Multi-scenario optimization approach for assessing the impacts of advanced traffic information under realistic stochastic capacity distributions

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Abstract

In this study, to incorporate realistic discrete stochastic capacity distribution over a large number of sampling days or scenarios (say 30 to 100 days), we propose a multi-scenario based optimization model with different types of traveler knowledge in an advanced traveler information provision environment. The proposed method categorizes commuters into two classes: (1) those with access to perfect traffic information every day, and (2) those with knowledge of the expected traffic conditions (and related reliability measure) across a large number of different sampling days. Using a gap function framework or describing the mixed user equilibrium under different information availability over a long-term steady state, a nonlinear programming model is formulated to describe the route choice behavior of the perfect information (PI) and expected travel time (ETT) user classes under stochastic day-dependent travel time. Driven by a computationally efficient algorithm suitable for large-scale networks, the model was implemented in a standard optimization solver and an open-source simulation package and further applied to medium-scale networks to examine the effectiveness of dynamic traveler information under realistic stochastic capacity conditions.

Keywords: stochastic road capacity, traffic assignment, travel time variability, value of dynamic traveler information, risk-sensitive route choice behavior

1. Introduction

Major sources of congestion include recurring bottlenecks, incidents, work zones, inclement weather, poor signal timing, special events, and day-to-day fluctuations in normal traffic demand. Considerable research efforts have been devoted to understanding and quantifying the effectiveness of different traffic mitigation strategies in addressing various sources of congestion. For instance, recurring congestion due to bottlenecks can be mitigated through road capacity enhancement, while real-time traffic information dissemination can reduce negative impacts of disruptions of non-recurring congestion due to traffic incidents and special events.

In our research, we specifically focus on how to evaluate the impact of providing traffic information within a congested network with a realistic stochastic capacity distribution over multiple sampling days. As shown in a recent research effort by Jia et al (2010) based on several bottleneck locations in the Bay Area, California, the sample histogram in Fig. 1 demonstrates the discrete probabilistic distribution of 100 lane capacity day samples, with a sample mean of 1837 vehicles/hour/lane and a coefficient of variation of 0.064. As most of samples range from 1400 to 2100 vehicles/hour/lane, which reveals the inherent randomness of stochastic road capacity (Brilon, et al. 2005, 2007). Even under constant geometric, traffic, environmental, and/or operational conditions, road capacities vary with time over a certain range around a mean value. More precisely, highway capacity is the result of complex driver behavioral interaction. It also varies according to many external factors such as accidents, incidents, severe weather, or work zones. The inclusion of stochastic capacity at the critical points in a highway network produces a more realistic modeling of travel time variability and introduces the concept of sustainable flow rates.

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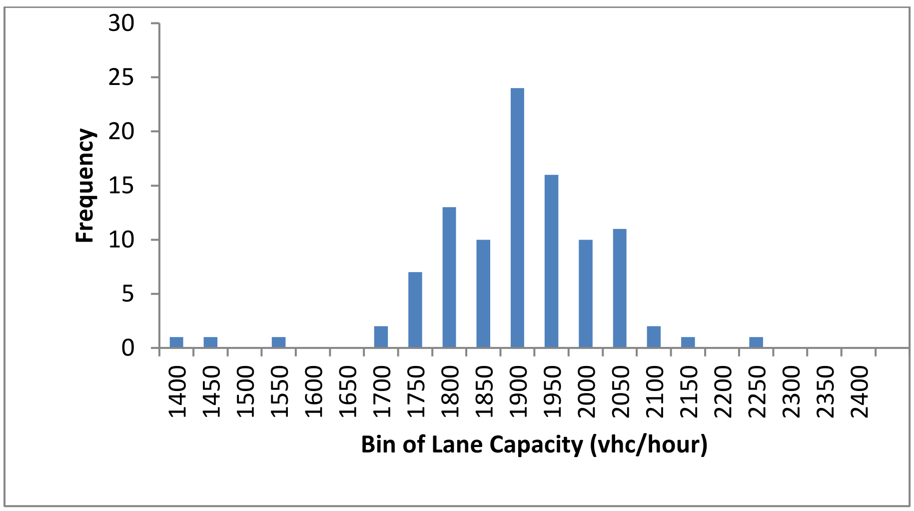


Fig. 1. Example histogram of 100 stochastic road capacity samples.

In addition to *stochastic capacity*, four other sources also contribute to increase travel time variability and unreliability: 1) *Stochastic input demand*, 2) *Random departure time choice and route choice* which can lead to uncertain flow inputs for a certain set of links (Noland & Polak, 2002), 3) *The absence of precise and real-time traffic information* due to the sensor coverage or limited traveler knowledge/experience, which can further compound the issue of travel time uncertainty, and 4) *Traveler perception error*. In fact, there are only a small fraction of travelers who currently have full access or are willing to always retrieve pre-trip or en-route traveler information through web-based traveler information sites, car radio, dynamic message signs, or Internet-connected navigation devices (Khattak et al. 2008). When making route choices, the majority of travelers still rely on their personal knowledge and driving experiences that have been gained over a long time period of time, which can be described as the expected travel time (caused by stochastic demand and capacity). When there is a significant variation in capacity, the resulting network conditions could deviate considerably from the average traffic pattern. In this case, the expected value-based travel knowledge should be treated as a biased estimate to the current traffic state.

Based on the current active demand management program from US Federal Highway Administration (FHWA, 2016), we can find a wide range of emerging traffic mitigation applications (shown in Table 1) are closely related to dynamic traveler information provision. The effectiveness of those emerging strategies cannot be simply evaluated under a perfect equilibrium condition, as those demand-responsive and supply-responsive strategies are very sensitive to the underlying capacity and congestion conditions. To consider a long-term steady state under both recurring and nonrecurring traffic congestion, we need to realistically model those strategies under stochastic capacity/demand variations over a large number of sampling days.

Table 1 List of emerging active traffic management strategies related to traffic information provision (Adapted from FHWA, 2016).

Strategy	Description	Expected benefits of providing dynamic information for travelers
Dynamic Fare Reduction (e.g. Savage, 2010)	This strategy reduces the transit fare in a specific corridor when congestion on that corridor increases.	Real-time transit fare adjustment to distribute the traffic spatially and promote mode shift
Dynamic High-Occupancy Vehicle (HOV)/Managed Lanes (e.g., Zhou, et al. 2008, Lou et al. 2011)	This strategy dynamically changes the restrictions on the number of occupants of a vehicle driving in an HOV lane.	Providing a reliable travel time for carpoolers and carpooling promotion
Dynamic Pricing (e.g. Lou et al. 2011)	This strategy dynamically changes the tolls in response to changing congestion levels.	Encouraging motorists to take unnecessary trips at a different time, or choose alternative routes or transportation modes if available
Dynamic Ridesharing (e.g. Mahmoudi & Zhou, 2016)	This strategy involves passengers using technologies to set a short-notice shared ride.	Utilizing vehicles only when needed (reduces car ownership)
Transfer Connection Protection (e.g. Chung & Shalaby, 2007)	This strategy improves the reliability of transfers from a high-frequency transit mode to a low-frequency transit service.	Guaranteeing that the connections are not missed
Dynamic Routing (e.g. Chatterjee & McDonald, 2004)	This strategy disseminates traffic information among road-users in order to better utilize roads capacity by directing vehicles to less congested areas.	Distributing the traffic spatially and providing route guidance

Many studies in the literature (e.g. Yang, 1998; Yang and Meng, 2001; Yin and Yang, 2003) use a general stochastic user equilibrium (SUE) traffic assignment model to quantify the value of traveler information under deterministic and time-invariant road capacity, where all travelers are assumed to have unbiased travel time estimates but with different degrees of uncertainty with respect to their own information user group (e.g. equipped with an ATIS or not). The uncertainty levels are modeled through the perception error term in SUE. That is, commuters with advanced traveler information have smaller perception errors, compared to travelers without access to ATIS channels. It should be noted that the majority of the related references assume static deterministic road capacity. Bell & Cassir (2002) considered equilibrium traffic assignment as a non-cooperative, n-player game. Recently, in order to reformulate the traditional static traffic assignment problem under stochastic capacity conditions, different numerical approximation methods have been proposed to describe the travel time variability for a single-day traffic equilibrium solution. For example, Lo and Tung (2003) and Lo et al. (2006) adopted Mellin transforms to describe network performance caused by stochastic link capacities. Chen and Zhou (2010) proposed a α-reliable mean-excess traffic equilibrium (METE) model to represent equilibrium route choice assignment under stochastic demand and supply. Their model assumes that travelers are willing to minimize their conditional expectation of travel times, with specific risk-taking consideration in their route choice decisions. According to Mirchandani and Soroush (1987), travelers' risk-taking behavior under an uncertain environment can be categorized into risk-prone (Chen & Zhou, 2010), risk-neutral (Arnott, et al., 1991 & 1999; Gao, 2012; Tan & Yang, 2012; Ban, et al., 2013; Lu, et al.,2014; Rapoport, et al.,2014) and risk-averse (Bell, et al.,2002; Chen, et al., 2011; Chorus, et al., 2006 &2010; Connors & Sumalee, 2009; Lam, et al., 2008). The non-uniqueness of path flow solutions was well recognized in the past, and was circumvented by imposing additional constraints or assumptions that limit the set of feasible solutions to remain unique path flows (He, et al., 2010; Guo 2013; Ban, 2013). Under uncertain traffic situation, adding more assumptions or constraints to guarantee unique equilibrium is more desirable for a specific case with particular emphasis on ensuring a unique equilibrium.

Table 2 Comparison between the related studies and this paper (presented in chronological order).

