Monads and their applications
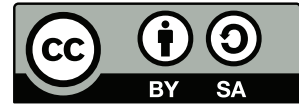
Dr. Daniel Schäppi's course lecture notes

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Chapter 1

Categorical preliminaries

Definition 1.0.1 (Categories)**.** A *category* $\mathbf{C}$ consists of:

1. a collection of objects $\mathrm{Ob}(\mathbf{C})$;

2. a collection of arrows $\mathrm{Ar}(\mathbf{C})$;

3. two maps $\mathrm{dom}, \mathrm{cod} \colon \mathrm{Ar}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$;

4. a map $\mathrm{id}_- \colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ar}(\mathbf{C})$ with $\mathrm{dom}(\mathrm{id}_c) = c = \mathrm{cod}(\mathrm{id}_c)$;

5. for every $f, g \in \mathrm{Ar}(\mathbf{C})$ such that $\mathrm{cod}(f) = \mathrm{dom}(g)$ a unique composite morphism gf such that $\mathrm{cod}(gf) = \mathrm{cod}(g)$, $\mathrm{dom}(gf) = f$.

This data has to satisfy the following axioms

1. given $f \in \mathrm{Ar}(\mathbf{C})$, $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$, $\mathrm{id}_{c'}\, f = f = \mathrm{id}_c$, that is the composition is unital;

2. given a composable triple $f, g, h \in \mathrm{Ar}(\mathbf{C})$, $h(gf) = (hg)f$, that is the composition is associative.

An arrow f such that $c = \mathrm{dom}(f)$ and $c' = \mathrm{cod}(f)$ is denoted $f \colon c \to c'$.

Definition 1.0.2 (Functors)**.**

Definition 1.0.3 (Full functors, faithful functor)**.**

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Definition 1.0.5 (Equivalent functors)**.**

Definition 1.0.6 (Representable Functors)**.**

Definition 1.0.7 (Whiskering)**.**

Definition 1.0.8 (Horizontal and vertical composition of nat.transf.)**.**

Definition 1.0.9 (adjunctions)**.**

Lemma 1.0.10 (Yoneda)**.**

Proof.   □

We will denote by よ (the hiragana kata for "Yo") the Yoneda embedding $\mathbf{C} \hookrightarrow \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$.

Chapter 2

Monads and algebras

2.1 Introduction

Throughout mathematics we encounter structures defined by some action morphisms. Here we give some examples.

Example 2.1.1. Given a group G, we may consider a G-set X described by an action map $G \times X \to X$.

Example 2.1.2. Given an abelian group M and a ring R, we can get an R-module M by fixing a group homomorphism $R \otimes_{\mathbb{Z}} M \to M$.

Example 2.1.3. Given a monoid M in **Set**, we get a map $\Pi_{k=1}^{n} M \to M$, $(m_1, \ldots, m_n) \mapsto ((\ldots((m_1 m_2)m_3)\ldots)m_{n-1})m_n$. This induces an action map from $W(M) = \amalg_{n \in \mathbb{N}} \Pi_{k=1}^{n} M$, the set of words on M, to M.

Example 2.1.4. Given a set X, let $\mathcal{U}X$ be the set of ultrafilters on it. Any compact T2 topology on X allows us to see each ultrafilter as a system of neighborhoods of a unique point in X, hence it gives us a unique map $\mathcal{U}X \to X$ sending each ultrafilter to the respective point.

Example 2.1.5. Given a directed graph $D = (V, E, E \begin{smallmatrix} -s \to \\ -t \to \end{smallmatrix} V)$, we can create its free category FD, where the objects are the vertices and $FD(v, w) = \{\text{finite paths } v \to \ldots \to w\}$. We set id_v to be the path of length 0, while composition is just the concatenation of paths.

In particular, if D is the directed graph with $V = \{0, \ldots, n\}$ and an edge $j \to k$ if and only if $k = j + 1$, we have $FD \cong [n]$.

If $D = \{*\}$ and $E = \{* \to *\}$, then $FD(*, *) \cong \mathbb{N}$.

Given a small category **C**, we may consider the underlying graph $U\,\mathbf{C} = D$ with $V = \mathrm{Ob}(\mathbf{C})$, $E = \mathrm{Ar}(\mathbf{C})$, $s = \mathrm{dom}$ and $t = \mathrm{cod}$. We get then an action map $UFU\,\mathbf{C} \to U\,\mathbf{C}$ sending a finite path to its composite. This map is a morphism of directed graphs.

Notice that we always have a category $\mathbf{C}$ and some functor $T\colon \mathbf{C} \to \mathbf{C}$ with an action map $T\,\mathbf{C} \to \mathbf{C}$. How can we see all of these examples as specific instances of a general phenomenon?

Definition 2.1.6. A *monad* on a category $\mathbf{C}$ is a triple (T, μ, η) where $T\colon \mathbf{C} \to \mathbf{C}$ is a functor, while $\mu\colon T^2 \Rightarrow T$ and $\eta\colon \mathrm{id}_{\mathbf{C}} \Rightarrow T$ are natural transformations such that the following diagrams commute:

$$
\begin{array}{ccc}
T^3 & \xRightarrow{\;T\mu\;} & T^2 \\
{\scriptstyle \mu T}\big\Downarrow & & \big\Downarrow{\scriptstyle \mu} \\
T^2 & \xRightarrow[\mu]{} & T
\end{array}
\qquad\qquad
\begin{array}{ccccc}
T & \xRightarrow{\;\eta T\;} & T^2 & \xLeftarrow{\;T\eta\;} & T \\
 & {\scriptstyle \mathrm{id}_T}\searrow & \big\Downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{id}_T} & \\
 & & T & &
\end{array}
$$

μ is called the *multiplicative map*, while η is the *unit* of T.

The commutativity of the first diagram is equivalent to stating that the following two diagrams are equal:

$$
\begin{array}{c}
\mathbf{C} \xrightarrow{\;T\;} \mathbf{C} \\
{\scriptstyle T}\nearrow \;\Downarrow\mu\; \Downarrow\mu\; {\scriptstyle T} \\
\mathbf{C} \xrightarrow[T]{\quad T \quad} \mathbf{C}
\end{array}
\quad = \quad
\begin{array}{c}
\mathbf{C} \xrightarrow{\;T\;} \mathbf{C} \\
{\scriptstyle T}\nearrow \;\Downarrow\mu\; \Downarrow\mu\; {\scriptstyle T} \\
\mathbf{C} \xrightarrow[T]{\quad T \quad} \mathbf{C}
\end{array}
$$

On the other hand, the second diagram can be rephrased as follows:

$$
\begin{array}{c}
{\scriptstyle \mathrm{id}_{\mathbf{C}}} \quad \mathbf{C} \\
\Downarrow\eta \;{\scriptstyle T}\; \Downarrow\mu \;{\scriptstyle T} \\
\mathbf{C} \xrightarrow[T]{} \mathbf{C}
\end{array}
\;\; = \;\;
T\left(\;=\;\right)T
\;\; = \;\;
\begin{array}{c}
\mathbf{C} \quad {\scriptstyle \mathrm{id}_{\mathbf{C}}} \\
{\scriptstyle T}\nearrow \;\Downarrow\mu\;{\scriptstyle T}\; \Downarrow\eta \\
\mathbf{C} \xrightarrow[T]{} \mathbf{C}
\end{array}
$$

A monad naturally defines other algebraic structures, which we now introduce.

Definition 2.1.7. Given a monad (T, μ, η), a *T-algebra* or *T-module* is a pair (a, α), where $a \in \mathrm{Ob}(\mathbf{C})$ and $\alpha\colon Ta \to a$ is such that the following diagrams commute:

$$
\begin{array}{ccc}
T^2 a & \xrightarrow{\;T\alpha\;} & Ta \\
{\scriptstyle \mu_a}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\
Ta & \xrightarrow[\alpha]{} & a
\end{array}
\qquad\qquad
\begin{array}{ccc}
a & \xrightarrow{\;\eta_a\;} & Ta \\
 & {\scriptstyle \mathrm{id}_a}\searrow & \big\downarrow{\scriptstyle \alpha} \\
 & & a
\end{array}
$$

Definition 2.1.8. A *morphism of T-algebras* $(a, \alpha) \to (b, \beta)$ is a morphism $f\colon a \to b$ such that the following diagram commutes:

$$
\begin{array}{ccc}
Ta & \xrightarrow{\;Tf\;} & Tb \\
{\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\
a & \xrightarrow[f]{} & b
\end{array}
$$

T-algebras form a category T-$\mathbf{Alg}$, which has a natural forgetful functor $U^T\colon T$-$\mathbf{Alg} \to \mathbf{C}$.

