量子场论期末考试参考答案
（2009秋季）

❶ 带电标量粒子的电磁相互作用拉氏量为:

$$\mathscr{L} = (D_\mu\phi)^+ D^\mu\phi - m^2\phi^+\phi - \frac{\lambda}{4}(\phi^+\phi)^2 \tag{1}$$

其中$D_\mu = \partial_\mu - ieA_\mu$。

(a) 推导出箱子变换所对应的Nother流;

(b) 写出费曼顶角。

- 解答:

拉氏量写成

$$\begin{aligned}
\mathscr{L} &= (D_\mu\phi)^+ D^\mu\phi - m^2\phi^+\phi - \frac{\lambda}{4}(\phi^+\phi)^2 \\
&= [(\partial_\mu - ieA_\mu)\phi]^+(\partial^\mu - ieA^\mu)\phi - m^2\phi^+\phi - \frac{\lambda}{4}(\phi^+\phi)^2 \qquad (2) \\
&= \underbrace{(\partial_\mu\phi)^+\partial^\mu\phi - m^2\phi^+\phi}_{(1)} + \underbrace{ieA_\mu[\phi^+\partial^\mu\phi - (\partial^\mu\phi)^+\phi]}_{(2)} + \underbrace{e^2 g^{\mu\nu}A_\mu A_\nu\phi^+\phi}_{(3)} - \underbrace{\frac{\lambda}{4}(\phi^+\phi)^2}_{(4)}.
\end{aligned}$$

(a)在相因子变换下,

$$\begin{cases} \phi \to e^{i\alpha}\phi \\ \phi^+ \to e^{-i\alpha}\phi^+ \end{cases} \Rightarrow \begin{cases} \delta\phi = i\alpha\phi \\ \delta\phi^+ = -i\alpha\phi^+ \end{cases} . \tag{3}$$

拉氏量保持不变:

$$\begin{aligned}
0 = \delta\mathscr{L} &= \frac{\partial\mathscr{L}}{\partial\phi}\delta\phi + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \delta\phi^+\frac{\partial\mathscr{L}}{\partial\phi^+} + \delta(\partial_\mu\phi^+)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^+)} \\
&= \partial_\mu\left\{\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi + \delta\phi^+\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^+)}\right\} \quad (EOM) \\
&= \partial_\mu\left\{(\partial^\mu\phi - ieA^\mu\phi)^+ i\alpha\phi + (-i\alpha\phi^+)(\partial^\mu\phi - ieA^\mu\phi)\right\} \\
&= \alpha\partial_\mu\left\{i[(\partial^\mu\phi^+)\phi - \phi^+(\partial^\mu\phi)] - 2eA^\mu\phi^+\phi\right\}. \qquad (4)
\end{aligned}$$

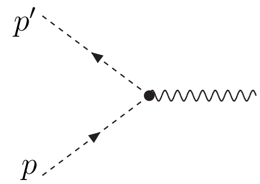
所以Noether流为

$$J^\mu = i[(\partial^\mu\phi^+)\phi - \phi^+(\partial^\mu\phi)] - 2eA^\mu\phi^+\phi. \tag{5}$$

(b)拉氏量中的(2)(3)(4)部分为顶角项，ϕ取负频，规定向内为正。在动量空间$\partial_\mu \to -ip_\mu$，将顶角中出现的场收缩掉，剩余部分乘以i（来自微扰展开）以及对称因子S（收缩方式不同导致）便得顶角费曼规则。

- 顶角(2):
  场的收缩只有一种方式，对称因子S=1.

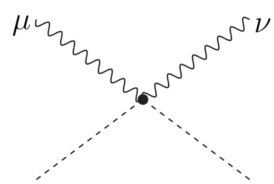
$$
\begin{aligned}
ieA_\mu[\phi^+\partial^\mu\phi-(\partial^\mu\phi)^+\phi] &\rightarrow i(ie)[-ip^\mu-(-ip'^\mu)^+]S \\
&= ie(p+p')^\mu.
\end{aligned} \tag{6}
$$