	Sources of Travel Time Variability	**Analysis Horizon**	**Route Choice Disutility Function**	**Approach to Model and Find Equilibrium**
Arnott et al. (1991)	Stochastic road capacity, unpredictable demand fluctuations	Single day	Mean travel cost, early arrival costs and late arrival costs	Deterministic solution with the values of capacity and demand on that day
Ben-Akiva et al. (1991)	Capacity levels, demand characteristics and driver intrinsic characteristics	Day-to-day	Travel time costs, and psychological costs	The exogenous information follows a stationary stochastic process
Arnott et al. (1999)	Stochastic capacity and demand	Within-day dynamic	Cost of queuing and schedule delay	Nash equilibrium with arrival times as the choice variables
Bell and Cassir (2002)	Variation in cost perception of network users	non-cooperative game	Mean travel time plus safety margin	Method of Successive Average (MSA)
Chorus et al. (2006)	Information acquisition and reaction to travel information	N/A	Expected regret induced by a choice situation	Expectations of regret
Siu et al. (2006, 2008)	Random link capacity degradations Perception variations	Steady state under probability distribution	Provision of quality information being effective only in certain situations	Gap function
Lam et al. (2008)	Day-to-day demand variation, Road capacity degradation	Fixed point problem	A generalized travel time function	Reliability-based fixed point problem by MSA
Connors &Sumalee (2009)	Variability of travel times	Equilibrium analysis	Path costs and the path utility variances	Perceived value (of link flow) based User Equilibrium
Chen &Zhou (2010)	Demand fluctuation and capacity degradation	Steady state under probability distribution	Traveler's confidence level of on-time arrival	Self-adaptive alternating direction method
Chen et al. (2011)	Link and path travel time uncertainties	Steady state under probability distribution	Expected travel time and standard deviation	Gap function, path based solution, reliable route algorithms for restricted problem
Zhang et al. (2011)	Uncertain demand Random road capacity	Steady state under probability distribution	Both probability and magnitude of flow and minimum cost	Smoothing projected gradient (SPG) method
de Palma (2012)	Stochastic travel time	Good and bad days	Probability distribution of travel times	Critical value for information
Gao S. (2012)	Stochastic dynamic demand and supply	Fixed point solution	Routing policies	Routing policy based dynamic network loading, MSA heuristics
Ban et al. (2013)	Demand or supply or toll solution uncertainties	Variational inequality	Weighted summation of link travel time and the toll	quadratic approximation of the objective, simulation optimization based algorithm
Lu et al. (2014)	Incidents, adverse conditions, unpredictable behavior of other travelers	Day-to-day	Risky branch based utility function	All used routing policies have the same and minimum mean travel times
Rapoport et al. (2014)	Stochastic travel conditions	Good and bad states, equilibria with pure and mixed strategies	travel cost with welfare effects	No systematic changes in behavior over a large number of iterations
This paper	Realistic multiday stochastic link capacity degradation	Large number of sampling days	Day-dependent travel time plus the dollar value of road toll	Gap function-based reformulation for approximating user equilibrium, solved by MSA

By considering time-dependent demands in a dynamic traffic assignment process as random variables with known probability distributions, Waller and Ziliaskopoulos (2001) offered one of the first modeling frameworks to fully take into the impact of stochastic demand variability for network design applications. Ng and Waller (2010) further presented a fast Fourier transform-based method to characterize travel time variability distributions due to random capacity. It should be noticed that in the context of ATIS benefit evaluation, the single-day steady-state representation is insufficient to capture the day-varying route choice behaviors by ATIS equipped drivers.

Lam et al. (2008) developed a generalized link travel time function to capture the uncertainties of random travel time on each path in their path choice decisions due to adverse weather conditions. Siu and Lo (2006, 2008) studied numerical formulations to link travel time variability with random link capacity degradations and imperfect traffic information, and they used a stochastic equilibrium model to address both types of uncertainty. Zhang et al. (2011) developed an expected residual minimization model to provide the "robust" prediction of future traffic flows for the planning and evaluation purposes. Based on a link-based variational inequality formulation, Dixit et al. (2013) proposed a strategic traffic assignment model which assumes deterministic road capacity but stochastic OD demand (e.g., through a lognormal distribution with known mean and variance). To compute the expected mean and variance of system-wide travel times, they developed an iterative solution procedure that involves Monte Carlo sampling for generating demand realizations/samples and the use of Frank-Wolfe algorithm for finding equilibrium link flow. Duell et al. (2014) extended the strategic user equilibrium model to evaluate a first-best tolling strategy. Recently, Li et al. (2016) used a point queue model to demonstrate how day-to-day travel time variability can be explained from the underlying demand and capacity variations.

A systematic comparison between our proposed approach and the related papers is provided in Table 2. Most of the related papers do not consider a discrete capacity distribution across a large number of sampling days in a stochastic traffic evolution environment, while the main focus has been on how to embed and reformulate a quite complex travel time function where the standard deviation of travel time is typically derived numerically.

In order to distinguish different system throughout states (i.e., stochastic capacity) in a user equilibrium framework, de Palma and Picard (2005) used a graphical method to consider two types of information user classes, including (1) those with perfect information on good days and bad days (i.e. under normal and reduced capacity with higher travel time); and (2) those with information on expected travel times on different days. Their pioneering investigation, together with their recent work that integrates information-acquisition and route-choice decisions (de Palma et al., 2012), provide theoretical insights into analyzing traveler behavior under stochastic capacity. Along these lines, this research has focused on developing a mathematical programming model and efficient solution algorithms for general traffic networks with realistic stochastic capacity distributions. In the studies by Arnott et al. (1991, 1996) and de Palma and Picard (2006), which compared the extremes of zero information and perfect information and used them as a benchmark to formalize the "better" information. Compared to their approach only focusing on typical good days and bad days, our proposed formulation specifically highlights the need of establishing a multi-scenario or multi-day framework to systematically consider stochastic capacity variations over a large number of sampling days, and further evaluate the behavior or expected value information users across different days, and accordingly evaluating the impact of perfect and dynamic traveler information users.

With a special focus on approximating the long-term steady-state user equilibrium problem with realistic stochastic capacity distribution, this research aims to find a network flow pattern that satisfies a generalization of Wardrop's first principle: travelers with the same origin-destination pair experience the same and minimum expected travel time along any used paths on different days, with no unused path offering a shorter expected travel time. The classical gap function-based framework for user equilibrium by Smith (1993) and Lo and Chen (2000 a, 2000 b) reformulated the nonlinear complementarity problem for traffic user equilibrium with fixed demand and capacity to a route flow-based mathematical program through a convex and smooth gap function. Through the gap function-based reformulation for user equilibrium, the proposed model explicitly considers the stochastic nature of network capacity over different days and represents travelers' imperfect information and general knowledge about the random travel time variations.

The paper is organized as follows. Starting in section 2, we illustrate the multi-day evaluation framework conceptually, and then further consider the route choice behavior of risk-sensitive travelers in Section 3 using nonlinear optimization models which can be solvable by a standard optimization package (e.g., GAMS

(Rosenthal, 2015) or MINOS (Murtagh and Saunders, 2003). In section 4, a computationally efficient solution method is developed to find the equilibrium path flow distribution, while stochastic link capacities on a large number of days and the effectiveness of ATIS are evaluated from a sampling-based simulation framework in section 5.

2. Conceptual framework with 5 sampling days

The conceptual modeling framework is illustrated in Fig. 2 using a simple corridor with a single origin-to-destination pair and two paths p=1 for the primary path, p=2 for the alternative path, where p is the path index. As each path only has one link, path 1 is denoted as link $a=1$ with a free-flow travel time of 20 minutes, and path 2 is denoted as link a=2 with a free-flow travel time of 30 minutes, where a is the link index. This example considers five different days d = 1, 2, 3, 4 and 5, and the peak hour demand is Q=8000 vehicles per hour on each day.

Following a similar analysis setting by de Palma and Picard (2005) but with specific highlights on the role of expected information users, we use the first illustrative example to consider day 1 as the "bad" day on path 1, with a reduced capacity for the primary route, and days 2, 3, 4, and 5 as good days with the full capacity available.
As detailed in Table 3, the primary path has the following capacity values: On the bad day (d=1) it is 3,000 vehicles per hour (vph) per link; on the good days (d=2, 3, 4, 5) it is 4,500 vph per link.
The alternative path is assumed to have a fixed capacity of $c_{a,d}$ = 3,000 on days d=1, 2, 3, 4 and 5, where $c_{a,d}$ is defined as the capacity of link a on day d.

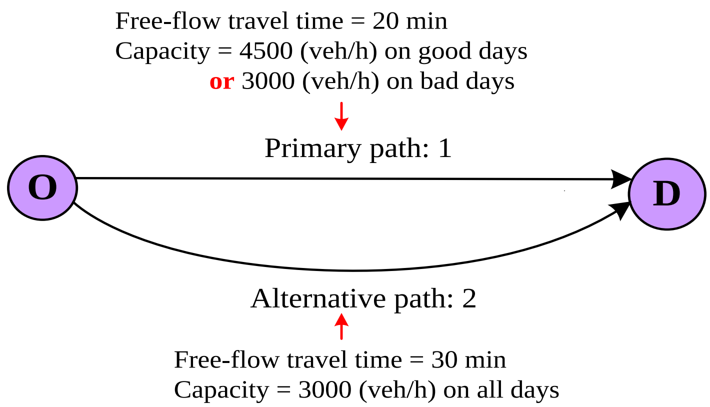


Fig. 2. Simple network used as an illustrative example of the framework.

Table 3 Day-dependent path demand, capacity and travel time values

			Day 1	Day 2	Day 3	Day 4	Day 5	Avg.
Day-Dependent Capacity	Daily Capacity on Path/Link 1 (veh/h) $c_{a=1,d}$	Path 1	3000*	4500	4500	4500	4500	4200
	Daily Capacity on Path/Link 2 (veh/h) $c_{a=2,d}$	Path 2	3000	3000	3000	3000	3000	3000
PI-Based User Equilibrium	Flow (Veh/hour/link)	Path 1	4636	6172	6172	6172	6172	5865
		Path 2	3364	1828	1828	1828	1828	2135
	Travel Time (min)	Path 1	37.1	30.6	30.6	30.6	30.6	31.9
		Path 2	37.1	30.6	30.6	30.6	30.6	31.9
ETT-Based User	Flow (Veh/hour/link)	Path 1	5503	5503	5503	5503	5503	5503
		Path 2	2497	2497	2497	2497	2497	2497

Equilibrium	Travel Time (min)	Path 1	54.0	26.7	26.7	26.7	26.7	32.2
		Path 2	32.2	32.2	32.2	32.2	32.2	32.2

*reduced capacity

To setup a mathematical programming model for steady-state traffic equilibrium, the non-negative flow variable $f_{p,d}$ is considered as the traffic flow using path p on day d. Obviously, the path flow distribution should ensure the total demand constraint on each day:

$$f_{1,d} + f_{2,d} = Q \quad \forall d \tag{1}$$

Let $T_{p,d}$ be defined as the travel time on path p on day d, which can be calculated from the BPR function such as

$$T_{p,d} = FFTT_a \times \left(1 + \alpha \times \left[\frac{f_{a,d}}{c_{a,d}}\right]^{\beta}\right) \tag{2}$$

where $FFTT_a$ is the free-flow travel time of link a. Coefficients α and β are set to commonly used default values 0.15 and 4, respectively. Now the two different degrees of traveler knowledge can be examined. In our numerical experiments, we use an open source simulator DTALite (Zhou, Taylor, 2014) to implement the above static travel time function and possible time-dependent queue-based models to evaluate the system performance.

Perfect Information (PI) based user equilibrium

Every day, perfect travel time estimates (i.e. zero prediction error) for all links are available to travelers to make route decisions, and travelers can switch routes every day. This perfect and complete information assumption for each day is consistent with deterministic static traffic assignment, which usually considers a typical weekday. It should be cautioned that this assumption might not be realistic from a dynamic traffic assignment perspective, as both pre-trip and en-route traveler information available to commuters are essentially forecasted estimates of traffic conditions unfolding in the future, with always some degree of prediction errors. According to Wardrop's first principle of user equilibrium, for a specific origin-destination pair, travelers with perfect information experience the same and minimum travel time along any used paths on each day *d*, with no unused path offering a shorter travel time.

