FRE7251 Algo Trading & High-frequency Finance HW2

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1 Problem 1

Draw graphs of AR(1) process $y_k = 0.75y_{k-1} + e_k$ for
a) $y_0 = 1$
b) $y_0 = 10$
Assume that e_k is uniformly distributed on the interval $[-0.5, 0.5]$.

Hint: use the function rand() (or similar) for uniform distribution on t $\in$ interval $[0, 1]$ available in modern languages (C/C++, Java) and computational software (Excel, Matlab, etc.) and subtract 0.5 from its values.

Describe qualitative difference between the graphs a) and b)

1.1 Answer

In [1]:
```
import numpy as np
import pandas as pd
import math
import random
import matplotlib.pyplot as plt
random.seed(1124)
```

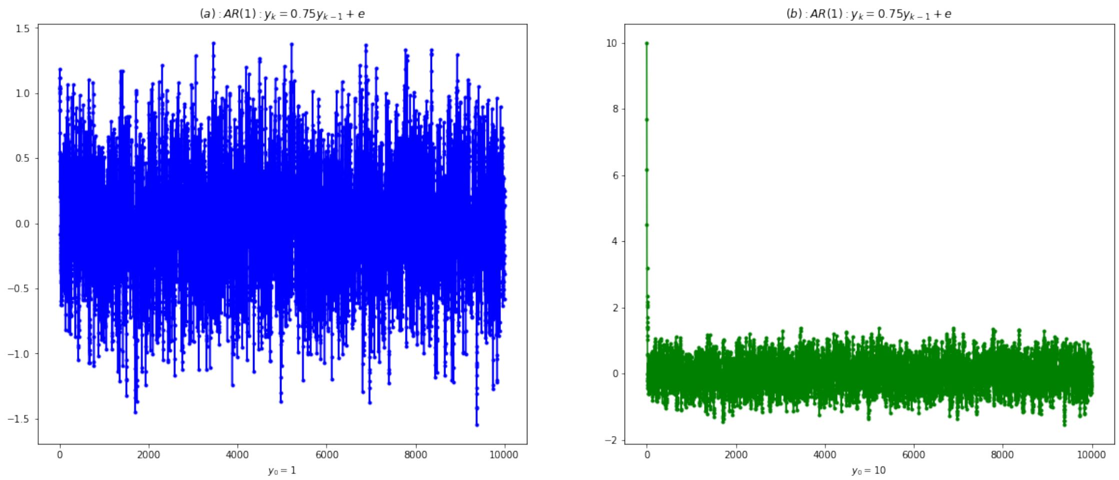
In [2]:
```
def AR_process(y0, e, N):
    y = np.zeros(N)
    y[0] = y0
    for i in range(1, N):
        y[i] = 0.75*y[i-1]+e[i]
    return y
```

In [3]:
```
N = 10000
e1 = np.random.uniform(-0.5, 0.5, N)
y1 = AR_process(1, e1, N)
y2 = AR_process(10, e1, N)

plt.figure(figsize=(20,8))
```

```
plt.subplot(121, title = '$(a): AR(1): y_k=0.75y_{k-1}+e$')
plt.xlabel('$y_0=1$')
plt.plot(y1, color = 'blue', marker = '.')

plt.subplot(122, title = '$(b): AR(1): y_k=0.75y_{k-1}+e$')
plt.xlabel('$y_0=10$')
plt.plot(y2, color = 'green', marker = '.')
plt.show()
```



1.2 Describe qualitative difference between the graphs a) and b)

In graph (a), AR(1) starts at $y_0 = 1$. Values fluctuate around 0. Most values are located in range (-1,1).
In graph (b), AR(1) starts at $y_0 = 10$ and keeps decreasing until y approaches approximately 1. Then values fluctuate around 0. Meanwhile, most values are located in range(-1,1) as graph(a).

