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Train scheduling for minimizing passenger waiting time with time-dependent demand and skip-stop patterns: Nonlinear integer programming models with linear constraints

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ABSTRACT

This paper focuses on how to minimize the total passenger waiting time at stations by computing and adjusting train timetables for a rail corridor with given time-varying origin-to-destination passenger demand matrices. Given predetermined train skip-stop patterns, a unified quadratic integer programming model with linear constraints is developed to jointly synchronize effective passenger loading time windows and train arrival and departure times at each station. A set of quadratic and quasi-quadratic objective functions are proposed to precisely formulate the total waiting time under both minute-dependent demand and hour-dependent demand volumes from different origin–destination pairs. We construct mathematically rigorous and algorithmically tractable nonlinear mixed integer programming models for both real-time scheduling and medium-term planning applications. The proposed models are implemented using general purpose high-level optimization solvers, and the model effectiveness is further examined through numerical experiments of real-world rail train timetabling test cases.

1. Introduction

The train timetabling problem aims to schedule the movements of trains over space and time and deliver passengers from their origin stations towards their destination stations in an efficient manner. In an increasingly competitive and highly dynamic multi-modal transportation market, train timetables are an essential bridge between a service provider and passengers for railroad and urban transit operators. For many heavily travelled passenger rail corridors, constructing a set of demand-responsive and user-centric timetables is difficult because of the complicated nature of the demand–supply interactions in both the temporal and spatial dimensions and because of the inherent difficulties in large-scale combinatorial optimization when a large number of trains are scheduled and tens thousands of passengers are served in a day.

As an important optimization application in transportation, the passenger train timetabling problem has received much attention in the past few decades. Two comprehensive surveys on the general train scheduling problem and on passenger train scheduling specifically are provided by Cordeau et al. (1998) and Caprara et al. (2007), respectively. In order to improve the level of service for passenger transportation, train service operators have gradually shifted their focus from operation-oriented decision making, which typically minimizes the total train travel time in a conflict-free timetable, to market-driven

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decision making, which highlights the convenience, reliability and waiting-time reduction measures. The researchers typically concentrate on one of two timetabling features: (I) a periodic schedule across multiple service time periods/days, which easily allow passengers to remember the exact departure times at stations and (ii) a time-dependent and noncyclic schedule throughout an extended period of the planning horizon during a day. To recognize time-varying demand loading/unloading rates, in practice, separate peak/off-peak hour timetables with different service frequencies or headways can also be designed first and then integrated into a daily schedule. An early study by Ghoneim and Wirasinghe (1986) examines how to optimize the transportation service along an existing rail line by adopting a zone-stop schedule during peak periods. Based on even headways, their model aims to reduce the generalized travel time costs to passengers and the related rail investment/operating costs. By applying event scheduling and graph-based modeling approaches, Serafini and Ukovich (1989) designed an implicit enumeration algorithm in order to design periodic train timetables. Focusing on resolving train conflicts while satisfying various operational constraints and objectives, Carey (1994) and Carey and Crawford (2007) developed models and heuristic algorithms for a single busy station and then extended the approach to a series of complex stations linked by multiple one-way lines. These studies provide a range of valuable methods for the passenger train timetabling problem. However, a unified modeling framework is critically needed for further fully considering the interaction of continuous passenger temporal/spatial distributions and discrete train arrival/departure events.

Regarding the periodic scheduling problem, many studies aim to provide a robust and economically efficient service schedule in response to different demand conditions. For predefined running times and stopping patterns of trains, Nachtigall and Voget's paper (1997) is concerned with building periodic timetables by minimizing the weighted sum of passenger waiting times at stations. By focusing on the benefit to both transit passengers and operators, Liebchen (2008) presented a periodic-event-scheduling approach to construct timetables for the Berlin urban rail system. Goverde (2007) proposed a railway timetable stability measure by introducing a max-plus system theory and analyzed train delay propagation processes. A stochastic optimization model was developed by Kroon et al. (2008) to allocate the time supplements and the buffer times in a given timetable to maximize the robust performance during stochastic disturbances. In order to minimize passengers' transfer times, Wong et al. (2008) concentrated on the synchronization between the different lines of an urban rail transit network. In general, the periodic-schedule-based timetables are still not fully sensitive and responsive to the time-varying passenger demands, which could result in long waiting times and reduced service reliability, particularly under irregular oversaturated conditions.