To construct the objective function in the optimization model, the following gap function (for each day d) can be used to characterize the Karush-Kuhn-Tucker optimality conditions (Wiki, 2010) required for reaching the user equilibrium for perfect information users.

$$gap_d^{PI} = f_{1,d}^{PI} \times (T_{1,d} - \pi_d) + f_{2,d}^{PI} \times (T_{2,d} - \pi_d) = 0, \forall d \tag{3}$$

Where $f_{1,d}^{PI}$ and $f_{2,d}^{PI}$ are path flow rate of PI users on paths 1 and 2, respectively, on day d, where π_d is minimum path travel time on day d

$$\pi_d = \min(T_{1,d}, T_{2,d}), \forall d \tag{4}$$

For the illustrative simple corridor, Table 3 shows the traffic assignment results when all travelers in the network have access to perfect information, and a standard deterministic user equilibrium state is reached every day. See also Fig. 3 for a graphical representation of the flows and travel times for PI users.

Expected travel time (ETT) knowledge-based user behavior based on multiple sampling days

As there are different realized capacity values on different days, the travel times on different links can be viewed as a set of random variables. In reality, most travelers are not equipped with advanced traveler information systems, so they rely on their expected travel times (based on their knowledge and experience) over different days to make route choices. The expected travel time can be considered as the long-run average, or more precisely, the probability-weighted sum of the possible travel time values from different days. Under a user equilibrium condition with ETT users, the expected travel times on used routes in the network are assumed to be the same, and accordingly, an ETT user selects the same route every day, regardless the actual traffic conditions.

This paper does not consider a more realistic day-to-day learning model to capture averaging behavior, as we want to find the long-term steady state solution first under the simple assumption of arithmetic average over multiple days. Interested readers are referred to a recent paper by Jia et al. (2011) on a day-to-day traffic simulation and traveler learning framework with stochastic road capacity. It should be noticed that, random link capacity such as severe incidents could result in outliers that introduce bias to the mean statistics, but finding percentile robust shortest path is another very difficult problem.

The expected travel time for link a with random capacity $\tilde{c}_a$ over different days can be represented as

$$\bar{T}_a(f_a,\tilde{c}_a)=\frac{\sum_d T_{a,d}(f_{a,d},c_{a,d})}{|d|}\ , \tag{5}$$

Where travel time on each day *d* for link *a*, $T_{a,d}(f_{a,d},c_{a,d})$ is a function of the prevailing flow and capacity on that particular day. More precisely, f_a in the left hand side of Eq. (5) is a sequence/vector of path flows over different days. $f_{a,d}$ is path flow rate on path a on day d. $\tilde{c}_a$ represents a vector of stochastic link capacities on different days generated from a sampling-based simulation framework.

For link *a*=1 in the illustrative example,

$$\begin{aligned}\bar{T}_a(f_a,c_a)&=0.2\times\bar{T}_a(f_a,c_a^R)+0.8\times\bar{T}_a(f_a,c_a^F)\\&=0.2\times FFTT_a\times\left(1+\alpha\times\left[\frac{f_a}{c_a^R}\right]^\beta\right)+0.8\times FFTT_a\times\left(1+\alpha\times\left[\frac{f_a}{c_a^F}\right]^\beta\right)\\&=FFTT_a\times\left(1+0.2\times\alpha\times\left[\frac{f_a}{c_a^R}\right]^\beta+0.8\times\alpha\times\left[\frac{f_a}{c_a^F}\right]^\beta\right)\end{aligned} \tag{6}$$

Where c_a^R and c_a^F correspond to the reduced and full capacity on link *a*.

Note that $\bar{T}_a$ is different from the expected value (EV) solution $T_a(f_a,\bar{c}_a)$ typically used in the context of stochastic optimization, which can be calculated using the expected value of capacity on link *a*, $\bar{c}_a$.

$$T_a(f_a,\overline{c_a})=FFTT_a\times\left(1+\alpha\times\left[\frac{f_a}{\overline{c_a}}\right]^\beta\right)=FFTT_a\times\left(1+\alpha\times\left[\frac{f_a}{0.2\times c_a^R+0.8\times c_a^F}\right]^\beta\right) \tag{7}$$

In this study, we generalize Wardrop's first principle to describe the equilibrium conditions for travelers relying on their expected travel time to make route decisions: travelers with the same origin-destination pair experience the same and minimum expected travel time along any used paths on different days, with no unused path offering a shorter expected travel time. Obviously, when there is a single capacity value, then the above conditions are consistent with the standard user equilibrium with deterministic capacity, as the expected travel time devolves to the travel time on the single day.

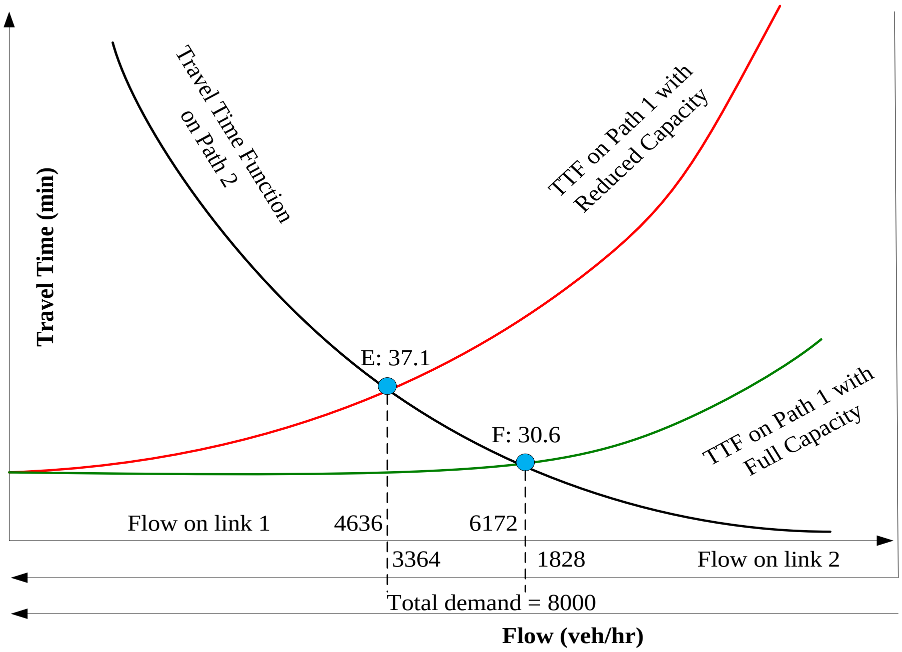


Fig. 3. Equilibrium solutions with 100% PI users.
Point E: reduced-capacity days, equilibrated travel times = 37.1 min, 4636 vehicles on link 1 and 3364 vehicles on link 2;
Point F: full-capacity days, equilibrated travel times = 30.6 min with 6172 vehicles on link 1 and 1828 vehicles on link 2.

The corresponding KKT condition can be re-written as

$$gap^{ETT} = f_1^{ETT} \times (\overline{T}_1 - \overline{\pi}) + f_2^{ETT} \times (\overline{T}_2 - \overline{\pi}) = 0 \quad (8)$$

where $\overline{\pi}$ is the least expected travel time between the given OD pair over a multi-day horizon satisfies

$$\overline{\pi} = \min(\overline{T}_1, \overline{T}_2) \quad (9)$$

An ETT knowledge-user uses the same route on different days, which leads to a day-invariant ETT flow pattern:

$$f_a^{ETT} = f_{a,d=1}^{ETT} = f_{a,d=2}^{ETT} = f_{a,d=3}^{ETT} = f_{a,d=4}^{ETT} = f_{a,d=5}^{ETT} \ \forall a \quad (10)$$

When $gap^{ETT} = 0$, it can be shown that if $f_a^{ETT} > 0$, then $\overline{T}_a = \overline{\pi}$. That is, the selected routes by expected travel time information users between an OD pair have equal and minimum costs. On the other hand, if $f_a^{ETT} = 0$, then $\overline{T}_a \geq \overline{\pi}$, which indicates that all unused routes by ETT users have greater or equal costs (compared to the used path costs). These two conditions further imply that no individual trip maker with expected travel time information can reduce his/her expected path costs by switching routes on any given day, under a user equilibrium condition.

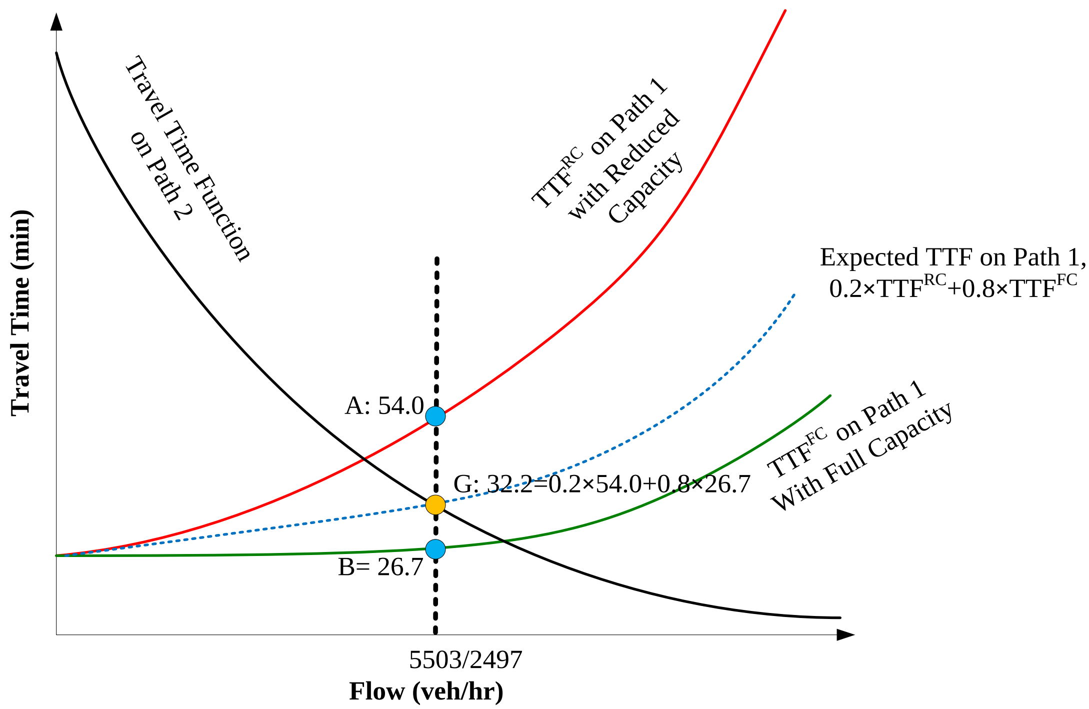


Fig. 4. Solutions with 100% ETT information users, the expected travel time function (TTF) is generated by assigning a 20% weight to TTF with reduced-capacity (RC) days and an 80% weight to TTF with full-capacity (FC). The ETT-based user equilibrium corresponds to the intersection (in orange) of expected TTF on path 2 and path 1. 5503 vehicles are using link 1 and 2497 vehicles are using link 2 each day.
Point A: travel time = 54.0 min on link 1, reduced-capacity days,
Point B: travel time = 26.7 min on link 1, full-capacity days.
Point G: travel time = 32.2 min on link 2 every day, and the expected travel time on link 1 is the same 32.2 min.