2 Problem 2

Derive the autocorrelation coefficients $\rho(k)$ for AR(2) (provide all details!) and draw their values (up to $k = 10$ for
a) $a_1 = 0.8, a_2 = 0.1$
b) $a_1 = 1.0, a_2 = -0.5$
Describe qualitative difference between these two cases

2.1 Answer

$$AR(2) : y_t = a_1 y_{t-1} + a_2 y_{t-2} + e_t$$

Let $\Phi(z) = 1 - a_1 z - a_2 z^2 - ... - a_p z^p$.
Denote G_j as the Green function. $G_0 = 1, G_j = \sum_{j=0}^{2} k_i \lambda_i^j (j = 1, 2, ...)$.
By Green function, the stationary AR(p) process is,

$$
\begin{aligned}
y_t &= \frac{\epsilon_t}{\Phi(z)} \\
&= \sum_{i=1}^{p} \frac{k_i}{1-\lambda_i z}\epsilon_t \\
&= \sum_{i=1}^{p} \sum_{j=0}^{\infty} k_i(\lambda_i z)^j \epsilon_t \\
&= \sum_{j=0}^{\infty} \sum_{i=1}^{p} k_i \lambda_i^j \epsilon_{t-j} \\
&= \sum_{j=0}^{\infty} G_j \epsilon_{t-j}
\end{aligned}
$$

where, λ_i is the eigenvalues of the AR(2) process.
Caculate autocovariance,
By the Green function above, we have

$Var(y_t) = \sum_{j=0}^{\infty} G_j^2 Var(\epsilon_{t-j})$
Hence,

$$
\gamma(0) = \frac{1-a_2}{(1+a_2)(1-a_1-a_2)(1+a_1-a_2)}\sigma_\epsilon^2
$$

Since cov$[e_t, y_{t-i}] = 0, i \geq 1$,

$$
\begin{aligned}
\gamma(1) &= cov[y_t, y_{t-1}] = cov[a_1 y_{t-1} + a_2 y_{t-2} + e_t, y_{t-1}] \\
&= a_1 cov[y_{t-1}, y_{t-1}] + a_2 cov[y_{t-2}, y_{t-1}] + cov[e_t, y_{t-1}] \\
&= a_1\gamma(0) + a_2\gamma(1) \\
\rightarrow \gamma(1) &= \frac{a_1\gamma(0)}{1-a_2} \\
\rightarrow \gamma(2) &= cov[y_t, y_{t-2}] = cov[a_1 y_{t-1} + a_2 y_{t-2} + e_t, y_{t-2}] \\
&= a_1 cov[y_{t-1}, y_{t-2}] + a_2 cov[y_{t-2}, y_{t-2}] + cov[e_t, y_{t-2}] \\
&= a_1\gamma(1) + a_2\gamma(0)
\end{aligned}
$$

In summary, we can have

$$
\begin{aligned}
\gamma(0) &= \frac{1-a_2}{(1+a_2)(1-a_1-a_2)(1+a_1-a_2)}\sigma_\epsilon^2 \\
\gamma(1) &= \frac{a_1\gamma(0)}{1-a_2} \\
\gamma(2) &= a_1\gamma(1) + a_2\gamma(0) \\
\gamma(k) &= a_1\gamma(k-1) + a_2\gamma(k-2), k \geq 2
\end{aligned}
$$

Then calculate autocorrelation function, which is $\rho(k) = \frac{\gamma(k)}{\gamma(0)}$

Apply $\gamma(0)$,$\gamma(k)$ to the equation, we obtain

$$\rho(0) = 1$$
$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{a_1}{1-a_2}$$
$$\rho(k) = \frac{\gamma(k)}{\gamma(0)} = a_1\rho(k-1) + a_2\rho(k-2), k \geq 2$$

In [4]:
```
df = pd.DataFrame()

def ACF(k, a1, a2):
    ACF = [1]
    ACF.append(a1/(1-a2))
    for i in range(2, k+1):
        ACF.append(a1*ACF[i-1]+a2*ACF[i-2])
    return ACF

df['a'] = ACF(10, 0.8, 0.1)
df['b'] = ACF(10, 1.0, -0.5)
df
```

Out[4]:

