CS 189: Homework 2

William Guss
26793499
wguss@berkeley.edu

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1.

2.

3. Let A be a positive definite matrix in $\mathbb{R}^{n\times n}$.

   (a) Cionsider the following derrivation:

$$x^T A x = x^T \begin{bmatrix} \sum_j^n a_{1j}x_j \\ \vdots \\ \sum_j^n a_{nj}x_j \end{bmatrix} = \sum_i^n \sum_j^n a_{ij}x_i x_j. \tag{1}$$

   (b)

Theorem 1. *If A is positive definite, then the diagonals of A are positive.*

Proof. Suppose that there is negative value on the diagonal, say a_{qq}. Then let $x = e_q$. If we apply the quadratic form we get $e_q A^T e_q = a_q q < 0$. This contradicts the positive semidefiniteness of A. □

4. Short Proofs.

   (a) Assume problem (b).

Lemma 1. *If A is a matrix with eigen values λ_n $A + \gamma I$ has eigenvalues $\gamma + \lambda_n$*

Proof. If λ_n is an eigenvalue, then $Av_n = \lambda_n v_n$ for a corresponding eigenvector v. Furthermore

$$(A + \gamma I)v = Av + \gamma I v = \lambda_n v + \gamma v = (\lambda_n + \gamma) \tag{2}$$

which implies that $\lambda_n + \gamma$ is an eigen value of $A + \gamma I$. This completes the proof. □

Theorem 2. *If A is positive semidefinite and $\gamma > 0$, then $A + \gamma I$ is positive definite.*

Proof. If A is positive definite then by the logic of the proof of (b),

$$x^T A x = \sum_i \lambda_i (x_i^T e_i)^2 \geq 0. \tag{3}$$

It follows that some $\lambda \geq 0$ since $x \neq 0$. Therefore by the previous lemma adding γ to the diagonal adds γ to every eigenvalue implying that all eigen values are positive. By (b), $A + I\gamma$ is positive definiute therefore. ☐

(b) Lolololol!

Theorem 3. *A is positive definite if and only if all of its eigen values are more than 0.*

Proof. Iff A is positive semidefinite then it is symmetric. Using spectral theorem we have that

$$\begin{aligned} x^T A x = \sum_i (x^T e_i) e_i^T A x = \sum_i &= x^T e_i e_i^T \lambda e_i^T x \\ &= \sum_i \lambda_i (x^T e_i)^2 > 0 \end{aligned} \tag{4}$$

which is true if and only if all λ_i are more than 0. ☐

(c)

Theorem 4. *If A is positive definite then it is invertible.*

Proof. The invertible matrix theorem statesa that a matrix is invertible if and only if all of its eigen values are more than 0. By the previous theorem if A is positive definite then all of its eigen values are positive and so it is invertible. ☐

(d)

Theorem 5. *If A is positive definite then there exist n linearly independent vectors so that $A_{ij} = x_i^T x_j$.*

Proof. The statement of the theorem is true if and only if $A = B^T B$ where B is invertible. By spectral theorem we have that $A = U \Lambda U^T$ where $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$. Furthermore $U^{-1} = U^T$. Let $\Omega = diag(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. Then, $\Omega^2 = \Lambda$. Let $W^T = U\Omega$ and $W = \Omega U^T$. So we have that W is still an orthonormal matrix and so $A = W^T W$. This completes the proof. ☐

5. DERIVATIONS :(Assuming theorems from Math 105

(a) Consider the followiong derivation

$$\frac{\partial(x^T a)}{\partial x} = \frac{\partial(x)}{\partial x}^T a + \left(\frac{\partial(a)}{\partial x}\right)^T x = a. \tag{5}$$

(b) Consider the following derivation

$$\frac{\partial(x^T A x)}{\partial x} = \frac{\partial(x^T)}{\partial x} A x + \frac{\partial(Ax)}{\partial x}^T X = Ax + Ax^T \tag{6}$$

(c) Consider the following derivation

(d)

Theorem 6. *If $x \in \mathbb{R}^n$*

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{7}$$

Proof. Squaring the first two terms of the inequality shows that $\|x\|_2^2$ has fewer terms than $\|x\|_1$.

Now define the following vector, e, so that $e_i = 1$ if x_i is positive and $e_i = -1$ if x is negative. Then $\langle x, e\rangle = \sum_i |x_i| = \|x\|_1$.

Cauchy schwartz says that $\langle x, e\rangle \leq \|x\|\|e\| = \|x\|_2\sqrt{n}$. This completes the proof. $\square$