Designing demand-responsible transit schedules is another research direction in the field of transportation service optimization. Ceder (1986) proposed an optimization framework for the transit timetable design problem that aims to synchronize vehicle-departure times under dynamic passenger demands. A number of early studies by Newell (1971) and Hurdle (1973) have adopted the techniques of cumulative counts from the general area of queuing theory to calculate the total passenger waiting time through analytical formulas. Using a continuum fluid-flow model to approximate the passenger loading patterns, Daganzo (1997) formulated a transient queues process to count the time-dependent passengers delay for a general scheduled transit system. A comprehensive modeling framework is provided by Ceder (2007) to adjust vehicle departure times and smooth the transitions between time periods through regular time headways or average passenger loads. Canca et al. (2011) tackled the train scheduling problem for middle and long-distance networks by considering a trade-off between the level of service and factors related to the network profitability. They proposed a nonlinear integer programming model which fits the schedules to a dynamic behavior of demand. To maximize the demand captured by a regular timetable with constant train arrival/departure intervals, Cordone and Redaelli (2011) developed a nonlinear mixed integer model and used a branch-and-bound algorithm to consider a piecewise-linear approximation of the objective function so as to provide both upper and lower bounds of optimal solutions. Using the time-dependent origin–destination (OD) passenger demand matrix directly from ridership data, Niu and Zhou (2013) developed integer programming models to design train timetables for a heavily congested urban rail line. By assuming trains stop at every station and focusing on mitigating oversaturated congestion at different intermediate passing stations, the authors develop a dynamic programming algorithm (for a single arrival station case) and a genetic algorithm (for a complex multi-arrival station case) that can adjust the starting time of the trains from the terminals. A nonlinear integer programming model is developed by Canca et al. (2014) that aims to determine the train arrival and departure times under dynamic passenger demands, with objective functions jointly considering both passenger waiting times and system operator costs.

There are also a number of studies related to many general aspects of demand-responsive train service design. For example, within a user equilibrium assignment framework with elastic demands, Hsieh (2003) studied the problem of a train service design for a high-speed rail line. Albrecht (2009) proposed a two-level approach for designing a demand-oriented subway timetable with a branch-and-bound algorithm for train frequency optimization in the first stage and a genetic algorithm for timetable design in the second stage. Recently, Barrena et al. (2014a,b) proposed two non-linear programming formulations for the train timetabling problem in the context of a dynamic demand pattern, with the objective of minimizing the average passenger waiting time at each time interval. A fast algorithm based on adaptive large neighborhood search is developed by removing or adding a train run iteratively.

In this paper, we first focus on how to derive mathematically tractable formulas for calculating passenger waiting times under time-varying demand volumes from different OD pairs and non-cyclic timetables. As indicated by many transportation demand modeling empirical studies, such as Bhat (1995), passenger waiting times at stations and in-vehicle travel times are two major attributes for which travelers can evaluate the level of service for rail and transit-related transportation modes, while a greater weight is given by travellers to at-station waiting times on a per unit time basis. Although various nonlinear models have been proposed to compute passenger waiting times for general continuous or discretized cumulative flow counts, this paper aims to develop a number of relatively simpler functions that can balance the mathematical tractability and representation completeness in capturing the within-day variability in OD demand data.

To balance in-vehicle travel times, passenger waiting times and operating costs, different skip-stop patterns are widely applied in practice, e.g., regular trains stop at each station to pick up all passengers waiting at stations, express trains skip stops along a route to reduce total travel time during peak hours, and off-peak trains skip stations with low demand to reduce operating costs. In order to simplify the problem, this paper considers the case that trains just follow one another. This operation experience is commonly adopted for a medium- or short-distance rail corridor, or in an urban rail transit system. Obviously, skip-stop services would lead to efficiency gains if the overtaking operations are allowed. In addition, compared to all-stop patterns, the skip-stop pattern allowed in our formation can clearly reduce the end-to-end travel times of boarding passengers and the energy consumptions of trains that skip stops. This study is intended to calculate time-dependent passenger waiting times precisely by jointly considering skip-stop patterns and spatial and temporal distributed OD passenger demands.