We now show how to recover the examples previously given with this language.

Example 2.1.9.

$$\begin{aligned}
T = G \times - \colon\ & \mathbf{Set} \to \mathbf{Set} \\
\mu_A\colon\ & G \times (G \times A) \to G \times A \\
& (g, (h, a)) \mapsto (gh, a) \\
\eta_A\colon\ & A \to G \times A \\
& a \mapsto (e, a)
\end{aligned}$$

is a monad and (A, α) is a T-algebra if and only if A is a G-set. It follows that T-$\mathbf{Alg} \cong G$-$\mathbf{Set}$.

Example 2.1.10. Given a ring R, $T = R \otimes_{\mathbb{Z}} - \colon \mathbf{Ab} \to \mathbf{Ab}$ is a monad when considered with the following natural transformations:

$$\begin{aligned}
\mu_-\colon\ & R \otimes_{\mathbb{Z}} (R \otimes_{\mathbb{Z}} -) \cong (R \otimes_{\mathbb{Z}}) \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \\
\eta_-\colon\ & - \cong \mathbb{Z} \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} -
\end{aligned}$$

We have that $(R \otimes_{\mathbb{Z}} -)$-$\mathbf{Alg} \cong \mathbf{Mod}_R$.

Example 2.1.11. Consider $W\colon \mathbf{Set} \to \mathbf{Set}$ given by $WX = \amalg_{n\in\mathbb{N}}\Pi_{k=1}^n X$. Multiplication $\mu_X\colon WWX \to WX$ is given by concatenation of words, while the unit $\eta_X\colon X \to WX$ is just $x \mapsto (x)$. With this, W-$\mathbf{Alg} \cong \mathrm{Mon}(\mathbf{Set})$.

Example 2.1.12. The functor $\mathcal{U}$ defined in Example 2.1.4, equipped with suitable natural transformations, is a monad on $\mathbf{Set}$ and $\mathcal{U}$-$\mathbf{Alg} \cong \mathbf{CHTop}$, the category of compact T2 spaces.

Example 2.1.13. The free-forgetful adjunction $F \dashv U$ between categories and directed graphs induces a monad on the latter, with UF-$\mathbf{Alg} \cong \mathbf{Cat}$.

2.2 Monadic functors

Now that we have introduced these structures, our aim is to characterize *monadic functors*, that is functors $U\colon \mathbf{A} \to \mathbf{C}$ which are equivalent to $U^T\colon T$-$\mathbf{Alg} \to \mathbf{C}$ for some monad (T, μ, η) on $\mathbf{C}$.

First of all, notice that U^T is faithful by construction, hence U must be faithful, but more is true.

Lemma 2.2.1. The functor U^T is conservative, that is if $U^T f$ is an isomorphism then f is an isomorphism of T-algebras.

Proof. Suppose that g is the inverse of $f\colon a \to b$ and f is a morphism $(a, \alpha) \to (b, \beta)$. We only need to prove that the square on the left commutes, that is $g\beta = \alpha Tg$:

$$Tb \xrightarrow{Tg} Ta \xrightarrow{Tf} Tb$$
$$\beta\downarrow \qquad \alpha\downarrow \qquad \qquad \downarrow\beta$$
$$b \xrightarrow{g} a \xrightarrow{f} b$$

We see that $fg\beta = \beta$ and $f\alpha Tg = \beta TfTg = \beta T(fg) = \beta T\,\mathrm{id}_b = \beta$, hence the thesis. $\qquad\square$

Remark 2.2.2. Notice that the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ can't be monadic since it does not reflect isomorphisms. However, if we restrict it to the full subcategory of $\mathbf{Top}$ spanned by compact Hausdorff spaces we indeed obtain a monadic functor.

Proposition 2.2.3. The functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ has a left adjoint $F^T\colon \mathbf{C} \to T\text{-}\mathbf{Alg}$ such that $F^T c = (Tc, \mu_c)$, $F^T f\colon (Tc, \mu_c) \xrightarrow{Tf} (Td, \mu_d)$ and $U^T F^T = T$. Furthermore, the unit of this adjunction is given by $\gamma_c = \eta_c\colon c \to U^T F^T c = Tc$ and the counit has components $\epsilon_{(a,\alpha)} = \alpha\colon (Ta, \mu_a) \to (a, \alpha)$.

Proof.   (i) To show that (Tc, μ_c) is a T-algebra we need the following diagrams to be commutative.

$$T^3c \xrightarrow{T\mu_c} T^2c \qquad\qquad Tc \xrightarrow{\eta_{Tc}} T^2c$$
$$\mu_{Tc}\downarrow \qquad\quad \downarrow\mu_c \qquad\qquad\qquad\qquad \downarrow\mu_c$$
$$T^2c \xrightarrow{\mu_c} Tc \qquad\qquad\qquad\qquad Tc$$

These are exactly the associativity and one of the unit laws for (T, μ, η).

(ii) For every $f\colon c \to c'$, Tf is a morphism of algebras $(Tc, \mu_c) \to (Tc', \mu_{c'})$. The diagram

$$T^2c \xrightarrow{T^2f} T^2c'$$
$$\mu_c\downarrow \qquad\qquad \downarrow\mu_{c'}$$
$$Tc \xrightarrow{Tf} Tc'$$

is commutative because of the naturality of μ. Hence F^T is defined on morphisms. It is a functor by the functoriality of T.

(iii) The unit is natural by assumption. We claim that $\epsilon_{(a,\alpha)} = \alpha$ is a morphism of algebras

$$F^T U^T(a, \alpha) = F^T a = (Ta, \mu_a) \to \mathrm{id}_{T\text{-}\mathbf{Alg}}(a, \alpha) = (a, \alpha)$$

and ϵ is a natural transformation $F^T U^T \Rightarrow \mathrm{id}_{T\text{-}\mathbf{Alg}}$. Let's check it. We know that α is a morphism of algebras if and only if

$$
\begin{array}{ccc}
T^2 a & \xrightarrow{T\alpha} & Ta \\
{\scriptstyle \mu_a}\downarrow & & \downarrow{\scriptstyle \alpha} \\
Ta & \xrightarrow{\alpha} & a
\end{array}
$$

is commutative. But this is one of the two T-algebra axioms! Moreover, to prove that ϵ is natural, we need to show that

$$
\begin{array}{ccc}
(Ta, \mu_a) & \xrightarrow{\alpha = \epsilon_{(a,\alpha)}} & (a, \alpha) \\
{\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle f} \\
(Tb, \mu_b) & \xrightarrow[\beta = \epsilon_{(b,\beta)}]{} & (b, \beta)
\end{array}
$$

is commutative, but this is the axiom for f to be a morphism of T-algebras!

(iv) It remains to check the two triangular identities $\epsilon F^T \cdot F^T \eta = \mathrm{id}_{F^T}$ and $U^T \epsilon \cdot \eta U^T = \mathrm{id}_{U^T}$. These are to be checked on the components at c and (a, α), respectively.

$$
\begin{array}{ccc}
(Tc, \mu_c) & \xrightarrow{T\eta_c} & (T^2 c, \mu_{Tc}) \\
& \diagdown\diagdown & \downarrow{\scriptstyle \mu_{Tc}} \\
& & (Tc, \mu_c)
\end{array}
\qquad
\begin{array}{ccc}
a & \xrightarrow{\eta_a} & Ta \\
& \diagdown\diagdown & \downarrow{\scriptstyle \alpha} \\
& & a
\end{array}
$$

The commutativity of these diagrams is ensured by the second unit law for a monad and the unit law for the T-algebra (a, α), respectively. $\quad\square$

Definition 2.2.4. Algebras of the form (Tc, μ_c) are called *free T-algebras*.

Thanks to the proposition above we can prove that, given a monad T we can always find an adjunction that generates it. Actually, the converse holds too.

Proposition 2.2.5. If $U\colon \mathbf{D} \to \mathbf{C}$ has a left adjoint F with unit η and counit ϵ, then $(UF, U\epsilon F, \eta)$ is a monad on $\mathbf{C}$. Also, if (T, μ, η) is a monad on $\mathbf{C}$, then $(U^T F^T, U^T \epsilon F^T, \eta) = (T, \mu, \eta)$.

