Functional Analysis Final, 2004-2005

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1. Proof (1): For any $x, y \in H$, we have

$$< Ax, y >= \lim_{n\to\infty} < A_n x, y >= \lim_{n\to\infty} < x, A_n y >=< x, Ay >,$$

which means A is self-adjoint.

(2): Since $A_n \to A$, so $A_n^* \to A^*$. Then $A^*A = \lim_{n\to\infty} A_n^* A_n = I$. In the same way, $AA^* = I$, which means A is unitary.

2. Proof : For any bounded $\{f_n\} \subset C[0,1]$, let $M = \sup_{n\in\mathbb{N}} \|f_n\|_{L_\infty[0,1]}$, then we have

$$|(Tf)(x) - (Tf)(y)| = \left| \int_y^x f(s)ds \right| \le M|x-y|,$$

which means $\{f_n\}$ is equicontinuous. By Ascoli-Arzela Theorem, there exists a convergent subsequence, which means T is compact.

For any λ, let $f \in N(T - \lambda I)$, we have

$$\int_0^t f(s)ds = \lambda f(t),\ \forall t \in [0,1].$$

If $\lambda = 0$, then $f = 0$; If not, then $f \in C^1[0,1]$ and take the derivative to have $f = \lambda f'$, which means $f(t) = f(0)e^{t/\lambda}$, where $f(0) = 0$, and so $f = 0$. Anyway, we have $\sigma_p(T) = \emptyset$. Since T is compact, then $\sigma(T)\backslash\{0\} \subset \sigma_p(T) = \emptyset$, which means $\sigma(T) = \{0\}$ since $\sigma(T) \ne \emptyset$. (Actually, $\sigma(T) = \sigma_r(T) = \{0\}$)

Then the spectral radius $R = 0 = \lim_{n\to\infty} \|T^n\|^{1/n}$, which means $\exists N \in \mathbb{N}$ such that $\|T^n\|^{1/n} < 1/2,\ \forall n > N$. So we have $\|T^n\| = (\|T^n\|^{1/n})^n \to 0$ as $n \to \infty$, which means $T^n \to 0$.

3. Proof (1): It's trivial to show $\mathcal{A}' \subset L(H)$ is a subspace.

For any $S, T \in \mathcal{A}'$, $A \in \mathcal{A}$, we have $(ST)A = SAT = A(ST)$, which means $ST \in \mathcal{A}'$;

And also $T^*A = (A^*T)^* = (TA^*)^* = AT^*$, which means $T^* \in \mathcal{A}'$.

So $\mathcal{A}'$ is a $*$-algebra.

(2): Since $\{B_n x\}$ is weakly convergent, so it's bounded for any $x \in H$, then $\sup_{n\in\mathbb{N}} \|B_n\| \triangleq M < \infty$ by principle of uniform boundedness.

Define linear operator $Bx \triangleq w\text{-}\lim_{n\to\infty} B_n x$, $\forall x \in H$, then

$$\|Bx\| \leq \liminf_{n\to\infty} \|B_n x\| \leq \liminf_{n\to\infty} \|B_n\|\|x\| \leq M\|x\|,\ \forall x \in H$$

which means $B \in L(H)$. For any $A \in \mathcal{A}$, we have

$BAx = w\text{-}\lim_{n\to\infty} B_n(Ax) = w\text{-}\lim_{n\to\infty} AB_n x = A(w\text{-}\lim_{n\to\infty} B_n x) = ABx$, $\forall x \in H$, which means $AB = BA$ and so $B \in \mathcal{A}'$.

4. Proof : Since A is compact and self-adjoint, then A has at most countably many nonzero eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ (up to multiplicity). Say $\lambda_1 = \max_{n\in\mathbb{N}} \lambda_n$.

