P_1

Let $A = \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_{,1} & a_{,2} & \cdots & a_{,k} \\ \vdots & \vdots & & \vdots \end{bmatrix}$ $B = \begin{bmatrix} \cdots & b_{1,} & \cdots \\ \cdots & b_{2,} & \cdots \\ & \vdots & \\ \cdots & b_{k,} & \cdots \end{bmatrix}$

rewrite $A = \begin{bmatrix} \vdots & 0 & & 0 \\ a_{,1} & 0 & \cdots & 0 \\ \vdots & 0 & & 0 \end{bmatrix} + \begin{bmatrix} 0 & \vdots & & 0 \\ 0 & a_{,2} & \cdots & \vdots \\ 0 & \vdots & & 0 \end{bmatrix} + \cdots \begin{bmatrix} 0 & & \vdots \\ \vdots & \cdots & a_{,k} \\ 0 & & \vdots \end{bmatrix}$

$B = \begin{bmatrix} \cdots & b_{1,} & \cdots \\ 0\ 0 & \cdots & 0\ 0 \\ 0\ 0 & \cdots & 0\ 0 \end{bmatrix} + \begin{bmatrix} 0\ 0 & \cdots & 0 \\ \cdots & b_{2,} & \cdots \\ 0\ 0 & 0 & 0 \cdots 0 \end{bmatrix} + \cdots \begin{bmatrix} 0\ 0 & \vdots & 0 \\ & \vdots & \\ \cdots & b_{k,} & \cdots \end{bmatrix}$

$\Rightarrow A \times B = \left(\begin{bmatrix} a_{,1} & \cdots & 0 \end{bmatrix} + \cdots \begin{bmatrix} 0 & \cdots & a_{,k} \end{bmatrix} \right) \times \left(\begin{bmatrix} \cdots & b_{1,} & \cdots \\ 0 & \cdots & 0 \end{bmatrix} + \cdots \begin{bmatrix} 0 & \cdots & 0 \\ \cdots & b_{k,} & \cdots \end{bmatrix} \right)$

By distributive propert of matrix multiplication

$\Rightarrow A \times B = \begin{bmatrix} a_{,1} & \cdots & 0 \end{bmatrix} \times \begin{bmatrix} \cdots & b_{1,} & \cdots \\ 0 & \cdots & 0 \end{bmatrix} + \cdots \begin{bmatrix} 0 & \cdots & a_{,k} \end{bmatrix} \times \begin{bmatrix} 0 & \cdots & 0 \\ \cdots & b_{k,} & \cdots \end{bmatrix}$

$= \text{outer}(a_{,1}, b_{1,}) + \text{outer}(a_{,2}, b_{1,}) \cdots \text{outer}(a_{,k}, b_{k,})$

$= \sum_{i=1}^{k} (a_{,i}, b_{i,})$

Let $M = U \Sigma V^T$

P_2

$$M = \begin{pmatrix} 3 & 0 \\ 1 & 0 \\ 0 & -2 \end{pmatrix} \quad M^T = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$M^TM = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 4 \end{pmatrix}$$

then we find the eigenvalues and eigenvectors

$\lambda_1 = 10 \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 4 \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow V^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$U = \left(\frac{1}{\sqrt{10}} \begin{pmatrix} 3 & 0 \\ 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \frac{1}{2} \begin{pmatrix} 3 & 0 \\ 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \frac{NS(M^T)}{\|NS(M^T)\|} \right)$$

where $NS(M) \Rightarrow Mx = 0 \Rightarrow u_3 = \begin{pmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix}$

$$U = \begin{pmatrix} \frac{3}{\sqrt{10}} & 0 & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{3}{\sqrt{10}} & 0 & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

P3 since Q is orthogonal

$\Rightarrow Q = U \Sigma V^T$

$= Q I I$

than $\|Q\|_2 = 1$

P4

WTS: $V_k = \underset{C: C^TC = I_k}{\operatorname{argmax}} Tr(C^T X^T X C)$

Since X^TX is diagonalizable with orthonormal eigenvectors $V_1 \dots V_n$, eigenvalue $\lambda_1 \geq \dots \geq \lambda_n \geq 0$

and $C \in \mathbb{R}^n \Rightarrow C = \sum_i a_i V_i$, $\sum a_i^2 = 1$

$\Rightarrow C^T X^T X C = \sum_{i,j} a_i a_j V_i^T \lambda_j V_j$

$= \sum_i a_i^2 \lambda_i$

is maximized when $a_1 = 1$, $a_2 \dots a_n = 0$

By induction

when $a_k = 1$, and the rest $= 0$,

$\sum_i a_i^2 \lambda_i$ is maximized with the

kth eigenvalue.

$\Rightarrow V_k = \underset{C}{\operatorname{argmax}} Tr(C^T X^T X C)$