Functional Analysis Final, 2004-2005

1. (1): For any $x, y \in H$, we have

$$< Ax, y >= \lim_{n\to\infty} < A_n x, y >= \lim_{n\to\infty} < x, A_n y >=< x, Ay >,$$

which means A is self-adjoint.

(2): Since $A_n \to A$, so $A_n^* \to A^*$. Then $A^*A = \lim_{n\to\infty} A_n^* A_n = I$. In the same way, $AA^* = I$, which means A is unitary.

2: For any bounded $\{f_n\} \subset C[0,1]$, let $M = \sup_{n\in\mathbb{N}} \|f_n\|_{L_\infty[0,1]}$, then we have

$$|(Tf)(x) - (Tf)(y)| = \left| \int_y^x f(s)ds \right| \le M|x-y|,$$

which means $\{f_n\}$ is equicontinuous. By Ascoli-Arzela Theorem, there exists a convergent subsequence, which means T is compact.

For any λ, let $f \in N(T - \lambda I)$, we have

$$\int_0^t f(s)ds = \lambda f(t), \ \forall t \in [0,1].$$

If $\lambda = 0$, then $f = 0$; If not, then $f \in C^1[0,1]$ and take the derivative to have $f = \lambda f'$, which means $f(t) = f(0)e^{t/\lambda}$, where $f(0) = 0$, and so $f = 0$. Anyway, we have $\sigma_p(T) = \emptyset$. Since T is compact, then $\sigma(T)\backslash\{0\} \subset \sigma_p(T) = \emptyset$, which means $\sigma(T) = \{0\}$ since $\sigma(T) \neq \emptyset$. (Actually, $\sigma(T) = \sigma_r(T) = \{0\}$)

Then the spectral radius $R = 0 = \lim_{n\to\infty} \|T^n\|^{1/n}$, which means $\exists N \in \mathbb{N}$ such that $\|T^n\|^{1/n} < 1/2, \ \forall n > N$. So we have $\|T^n\| = (\|T^n\|^{1/n})^n \to 0$ as $n \to \infty$, which means $T^n \to 0$.

3 (1): It's trivial to show $\mathcal{A}' \subset L(H)$ is a subspace.

For any $S, T \in \mathcal{A}'$, $A \in \mathcal{A}$, we have $(ST)A = SAT = A(ST)$, which means $ST \in \mathcal{A}'$;

And also $T^*A = (A^*T)^* = (TA^*)^* = AT^*$, which mean $T^* \in \mathcal{A}'$.

So $\mathcal{A}'$ is a $*$-algebra.

(2): Since $\{B_n x\}$ is weakly convergent, so it's bounded for any $x \in H$, then $\sup_{n\in\mathbb{N}} \|B_n\| \triangleq M < \infty$ by principle of uniform boundedness.

Define linear operator $Bx \triangleq w\text{-}\lim_{n\to\infty} B_n x$, $\forall x \in H$, then

$$\|Bx\| \le \liminf_{n\to\infty} \|B_n x\| \le \liminf_{n\to\infty} \|B_n\|\|x\| \le M\|x\|, \ \forall x \in H$$

which means $B \in L(H)$. For any $A \in \mathcal{A}$, we have

$BAx = w\text{-}\lim_{n\to\infty} B_n(Ax) = w\text{-}\lim_{n\to\infty} AB_n x = A(w\text{-}\lim_{n\to\infty} B_n x) = ABx$, $\forall x \in H$, which means $AB = BA$ and so $B \in \mathcal{A}'$.

4: Since A is compact and self-adjoint, then A has at most countably many nonzero eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ (up to multiplicity). Say $\lambda_1 = \max_{n\in\mathbb{N}} \lambda_n$.

Let $\{e_n\}_{n\in\mathbb{N}} \subset H$ be the corresponding orthonormal eigenvectors, then for any $x \in H$, we have

$$< Ax, x >=< \sum_{n=1}^{\infty} \lambda_n < x, e_n > e_n, \sum_{m=1}^{\infty} < x, e_m > e_m >= \sum_{n=1}^{\infty} \lambda_n | < x, e_n > |^2 \leq \lambda_1 \|x\|^2,$$

which means $\sup_{\|x\|=1} < Ax, x > \leq \lambda_1$.

Any $< Ae_1, e_1 >= \lambda_1$, so $\sup_{\|x\|=1} < Ax, x >= \lambda_1$.

5: Claim: A_n is unitary.

For any $x, y \in l^2$, we have

$$< A_n^* x, y >=< x, A_n y >= \sum_{i=1}^{\infty} x_i y_{\sigma_n(i)} = \sum_{i=1}^{\infty} x_{\sigma_n^{-1}(i)} y_i =< A_n^{-1} x, y >,$$

which means $A_n^* = A^{-1}$, and so A_n is unitary.

For any $x \in H$, define $R \in L(l^2)$ as the right translation operator, then

$$\|A_n - Rx\|^2 = \sum_{i=1}^{\infty} |x_{\sigma_n(i)} - x_{i-1}|^2 = |x_n|^2 + \sum_{i=n+1}^{\infty} |x_i - x_{i-1}|^2 \to 0, (n \to \infty)$$

where $x_0 = 0$. So $A_n x \to Rx$, $\forall x \in H$. But R is not unitary since it is not surjective.

6 (1): Only need to show $A^{-1}B$ is a closed operator.

Let $x_n \to x$ and $A^{-1}Bx_n \to y$, then $Bx \leftarrow Bx_n \to Ay$, so $Bx = Ay$ and then $A^{-1}Bx = y$, which means $A^{-1}B$ is closed.

(2): $B = A(A^{-1}B) \in C(H)$.

7: Let $P_E : H \to H$ be the orthogonal projection onto E. Since $E \subset H$ is closed, then $P_E \in L(H)$.

If $N(A) = \{0\}$, then $R(P_E) = E \subset R(A)$, by problem 6, part(2), we know $P_E \in C(H)$, which means $P_E|_E = I_E \in C(E)$, and so $\dim E < \infty$.

If $N(A) \neq \{0\}$, consider operator $\tilde{A}[x] \triangleq Ax$, $\forall [x] \in H/N(A)$, then $N(\tilde{A}) = \{[0]\}$ and repeat the case when $N(A) = \{0\}$.