$= ie(p+p')^\mu$

- 顶角(3):
  $<0|T\{A_\alpha(x_1)A_\beta(x_2)\phi(x_3)\phi^+(x_4)A_\mu(x)A_\nu(x)\phi^+(x)\phi(x)\}|0>$中$\phi$场只有一种收缩方式，剩下的光子场的收缩有两种方式，所以对称因子S=2。

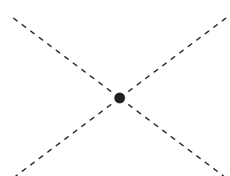
$$
\begin{aligned}
e^2g^{\mu\nu}A_\mu A_\nu\phi^+\phi &\rightarrow ie^2g^{\mu\nu}S \\
&= 2ie^2g^{\mu\nu}.
\end{aligned} \tag{7}
$$



$= 2ie^2g^{\mu\nu}$

- 顶角(4):
  $<0|T\{\phi(x_1)\phi^+(x_2)\phi(x_3)\phi^+(x_4)\phi(x)\phi^+(x)\phi(x)\phi^+(x)\}|0>$，总共有$2\times 2$中收缩方式，对称因子S=4。

$$
\begin{aligned}
-\frac{\lambda}{4}(\phi^+\phi)^2 &\rightarrow i(-\frac{\lambda}{4})S \\
&= -i\lambda.
\end{aligned} \tag{8}
$$



$= -i\lambda$

❷ 令$j_\mu(x)=\overline{\psi}\gamma_\mu\psi$，利用费米子反对易关系，在QED理论中证明下面等式:

$$
\begin{aligned}
&\partial_{x,\mu}<0|T\{j^\mu(x)\psi(x_1)\overline{\psi}(x_2)\}|0>=\delta^{(4)}(x-x_2)<0|T\{\psi(x_1)\overline{\psi}(x_2)\}|0> \\
&-\delta^{(4)}(x-x_1)<0|T\{\psi(x_1)\overline{\psi}(x_2)\}|0>.
\end{aligned} \tag{9}
$$

- 解答:

$$
\begin{aligned}
& \partial_{x,\mu} < 0|T\{j^\mu(x)\psi_\rho(x_1)\overline{\psi}_\sigma(x_2)\}|0> \\
= \; & \partial_{x,\mu} < 0|\theta(x^0-x_1^0)\theta(x_1^0-x_2^0)j^\mu(x)\psi_\rho(x_1)\overline{\psi}_\sigma(x_2) - \theta(x^0-x_2^0)\theta(x_2^0-x_1^0)j^\mu(x)\overline{\psi}_\sigma(x_2)\psi_\rho(x_1) \\
& +\theta(x_1^0-x^0)\theta(x^0-x_2^0)\psi_\rho(x_1)j^\mu(x)\overline{\psi}_\sigma(x_2) - \theta(x_2^0-x^0)\theta(x^0-x_1^0)\overline{\psi}_\sigma(x_2)j^\mu(x)\psi_\rho(x_1) \\
& +\theta(x_1^0-x_2^0)\theta(x_2^0-x^0)\psi_\rho(x_1)\overline{\psi}_\sigma(x_2)j^\mu(x) - \theta(x_2^0-x_1^0)\theta(x_1^0-x^0)\overline{\psi}_\sigma(x_2)\psi_\rho(x_1)j^\mu(x)|0> \\
= \; & < 0|\delta(x^0-x_1^0)\theta(x_1^0-x_2^0)j^0(x)\psi_\rho(x_1)\overline{\psi}_\sigma(x_2) - \delta(x^0-x_2^0)\theta(x_2^0-x_1^0)j^0(x)\overline{\psi}_\sigma(x_2)\psi_\rho(x_1) \\
& -\delta(x_1^0-x^0)\theta(x^0-x_2^0)\psi_\rho(x_1)j^0(x)\overline{\psi}_\sigma(x_2) + \theta(x_1^0-x^0)\delta(x^0-x_2^0)\psi_\rho(x_1)j^0(x)\overline{\psi}_\sigma(x_2) \\
& +\delta(x_2^0-x^0)\theta(x^0-x_1^0)\overline{\psi}_\sigma(x_2)j^0(x)\psi_\rho(x_1) - \theta(x_2^0-x^0)\delta(x^0-x_1^0)\overline{\psi}_\sigma(x_2)j^0(x)\psi_\rho(x_1) \\
& -\theta(x_1^0-x_2^0)\delta(x_2^0-x^0)\psi_\rho(x_1)\overline{\psi}_\sigma(x_2)j^0(x) + \theta(x_2^0-x_1^0)\delta(x_1^0-x^0)\overline{\psi}_\sigma(x_2)\psi_\rho(x_1)j^0(x) \\
& +(\text{含}\partial_\mu j^\mu\text{的项=0})|0> \\
= \; & < 0|\delta(x^0-x_1^0)\theta(x_1^0-x_2^0)[j^0(x),\psi_\rho(x_1)]\overline{\psi}_\sigma(x_2) - \delta(x^0-x_2^0)\theta(x_2^0-x_1^0)[j^0(x),\overline{\psi}_\sigma(x_2)]\psi_\rho(x_1) \\
& +\delta(x^0-x_2^0)\theta(x_1^0-x_2^0)\psi_\rho(x_1)[j^0(x),\overline{\psi}_\sigma(x_2)] - \delta(x^0-x_1^0)\theta(x_2^0-x_1^0)\overline{\psi}_\sigma(x_2)[j^0(x),\psi_\rho(x_1)]|0> \\
= \; & \delta(x^0-x_1^0) < 0|T\Big\{[j^0(x),\psi_\rho(x_1)]\overline{\psi}_\sigma(x_2)\Big\}|0> +\delta(x^0-x_2^0) < 0|T\Big\{\psi_\rho(x_1)[j^0(x),\overline{\psi}_\sigma(x_2)]\Big\}|0>
\end{aligned} \tag{10}
$$