Proof. Let us check the axioms. First of all, the associativity holds due to the following equations.

$$
\begin{array}{c}
\mathbf{C} \\
U\nearrow \;{\scriptstyle \epsilon\Downarrow}\; \searrow F \\
\mathbf{C} \xrightarrow{F} \mathbf{D} \Longrightarrow \mathbf{D} \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{C}
\end{array}
\qquad
\begin{array}{c}
\mathbf{C} \\
U\nearrow \;{\scriptstyle \epsilon\Downarrow}\; \searrow F \\
\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D} \Longrightarrow \mathbf{D} \xrightarrow{U} \mathbf{C}
\end{array}
$$

$$=$$

$$\mathbf{C} \xrightarrow{F} \mathbf{D} =\!=\!=\!=\!= \mathbf{D} =\!=\!=\!=\!= \mathbf{D} \xrightarrow{U} \mathbf{C}$$

(with the $\mathbf{C}$-apex triangles $U,\ \epsilon\!\Downarrow,\ F$ over each $=\!=$ segment)

$$=$$

$$\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D} =\!=\!=\!= \mathbf{D} \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D} =\!=\!=\!= \mathbf{D} \xrightarrow{U} \mathbf{C}$$

(with the $\mathbf{C}$-apex triangles $U,\ \epsilon\!\Downarrow,\ F$ over each $=\!=$ segment)

Unit laws:

$$\mathbf{C} \xrightarrow{F} \mathbf{D} =\!=\!=\!= \mathbf{D} \xrightarrow{U} \mathbf{C}$$

(with $\eta\!\Downarrow$, U, $\epsilon\!\Downarrow$, F over the apex $\mathbf{C}$)

is equal to 1_{UF}, since $\epsilon F \cdot F\eta = 1_F$ by one of the triangular identities of the adjunction $F \dashv U$. Furthermore,

$$\mathbf{C} \xrightarrow{F} \mathbf{D} =\!=\!=\!= \mathbf{D} \xrightarrow{U} \mathbf{C}$$

(with U, $\epsilon\!\Downarrow$, F, $\eta\!\Downarrow$ over the apex $\mathbf{C}$)

is equal to 1_{UF}. This follows from the explicit description of the unit and the counit of the adjunction $F^T \dashv U^T$, in fact

$$U^T \epsilon F^T c = U^T \epsilon_{(Tc,\eta_c)} = \mu_c \qquad\qquad \square$$

Example 2.2.6 (Interesting adjunction, boring monad)**.** Let us consider the adjunction $U\colon \mathbf{Top} \rightleftarrows \mathbf{Set} \colon \mathsf{Disc} =: F$, whose left adjoint assigns to every set X the discrete topological space $FX = (X, 2^X)$. It's immediate to see that $UFX = X$, hence $UF = \mathrm{id}_{\mathbf{Set}}$. How many natural transformations $\mathrm{id}_{\mathbf{Set}} = UF \overset{\alpha}{\Rightarrow} UF = \mathrm{id}_{\mathbf{Set}}$ are there? We know that $\mathrm{id}_{\mathbf{Set}} \cong \mathrm{Hom}(*, -)$, so $\mathrm{Nat}(\mathrm{id}_{\mathbf{Set}}, \mathrm{id}_{\mathbf{Set}}) \cong \mathrm{Nat}(\mathrm{Hom}(*, -), \mathrm{Hom}(*, -)) \cong \mathrm{Hom}(*, *) = \{\mathrm{id}_*\}$ by Yoneda, hence $\alpha = \mathrm{id}$. Therefore $(UF, U\epsilon F, \eta) = (\mathrm{id}_{\mathbf{Set}}, \mathrm{id}, \mathrm{id})$

Example 2.2.7. If S is a set, $\mathbf{Set}(S, -)\colon \mathbf{Set} \to \mathbf{Set}$ is right adjoint to $S \times -\colon \mathbf{Set} \to \mathbf{Set}$, so we get a monad $X \mapsto \mathbf{Set}(S, S \times X)$. This is called *the state monad* and is important in Computer Science.

There is always a comparison morphism $\mathbf{D} \xrightarrow{\overline{U}} UF\text{-}\mathbf{Alg}$ s.t.

$$\mathbf{D} \xrightarrow{\ \overline{U}\ } UF\text{-}\mathbf{Alg}$$

(with U down-left to $\mathbf{C}$ and U^{UF} down-right to $\mathbf{C}$)

commutes. We set $\overline{U}f = (Ud, UFUd \xrightarrow{U\epsilon_d} Ud) = (Ud, U\epsilon_d)$. More specifically, for a given functor $G: \mathbf{D} \to \mathbf{C}$ we can ask what do we need to get an equivalence $\overline{G}: \mathbf{D} \to T\text{-}\mathbf{Alg}$. To get there, we will need a few more definitions and lemmas.

2.3 The category of T-actions

Just like a monad (T, μ, η) defines a category $T\text{-}\mathbf{Alg}$, it also allows us to construct another category from functors $\mathbf{D} \to \mathbf{C}$.

Definition 2.3.1. Given a monad (T, μ, η) on a category $\mathbf{C}$ and fixed another category $\mathbf{D}$, a T-*action* on a functor $G: \mathbf{D} \to \mathbf{C}$ is a natural transformation $\gamma: TG \Rightarrow G$ such that the diagrams

$$
\begin{array}{ccc}
T^2G & \xRightarrow{T\gamma} & TG \\
{\scriptstyle \mu G}\Big\Vert & & \Big\Vert{\scriptstyle \gamma} \\
TG & \xRightarrow[\gamma]{} & G
\end{array}
\qquad
\begin{array}{ccc}
G & \xRightarrow{\eta G} & TG \\
& \searrow & \Big\Vert{\scriptstyle \gamma} \\
& & G
\end{array}
$$

commute.

A *morphism of T-actions* $(G, \gamma) \xRightarrow{\varphi} (K, \kappa)$ is a natural transformation $\varphi: G \Rightarrow K$ such that

$$
\begin{array}{ccc}
TG & \xRightarrow{T\varphi} & TK \\
{\scriptstyle \gamma}\Big\Vert & & \Big\Vert{\scriptstyle \kappa} \\
G & \xRightarrow[\varphi]{} & K
\end{array}
$$

commutes.

Up to size, T-actions and their morphisms assemble into a category $T\text{-}\mathsf{Act}(\mathbf{D})$.

Example 2.3.2. The functor $U^T: T\text{-}\mathbf{Alg} \to \mathbf{C}$ has a T-action given by $(U^T, \alpha: TU^T \Rightarrow U^T)$, where $\alpha_{(b,\beta)} \coloneqq \beta: Tb \to b$.

Example 2.3.3. Given an adjunction $F \dashv U: \mathbf{C} \rightleftarrows \mathbf{D}$ with unit $\eta: \mathrm{id}_{\mathbf{C}} \Rightarrow UF$ and counit $\epsilon: FU \Rightarrow \mathrm{id}_{\mathbf{D}}$, we get a monad on $(UF, U\epsilon F, \eta)$ on $\mathbf{C}$. We have then a UF-action $U\epsilon: UFU \Rightarrow U$, where the axioms follow from the triangular identities and the naturality of $U\epsilon$.

Proposition 2.3.4. (U^T, α) is the universal T-action, that is for any category $\mathbf{D}$ the functor $\mathbf{Cat}(\mathbf{D}, T\text{-}\mathbf{Alg}) \to T\text{-}\mathsf{Act}(\mathbf{D})$ sending G to $(U^TG, \alpha G)$ and $\beta: G \Rightarrow H$ to $U^T\beta: (U^TG, \alpha G) \Rightarrow (U^TH, \alpha H)$ is an isomorphism of categories.

Proof. In other words, for every T-action (G, γ) there exists a unique lift $\overline{G}: \mathbf{D} \to T\text{-}\mathbf{Alg}$ such that $(U^T\overline{G}, \alpha\overline{G}) = (G, \gamma)$ and for every $\phi: (G, \gamma) \Rightarrow (K, \kappa)$ there is a unique $\overline{\phi}: \overline{G} \Rightarrow \overline{K}$ with $U^T\overline{\phi} = \phi$.

It is enough to set $\overline{G}d := (Gd, \gamma_d)$ on objects, $\overline{G}f := Gf$ on morphisms, $\overline{\phi}_d := \phi_d$ and check the axioms.

$$
\begin{array}{ccc}
\mathbf{D} & \xdashrightarrow{\ \exists!\overline{G}\ } & T\text{-}\mathbf{Alg} \\
& \searrow{\scriptstyle G} & \big\downarrow{\scriptstyle U^T} \\
& & \mathbf{C}
\end{array}
$$

$\square$

Following the construction in this proof, from the last example we get the comparison functor for the adjunction $F \dashv U$. In particular, $\overline{U}d = (Ud, U\epsilon_d)$. Furthermore, this means that $U\colon \mathbf{Top} \to \mathbf{Set}$ factors through identities.