To specifically minimize the passenger waiting times, we consider and distinguish short-term dispatching and long-term timetabling applications. The former typically uses one hour as the planning horizon, and a published train timetable can be dynamically adjusted to adapt predicted passenger demands in the near future, e.g., after special events and irregular weather conditions. In the latter case, a within-day time-dependent OD demand table is generated by averaging multiple days of ridership records. Handling these two cases in a unified framework allows us to construct on-line and off-line scheduling models that can respond to the variation of temporal and spatial demand distributions. For the real-time scheduling and daily operational cases, how to seamlessly integrate the buffer time assignment within the given train skip-stop pattern is another important research topic of this study that helps design a robust rail schedule against stochastic disturbances during the train operation process.

In addition, this research adopts a general purpose high-level mathematical optimization modeling framework, which is different from special purpose algorithmic development, such as genetic algorithms. In our study, we propose a number of mathematical reformulation methods to handle the complex relationship between the passenger waiting time and time-dependent demand to create mixed-integer programming models compatible with standard high-level optimization platforms. Specifically, we use the General Algebraic Modeling System (GAMS) and related packages, such as AMPL (Fourer et al., 2002), which are specifically designed for modeling linear, nonlinear and mixed integer optimization problems. The general purpose modeling framework requires a number of particular structural elements (e.g., if statements cannot be arbitrarily enforced), but the resulting mathematically tractable model allows users to focus on the high-level modeling issues first and then select a variety of optimization solvers to effectively handle the computational challenges. An early application of GAMS for train scheduling problems can be found in a study by Carey (1994).

This paper aims to optimize train timetables for a rail corridor with given time-varying origin-to-destination passenger demand data and train skip-stop patterns. Compared to the existing literature, this study offers the two main contributions as below. First, a unified quadratic integer programming model with linear constraints is developed to jointly synchronize effective passenger loading time windows and train arrival and departure times at each station. Second, a class of mathematically rigorous and algorithmically tractable nonlinear mixed integer programming models are developed and solved by state-of-art solvers for both real-time scheduling and medium-term planning applications.

The remainder of this paper is organized as follows. A detailed problem statement and assumptions are presented in Section 2. In Section 3, a unified quadratic integer programming model with linear constraints is developed to design a train timetable using the existing and predicted passenger demands in the future. By reformulating the unified model and adding the required constraints with mathematically tractable forms in GAMS, two improved models for both the real-time scheduling and daily operational applications are proposed in Section 4 and Section 5. In Section 6, numerical examples with real-world data are provided to demonstrate the effectiveness of the proposed models and algorithms. The last section concludes the paper and discusses future research topics.

2. Problem statement

As shown in Fig. 1, we consider how to design a demand-responsive train timetable along one train direction of a two-direction rail line, where the stations are sequentially numbered as $1, 2, \ldots, S$, and the train services move from the first station to the last station.

The planning time horizon is denoted as $[0, T]$ with equal one-minute intervals. We hereafter refer to any particular *time interval* as *time*. The notations, input parameters and variables for the optimization problems under consideration are listed as follows.

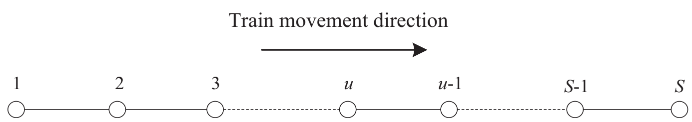

Fig. 1. Illustration of a rail corridor.

2.1. Notations

t	index of times, $t \in [0, T]$;
j	index of trains;
u, v	index of stations.
Input parameters	
$P^{u,v}(t)$	number of passengers who arrive at station u traveling to station v at time t on the line, $t \in [0, T]$;
N	number of trains that departed from the first station during the study horizon;
r^u	free-flow running time from station u to station $u+1$ on the line;
ε	additional acceleration/deceleration time associated with train stops;
$\theta_{\min}^{\text{section}}$	predetermined minimum interval between two consecutive trains at the same section;
$\theta_{\min}^{\text{station}}$	predetermined minimum interval between two consecutive trains at the same station;
$\lambda_{\min}$	predetermined minimum dwell time of trains at stations;
$\lambda_{\max}$	predetermined maximum dwell time of trains at stations;
$\Delta_{\max}$	predetermined maximum sum of buffer times at segments of a train;
c_j	capacity of train j, i.e., the maximum number of passengers accommodated by train j;
μ_j^u	skip-stop index, which is equal to 1 if train j stops at station u; otherwise, it is 0;
$\tau_j^{u,v}$	coupled-stop index, which is equal to 1 if train j stops at both station u and station v; otherwise, it is 0;
TD_j^E	pre-specified earliest departure time of train j at the first station;
TD_j^L	pre-specified latest departure time of train j at the first station.
Variables	
TA_j^u	arrival time of train j at station u on the line;
TD_j^u	departure time of train j from station u on the line;
TS_j^u	stopping time of train j at station u on the line;
TR_j^u	buffer time of train j traveling from station u to station $u+1$;
Q_j^u	number of passengers remaining on train j after the train departs from station u;
$A^{u,v}(t)$	cumulative number of arrival passengers at station u heading to station v by time t;
$D^{u,v}(t)$	cumulative number of departure passengers at station u heading to station v by time t;
$W_j^{u,v}$	total passenger waiting time at station u towards station v on train j.