已知狄拉克场的等时反对易关系:

$$\{\psi_\alpha(\overrightarrow{x}_1,t),\psi_\beta^+(\overrightarrow{x}_2,t)\} = \delta_{\alpha\beta}\delta^{(3)}(\overrightarrow{x}_1-\overrightarrow{x}_2) \tag{11}$$

并且流可以写成

$$j^0(x) = \psi_\alpha^+\psi_\alpha. \tag{12}$$

同时利用关系式$[AB,C] = A\{B,C\}-\{A,C\}B$，所以

$$
\begin{aligned}
[j^0(x),\psi_\rho(x_1)] &= [\phi_\alpha^+(x)\psi_\alpha(x),\psi_\rho(x_1)] \\
&= -\{\psi_\alpha^+(x),\psi_\rho(x_1)\}\psi_\alpha(x) \\
&= -\delta_{\alpha\rho}\delta^{(3)}(\overrightarrow{x}-\overrightarrow{x}_1)\psi_\alpha(x) \\
&= -\delta^{(3)}(\overrightarrow{x}-\overrightarrow{x}_1)\psi_\rho(x).
\end{aligned} \tag{13}
$$

同理

$$[j^0(x),\overline{\psi}_\sigma(x_2)] = \delta^{(3)}(\overrightarrow{x}-\overrightarrow{x}_2)\overline{\psi}_\sigma(x). \tag{14}$$

将(13)(14)代回(10)得

$$
\begin{aligned}
& \partial_{x,\mu} < 0|T\{j^\mu(x)\psi_\rho(x_1)\overline{\psi}_\sigma(x_2)\}|0> \\
= \; & \delta(x^0-x_1^0) < 0|T\Big\{[-\delta^{(3)}(\overrightarrow{x}-\overrightarrow{x}_1)\psi_\rho(x)]\overline{\psi}_\sigma(x_2)\Big\}|0> \\
& +\delta(x^0-x_2^0) < 0|T\Big\{\psi_\rho(x_1)[\delta^{(3)}(\overrightarrow{x}-\overrightarrow{x}_2)\overline{\psi}_\sigma(x)]\Big\}|0> \\
= \; & -\delta^{(4)}(x-x_1) < 0|T\Big\{\psi_\rho(x_1)\overline{\psi}_\sigma(x_2)\Big\}|0> +\delta^{(4)}(x-x_2) < 0|T\Big\{\psi_\rho(x_1)\overline{\psi}_\sigma(x_2)\Big\}|0>.
\end{aligned} \tag{15}
$$