2.4 Limits and colimits in the category of algebras

We have shown that the forgetful functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ is a right adjoint, and as such it preserves limits. However, more is true.

Proposition 2.4.1. For any monad (T, μ, η) on $\mathbf{C}$, the forgetful functor $U^T\colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ strictly creates limits.

Proof. This statement means that, for any diagram $D\colon I \to T\text{-}\mathbf{Alg}$ such that $U^T D\colon I \to \mathbf{C}$ has a limit (l, κ_i) in $\mathbf{C}$, there is a unique T-algebra structure $\lambda\colon Tl \to l$ such that κ_i is a morphism of T-algebras for all $i \in I$ and this makes $((l, \lambda), \kappa_i)$ into a limit of D.

Now we begin the proof.

First of all, remember that $D\phi\colon D_i \to D_j$ is a morphism of T-algebras for all $\phi\colon i \to j$ by assumption, hence the morphisms $\delta_i T\kappa_i\colon Tl \to D_i$ define a cone over D, where δ_i is the T-algebra structure on D_i. By the universal property of the limit, there is a unique morphism $\lambda\colon Tl \to l$ making the following diagram commute for all i.

$$
\begin{array}{ccc}
Tl & \xrightarrow{\ T\kappa_i\ } & TD_i \\
{\scriptstyle \lambda}\big\downarrow & & \big\downarrow{\scriptstyle \delta_i} \\
l & \xrightarrow[\ \kappa_i\ ]{} & D_i
\end{array}
$$

This tells us that, if the limit $((l, \lambda), \kappa_i)$ of D exists, it is unique. We have to check that (l, λ) is a T-algebra.

Notice that for all i all of the faces of the following diagrams, except for possibly the top ones, commute:

Since the κ_i are jointly monic, the upper face commutes and therefore (l, λ) is a T-algebra. It remains to check that $((l, \lambda), \kappa_i)$ factors every other cone over D.

Let $\gamma_i \colon (x, \zeta) \to (D_i, \delta_i)$ be a cone over D. Then, there is a unique $f \colon x \to l$ in $\mathbf{C}$ such that $\kappa_i f = \gamma_i$. We only have to show that f is a morphism of T-algebras $(x, \zeta) \to (l, \lambda)$.

Consider the following diagram and notice that the outer square, the one on the right and the two triangles commute, hence the square on the left commutes as well since the κ_i are jointly monic.

$\square$

A similar statement holds for colimits.

Proposition 2.4.2. Given a monad (T, μ, η) on $\mathbf{C}$, the forgetful functor $U^T \colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ strictly creates any colimit preserved by both T and T^2.

Proof. Similarly to the dual situation, this means that for any diagram $D \colon I \to T\text{-}\mathbf{Alg}$ such that $U^T D \colon I \to \mathbf{C}$ has a colimit (c, κ_i) preserved by both T and T^2, there is a unique T-algebra structure $\lambda \colon Tc \to c$ such

that κ_i is a morphism of T-algebras for all $i \in I$. This makes $((c, \lambda), \kappa_i)$ into a colimit of D.

The proof is essentially dual to the one given earlier, in the sense that we find again a unique $\lambda\colon Tc \to c$ using the universal property of the colimit $(Tc, T\kappa_i)$ of TD.

$$
\begin{array}{ccc}
TD_i & \xrightarrow{\ T\kappa_i\ } & Tc \\
{\scriptstyle \delta_i}\downarrow & & \downarrow{\scriptstyle \lambda} \\
D_i & \xrightarrow[\ \kappa_i\ ]{} & c
\end{array}
$$

To check that (c, λ) is an algebra we use the universal property of $(T^2 c, T^2 \kappa_i)$, for μ, and the one of (c, κ_i), for η. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.4.3. The same statements hold for monadic functors, except for the fact that they might not create limits and colimits strictly since they are just equivalent to a U^T.

Remark 2.4.4. If T is a monad on a complete category $\mathbf{C}$, then T-$\mathbf{Alg}$ is complete. If $\mathbf{C}$ is cocomplete and T is cocontinuous, then T-$\mathbf{Alg}$ is cocomplete.

Example 2.4.5. Let $\mathbf{C}$ be a small category. There is a cocontinuous monad on the category of $\mathrm{Ob}(\mathbf{C})$-indexed collections of sets whose category of algebras is the functor category $[\mathbf{C}, \mathbf{Set}]$. The underlying endofunctor of this monad is defined as

$$
T\colon [\mathrm{Ob}(\mathbf{C}), \mathbf{Set}] \to [\mathrm{Ob}(\mathbf{C}), \mathbf{Set}]
$$

$$
(X_c)_{c \in \mathbf{C}} \mapsto \left(\coprod_{d \in \mathbf{C}} \mathbf{C}(d, c) \times X_d \right)_{c \in \mathbf{C}}
$$

Since $[\mathrm{Ob}(\mathbf{C}), \mathbf{Set}]$ is complete and cocomplete, so is $[\mathbf{C}, \mathbf{Set}]$ (with limits and colimits computed pointwise).

Chapter 3

Beck's monadicity theorem

The final ingredient we need is the observation that T-algebras admit canonical presentations using free algebras.

Example 3.0.1. Pick an epi $F \twoheadrightarrow G$ in the category of groups **Grp**, where F is a free group. The kernel of this homomorphism defines a (normal) subgroup K of F, giving rise to the sequence $K \rightarrowtail F \twoheadrightarrow G$. We can take another epi $F' \twoheadrightarrow K$, with F' again a free group. Therefore G is the cokernel of some morphism $F' \to F$. This argument applies to rings, algebras etc.

It is natural to ask if we can do this systematically for general T-algebras. Given (a, α) in T-**Alg**, we have $F^T U^T (a, \alpha) \to (a, \alpha)$ i.e. $(Ta, \mu_a) \xrightarrow{\alpha} (a, \alpha)$. A candidate[1] for F' would be $F^T U^T (Ta, \mu_a) = (T^2 a, \mu_{Ta})$. What are the "elements" of Ta? Notice that

$$(T^2 a, \mu_{Ta}) \underset{\mu_a}{\overset{T\alpha}{\rightrightarrows}} (Ta, \mu_a) \xrightarrow{\alpha} (a, \alpha)$$

is a well defined diagram in T-**Alg**, with $\alpha\mu_a = \alpha T\alpha$. Moreover, this is a coequalizer. We can use Proposition 2.4.2 to prove it, so that we need to check whether U^T sends the diagram above into a coequalizer preserved by T and T^2. In **C**, we get the diagram

$$T^2 a \underset{\mu_a}{\overset{T\alpha}{\rightrightarrows}} Ta \xrightarrow{\alpha} a$$

in which the following equations hold true: $\alpha\eta_a = \mathrm{id}_a$, $\mu_a \eta_{Ta} = \mathrm{id}_{Ta}$ and $\eta_a \alpha = T\alpha\, \eta_{Ta}$ by naturality. It is a particular case of a more general concept.

Definition 3.0.2. A *split coequalizer* is a diagram of the form

$$a \underset{g}{\overset{f}{\rightrightarrows}} b \xrightarrow{h} c$$

so that $hf = hg$, $hs = \mathrm{id}_c$, $gt = \mathrm{id}_b$, and $ft = sh$.

---

[1] Think about free groups: in that case we take words on Ta.

Proposition 3.0.3. In the above situation,

$$a \underset{g}{\overset{f}{\rightrightarrows}} b \xrightarrow{h} c$$

is a coequalizer. In particular, any functor preserves this coequalizer.

Proof. Take $k \colon b \to d$ such that $kf = kg$ and define $\overline{k} := ks$. Then we have

$$\overline{k}h = ksh = kft = kgt = k.$$

Uniqueness is clear since h is a (split) epi. $\square$

T and T^2 preserve split coequalizers, so they preserve our coequalizer.

Corollary 3.0.4. Let T be a monad on $\mathbf{C}$ and (a, α) a T-algebra. Then

$$(T^2 a, \mu_{Ta}) \underset{\mu_a}{\overset{T\alpha}{\rightrightarrows}} (Ta, \mu_a) \xrightarrow{\alpha} (a, \alpha)$$

is a coequalizer in $T\text{-}\mathbf{Alg}$, which $U^T \colon T\text{-}\mathbf{Alg} \to \mathbf{C}$ sends to a split coequalizer in $\mathbf{C}$.