The optimization problem for train timetabling studied in this paper is formally stated as follows. Given (1) time-dependent origin-to-destination demand, $P^{u,v}(t)$, (2) the total number of trains departing from the first station N with predetermined free-flow segment running times r^u, and (3) the skip-stop patterns represented through μ_j^u, the proposed model aims to minimize the total passenger waiting time by determining train arrival and departure times at stations. As the free-flow segment running time is deterministic as given input, the optimized arrival and departure times TA_j^u and TD_j^u jointly determine the buffer times TR_j^u for each train at different segments and imply that our model can handle the train scheduling problem with variable train running speeds. The time-varying origin-to-destination demand information along the corridor can be collected through rail automatic fare collection systems (RAFC) or generated by other demand-forecasting methods. To build a generic optimization model, we further introduce the following assumptions.

Assumption 1. First, we assume that the provided total train capacity is greater than or equal to the total passenger demand, and the last train stops at every station and departs from the first station at the exact ending time T of the study period. This assumption is to ensure that all passengers arrive at their destination stations during the planning horizon.

Assumption 2. At each station, we further assume that the oversaturation situation is not permitted, and each train should use its capacity c_j to accommodate all waiting passengers. If oversaturation is allowed at stations, then we refer readers to our previous paper (Niu and Zhou, 2013) for various models on calculating the effective loading time periods for passengers who cannot board the next train. When oversaturation is not allowed, we do not need to consider complex conditions in which many effective loading time periods lag behind corresponding train departure times. When applying the proposed models in this paper to real-world cases with a certain possibility of oversaturation (due to limited train resources), one can relax the strict train capacity constraint, and add an additional penalty to reduce the chances of violations.

3. Unified quadratic integer programming model

3.1. Effective loading time period

In this section, we consider how to calculate the effective loading time period, and accordingly, the overall waiting time for passengers from different OD pairs on a corridor with a given skip-stop pattern; note that our previous paper (Niu and Zhou, 2013) assumes that trains stop at each station.

Initially, this study uses binary parameter $\tau_j^{u,v}$ to indicate whether train j stops at both station u and station v; the value of the parameter can be obtained by $\tau_j^{u,v} = \mu_j^u \cdot \mu_j^v$. Namely, binary parameter $\tau_j^{u,v} = 1$ indicates train j stops at both stations u and v; otherwise, it is equal to 0. As shown in Fig. 2, trains $\bar{j}, j-2, j-1$ and j have different stopping patterns at stations u and v. To calculate the passenger waiting time for a given OD pair, e.g., from u to v, we introduce the notation of the matched train indicator, denoted as $\bar{j}$, which is the nearest preceding train that stops at station u and station v same as train j.

Obviously, train indicator $\bar{j}$ is dependent on the specific train j and the station OD pair (u, v) associated with a set of passengers. To optimize any train j within the study horizon, we assume there is always a matched train $\bar{j}$ exiting, and their schedules at each station have been included in our optimization model. Another way to permit a matched train $\bar{j}$ for each train is to assume that, in the beginning of the horizon, there is a dummy train $\bar{j} = 0$ that stops at every station. Therefore, the matched train can be formally expressed as $\bar{j} = \max\{j' | j' < j, \tau_{j'}^{u,v} = \tau_j^{u,v} = 1 \text{ or } j' = 0\}$.

Given the train skip-stop patterns, we can use a pre-processing step to obtain the matched-train index in an easy manner. Specifically, for a given station OD pair (u, v) and all trains j that stop at both station u and station v $(\tau_j^{u,v} = 1)$, we can determine the matched train $\bar{j}$ station by station. When all trains stop at every station $(\mu_j^u = 1,\ \forall u, j)$, the matched train is then exactly the corresponding preceding train j $(\bar{j} = j-1)$.