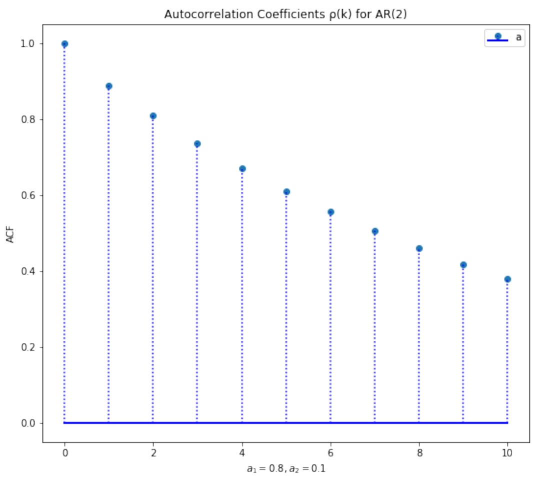
	a	b
0	1.000000	1.000000
1	0.888889	0.666667
2	0.811111	0.166667
3	0.737778	-0.166667
4	0.671333	-0.250000
5	0.610844	-0.166667
6	0.555809	-0.041667
7	0.505732	0.041667
8	0.460166	0.062500
9	0.418706	0.041667
10	0.380981	0.010417

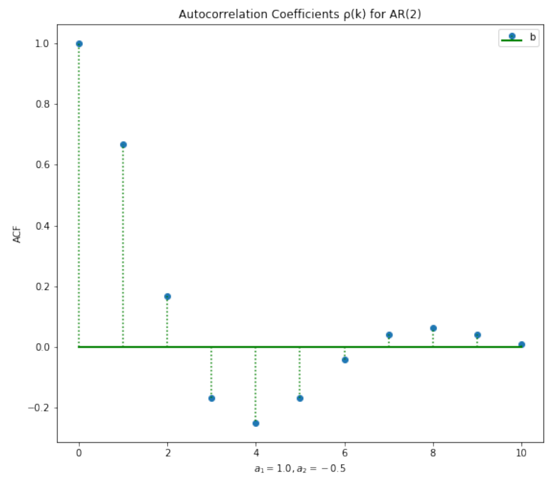
In [5]:
```
bottom = 0
plt.figure(figsize=(20,8))

plt.subplot(121, title = 'Autocorrelation Coefficients (k) for AR(2)')
plt.xlabel('$a_1=0.8,a_2=0.1$')
plt.ylabel('ACF')
markerline1, stemlines1, baseline1, = plt.stem(df['a'], bottom=bottom, label='a')
plt.setp(stemlines1, color='blue', linestyle=':')
plt.setp(baseline1, color='blue', linewidth=2, linestyle='-')
plt.legend(loc='best')


plt.subplot(122, title = 'Autocorrelation Coefficients (k) for AR(2)')
plt.xlabel('$a_1=1.0,a_2=-0.5$')
```

```
plt.ylabel('ACF')
markerline2, stemlines2, baseline2, = plt.stem(df['b'], bottom=bottom, label='b')
plt.setp(stemlines2, color='green', linestyle=':')
plt.setp(baseline2, color='green', linewidth=2, linestyle='-')
plt.legend(loc='best')
plt.show()
```





2.2 Describe qualitative difference between these two cases

- In the left graph above, autocorrelation coefficients are all positive and decrease to 0 slowly.
- In the right graph above, autocorrelation coefficients oscillate around 0. If increasing k enough, it will converge to 0 finally.

3 Problem 3

Verify if the process $y_k = 0.6y_{k-1} - 0.08y_{k-2} + e_k$ has unit root. If it does, modify the coefficient a_2 so that it does not. If the process does not have unit root, modify a_2 so that it has unit root. Draw graphs of both processes with $y_0 = y_1 = 1$ and e_k uniformly distributed on the interval $[-0.5, 0.5]$.

3.1 Answer

The characteristic polynomial for the AR(2) is:

$1 - a_1 z - a_2 z^2 = 0$
where $a_1 = 0.6, a_2 = -0.08$.
The solution for function $0.08z^2 - 0.6z + 1 = 0$ is:
$z = \frac{0.6 \pm \sqrt{(-0.6)^2 - 4(0.08)(1)}}{2(0.08)} = \frac{0.6 \pm \sqrt{(-0.6)^2 - 4(0.08)(1)}}{2(0.08)}$
Hence,
$z_1 = 5.0 > 1, z_2 = 2.5 > 1$
The process $y_k = 0.6y_{k-1} - 0.08y_{k-2} + e_k$ doesn't have unit root. This AR(2) is stationary.

```
In [6]: (0.6+math.sqrt(0.6**2-4*(-0.5)))/(-1)
```

```
Out[6]: -2.1362291495737216
```

```
In [7]: (0.6-math.sqrt(0.6**2-4*(-0.5)))/(-1)
```

```
Out[7]: 0.9362291495737217
```

Modify a_2 so that it has unit root.