Proof. We have already observed that the second statement holds, so that $\mathrm{coeq}(U^T(T\alpha), U^T(\mu_a))$ is preserved by T and T^2. Hence there exists a unique lift of the (split) coequalizer in $\mathbf{C}$ to a coequalizer in $T\text{-}\mathbf{Alg}$. $\square$

Results like the previous one inspire us to look at the parallel pairs of morphisms in a category which are sent to split coequalizers or, to say it better, to a parallel pair of morphisms that can be extended to a split coequalizer diagram. Those kinds of pairs will be of crucial importance in the following.

Definition 3.0.5. Let $U \colon \mathbf{D} \to \mathbf{C}$ be a functor. A pair of morphisms $f, g \colon d \rightrightarrows d'$ in $\mathbf{D}$ is U-*split* if $Uf, Ug \colon Ud \rightrightarrows Ud'$ is part of a split coequalizer in $\mathbf{C}$.

Remark 3.0.6. $T\alpha, \mu_a \colon (T^2 a, \mu_{Ta}) \rightrightarrows (Ta, \mu_a)$ is a U^T-split pair. Moreover, $T\text{-}\mathbf{Alg}$ has coequalizers of U^T-split pairs and U^T preserves them. Hence, functors equivalent to U^T satisfy three conditions:

1. they have a left adjoint;

2. they are conservative;

3. U-split pairs have coequalizers which are preserved by U.

Theorem 1 (Beck). Let $U\colon \mathbf{D} \to \mathbf{C}$ be a right adjoint to $F\colon \mathbf{C} \to \mathbf{D}$. Let $(T = UF, U\epsilon F, \eta)$ be the induced monad and $\overline{U}\colon \mathbf{D} \to T\text{-}\mathbf{Alg}$ be the comparison functor.

1. If $\mathbf{D}$ has coequalizers of U-split pairs, then $\overline{U}$ has a left adjoint $\overline{F}\colon T\text{-}\mathbf{Alg} \to \mathbf{D}$;

2. if, in addition, U preserves coequalizers of U-split pairs, the unit $\overline{\eta}\colon \mathrm{id}_{T\text{-}\mathbf{Alg}} \Rightarrow \overline{U}\,\overline{F}$ is an isomorphism;

3. if U is also conservative, then $\overline{U}$ is an equivalence of categories.

Proof.   1. For each free T-algebra (Ta, μ_a) we have

$$T\text{-}\mathbf{Alg}((Ta, \mu_a), \overline{U}-) = T\text{-}\mathbf{Alg}(F^T a, \overline{U}-)$$
$$\cong \mathbf{C}(a, U^T \overline{U}-)$$
$$= \mathbf{C}(a, U-)$$
$$\cong \mathbf{D}(Fa, -)$$

therefore the value of $\overline{F}$ at (Ta, μ_a) has to be Fa. Since every T-algebra is a coequalizer of free algebras which is preserved by U^T, we may define $\overline{F}(a, \alpha)$ as the coequalizer of a pair of morphisms $FTa \rightrightarrows Fa$. We write this as $FUFU^T(a, \alpha) \rightrightarrows FU^T(a, \alpha)$. Consider the following pair of morphisms of functors

$$FUFU^T \xrightarrow[\epsilon FU^T]{F\alpha} FU^T$$

in the functor category $[T\text{-}\mathbf{Alg}, \mathbf{D}]$. We claim that this pair has a coequalizer and $\overline{F}\colon T\text{-}\mathbf{Alg} \to \mathbf{D}$ is left adjoint to $\overline{U}$. Note that the pair of morphisms just above becomes split after the composition with $U\colon \mathbf{D} \to \mathbf{C}$. In fact

$$UFUFU^T \xrightarrow[U\epsilon FU^T]{UF\alpha} UFU^T \xrightarrow{\alpha} U^T$$

with ηUFU^T and ηU^T.

is a split coequalizer in $[T\text{-}\mathbf{Alg}, \mathbf{C}]$, given that it holds pointwise since $UF = T$. Let us denote by $\beta\colon FU^T \to \overline{F}$ the colimit (computed pointwise) of the pair $F\alpha, \epsilon FU^T\colon FUFU^T \rightrightarrows FU^T$. Precomposing this pair with $\overline{U}$ and recalling that $\alpha\overline{U} = U\epsilon$, $U^T\overline{U} = U$, we get the pair

$$FUFU \xrightarrow[\epsilon FU]{FU\epsilon} FU,$$

which is coequalized by $\epsilon\colon FU \Rightarrow \mathrm{id}_{\mathbf{D}}$.

$$FUFU \underset{\epsilon FU}{\overset{FU\epsilon}{\rightrightarrows}} FU \xrightarrow{\beta\overline{U}} \overline{F}\,\overline{U}$$

with ϵ going to $\mathrm{id}_{\mathbf{D}}$ and $\exists!\overline{\epsilon}$

Since $\overline{F}\,\overline{U}$ is the coequalizer of the diagram above, there exists a unique $\overline{\epsilon}\colon \overline{F}\,\overline{U} \Rightarrow \mathrm{id}_{\mathbf{D}}$ such that $\overline{\epsilon} \cdot \beta\overline{U} = \epsilon$. To get the unit $\overline{\eta}\colon \mathrm{id}_{T\text{-}\mathbf{Alg}} \Rightarrow \overline{U}\,\overline{F}$ we need to describe a morphism of T-actions $(U^T, \alpha) \to (U^T\overline{U}\,\overline{F}, \alpha\overline{U}\,\overline{F})$. We claim that the natural transformation induced by the universal property of the split coequalizer in the first row

$$\begin{array}{ccccc}
UFUFU^T & \overset{UF\alpha}{\underset{U\epsilon FU^T}{\rightrightarrows}} & UFU^T & \xrightarrow{\alpha} & U^T \\
\| & & \| & & \Big\downarrow \exists!\overline{\eta} \\
U^T\overline{U}FUFU^T & \overset{U^T\overline{U}F\alpha}{\underset{U^T\overline{U}\epsilon FU^T}{\rightrightarrows}} & U^T\overline{U}FU^T & \xrightarrow{U^T\overline{U}\beta} & U^T\overline{U}\,\overline{F}
\end{array}$$

is a morphism of T-actions[2].

Unraveling what this means, we have to check that the diagram

$$\begin{array}{ccc}
UFa & \xrightarrow{UF\overline{\eta}_{(a,\alpha)}} & UFU\overline{F}(a,\alpha) \\
\Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle U\epsilon_{\overline{F}(a,\alpha)}} \\
a & \xrightarrow{\overline{\eta}_{(a,\alpha)}} & U\overline{F}(a,\alpha)
\end{array}$$

is commutative. We know that $\overline{\eta}\alpha = U\beta$ by the definition of $\overline{\eta}$. Moreover, α is a split epi in $\mathbf{C}$, hence we can precompose with $UF\alpha$ (again a split epi) and check the commutativity of the resulting diagram. We get the diagram

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[2]In fact, this tells us that the morphism $\overline{\eta}_{(a,\alpha)}\colon a \to U^T\overline{U}\,\overline{F}(a,\alpha)$ in $\mathbf{C}$ lifts uniquely to a morphism of T-algebras $\overline{\eta}_{(a,\alpha)}\colon (a,\alpha) \to \overline{U}\,\overline{F}(a,\alpha)$.

The definition of β as a coequalizer implies that $\beta_{(a,\alpha)}F\alpha = \beta_{(a,\alpha)}\epsilon_{Fa}$, so we get the natural transformation $\overline{\eta}\colon \mathrm{id}_{T\text{-}\mathbf{Alg}} \Rightarrow \overline{U}\,\overline{F}$. The only thing left to do is checking the triangular identities, which is left to the reader.

2. If U preserves coequalizers of U-split pairs, both $U\overline{F}$ and U^T are coequalizers of the above diagram, hence $\overline{\eta}$ is an isomorphism.

3. From the triangular identities, $\overline{U}\overline{\epsilon} \cdot \overline{\eta}\overline{U} = \mathrm{id}_{\overline{U}}$, hence $\overline{U}\overline{\epsilon}$ is an isomorphism. Being $U^T\overline{U} = U$ conservative, $\overline{\epsilon}$ is an isomorphism as well. $\qquad\square$

Definition 3.0.7. A pair $f, g\colon c \rightrightarrows d$ in a category $\mathbf{C}$ is *reflexive* if there exists a common section $i\colon d \to c$, that is $fi = gi = \mathrm{id}_d$.

A coequalizer of a reflexive pair is a *reflexive coequalizer*.

Remark 3.0.8. To give a cone of a reflexive pair it is enough to give a map $h\colon d \to x$ such that $hf = hg$, hence $\mathrm{colim}(c \rightrightarrows d) \cong \mathrm{colim}(c \to d)$.