Furthermore, we can define the effective time period for passengers boarding train j at station u heading to station v as $[TD_{\bar{j}}^u, TD_j^u)$. Note that the matched train indexes are treated as predetermined input for our proposed optimization model, while the effective time period associated with departure times of two related trains are variables to be optimized.

3.2. Passenger waiting time

The next task is to derive a mathematically tractable form for the total waiting time for different OD pairs. First, the cumulative number of arrival passengers at station u towards station v by time t can be calculated as below.

$$A^{u,v}(t) = \sum_{t'=0}^{t} P^{u,v}(t') \tag{1}$$

The number of passengers boarding train j at station u and heading towards station v, $B_j^{u,v}$, can be calculated by the following equation.

$$B_j^{u,v} = \sum_{t \in [TD_{\bar{j}}^u, TD_j^u)} \tau_j^{u,v} \cdot P^{u,v}(t) \tag{2}$$

Because passenger queues in the train scheduling problem can be viewed as a bulk queue, the cumulative number of departure passengers at station u and heading towards station v by time t, $D^{u,v}(t)$, is a linear step function.

$$D^{u,v}(t) = \begin{cases} D^{u,v}(t-1) & \text{if } TD_{\bar{j}}^u \leqslant t < TD_j^u \\ D^{u,v}(t-1) + B_j^{u,v} & \text{if } t = TD_j^u \end{cases} \tag{3}$$

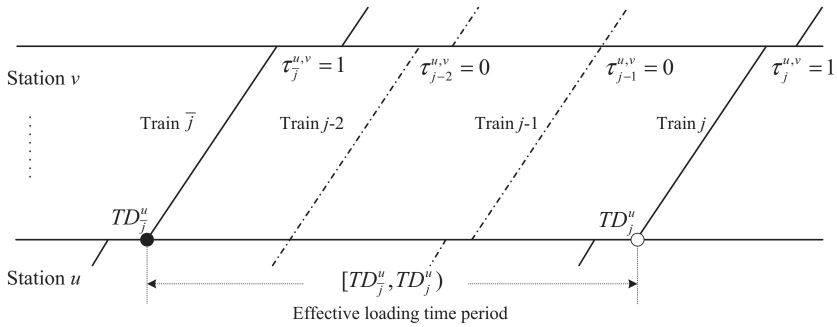

Fig. 2. Illustration of two matched trains.

As illustrated in the shaded area in Fig. 3, we can derive the following properties for the passengers arriving during the effective loading time period $[TD_{\bar{j}}^u, TD_j^u)$ for train j.

Property 1. *The total passenger waiting time at station u toward station v boarding train j is:*

$$\sum_{t \in [TD_{\bar{j}}^u, TD_j^u)} [A^{u,v}(t) - D^{u,v}(t)] = \sum_{t \in [TD_{\bar{j}}^u, TD_j^u)} P^{u,v}(t) \cdot (TD_j^u - t) \tag{4}$$

Proof. The waiting time for a passenger arriving at time t to board train j is $TD_j^u - t$, so the right-hand term of Eq. (4) can be derived by taking the summation across the effective loading period for train j, weighted by the given time-dependent OD demand $P^{u,v}(t)$. Additionally, the waiting delay at time t is $A^{u,v}(t) - D^{u,v}(t)$ when the sum is calculated in the vertical direction. □

It should be noted that the mathematical structure of Eq. (4) is dependent on the given demand framework and the discretized time unit. Furthermore, the expression is also linked with the train index $\bar{j}$, which is subject to the given train skip-stop patterns and can be determined by a pre-processing procedure. The generic cost associated with the waiting time, as discussed below, can be changed with different demand conditions and application contexts.

3.3. Objective function with and without time-dependent OD demands

The objective function considered in this paper minimizes the total waiting times of passengers at stations, as shown in Eq. (5).

$$\min \sum_{j=1}^{N} \sum_{u=1}^{S-1} \sum_{v=u+1}^{S} \sum_{t \in [TD_{\bar{j}}^u, TD_j^u)} \tau_j^{u,v} \cdot P^{u,v}(t) \cdot (TD_j^u - t) \tag{5}$$

Property 2. *If the passengers arrive at station u with a uniform rate between two trains $\bar{j}$ and j, under time-invariant demand conditions, the total passenger waiting time at station u for train j towards station v is a function of the train departure times and the associated passenger arrival rate.*

$$W_j^{u,v} = 0.5 \cdot \tau_j^{u,v} \cdot [TD_j^u - TD_{\bar{j}}^u]^2 \cdot P^{u,v} \tag{6}$$

where $P^{u,v}$ is the constant demand flow rate of passengers for a station pair (u, v).