$z = \frac{a_1 \pm \sqrt{(-a_1)^2 - 4(-a2)(1)}}{2(-a_2)} = \frac{0.6 \pm \sqrt{(-0.6)^2 - 4(-a2)(1)}}{2(-a_2)}$
Let $|z| \leq 1$
Hence, $a_2 \in (-0.09, 0) \cup [0.4, 0.6]$ if a_2 is a real number.

- For example, modify $a_2 = 0.4$ so that it has unit root

```
In [8]: z1 = (0.6-math.sqrt(0.6**2-4*(-0.5)))/(-1)
        z2 = (0.6+math.sqrt(0.6**2-4*(-0.5)))/(-1)
        print(z1, z2)
```

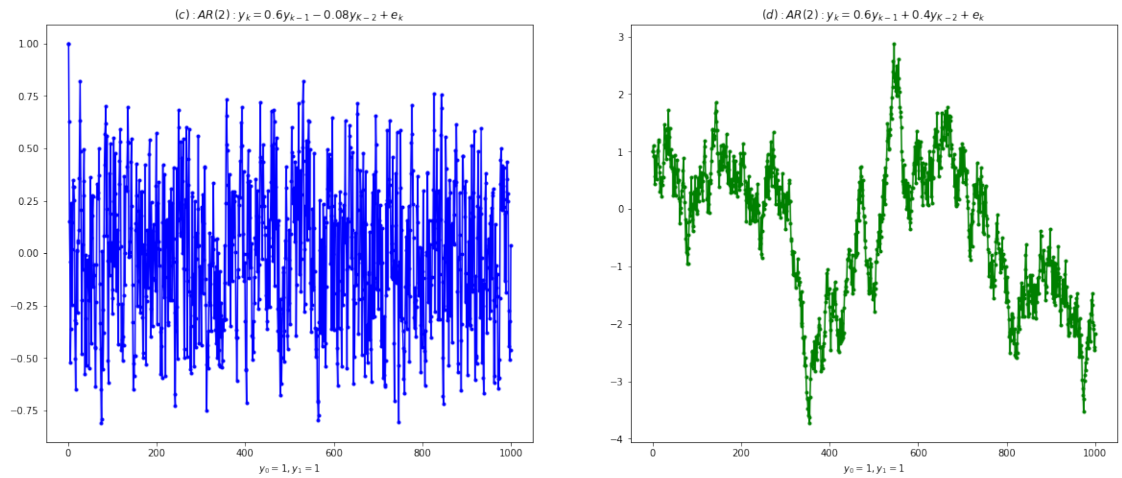
```
0.9362291495737217 -2.1362291495737216
```

```
In [9]: def AR2_process(y0, y1, a1, a2, e, N):
            y = np.zeros(N+1)
            y[0] = y0
            y[1] = y1
            for i in range(2, N+1):
                y[i] = a1*y[i-1]+a2*y[i-2]+e[i-2]
            return y
```

```
In [10]: N=1000
         e1 = np.random.uniform(-0.5, 0.5, N-1)
         y1 = AR2_process(1, 1, 0.6, -0.08, e1, N)
         # modify a2=0.4 so that it has unit root
         y2 = AR2_process(1, 1, 0.6, 0.4, e1, N)

         plt.figure(figsize=(20,8))
         plt.subplot(121, title = '$(c):AR(2): y_k=0.6y_{k-1}-0.08y_{K-2}+e_k$')
         plt.xlabel('$y_0=1, y_1=1$')
         plt.plot(y1, color = 'blue', marker = '.')

         plt.subplot(122, title = '$(d):AR(2): y_k=0.6y_{k-1}+0.4y_{K-2}+e_k$')
         plt.xlabel('$y_0=1, y_1=1$')
         plt.plot(y2, color = 'green', marker = '.')
         plt.show()
```



4 Problem 4

Impelment an algorithm for simulating the normal distribution. Repeat the exercise (1) with $e_k \sim N(0, 0.5)$. Describe the difference between results with uniform and normal distributions.
Hint: use the Box-Miller transfrom:
(e.g.https://en.wikipedia.org/wiki/Box–Muller_transform)
or the generic inverse transform:
https://en.wikipedia.org/wiki/Inverse_transform_sampling

4.1 Answer

Box–Muller transform
Suppose U_1 and U_2 are independent samples chosen from the uniform distribution on the unit interval $(0, 1)$. Let
$Z_0 = R\cos(\Theta) = \sqrt{-2lnU_1}\cos(2\pi U_2)$
$Z_1 = R\sin(\Theta) = \sqrt{-2lnU_1}\sin(2\pi U_2)$
where $R^2 = -2 \cdot lnU_1$ and $\Theta = 2\pi U_2$ Then Z_0 and Z_1 are independent random variables with a standard normal distribution.