Proposition 3.0.9. In Beck's monadicity theorem it suffices for (1) that coequalizers of reflexive U-split pairs exist, while in (2) and (3) we only need for them to be preserved.

Proof. The pair

$$FUFU^T \xrightarrow[\epsilon FU^T]{F\alpha} FU^T$$

has $F\eta U^T$ as common section. In fact, $\alpha \cdot \eta U^T = \mathrm{id}_{U^T}$ by the unit law of the T-action $\alpha\colon TU^T \Rightarrow U^T$ and $\epsilon F \cdot F\eta = \mathrm{id}_F$ by the triangular identities. $\qquad\square$

Example 3.0.10. Let $\mathbf{A}$ and $\mathbf{B}$ be small categories, $\mathbf{C}$ a category which is both complete and cocomplete, and $G\colon \mathbf{A} \to \mathbf{B}$ a functor. The restriction along G, G^*, has both adjoints, given by left and right Kan extensions. Notice that the induced monad is cocontinuous since G^* is a left adjoint. Moreover, G^* is conservative if G is essentially surjective, thus any essentially surjective functor G induces a monadic adjunction as follows:

$$
\begin{array}{ccc}
[\mathbf{B}, \mathbf{C}] & \xrightarrow{\;\overline{G^*}\;} & G^*\mathrm{Lan}_G\text{-}\mathbf{Alg} \\
& \overset{\sim}{\Longrightarrow} & \\
{\scriptstyle G^*}\searrow & & \swarrow \\
& [\mathbf{A}, \mathbf{C}] &
\end{array}
$$

We are going to show why reflexive coequalizers are useful, but first we need some preliminary definitions and results.

Definition 3.0.11. A functor $F\colon \mathbf{I} \to \mathbf{J}$ between small categories is called *final* if for any diagram $D\colon \mathbf{J} \to \mathbf{C}$ the comparison morphism $\mathrm{colim}_{\mathbf{I}} DF \to \mathrm{colim}_{\mathbf{J}} D$ is an isomorphism whenever both colimits exist.

Proposition 3.0.12. Let $F\colon \mathbf{I} \to \mathbf{J}$ be a functor between small categories. The following are equivalent:

(i) F is final;

(ii) the unique isomorphism

$$
\begin{array}{ccc}
\mathbf{I}^{\mathrm{op}} & \xrightarrow{\;F^{\mathrm{op}}\;} & \mathbf{J}^{\mathrm{op}} \\
{\scriptstyle *}\searrow & \stackrel{\sim}{\Longrightarrow} & \swarrow{\scriptstyle *} \\
& \mathbf{Set} &
\end{array}
$$

exibits $*$ as $\mathrm{Lan}_{F^{\mathrm{op}}} *$;

(iii) for each $j \in \mathbf{J}$, the category $(j \downarrow F)$ is connected.

Proof. (ii) $\Longleftrightarrow$ (iii) We have $\mathrm{Lan}_{F^{\mathrm{op}}} *(j) \cong \mathrm{colim}_{(j\downarrow F)} *$ by the formula for Kan extensions. A colimit of $(j \downarrow F) \to \mathbf{Set}$, $(\phi, j') \mapsto *$ is terminal if and only if $(j \downarrow F)$ is connected, hence the thesis.

(ii) $\Longrightarrow$ (i) Let $D\colon \mathbf{J} \to \mathbf{C}$ be a diagram. We can then write $\mathrm{Cocone}(D, -)$ as follows:

$$\mathrm{Cocone}(D, X) \cong \mathrm{Nat}(*, \mathbf{C}(D-, X)) \cong [\mathbf{J}^{\mathrm{op}}, \mathbf{Set}](*, \mathbf{C}(D-, X))$$

By definition of left Kan extension, we also have

$$[I^{\mathrm{op}}, \mathbf{Set}(*, \mathbf{C}(DF-, X))] \cong [\mathbf{J}^{\mathrm{op}}, \mathbf{Set}](\mathrm{Lan}_{F^{\mathrm{op}}} *, \mathbf{C}(D-, X))$$

If $\mathrm{Lan}_{F^{\mathrm{op}}} * \cong *$, this shows that $\mathrm{colim}_{\mathbf{I}} DF \cong \mathrm{colim}_{\mathbf{J}} D$.

(i) $\Longrightarrow$ (iii) Left as an exercise.

$\square$

Definition 3.0.13. A small category $\mathbf{I}$ is *sifted* if the diagonal $\Delta\colon \mathbf{I} \to \mathbf{I} \times \mathbf{I}$ is final. A colimit is sifted if the domain category is.

Example 3.0.14. For any filtered category $\mathbf{I}$, the category $((i, i') \downarrow \Delta)$ is again filtered for any $i, i' \in \mathbf{I}$ and hence connected, thus filtered colimits are sifted.

Example 3.0.15. Reflexive coequalizers are sifted. Checking it for yourself may be a tedious yet useful exercise.

However, coequalizers are not sifted. Indeed, their indexing category $\mathbf{I} = \{1 \rightrightarrows 0\}$ is such that $((0, 1) \downarrow \Delta)$ is not connected.

Example 3.0.16. Coproducts and initial objects are not sifted, for their slice categories may be either empty or have several connected components.

Example 3.0.17. Pushouts are not sifted.

Proposition 3.0.18. If $F\colon \mathbf{A}\times\mathbf{B}\to\mathbf{C}$ is a functor preserving reflexive coequalizers in each variable, that is for any $a\in\mathbf{A}$, $b\in\mathbf{B}$ the functors $F(a,-)\colon \mathbf{B}\to\mathbf{C}$ and $F(-,b)\colon \mathbf{A}\to\mathbf{C}$ preserve reflexive coequalizers, then F preserves reflexive coequalizers as well.

Proof. We need to check that, given a reflexive coequalizer $a_0 \rightrightarrows a_1 \longrightarrow a_2$ in $\mathbf{A}$ and $b_0 \rightrightarrows b_1 \longrightarrow b_2$ in $\mathbf{B}$, the diagonal of the following diagram is a coequalizer diagram in $\mathbf{C}$.

$$
\begin{array}{ccc}
F(a_0,b_0) \rightrightarrows F(a_1,b_0) \longrightarrow F(a_2,b_0) \\
F(a_0,b_1) \rightrightarrows F(a_1,b_1) \longrightarrow F(a_2,b_1) \\
F(a_0,b_2) \rightrightarrows F(a_1,b_2) \longrightarrow F(a_2,b_2)
\end{array}
$$

From general facts, $F(a_2,b_2)$ is the colimit of the square in the top left. We may prove this using the sections, however in this case we can use the fact that, under certain hypothesis, given a diagram $D\colon \mathbf{I}\times\mathbf{I}\to\mathbf{C}$ we have $\operatorname{colim}_{\mathbf{I}\times\mathbf{I}} D \cong \operatorname{colim}_{\mathbf{I}} D\Delta$. Specifically, we want this to hold when $\mathbf{I}=\{a_0 \rightrightarrows a_1\}$. But the category $((i,i')\downarrow\Delta)$ is connected for any $i\in\mathbf{I}$, hence Δ is final and we have the thesis. $\square$

Example 3.0.19. The functor $\mathbf{Set}\times\mathbf{Set}\xrightarrow{-\times-}\mathbf{Set}$ satisfies the hypothesis of the theorem since $\mathbf{Set}$ is cartesian closed, hence $X\mapsto X\times X$ preserves reflexive coequalizers by the proposition. This shows that $\mathbf{Set}(X,-)\cong \Pi_{x\in X}\mathbf{Set}(*,-)$ preserves reflexive coequalizers if X is finite, hence the functor $\mathbf{Set}(X,-)\colon \mathbf{Set}\to\mathbf{Set}$ is monadic with $T=\mathbf{Set}(X,X\times-)$ for X finite.

$$
\begin{array}{ccc}
\mathbf{Set} & \xrightarrow{\ \cong\ } & T\text{-}\mathbf{Alg} \\
 {\scriptstyle\mathbf{Set}(X,-)}\searrow & & \swarrow{\scriptstyle U^T} \\
 & \mathbf{Set} &
\end{array}
$$

We also have the following result.

Proposition 3.0.20. If $F\colon \mathbf{B}\times\mathbf{C}\to\mathbf{D}$ is a functor preserving sifted colimits in each variable, then it preserves them as a functor $\mathbf{B}\times\mathbf{C}\to\mathbf{D}$.