If it is assumed that all users rely on ETT information in the simple corridor, then the ETT-based user equilibrium assigns about 5503 vehicles on path 1, and about 2497 vehicles on path 2, leading to different travel time on 5 different days shown in Table 3. As both paths carry positive flows, their average travel times over the 5-day horizon are the same at 32.2 min.

Quantification of the value of information

In Fig. 5, the relative travel time savings were examined under different market penetration rates of perfect information users on a reduced-capacity day when there are both PI and ETT users. At points A and B, there is no PI users, so the ETT users on each path is shown as the previous. However, when there are 5% PI users in travel demand, 5284 ETT vehicles will be on link 1 with travel time of 48.9min, and 2716 vehicles (2316 ETT users + 400 PI users) will choose link 2 with travel time of 33min, as shown at points C5 and D5. The detailed modelling and algorithm will be presented in the following sections. The average travel time for ETT user is (5284*48.9+2316*33.0)/7,600=44.0 min, so the travel time saving for PI users is 44.0-33.0=11.0 min. Similarly, points C10 and D10 represent the assignment result when there are 10% PI users, and the corresponding time saving for PI users will be 7.1 min. As the flow of PI users increases, the time saving will decreases and the traffic condition finally will reach equilibrium point where no PI user can reduce his/her travel time by switching routes. Point E20 shows that the traffic is at equilibrium condition when there are 20% PI users. The travel time on two links is same of 37.1 min, and it turns out to be no travel time saving for PI users.

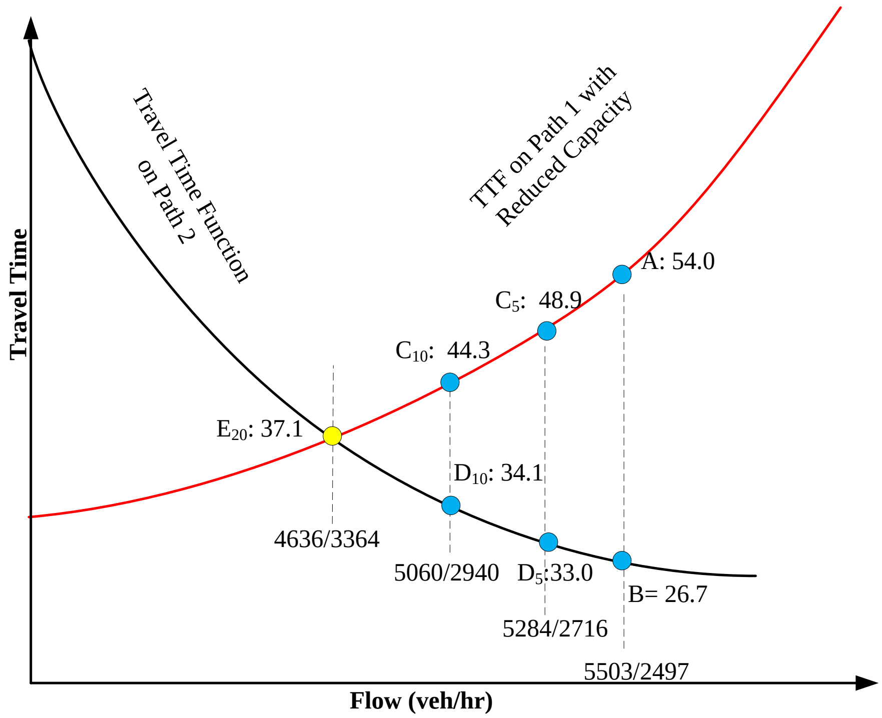


Fig. 5. Solutions on a reduced-capacity day.

As the proposed model approximates a long-term stochastic steady state under stochastic capacity, Fig. 6 further shows a possible sequence of the corresponding travel times of ETT users on 2 different routes over a 20-day horizon. Note that, even though there still exists a 5-day cycle with 4 good days and 1 bad day with reduced capacity, the impaired capacity conditions occur on day 1, day 7, day 11 and day 19 in this example. This irregularity, which is permitted by the model, shows the un-predictability of stochastic travel times, so the ETT knowledge users simply consider the average travel time on path 1 (with a 20% chance or risk of severe traffic congestion) in their long-term route choice decisions. It should also be noted that the behavioral model used here is structurally different from the commonly used day-to-day learning model in a DTA framework, even though the underlying traffic states are represented within the same multi-day structure. In a day-to-day dynamic learning model, the perceived travel time on day d +1 is updated using experienced travel times from previous days d, d -1, d -2 and so on, and the path can also be changed on a daily basis. In comparison, the ETT knowledge users consider average traffic conditions over all the days as a whole, and always stick to the same route, which means that the ETT flow assigned on each route is fixed on different days.

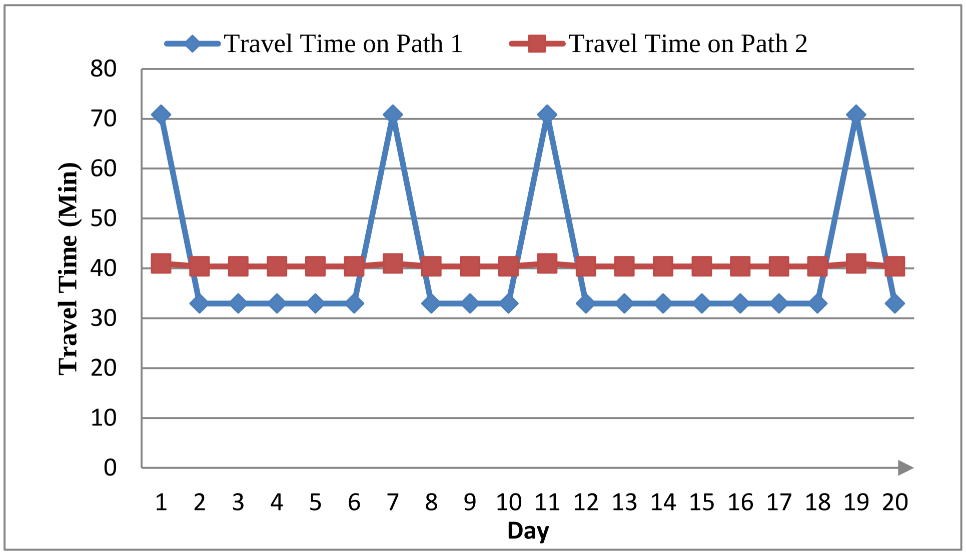


Fig. 6. Day-dependent travel times on different routes.

3. General nonlinear programming formulation for a large number of sampling days

This section extends the above conceptual framework to a general network with multiple origin-destination pairs and with road pricing strategies. A systematic comparison between our proposed approach in Section 2 and the related papers is highlighted in the conclusion part of our paper.

Formulation

The sets and subscripts, parameters and decision variables in the proposed flow assignment model are introduced as follows:

Indices:

i = index of origins, $i = 1, \ldots, I$, where I is the number of origins

j = index of destinations, $j = 1, \ldots, J$, where J is the number of destinations

p = index of paths, $p=1, \ldots, P$, where P is the number of paths between OD pair i and j

a = index of links, $a=1, \ldots, A$, where A is the number of links in networks

d = index of days, $d=1, \ldots, D$, where D is the number of days over analysis horizon

Input Parameters:

$c_{a,d}$ = capacity of link a on day d

$s_{a,d}$ = toll value charged on link a on day d

$q^{i,j}$ = OD demand volume between an OD pair i and j

$\delta_{p,a}$ = path-link incidence coefficient, $\delta_{p,a}=1$, if path p passes through link a, and 0 otherwise

γ = market penetration rate of the perfect information (PI) users as a function of the total OD demand

VOT = value of time in dollars per minute

Decision variables:

$f_{p,d}^{PI,i,j}$	=	flow of PI users on path *p* for OD pair (*i*, *j*) on day *d*
$f_{p}^{ETT,i,j}$	=	flow of ETT users on path *p* for OD pair (*i*, *j*) (flow rates are the same across different days)
$v_{a,d}$	=	total flow on link *a* on day *d*
$T_{a,d}$	=	travel time on link *a* on day *d*
$U_{a,d}$	=	generalized disutility on link *a* on day *d*, which is a function of capacity $c_{a,d}$ and link flow $v_{a,d}$
$U_{p,d}^{i,j}$	=	generalized disutility of path p between OD pair (*i*, *j*) on day *d*
$\overline{U_{p}^{i,j}}$	=	expected disutility of path p between OD pair (*i*, *j*) over the multi-day horizon
$\pi_{d}^{i,j}$	=	day-dependent least path disutility between OD pair (*i*, *j*) on day *d*
$\overline{\pi^{i,j}}$	=	least expected disutility between OD pair (*i*, *j*) over the multi-day horizon

It should be noted that the analytical approaches generally allow the theoretical claims of properties such as global optimality and uniqueness and produce superior solutions to those obtained utilizing the simulation-based heuristic approaches which seek close-to-optimal solutions. Due to the mathematical intractability of the analytical DTA formulation, the route-cost functions in DTA traffic flow models are generally non-differentiable (Han & Lo, 2004). Hence, measures of gap for simulation-based heuristic algorithms, which do not require derivative information, have been widely used to measure the difference between the current iteration solution and the ideal solution and gauge the effectiveness and deployment efficiency of a solution.

Objective function:

$$\min Gap = \sum_{d}\sum_{i}\sum_{j}\sum_{p}\left[f_{p,d}^{PI,i,j} \times \left(U_{p,d}^{i,j} - \pi_{d}^{i,j}\right) + f_{p}^{ETT,i,j} \times \left(\overline{U_{p}^{i,j}} - \overline{\pi^{i,j}}\right)\right] \tag{11}$$

Subject to

PI flow constraints:

$$\gamma \times q^{i,j} = \sum_{p} f_{p,d}^{PI,i,j} \qquad \forall i, j, d \tag{12}$$

ETT flow constraints

$$(1-\gamma) \times q^{i,j} = \sum_{p} f_{p}^{ETT,i,j} \qquad \forall i, j \tag{13}$$

Path - link flow balance constraints

$$v_{a,d} = \sum_{i}\sum_{j}\sum_{p}\left(f_{p,d}^{PI,i,j} \cdot \delta_{p,a}\right) + \sum_{i}\sum_{j}\sum_{p}\left(f_{p}^{ETT,i,j} \cdot \delta_{p,a}\right) \qquad \forall a, d \tag{14}$$

Path- link cost connection

$$U_{a,d} = T_{a,d}(v_{a,d}, c_{a,d}) + \frac{s_{a,d}}{VOT} \qquad \forall a, d \tag{15}$$

$$U_{p,d}^{i,j} = \sum_{a}\left(U_{a,d} \cdot \delta_{p,a}\right) \qquad \forall i, j, d, p \tag{16}$$

Definitional constraint of expected path disutility: average disutility

$$\overline{U_{p}^{i,j}} = \frac{1}{D}\sum_{d} U_{p,d}^{i,j} \qquad \forall i, j, p \tag{17}$$

Least disutility definitional constraints:

$$\pi_d^{i,j} \le U_{p,d}^{i,j}, \quad \forall i,j,p,d \tag{18}$$

$$\bar{\pi}^{i,j} \le \bar{U}_p^{i,j} \quad \forall i,j,p \tag{19}$$

Non-negativity constraints on path flow variables.

