{k=1}^{n} M \to M$, $(m_1, \ldots, m_n) \mapsto ((\ldots((m_1 m_2) m_3) \ldots) m_{n-1}) m_n$. This induces an action map from $W(M) = \amalg_{n \in \mathbb{N}} \Pi_{k=1}^{n} M$, the set of words on M, to M.

Example 1.1.4. Given a set X, let $\mathcal{U}X$ be the set of ultrafilters on it. Any compact T2 topology on X allows us to see each ultrafilter as a system of neighborhoods of a unique point in X, hence it gives us a unique map $\mathcal{U}X \to X$ sending each ultrafilter to the respective point.

Example 1.1.5. Given a directed graph $D = (V, E, E \underset{t}{\overset{s}{\rightrightarrows}} V)$, we can create its free category FD, where the objects are the vertices and $FD(v, w) = \{\text{finite paths } v \to \ldots \to w\}$. We set id_v to be the path of length 0, while composition is just the concatenation of paths.

In particular, if D is the directed graph with $V = \{0, \ldots, n\}$ and an edge $j \to k$ if and only if $k = j + 1$, we have $FD \cong [n]$.

If $D = \{*\}$ and $E = \{* \to *\}$, then $FD(*, *) \cong \mathbb{N}$.

Given a small category $\mathcal{C}$, we may consider the underlying graph $U\mathcal{C} = D$ with $V = \mathrm{Ob}(\mathcal{C})$, $E = \mathrm{Ar}(\mathcal{C})$, $s = \mathrm{dom}$ and $t = \mathrm{cod}$. We get then an action map $UFU\mathcal{C} \to U\mathcal{C}$ sending a finite path to its composite. This map is a morphism of directed graphs.

Notice that we always have a category $\mathcal{C}$ and some functor $T \colon \mathcal{C} \to \mathcal{C}$ with an action map $T\mathcal{C} \to \mathcal{C}$. How can we see all of these examples as specific instances of a general phenomenon?

Definition 1.1.6. A *monad* on a category $\mathcal{C}$ is a triple (T, μ, η) where $T \colon \mathcal{C} \to \mathcal{C}$ is a functor, while $\mu \colon T^2 \Rightarrow T$ and $\eta \colon \mathrm{id}_{\mathcal{C}} \Rightarrow T$ are natural transformations such that the following diagrams commute.

$$
\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "\mathrm{id}_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "\mathrm{id}_T"] \\
& T &
\end{tikzcd}
$$

μ is called the *multiplicative map*, while η is the *unit* of T.

The commutativity of the first diagram is equivalent to stating that the following two diagrams are equal.

$$
\begin{tikzcd}
& \mathcal{C} \arrow[r, "T"] \arrow[drr, bend right, "T"'] \arrow[d, phantom, "\Downarrow\mu"] & \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathcal{C} \arrow[ur, "T"] \arrow[rrr, "T"'] & {} & {} & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{C} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathcal{C} \arrow[ur, "T"] \arrow[urr, bend right, "T"'] \arrow[rrr, "T"'] & {} & {} & \mathcal{C}
\end{tikzcd}
$$

On the other hand, the second diagram can be rephrased as follows:

$$
\begin{tikzcd}
& \mathcal{C} \arrow[dr, "T"] & \\
\mathcal{C} \arrow[ur, equal, bend left=60] \arrow[ur, bend right, "T"'] \arrow[rr, "T"'] \arrow[ur, phantom, "\Downarrow\eta"] & {} \arrow[u, phantom, "\Downarrow\mu"] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[d, bend right=50, "T"'] \arrow[d, bend left=50, "T"] \arrow[d, phantom, "="] \\
\mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{C} \arrow[dr, equal, bend left=60] \arrow[dr, bend right, "T"'] & \\
\mathcal{C} \arrow[ur, "T"] \arrow[rr, "T"'] & {} \arrow[u, phantom, "\Downarrow\mu"] & \mathcal{C} \arrow[ul, phantom, "\Downarrow\eta"]
\end{tikzcd}
$$

A monad naturally defines other algebraic structures, which we now introduce.

Definition 1.1.7. Given a monad (T, μ, η), a *T-algebra* or *T-module* is a pair (a, α), where $a \in \mathrm{Ob}(\mathcal{C})$ and $\alpha\colon Ta \to a$ is such that the following diagrams commute.

$$
\begin{tikzcd}
T^2a \arrow[r, "T\alpha"] \arrow[d, "\mu_a"'] & Ta \arrow[d, "\alpha"] \\
Ta \arrow[r, "\alpha"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[dr, equal] & Ta \arrow[d, "\alpha"] \\
& a
\end{tikzcd}
$$

Definition 1.1.8. A *morphism of T-algebras* $(a, \alpha) \to (b, \beta)$ is a morphism $f\colon a \to b$ such that the following diagram commutes:

$$
\begin{tikzcd}
Ta \arrow[r, "Tf"] \arrow[d, "\alpha"'] & Tb \arrow[d, "\beta"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}
$$

T-algebras form a category $T\text{-}\mathsf{Alg}$, which has a natural forgetful functor $U^T\colon T\text{-}\mathsf{Alg} \to \mathcal{C}$. We now show how to recover the examples previously given with this language.

Example 1.1.9.

$$
\begin{aligned}
T = G \times -\colon &\ \mathbf{Set} \to \mathbf{Set} \\
\mu_A\colon &\ G \times (G \times A) \to G \times A \\
&\ (g, (h, a)) \mapsto (gh, a) \\
\eta_A\colon &\ A \to G \times A \\
&\ a \mapsto (e, a)
\end{aligned}
$$

is a monad and (A, α) is a T-algebra if and only if A is a G-set. It follows that $T\text{-}\mathsf{Alg} \cong G\text{-}\mathbf{Set}$.

Example 1.1.10. Given a ring R, $T = R \otimes_{\mathbb{Z}} -\colon \mathbf{Ab} \to \mathbf{Ab}$ is a monad when considered with the following natural transformations:

$$
\begin{aligned}
\mu_-\colon &\ R \otimes_{\mathbb{Z}} (R \otimes_{\mathbb{Z}} -) \cong (R \otimes_{\mathbb{Z}} R) \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \\
\eta_-\colon &\ - \cong \mathbb{Z} \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} -
\end{aligned}
$$

We have that $(R \otimes_{\mathbb{Z}} -)\text{-}\mathsf{Alg} \cong \mathbf{Mod}_R$.

Example 1.1.11. Consider $W\colon \mathbf{Set} \to \mathbf{Set}$ given by $WX = \amalg_{n\in\mathbb{N}}\Pi_{k=1}^{n}X$. Multiplication $\mu_X\colon WWX \to WX$ is given by concatenation of words, while the unit $\eta_X\colon X \to WX$ is just $x \mapsto (x)$. With this, $W\text{-}\mathsf{Alg} \cong \mathrm{Mon}(\mathbf{Set})$.

Example 1.1.12. The functor $\mathcal{U}$ defined in Example 1.1.4, equipped with suitable natural transformations, is a monad on **Set** and $\mathcal{U}\text{-}\mathsf{Alg} \cong \mathbf{CHTop}$, the category of compact T2 spaces.

Example 1.1.13. The free-forgetful adjunction $F \dashv U$ between categories and directed graphs induces a monad on the latter, with $UF\text{-}\mathsf{Alg} \cong \mathbf{Cat}$.

1.2 Monadic functors

Now that we have introduced these structures, our aim is to characterize *monadic functors*, that is functors $U\colon \mathcal{A} \to \mathcal{C}$ which are equivalent to $U^T\colon T\text{-}\mathsf{Alg} \to \mathcal{C}$ for some monad (T, μ, η) on $\mathcal{C}$.

First of all, notice that U^T is faithful by construction, hence U must be faithful, but more is true.

Lemma 1.2.1. The functor U^T is conservative, that is if $U^T f$ is an isomorphism then f is an isomorphism of T-algebras.

Proof. Suppose that g is the inverse of $f\colon a \to b$ and f is a morphism $(a, \alpha) \to (b, \beta)$. We only need to prove that the square on the left commutes, that is $g\beta = \alpha Tg$:

$$\begin{tikzcd} Tb \arrow[r, "Tg"] \arrow[d, "\beta"'] & Ta \arrow[r, "Tf"] \arrow[d, "\alpha"'] & Tb \arrow[d, "\beta"] \\ b \arrow[r, "g"'] & a \arrow[r, "f"'] & b \end{tikzcd}$$

We see that $fg\beta = \beta$ and $f\alpha Tg = \beta TfTg = \beta T(fg) = \beta T\,\mathrm{id}_b = \beta$, hence the thesis. $\square$

Remark 1.2.2. Notice that the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ can't be monadic since it does not reflect isomorphisms. However, if we restrict it to the full subcategory of **Top** spanned by compact Hausdorff spaces we indeed obtain a monadic functor.

Proposition 1.2.3. The functor $U^T\colon T\text{-}\mathsf{Alg} \to \mathcal{C}$ has a left adjoint $F^T\colon \mathcal{C} \to T\text{-}\mathsf{Alg}$ such that $F^T c = (Tc, \mu_c)$, $F^T f\colon (Tc, \mu_c) \xrightarrow{Tf} (Td, \mu_d)$ and $U^T F^T = T$. Furthermore, the unit of this adjunction is given by $\gamma_c = \eta_c\colon c \to U^T F^T c = Tc$ and the counit has components $\epsilon_{(a,\alpha)} = \alpha\colon (Ta, \mu_a) \to (a, \alpha)$.

Proof. (i) To show that (Tc, μ_c) is a T-algebra we need the following diagrams to be commutative.

$$\begin{tikzcd} T^3c \arrow[r, "T\mu_c"] \arrow[d, "\mu_{Tc}"'] & T^2c \arrow[d, "\mu_c"] \\ T^2c \arrow[r, "\mu_c"'] & Tc \end{tikzcd} \qquad \begin{tikzcd} Tc \arrow[r, "\eta_{Tc}"] \arrow[dr, equal] & T^2c \arrow[d, "\mu_c"] \\ & Tc \end{tikzcd}$$

These are exactly the associativity and one of the unit laws for (T, μ, η).

(ii) For every $f\colon c \to c'$, Tf is a morphism of algebras $(Tc, \mu_c) \to (Tc', \mu_{c'})$. The diagram

$$\begin{tikzcd} T^2c \arrow[r, "T^2f"] \arrow[d, "\mu_c"'] & T^2c' \arrow[d, "\mu_{c'}"] \\ Tc \arrow[r, "Tf"'] & Tc' \end{tikzcd}$$

is commutative because of the naturality of μ. Hence F^T is defined on morphisms. It is a functor by the functoriality of T.

(iii) The unit is natural by assumption. We claim that $\epsilon_{(a,\alpha)} = \alpha$ is a morphism of algebras

$$F^TU^T(a, \alpha) = F^Ta = (Ta, \mu_a) \to \mathrm{id}_{T\text{-}\mathsf{Alg}}(a, \alpha) = (a, \alpha)$$

and ϵ is a natural transformation $F^TU^T \Rightarrow \mathrm{id}_{T\text{-}\mathsf{Alg}}$. Let's check it. We know that α is a morphism of algebras if and only if

$$\begin{tikzcd} T^2a \arrow[r, "T\alpha"] \arrow[d, "\mu_a"'] & Ta \arrow[d, "\alpha"] \\ Ta \arrow[r, "\alpha"'] & a \end{tikzcd}$$

is commutative. But this is one of the two T-algebra axioms! Moreover, to prove that ϵ is natural, we need to show that

$$\begin{tikzcd} (Ta, \mu_a) \arrow[r, "\alpha=\epsilon_{(a,\alpha)}"] \arrow[d, "Tf"'] & (a, \alpha) \arrow[d, "f"] \\ (Tb, \mu_b) \arrow[r, "\beta=\epsilon_{(b,\beta)}"'] & (b, \beta) \end{tikzcd}$$

is commutative, but this is the axiom for f to be a morphism of T-algebras!

(iv) It remains to check the two triangular identities $\epsilon F^T \cdot F^T\eta = \mathrm{id}_{F^T}$ and $U^T\epsilon \cdot \eta U^T = \mathrm{id}_{U^T}$. These are to be checked on the components at c and (a, α), respectively.

$$\begin{tikzcd} (Tc, \mu_c) \arrow[r, "T\eta_c"] \arrow[dr, equal] & (T^2c, \mu_{Tc}) \arrow[d, "\mu_{Tc}"] \\ & (Tc, \mu_c) \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[dr, equal] & Ta \arrow[d, "\alpha"] \\ & a \end{tikzcd}$$

The commutativity of these diagrams is ensured by the second unit law for a monad and the unit law for the T-algebra (a, α), respectively. □

Definition 1.2.4. Algebras of the form (Tc, μ_c) are called *free T-algebras.*

Thanks to the proposition above we can prove that, given a monad T we can always find an adjunction that generates it. Actually, the converse holds too.

Proposition 1.2.5. *If $U\colon \mathcal{D} \to \mathcal{C}$ has a left adjoint F with unit η and counit ϵ, then $(UF, U\epsilon F, \eta)$ is a monad on $\mathcal{C}$. Also, if (T, μ, η) is a monad on $\mathcal{C}$, then $(U^TF^T, U^T\epsilon F^T, \eta) = (T, \mu, \eta)$.*

Proof. Let us check the axioms. First of all, the associativity holds due to the following equations.

$$\begin{tikzcd}
& & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & & & & & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[ur, "U"] \arrow[rr, equal] & {} & \mathcal{D} \arrow[r, "U"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, "U"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[ur, "U"] \arrow[rr, equal] & {} & \mathcal{D} \arrow[r, "U"] & \mathcal{C}
\end{tikzcd}$$

$$= \begin{tikzcd}
& & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[ur, "U"] \arrow[rr, equal] & {} & \mathcal{D} \arrow[ur, "U"] \arrow[rr, equal] & {} & \mathcal{D} \arrow[r, "U"] & \mathcal{C}
\end{tikzcd}$$

$$= \begin{tikzcd}
& & & & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & & & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, "U"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[ur, "U"] \arrow[rr, equal] & {} & \mathcal{D} \arrow[r, "U"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[ur, "U"] \arrow[rr, equal] & {} & \mathcal{D} \arrow[r, "U"] & \mathcal{C}
\end{tikzcd}$$

Unit laws:

$$\begin{tikzcd}
& {} \arrow[d, phantom, "\eta\Downarrow"] & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \\
\mathcal{C} \arrow[urr, equal, bend left] \arrow[r, "F"] & \mathcal{D} \arrow[ur, "U"] \arrow[rr, equal] & {} & \mathcal{D} \arrow[r, "U"] & \mathcal{C}
\end{tikzcd}$$

is equal to 1_{UF}, since $\epsilon F \cdot F\eta = 1_F$ by one of the triangular identities of the adjunction $F \dashv U$. Furthermore,

$$\begin{tikzcd}
& & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] \arrow[drr, equal, bend left] & {} \arrow[d, phantom, "\eta\Downarrow"] & \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[ur, "U"] \arrow[rr, equal] & {} & \mathcal{D} \arrow[r, "U"] & \mathcal{C}
\end{tikzcd}$$

is equal to 1_{UF}. This follows from the explicit description of the unit and the counit of the adjunction $F^T \dashv U^T$, in fact $U^T \epsilon F^T c = U^T \epsilon_{(Tc,\eta_c)} = \mu_c$. □

Example 1.2.6 (Interesting adjunction, boring monad)**.** Let us consider the adjunction $U\colon \mathbf{Top} \rightleftarrows \mathbf{Set} \colon \mathsf{Disc} =: F$, whose left adjoint assigns to every set X the discrete topological space $FX = (X, 2^X)$. It's immediate to see that $UFX = X$, hence $UF = \mathrm{id}_{\mathbf{Set}}$. How many natural transformations $\mathrm{id}_{\mathbf{Set}} = UF \overset{\alpha}{\Rightarrow} UF = \mathrm{id}_{\mathbf{Set}}$ are there? We know that $\mathrm{id}_{\mathbf{Set}} \cong \mathrm{Hom}(*, -)$, so $\mathrm{Nat}(\mathrm{id}_{\mathbf{Set}}, \mathrm{id}_{\mathbf{Set}}) \cong \mathrm{Nat}(\mathrm{Hom}(*, -), \mathrm{Hom}(*, -)) \cong \mathrm{Hom}(*, *) = \{\mathrm{id}_*\}$ by Yoneda, hence $\alpha = \mathrm{id}$. Therefore $(UF, U\epsilon F, \eta) = (\mathrm{id}_{\mathbf{Set}}, \mathrm{id}, \mathrm{id})$

Example 1.2.7. If S is a set, $\mathbf{Set}(S, -)\colon \mathbf{Set} \to \mathbf{Set}$ is right adjoint to $S \times -\colon \mathbf{Set} \to \mathbf{Set}$, so we get a monad $X \mapsto \mathbf{Set}(S, S \times X)$. This is called *the state monad* and is important in Computer Science.

There is always a comparison morphism $\mathcal{D} \xrightarrow{\overline{U}} UF\text{-}\mathsf{Alg}$ such that

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "\overline{U}"] \arrow[dr, "U"'] & & UF\text{-}\mathsf{Alg} \arrow[dl, "U^{UF}"] \\
& \mathcal{C} &
\end{tikzcd}$$

commutes. We set $\overline{U} f = (Ud, UFUd \xrightarrow{U\epsilon_d} Ud) = (Ud, U\epsilon_d)$. More specifically, for a given functor $G\colon \mathcal{D} \to \mathcal{C}$ we can ask what do we need to get an equivalence $\overline{G}\colon \mathcal{D} \to T\text{-}\mathsf{Alg}$. To get there, we will need a few more definitions and lemmas.

1.3 The category of T-actions

Just like a monad (T, μ, η) defines a category $T\text{-}\mathsf{Alg}$, it also allows us to construct another category from functors $\mathcal{D} \to \mathcal{C}$.

Definition 1.3.1. Given a monad (T, μ, η) on a category $\mathcal{C}$ and fixed another category $\mathcal{D}$, a T*-action* on a functor $G\colon \mathcal{D} \to \mathcal{C}$ is a natural transformation $\gamma\colon TG \Rightarrow G$ such that the diagrams

$$\begin{tikzcd} T^2G \arrow[r, Rightarrow, "T\gamma"] \arrow[d, Rightarrow, "\mu G"'] & TG \arrow[d, Rightarrow, "\gamma"] \\ TG \arrow[r, Rightarrow, "\gamma"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, equal] & TG \arrow[d, Rightarrow, "\gamma"] \\ & G \end{tikzcd}$$

commute.