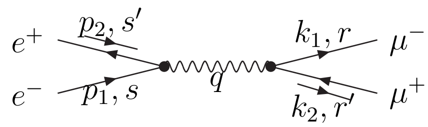
得证。

❸ 计算$e^+ + e^- \to \mu^+\mu^-$反应

(a) 画出费曼图。
(b) 计算不变振幅。
(c) 质心系中算出散射总截面。

- 解答:

(a)费曼图如下:



(b)费曼振幅:

$$
\begin{aligned}
i\mathscr{M} &= \overline{v}(p_2,s')(-ie\gamma^\mu)u(p_1,s)\frac{-ig_{\mu\nu}}{q^2}\overline{u}(k_1,r)(-ie\gamma^\nu)v(k_2,r') \\
&= \frac{ie^2}{q^2}\overline{v}(p_2,s')\gamma^\mu u(p_1,s)\overline{u}(k_1,r)\gamma_\mu v(k_2,r'). \qquad (16)
\end{aligned}
$$

其共轭为:

$$
-i\mathscr{M}^+ = \frac{-ie^2}{q^2}\overline{v}(k_2,r')\gamma^\nu u(k_1,r)\overline{u}(p_1,s)\gamma_\nu v(p_2,s'). \qquad (17)
$$

对末态自旋求和，初态自旋求和取平均:

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathscr{M}|^2 &= \frac{e^4}{4q^4}\sum_{spins}\overline{v}(k_2,r')\gamma^\nu u(k_1,r)\overline{u}(p_1,s)\gamma_\nu v(p_2,s') \\
&\quad \times\overline{v}(p_2,s')\gamma^\mu u(p_1,s)\overline{u}(k_1,r)\gamma_\mu v(k_2,r') \\
&= \frac{e^4}{4q^4}\sum_{spins}\overline{u}(p_1,s)\gamma_\nu v(p_2,s')\overline{v}(p_2,s')\gamma_\mu u(p_1,s) \\
&\quad \times\overline{v}(k_2,r')\gamma^\nu u(k_1,r)\overline{u}(k_1,r)\gamma^\mu v(k_2,r') \\
&= \frac{e^4}{4q^4}\sum_{spins}tr\Big\{u(p_1,s)\overline{u}(p_1,s)\gamma_\nu v(p_2,s')\overline{v}(p_2,s')\gamma_\mu\Big\} \\
&\quad \times tr\Big\{v(k_2,r')\overline{v}(k_2,r')\gamma^\nu u(k_1,r)\overline{u}(k_1,r)\gamma^\mu\Big\} \\
&= \frac{e^4}{4q^4}tr\Big\{(\not{p}_1+m_e)\gamma_\nu(\not{p}_2-m_e)\gamma_\mu\Big\}tr\Big\{(\not{k}_2-m_\mu)\gamma^\nu(\not{k}_1+m_\mu)\gamma^\mu\Big\} \\
&= \frac{e^4}{4q^4}tr\Big\{\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu - m_e^2\gamma_\nu\gamma_\mu\Big\}tr\Big\{\not{k}_2\gamma^\nu\not{k}_1\gamma^\mu - m_\mu^2\gamma^\nu\gamma^\mu\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^4}{4q^4}4[p_{1\nu}p_{2\mu}+p_{1\mu}p_{2\nu}-g_{\nu\mu}(p_1\cdot p_2+m_e^2)]4[k_2^\nu k_1^\mu+k_2^\mu k_1^\nu-g^{\nu\mu}(k_2\cdot k_1+m_\mu^2)]\\
&= \frac{8e^4}{q^4}\Big\{(p_1\cdot k_2)(p_2\cdot k_1)+(p_1\cdot k_1)(p_2\cdot k_2)+(p_1\cdot p_2)m_\mu^2+(k_2\cdot k_1 m_e^2)+2m_\mu^2 m_e^2\Big\}\\
&= \frac{8e^4}{q^4}\Big\{(p_1\cdot k_2)(p_2\cdot k_1)+(p_1\cdot k_1)(p_2\cdot k_2)+(p_1\cdot p_2)m_\mu^2\Big\} \qquad (For\ \frac{m_e}{m_\mu}\sim\frac{1}{200})
\end{aligned}
$$