Remark 3.0.21. This proposition is false if we drop the siftedness condition, for under the functor $\mathbf{Set}\times\mathbf{Set}\xrightarrow{-\times-}\mathbf{Set}$ we have $(*+*)\times(*+*)\not\cong *+*\cong (*\times *)+(*\times *)$

3.1   Algebraic Theories and Finitary Monads

Many objects in algebra can be described as sets X with some finitary operations $X^n \xrightarrow{m_i} X$ subject to a list of axioms involving the m_i and their products.

Example 3.1.1. (Commutative) monoids and groups, rings and, fixed a ring R, R-modules, where for each $r \in R$ we specify an operation $X \xrightarrow{r} X$.

We can construct categories whose objects are sets paired with operations fullfilling the axioms and functions commuting with the operations as morphisms. These are called models for single-sorted finitary theories.

Proposition 3.1.2. The forgetful functor from a category $\mathbf{C}$ of models of a single-sorted finitary theory to $\mathbf{Set}$ creates sifted colimits.

Proof. Notice that the n-fold product $\mathbf{Set} \xrightarrow{(-)^n} \mathbf{Set}$ factors as $\mathbf{Set} \xrightarrow{\Delta} \mathbf{Set}^n \xrightarrow{- \times \cdots \times -} \mathbf{Set}$, hence it preserves sifted colimits. This gives us unique candidates for operations on the colimit such that the cocone in $\mathbf{Set}$ is a morphism of models. These operations satisfy the axioms because the domain is in each case again of the form colim^n.

$$
\begin{array}{ccc}
X^n \rightrightarrows Y^n \longrightarrow Z^n \\
\downarrow m_i \qquad \downarrow m_i \qquad \vdots \exists! m_i \\
X \rightrightarrows Y \longrightarrow Z
\end{array}
$$

$\square$

Corollary 3.1.3. Let $\mathbf{C}, \mathbf{D}$ be categories of models of single sorted finitary theories. Any functor $U \colon \mathbf{C} \to \mathbf{D}$ which commutes with the forgetful functor to $\mathbf{Set}$ preserves reflexive coequalizers. If U has a left adjoint, then it is monadic.

Proof. The forgetful functors $\mathbf{C} \to \mathbf{Set}$, $\mathbf{D} \to \mathbf{Set}$ are both conservative, hence U is as well. The statement about reflexive coequalizers was just proved and the last claim follows from Beck's theorem. $\square$

Example 3.1.4. The forgetful functor $\mathbf{CAlg}_R \to \mathbf{Mod}_R$ is monadic for any commutative ring R. The same goes for the forgetful functors from $\mathbf{Grp}$, $\mathbf{Mod}_R$ and $\mathbf{Ring}$ to $\mathbf{Set}$.

Definition 3.1.5. A functor $F \colon \mathbf{C} \to \mathbf{D}$ is finitary if it commutes with filtered colimits. A monad is finitary if its underlying endofunctor T is finitary.

Remark 3.1.6. Asking for the underlying endofunctor T to be finitary is equivalent to asking for its forgetful functor U^T to be finitary.

Example 3.1.7. All the forgetful functors from single sorted finitary theories which have a left adjoint (fact: all of them do) come from finitary monads on **Set**.

In general, preserving filtered colimits is a weaker condition than preserving sifted colimits. However, for endofunctors on **Set** the two coincide.

In order to prove this, we need to know how such an endofunctor is determined by its action on finite sets and the idea is to check its behaviour on finite sets, since every set is a directed union of its finite subsets.

We write $[\mathbf{C}, \mathbf{D}]_{\text{fin}}$ for the full subcategory of $[\mathbf{C}, \mathbf{D}]$ whose objects are finitary functors, $\mathbf{Set}_{\text{fin}}$ for the category of finite sets.

Theorem 3.1.8. Restriction along the inclusion $K \colon \mathbf{Set}_{\text{fin}} \to \mathbf{Set}$ induces an equivalence of $[\mathbf{Set}, \mathbf{Set}]_{\text{fin}} \to [\mathbf{Set}_{\text{fin}}, \mathbf{Set}]$ whose inverse send F to $\text{Lan}_K F$.

Before proving this theorem, we present some consequences.

Corollary 3.1.9. Any finitary functor $\mathbf{Set} \to \mathbf{Set}$ preserves sifted colimits. In particular, if $T \colon \mathbf{Set} \to \mathbf{Set}$ is the underlying endofunctor of a finitary monad, then $U^T \colon T\text{-}\mathbf{Alg} \to \mathbf{Set}$ creates sifted colimits.

Proof. Recall that $\text{Lan}_K \colon [\mathbf{Set}_{\text{fin}}, \mathbf{Set}] \to [\mathbf{Set}, \mathbf{Set}]_{\text{fin}}$ is a left adjoint and on both sides colimits are computed pointwise. Moreover, functors preserving colimits of a given class are closed under formation of pointwise colimits, hence it is enough to check on a generating set.

If $\text{Lan}_K F_i$ preserves sifted colimits for all $F_i \colon \mathbf{Set}_{\text{fin}} \to \mathbf{Set}$, then the functor $\text{colim} \, \text{Lan}_K F_i \cong \text{Lan}_K \text{colim} \, F_i$ preserves sifted colimits.

Any functor $F \colon \mathbf{Set}_{\text{fin}} \to \mathbf{Set}$ is a colimit of representable functors $\mathbf{Set}_{\text{fin}}(X, -)$. Indeed, it is enough to consider $(\text{よ} \downarrow F) \to [\mathbf{Set}_{\text{fin}}, \mathbf{Set}]$, $(\mathbf{Set}_{\text{fin}}(X, -) \Rightarrow F) \mapsto \mathbf{Set}_{\text{fin}}(X, -)$ and notice that this being a colimit diagram essentially follows from Yoneda as $F \cong \text{colim}_{(\text{よ} \downarrow F)} \text{dom}$.

By the previous theorem, it is enough to check now that the functor $\text{Lan}_K \mathbf{Set}_{\text{fin}}(X, -)$ preserves sifted colimits.

Observe the following diagram:

$$
\begin{array}{ccc}
\mathbf{Set}_{\text{fin}} & \xrightarrow{\quad K \quad} & \mathbf{Set} \\
\ {}_{\mathbf{Set}_{\text{fin}}(X, -)} \searrow & \Longrightarrow & \swarrow {}_{\text{Lan}_K \mathbf{Set}_{\text{fin}}(X, -)} \\
& \mathbf{Set} &
\end{array}
$$

A natural transformation $\text{Lan}_K \mathbf{Set}_{\text{fin}}(X, -) \Rightarrow G$ by definition is equivalent to a natural transformation $\mathbf{Set}_{\text{fin}}(X, -) \Rightarrow GK$, which by Yoneda is equivalent to a map $* \to GKX$, which again by Yoneda corresponds to a natural transformation $\mathbf{Set}(KX, -) \Rightarrow G$, hence $\text{Lan}_K \mathbf{Set}_{\text{fin}}(X, -) \cong \mathbf{Set}(KX, -)$.

We only have to check now that $\mathbf{Set}(KX,-) \cong \Pi_{x \in X}\mathbf{Set}(*,-)$ preserves sifted colimits, but this is just the functor $Y \mapsto \Pi_{x \in X}Y$, which as we know commutes with sifted colimits. □

Proposition 3.1.10. A category $\mathbf{C}$ with sifted colimits is cocomplete if and only if it has an initial object and binary coproducts.

Proof. One implication is obvious. For the other one, notice that we get finitary coproducts immediately. An infinite coproduct can be written as a filtered colimit of finite coproducts. We are only missing coequalizers. If $f,g \colon a \rightrightarrows b$ is a pair, then $f + \mathrm{id}, g + \mathrm{id} \colon a + b \rightrightarrows b \colon \mathrm{in}_b$ (where in_b is the inclusion of b in the coproduct) is a reflexive pair hence the coequalizer $K = \mathrm{coeq}(a + b \rightrightarrows b)$ exists and coequalizes the original pair. The universal $h \colon b \to K$ is also a coequalizer of $f,g \colon a \rightrightarrows b$. □

Theorem 3.1.11. Let X be a set, $T \colon \prod_{x \in X}\mathbf{Set} \to \prod_{x \in X}\mathbf{Set}$ be a finitary monad. Then $T\text{-}\mathbf{Alg}$ is cocomplete.