Constraints (12) and (13) show the relationship between OD demand and path flows for each information class. Eq. (14) aggregates path flows from two different user classes to link flows. Eqs. (15-16) calculate the path disutility for each path on day d, where the dollar value of road toll is incorporated as equivalent travel time through value of time (VOT). Eq. (17a) defines the average disutility for each path across different days, which will be used in the gap function for ETT users in objective function (11).

To consider risk-sensitive users, one can use a disutility function that incorporates both mean and standard deviation of multi-day travel time as $\bar{U}_p^{i,j} = \frac{1}{D}\sum_d U_{p,d}^{i,j} + \beta \times \sqrt{\frac{1}{D}\sum_d [U_{p,d}^{i,j} - \frac{1}{D}\sum_{d'} U_{p,d'}^{i,j}]^2}$ where $\frac{1}{D}\sum_{d'} U_{p,d'}^{i,j}$ corresponds to the mean value of travel time and β is the travel time reliability coefficient. As discussed by Xing and Zhou (2011), the form of $mean + \beta \times \sqrt{Variance}$ is not a convex and additive function, which leads to difficulties for further examining the solution uniqueness properties.

In addition, to consider risk-averse users (absolute robustness), the generalized path disutility can have the worst-case travel time defined by $\bar{U}_p^{i,j} \ge U_{p,d}^{i,j}$ for each path of each OD pair. This worst-case disutility function might be the extreme case for risk-averse behavior. By assuming continuous probability distribution, Zhang et al. (2011) considered a similar worst-base cost function to construct a robust Wardrop's user equilibrium assignment model. To reformulate a more realistic case of percentile travel time robustness across different day samples, one can check the min-max expression and additional integer programming method proposed in the paper by Xing and Zhou (2013). It should be remarked, in one of the important studies along this direction, Chen and Zhou (2010) considered a α-reliable mean-excess utility function, while their model is built on the assumption of continuous probabilistic travel time distribution (e.g. log-normal distribution). In comparison, our proposed scenario-based probabilistic travel time representation focuses on how to distinguish the day-dependent traffic route choice behavior for different classes of user information, under recurring vs. non-recurring traffic conditions. Overall, $\bar{U}_p^{i,j} \ge U_{p,d}^{i,j}$ involves only linear inequalities, but it introduces a large number of additional constraints across different days.

For comparison purposes, the system optimal benchmark can also be defined as:

$$\min z = \sum_d \sum_a \left[v_{a,d} \times T_{a,d}(v_{a,d}, c_{a,d}) \right] \tag{20}$$

, or minimizing the total system-wide disutility for all travelers with defined mean+standard deviation or absolute robust travel time route choice measures, where the flow to be optimized $f_{a,d}$ can be day-varying.

To discuss the existence and uniqueness of solution(s) of the risk-sensitive traffic equilibrium model under stochastic capacity conditions, one needs to carefully examine the underlying stochastic link performance functions and its relationship with the assigned path flow. In the development of a multi-class percentile user equilibrium model with stochastic service flow rate conditions, Nie (2011) thoroughly discussed how the percentile route travel time function derived from convolution does not have a closed form, and the monotonicity of this function with respective to the route flow only holds under a very restrictive case. Thus, the resulting implications need to be carefully understood when studying the solution properties of reliability-based traffic assignment models. In recent papers by Ban et al. (2013) and Rapoport et al. (2014), the solution properties for equilibria with stochastic traffic conditions are discussed, under different monotonic travel cost functions and behavioral assumptions.

To understand the theoretical properties of our proposed model with a sample-based modeling framework, we could also examine the model structure from a side constrained traffic equilibrium perspective. This model can be viewed as an extension of standard static traffic assignment for each individual day, but with side constraints that enforce the ETT path flow to be the same across different days. Without side constraints, as discussed by Patriksson (1994), the gap based formulation (adopted in this paper) and BMW formulation (Beckman et al, 1956) are equivalent to the NCP formulation with following form, $f_p^{i,j} \times (U_p^{i,j} - \pi^{i,j}) = 0$ and $U_p^{i,j} - \pi^{i,j} \ge 0$.

However, enforcing the side constraints in traffic equilibrium models could lead to a number of theoretical difficulties. Specifically, in our paper, we consider a non-anticipativity constraint associated with a priori path, $f_{p,d1}^{ETT,i,j} = f_{p,d2}^{ETT,i,j} = ... = f_{p,D}^{ETT,i,j}$, where $f_{p,d}^{ETT,i,j}$ is path flow for ETT users on day d. In comparison, the flow choice defined by $f_{p,d}^{\mathrm{PI},i,j}$ could be different across different days for PI users. One could consider the use of generalized user equilibrium (GUE) to characterize the solution, while additional Lagrangian multipliers associated with the side constraints have to be included in the generalized cost function $U_p^{i,j}$ to reach the GUE condition. Note that, our gap-based formulation does not include such a Lagrangian multiplier variable, and in this case, the closest theoretical results for traffic assignment with side constraints can be found in the important work by Correa et al. (2004) where they show under a typical side constraint (i.e., road capacity constraints), the related formulation could lead to one of Nash equilibriums. To carefully examine how the side constraints affect the solution uniqueness and stability, we refer the readers to the seminal work by Larsson and Patriksson (1995, 1999) for further theoretical references. In our case, it is possible that there are Nash equilibrium solution(s) that still exist for our problem where two classes of travelers have non-cooperative route choice behavior. Our future research needs to study how the proposed algorithm would converge to a local minimum, and how we can check the gap between $U_p^{i,j}$ and $\pi^{i,j}$, based on experienced travel time, especially when the objective functions have nonconvex terms such as for the risk-sensitive users above.

4. Operational solution algorithm for large-scale network

The solution algorithm executing the above steps is depicted in Fig. 7.

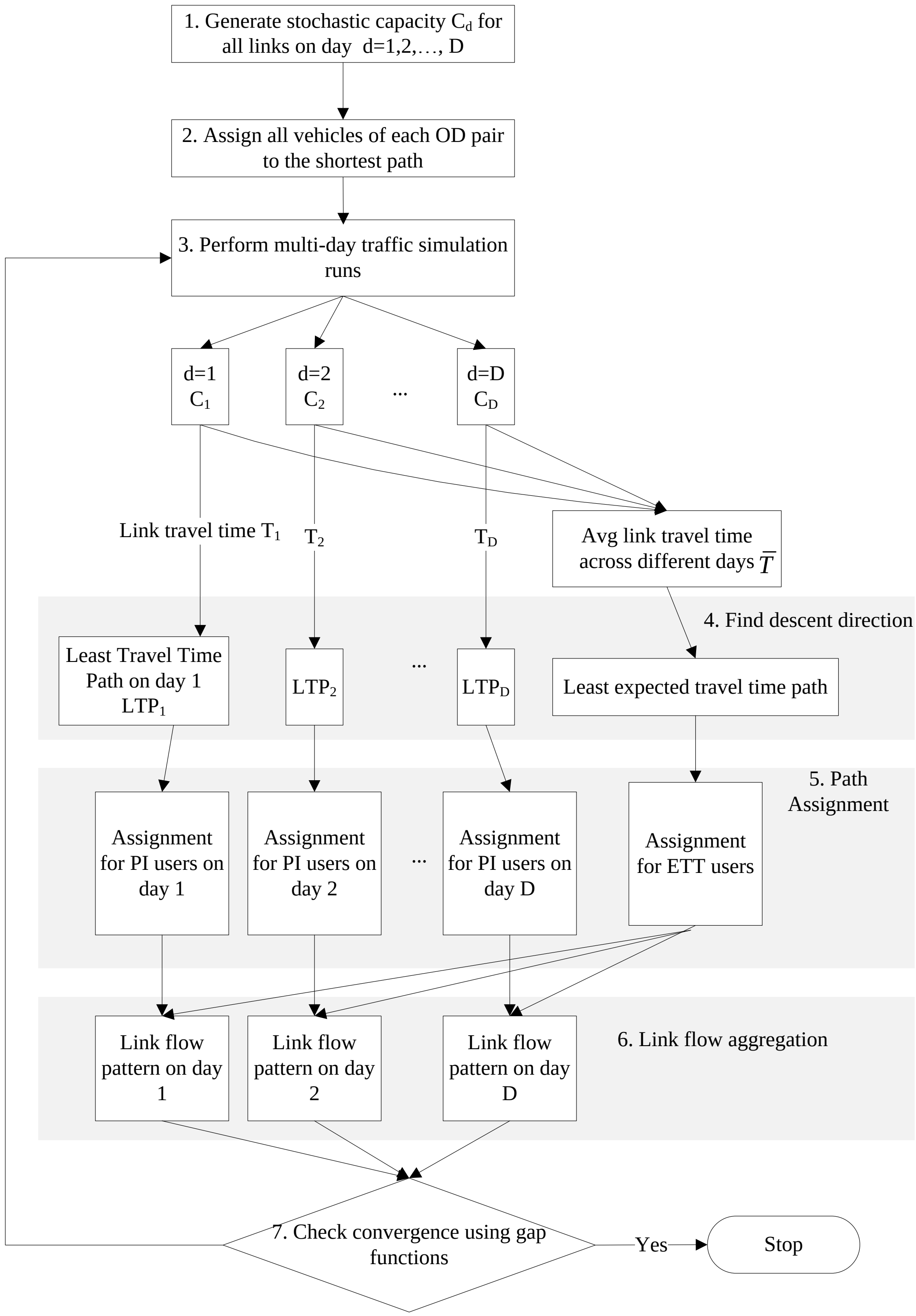


Fig. 7: Solution algorithm for static traffic assignment with both PI and ETT users

In order to iteratively reduce the overall gap in the proposed optimization problem for a general network with multiple origins and destinations, we extend a descent search solution framework developed by Lu et al. (2009), which also used a path-based gap function to describe the dynamic traffic equilibrium pattern. Fig. 7 presents the iterative procedure for solving the multi-class static traveler assignment problem under stochastic capacity conditions. The proposed procedure adds day-dependent simulation, path finding and assignment dimensions to the existing static traffic assignment algorithm that typically assumes deterministic road capacity conditions. In this study, we implement the proposed algorithm within a mesoscopic traffic assignment framework, which represents flow as vehicles with origin, destination and path attributes. Recall that, in conventional assignment programs, a vehicle is associated with a single path. In the proposed multi-day traffic assignment algorithm, an ETT vehicle still follows a single path across different days, but a PI vehicle can use and store different (day-dependent) paths on different days. To consider risk-sensitive or risk-averse users, one could use customized shortest path algorithms (such as Xing and Zhou (2011, 2013)) for finding the least cost path across different sampled travel times in Step 4.