A *morphism of* T*-actions* $(G, \gamma) \overset{\varphi}{\Rightarrow} (K, \kappa)$ is a natural transformation $\varphi\colon G \Rightarrow K$ such that

$$\begin{tikzcd} TG \arrow[r, Rightarrow, "T\varphi"] \arrow[d, Rightarrow, "\gamma"'] & TK \arrow[d, Rightarrow, "\kappa"] \\ G \arrow[r, Rightarrow, "\varphi"'] & K \end{tikzcd}$$

commutes.

Up to size, T-actions and their morphisms assemble into a category $T\text{-}\mathsf{Act}(\mathcal{D})$.

Example 1.3.2. The functor $U^T\colon T\text{-}\mathsf{Alg} \to \mathcal{C}$ has a T-action given by $(U^T, \alpha\colon TU^T \Rightarrow U^T)$, where $\alpha_{(b,\beta)} \coloneqq \beta\colon Tb \to b$.

Example 1.3.3. Given an adjunction $F \dashv U\colon \mathcal{C} \rightleftarrows \mathcal{D}$ with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow UF$ and counit $\epsilon\colon FU \Rightarrow \mathrm{id}_{\mathcal{D}}$, we get a monad on $(UF, U\epsilon F, \eta)$ on $\mathcal{C}$. We have then a UF-action $U\epsilon\colon UFU \Rightarrow U$, where the axioms follow from the triangular identities and the naturality of $U\epsilon$.

Proposition 1.3.4. (U^T, α) is the universal T-action, that is for any category $\mathcal{D}$ the functor $\mathbf{Cat}(\mathcal{D}, T\text{-}\mathsf{Alg}) \to T\text{-}\mathsf{Act}(\mathcal{D})$ sending G to $(U^TG, \alpha G)$ and $\beta\colon G \Rightarrow H$ to $U^T\beta\colon (U^TG, \alpha G) \Rightarrow (U^TH, \alpha H)$ is an isomorphism of categories.

Proof. In other words, for every T-action (G, γ) there exists a unique lift $\overline{G}\colon \mathcal{D} \to T\text{-}\mathsf{Alg}$ such that $(U^T\overline{G}, \alpha\overline{G}) = (G, \gamma)$ and for every $\phi\colon (G, \gamma) \Rightarrow (K, \kappa)$ there is a unique $\overline{\phi}\colon \overline{G} \Rightarrow \overline{K}$ with $U^T\overline{\phi} = \phi$.

It is enough to set $\overline{G}d \coloneqq (Gd, \gamma_d)$ on objects, $\overline{G}f \coloneqq Gf$ on morphisms, $\overline{\phi}_d \coloneqq \phi_d$ and check the axioms.

$$\begin{tikzcd} \mathcal{D} \arrow[r, dashed, "\exists!\overline{G}"] \arrow[dr, "G"'] & T\text{-}\mathsf{Alg} \arrow[d, "U^T"] \\ & \mathcal{C} \end{tikzcd}$$

∎

Following the construction in this proof, from the last example we get the comparison functor for the adjunction $F \dashv U$. In particular, $\overline{U}d = (Ud, U\epsilon_d)$. Furthermore, this means that $U\colon \mathbf{Top} \to \mathbf{Set}$ factors through identities.

1.4 Limits and colimits in the category of algebras

We have shown that the forgetful functor $U^T : T\text{-}\mathsf{Alg} \to \mathcal{C}$ is a right adjoint, and as such it preserves limits. However, more is true.

Proposition 1.4.1. For any monad (T, μ, η) on $\mathcal{C}$, the forgetful functor $U^T : T\text{-}\mathsf{Alg} \to \mathcal{C}$ strictly creates limits.

Proof. This statement means that, for any diagram $D\colon I \to T\text{-}\mathsf{Alg}$ such that $U^T D\colon I \to \mathcal{C}$ has a limit (l, κ_i) in $\mathcal{C}$, there is a unique T-algebra structure $\lambda\colon Tl \to l$ such that κ_i is a morphism of T-algebras for all $i \in I$ and this makes $((l, \lambda), \kappa_i)$ into a limit of D.

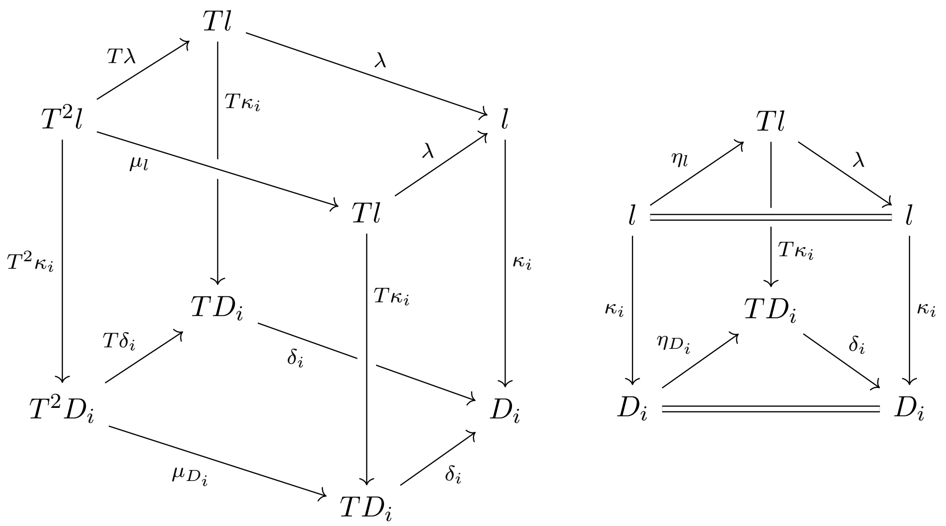
Now we begin the proof.

First of all, remember that $D\phi\colon D_i \to D_j$ is a morphism of T-algebras for all $\phi\colon i \to j$ by assumption, hence the morphisms $\delta_i T\kappa_i\colon Tl \to D_i$ define a cone over D, where δ_i is the T-algebra structure on D_i. By the universal property of the limit, there is a unique morphism $\lambda\colon Tl \to l$ making the following diagram commute for all i.

$$\begin{tikzcd}
Tl \arrow[r, "T\kappa_i"] \arrow[d, dashed, "\lambda"'] & TD_i \arrow[d, "\delta_i"] \\
l \arrow[r, "\kappa_i"'] & D_i
\end{tikzcd}$$

This tells us that, if the limit $((l, \lambda), \kappa_i)$ of D exists, it is unique. We have to check that (l, λ) is a T-algebra.

Notice that for all i all of the faces of the following diagrams, except for possibly the top ones, commute.



Since the κ_i are jointly monic, the upper face commutes and therefore (l, λ) is a T-algebra. It remains to check that $((l, \lambda), \kappa_i)$ factors every other cone over D.

Let $\gamma_i\colon (x, \zeta) \to (D_i, \delta_i)$ be a cone over D. Then, there is a unique $f\colon x \to l$ in $\mathcal{C}$ such that $\kappa_i f = \gamma_i$. We only have to show that f is a morphism of T-algebras $(x, \zeta) \to (l, \lambda)$.

Consider the following diagram and notice that the outer square, the one on the right and the two triangles commute, hence the square on the left commutes as well since the κ_i are jointly monic.

$$\begin{tikzcd} Tx \arrow[r, "Tf"'] \arrow[d, "\zeta"] \arrow[rr, bend left, "T\gamma_i"] & Tl \arrow[r, "T\kappa_i"'] \arrow[d, "\lambda"] & TD_i \arrow[d, "\delta_i"] \\ x \arrow[r, "f"] \arrow[rr, bend right, "\gamma_i"'] & l \arrow[r, "\kappa_i"] & D_i \end{tikzcd}$$

□

A similar statement holds for colimits.

Proposition 1.4.2. Given a monad (T, μ, η) on $\mathcal{C}$, the forgetful functor $U^T : T\text{-}\mathsf{Alg} \to \mathcal{C}$ strictly creates any colimit preserved by both T and T^2.

Proof. Similarly to the dual situation, this means that for any diagram $D\colon I \to T\text{-}\mathsf{Alg}$ such that $U^T D\colon I \to \mathcal{C}$ has a colimit (c, κ_i) preserved by both T and T^2, there is a unique T-algebra structure $\lambda\colon Tc \to c$ such that κ_i is a morphism of T-algebras for all $i \in I$. This makes $((c, \lambda), \kappa_i)$ into a colimit of D.

The proof is essentially dual to the one given earlier, in the sense that we find again a unique $\lambda\colon Tc \to c$ using the universal property of the colimit $(Tc, T\kappa_i)$ of TD.

$$\begin{tikzcd} TD_i \arrow[r, "T\kappa_i"] \arrow[d, "\delta_i"'] & Tc \arrow[d, dashed, "\lambda"] \\ D_i \arrow[r, "\kappa_i"'] & c \end{tikzcd}$$

To check that (c, λ) is an algebra we use the universal property of $(T^2c, T^2\kappa_i)$, for μ, and the one of (c, κ_i), for η. □

Remark 1.4.3. The same statements hold for monadic functors, except for the fact that they might not create limits and colimits strictly since they are just equivalent to a U^T.

Remark 1.4.4. If T is a monad on a complete category $\mathcal{C}$, then $T\text{-}\mathsf{Alg}$ is complete. If $\mathcal{C}$ is cocomplete and T is cocontinuous, then $T\text{-}\mathsf{Alg}$ is cocomplete.

Example 1.4.5. Let $\mathcal{C}$ be a small category. There is a cocontinuous monad on the category of $\mathrm{Ob}(\mathcal{C})$-indexed collections of sets whose category of algebras is the functor category $[\mathcal{C}, \mathbf{Set}]$. The underlying endofunctor of this monad is defined as

$$\begin{aligned} T\colon [\mathrm{Ob}(\mathcal{C}), \mathbf{Set}] &\to [\mathrm{Ob}(\mathcal{C}), \mathbf{Set}] \\ (X_c)_{c\in\mathcal{C}} &\mapsto \left(\coprod_{d\in\mathcal{C}} \mathcal{C}(d, c) \times X_d\right)_{c\in\mathcal{C}} \end{aligned}$$

Since $[\mathrm{Ob}(\mathcal{C}), \mathbf{Set}]$ is complete and cocomplete, so is $[\mathcal{C}, \mathbf{Set}]$ (with limits and colimits computed pointwise).

2 Beck's monadicity theorem

The final ingredient we need is the observation that T-algebras admit canonical presentations using free algebras.

2.1 Split coequalizers and Beck's theorem

Example 2.1.1. Pick an epi $F \twoheadrightarrow G$ in the category of groups **Grp**, where F is a free group. The kernel of this homomorphism defines a (normal) subgroup K of F, giving rise to the sequence $K \rightarrowtail F \twoheadrightarrow G$. We can take another epi $F' \twoheadrightarrow K$, with F' again a free group. Therefore G is the cokernel of some morphism $F' \to F$. This argument applies to rings, algebras etc.

It is natural to ask if we can do this systematically for general T-algebras. Given (a, α) in $T\text{-}\mathsf{Alg}$, we have $F^T U^T(a, \alpha) \to (a, \alpha)$ i.e. $(Ta, \mu_a) \xrightarrow{\alpha} (a, \alpha)$. A candidate[1] for F' would be $F^T U^T(Ta, \mu_a) = (T^2 a, \mu_{Ta})$. What are the "elements" of Ta? Notice that

$$(T^2 a, \mu_{Ta}) \overset{T\alpha}{\underset{\mu_a}{\rightrightarrows}} (Ta, \mu_a) \xrightarrow{\alpha} (a, \alpha)$$

is a well defined diagram in $T\text{-}\mathsf{Alg}$, with $\alpha\mu_a = \alpha T\alpha$. Moreover, this is a coequalizer. We can use Proposition 1.4.2 to prove it, so that we need to check whether U^T sends the diagram above into a coequalizer preserved by T and T^2. In $\mathcal{C}$, we get the diagram

$$\begin{tikzcd} T^2 a \arrow[r, shift left, "T\alpha"] \arrow[r, shift right, "\mu_a"'] & Ta \arrow[r, "\alpha"] \arrow[l, bend left=60, "\eta_{Ta}"] & a \arrow[l, bend left=60, "\eta_a"] \end{tikzcd}$$

in which the following equations hold true: $\alpha T\alpha = \alpha\mu_a$, $\alpha\eta_a = \mathrm{id}_a$, $\mu_a \eta_{Ta} = \mathrm{id}_{Ta}$ and $\eta_a \alpha = T\alpha\, \eta_{Ta}$ by naturality. It is a particular case of a more general concept.

Definition 2.1.2. A *split coequalizer* is a diagram of the form

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "h"] \arrow[l, bend left=60, "t"] & c \arrow[l, bend left=60, "s"] \end{tikzcd}$$

so that $hf = hg,\ hs = \mathrm{id}_c,\ gt = \mathrm{id}_b$, and $ft = sh$.

Proposition 2.1.3. In the above situation,

$$a \overset{f}{\underset{g}{\rightrightarrows}} b \xrightarrow{h} c$$

is a coequalizer. In particular, any functor preserves this coequalizer.

[1] Think about free groups: in that case we take words on Ta.

Proof. Take $k\colon b \to d$ such that $kf = kg$ and define $\overline{k} := ks$. Then we have

$$\overline{k}h = ksh = kft = kgt = k.$$

Uniqueness is clear since h is a (split) epi. □

T and T^2 preserve split coequalizers, so they preserve our coequalizer.

Corollary 2.1.4. Let T be a monad on $\mathcal{C}$ and (a, α) a T-algebra. Then

$$(T^2 a, \mu_{Ta}) \underset{\mu_a}{\overset{T\alpha}{\rightrightarrows}} (Ta, \mu_a) \xrightarrow{\alpha} (a, \alpha)$$

is a coequalizer in $T\text{-}\mathsf{Alg}$, which $U^T\colon T\text{-}\mathsf{Alg} \to \mathcal{C}$ sends to a split coequalizer in $\mathcal{C}$.

Proof. We have already observed that the second statement holds, so that $\mathrm{coeq}(U^T(T\alpha), U^T(\mu_a))$ is preserved by T and T^2. Hence there exists a unique lift of the (split) coequalizer in $\mathcal{C}$ to a coequalizer in $T\text{-}\mathsf{Alg}$. □

Results like the previous one inspire us to look at the parallel pairs of morphisms in a category which are sent to split coequalizers or, to say it better, to a parallel pair of morphisms that can be extended to a split coequalizer diagram. This kind of pairs will be of crucial importance in the following.

Definition 2.1.5. Let $U\colon \mathcal{D} \to \mathcal{C}$ be a functor. A pair of morphisms $f, g\colon d \rightrightarrows d'$ in $\mathcal{D}$ is *U-split* if $Uf, Ug\colon Ud \rightrightarrows Ud'$ is part of a split coequalizer in $\mathcal{C}$.

Remark 2.1.6. $T\alpha, \mu_a\colon (T^2 a, \mu_{Ta}) \rightrightarrows (Ta, \mu_a)$ is a U^T-split pair. Moreover, $T\text{-}\mathsf{Alg}$ has coequalizers of U^T-split pairs and U^T preserves them. Hence, functors equivalent to U^T satisfy three conditions:

1. they have a left adjoint;
2. they are conservative;
3. U-split pairs have coequalizers which are preserved by U.

Theorem 1 (Beck). Let $U\colon \mathcal{D} \to \mathcal{C}$ be a right adjoint to $F\colon \mathcal{C} \to \mathcal{D}$. Let $(T = UF, U\epsilon F, \eta)$ be the induced monad and $\overline{U}\colon \mathcal{D} \to T\text{-}\mathsf{Alg}$ be the comparison functor.

1. If $\mathcal{D}$ has coequalizers of U-split pairs, then $\overline{U}$ has a left adjoint $\overline{F}\colon T\text{-}\mathsf{Alg} \to \mathcal{D}$;
2. if, in addition, U preserves coequalizers of U-split pairs, the unit $\overline{\eta}\colon \mathrm{id}_{T\text{-}\mathsf{Alg}} \Rightarrow \overline{U}\,\overline{F}$ is an isomorphism;
3. if U is also conservative, then $\overline{U}$ is an equivalence of categories.

Proof. 1. For each free T-algebra (Ta, μ_a) we have

$$\begin{aligned} T\text{-}\mathsf{Alg}((Ta, \mu_a), \overline{U}-) &= T\text{-}\mathsf{Alg}(F^T a, \overline{U}-) \\ &\cong \mathcal{C}(a, U^T \overline{U}-) \\ &= \mathcal{C}(a, U-) \\ &\cong \mathcal{D}(Fa, -) \end{aligned}$$

therefore the value of $\overline{F}$ at (Ta, μ_a) has to be Fa. Since every T-algebra is a coequalizer of free algebras which is preserved by U^T, we may define $\overline{F}(a, \alpha)$ as the coequalizer of a pair of morphisms $FTa \rightrightarrows Fa$. We write this as $FUFU^T(a, \alpha) \rightrightarrows FU^T(a, \alpha)$. Consider the following pair of morphisms of functors

$$\begin{tikzcd} FUFU^T \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\epsilon FU^T"'] & FU^T \end{tikzcd}$$

in the functor category $[T\text{-}\mathsf{Alg}, \mathcal{D}]$. We claim that this pair has a coequalizer and $\overline{F}\colon T\text{-}\mathsf{Alg} \to \mathcal{D}$ is left adjoint to $\overline{U}$. Note that the pair of morphisms just above becomes split after the composition with $U\colon \mathcal{D} \to \mathcal{C}$. In fact

$$\begin{tikzcd} UFUFU^T \arrow[r, shift left, "UF\alpha"] \arrow[r, shift right, "U\epsilon FU^T"'] & UFU^T \arrow[r, "\alpha"] \arrow[l, bend left=60, "\eta UFU^T"] & U^T \arrow[l, bend left=60, "\eta U^T"] \end{tikzcd}$$

is a split coequalizer in $[T\text{-}\mathsf{Alg}, \mathcal{C}]$, given that it holds pointwise since $UF = T$. Let us denote by $\beta\colon FU^T \to \overline{F}$ the colimit (computed pointwise) of the pair $F\alpha, \epsilon FU^T\colon FUFU^T \rightrightarrows FU^T$. Precomposing this pair with $\overline{U}$ and recalling that $\alpha\overline{U} = U\epsilon$, $U^T\overline{U} = U$, we get the pair

$$\begin{tikzcd} FUFU \arrow[r, shift left, "FU\epsilon"] \arrow[r, shift right, "\epsilon FU"'] & FU, \end{tikzcd}$$

which is coequalized by $\epsilon\colon FU \Rightarrow \mathrm{id}_{\mathcal{D}}$.

$$\begin{tikzcd} FUFU \arrow[r, shift left, "FU\epsilon"] \arrow[r, shift right, "\epsilon FU"'] & FU \arrow[r, "\beta\overline{U}"] \arrow[dr, "\epsilon"'] & \overline{F}\,\overline{U} \arrow[d, dashed, "\exists!\overline{\epsilon}"] \\ & & \mathrm{id}_{\mathcal{D}} \end{tikzcd}$$

Since $\overline{F}\,\overline{U}$ is the coequalizer of the diagram above, there exists a unique $\overline{\epsilon}\colon \overline{F}\,\overline{U} \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that $\overline{\epsilon} \cdot \beta\overline{U} = \epsilon$. To get the unit $\overline{\eta}\colon \mathrm{id}_{T\text{-}\mathsf{Alg}} \Rightarrow \overline{U}\,\overline{F}$ we need to describe a morphism of T-actions $(U^T, \alpha) \to (U^T\overline{U}\,\overline{F}, \alpha\overline{U}\,\overline{F})$. We claim that the natural transformation induced by the universal property of the split coequalizer in the first row

$$\begin{tikzcd} UFUFU^T \arrow[r, shift left, "UF\alpha"] \arrow[r, shift right, "U\epsilon FU^T"'] \arrow[d, equal] & UFU^T \arrow[r, "\alpha"] \arrow[d, equal] & U^T \arrow[d, dashed, "\exists!\overline{\eta}"] \\ U^T\overline{U}FUFU^T \arrow[r, shift left, "U^T\overline{U}F\alpha"] \arrow[r, shift right, "U^T\overline{U}\epsilon FU^T"'] & U^T\overline{U}FU^T \arrow[r, "U^T\overline{U}\beta"'] & U^T\overline{U}\,\overline{F} \end{tikzcd}$$

is a morphism of T-actions[2].