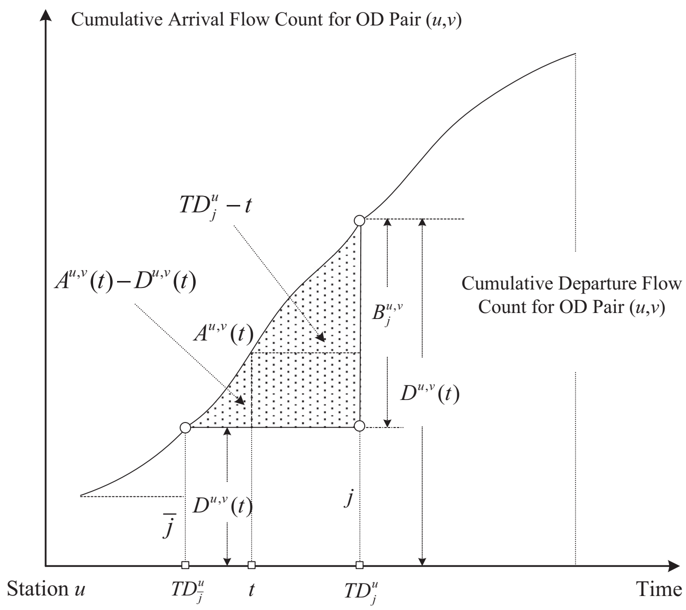

Fig. 3. Illustration of cumulative arrival and departure flow counts.

Proof. Given the constant flow rate, the total waiting time at station u for train j (and for heading to station v) is a triangle with a vertical height of $B_j^{u,v} = [TD_j^u - TD_{\bar{j}}^u] \cdot P^{u,v}$ and a base width of the effective time window $TD_j^u - TD_{\bar{j}}^u$ (Fig. 3), which leads to Eq. (6). As $P^{u,v}$ is given and departure times TD_j^u and $TD_{\bar{j}}^u$ are variables, Eq. (6) is a quadratic function of the variables to be optimized. □

In a special case in which all trains stop at each station, i.e., $\eta_j^u = 1$ and $\bar{j} = j - 1$, the objective function can be represented as below.

$$\min \sum_{j=1}^{N}\sum_{u=1}^{S-1} 0.5 \cdot [TD_j^u - TD_{j-1}^u]^2 \cdot P^u \tag{7}$$

where $P^u = \sum_v P^{u,v}$ denotes the arrival rate of passengers at station u.

Eq. (7) is commonly adopted in many existing studies to count the total passenger waiting times at stations in transit systems. For a train scheduling problem with a general skip-stop pattern and time-varying demand, we must use Eq. (5) to determine the total waiting time at stations precisely.

3.4. Constraints

(1) Train capacity constraints

Using the time-dependent demand matrix $P^{u,v}(t)$, the number of passengers boarding train j at station u and heading towards station v can be determined as $\sum_{t \in [TD_{\bar{j}}^u, TD_j^u)} \tau_j^{u,v} \cdot P^{u,v}(t)$. Thus, the number of passengers remaining in train j after the train departs from station u can be represented as,

$$Q_j^u = \sum_{u'=1}^{u}\sum_{v=u+1}^{S}\sum_{t \in [TD_{\bar{j}}^{u'}, TD_j^{u'})} \tau_j^{u',v} \cdot P^{u',v}(t) \tag{8}$$

This paper assumes that the strict capacity constraint is required for each train. Therefore, the number of in-train passengers is no more than the given capacity during train j's runs on the line.

$$Q_j^u \leqslant c_j \quad \forall u, j \tag{9}$$

(2) Linking constraints

$$TA_j^u = TD_j^{u-1} + r^{u-1} + \varepsilon \cdot (\mu_j^{u-1} + \mu_j^u) + TR_j^{u-1} \qquad \forall u > 1, j \tag{10}$$

$$TD_j^u = TA_j^u + TS_j^u \qquad \forall u, j \tag{11}$$

The above constraints can be interpreted as follows. Fig. 4 illustrates the linkages expressed in Eqs. (10) and (11). Using the departure time TD_j^1 at the first station, the stopping time TS_j^u at station u and the buffer time TR_j^u at segment $u \to u+1$ for train j, we can determine the arrival and departure times TA_j^u and TD_j^u recursively from the first station step by step.