The main steps of the solution procedure are described as follows:

Step 1: Day-dependent capacity generation.

Generate road capacity vector $C_d = \left[c_{a,d} \right]$, for all link a=1, 2, …, A, on day d=1, 2, …, D, according to given stochastic capacity distributions.

Step 2: Initialization.

Start with iteration number n=0. Generate PI and ETT vehicles according to given market penetration rate γ. For each OD pair, compute the shortest path (in distance) and assign both PI and ETT vehicles to the corresponding shortest path.

Step 3: Multi-day traffic simulation with stochastic capacity.

On each day d =1, 2, …, D, for given link flow patterns, generate day-dependent link travel times according to stochastic capacity vector C_d. The simulation results generate link travel time $T_{a,d}$ for link a=1, 2, …, A, on day d=1, 2, …, D.

Step 4: Find descent directions for traffic assignment

Find the Least Travel time Path (LTP) using day-dependent link travel time $T_{a,d}$ on each day d, for link a=1, 2, …, A.

Find the Least Expected Travel time Path (LETP) using average link travel time $\overline{T_a} = \frac{\sum_d T_{a,d}}{D}$, for link a=1, 2, …, A.

Step 5: Path assignment for PI and ETT vehicles

For each day d, a certain percentage of PI vehicles are assigned to the least travel time path.

By adapting the path-swapping method proposed by Lu et al. (2009), this study uses the following probabilistic ratio for a vehicle on path p to switch to the least travel time path at iteration n:

$$\frac{1}{n+1} \times \frac{U_{p,d}^{i,j} - \pi_d^{i,j}}{U_{p,d}^{i,j}} \tag{21}$$

The first term 1/(n+1) is equivalent to the fixed step size in the Method of Successive Average (MSA). The second term ensures that, the path swapping probability is proportional to the relative difference between the experienced path travel time $U_{p,d}^{i,j}$ and the minimum path travel time $\pi_d^{i,j}$. An intuitive interpretation for this heuristic swapping rule is that, travelers on longer paths (i.e. farther from the equilibrium solution) are more likely to switch to the least travel time path than those on paths with travel cost closer to the least travel time path.

Similarly, a certain percentage of ETT vehicles are swapped to the least **expected** travel time path, the route swapping probability at iteration n can be determined by

$$\frac{1}{n+1} \times \frac{\overline{U}_p^{i,j} - \overline{\pi}^{i,j}}{\overline{U}_p^{i,j}} \tag{22}$$

As shown in Lu et al. (2009), search directions specified by Eqs. (21-22) can be proven to be in the descent direction of the gap function in Eq. (11) for $f_{p,d}^{PI,i,j}$, $f_p^{ETT,i,j}$, respectively, at iteration n.

Step 6: Link flow aggregation

For each day d, calculate the aggregated link volume $v_{a,d}$ using PI flow volume on day d and ETT flow (across every day), using Eq. (14).

Step 7: Convergence checking

Calculate the gap function as shown in Eq. (11), if $Gap < \delta$ convergence is achieved, where δ is a pre-specified parameter. If convergence is attained, stop. Otherwise, go to Step 3.

5. Experimental results with stochastic capacity distribution

The first set of experiments uses the simple corridor with two routes in the illustrative example, shown in Table 3. In this set of experiments, it is further assumed that links 1 and 2 have 3 and 2 lanes, respectively, and on this basis 100 days of random lane capacity are generated. The headway data used in this analysis were obtained from a recent research effort by Jia et al (2010). In their study, the pre-breakdown time headways are found to follow a shifted log-normal distribution with the following probability density function:

$$f_X = (x; \mu, \sigma) = \frac{1}{(x-c)\sigma\sqrt{2\pi}} e^{-\frac{(lnx-\mu)^2}{2\sigma^2}}, \quad x > 0 \tag{23}$$

Where,
x is the average pre-breakdown headway (in second) for 15-minute interval,
c is the minimum pre-breakdown headway (in second),
μ is the mean of the variable's natural logarithm, and
σ is the standard deviation of the variable's natural logarithm.

Their calibration results based on data from several bottleneck locations in the Bay Area, California show that c=1.5 seconds, μ=-0.97, and σ=0.68. To convert 15-minute breakdown flow rates to hourly capacity, we take an average value of 4 samples from the above random distribution. The histogram in Fig. 1 shows the probabilistic distribution of 100 lane capacity samples on link 1. The hourly lane capacity is multiplied by the number of lanes to generate link capacity, and the resulting average total capacity of both links is 9,298 vehicles per hour.

Measure of Effectiveness (MOE)

The system-wide travel time is defined as the average travel time for all users. Mean travel time for users using ETT knowledge over different days can be represented as

$$\bar{T}^{ETT} = \frac{1}{|d|} \times \sum_d T_d^{ETT} \tag{24}$$

and the average travel time for ETT users on day d is

$$T_d^{ETT} = \frac{\sum_p \sum_{i,j} \left(f_p^{ETT} \times T_{p,d}^{ETT,i,j} \right)}{q \times (1-\gamma)} \tag{25}$$

where $T_{p,d}^{ETT,i,j}$ is the travel time experienced by ETT users on day d along path p for OD pair (i,j).

Travel time standard deviation is used to represent day-to-day travel time variability:

$$STD^{ETT} = \sqrt{\frac{\sum_d (T_d^{ETT} - \bar{T}^{ETT})^2}{(|d|-1)}} \tag{26}$$

Similarly, $\bar{T}^{PI}$, $\bar{T}_d^{PI}$ and $\mathrm{STD}^{\mathrm{ETT}}$ can be calculated for PI users and $\bar{T}$, $\bar{T}_d$, STD for different classes of users.

Relative travel time improvement is defined as an indicator of the value of information:

$$\frac{\bar{T}^{ETT} - \bar{T}^{PI}}{\bar{T}^{ETT}} \tag{27}$$

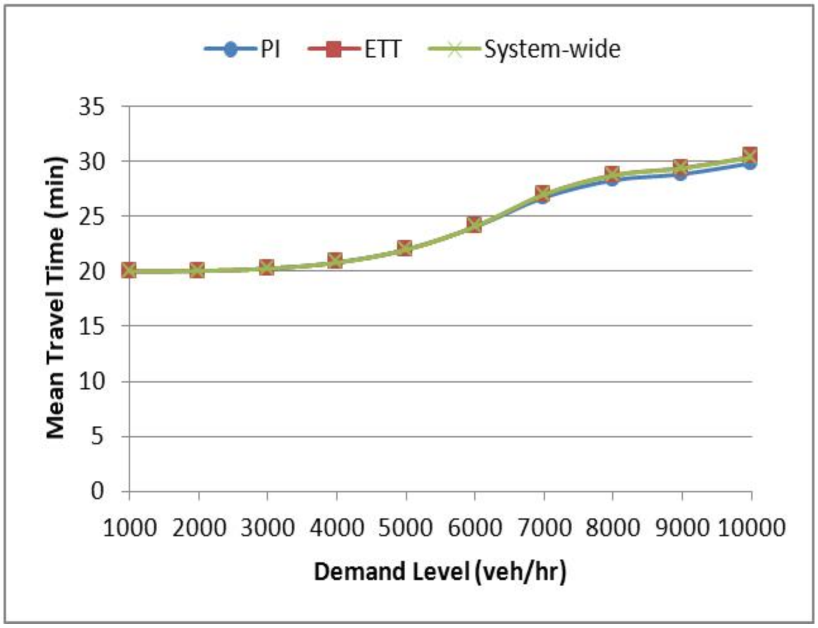
This measure compares the relative difference of mean travel time savings between PI and ETT users across all days.

5.1 Sensitivity analysis with 100 days of stochastic capacity distribution

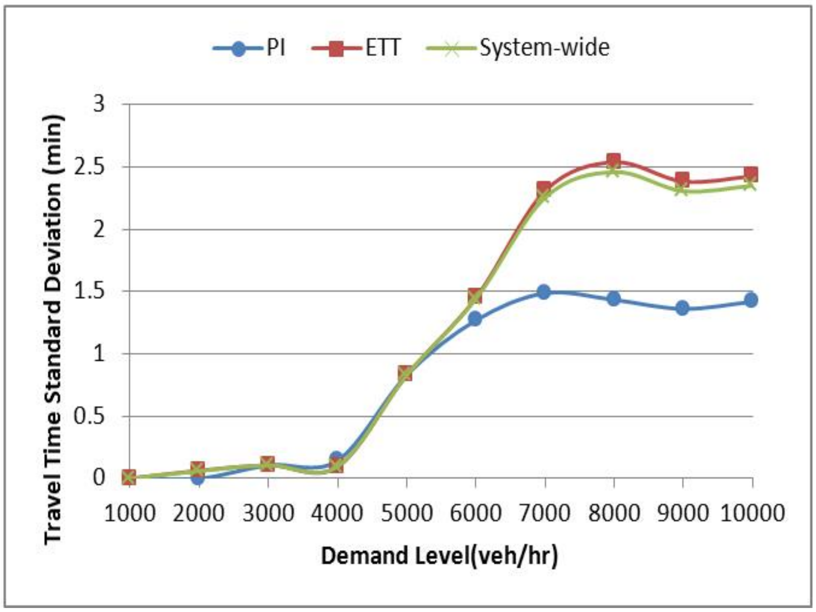
The following experiments describe the results of a sensitivity analysis for three major inputs: the total demand q, the market penetration rate of PI users, and the toll value imposed on the primary route. In the baseline configuration, q =8,000 vehicle/ hour, γ =0.05 and there is no tolling on the primary routes.