Unraveling what this means, we have to check that the diagram

$$\begin{tikzcd} UFa \arrow[r, "UF\overline{\eta}_{(a,\alpha)}"] \arrow[d, "\alpha"'] & UFU\overline{F}(a, \alpha) \arrow[d, "U\epsilon_{\overline{F}(a,\alpha)}"] \\ a \arrow[r, "\overline{\eta}_{(a,\alpha)}"'] & U\overline{F}(a, \alpha) \end{tikzcd}$$

[2] In fact, this tells us that the morphism $\overline{\eta}_{(a,\alpha)}\colon a \to U^T\overline{U}\,\overline{F}(a, \alpha)$ in $\mathcal{C}$ lifts uniquely to a morphism of T-algebras $\overline{\eta}_{(a,\alpha)}\colon (a, \alpha) \to \overline{U}\,\overline{F}(a, \alpha)$.

is commutative. We know that $\overline{\eta}\alpha = U\beta$ by the definition of $\overline{\eta}$. Moreover, α is a split epi in $\mathcal{C}$, hence we can precompose with $UF\alpha$ (again a split epi) and check the commutativity of the resulting diagram. We get the diagram

$$\begin{tikzcd}
UFUFa \arrow[d, "UF\alpha"'] \arrow[dr, bend left, "UFU\beta_{(a,\alpha)}"] \arrow[drr, bend left=30, "U\epsilon_{Fa}"] \arrow[drr, phantom, "\text{nat. of } \epsilon"] & & \\
UFa \arrow[r, "UF\overline{\eta}_{(a,\alpha)}"] \arrow[d, "\alpha"'] \arrow[dr, "U\beta_{(a,\alpha)}"] & UFU\overline{F}(a,\alpha) \arrow[d, "U\epsilon_{\overline{F}(a,\alpha)}"] & UFa \arrow[dl, bend left, "U\beta_{(a,\alpha)}"] \\
a \arrow[r, "\overline{\eta}_{(a,\alpha)}"'] & U\overline{F}(a,\alpha) &
\end{tikzcd}$$

The definition of β as a coequalizer implies that $\beta_{(a,\alpha)}F\alpha = \beta_{(a,\alpha)}\epsilon_{Fa}$, so we get the natural transformation $\overline{\eta}\colon \mathrm{id}_{T\text{-}\mathsf{Alg}} \Rightarrow \overline{U}\,\overline{F}$. The only thing left to do is checking the triangular identities, which is left to the reader.

2. If U preserves coequalizers of U-split pairs, both $U\overline{F}$ and U^T are coequalizers of the above diagram, hence $\overline{\eta}$ is an isomorphism.

3. From the triangular identities, $\overline{U}\overline{\epsilon}\cdot\overline{\eta}\overline{U} = \mathrm{id}_{\overline{U}}$, hence $\overline{U}\overline{\epsilon}$ is an isomorphism. Being $U^T\overline{U} = U$ conservative, $\overline{\epsilon}$ is an isomorphism as well. ☐

Definition 2.1.7. A pair $f, g\colon c \rightrightarrows d$ in a category $\mathcal{C}$ is *reflexive* if there exists a common section $i\colon d \to c$, that is $fi = gi = \mathrm{id}_d$.

A coequalizer of a reflexive pair is a *reflexive coequalizer*.

Remark 2.1.8. To give a cone of a reflexive pair it is enough to give a map $h\colon d \to x$ such that $hf = hg$, hence $\mathrm{colim}(c \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}} d) \cong \mathrm{colim}(c \rightrightarrows d)$.

Proposition 2.1.9. In Beck's monadicity theorem it suffices for (1) that coequalizers of reflexive U-split pairs exist, while in (2) and (3) we only need for them to be preserved.

Proof. The pair

$$FUFU^T \underset{\epsilon FU^T}{\overset{F\alpha}{\rightrightarrows}} FU^T$$

has $F\eta U^T$ as common section. In fact, $\alpha \cdot \eta U^T = \mathrm{id}_{U^T}$ by the unit law of the T-action $\alpha\colon TU^T \Rightarrow U^T$ and $\epsilon F \cdot F\eta = \mathrm{id}_F$ by the triangular identities. ☐

Example 2.1.10. Let $\mathcal{A}$ and $\mathcal{B}$ be small categories, $\mathcal{C}$ a category which is both complete and cocomplete, and $G\colon \mathcal{A} \to \mathcal{B}$ a functor. The restriction along G, G^*, has both adjoints, given by left and right Kan extensions. Notice that the induced monad is cocontinuous since G^* is a left adjoint. Moreover, G^* is conservative if G is essentially surjective, thus any essentially surjective functor G induces a monadic adjunction as follows:

$$\begin{tikzcd}
{[\mathcal{B},\mathcal{C}]} \arrow[rr, "\overline{G^*}"] \arrow[dr, "G^*"'] \arrow[rr, phantom, shift right=6, "\overset{\simeq}{\Rrightarrow}"] & & G^*\,\mathrm{Lan}_G\text{-}\mathsf{Alg} \arrow[dl] \\
& {[\mathcal{A},\mathcal{C}]} &
\end{tikzcd}$$

We are going to show why reflexive coequalizers are useful, but first we need some preliminary definitions and results.

Definition 2.1.11. A functor $F\colon \mathcal{I} \to \mathcal{J}$ between small categories is called *final* if for any diagram $D\colon \mathcal{J} \to \mathcal{C}$ the comparison morphism $\operatorname{colim}_{\mathcal{I}} DF \to \operatorname{colim}_{\mathcal{J}} D$ is an isomorphism whenever both colimits exist.

Proposition 2.1.12. Let $F\colon \mathcal{I} \to \mathcal{J}$ be a functor between small categories. The following are equivalent:

(i) F is final;

(ii) the unique isomorphism

$$\begin{tikzcd} \mathcal{I}^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}"] \arrow[dr, "*"'] & \arrow[d, phantom, "\overset{\sim}{\Longrightarrow}"] & \mathcal{J}^{\mathrm{op}} \arrow[dl, "*"] \\ & \mathbf{Set} & \end{tikzcd}$$

exibits $*$ as $\operatorname{Lan}_{F^{\mathrm{op}}} *$;

(iii) for each $j \in \mathcal{J}$, the category $(j \downarrow F)$ is connected.

Proof. (ii) $\iff$ (iii) We have $\operatorname{Lan}_{F^{\mathrm{op}}} *(j) \cong \operatorname{colim}_{(j\downarrow F)} *$ by the formula for Kan extensions. A colimit of $(j \downarrow F) \to \mathbf{Set}$, $(\phi, j') \mapsto *$ is terminal if and only if $(j \downarrow F)$ is connected, hence the thesis.

(ii) $\implies$ (i) Let $D\colon \mathcal{J} \to \mathcal{C}$ be a diagram. We can then write $\operatorname{Cocone}(D, -)$ as follows:

$$\operatorname{Cocone}(D, X) \cong \operatorname{Nat}(*, \mathcal{C}(D-, X)) \cong [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}](*, \mathcal{C}(D-, X))$$

By definition of left Kan extension, we also have

$$[\mathcal{I}^{\mathrm{op}}, \mathbf{Set}](*, \mathcal{C}(DF-, X)) \cong [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}](\operatorname{Lan}_{F^{\mathrm{op}}} *, \mathcal{C}(D-, X))$$

If $\operatorname{Lan}_{F^{\mathrm{op}}} * \cong *$, this shows that $\operatorname{colim}_{\mathcal{I}} DF \cong \operatorname{colim}_{\mathcal{J}} D$.

(i) $\implies$ (iii) Left as an exercise. $\square$

Definition 2.1.13. A small category $\mathcal{I}$ is *sifted* if the diagonal $\Delta\colon \mathcal{I} \to \mathcal{I} \times \mathcal{I}$ is final. A colimit is sifted if the domain category is.

Example 2.1.14. For any filtered category $\mathcal{I}$, the category $((i, i') \downarrow \Delta)$ is again filtered for any $(i, i') \in \mathcal{I} \times \mathcal{I}$ and hence connected, thus filtered colimits are sifted.

Example 2.1.15. Coequalizers are not sifted. Indeed, their indexing category $\mathcal{I} = \{1 \rightrightarrows 0\}$ is such that $((0, 1) \downarrow \Delta)$ is not connected. However, reflexive coequalizers are sifted. Checking it for yourself may be a tedious yet useful exercise.

Example 2.1.16. Coproducts and initial objects are not sifted, for their slice categories may be either empty or have several connected components.

Example 2.1.17. Pushouts are not sifted.

Proposition 2.1.18. If $F\colon \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ is a functor preserving reflexive coequalizers in each variable, that is for any $a\in\mathcal{A}$, $b\in\mathcal{B}$ the functors $F(a,-)\colon \mathcal{B}\to\mathcal{C}$ and $F(-,b)\colon \mathcal{A}\to\mathcal{C}$ preserve reflexive coequalizers, then F preserves reflexive coequalizers as well.

Proof. We need to check that, given a reflexive coequalizer $a_0 \rightleftarrows a_1 \longrightarrow a_2$ in $\mathcal{A}$ and $b_0 \rightleftarrows b_1 \longrightarrow b_2$ in $\mathcal{B}$, the diagonal of the following diagram is a coequalizer diagram in $\mathcal{C}$.

$$\begin{tikzcd}
F(a_0,b_0) \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[d, shift left=1.5] \arrow[d, shift right=1.5] \arrow[dr, shift left=1.5] \arrow[dr, shift right=1.5] & F(a_1,b_0) \arrow[l] \arrow[r] \arrow[d, shift left=1.5] \arrow[d, shift right=1.5] & F(a_2,b_0) \arrow[d, shift left=1.5] \arrow[d, shift right=1.5] \\
F(a_0,b_1) \arrow[u] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[d] & F(a_1,b_1) \arrow[ul] \arrow[u] \arrow[l] \arrow[r] \arrow[d] \arrow[dr] & F(a_2,b_1) \arrow[u] \arrow[d] \\
F(a_0,b_2) \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] & F(a_1,b_2) \arrow[l] \arrow[r] & F(a_2,b_2)
\end{tikzcd}$$

From general facts, $F(a_2,b_2)$ is the colimit of the square in the top left. We may prove this using the sections, however in this case we can use the fact that, under certain hypothesis, given a diagram $D\colon \mathcal{I}\times\mathcal{I}\to\mathcal{C}$ we have $\mathrm{colim}_{\mathcal{I}\times\mathcal{I}} D \cong \mathrm{colim}_{\mathcal{I}} D\Delta$. Specifically, we want this to hold when $\mathcal{I} = \{a_0 \rightleftarrows a_1\}$. But the category $((i,i')\downarrow\Delta)$ is connected for any $i\in\mathcal{I}$, hence Δ is final and we have the thesis. $\square$

Example 2.1.19. The functor $\mathbf{Set}\times\mathbf{Set} \xrightarrow{-\times-} \mathbf{Set}$ satisfies the hypothesis of the theorem since $\mathbf{Set}$ is cartesian closed, hence $X\mapsto X\times X$ preserves reflexive coequalizers by the proposition. This shows that $\mathbf{Set}(X,-)\cong \Pi_{x\in X}\mathbf{Set}(*,-)$ preserves reflexive coequalizers if X is finite, hence the functor $\mathbf{Set}(X,-)\colon \mathbf{Set}\to\mathbf{Set}$ is monadic with $T=\mathbf{Set}(X,X\times -)$ for X finite.

$$\begin{tikzcd}
\mathbf{Set} \arrow[rr, "\cong"] \arrow[dr, "\mathbf{Set}(X,-)"'] & & T\text{-}\mathsf{Alg} \arrow[dl, "U^T"] \\
& \mathbf{Set} &
\end{tikzcd}$$

We also have the following result.

Proposition 2.1.20. If $F\colon \mathcal{B}\times\mathcal{C}\to\mathcal{D}$ is a functor preserving sifted colimits in each variable, then it preserves them as a functor $\mathcal{B}\times\mathcal{C}\to\mathcal{D}$.

Remark 2.1.21. This proposition is false if we drop the siftedness condition, for under the functor $\mathbf{Set}\times\mathbf{Set} \xrightarrow{-\times-} \mathbf{Set}$ we have $(*+*)\times(*+*)\ncong *+*\cong(*\times*)+(*\times*)$

2.2 Algebraic Theories and Finitary Monads

Many objects in algebra can be described as sets X with some finitary operations $X^n \xrightarrow{m_i} X$ subject to a list of axioms involving the m_i and their products.

Example 2.2.1. (Commutative) monoids and groups, rings and, fixed a ring R, R-modules, where for each $r\in R$ we specify an operation $X\xrightarrow{r}X$.

We can construct categories whose objects are sets paired with operations fullfilling the axioms and functions commuting with the operations as morphisms. These are called models for single-sorted finitary theories.

Proposition 2.2.2. The forgetful functor from a category $\mathcal{C}$ of models of a single-sorted finitary theory to **Set** creates sifted colimits.

Proof. Notice that the n-fold product $\mathbf{Set} \xrightarrow{(-)^n} \mathbf{Set}$ factors as $\mathbf{Set} \xrightarrow{\Delta} \mathbf{Set}^n \xrightarrow{-\times\cdots\times-} \mathbf{Set}$, hence it preserves sifted colimits. This gives us unique candidates for operations on the colimit such that the cocone in **Set** is a morphism of models. These operations satisfy the axioms because the domain is in each case again of the form colim^n.

$$\begin{tikzcd} X^n \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "m_i"] & Y^n \arrow[l] \arrow[r] \arrow[d, "m_i"] & Z^n \arrow[d, dotted, "\exists! m_i"] \\ X \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[l] \arrow[r] & Z \end{tikzcd}$$

□

Corollary 2.2.3. Let $\mathcal{C}, \mathcal{D}$ be categories of models of single sorted finitary theories. Any functor $U\colon \mathcal{C} \to \mathcal{D}$ which commutes with the forgetful functor to **Set** preserves reflexive coequalizers. If U has a left adjoint, then it is monadic.

Proof. The forgetful functors $\mathcal{C} \to \mathbf{Set}$, $\mathcal{D} \to \mathbf{Set}$ are both conservative, hence U is as well. The statement about reflexive coequalizers was just proved and the last claim follows from Beck's theorem. □

Example 2.2.4. The forgetful functor $\mathbf{CAlg}_R \to \mathbf{Mod}_R$ is monadic for any commutative ring R. The same goes for the forgetful functors from **Grp**, **Ab**, $\mathbf{Mod}_R$, **Rng**, **CRng**, **Ring** and **CRing** to **Set**.

Definition 2.2.5. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is finitary if it commutes with filtered colimits. A monad is finitary if its underlying endofunctor T is finitary.

Remark 2.2.6. Asking for the underlying endofunctor T to be finitary is equivalent to asking for its forgetful functor U^T to be finitary.

Example 2.2.7. All the forgetful functors from single sorted finitary theories which have a left adjoint (fact: all of them do) come from finitary monads on **Set**.

In general, preserving filtered colimits is a weaker condition than preserving sifted colimits. However, for endofunctors on **Set** the two coincide.

In order to prove this, we need to know how such an endofunctor is determined by its action on finite sets and the idea is to check its behaviour on finite sets, since every set is a directed union of its finite subsets.

We write $[\mathcal{C}, \mathcal{D}]_{\mathrm{fin}}$ for the full subcategory of $[\mathcal{C}, \mathcal{D}]$ whose objects are finitary functors, $\mathbf{Set}_{\mathrm{fin}}$ for the category of finite sets.

Theorem 2.2.8. Restriction along the inclusion $K\colon \mathbf{Set}_{\mathrm{fin}} \to \mathbf{Set}$ induces an equivalence $[\mathbf{Set}, \mathbf{Set}]_{\mathrm{fin}} \to [\mathbf{Set}_{\mathrm{fin}}, \mathbf{Set}]$ whose inverse send F to $\mathrm{Lan}_K F$.

Before proving this theorem, we present some consequences.

Corollary 2.2.9. Any finitary functor $\mathbf{Set} \to \mathbf{Set}$ preserves sifted colimits. In particular, if $T\colon \mathbf{Set} \to \mathbf{Set}$ is the underlying endofunctor of a finitary monad, then $U^T\colon T\text{-}\mathsf{Alg} \to \mathbf{Set}$ creates sifted colimits.

Proof. Recall that $\mathrm{Lan}_K\colon [\mathbf{Set}_{\mathrm{fin}}, \mathbf{Set}] \to [\mathbf{Set}, \mathbf{Set}]_{\mathrm{fin}}$ is a left adjoint and on both sides colimits are computed pointwise. Moreover, functors preserving colimits of a given class are closed under formation of pointwise colimits, hence it is enough to check on a generating set.

