

CS 189: Homework 2

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February 18, 2016

- 1.
- 2.
3. Let A be a positive definite matrix in $\mathbb{R}^{n \times n}$.
 - (a) Consider the following derivation:

$$x^T A x = x^T \begin{bmatrix} \sum_j^n a_{1j} x_j \\ \vdots \\ \sum_j^n a_{nj} x_j \end{bmatrix} = \sum_i^n \sum_j^n a_{ij} x_i x_j. \quad (1)$$

(b)

Theorem 1. *If A is positive definite, then the diagonals of A are positive.*

Proof. Suppose that there is negative value on the diagonal, say a_{qq} . Then let $x = e_q$. If we apply the quadratic form we get $e_q^T A e_q = a_{qq} < 0$. This contradicts the positive semidefiniteness of A . \square

4. Short Proofs.

- (a) Assume problem (b).

Lemma 1. *If A is a matrix with eigen values λ_n $A + \gamma I$ has eigenvalues $\gamma + \lambda_n$*

Proof. If λ_n is an eigenvalue, then $Av_n = \lambda_n v_n$ for a corresponding eigenvector v . Furthermore

$$(A + \gamma I)v = Av + \gamma Iv = \lambda_n v + \gamma v = (\lambda_n + \gamma) \quad (2)$$

which implies that $\lambda_n + \gamma$ is an eigen value of $A + \gamma I$. This completes the proof. \square

Theorem 2. *If A is positive semidefinite and $\gamma > 0$, then $A + \gamma I$ is positive definite.*

Proof. If A is positive definite then by the logic of the proof of (b),

$$x^T A x = \sum_i \lambda_i (x_i^T e_i)^2 \geq 0. \quad (3)$$

It follows that some $\lambda \geq 0$ since $x \neq 0$. Therefore by the previous lemma adding γ to the diagonal adds γ to every eigenvalue implying that all eigen values are positive. By (b), $A + I\gamma$ is positive definite therefore. \square

(b) Lolololol!

Theorem 3. *A is positive definite if and only if all of its eigen values are more than 0.*

Proof. If A is positive semidefinite then it is symmetric. Using spectral theorem we have that

$$\begin{aligned} x^T A x &= \sum_i (x^T e_i) e_i^T A x = \sum_i x^T e_i e_i^T \lambda_i e_i^T x \\ &= \sum_i \lambda_i (x^T e_i)^2 > 0 \end{aligned} \quad (4)$$

which is true if and only if all λ_i are more than 0. \square

(c)

Theorem 4. *If A is positive definite then it is invertible.*

Proof. The invertible matrix theorem states that a matrix is invertible if and only if all of its eigen values are more than 0. By the previous theorem if A is positive definite then all of its eigen values are positive and so it is invertible. \square

(d)

Theorem 5. *If A is positive definite then there exist n linearly independent vectors so that $A_{ij} = x_i^T x_j$.*

Proof. The statement of the theorem is true if and only if $A = B^T B$ where B is invertible. By spectral theorem we have that $A = U \Lambda U^T$ where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$. Furthermore $U^{-1} = U^T$. Let $\Omega = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$. Then, $\Omega^2 = \Lambda$. Let $W^T = U \Omega$ and $W = \Omega U^T$. So we have that W is still an orthonormal matrix and so $A = W^T W$. This completes the proof. \square

5. DERIVATIONS :(Assuming theorems from Math 105

(a) Consider the following derivation

$$\frac{\partial(x^T a)}{\partial x} = \frac{\partial(x)}{\partial x}^T a + \left(\frac{\partial(a)}{\partial x} \right)^T x = a. \quad (5)$$

(b) Consider the following derivation

$$\frac{\partial(x^T A x)}{\partial x} = \frac{\partial(x^T)}{\partial x} A x + \frac{\partial(A x)}{\partial x}^T X = A x + A x^T \quad (6)$$

(c) Consider the following derivation

(d)

Theorem 6. *If $x \in \mathbb{R}^n$*

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}. \quad (7)$$

Proof.

□