Let $\{e_n\}_{n\in\mathbb{N}} \subset H$ be the corresponding orthonormal eigenvectors, then for any $x \in H$, we have

$$< Ax, x >=< \sum_{n=1}^{\infty} \lambda_n < x, e_n > e_n, \sum_{m=1}^{\infty} < x, e_m > e_m >= \sum_{n=1}^{\infty} \lambda_n | < x, e_n > |^2 \leq \lambda_1 \|x\|^2,$$

which means $\sup_{\|x\|=1} < Ax, x > \leq \lambda_1$.

Any $< Ae_1, e_1 >= \lambda_1$, so $\sup_{\|x\|=1} < Ax, x >= \lambda_1$.

5. Proof : Claim: A_n is unitary.

For any $x, y \in l^2$, we have

$$< A_n^* x, y >=< x, A_n y >= \sum_{i=1}^{\infty} x_i y_{\sigma_n(i)} = \sum_{i=1}^{\infty} x_{\sigma_n^{-1}(i)} y_i =< A_n^{-1} x, y >,$$

which means $A_n^* = A^{-1}$, and so A_n is unitary.

For any $x \in H$, define $R \in L(l^2)$ as the right translation operator, then

$$\|A_n - Rx\|^2 = \sum_{i=1}^{\infty} |x_{\sigma_n(i)} - x_{i-1}|^2 = |x_n|^2 + \sum_{i=n+1}^{\infty} |x_i - x_{i-1}|^2 \to 0, (n \to \infty)$$

where $x_0 = 0$. So $A_n x \to Rx$, $\forall x \in H$. But R is not unitary since it is not surjective.

6. Proof (1): Only need to show $A^{-1}B$ is a closed operator.

Let $x_n \to x$ and $A^{-1}Bx_n \to y$, then $Bx \leftarrow Bx_n \to Ay$, so $Bx = Ay$ and then $A^{-1}Bx = y$, which means $A^{-1}B$ is closed.

(2): $B = A(A^{-1}B) \in C(H)$.

7. Proof : Let $P_E : H \to H$ be the orthogonal projection onto E. Since $E \subset H$ is closed, then $P_E \in L(H)$.

If $N(A) = \{0\}$, then $R(P_E) = E \subset R(A)$, by problem 6, part(2), we know $P_E \in C(H)$, which means $P_E|_E = I_E \in C(E)$, and so $\dim E < \infty$.

If $N(A) \neq \{0\}$, consider operator $\tilde{A}[x] \triangleq Ax$, $\forall [x] \in H/N(A)$, then $N(\tilde{A}) = \{[0]\}$ and repeat the case when $N(A) = \{0\}$.

8. Proof (1): For any $A \in \mathcal{A}$, $x \in H$, we have $P_{E_\xi}x \in E_\xi$, which means $\exists \{A_n\} \subset \mathcal{A}$ such that $A_n\xi \to P_{E_\xi}x$. Then

$$AP_{E_\xi}x = A(\lim_{n\to\infty} A_n\xi) = \lim_{n\to\infty} AA_n\xi \in E_\xi,$$

which means $R(AP_{E_\xi}) \subset E_\xi$. So $P_{E_\xi}AP_{E_\xi} = AP_{E_\xi}$, $\forall A \in \mathcal{A}$. Then we have

$$P_{E_\xi}A = (A^*P_{E_\xi})^* = (P_{E_\xi}A^*P_{E_\xi})^* = P_{E_\xi}AP_{E_\xi} = AP_{E_\xi} \ \forall A \in \mathcal{A},$$

which means $P_{E_\xi} \in \mathcal{A}'$.

(2): For any $A \in \mathcal{A}$, by part(1), we know $P_{E_\xi}A = AP_{E_\xi}$, then we have $E_\xi \ni A\xi = P_{E_\xi}A\xi = AP_{E_\xi}\xi$, which means $A(\xi - P_{E_\xi}\xi) = 0$, $\forall A \in \mathcal{A}$. By assumption, we have $\xi = P_{E_\xi}\xi$, so for any $S \in \mathcal{A}''$, we have

$$S\xi = S(P_{E_\xi}\xi) = P_{E_\xi}(S\xi),$$

which means $S\xi \in E_\xi$.

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