(c)质心系中计算散射截面。参数化动量如下

$$
\begin{aligned}
p_1 &= (E, E\hat{z})\\
p_2 &= (E, -E\hat{z})\\
k_1 &= (E, \overrightarrow{k})\\
k_2 &= (E, -\overrightarrow{k}).
\end{aligned} \tag{18}
$$

则费曼振幅中出现的四动量点乘可写成,

$$
\begin{aligned}
p_1\cdot k_2 &= p_2\cdot k_1 = E^2+E\overrightarrow{k}\cdot\hat{z} = E^2+E|\overrightarrow{k}|cos\theta,\\
p_1\cdot k_1 &= p_2\cdot k_2 = E^2-E\overrightarrow{k}\cdot\hat{z} = E^2-E|\overrightarrow{k}|cos\theta,\\
p_1\cdot p_2 &= E^2+E^2=2E^2, \qquad q^2=(p_1+p_2)^2=4E^2,\\
u_{12} &= |\frac{\overrightarrow{p}_1}{E_1}-\frac{\overrightarrow{p}_2}{E_2}|=2. \qquad (\text{初态粒子相对速度})
\end{aligned} \tag{19}
$$

代入费曼振幅化简

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathscr{M}|^2 &= \frac{8e^4}{16E^4}\Big\{[E^2+E|\overrightarrow{k}|cos\theta]^2+[E^2-E|\overrightarrow{k}|cos\theta]^2+2E^2m_\mu^2\Big\}\\
&= e^4\Big\{(1-\frac{m_\mu^2}{E^2})cos^2\theta+(1+\frac{m_\mu^2}{E^2})\Big\}.
\end{aligned} \tag{20}
$$

同时在质心系中计算二体末态相空间积分,

$$
\begin{aligned}
\int d\Gamma_2 &= \int_{-\infty}^{\infty}(2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2)\frac{d^3\overrightarrow{k}_1}{(2\pi)^3 2E_3}\frac{d^3\overrightarrow{k}_2}{(2\pi)^3 2E_4}\\
&= \int_{-\infty}^{\infty}(2\pi)\delta(2E_1-2E_3)\frac{d^3\overrightarrow{k}_1}{(2\pi)^3 2E_3}\frac{}{2E_3}\\
&= \int_0^{\infty}(2\pi)\frac{1}{2}\delta(E_1-E_3)\frac{k_1^2dk_1d\Omega}{(2\pi)^3 4E_3^2}\\
&= \frac{\sqrt{E^2-m_\mu^2}}{32\pi^2}\int d\Omega.
\end{aligned} \tag{21}
$$

微分截面:

$$
d\sigma = \frac{1}{2E_1 2E_2 u_{12}}\Big\{\frac{1}{4}|\mathscr{M}|^2\Big\}d\Gamma_2
$$

$$
\begin{aligned}
&= \frac{1}{8E^2}e^4\Big\{(1-\frac{m_\mu^2}{E^2})cos^2\theta+(1+\frac{m_\mu^2}{E^2})\Big\}\frac{\sqrt{E^2-m_\mu^2}}{32\pi^2}d\Omega \\
&= \frac{\alpha^2\sqrt{E^2-m_\mu^2}}{16E^2}\Big\{(1-\frac{m_\mu^2}{E^2})cos^2\theta+(1+\frac{m_\mu^2}{E^2})\Big\}d\Omega \\
&= \frac{\alpha^2}{4E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\Big\{(1-\frac{m_\mu^2}{E^2})cos^2\theta+(1+\frac{m_\mu^2}{E^2})\Big\}d\Omega. \qquad (22)
\end{aligned}
$$

总截面:

$$
\begin{aligned}
\sigma_{total} &= \int_0^{2\pi}d\varphi\int_{-1}^{1}dcos\theta\frac{\alpha^2}{4E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\Big\{(1-\frac{m_\mu^2}{E^2})cos^2\theta+(1+\frac{m_\mu^2}{E^2})\Big\} \\
&= \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\Big\{1+\frac{m_\mu^2}{2E^2}\Big\} \qquad (23)
\end{aligned}
$$

其中$\alpha=\frac{e^2}{4\pi}$为精细结构常数，$E_{cm}=2E$为质心系能量。