If $\mathrm{Lan}_K F_i$ preserves sifted colimits for all $F_i\colon \mathbf{Set}_{\mathrm{fin}} \to \mathbf{Set}$, then the functor $\operatorname{colim} \mathrm{Lan}_K F_i \cong \mathrm{Lan}_K \operatorname{colim} F_i$ preserves sifted colimits.

Any functor $F\colon \mathbf{Set}_{\mathrm{fin}} \to \mathbf{Set}$ is a colimit of representable functors $\mathbf{Set}_{\mathrm{fin}}(X,-)$. Indeed, it is enough to consider $(よ \downarrow F) \to [\mathbf{Set}_{\mathrm{fin}}, \mathbf{Set}]$, $(\mathbf{Set}_{\mathrm{fin}}(X,-) \Rightarrow F) \mapsto \mathbf{Set}_{\mathrm{fin}}(X,-)$ and notice that this being a colimit diagram essentially follows from Yoneda as $F \cong \operatorname{colim}_{(よ\downarrow F)} \mathrm{dom}$.

By the previous theorem, it is enough to check now that the functor $\mathrm{Lan}_K \mathbf{Set}_{\mathrm{fin}}(X,-)$ preserves sifted colimits.

Observe the following diagram:

$$\begin{tikzcd} \mathbf{Set}_{\mathrm{fin}} \arrow[rr, "K"] \arrow[dr, "\mathbf{Set}_{\mathrm{fin}}(X,-)"'] & \arrow[d, phantom, "\Longrightarrow"] & \mathbf{Set} \arrow[dl, "\mathrm{Lan}_K \mathbf{Set}_{\mathrm{fin}}(X,-)"] \\ & \mathbf{Set} & \end{tikzcd}$$

A natural transformation $\mathrm{Lan}_K \mathbf{Set}_{\mathrm{fin}}(X,-) \Rightarrow G$ by definition is equivalent to a natural transformation $\mathbf{Set}_{\mathrm{fin}}(X,-) \Rightarrow GK$, which by Yoneda is equivalent to a map $* \to GKX$, which again by Yoneda corresponds to a natural transformation $\mathbf{Set}(KX,-) \Rightarrow G$, hence $\mathrm{Lan}_K \mathbf{Set}_{\mathrm{fin}}(X,-) \cong \mathbf{Set}(KX,-)$.

We only have to check now that $\mathbf{Set}(KX,-) \cong \Pi_{x\in X}\mathbf{Set}(*,-)$ preserves sifted colimits, but this is just the functor $Y \mapsto \Pi_{x\in X} Y$, which as we know commutes with sifted colimits. $\square$

Proposition 2.2.10. A category $\mathcal{C}$ with sifted colimits is cocomplete if and only if it has an initial object and binary coproducts.

Proof. One implication is obvious. For the other one, notice that we get finitary coproducts immediately. An infinite coproduct can be written as a filtered colimit of finite coproducts. We are only missing coequalizers. If $f, g\colon a \rightrightarrows b$ is a pair, then $f + \mathrm{id}, g + \mathrm{id}\colon a + b \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} b\colon \mathrm{in}_b$ (where in_b is the inclusion of b in the coproduct) is a reflexive pair hence the coequalizer $k = \mathrm{coeq}(a + b \rightrightarrows b)$ exists and coequalizes the original pair. The universal $h\colon b \to k$ is also a coequalizer of $f, g\colon a \rightrightarrows b$. $\square$

Theorem 2.2.11. Let X be a set, $T\colon \prod_{x\in X} \mathbf{Set} \to \prod_{x\in X} \mathbf{Set}$ be a finitary monad. Then T-Alg is cocomplete.

Proof. T-Alg has always an initial object, namely the free algebra of the initial object $\left(T \prod_{x\in X} \emptyset, \mu_{\prod_{x\in X}\emptyset}\right)$. Similarly, for $a, b \in \prod_{x\in X} \mathbf{Set}$, using that F^T is a left adjoint (hence it preserves colimits) we find that $\left(T\left(a \coprod b\right), \mu_{a \coprod b}\right)$ is a coproduct of (Ta, μ_a) and (Tb, μ_b). Therefore T-Alg has coproducts of free algebras. We want to check that we have binary coproducts of T-algebras (a, α) and (b, β). We have reflective coequalizers

$$\begin{tikzcd} (T^2a, \mu_{Ta}) \arrow[r, shift left=2, "T\alpha"] \arrow[r, shift right=2, "\mu_a"'] & (Ta, \mu_a) \arrow[l, "T\eta_a" description] \arrow[r, "\alpha"] & (a, \alpha) \end{tikzcd}$$

$$\begin{tikzcd} (T^2b, \mu_{Tb}) \arrow[r, shift left=2, "T\beta"] \arrow[r, shift right=2, "\mu_b"'] & (Tb, \mu_b) \arrow[l, "T\eta_b" description] \arrow[r, "\beta"] & (b, \beta) \end{tikzcd}$$

so we get a new reflective pair by taking coproducts of the free algebras

$$(T^2a, \mu_{Ta}) + (T^2b, \mu_{Tb}) \underset{\mu_a+\mu_b}{\overset{T\alpha+T\beta}{\underset{\longleftarrow T\eta_b \longrightarrow}{\longrightarrow}}} (Ta, \mu_a) + (Tb, \mu_b)$$

From the corollary, T preserves sifted colimits, hence T-Alg has reflexive coequalizers. Then the diagram above has a coequalizer, which is a coproduct of (a, α) and (b, β). $\square$

Remark 2.2.12. This shows that $\mathbf{Ab}, \mathbf{Grp}, \mathbf{Rng}$ etc. are cocomplete.

Remark 2.2.13. We only used the fact that T preserves sifted colimits, hence a monad on a cocomplete category $\mathcal{C}$ preserving sifted colimits has a cocomplete category of algebras.

Remark 2.2.14. In fact, we only need that reflexive coequalizers and filtered colimits exist in T-Alg.

2.3 Dense generators

The aim of this section is to prove the theorem about finitary endofunctors of **Set**. We want to identify "nice" generating subcategories like $\mathbf{Set}_{\text{fin}} \to \mathbf{Set}$.

Definition 2.3.1. Let $K\colon \mathcal{A} \to \mathcal{C}$ be the inclusion of a full subcategory or, equivalently, a fully faithful functor. We define the *restricted Yoneda functor* $\widetilde{K}\colon \mathcal{C} \to [\mathcal{A}^{\text{op}}, \mathbf{Set}]$ as the functor sending $c \in \mathcal{C}$ to $\mathcal{C}(K-, c)$.

The *canonical cocone* on the domain functor

$$\begin{aligned} \text{dom}\colon (K \downarrow c) &\to \mathcal{C} \\ (a, \varphi) &\mapsto Ka \end{aligned}$$

has components $\varphi\colon \text{dom}(a, \varphi) = Ka \xrightarrow{\varphi} c$.

Definition 2.3.2. A colimit of a diagram $D\colon I \to \mathcal{C}$ is K*-absolute* if it is preserved by $\widetilde{K}\colon \mathcal{C} \to [\mathcal{A}^{\text{op}}, \mathbf{Set}]$.

Definition 2.3.3. A full subcategory/fully faithful functor $K\colon \mathcal{A} \to \mathcal{C}$ is called *dense* if $\widetilde{K}$ is fully faithful.

Theorem 2.3.4. Let $\mathcal{C}$ be a cocomplete category, $\mathcal{A}$ a small category and $K\colon \mathcal{A} \to \mathcal{C}$ a fully faithful functor. Then $\widetilde{K}$ has a left adjoint given by $\text{Lan}_{よ} K\colon [\mathcal{A}^{\text{op}}, \mathbf{Set}] \to \mathcal{C}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "よ"] \arrow[dr, "K"'] & {[\mathcal{A}^{\text{op}}, \mathbf{Set}]} \arrow[d, "\text{Lan}_{よ} K"] \\ & \mathcal{C} \end{tikzcd}$$

Moreover, the following conditions are equivalent:

1. $\widetilde{K}$ is fully faithful;

2. for every $c \in \mathcal{C}$, the canonical cocone on $\text{dom}\colon (\mathcal{A} \downarrow c) \to \mathcal{C}$ exhibits c as colimit of the diagram $\text{dom}\colon (K \downarrow c) \to \mathcal{C}$;

3. every object is a K-absolute colimit of a diagram of the form $\mathcal{I} \xrightarrow{D} \mathcal{A} \xrightarrow{K} \mathcal{C}$;

4. there exists some family of diagrams $D_i \colon \mathcal{I}_i \to \mathcal{C}$ which have K-absolute colimits and $\mathcal{C}$ is the closure[3] of $\mathcal{A}$ under the colimits of D_i;

5. the counit of $\operatorname{Lan}_{\text{よ}} K \dashv \widetilde{K}$ is an isomorphism.

Proof. We have $\operatorname{Lan}_{\text{よ}} K(F) = \operatorname*{colim}_{(\text{よ}\downarrow F)} Ka$, hence

$$\begin{aligned} \mathcal{C}(\operatorname{Lan}_{\text{よ}} K(F), c) &\cong \lim_{(\text{よ}\downarrow F)} \mathcal{C}(Ka, c) \\ &\overset{\text{Yoneda}}{\cong} \lim_{(\text{よ}\downarrow F)} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\mathcal{A}(-,a), \mathcal{C}(K-,c)) \\ &\cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \left(\operatorname*{colim}_{(\text{よ}\downarrow F)} \mathcal{A}(-,a), \mathcal{C}(K-,c) \right) \\ &\cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(F, \mathcal{C}(K-,c)) \end{aligned}$$

hence $\operatorname{Lan}_{\text{よ}} K \dashv \widetilde{K}$. It remains to check the implications.

(1) $\implies$ (2) If $\widetilde{K}$ is fully faithful, it suffices to check that the image of the canonical cocone under $\widetilde{K}$ is a colimit. Since K is fully faithful, this image is precisely the diagram $\mathcal{A}(-,a) \cong \mathcal{C}(K-,Ka) \Rightarrow \mathcal{C}(K-,c)$. Then

$$\operatorname*{colim}_{(K\downarrow c)} \widetilde{K}\,\mathrm{dom} = \operatorname*{colim}_{(\text{よ}\downarrow \mathcal{C}(K-,c))} \mathcal{A}(-,a) \cong \mathcal{C}(K-,c).$$

This proves 2. and the fact that the colimit of $(K \downarrow c) \to \mathcal{C}$ is preserved by $\widetilde{K} \colon \mathcal{C} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$.

(2) $\implies$ (3) As we just observed, $\operatorname*{colim}_{(K\downarrow c)} \mathrm{dom}$ is $\widetilde{K}$-absolute and $\mathrm{dom} \colon (K \downarrow c) \to \mathcal{C}$ factors through $\mathcal{A}$.

(3) $\implies$ (4) Clear.

(4) $\implies$ (5) Let $\mathcal{B}$ be full subcategory spanned by the elements $b \in \mathcal{C}$ such that $\epsilon_b \colon \operatorname{Lan}_{\text{よ}} K\widetilde{K}(b) \to b$ is an isomorphism. It is closed under K-absolute colimits since they are preserved by $\widetilde{K}$ (by definition), by the left adjoint $\operatorname{Lan}_{\text{よ}} K$ and by $\mathrm{id}_{\mathcal{C}}$. It remains to check that the counit at $a \in \mathcal{A}$ is an isomorphism. But $\widetilde{K}a = \mathcal{C}(K-,Ka) \cong \mathcal{A}(-,a)$, so $\operatorname{Lan}_{\text{よ}} K(\mathcal{A}(-,a)) \cong a$, where the last isomorphism is a consequence of the fact that $(\text{よ} \downarrow \mathcal{A}(-,a))$ has a terminal object $\mathrm{id} \colon \mathcal{A}(-,a) \Rightarrow \mathcal{A}(-,a)$.

(5) $\implies$ (1) Any right adjoint whose counit is an isomorphism is fully faithful. $\square$

Remark 2.3.5. Notice that the first four points of the Theorem 2.3.4 are equivalent even when $\mathcal{C}$ is not cocomplete. In fact, we also have the implication (4) $\implies$ (1) without assumptions on the cocompleteness of $\mathcal{C}$.

[3] That is, the smallest full subcategory $\mathcal{B}$ of $\mathcal{C}$ which contains $\mathcal{A}$ and which satisfies the following

if some diagram $D_i \colon \mathcal{I}_i \to \mathcal{C}$ factors through $\mathcal{B}$, then $\operatorname{colim} D_i \in \mathcal{B}$

is $\mathcal{C}$ itself.

Proof. We give an alternative proof of the implication (4) $\implies$ (1).

Consider the (full) subcategory $\mathcal{B}$ of $\mathcal{C}$ with objects

$$\{b \mid \widetilde{K}_{Ka,b}\colon \mathcal{C}(Ka,b) \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\widetilde{K}Ka, \widetilde{K}b) \text{ is bijective } \forall a \in \mathcal{A}\}$$

Since $\widetilde{K}Ka = \mathcal{A}(-,a)$, by Yoneda, the target is given by $\mathrm{ev}_a \circ \widetilde{K}(b)$. Therefore it preserves K-absolute colimits. The domain is also equal to $\mathrm{ev}_a \circ \widetilde{K}(b)$, by definition. Hence it preserves K-absolute colimits and contains Ka' for every a', therefore $\mathcal{B} = \mathcal{C}$. Consider now

$$\mathcal{B}' = \{b' \mid \mathcal{C}(b',b) \xrightarrow{\widetilde{K}} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\widetilde{K}b', \widetilde{K}b) \text{ is bijective}\}$$

This is closed under all K-absolute colimits and contains Ka by the above argument, hence it is all of $\mathcal{C}$. $\square$

Definition 2.3.6. A fully faithful $K\colon \mathcal{A} \to \mathcal{C}$ is *dense* if $\widetilde{K}$ is fully faithful. A collection of diagrams $\{D_j\colon \mathcal{I}_j \to \mathcal{C}\}$ s.t. $\mathcal{C}$ is the closure of K under colimits of D_j and $\operatorname{colim} D_j$ is K-absolute is a *density presentation.*

Remark 2.3.7. The definition of density makes sense for arbitrary K, but the implication (4) $\implies$ (1) does not work in general.

Example 2.3.8.

1. よ$\colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is dense. In fact, we have

$$\widetilde{よ}(G) = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(よ-, G) \overset{\text{Yoneda}}{\cong} G,$$

thus $\widetilde{よ} \cong \mathrm{id}$ preserves all colimits. The colimit indexed by the slice $(よ \downarrow F)$ gives a density presentation.

2. $K\colon \mathbf{Set}_{\mathrm{fin}} \hookrightarrow \mathbf{Set}$ is dense: S finite implies $\mathbf{Set}(S,-)$ preserves sifted, hence filtered, colimits. Filtered (and sifted) colimits are K-absolute, and a density presentation for K can be found once one writes an arbitrary set as union of its finite subset.

3. $K\colon \{*\} \subseteq \mathbf{Set}$ is dense: $\widetilde{K}\colon \mathbf{Set} \to [*, \mathbf{Set}] \cong \mathbf{Set}$, $X \mapsto \mathbf{Set}(*, S) \cong S$, hence we find that $\widetilde{K}$ preserves all colimits. We can use coproducts for density presentation.

4. $\{k\} \subset \mathbf{Vect}_k$ is not dense even though every vector space is a coproduct of copies of k.

Definition 2.3.9. Given *any* functor $F\colon \mathcal{A} \to \mathcal{D}$, we can talk about the *restricted Yoneda embedding* $\widetilde{F}\colon \mathcal{D} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, sending $d \in \mathcal{D}$ to $\mathcal{D}(F-, d)$.

Proposition 2.3.10. Let $K\colon \mathcal{A} \to \mathcal{C}$ be fully faithful and $F\colon \mathcal{A} \to \mathcal{D}$ any functor. Suppose there exists $L\colon \mathcal{C} \to \mathcal{D}$ and bijections $\mathcal{D}(Lc, d) \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\mathcal{C}(K-,c), \mathcal{D}(F-,d))$ natural both in c and d. Then there is an isomorphism $\eta\colon F \overset{\sim}{\Rightarrow} L \circ K$ exibiting L as left Kan extension of F along K.

Proof. The presheaf $\mathcal{C}(K-,c)$ is the colimit of the canonical cone on $(よ \downarrow \mathcal{C}(K-,c))$. By Yoneda, this is equivalent to $(K \downarrow c)$ with objects $(a \in \mathcal{A}, \varphi\colon Ka \to c)$ and the evident morphisms. If $c \cong Ka$, this has (a, id_a) as terminal object. Therefore the colimit is $\mathcal{C}(K-, Ka) \cong \mathcal{A}(-,a)$. Moreover, the formula above in this case gives

$$\mathcal{D}(LKa, d) \cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\mathcal{A}(-,c), \mathcal{D}(F-,d)) \cong \mathcal{D}(Fa, d).$$

This shows that $L \circ K \cong F$ naturally. Then $Lc = \operatorname*{colim}_{(K\downarrow c)} Fa$, by the classical formula for left Kan extensions. $\square$

Definition 2.3.11. Any such Kan extension is called pointwise.

Theorem 2.3.12. Let $K\colon \mathcal{A} \to \mathcal{C}$ be a fully faithful dense functor with density presentation $\{D_j\colon \mathcal{I}_j \to \mathcal{C}\}_{j\in\mathcal{J}}$. Let $F\colon \mathcal{A} \to \mathcal{D}$ be a functor and assume that $\mathcal{D}$ has colimits of shape $\mathcal{I}_j$ $\forall j \in \mathcal{J}$. Then the pointwise left Kan extension of F along K exists. In particular, the unit $F \Rightarrow \mathrm{Lan}_K F \circ K$ is an isomorphism.