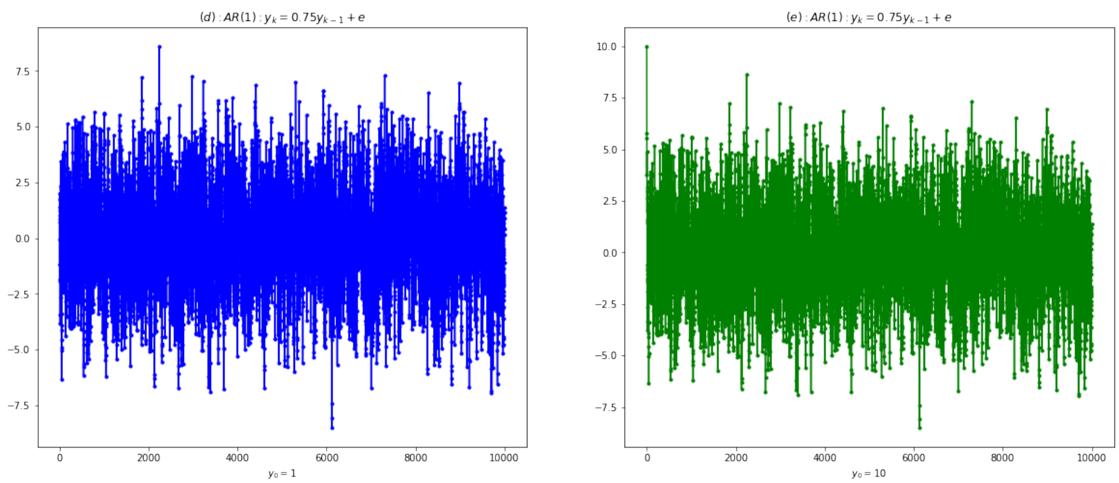
```
In [11]: # We also can use random.normal to get z1, z2
         # transformation function
         def generateGaussian(u1, u2):
           z1 = np.sqrt(-2*np.log(u1))*np.cos(2*math.pi*u2)
           z2 = np.sqrt(-2*np.log(u1))*np.sin(2*math.pi*u2)
           return z1,z2
```

```
In [12]: N = 10000
         #     uniformly distributed values between 0 and 1
         u1 = np.random.uniform(0, 1, N)
         u2 = np.random.uniform(0, 1, N)
```

```python
z1, z2 = generateGaussian(u1, u2)
y1 = AR_process(1, z1/math.sqrt(0.5), N)
y2 = AR_process(10, z1/math.sqrt(0.5), N)


plt.figure(figsize=(20,8))
plt.subplot(121, title = '$(d):AR(1): y_k=0.75y_{k-1}+e$')
plt.xlabel('$y_0=1$')
plt.plot(y1, color = 'blue', marker = '.')

plt.subplot(122, title = '$(e):AR(1): y_k=0.75y_{k-1}+e$')
plt.xlabel('$y_0=10$')
plt.plot(y2, color = 'green', marker = '.')
plt.show()
```



4.2 Describe the difference between results with uniform and normal distributions.

In graph(a) and (d), AR(1) starts at $y_0 = 1$.

For the graph(a) with uniform distribution, values fluctuate around 0 and most values are located in range (-1,1).

While for the graph(d) with normal distribution, values fluctuate around 0 and most values are located in range (-3,3).

In graph(b) and (e), AR(1) starts at $y_0 = 10$.

For the graph(b) with uniform distribution, values fluctuate around 0 and most values are located in range (-1,1).

While for the graph(e) with normal distribution, values fluctuate around 0 and most values are located in range (-3,3).

In summary, - the fluctuation range using normal distribution is larger than that of uniform distribution. - No matter what the start value is, the AR process approaches to around 0 eventually.