Proof. $T\text{-}\mathbf{Alg}$ has always an initial object, namely the free algebra of the initial object $\left(T\prod_{x \in X}\emptyset, \mu_{\prod_{x \in X}\emptyset}\right)$. Similarly, for $a,b \in \prod_{x \in X}\mathbf{Set}$, using that F^T is a left adjoint (hence it preserves colimits) we find that $\left(T\left(a\coprod b\right), \mu_{a\coprod b}\right)$ is a coproduct of (Ta, μ_a) and (Tb, μ_b). Therefore $T\text{-}\mathbf{Alg}$ has coproducts of free algebras. We want to check that we have binary coproducts of T-algebras (a, α) and (b, β). We have reflective coequalizers

$$(T^2a, \mu_{Ta}) \underset{\underset{\mu_a}{\longrightarrow}}{\overset{\overset{T\alpha}{\longrightarrow}}{\xleftarrow{\ T\eta_a\ }}} (Ta, \mu_a) \overset{\alpha}{\longrightarrow} (a, \alpha)$$

$$(T^2b, \mu_{Tb}) \underset{\underset{\mu_b}{\longrightarrow}}{\overset{\overset{T\beta}{\longrightarrow}}{\xleftarrow{\ T\eta_b\ }}} (Tb, \mu_b) \overset{\beta}{\longrightarrow} (b, \beta)$$

so we get a new reflective pair by taking coproducts of the free algebras

$$(T^2a, \mu_{Ta}) + (T^2b, \mu_{Tb}) \underset{\underset{\mu_a + \mu_b}{\longrightarrow}}{\overset{\overset{T\alpha + T\beta}{\longrightarrow}}{\xleftarrow{\ T\eta_b\ }}} (Ta, \mu_a) + (Tb, \mu_b)$$

From the corollary, T preserves sifted colimits, hence $T\text{-}\mathbf{Alg}$ has reflexive coequalizers. Then the diagram above has a coequalizer, which is a coproduct of (a, α) and (b, β). □

Remark 3.1.12. This shows that $\mathbf{Ab}, \mathbf{Grp}, \mathbf{Rng}$ etc. are cocomplete.

Remark 3.1.13. We only used the fact that T preserves sifted colimits, hence a monad on a cocomplete category $\mathbf{C}$ preserving sifted colimits has a cocomplete category of algebras.

Remark 3.1.14. In fact, we only need that reflexive coequalizers and filtered colimits exist in $T\text{-}\mathbf{Alg}$.

3.2   Dense generators

The aim of this section is to prove the theorem about finitary endo-functors of **Set**. We want to identify "nice" generating subcategories like $\mathbf{Set}_{\mathrm{fin}} \to \mathbf{Set}$.

Definition 3.2.1. Let $K\colon \mathbf{A} \to \mathbf{C}$ be the inclusion of a full subcategory or, equivalently, a fully faithful functor. We define the *restricted Yoneda functor* $\widetilde{K}\colon \mathbf{C} \to [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ as the functor sending $c \in \mathbf{C}$ to $\mathbf{C}(K-, c)$.

The *canonical cocone* on the domain functor

$$\mathrm{dom}\colon (K \downarrow c) \to \mathbf{C}$$
$$(a, \varphi) \mapsto Ka$$

has components $\varphi\colon \mathrm{dom}(a, \varphi) = Ka \xrightarrow{\varphi} c$.

Definition 3.2.2. A colimit of a diagram $D\colon I \to \mathbf{C}$ is *K-absolute* if it is preserved by $\widetilde{K}\colon \mathbf{C} \to [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$.

Definition 3.2.3. A full subcategory/fully faithful functor $K\colon \mathbf{A} \to \mathbf{C}$ is called *dense* if $\widetilde{K}$ is fully faithful.

Theorem 3.2.4. Let $\mathbf{C}$ be a cocomplete category, $\mathbf{A}$ a small category and $K\colon \mathbf{A} \to \mathbf{C}$ a fully faithful functor. Then $\widetilde{K}$ has a left adjoint given by $\mathrm{Lan}_{\mathrm{よ}} K\colon [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{C}$.

$$\mathbf{A} \xrightarrow{\;\mathrm{よ}\;} [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$$

$$K \searrow \quad \swarrow \mathrm{Lan}_{\mathrm{よ}} K$$

$$\mathbf{C}$$

Moreover, the following conditions are equivalent:

1. $\widetilde{K}$ is fully faithful;

2. $\forall c \in \mathbf{C}$, the canonical cocone on $\mathrm{dom}\colon (\mathbf{A} \downarrow c) \to \mathbf{C}$ exibits it as colimit of the diagram $\mathrm{dom}\colon (K \downarrow c) \to \mathbf{C}$;

3. every object is a K-absolute colimit of a diagram of the form $\mathbf{I} \xrightarrow{D} \mathbf{A} \xrightarrow{K} \mathbf{C}$;

4. there exists some family of diagrams $D_i\colon \mathbf{I}_i \to \mathbf{C}$ which have K-absolute colimits and $\mathbf{C}$ is the closure[3] of $\mathbf{A}$ under the colimits of D_i;

---

[3]That is, the smallest subcategory $\mathbf{B}$ of $\mathbf{C}$ which contains $\mathbf{A}$ and which satisfies the following

if some diagram $D_i\colon \mathbf{I}_i \to \mathbf{C}$ factors through $\mathbf{B}$, then $\mathrm{colim}\, D_i \in \mathbf{B}$

is $\mathbf{C}$ itself.

5. the counit of $\mathrm{Lan}_{\text{よ}} K \dashv \widetilde{K}$ is an isomorphism.

Proof. We have $\mathrm{Lan}_{\text{よ}} K(F) = \operatorname*{colim}_{(\text{よ}\downarrow F)} Ka$, hence

$$
\begin{aligned}
\mathbf{C}(\mathrm{Lan}_{\text{よ}} K(F), c) &\cong \operatorname*{lim}_{(\text{よ}\downarrow F)} \mathbf{C}(Ka, c) \\
&\overset{\text{Yoneda}}{\cong} \operatorname*{lim}_{(\text{よ}\downarrow F)} [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}] \left(\mathbf{A}(-, a), \mathbf{C}(K-, c)\right) \\
&\cong [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}] \left(\operatorname*{colim}_{(\text{よ}\downarrow F)} \mathbf{A}(-, a), \mathbf{C}(K-, c)\right) \\
&\cong [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}] \left(F, \mathbf{C}(K-, c)\right)
\end{aligned}
$$

hence $\mathrm{Lan}_{\text{よ}} K \dashv \widetilde{K}$. It remains to check the implications.

1. $\implies$ 2. If $\widetilde{K}$ is fully faithful, it suffices to check that the image of the canonical cocone under $\widetilde{K}$ is a colimit. Since K is fully faithful, this image is precisely the diagram $\mathbf{A}(-, a) \cong \mathbf{C}(K-, Ka) \Rightarrow \mathbf{C}(K-, c)$. Then
$$
\mathrm{colim}\,\widetilde{K} \circ \mathrm{dom} = \operatorname*{colim}_{(\text{よ}\downarrow\mathbf{C}(K-,c))} \mathbf{A}(-, a) \cong \mathbf{C}(K-, c).
$$
This proves 2. and the fact that the colimit of $(K \downarrow c) \to \mathbf{C}$ is preserved by $\widetilde{K}\colon \mathbf{C} \to [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$.

2. $\implies$ 3. As we just observed, $\operatorname*{colim}_{(K\downarrow c)} \mathrm{dom}$ is $\widetilde{K}$-abs. and $\mathrm{dom}\colon (K \downarrow c) \to \mathbf{C}$ factors through $\mathbf{A}$.

3. $\implies$ 4. Clear.

4. $\implies$ 5. Let $\mathcal{B}$ be the class of objects b in $\mathbf{C}$ s.t. $\epsilon_b\colon \mathrm{Lan}_{\text{よ}} K\widetilde{K}(b) \to b$ is an isomorphism. The subcategory $\mathbf{B}$ spanned by $\mathcal{B}$ is closed under K-absolute colimits since they are preserved by $\widetilde{K}$ (by definition), by the left adjoint $\mathrm{Lan}_{\text{よ}} K$ and by $\mathrm{id}_{\mathbf{C}}$. It remains to check that the counit at $a \in \mathbf{A}$ is an isomorphism. But $\widetilde{K}a = \mathbf{C}(K-, Ka) \cong \mathbf{A}(-, a)$, so $\mathrm{Lan}_{\text{よ}} K(\mathbf{A}(-, a)) \cong a$, where the last isomorphism is a consequence of the fact that $(\text{よ} \downarrow \mathbf{A}(-, a))$ has a terminal object $\mathrm{id}\colon \mathbf{A}(-, a) \Rightarrow \mathbf{A}(-, a)$.

5. $\implies$ 1. Any right adjoint whose counit is an isomorphism is fully faithful. $\square$

Chapter 4

Monads in 2-category theory

Chapter 5

Monads in ∞-category theory