Proof. We want a L as in the above proposition $\mathcal{D}(Lc, d) \cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\mathcal{C}(K-, c), \mathcal{D}(F-, d))$ natural in c, d. This simply says that $\forall c \in \mathcal{C}$ the functor $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\widetilde{K}c, \widetilde{F}-)\colon \mathcal{D} \to \mathbf{Set}$ is representable. Take $\mathcal{B} := \{b \in \mathcal{C} \mid [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\widetilde{K}b, \widetilde{F}-) \text{ is representable}\}$. If $b = Ka$, then $\widetilde{K}b = \mathcal{C}(K-, Ka) \cong \mathcal{A}(-, a)$. Therefore

$$\begin{aligned} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\widetilde{K}Ka, \widetilde{F}d) &\cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\mathcal{A}(-, a), \widetilde{F}d) \\ &\overset{\text{Yoneda}}{\cong} \widetilde{F}d(a) \\ &\cong \mathcal{D}(Fa, d) \end{aligned}$$

so it is represented by Fa and $Ka \in \mathcal{B}$. Furthermore $\mathcal{B}$ is closed under $\widetilde{K}$-absolute colimits of shape $\mathcal{I}_j$. Let $D\colon \mathcal{I}_j \to \mathcal{B}$ be a diagram such that $\mathrm{colim}_{\mathcal{I}_j} D$ exists in $\mathcal{C}$ and is preserved by $\widetilde{K}$. We claim that $\mathrm{colim}_{\mathcal{I}_j} D \in \mathcal{B}$.

$$\begin{aligned} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\widetilde{K} \operatorname*{colim}_{\mathcal{I}_j} D, \widetilde{F}d) &\cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\operatorname*{colim}_{\mathcal{I}_j} \widetilde{K}D, \widetilde{F}d) \\ &\cong \lim_{\mathcal{I}_j} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\widetilde{K}D, \widetilde{F}d) \\ &\cong \lim_{\mathcal{I}_j} \mathcal{D}(LD, d) \\ &\cong \mathcal{D}(\operatorname*{colim}_{\mathcal{I}_j} LD, d) \end{aligned}$$

Hence $\mathcal{B} = \mathcal{C}$ and we get the functor $L \cong \mathrm{Lan}_K F$. □

Lemma 2.3.13. On the same conditions as before, pointwise Kan extensions along K preserve K-absolute colimits.

Proof. By definition, $\mathcal{D}(\mathrm{Lan}_K F(c), d) \cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\widetilde{K}c, \widetilde{F}d)$. Let $\operatorname*{colim}_{i\in\mathcal{I}_j} D_i$ be K-absolute. Then the claim is shown by the following chain of isomorphisms.

$$\begin{aligned} \mathcal{D}(\mathrm{Lan}_K F(\mathrm{colim}\, D_i), d) &\cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\widetilde{K}\,\mathrm{colim}\, D_i, \widetilde{F}d) \\ &\cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\mathrm{colim}\, \widetilde{K}D_i, \widetilde{F}d) \\ &\cong \lim\, [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\widetilde{K}D_i, \widetilde{F}d) \\ &\cong \lim \mathcal{D}(\mathrm{Lan}_K F(D_i), d) \\ &\cong \mathcal{D}(\mathrm{colim}\, \mathrm{Lan}_K F(D_i), d). \end{aligned}$$ □

Theorem 2.3.14. Let $K\colon \mathcal{A} \to \mathcal{C}$ be fully faithful, φ a class of colimit shapes and assume there exists a density presentation for K with colimits of shape $\mathcal{I}_j \in \varphi$. Let $\mathcal{D}$ be a category with colimits of shape φ. We write φ-$\mathbf{Cocts}(\mathcal{C}, \mathcal{D})$ for the category of functors $\mathcal{C} \to \mathcal{D}$ which preserve φ-colimits. If all colimits of shape φ are K-absolute, then

$$\begin{tikzcd} {[\mathcal{A}, \mathcal{D}]} \arrow[r, shift left, "\mathrm{Lan}_K"] \arrow[r, phantom, "\perp"] & \varphi\text{-}\mathbf{Cocts}(\mathcal{C}, \mathcal{D}) \arrow[l, shift left, "K^*"] \end{tikzcd}$$

is an equivalence.

Proof. The existence of Lan_K is guaranteed by the fact that $\mathcal{D}$ has φ-colimits and K has density presentation with colimits of shape $\mathcal{I}_j \in \varphi$. By definition, Lan_K is left adjoint to $K^*\colon [\mathcal{C},\mathcal{D}] \to [\mathcal{A},\mathcal{D}]$ and by the lemma it lands in φ-$\mathbf{Cocts}(\mathcal{C},\mathcal{D})$. Then Lan_K is a left adjoint to the restriction. The unit is an isomorphism since $\mathrm{Lan}_K F$ is pointwise, so it suffice to check that K^* is conservative. Let $G, H\colon \mathcal{C} \to \mathcal{D}$ be φ-cocts, $\alpha\colon G \Rightarrow H$ natural transformation such that αK is an isomorphism. Then, $\{c \mid \alpha_c \text{ is an isomorphism}\}$ contains Ka for every a and is closed under colimits of density presentations. Hence α is an isomorphism. □

Corollary 2.3.15. For $K\colon \mathbf{Set}_{\mathrm{fin}} \hookrightarrow \mathbf{Set}$, we get that

$$\begin{tikzcd}
{[\mathbf{Set}_{\mathrm{fin}}, \mathbf{Set}]} \arrow[r, shift left=1.5, "\mathrm{Lan}_K"] \arrow[r, phantom, "\perp"] & {[\mathbf{Set}, \mathbf{Set}]_{\mathrm{fin}}} \arrow[l, shift left=1.5, "K^*"]
\end{tikzcd}$$

is an equivalence.

Definition 2.3.16. Let Φ be a class of colimit shapes and $\mathcal{A}$ be a small category. We write $\Phi(\mathcal{A})$ for the closure of the representable presheaves in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ under Φ-colimits. We have a functor $\text{よ}\colon \mathcal{A} \to \Phi(\mathcal{A})$.

Remark 2.3.17. By construction, there exists a density presentation for $\text{よ}\colon \mathcal{A} \to \Phi(\mathcal{A})$ consisting of Φ-colimits. This follows from $\widetilde{\text{よ}}\colon \Phi(\mathcal{A}) \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ being simply the inclusion.

Theorem 2.3.18. Let Φ be a class of small colimit shapes, $\mathcal{A}$ a small category. Then $\mathcal{A} \xrightarrow{\text{よ}} \Phi(\mathcal{A})$ is the *free cocompletion* of $\mathcal{A}$ under Φ-colimits, that is

$$\begin{tikzcd}
{[\mathcal{A}, \mathcal{C}]} \arrow[r, shift left=1.5, "\mathrm{Lan}_{\text{よ}}"] \arrow[r, phantom, "\perp"] & \Phi\text{-}\mathbf{Cocts}(\Phi(\mathcal{A}), \mathcal{C}) \arrow[l, shift left=1.5, "\text{よ}^*"]
\end{tikzcd}$$

is an equivalence for every Φ-cocomplete $\mathcal{C}$. In particular, if Φ is the class of all small colimit shapes, then $\Phi(\mathcal{A}) = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is the free cocompletion of $\mathcal{A}$. In this case, every $L \cong \mathrm{Lan}_{\text{よ}} F \in \mathbf{Cocts}([\mathcal{A}^{\mathrm{op}}, \mathbf{Set}], \mathcal{C})$ has a right adjoint: $\mathrm{Lan}_{\text{よ}} F \dashv \widetilde{F}$.

Proof. Since $\text{よ}\colon \mathcal{A} \to \Phi(\mathcal{A})$ has density presentation consisting of Φ-colimits, the two equivalences follow from the previous theorem. To see that L has right adjoint, it suffices to check $L \cong \mathrm{Lan}_{\text{よ}} F \dashv \widetilde{F}$:

$$\mathcal{C}(\mathrm{Lan}_{\text{よ}} F(G), c') \cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\widetilde{\text{よ}}(G), \widetilde{F}c') \cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(G, \widetilde{F}c')$$

since $\mathrm{Lan}_{\text{よ}} \Delta_\bullet\colon F$ is pointwise. □

Example 2.3.19. Let Δ be the category of finite non-empty ordinals $[0], [1], \ldots$ and order preserving maps. We have a functor $\Delta_\bullet\colon \Delta \to \mathbf{Top}$, sending $[n]$ to the standard (geometric) n-simplex Δ_n so we get and adjunction $\mathrm{Lan}_{\text{よ}} \Delta_\bullet\colon [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \rightleftarrows \mathbf{Top}\colon \Delta_\bullet^*$. $\mathrm{Lan}_{\text{よ}} \Delta_\bullet$ is called the *geometric realization* and $\Delta_\bullet^* =: \mathrm{Sing}(-)$ is called the *singular complex*. In pictures:

$[2]$

$$\begin{tikzcd}
& 1 \arrow[dr] & \\
0 \arrow[ur] \arrow[rr] & & 2
\end{tikzcd}$$

$$[3] \qquad \begin{tikzcd} & 1 \arrow[dd] \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] \arrow[dr] & & 3 \\ & 2 \arrow[ur] & \end{tikzcd}$$

$$\vdots$$

$[\Delta^{\mathrm{op}}, \mathbf{Set}]$ is called the category of *simplicial sets* and it is denoted by $\mathbf{sSet}$. $\operatorname{Lan}_{よ} \Delta_\bullet$ is denoted by $|\cdot|$.

2.4 Locally presentable categories

Definition 2.4.1. Let κ be a regular cardinal[4]. A category $\mathcal{C}$ is *κ-filtered* if any diagram in $\mathcal{C}$ of size $< \kappa$ has a cocone. Equivalently, if it is non-empty and for any set of objects $\{x_i\}$ of cardinality $< \kappa$ there exists $x \in \mathcal{C}$ and $x_i \to x$ such that

$$x_i \underset{<\kappa}{\overset{\rightrightarrows}{\vdots}} x_j \longrightarrow x$$

is coequalizing. If $\kappa = \aleph_0$ a κ-filtered category is just a filtered one.

Definition 2.4.2. An object $c \in \mathcal{C}$ is called *κ-presentable* if $\mathcal{C}(c, -)$ preserves κ-filtered colimits. If $\kappa = \aleph_0$, c is called *finitely presentable.*

Definition 2.4.3. A category $\mathcal{A}$ is called *κ-accessible* if there exists a small subcategory $\mathcal{A}_0$ of κ-presentable objects such that $\mathcal{A}$ is the closure of $\mathcal{A}_0$ under κ-filtered colimits. The category $\mathcal{A}$ is called *locally κ-presentable* if it is κ-accessible and cocomplete.

Example 2.4.4. Take $\mathcal{A} = \mathbf{Set}$, $\mathcal{A}_0 = \mathbf{Set}_{\mathrm{fin}}$ and $S \in \mathbf{Set}_{\mathrm{fin}}$. Then

$$\mathbf{Set}(S, -) = \prod_{|S| \text{ finite}} \mathbf{Set}(*, -)$$

preserves sifted, hence filtered, colimits. Thus $\mathbf{Set}$ is locally finitely presentable.

Remark 2.4.5. For $\mathcal{A}_0$ as in the definition, $K \colon \mathcal{A}_0 \to \mathcal{A}$ the inclusion, we find that κ-filtered colimits are K-absolute.

$$\begin{aligned} \mathcal{A} &\xrightarrow{\widetilde{K}} [\mathcal{A}_0^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\mathrm{ev}_a} \mathbf{Set} \\ a' &\longmapsto \mathcal{A}(K-, a') \longmapsto \mathcal{A}(a, a') \end{aligned}$$

so that $\mathcal{A}_0 \overset{K}{\hookrightarrow} \mathcal{A}$ has density presentation consisting of κ-filtered colimits. Thus $\widetilde{K} \colon \mathcal{A} \to [\mathcal{A}_0^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful and preserves κ-filtered colimits.

Definition 2.4.6. A functor is *κ-accessible* if it commutes with κ-filtered colimits. We write $[\mathcal{A}, \mathcal{B}]_\kappa$ for the subcategory of κ-accessible functors ($[\mathcal{A}, \mathcal{B}]_{\mathrm{fin}}$ if $\kappa = \aleph_0$).

[4]Namely, a union of $< \kappa$ sets of cardinality $< \kappa$ has cardinality $< \kappa$.

It follows that

$$[\mathcal{A}_0, \mathcal{B}] \underset{K^*}{\overset{\mathrm{Lan}_K}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} [\mathcal{A}, \mathcal{B}]_\kappa$$

if $\mathcal{A}$, $\mathcal{A}_0$ are as above and $\mathcal{B}$ has κ-filtered colimits.

Remark 2.4.7. A κ-accessible category $\mathcal{A}$ is locally κ-presentable if and only if $\mathcal{A} \xrightarrow{\widetilde{K}} [\mathcal{A}_0^{\mathrm{op}}, \mathbf{Set}]$ has a left adjoint for any choice of $\mathcal{A}_0 \subseteq \mathcal{A}$ that defines it as the closure under κ-filtered colimits.

Definition 2.4.8. A monad (T, μ, η) is said to have rank κ if T is a κ-accessible endofunctor.

Definition 2.4.9. A category $\mathcal{A}$ is called *accessible* (resp. *locally presentable*) if it is κ-accessible (resp. locally κ-presentable) for some regular cardinal κ. A functor is accessible if it is κ-accessible for some κ. A monad has rank if it is accessible.

Our next goal is to prove that if (T, μ, η) is an accessible monad on a locally presentable category, then T-Alg is locally presentable.

Proposition 2.4.10. Let $\mathcal{A}$ be κ-accessible, $\mathcal{A}_0 \subseteq \mathcal{A}$ be the small subcategory of κ-presentable objects such that $\mathcal{A}$ is the closure of $\mathcal{A}_0$ under κ-filtered colimits, and (T, μ, η) be a monad of rank κ on $\mathcal{A}$. Then $\mathcal{B} = \{(Ta_0, \mu_{a_0}) \mid a_0 \in \mathcal{A}_0\}$ is a dense generator of T-Alg.

Proof. First note that (Ta_0, μ_{a_0}) is κ-presentable. Indeed, we have $T\text{-}\mathsf{Alg}((Ta_0, \mu_{a_0}), -) \cong \mathcal{A}(a_0, U^T-)$ and U^T creates all colimits that T preserves, in particular κ-filtered ones. Hence, LHS preserves κ-filtered colimits and for this reason (Ta_0, μ_{a_0}) is κ-presentable. Let Φ_1 be the class of κ-filtered diagrams. Writing $K\colon \mathcal{B} \to T\text{-}\mathsf{Alg}$ for the inclusion, we have just shown that Φ_1-colimits are K-absolute. Note that the closure of $\mathcal{B}$ under Φ_1 contains all free algebras (Ta, μ_a) since $\mathcal{A}$ is the closure of $\mathcal{A}_0$ under Φ_1-colimits. Let Φ_2 be the class of diagrams $(T^2a, \mu_{Ta}) \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} (Ta, \mu_a)$ for all $(a, \alpha) \in T\text{-}\mathsf{Alg}$. The closure under $\Phi_1 \cup \Phi_2$ is clearly all of T-Alg, so we just need to show that Φ_2-colimits are K-absolute, that is preserved by each $T\text{-}\mathsf{Alg}((Ta_0, \mu_{a_0}), -)$. Since $T\text{-}\mathsf{Alg}((Ta_0, \mu_{a_0}), -) \cong \mathcal{A}(a_0, U^T-)$ and U^T sends a coequalizer in question to a split coequalizer, the colimit is indeed K-absolute. □

Example 2.4.11. *Finite* free groups, abelian groups, commutative rings etc. form dense generators of **Grp**, **Ab**, **CRng** etc.

Recall that the category of T-algebras of a finitary monad $T\colon \mathbf{Set} \to \mathbf{Set}$ is cocomplete. We would like to know that T-Alg is locally finitely presentable. This result can be proved using the following fact (which is in turn an easy consequence of the result about the commutativity of κ-filtered colimits with κ-small limits in **Set**):

κ-presentable objects are closed under κ-small colimits.

Proposition 2.4.12. Let $\mathcal{C}$ be a cocomplete category which has a *small* dense subcategory consisting of κ-presentable objects. Then $\mathcal{C}$ is locally κ-presentable.

Proof. Let $\mathcal{A}'$ be the closure of $\mathcal{A}$ under κ-small colimits. This is constructed as follows: $\mathcal{A}_0 = \mathcal{A}$. For any ordinal i we set

$$\mathcal{A}_{i+1} = \{\text{colimits of } \kappa\text{-small diagrams in } \mathcal{A}_i\}$$

and for a limit-ordinal λ we set $\mathcal{A}_\lambda = \bigcup_{\mu<\lambda} \mathcal{A}_\mu$. This terminates when $\lambda = \kappa$, so $\mathcal{A}_\kappa$ is the colimit closure and thus small. From the above mentioned fact we know that $\mathcal{A}'$ consists of κ-presentable objects. Since it contains $\mathcal{A}$, the inclusion $K\colon \mathcal{A}' \to \mathcal{C}$ is dense. If $\mathcal{A}'$ is dense, then each object in $\mathcal{C}$ is a colimit of $(\mathcal{A}' \downarrow c)$ which is a κ-filtered diagram by construction. □

Corollary 2.4.13. For each finitary monad T on **Set**, the category T-Alg is locally finitely presentable. Moreover if T is a monad of rank κ on a locally κ-presentable category, then T-Alg is locally κ-presentable if and only if it is cocomplete.

Theorem 2.4.14. Let $\mathcal{I}$ be a filtered category and

$$X\colon \mathcal{I} \to \mathbf{Set}, \quad i \mapsto X_i$$

a diagram and $(X_i \xrightarrow{n_i} X)_i$ a cocone. Then $(X_i \to X)_i$ is a colimit cocone if and only if

i) For all $x \in X$ there exists an $i \in \mathcal{I}$ and an $\tilde{x} \in X_i$ such that $x = n_i(\tilde{x})$.

ii) If $x, y \in X_i$ satisfy $n_i(x) = n_i(y)$, then there is some $\phi\colon i \to j$ such that $X_\phi(x) = X_\phi(y)$. (Informally: "all equalities that hold in X hold in some X_j.")

Proof. Given any other cone $\lambda_i \to X_i$ we define $f\colon X \to Y$ by sending $x \to \lambda_i(\tilde{x})$ for any $\tilde{x}$ in i). This is well defined by ii) and filteredness. It only remains to who that there exists such a cocone. Take $X = (\coprod X_i)/\sim$ with $(x,i) \sim (y,j)$ if there is some some diagram $i \xrightarrow{\phi} k \xleftarrow{\psi} j$ in $\mathcal{I}$ with $X_\phi(x) = X_\psi(y)$. □

Corollary 2.4.15. In **Set** filtered colimits commute with finite limits and κ-filtered colimits commute with κ-small limits.

Proof. Check that a levelwise equalizer of cones satisfying i) and ii) above still satisfies i) and ii). This can be done by chasing through the following diagram

$$\begin{tikzcd} X_i \arrow[r, hook] \arrow[d, dashed] & Y_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & Z_i \arrow[d] \\ X \arrow[r, hook] & Y \arrow[r, shift left] \arrow[r, shift right] & Z \end{tikzcd}$$

For $< \kappa$-fold products, we use κ-filteredness to extend ii) to any set of pairs of size $< \kappa$. Then check that the product of cones satisfying i) and ii) of size $< \kappa$ still satisfies i) and ii). □

Corollary 2.4.16. In any category, κ-presentable objects are closed under κ-small colimits.