MOE at varying demand levels

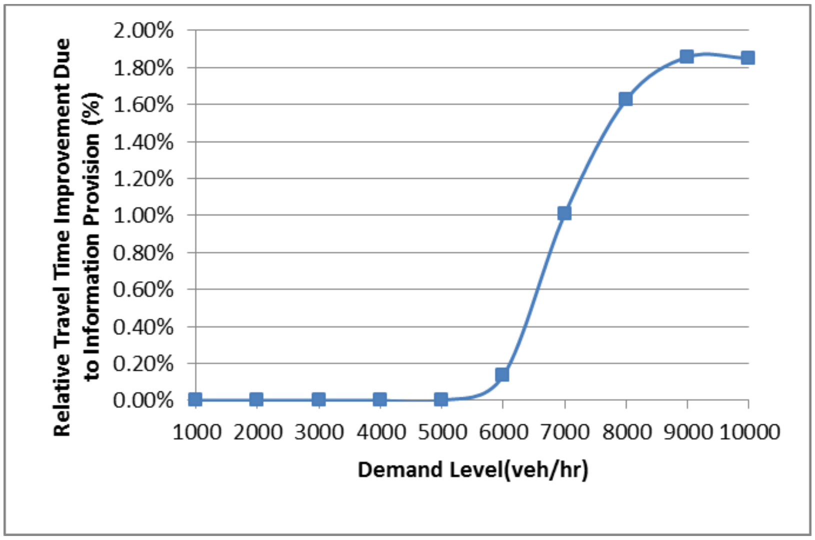
Fig. 8 shows the different MOEs, when the total demand level q is varied between 1,000 and 10,000 vehicles per hour. Fig. 8a shows that the average travel time dramatically increases after the total demand is raised above 4,000 vehicles per hour, which is close to the capacity of the primary route. Interestingly, Fig. 8b shows that PI users experience significantly lower travel time variability compared to ETT users. The same pattern applies to the relative travel time improvement (Fig. 8c), although the value is quite marginal. In general, however it is clear that information provision is much more beneficial at higher demand or congestion levels.



(a)



(b)



(c)

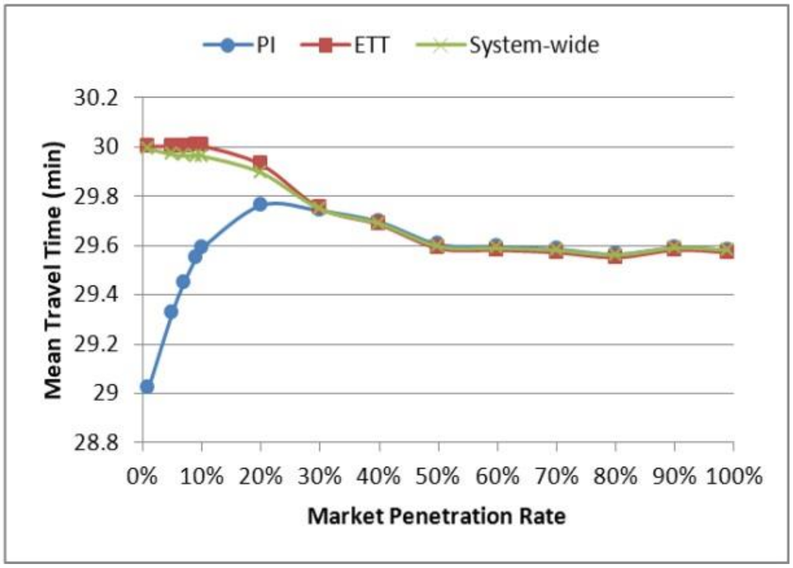
Fig. 8. Effectiveness of information provision under stochastic capacity at varying demand levels.

MOE's at different market penetration rates

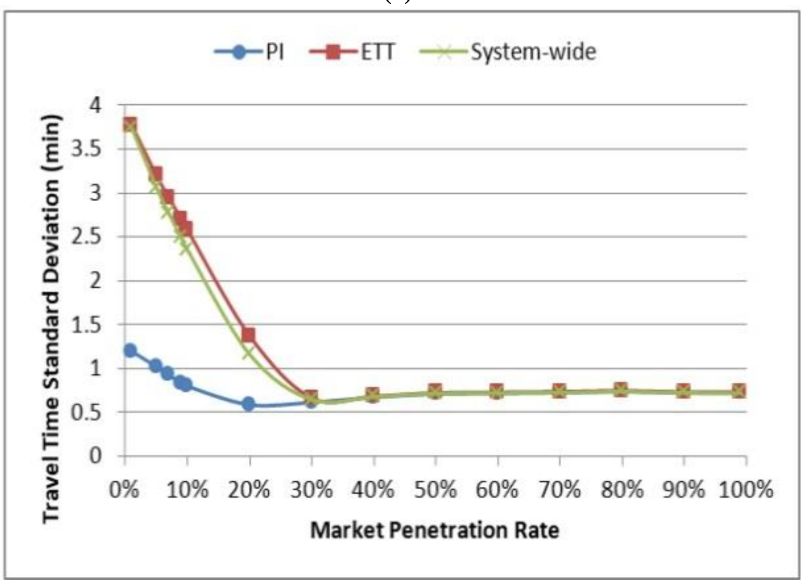
Fig. 9 shows the sensitivity analysis results of travel time under different PI users market penetration rates. In many previous studies, the proposed models' aim was to evaluate the effect of varying market penetration rates and identify the saturation level of market penetration of ATIS services. In this research the objective is to understand the relationship between market penetration rate and the value of information.

When the market penetration rate is below 30%, all PI users are able to switch from a congested route (typically route 1) to a less congested route (typically route 2), so that travel information provision strategies yield meaningful savings in terms of mean travel time and travel time variability (Fig. 9a, Fig. 9b). However, when the market penetration rate exceeds a certain threshold (30% in our example), a large number of PI users can take the detour, so the previously less congested route becomes crowded. Under the assumption of user equilibrium with perfect information, both routes at this point should have the same travel time so that no PI user can reduce his/her travel time by switching routes. This implies no additional benefit in terms of travel time reliability is available by using traveler information strategies.

Although the finding on the diminishing value of information as a function of market penetration rate is similar to a number of previous studies (for example, Yang et al., 1993; Yang et al., 1999; Arnott et al., 1996; Arnott et al., 1999), those findings are based on two fundamentally different settings. Thus, for travelers not equipped with ATIS, previous studies have considered different levels of perception errors under deterministic capacity for a single day, while this proposed approach assumes no perception error on the expected travel time of multiple days with stochastic capacity.



(a)



(b)

Fig. 9. Effectiveness of information provision under stochastic capacity with different market penetration rate.

5.2 Experiments on medium-scale networks with stochastic capacity distributions over 30 days

The following numerical experiments are performed on two medium-scale network data sets, which are publicly available at a website maintained by Bar-Gera (2001). The proposed algorithm is implemented in C++ on the Windows Vista 64-bit platform and evaluated on a computer with a PC with 9 GB memory. The proposed algorithm has been incorporated into an open-source traffic assignment package available at https://sites.google.com/site/dtalite. To fully utilize the available parallel computing capability, all the shortest path calculations and path assignment computations for different origin zones (at steps 4 and 5) are migrated to different processors. The above parallelization scheme is implemented through an Open Multi-Processing (OpenMP) shared-memory parallel programming interface. As shown in Fig. 7, the parallelization of shortest path calculation can be also carried out for different days.

Table 4. Test network characteristics and computational performance with 30 day samples, 20 iterations and 10% PI vehicles.

	Anaheim, California	**Chicago Sketch Network**
# of nodes	416	933
# of links	914	2950
# of OD zones	38	387
Total OD Volume	104K	1,261K
Computational time	34 min	8 h 16 min

As shown in Table 4 and Fig. 10, the Anaheim, California network contains about 38 zones, and 0.1 million vehicles, and the Chicago sketch network, an aggregated representation of the Chicago region, has 387 zones with 1.2 million vehicles. Under a setting of 10% PI users, 20 assignment iterations and 30 days of random road capacity, the Anaheim network uses about 30 minutes, and the Chicago sketch network takes about 8 hours of CPU time and 2.6G memory. There are three major factors affecting the computational complexity of the proposed algorithm: (1) the number of OD zones, (2) the number of days in the random capacity representation, and (3) the number of PI vehicles. Specifically, the first two factors are related to the number of path finding calculations, as the algorithm must find the least travel time routes using day-dependent travel time for PI vehicles originating from each origin zone. The other two factors, namely the number of days and the number of PI vehicles, jointly determine the complexity of path swapping operations, as each PI vehicle must carry and update individual paths on different days in the proposed path-based and mesoscopic representation. In addition, as steps 4 and 5 are the most computationally intensive in the proposed algorithm, additional CPU cores can also accordingly speed up the overall computational process through parallel computing.

To measure the convergence of the proposed algorithm, we use the following average optimality gap as the solution quality indicator:

$$AvgGap = \frac{1}{D \times \sum_{i,j} q^{i,j}} \times \sum_d \sum_i \sum_j \sum_p \left[f_{p,d}^{PI,i,j} \times \left(U_{p,d}^{i,j} - \pi_d^{i,j} \right) + f_p^{ETT,i,j} \times \left(\overline{U}_p^{i,j} - \overline{\pi}^{i,j} \right) \right]. \quad (28)$$

Focusing on the first 20 iterations, Fig. 11 compares the numerical convergence of two algorithms in the Chicago sketch network: (i) the proposed route- swapping algorithm that uses the swapping ratio in Eq. (22), (ii) the algorithm uses the conventional MSA method with a fixed step size of 1/(n+1), where *n* is the iteration counter. Both algorithms can steadily reduce the optimality gap toward zero, and the average gap measures reach within 0.5 minutes per vehicle within the first 10 iterations. In comparison, the proposed route swapping rule is able to more quickly decrease the optimality gap within first 5-7 iterations. Shown in Table 4, the two test networks have quite small optimality gaps at iteration 20, namely 0.289 and 0.344 minutes. To reduce the overall computational efforts, one might consider using fewer iterations as long as the solution quality reaches an acceptable level.

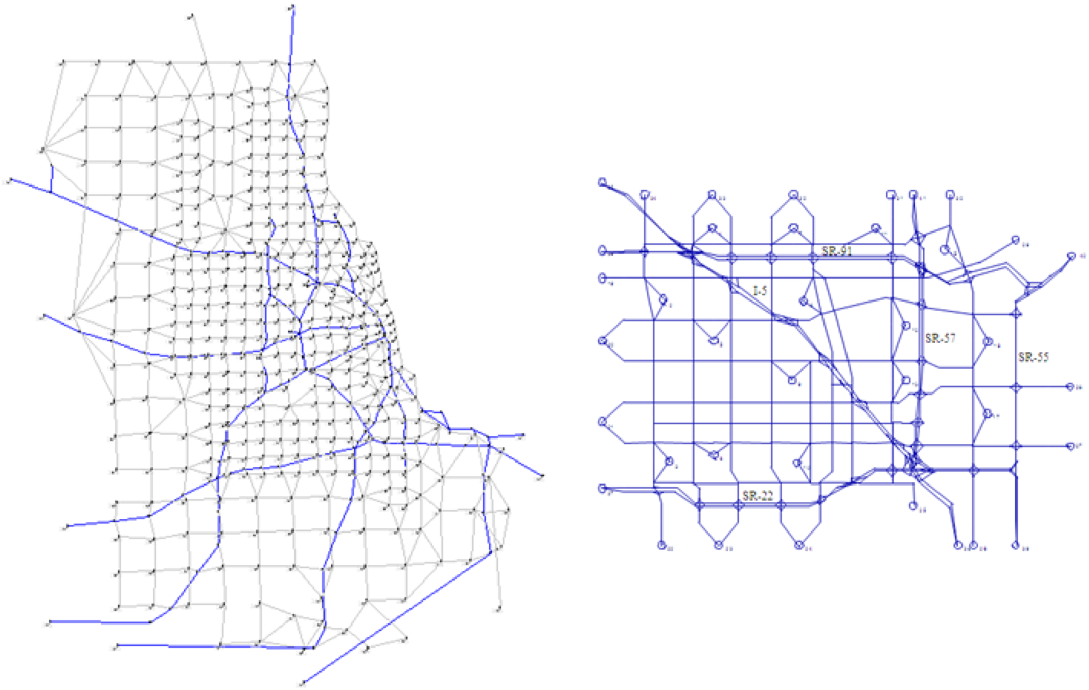


Fig. 10. Chicago sketch network (left) and Anaheim, California network (right).