Proof. Let $C\colon \mathcal{I} \to \mathcal{C}$ be a filtered diagram and $D\colon \mathcal{J} \to \mathcal{C}$ a κ-small diagram of κ-presentable objects D_J.

$$\begin{aligned} \mathcal{C}(\operatorname*{colim}_{\mathcal{J}} D_j, \operatorname*{colim}_{\mathcal{I}} C_i) &\cong \lim_{\mathcal{J}} \mathcal{C}(D_j, \operatorname*{colim}_{\mathcal{I}} C_i) \\ &\cong \lim_{\mathcal{J}} (\operatorname*{colim}_{\mathcal{I}} \mathcal{C}(D_j, C_i)) \\ \text{Explicit check in Set} &\cong \operatorname*{colim}_{\mathcal{I}} (\lim_{\mathcal{J}} \mathcal{C}(D_j, C_i)) \\ &\cong \operatorname*{colim}_{\mathcal{I}} \mathcal{C}(\operatorname*{colim}_{\mathcal{J}} D_j, C_i). \end{aligned}$$

□

Proposition 2.4.17. Each object in a locally presentable category is λ-presentable for $\lambda \gg 0$.

Proof. Let $\mathcal{C}$ be locally κ-presentable, choose a small dense subcategory $\mathcal{A}$ of κ-presentable objects. So, any object $c \in \mathcal{C}$ we have is a colimit of $\mathrm{dom}\colon (\mathcal{A} \downarrow c) \to \mathcal{C}$. Choose λ such that $\lambda > \kappa$ and $\lambda > |\mathrm{Arr}(\mathcal{A}/c)|$. □

The characterization of filtered colimits in **Set** gives the following characterization of finitely presentable objects: a is finitely presentable if for all filtered colimits $k_i\colon c_i \to c$ in $\mathcal{C}$ and all $f\colon a \to c$ there exists a factorization

$$\begin{tikzcd} & c_i \arrow[dr, "k_i"] & \\ a \arrow[ur, dashed, "f'"] \arrow[rr, "f"'] & & c \end{tikzcd}$$

and any two such lifts $f', f"$ satisfying $k_i \circ f' = k_i \circ f"$ become equal after composing with some $c_\phi\colon c_i \to c_j$.

Corollary 2.4.18. Let $\mathcal{C}$ be a locally κ-presentable category. We have that κ-filtered colimits commute with κ-small limits in $\mathcal{C}$.

Proof. Choose a small dense subcategory $\mathcal{A} \subset \mathcal{C}$ of κ-presentable objects. The inclusion $K\colon \mathcal{A} \to \mathcal{C}$ induces a fully faithful functor $\widetilde{K}\colon \mathcal{C} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ with left adjoint $\mathrm{Lan}_{\text{よ}} K$, hence it preserves all limits. This implies that $\mathcal{C}$ is complete as a reflective subcategory of the complete category $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. Moreover, $\widetilde{K}$ preserves κ-filtered colimits, given that $\mathrm{ev}_a \circ \widetilde{K} = \mathcal{C}(Ka, -)$ hence this reduces the problem to limits and κ-filtered colimits in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, where both are computed levelwise. □

Proposition 2.4.19. Let $\mathcal{C}$, $\mathcal{D}$ be locally κ-presentable, $\lambda \geq \kappa$ a regular cardinal. Then the category $[\mathcal{C}, \mathcal{D}]_\lambda$ of λ-accessible functors and natural transformations is locally small, cocomplete and the inclusion $[\mathcal{C}, \mathcal{D}]_\lambda \to [\mathcal{C}, \mathcal{D}]$ preserves colimits. In fact, $[\mathcal{C}, \mathcal{D}]_\lambda$ is locally presentable.

Proof. The category $\mathcal{C}_\lambda$ of λ-presentable objects in $\mathcal{C}$ is essentially small and each $(\mathcal{C}_\lambda \downarrow c)$ is λ-filtered, so $\mathcal{C}_\lambda \hookrightarrow \mathcal{C}$ is dense with density presentation consisting of λ-filtered colimits. From a general fact, the left adjoint of

$$\begin{tikzcd} {[\mathcal{C}_\lambda, \mathcal{D}]} \arrow[r, shift left=1.5, "\mathrm{Lan}_K"] \arrow[r, phantom, "\perp"] & {[\mathcal{C}, \mathcal{D}]} \arrow[l, shift left=1.5, "K^*"] \end{tikzcd}$$

induces an equivalence onto its essential image, which is precisely $[\mathcal{C}, \mathcal{D}]_\lambda$. In other words, $[\mathcal{C}, \mathcal{D}]_\lambda \cong [\mathcal{C}_\lambda, \mathcal{D}]$ is locally small and the inclusion preserves all colimits. Furthermore, $[\mathcal{C}, \mathcal{D}]_\lambda$ is locally presentable since $[\mathcal{C}_\lambda, \mathcal{D}]$ is locally κ-presentable. □

Corollary 2.4.20. The category of accessible functors $[\mathcal{C}, \mathcal{D}]_{\mathrm{acc}}$ is closed under small colimits in $[\mathcal{C}, \mathcal{D}]$.

Proof. This is clear, since $[\mathcal{C}, \mathcal{D}]_{\mathrm{acc}} = \bigcup\limits_\lambda [\mathcal{C}, \mathcal{D}]_\lambda$. □

The following theorem about dense functor has already been secretly used previously. Let us prove it once and for all.

Theorem 2.4.21. Consider two small categories $\mathcal{A}$ and $\mathcal{A}'$ and two fully faithful functors $\mathcal{A} \xrightarrow{P} \mathcal{A}' \xrightarrow{J} \mathcal{C}$. If the composite $K = JP$ is dense, then both P and J are dense.

Proof. It is immediate for P, since $\widetilde{P} = \widetilde{K}|_{\mathcal{A}'}$. Let us show J is dense. Note that we have

$$\begin{aligned} \mathcal{C}(Jd, c) &\xrightarrow[\sim]{\widetilde{JP}} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\mathcal{C}(JP-, Jd), \mathcal{C}(JP-, c)) \\ &\xrightarrow[\sim]{(J_{P-,d})^*} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\,(\mathcal{A}'(P-, d), \mathcal{C}(JP-, c)) \end{aligned}$$

where the first isomorphism holds because JP is dense and the second one because J is fully faithful. Consequently J is the pointwise left Kan extension of JP along P:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "P"] \arrow[dr, "JP"'] & \arrow[d, phantom, "\overset{\sim}{\Rightarrow}"] & \mathcal{A}' \arrow[dl, "J"] \\
& \mathcal{C} &
\end{tikzcd}$$

Since the left Kan extension is pointwise, it is preserved by any cocontinuous functor out of $\mathcal{C}$. In particular, for every $c \in \mathcal{C}$ we can apply $\mathcal{C}(-,c)\colon \mathcal{C} \to \mathbf{Set}^{\mathrm{op}}$ and we get that

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{op}} \arrow[rr, "P^{\mathrm{op}}"] \arrow[dr, "\mathcal{C}(JP-,c)"'] & \arrow[d, phantom, "\overset{\sim}{\Leftarrow}"] & (\mathcal{A}')^{\mathrm{op}} \arrow[dl, "\mathcal{C}(J-,c)"] \\
& \mathbf{Set} &
\end{tikzcd}$$

is a right Kan extension for every $c \in \mathcal{C}$. In particular, each $\alpha\colon \mathcal{C}(J-,c) \Rightarrow \mathcal{C}(J-,c')$ is uniquely given by $\alpha P^{\mathrm{op}}\colon \mathcal{C}(JP-,c) \Rightarrow \mathcal{C}(JP-,c')$. By density of $JP = K$, αP^{op} must be of the form g_* for a unique $g\colon c \to c'$. By uniqueness, $\alpha = \mathcal{C}(J-,g)$, hence $\widetilde{J}$ is full. Moreover, $\widetilde{JP}$ is equal to the composition $\mathcal{C} \xrightarrow{\widetilde{J}} [(\mathcal{A}')^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{(P^{\mathrm{op}})^*} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ and then $\widetilde{J}$ is also faithful. $\square$

2.5 Cocompleteness of categories of algebras

The goal of this section is to show that, if T is a monad on a locally presentable category $\mathcal{C}$ and T has rank (it is accessible), then T-Alg is cocomplete and thus locally presentable (the last bit is a consequence of a previous result).

There exists a single construction which admits the following as special cases:

- free monad on an endofunctor,
- free monad on a pointed endofunctor,
- free monoid on an object in a monoidal category,
- orthogonal factorization system generated by a set of morphisms,
- reflectiveness of a small orthogonality class,
- cocompletion of T-Alg for suitable monads T,
- existence of colimits of diagrams of accessible monads.

This was observed by G. M. Kelly in *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on* (1980), which is "hard to read" but simplifies greatly in the context of locally presentable categories.

Throughout this section we will work with locally presentable categories and accessible functors.

Kelly's main observation is that all obvious constructions can be reduced to the case of algebras for a well pointed endofunctor.

Definition 2.5.1. Let $S\colon \mathcal{C} \to \mathcal{C}$ be a functor. We call S *pointed* if there exists $\sigma\colon \mathrm{id}_{\mathcal{C}} \Rightarrow S$. The pair (S,σ) is *well pointed* if $S\sigma = \sigma S\colon S \Rightarrow S^2$.

Definition 2.5.2. Given a pointed endofunctor (S,σ), a (S,σ)-algebra is a pair (a,α), where $\alpha\colon Sa \to a$ is a morphism in $\mathcal{C}$ s.t. $\alpha\sigma_a = \mathrm{id}_a$ (basically a monad without multiplication gives an example). A morphism of algebras $(a,\alpha) \to (b,\beta)$ is a morphism $f\colon a \to b$ in $\mathcal{C}$ such that

$$\begin{tikzcd} Sa \arrow[r, "Sf"] \arrow[d, "\alpha"'] & Sb \arrow[d, "\beta"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

commutes. We write $(S,\sigma)\text{-}\mathsf{Alg}$ for the resulting category and $U^S\colon (S,\sigma)\text{-}\mathsf{Alg} \to \mathcal{C}$ for the forgetful functor.

Lemma 2.5.3. If (S,σ) is a well pointed endofunctor, then there exists at most one algebra structure for any object and it exists if and only if σ_a is invertible, in which case $\alpha = \sigma_a^{-1}$. Moreover, $U^S\colon (S,\sigma)\text{-}\mathsf{Alg} \to \mathcal{C}$ is fully faithful[5]. In other words, $(S,\sigma)\text{-}\mathsf{Alg}$ is isomorphic to the full subcategory of $\mathcal{C}$ given by $\{a \in \mathcal{C} \mid \sigma_a \text{ is invertible}\}$.

Proof. For fixed $(a,\alpha) \in (S,\sigma)\text{-}\mathsf{Alg}$, the diagram

$$\begin{tikzcd} Sa \arrow[r, "\sigma_{Sa}"] \arrow[d, "\alpha"'] & S^2a \arrow[d, "S\alpha"] \\ a \arrow[r, "\sigma_a"'] & Sa \end{tikzcd}$$

commutes by the naturality of σ. Since S is well pointed, this implies $\sigma_a\alpha = S\alpha \cdot \sigma_{Sa} = S\alpha \cdot S\sigma_a = S(\alpha\sigma_a) = S(\mathrm{id}_a) = \mathrm{id}_{Sa}$, therefore $\alpha = \sigma_a^{-1}$. On the other hand, if σ_a is invertible then (a, σ_a^{-1}) is a (S,σ)-algebra.

If $f\colon a \to b$ is any morphism, and both σ_a and σ_b are invertible, then

$$\begin{tikzcd} Sa \arrow[r, "Sf"] \arrow[d, "\sigma_a^{-1}"'] & Sb \arrow[d, "\sigma_b^{-1}"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

commutes by naturality of σ, so U^S is full (being faithful by construction). It follows that $U^S\colon (S,\sigma)\text{-}\mathsf{Alg} \to \{a \in \mathcal{C} \mid \sigma_a \text{ is invertible}\}$ is bijective on objects and fully faithful, so it is an isomorphism. $\square$

Lemma 2.5.4. Let (S,σ) be a pointed endofunctor, then $U^S\colon (S,\sigma)\text{-}\mathsf{Alg} \to \mathcal{C}$ is monadic if and only if it has a left adjoint.

Proof. U^S is conservative and creates all colimits preserved by S. In particular, it preserves coequalizers of U^S-split pairs. $\square$

Definition 2.5.5. For an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$ (or a pointed endofunctor (S,σ)), we say that the *algebraically free monad* on F (respectively (S,σ)) exists if $U^F\colon F\text{-}\mathsf{Alg} \to \mathcal{C}$ (or $U^S\colon (S,\sigma)\text{-}\mathsf{Alg} \to \mathcal{C}$) has a left adjoint.

We will denote by **Ord** the category of ordinals.

[5]For instance, it is easy to show that if the unit of a monad satisfies the condition of well pointedness, then the monad is idempotent.

Theorem 2.5.6. Let $\mathcal{C}$ be a category with colimits of chains (that is the domain of the diagram is an ordinal). Let (S,σ) be a well pointed endofunctor such that S preserves κ-filtered colimits. Then, the algebraically free monad on (S,σ) exists. In particular, $\{c \in \mathcal{C} \mid \sigma_c \text{ is an isomorphism}\}$ is a reflective subcategory.

Proof. For a given object $c \in \mathcal{C}$ we define a functor $S^\bullet c\colon \mathbf{Ord} \to \mathcal{C}$ by setting $S^0 c := c$, while $S^{\lambda+1}c := S(S^\lambda c)$, with $S^\lambda c \to S^{\lambda+1}c$ given by $\sigma_{S^\lambda c}$ for $\lambda \in \mathbf{Ord}$. Given a limit ordinal μ, we set $S^\mu c = \operatorname{colim}_{\lambda<\mu} S^\lambda c$.

We claim that $S^\kappa c$ lies in (S,σ)-Alg, that is $\sigma_{S^\kappa c}$ is an isomorphism. We will prove this by constructing an inverse $\alpha\colon S(S^\kappa c) \to S^\kappa c$.

Since S is κ-accessible, $S^{\kappa+1}c = S(S^\kappa c) = \operatorname{colim}_{\lambda<\kappa} S(S^\lambda c)$. We construct a cocone on $S(S^\bullet c)$ by considering the maps $l_{\lambda+1}\colon S(S^\lambda c) = S^{\lambda+1}c \to S^\kappa c$ exhibiting $S^\kappa c$ as a colimit.

$$\begin{tikzcd}
S(S^\lambda c) = S^{\lambda+1}c \arrow[r, "S\sigma_{S^\lambda c} = \sigma_{S^{\lambda+1}c}"] \arrow[d, "l_{\lambda+1}"] & S(S^{\lambda+1}c) \arrow[d, "l_{\lambda+2}"] \\
S^\kappa c \arrow[r, equal] & S^\kappa c
\end{tikzcd}$$

Well pointedness gives us the upper equality and the diagram commutes, hence we get a cocone culminating in $S^\kappa c$, which will then factor uniquely through the cocone culminating in $S(S^\kappa c)$ as $\alpha\colon S(S^\kappa c) \to S^\kappa c$. By construction, the following diagram commutes and $l_{\mu+1}\sigma_{S^\mu c} = l_\mu$.

$$\begin{tikzcd}
S^\mu c \arrow[r, "\sigma_{S^\mu c}"] \arrow[d, "l_\mu"] & S(S^\mu c) \arrow[d, "Sl_\mu"] \arrow[dr, "l_{\mu+1}"] & \\
S^\kappa c \arrow[r, "\sigma_{S^\kappa c}"] & S(S^\kappa c) \arrow[r, "\alpha"] & S^\kappa c
\end{tikzcd}$$

Passing to the colimit, this implies that $\alpha\sigma_{S^\kappa c} = \mathrm{id}_{S^\kappa c}$ because the l_μ on the left and $l_{\mu+1}\sigma_{S^\mu c} = l_\mu$ become identities, hence (S^κ, α) is indeed a (S,σ)-algebra.

We now claim that $l_0\colon c \to S^\kappa c$ defines a reflection into the full subcategory given by $\mathcal{B} := \{c \in \mathcal{C} \mid \sigma_c \text{ is an isomorphism}\}$. Firstly, we have shown that $S^\kappa c \in \mathcal{B}$, hence we only need $l_0^*\colon \mathcal{B}(S^\kappa c, b) \to \mathcal{C}(c,b)$ to be a bijection for all $b \in \mathcal{B}$.

Since representable functors $\mathcal{C}(-,b)$ send colimits to limits, this immediately reduces to the following: given $b \in \mathcal{B}$, $c \in \mathcal{C}$, the map $\sigma_c^*\colon \mathcal{C}(Sc,b) \to \mathcal{C}(c,b)$ is a bijection.

Using well pointedness, we can write the inverse to σ_c^* as $\mathcal{C}(c,b) \to \mathcal{C}(c,b)$, $f \mapsto \sigma_b^{-1}Sf$. $\square$

Theorem 2.5.7. Let $\mathcal{C}$ be a cocomplete category, $F\colon \mathcal{C} \to \mathcal{C}$ an endofunctor. The slice category $(F \downarrow \mathcal{C})$ is cocomplete. Moreover, all colimits preserved by F are computed pointwise, that is $\operatorname{colim}_{\mathcal{I}}(a_i, b_i, \alpha_i\colon Fa_i \to b_i) = (\operatorname{colim}_{\mathcal{I}} a_i, \operatorname{colim}_{\mathcal{I}} b_i, \operatorname{colim}_{\mathcal{I}} \alpha_i\colon F(\operatorname{colim}_{\mathcal{I}} a_i) \to \operatorname{colim}_{\mathcal{I}} b_i)$.

Proof. Giving a diagram $D\colon \mathcal{I} \to (F \downarrow \mathcal{C})$ amounts to giving diagrams $a_\bullet\colon \mathcal{I} \to \mathcal{C}$, $b_\bullet\colon \mathcal{I} \to \mathcal{C}$ and a natural transformation $\alpha_\bullet\colon Fa_\bullet \Rightarrow b_\bullet$.

Giving a cocone on this with vertex $(c, d, \gamma\colon Fc \to d)$ is equivalent to giving morphisms $\operatorname{colim}_{\mathcal{I}} a_i \to c$, $\operatorname{colim}_{\mathcal{I}} b_i \to c$ such that the following diagram commutes for all i.

$$\begin{tikzcd}
Fa_i \arrow[r] \arrow[d] & F(\operatorname{colim}_{\mathcal{I}} a_i) \arrow[r] \arrow[d] & Fc \arrow[d, "\gamma"] \\
b_i \arrow[r] & \operatorname{colim}_{\mathcal{I}} b_i \arrow[r] & d
\end{tikzcd}$$

Equivalently, we can give a morphism $\operatorname{colim}_{\mathcal{I}} a_i \to c$ and a morphism from the pushout p to d making the following diagram commute.

$$\begin{tikzcd}
\operatorname{colim}_{\mathcal{I}} Fa_i \arrow[r] \arrow[d, "\operatorname{colim}_{\mathcal{I}} \alpha_i"] & F(\operatorname{colim}_{\mathcal{I}} a_i) \arrow[r] \arrow[d] & Fc \arrow[d, "\gamma"] \\
\operatorname{colim}_{\mathcal{I}} b_i \arrow[r] & p \arrow[r] & d
\end{tikzcd}$$

We have then the colimit $(\operatorname{colim}_{\mathcal{I}} a_i, p, F(\operatorname{colim}_{\mathcal{I}} \alpha_i) \to p)$ in $(F \downarrow \mathcal{C})$. In particular, if F preserves this colimit, then the top map $F(\operatorname{colim}_{\mathcal{I}} a_i) \to Fc$ is an isomorphism, in which case we may take $p = d$ and the identity as the map from p to d. $\square$

Proposition 2.5.8. If in the theorem above $\mathcal{C}$ is locally presentable and F is accessible, then $(F \downarrow \mathcal{C})$ is locally presentable.

Proof. There exists a regular cardinal κ such that $\mathcal{C}$ is locally κ-presentable and $F(\mathcal{C}_\lambda) \subset \mathcal{C}_\kappa$, with F λ-accessible and $\lambda \leq \kappa$. We claim that the full subcategory $\mathcal{A} := \{(a, b, \alpha\colon Fa \to b) \mid a \in \mathcal{C}_\lambda,\ b \in \mathcal{C}_\kappa\}$ is dense and consists of κ-presentable objects in $(F \downarrow \mathcal{C})$.

The fact that it consists of κ-presentable objects follows from the facts that κ-filtered colimits in $(F \downarrow \mathcal{C})$ are computed pointwise.

To prove density, we want that for each $(c, d, \gamma\colon Fc \to d)$ the canonical cocone $(\mathcal{A} \downarrow (c, d, \gamma))$ exhibits (c, d, γ) as a colimit. In the arrow category $[[1], \mathcal{C}]$, $Fc \to d$ is a colimit of all κ-presentable pairs $c_0, c_1 \in \mathcal{C}_\lambda$.

$$\begin{tikzcd}
c_0 \arrow[r] \arrow[d] & Fc \arrow[d] \\
c_1 \arrow[r] & d
\end{tikzcd}$$

We need to check that the natural functor $(\mathcal{A} \downarrow (c, d, \gamma)) \to ([[1], \mathcal{C}]_\lambda \downarrow \gamma)$ is final.

Check for yourself that the category $([[1], \mathcal{C}]_\lambda \downarrow \gamma)$ we are considering is actually filtered and specifically can always find a pair of morphisms completing the following commutative diagram, where $Fa \to b$ comes from $\mathcal{A}$.

$$\begin{tikzcd}
& Fa \arrow[dr] \arrow[dd] & \\
c_0 \arrow[ur, dotted] \arrow[rr] \arrow[dd] & & Fc \arrow[dd] \\
& b \arrow[dr] & \\
c_1 \arrow[ur, dotted] \arrow[rr] & & d
\end{tikzcd}$$

The codomains form a colimit diagram in $\mathcal{C}$, hence we are left with checking that the domains form a colimit diagram as well. To do this, we use the fact that $(\mathcal{C}_\lambda \downarrow c) \to \mathcal{C}$ has colimit c and an argument similar to the previous one. $\square$

2.6 Algebraically free Monads on a pointed Endofunctor

Let $T\colon \mathcal{C} \to \mathcal{C}$ a κ-accessible endofunctor, with $\mathcal{C}$ cocomplete. As we have already shown, the category $(T \downarrow \mathcal{C})$ is cocomplete and κ-filtered colimits in $(T \downarrow \mathcal{C})$ are computed objectwise.

Given a natural transformation $\alpha\colon T' \to T$, we get an adjunction

$$\begin{tikzcd} (T' \downarrow \mathcal{C}) \arrow[r, bend left, "\alpha_!"] \arrow[r, phantom, "\perp"] & (T \downarrow \mathcal{C}) \arrow[l, bend left, "\alpha^*"] \end{tikzcd}$$

where $\alpha^*(a, b, Ta \xrightarrow{\gamma} b) = (a, b, T'a \to Ta \xrightarrow{\gamma} b)$ and $\alpha_!$ is given by the pushout

$$\begin{tikzcd} T'a \arrow[r, "\beta"] \arrow[d, "\alpha_a"] & b \arrow[d] \\ \alpha_!(a,b,\beta) = (Ta \arrow[r, "\gamma"] & c) \end{tikzcd}$$

If T, T' are κ-accessible, then α^* is κ-accessible. If we apply this to the case $T' = \mathrm{id}_{\mathcal{C}}$, $\alpha = \tau\colon \mathrm{id}_{\mathcal{C}} \to T$, then $(\mathrm{id}_{\mathcal{C}} \downarrow C = \mathcal{C}^{[1]})$ is the arrow category, and τ^* sends $(a, b, Ta \xrightarrow{\alpha} b)$ to $a \to Ta \to b$.

Now use exercise 5.1.

Proposition 2.6.1. If

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "U"] \arrow[r, phantom, "\top"] & \mathcal{D} \arrow[l, bend left, "F"] \end{tikzcd}$$

is an adjunction, (S, σ) is a well-pointed endofunctor on $\mathcal{D}$ and $\mathcal{C}$ has pushouts, then

$$\begin{tikzcd} FU \arrow[r, "F\sigma U"] \arrow[d, "\epsilon"] & FSU \arrow[d] \\ \mathrm{id}_{\mathcal{C}} \arrow[r, "\sigma'"'] & S' \end{tikzcd}$$

defines a well-pointed endofunctor (S', σ') on $\mathcal{C}$ such that the square

$$\begin{tikzcd} (S', \sigma')\text{-}\mathsf{Alg}, \arrow[r, "\overline{U}"] \arrow[d] & (S, \sigma)\text{-}\mathsf{Alg} \arrow[d] \\ \mathcal{C} \arrow[r, "U"'] & \mathcal{D} \end{tikzcd}$$

is a pullback. □

Theorem 2.6.2. Let $\mathcal{C}$ be a cocomplete category, (T, τ) a well-pointed endofunctor on $\mathcal{C}$ with T κ-accesible. Then $(T, \tau)\text{-}\mathsf{Alg}$ is a reflexive subcategory of $(T \downarrow \mathcal{C})$ and the algebraically free monad on (T, τ) exists. In particular, by reflexiveness, $(T, \tau)\text{-}\mathsf{Alg}$ is cocomplete.

Proof. Notice that the functor

$$(T, \tau)\text{-}\mathsf{Alg} \to (T \downarrow \mathcal{C}), \quad (a, \alpha) \mapsto (a, a, Ta \xrightarrow{\alpha} a), \quad f \mapsto (f, f)$$

is fully faithful. It is clearly faithful, and if $(f, g)\colon (a, a, \alpha) \to (b, b, \beta)$ is a morphism in $(T \downarrow \mathcal{C})$, then we have a commutative diagram

$$\begin{tikzcd} a \arrow[r, "\tau_a"] \arrow[d, "f"] & Ta \arrow[r, "\alpha"] \arrow[d, "Tf"] & a \arrow[d, "g"] \\ b \arrow[r, "\tau_b"'] & Tb \arrow[r, "\beta"'] & b \end{tikzcd}$$

Since (a, α) and (b, β) are algebras, we have $\beta\tau_b = \mathrm{id}_b$ and $\alpha\tau_a = \mathrm{id}_a$. It immediately follows that $f = g$. Moreover, the essential image of this functor is

$$\{(a, b, \gamma) \in (T \downarrow \mathcal{C}) \mid \gamma\tau_a \text{ is an isomorphism}\}$$

Apply the previous proposition to the pullback

$$\begin{array}{ccc} (T,\tau)\text{-}\mathsf{Alg} & \longrightarrow & \mathrm{Iso}(\mathcal{C}) \\ \downarrow & & \downarrow \\ (T \downarrow \mathcal{C}) & \xrightarrow{\tau^*} & \mathcal{C}^{[1]} \\ (a, b, \gamma) & \longmapsto & \gamma\tau_a \end{array}$$

and the well-pointed endofunctor $S\colon C^{[1]} \to C^{[1]}$ given by $S(c \to d) = \mathrm{id}_d$ with $(S, \sigma)\text{-}\mathsf{Alg} = \mathrm{Iso}(\mathcal{C})$. We obtain (S', σ') such that $(T, \tau)\text{-}\mathsf{Alg} \cong (S', \sigma')\text{-}\mathsf{Alg}$. Since the pushout from the previous proposition consists of κ-accessible functors (here we use that τ^* is κ-accessible), (S', σ') is κ-accessible. It follows that $(S', \sigma')\text{-}\mathsf{Alg}$ is reflexive in $(T \downarrow \mathcal{C})$, as claimed.

Thus $(T, \tau)\text{-}\mathsf{Alg}$ is cocomplete. Note that the forgetful functor $U^T\colon (T, \tau)\text{-}\mathsf{Alg} \to \mathcal{C}$ factors as

$$\begin{array}{ccccc} (T,\tau)\text{-}\mathsf{Alg} & \longrightarrow & (T \downarrow \mathcal{C}) & \xrightarrow{\mathrm{dom}} & \mathcal{C} \\ (a, \alpha) & \longmapsto & (a, a, \alpha) & \longmapsto & a \end{array}$$

and $(T, \tau)\text{-}\mathsf{Alg} \to \mathcal{C}$ has a left adjoint. We only need to a left adjoint to dom which is given by $c \mapsto (c, Tc, \mathrm{id}_{Tc})$. $\square$

Theorem 2.6.3. Let $\mathcal{C}$ a cocomplete category and F a κ-accessible endofunctor. The category $F\text{-}\mathsf{Alg}$ is cocomplete and the algebraically free monad on F exists, that is the functor $U^F\colon F\text{-}\mathsf{Alg} \to \mathcal{C}$ has a left adjoint.

Proof. Let T be the coproduct $F + \mathrm{id}_{\mathcal{C}}$ and $\tau\colon \mathrm{id}_{\mathcal{C}} \to F + \mathrm{id}_{\mathcal{C}}$ the inclusion. Then $(T, \tau)\text{-}\mathsf{Alg} \cong F\text{-}\mathsf{Alg}$ is an isomorphism which is compatible with the forgetful functors. $\square$

For example, we can easily prove the following

Proposition 2.6.4. Let $\mathcal{C}$ a κ-presentable category and $T\colon \mathcal{C} \to \mathcal{C}$ κ-accessible. Then $(T \downarrow \mathcal{C})$ is locally κ-presentable.

Proof. Consider the functor $\mathcal{C} \times \mathcal{C} \xrightarrow{F} \mathcal{C} \times \mathcal{C}$, $(a, b) \mapsto (\emptyset, Ta)$. Then $F\text{-}\mathsf{Alg} \cong (T \downarrow \mathcal{C})$ and U^F is κ-accessible. Since U^F is monadic, the free objects on the κ-presentable objects form a dense generating set consisting of κ-presentable objects in $F\text{-}\mathsf{Alg} \cong (T \downarrow \mathcal{C})$. $\square$

Remark 2.6.5. An analysis of the construction of (S', σ') in the proof of the previous theorem shows that $S'\colon (T \downarrow \mathcal{C}) \to (T \downarrow \mathcal{C})$ sends $(a, b, \alpha\colon Ta \to b)$ to $(b, c, \gamma\colon Tb \to c)$ where

$$T^2a \underset{\tau_{Ta}}{\overset{T\tau_a}{\rightrightarrows}} T^2a \xrightarrow{T\alpha} Tb \xrightarrow{\gamma} c \tag{1}$$

is a coequalizer diagram in $\mathcal{C}$. Notice that γ is a coequalizer of $Ta \cdot T\tau_a$ and $T\alpha \cdot \tau_{Ta}$ (see the exercises for more details).

Proposition 2.6.6. Let (S, σ) be a well-pointed endofunctor on $\mathcal{C}$ and let $L\colon \mathcal{C} \to \mathcal{C}$ be a functor. If $\pi\colon S \to L$ is a natural transforamtion such that $\pi_c\colon Sc \to Lc$ is epic for all $c \in \mathcal{C}$, htne $(L, \pi \cdot \sigma)$ is a well-pointed endofunctor and $(L, \pi\sigma)$-Alg is equivalent to the full subcategory of (S, σ)-Alg on objects (a, α) such that $\pi_a\colon Sa \to La$ is an isomorphism.

Proof. Exercise. □

Now let (T, η, μ) be a monad on a cocomplete category $\mathcal{C}$ and assume T is κ-accessible. We define an endofunctor $L\colon (T \downarrow \mathcal{C}) \to (T \downarrow \mathcal{C})$ as follows: Given $(a, b, \alpha\colon Ta \to b)$ we set $L(a, b, \alpha) = (b, d, \gamma\colon Tb \to d)$ with γ defined by the following pushout in $\mathcal{C}$.

$$\begin{tikzcd}
T^2a \arrow[r, "\mu_a"] \arrow[d, "T\alpha"'] & Ta \arrow[d, "\delta"] \\
Tb \arrow[r, "\gamma"'] & d
\end{tikzcd}$$

Using this construction we can finally prove the following.

Theorem 2.6.7. Let $\mathcal{C}$ be a complete category, (T, η, μ) a monad over $\mathcal{C}$ with T κ-accessible. Then T-Alg is reflexive in $(T \downarrow \mathcal{C})$ and cocomplete.

Proof. Recall that (T, η)-Alg is reflexive in $(T \downarrow \mathcal{C})$ and we have a well-pointed endofunctor given by S' described in 1. Remember the functor L just defined.

Since μ_a has a section $T\eta_a$, it is the coequalizer of id_a and $T\eta_a \cdot \mu_a$.

$$\begin{tikzcd}
T^2a \arrow[rr, equal] \arrow[dr, "\mu_a"'] & & T^2a \arrow[r, "T\alpha"] & Tb \arrow[r, "\gamma"] & d \\
& Ta \arrow[ur, "T\eta_a"'] & & &
\end{tikzcd} \tag{2}$$

Given that $\mu_a \cdot T\eta_a = \mathrm{id}_{Ta}$, we have $\delta = \delta \cdot \mu_a \cdot T\eta_a = \gamma \cdot T\alpha \cdot T\eta_a$. Moreover, γ coequalizes $T\alpha \cdot T\eta_a$ and $T\alpha \cdot \eta_{Ta}$, hence there exists a unique $\pi\colon c \to d$ making the following diagram commute.

$$\begin{tikzcd}
Tb \arrow[r, "\beta"] \arrow[d, equal] & c \arrow[d, "\pi"] \\
Tb \arrow[r, "\gamma"] & d
\end{tikzcd}$$

This defines a natural transformation $(\mathrm{id}, \pi)\colon S(a, b, \alpha) \to L(a, b, \alpha)$, where the components are epimorphisms because γ is a coequalizer of the diagram in 1 and therefore an epimorphism.

We get then a well-pointed endofunctor $(L, \pi\sigma')$ over $(T \downarrow \mathcal{C})$ with $(L, \pi\sigma')$-Alg equivalent to the full subcategory of (S', σ')-Alg given by the objects b such that (id_b, π) is an isomorphism. We also have an equivalence (T, η)-$\mathsf{Alg} \to (S', \sigma')$-$\mathsf{Alg}$, $(a, \alpha) \mapsto (a, a, \alpha)$, hence we get that (S', σ')-Alg is isomorphic to the full subcategory of (T, η)-Alg given by $\{(a, \alpha) \in (T, \eta)\text{-}\mathsf{Alg} \mid (\mathrm{id}_a, \pi)\colon S'(a, a, \alpha) \to L(a, a, \alpha) \text{ is an isomorphism}\}$.

In this case, the coequalizer of 1 is actually $\alpha\colon Ta \to a$, hence our π looks as follows.

$$\begin{tikzcd}
Ta \arrow[r, "\alpha"] \arrow[d, equal] & a \arrow[d, "\pi"] \\
Ta \arrow[r, "\gamma"] & d
\end{tikzcd}$$

Having π invertible is then equivalent to α being the coequalizer of 2, where $b = a$. If it is a coequalizer diagram, $\alpha\mu_a = \alpha T\alpha$, which implies that (a, α) is a T-algebra. Conversely, if (a, α) is a T-algebra, then this is a split coequalizer in $\mathcal{C}$. It follows that $T\text{-}\mathsf{Alg}$ is equivalent to $(L, \pi\sigma')\text{-}\mathsf{Alg}$.

L is accessible since T is and κ-filtered colimits in $(T \downarrow \mathcal{C})$ are computed as in $\mathcal{C}$, thus $(L, \pi\sigma')\text{-}\mathsf{Alg} \to (T \downarrow \mathcal{C})$ has a left adjoint and therefore $T\text{-}\mathsf{Alg} \to (T \downarrow \mathcal{C})$, $(a, \alpha) \mapsto (a, a, \alpha)$ is fully faithful and has a left adjoint. □

We have the following result as a consequence.

Theorem 2.6.8. Given a locally κ-presentable category $\mathcal{C}$ and a monad (T, η, μ) of rank κ, $T\text{-}\mathsf{Alg}$ is locally κ-presentable.