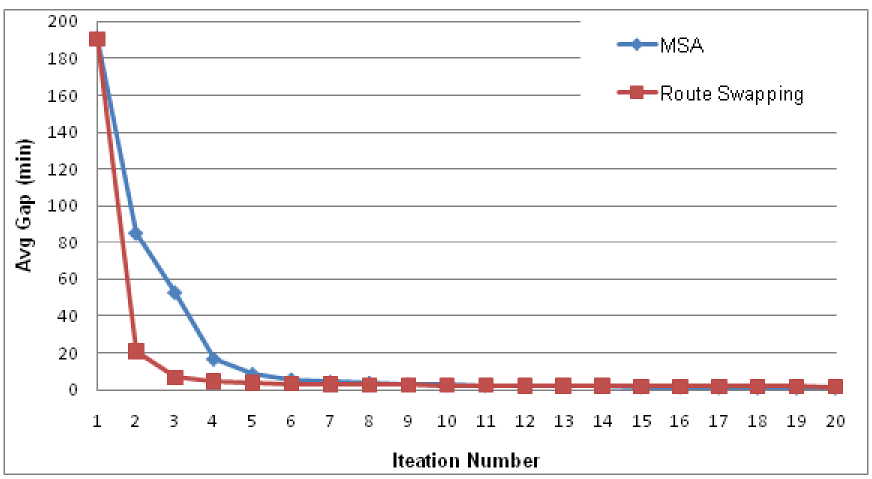


Fig. 11. Convergence patterns of route-swapping and MSA rules on Chicago sketch network.

The original data sets use the BPR function to describe travel time performance, and a single valued mean capacity is specified for each link. To generate random road capacity samples, we use the pre-breakdown headway distribution in Eq. (23) to generate multiple samples of 15-min pre-breakdown capacity first. To approximate the peak-hour capacity values used in the BPR function, we evaluate the impact of two alternative schemes:

(i) Possible multiple congestion periods within a peak hour, so we use the average of 4 pre-breakdown capacity values as the peak hour capacity, leading to a Coefficient of Variation (CV) = 6.4%

(ii) Single congestion period, where a single value of 15-min pre-breakdown capacity becomes the dominating factor for the whole peak hour, leading to CV = 12.8%

Table 5. Value of traveler information under different peak-hour capacity approximation schemes.

Peak-hour Capacity Approximation Scheme	**Anaheim network**			**Chicago Sketch Network**		
	ETT Travel Time	**PI Travel Time**	**Average Relative Travel Time Saving (%)**	**ETT Travel Time**	**PI Travel Time**	**Average Relative Travel time Saving (%)**
Average Pre-breakdown Capacity with CV= 0.064	12.903	12.864	0.302%	17.348	17.242	0.611%
Single-valued Pre-breakdown Capacity with CV= 0. 128	13.233	13.157	0.574%	17.794	17.469	1.827%

As shown in Table 5, the single valued pre-breakdown capacity implies larger variations compared to the average (aggregated) pre-breakdown capacity, so it slightly increases the average travel time for PI and ETT travelers on both networks. However, according to the above experimental results, the travel time savings obtainable for PI users seem to be not too significant even under the large link travel time variations. Similarly, for the hypothetical network in Fig. 2, the capacity variations also only lead to less than 1 minute travel time savings due to traveler information provision, shown in Fig. 10-(a).

To fully understand the benefit of traffic information provision, an analyst needs to better characterize travel time dynamics/variability, which is caused by a wide range of recurring and non-recurring delay sources, such as incidents, work zones, and random fluctuations in road capacity. Although the proposed framework allows and is naturally suited to consider any given random capacity distributions with a multi-day or sample-based representation scheme, the calibrated capacity distribution used in our study in fact only focuses on "normal" random capacity perturbations, while the "outliers" in the capacity distribution due to nonrecurring events such as incidents, work zone, severe weather, are not explicitly modeled in the given capacity distribution and should be also integrated in the future research to fully account for the benefits of traveler information provision strategies under random capacity breakdowns and "unplanned" events.

5.3 Experiments using nonlinear optimization software package

To examine if the proposed nonlinear optimization program is mathematically trackable using standard optimization solvers, we further develop a GAMS program (accessible at https://www.researchgate.net/publication/307923877_ETT_2_day_samples), for the commonly used Sioux Falls test network with 24*23= 552 OD pairs and 76 links. By considering a maximum of 15 possible paths between each OD pair and adopting a link elimination approach described in Ramming (2001) to generate possible alternative paths, we use 2, 3, 5 and 10 day samples, for incorporating three different route choice utility functions listed in Section 3. As the exact benefit of perfect information is heavily dependent on capacity breakdown assumption in this medium-scale network, our discussion below is not intended to examine the detailed travel time saving further (as we have done so in a number of numerical experiments in the early sections). Instead, we are interested in the computational performance of the formally defined nonlinear programming model in terms of the number of variables, constraints and computational time, to shed more light on how to select and improve the optimization models to systematically take into account travel time variabilities across different day samples. The number of variables are essentially the same, across three models with different utility functions. Specifically, there are 22369, 32489, 52729, 103329 variables for models with 2, 3, 5 and 10 day samples. Tables 6 and 7 list the number of constraints, and CPU time for both UE and SO models. As the number of day samples increases, the number of constraints and variables also dramatically increase as expected. The ETT model and mean+standard deviation model have the same number of constraints, and the only difference is the defined objective function. The ETT model is solvable with a 1,000 second CPU time limit in most cases. However, due to the introduced non-convex function associated with the standard deviation term, the model mean+standard deviation model has difficulties to reach solution convergence criteria defined by the optimization solver MINOS with a reasonable amount of time. The absolute travel time robustness model needs to add the worst-case travel time definitional constraints across each day *d*, so it becomes almost three times larger than the corresponding ETT model in the case of 10 day samples. This required definitional constraint sets makes a standard nonlinear optimization solver difficult to find converged solutions for medium case instances, as most of constrained nonlinear optimization solvers need to convert the constraints through penalty methods and it is challenging to search within a feasible region when there are a large number of active or inactive constraints. It should be remarked that the nonlinear optimization solvers in GAMS could have the built-in nonlinear programming based convergence criteria to evaluate if the iterative searching process has reached possible stable solutions within a given computing resource limit, while the simulation based experiment we conducted rely on Eq. (28) to measure the solution convergence externally and a more systematic assessment on how to define and select the gap functions for a DTA simulator can be found in the paper by Lu et al. (2009).

Table 6. The number of constraints for nonlinear model with different route choice utilities

	2 day samples	**3 day samples**	**5 day samples**	**10 day samples**
ETT	14529	16809	21369	32769
Mean + standard deviation	14529	16809	21369	32769
Absolute robustness	23169	32089	55929	110529

Table 7. CPU time for nonlinear model with different route choice utilities, two numbers in each cell represent (UE model CPU time, SO model CPU time), and * represents reaching the CPU time limit of 1000 seconds

	2 day samples	**3 day samples**	**5 day samples**	**10 day samples**
ETT	30s, 30s	1min, 58s	2min 21s, 2min 46s	15min 27s,*
Mean + standard deviation	*,7min11s	2min40s, 6min 41s	6min 2s, 11min 41s	*,*
Absolute robustness	*,2min 1s	*,*	*,*	*,*

6. Conclusions

In this paper, a non-linear optimization-based analysis method is proposed along with related modeling components pertaining to stochastic capacity, travel time performance functions and different degrees of traveler knowledge in an ATIS environment. Within a multi-day analysis framework, this proposed method categorizes commuters into two classes: (1) travelers with access to perfect traffic information every day, and (2) travelers with some degree of knowledge of average traffic conditions across different days. Within a gap function framework (for describing the user equilibrium under different information availability), a mathematical programming model is formulated to describe the route choice behavior of the perfect information (PI) and expected travel time (ETT) user classes under stochastic day-dependent travel time. We systematically implemented and tested the computational performance of the proposed models under a standard optimization solver, and the numerical results provide valuable information for (1) constructing nonlinear optimization models to capture the risk-sensitive traveler behavior across multiple scenarios/days, and (2) how to select a set of representative day or cluster samples to represent the system performance variables with compact optimization models.

Our paper aims to make two primary contributions to the existing vast literature on evaluating the impact of traveler information. (1) This paper takes into account traveler routing behavior with day-specific travel times over a large number of representative random scenarios. Most of the related papers mainly consider adaptive traveler behavior within a single day in a stochastic traffic evolution environment and many studies focus on how to embed and reformulate a quite complex travel time function where the standard deviation of travel time is typically derived numerically. (2) This paper is primarily concerned with the long-term steady state solution with different types of traveler information users over multiple days, while the effect of expected information and perfect day-dependent information on driver choices has not been addressed sufficiently in the literature especially for day-dependent capacity conditions.

The analysis in the paper could be extended in several directions. One is to incorporate stochastic demand fluctuations and different levels of information quality. Quantifying benefits of traveler information provision strategies in a stochastic environment places a great need for rigorous formulations and practical solution procedures for the traffic network assignment problem. Given a wide range of travel time variability sources, it is desirable to further enhance the proposed model to systemically evaluate the value of dynamic information and reliability associated with stochastic demand fluctuations and different levels of information quality, under both recurring and nonrecurring congestion conditions. Additionally, in this study, we selected several utility functions so as to capture the travel time statistics over multiple days. This framework can be further extended to a general utility for risk-averse travelers but might introduce more complexity in analyzing the impact of different behavior factors. It should be pointed out that driver route choice decisions are influenced by many factors including compliance rate. Since the issue of compliance introduces another different degree of complexity in our mathematical formulations, we used a simple 100% compliance rate assumption in this paper. It would also be of interest to extend the proposed modeling framework to the case with the compliance and inertia rules. Our future research will also examine computational challenges introduced by the proposed method stemming from the sampling-based representation of stochastic capacity distributions, to provide a more systematic assessment on the number of scenarios/days in estimating travel time variance as shown in Kim and Mahmassani (2014). Possible solutions along this line include using variance reduction techniques, such as importance sample, to reduce the required sample size, and applying distributed computing techniques, such as cloud computing, to improve the computational efficiency within a shared-memory or distributed computing environment.

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