Proof. We have shown that $\{(Ta, \mu_a) \mid a \in \mathcal{C}_\kappa\}$ is a dense generating system of κ-presentable objects, hence the claim follows from the fact that $T\text{-}\mathsf{Alg}$ is cocomplete. □

3 Monads in 2-category theory

3.1 Monads are monadic

Given an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$, an algebraically free monad on F exists if $U^F\colon F\text{-}\mathsf{Alg} \to \mathcal{C}$ has a left adjoint $F^F\colon \mathcal{C} \to F\text{-}\mathsf{Alg}$. We write then $T(F) = (U^F F^F, \eta, U^F \epsilon F^F)$ for the resulting monad. From Beck's theorem, we know that $J\colon F\text{-}\mathsf{Alg} \to T(F)\text{-}\mathsf{Alg}$, $(a, \alpha) \mapsto (U^F(a, \alpha) = a, U^F \epsilon_{(a,\alpha)})$ is an equivalence of categories. We also have a natural transformation $\psi\colon F \Rightarrow T(F)$ corresponding via adjunction to $\alpha\colon FU^F \Rightarrow U^F$. This gives us a functor $\psi^*\colon T(F)\text{-}\mathsf{Alg} \to F\text{-}\mathsf{Alg}$, $(a, \alpha) \mapsto (a, \alpha\psi_a)$ such that $\psi^* J = \mathrm{id}_{F\text{-}\mathsf{Alg}}$. We have the following result.

Proposition 3.1.1. In the described situation, ψ^* is an isomorphism of categories.

Proof. We still have to show that J is surjective on objects, which follows from the fact that both U^F and $U^{T(F)}$ are isofibrations and the fact that two $T(F)$-algebras isomorphic via id_a are equal. □

Definition 3.1.2. A *morphism of monads* $(T, \eta, \mu) \to (T', \eta', \mu')$ over a category $\mathcal{C}$ is a natural transformation $\phi\colon T \Rightarrow T'$ making the following diagrams commute.

$$\begin{tikzcd}
T^2 \arrow[r, Rightarrow, "\phi^2"] \arrow[d, Rightarrow, "\mu"] & (T')^2 \arrow[d, Rightarrow, "\mu'"] \\
T \arrow[r, Rightarrow, "\phi"] & T'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_{\mathcal{C}} \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta'"'] & T \arrow[d, Rightarrow, "\phi"] \\
 & T'
\end{tikzcd}$$

The first diagram is equivalent to equating the following two.

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathcal{C} \arrow[ur, "T"] \arrow[rr, "T" description] \arrow[rr, bend right=60, "T'"'] & {} \arrow[d, phantom, "\Downarrow\phi"] & \mathcal{C} \\
 & {} &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "T'"'] \arrow[dr, bend left=60, "T"] \arrow[dr, phantom, bend left=20, "\Downarrow\phi"] & \\
\mathcal{C} \arrow[ur, "T'"] \arrow[ur, bend left=60, "T"] \arrow[ur, phantom, bend left=20, "\Downarrow\phi"] \arrow[rr, bend right, "T'"'] & \mu'\Downarrow & \mathcal{C}
\end{tikzcd}$$

The second diagram amounts saying that the following two are equal.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "T" description] \arrow[rr, bend left=60, "\mathrm{id}_{\mathcal{C}}"] \arrow[rr, bend right=60, "T'"'] \arrow[rr, phantom, bend left=30, "\Downarrow\eta"] \arrow[rr, phantom, bend right=30, "\Downarrow\phi"] & & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "\mathrm{id}_{\mathcal{C}}"] \arrow[rr, bend right, "T'"'] \arrow[rr, phantom, "\Downarrow\eta'"] & & \mathcal{C}
\end{tikzcd}$$

We denote the category of monads over $\mathcal{C}$ by $\mathsf{Mnd}(\mathcal{C})$.

Proposition 3.1.3. The functor

$$\begin{aligned}\mathsf{Mnd}(\mathcal{C})^{\mathrm{op}} &\to (\mathbf{CAT} \downarrow \mathcal{C})\\ (T,\eta,\mu) &\mapsto (U^F : T\text{-}\mathsf{Alg} \to \mathcal{C})\\ \phi &\mapsto \begin{pmatrix}\phi^* : T'\text{-}\mathsf{Alg} \to T\text{-}\mathsf{Alg}\\ \qquad (a,\alpha) \mapsto (a, \alpha\phi_a)\end{pmatrix}\end{aligned}$$

is fully faithful.

Proof. Prove by yourself that this is a functor. Consider then two monads T, T' over $\mathcal{C}$. Giving a functor making the following diagram commute amounts to giving an action on $U^{T'}$, that is $\rho\colon TU^{T'} \Rightarrow U^{T'}$.

$$\begin{tikzcd} T'\text{-}\mathsf{Alg} \arrow[rr] \arrow[dr, "U^{T'}"'] & & T\text{-}\mathsf{Alg} \arrow[dl, "U^T"] \\ & \mathcal{C} & \end{tikzcd}$$

Using the adjunction $F^{T'} \dashv U^{T'}$, this corresponds to a unique natural transformation $T \Rightarrow U^{T'}F^{T'} = T'$. Notice that the T-action axioms for ρ correspond precisely to axioms for morphisms of monads, hence we are done. □

Proposition 3.1.4 (Algebraically free monads are free)**.** Let $F\colon\ \mathcal{C} \to \mathcal{C}$ be an endofunctor such that the algebraically free monad $T(F)$ exists. Then, for every monad T over $\mathcal{C}$, the natural transformation $\psi\colon F \Rightarrow T(F)$ induces a bijection $\psi^*\colon\ \mathsf{Mnd}(\mathcal{C}) \to [\mathcal{C},\mathcal{C}](F,T)$.

Proof. Consider the following commutative diagram. We want to prove that the horizontal arrows and the one on the right are bijection, which will give us the thesis.

— Placeholder —

The diagram commutes by Yoneda, while the top map are bijections by the previous proposition and the one on the right is a bijection given by composing with $\psi^*\colon T(F)\text{-}\mathsf{Alg} \xrightarrow{\sim} F\text{-}\mathsf{Alg}$.

Notice that giving the following commutative diagram amounts to giving a natural transformation $\rho\colon FG \Rightarrow G$ without requiring any additional property.

$$\begin{tikzcd} \mathcal{A} \arrow[rr] \arrow[dr, "G"'] & & F\text{-}\mathsf{Alg} \arrow[dl, "U^F"] \\ & \mathcal{C} & \end{tikzcd}$$

Here the natural transformations $FU^T \Rightarrow U^T$ correspond bijectively to natural transformations $F \Rightarrow U^TF^T = T$ by adjunction, hence the bottom map is bijective as well. □

Theorem 3.1.5. Let $\mathcal{C}$ be a locally κ-presentable category. We write $\mathsf{Mnd}_\kappa(\mathcal{C})$ for the full subcategory of $\mathsf{Mnd}(\mathcal{C})$ given by κ-accessible monads. Then, the forgeful functor $U\colon\ \mathsf{Mnd}_\kappa(\mathcal{C}) \to [\mathcal{C},\mathcal{C}]_\kappa$ is monadic and κ-accessible. In particular, $\mathsf{Mnd}_\kappa(\mathcal{C})$ is locally κ-presentable.

— Placeholder for Niklas —

3.2 Symmetric monoidal categories

Definition 3.2.1. A *monoidal category* is a tuple $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$, where $\mathcal{V}$ is a category, $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ a functor, $I \in \mathcal{V}$ an object, $\alpha\colon \otimes \circ (\otimes \times \mathrm{id}) \Rightarrow \otimes \circ (\otimes \times \mathrm{id})$, $\lambda\colon I \otimes - \Rightarrow \mathrm{id}$ and $\rho\colon - \otimes I \to \mathrm{id}$ natural isomorphisms such that for every $W, X, Y, Z \in \mathcal{V}$ the diagrams

$$\begin{tikzcd}
 & & (W \otimes X) \otimes (Y \otimes Z) \arrow[drr, "a_{W,X,Y \otimes Z}"] & & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[urr, "a_{W \otimes X,Y,Z}"] \arrow[dr, "a_{W,X,Y} \otimes \mathrm{id}_Z"'] & & & & W \otimes (X \otimes (Y \otimes Z)) \\
 & (W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a_{W,X \otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[ur, "\mathrm{id}_W \otimes a_{X,Y,Z}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[dr, "\rho_X \otimes \mathrm{id}_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes \lambda_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$

commute.

Example 3.2.2. We now list some monoidal categories.

1. If $\mathcal{E}$ is a category with finite products, then $(\mathcal{E}, \times, *)$ is monoidal, with α, λ and ρ induced by the universal property. Instances of this are **Set**, **Cat**, **Grp**, **sSet**, **Top**, **CGTop** and **CGHTop**.

2. $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ and, given a commutative ring R, $(\mathbf{Mod}_R, \otimes_R, R)$ and $(\mathbf{dgMod}_R, \otimes_R, R)$.

3. The order $\overline{\mathbb{R}}_+ = [0, \infty]$ with $\otimes = +$, $I = 0$.

4. A monoid in **Cat** or **CAT** is a monoidal category such that α, λ and ρ are identities. This is the case of $[\mathcal{C}, \mathcal{C}]$, $[\mathcal{C}, \mathcal{C}]_\kappa$, $\Phi\text{-}\mathsf{Cocts}[\mathcal{C}, \mathcal{C}]$.

We mention without proof the following fundamental theorem.

Theorem 3.2.3 (Mac Lane)**.** Any diagram built from $\otimes$, I, α, λ, ρ and their iterations is commutative.

Given a word of objects tensored among them, any two choices of bracketing are uniquely isomorphic. This result is plausible if $\otimes$ is derived from an universal property as in (1) – (3) and clear if $\mathcal{V}$ is strict, like in (3) and (4), while the general proof uses a rewriting argument which can be found in *Categories for the Working Mathematician*.

Definition 3.2.4. A *lax monoidal functor* from $(\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}}, \alpha^{\mathcal{V}}, \lambda^{\mathcal{V}}, \rho^{\mathcal{V}})$ to $(\mathcal{W}, \otimes_{\mathcal{W}}, I_{\mathcal{W}}, \alpha^{\mathcal{W}}, \lambda^{\mathcal{W}}, \rho^{\mathcal{W}})$ is a triple (F, ϕ_0, ϕ), where $F\colon \mathcal{V} \to \mathcal{W}$ is a functor, $\phi_0\colon I_{\mathcal{W}} \to FI_{\mathcal{V}}$ a morphism and $\phi\colon \otimes_{\mathcal{W}} \circ(F \times F) \Rightarrow F \circ \otimes_{\mathcal{W}}$ a natural transformation such that for all $X, Y, Z \in \mathcal{V}$ the diagrams

$$\begin{tikzcd}
(FX \otimes_{\mathcal{W}} FY) \otimes_{\mathcal{W}} FZ \arrow[r, "\alpha^{\mathcal{W}}"] \arrow[d, "\phi_{X,Y} \otimes_{\mathcal{W}} FZ"] & FX \otimes_{\mathcal{W}} (FY \otimes_{\mathcal{W}} FZ) \arrow[d, "FX \otimes_{\mathcal{W}} \phi_{X,Z}"] \\
F(X \otimes_{\mathcal{V}} Y) \otimes_{\mathcal{W}} FZ \arrow[d, "\phi_{X \otimes_{\mathcal{V}} Y, Z}"] & FX \otimes_{\mathcal{W}} F(Y \otimes_{\mathcal{V}} Z) \arrow[d, "\phi_{X, Y \otimes_{\mathcal{V}} Z}"] \\
F((X \otimes_{\mathcal{V}} Y) \otimes_{\mathcal{V}} Z) \arrow[r, "F\alpha^{\mathcal{V}}"] & F(X \otimes_{\mathcal{V}} (Y \otimes_{\mathcal{V}} Z))
\end{tikzcd}$$

$$\begin{tikzcd}
I_{\mathcal{W}} \otimes_{\mathcal{W}} FX \arrow[r, "\phi_0 \otimes_{\mathcal{W}} FX"] \arrow[d, "\lambda^{\mathcal{W}}_X"] & FI_{\mathcal{V}} \otimes_{\mathcal{W}} FX \arrow[d, "\phi_{I_{\mathcal{V}}, X}"] \\
FX & F(I_{\mathcal{V}} \otimes_{\mathcal{V}} X) \arrow[l, "F\lambda^{\mathcal{V}}_X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes_{\mathcal{W}} I_{\mathcal{W}} \arrow[r, "FX \otimes_{\mathcal{W}} \phi_0"] \arrow[d, "\rho^{\mathcal{W}}_X"] & FI_{\mathcal{V}} \otimes_{\mathcal{W}} FX \arrow[d, "\phi_{X, I_{\mathcal{V}}}"] \\
FX & F(I_{\mathcal{V}} \otimes_{\mathcal{V}} X) \arrow[l, "F\rho^{\mathcal{V}}_X"']
\end{tikzcd}$$

are commutative.

If we reverse the direction of ϕ_0 and ϕ we get *oplax monoidal functors*.

A *strong (strict) monoidal functor* is a lax monoidal functor such that ϕ_0 and ϕ are isomorphisms (respectively identities).

A *monoidal natural transformation* from (F, ϕ_0, ϕ) to (G, ψ_0, ψ) is a natural transformation $\gamma\colon F \Rightarrow G$ such that the diagrams

$$\begin{tikzcd}
FX \otimes_{\mathcal{W}} FY \arrow[r, "\gamma_X \otimes_{\mathcal{W}} \gamma_Y"] \arrow[d, "\phi_{X,Y}"] & GX \otimes_{\mathcal{W}} GY \arrow[d, "\psi_{X,Y}"] \\
F(X \otimes_{\mathcal{V}} Y) \arrow[r, "\gamma_{X \otimes_{\mathcal{V}} Y}"] & G(X \otimes_{\mathcal{V}} Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
& I_{\mathcal{W}} \arrow[dl, "\phi_0"'] \arrow[dr, "\psi_0"] & \\
FI_{\mathcal{V}} \arrow[rr, "\gamma_{I_{\mathcal{V}}}"] & & GI_{\mathcal{V}}
\end{tikzcd}$$

commute.

Proposition 3.2.5. Lax monoidal functors compose and monoidal natural transformations whisker.

Proposition 3.2.6. There is a finitary monad T on **Cat** such that T-Alg is the category of monoidal categories and strict monoidal functors.

Proof. We can write down a presentation of this monad using the finitary endofunctors $X \mapsto X \times X$, $X \mapsto X \times X \times X \times [1]$. □

Example 3.2.7. Given a locally small monoidal category $\mathcal{V}$, the functor $\mathcal{V}(I, -)\colon \mathcal{V} \to \mathbf{Set}$ is lax monoidal, with $\phi_0\colon \{*\} \to \mathcal{V}(I, I)$, $* \mapsto \mathrm{id}_I$ and $\phi_{X,Y}$ sending $(f, g) \in \mathcal{V}(I, X) \times \mathcal{V}(I, Y)$ to $(f \otimes g) \circ \lambda_I^{-1} = (f \otimes g) \circ \rho_I^{-1}\colon I \xrightarrow{\sim} I \otimes I \to X \otimes Y$. It is universally denoted by $V\colon \mathcal{V} \to \mathbf{Set}$ and, if $\mathcal{V}$ has coproducts, it has a left adjoint given by $F\colon \mathbf{Set} \to \mathcal{V}$, $S \mapsto \amalg_S I$.

Assuming for simplicity that $\mathcal{V}$ is cocomplete, it is easy to show that F is strong monoidal if $\otimes$ preserves colimits in each variable by using that **Set** is the free cocomplete category on $\{*\}$.

The previous example is an instance of a more general phenomenon, as shown by the following result.

Theorem 3.2.8. Let $F\colon \mathcal{V} \to \mathcal{W}$ be a left adjoint to U. If $\mathcal{V}$, $\mathcal{W}$ are monoidal, F, U lax and η, ϵ monoidal natural transformations, then F is strong monoidal. Conversely, if (F, ϕ_0, ϕ) is strong monoidal and U is any right adjoint, then

$$\begin{tikzcd} I_{\mathcal{V}} \arrow[r, "\eta_{I_{\mathcal{V}}}"] \arrow[dr, "\psi_0"'] & UFI_{\mathcal{V}} \arrow[d, "U\phi_0^{-1}"] \\ & UI_{\mathcal{W}} \end{tikzcd} \qquad \begin{tikzcd} UX \otimes_{\mathcal{V}} UY \arrow[r, "\psi_{X,Y}"] \arrow[d, "U(\epsilon_X \otimes_{\mathcal{V}} \epsilon_Y)"] & U(X \otimes_{\mathcal{V}} Y) \\ UF(UX \otimes_{\mathcal{V}} UY) \arrow[r, "U\phi^{-1}_{X,Y}"'] & U(FUX \otimes_{\mathcal{W}} FUY) \arrow[u, "\eta_{UX \otimes_{\mathcal{W}} UY}"'] \end{tikzcd}$$

define a lax monoidal structure on U which is unique with the property that η, ϵ are monoidal.

Proof. Exercise. □

Example 3.2.9. Given a homomorphism of commutative rings $R \to S$, then $S \otimes_R - \dashv U$, where U is the restriction on scalars, and $- \otimes_R S\colon \mathbf{Mod}_R \to \mathbf{Mod}_S$ is a monoidal adjunction.

The free module functor $\mathbf{Set} \to \mathbf{Mod}_R$ is strong monoidal by the previous example.

Remark 3.2.10. The last example still holds if we substitute to $\mathbf{Mod}_R$ any cocomplete monoidal category $\mathcal{V}$ with $- \otimes V$, $V \otimes -$ cocontinuous.

Definition 3.2.11. A monoid in a monoidal category $\mathcal{V}$ is a triple (M, m, u) where $m\colon M \otimes M \to M$ is the *multiplication*, $u\colon I \to M$ the *unit* and the diagrams

$$\begin{tikzcd} & (M \otimes M) \otimes M \arrow[r, "\alpha_{M,M,M}"] \arrow[dl, "m \otimes M"'] & M \otimes (M \otimes M) \arrow[dr, "M \otimes m"] & \\ M \otimes M \arrow[drr, "m"'] & & & M \otimes M \arrow[dl, "m"] \\ & & M & \end{tikzcd}$$

$$\begin{tikzcd} I \otimes M \arrow[r, "u \otimes M"] \arrow[dr, "\lambda_M"'] & M \otimes M \arrow[d, "m"] & M \otimes I \arrow[l, "M \otimes u"'] \arrow[dl, "\rho_M"] \\ & M & \end{tikzcd}$$

commute.

Morphisms of monoids are maps $f\colon M \to M'$ such that $m' \circ (f \otimes f) = fm$, $fu = u'$.

We write $\mathsf{Mon}(\mathcal{V})$ for the category of monoids over $\mathcal{V}$.

Remark 3.2.12. If $\mathcal{V}$ is additive, monoids are often called algebras as well because $\mathsf{Mon}(\mathbf{Mod}_R) = \mathbf{Alg}_R$, $\mathsf{Mon}(\mathbf{dgMod}_R) = \mathbf{dgAlg}_R$.