(3) Safety headway constraints

$$TD_j^u - TD_{j-1}^u \geqslant \theta_{\min}^{\text{section}} \qquad \forall u, j > 1 \tag{12}$$

$$TA_j^u - TA_{j-1}^u \geqslant \theta_{\min}^{\text{section}} \qquad \forall u, j > 1 \tag{13}$$

$$TA_j^u - TD_{j-1}^u \geqslant \theta_{\min}^{\text{station}} \qquad \forall u, j > 1 \tag{14}$$

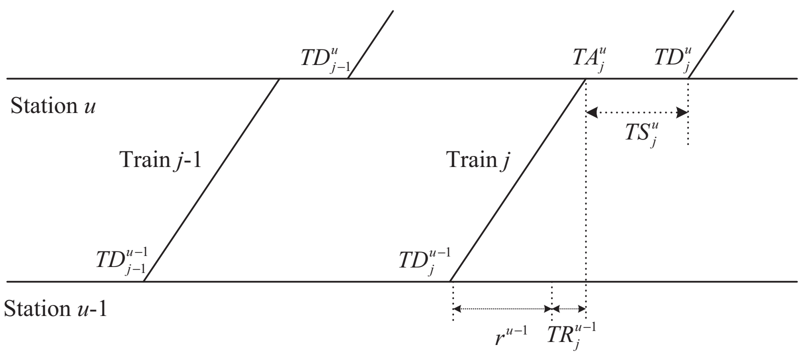

Fig. 4. Illustration of train operation activity.

Eqs. (12) and (13) enforce the safety headway constraints between two consecutive trains at the same section, by restricting the train entering/departure times and exiting/arrival times at that section. The station-based headway constraint (14) is to ensure at most one train stops at a station during a small time period.

(4) Feasible range constraints

$$\lambda_{\min} \leqslant TS_j^u \leqslant \lambda_{\max} \quad \forall u, j \tag{15}$$

$$\sum_{u=1}^{S-1} TR_j^u \leqslant \Delta_{\max} \quad \forall j \tag{16}$$

Eqs. (15) and (16) ensure that the train stopover time at each station has a feasible range, with an upper bound of $\Delta_{\max}$ for the total buffer time to maintain a reasonable speed.

(4) Departure time constraints

Theoretically, integer variables TD_j^1 can have any values between 0 and T. For a fixed train indicator j, however, a reasonable upper bound and lower bound are always imposed on the departure time TD_j^1 to consider a reasonable adjustment range, in practice.

$$TD_j^E \leqslant TD_j^1 \leqslant TD_j^L \quad \forall j \tag{17}$$

Adding inequality (17) can tighten the feasible space of the resulting optimization models and reduce the solution search time, where the values of TD_j^L and TD_j^E can be set by the experiences of the planners or the operational practices of an existing timetable.

We are now ready to examine the tractability of the proposed model as follows.

Model M1 (simple model with time-invariant demand and all-stop pattern and no train capacity constraint):

Objective function (7).

Subject to constraints (10)–(17).

Model M2 (generic model with time-dependent demand and skip-stop pattern):

Objective function (5).

Subject to constraints (8)–(17).

It is clear that model **M1** is a nonlinear integer programming with a quadratic objective function and linear constraints, which can be solved directly by standard optimization packages (e.g., GAMS). For general algebraic modeling languages, the indexed summations should be expressed in terms of pre-defined or constant subscript sets. However, in unified optimization framework **M2**, there are variables $[TD_{\bar{j}}^u, TD_j^u)$ in the index for the sums in Eqs. (5) and (8). Specifically, under time-dependent demand conditions, effective time periods need to be associated with the departure time variables of two related trains.

We now consider how to reformulate model **M2** to make it algorithmically tractable for both online scheduling and off-line planning applications. In the former case, we can directly use the minute-dependent OD demand input, which is assumed to be constant during each minute. In the offline planning applications, the demand interval could be one hour by averaging the highly dynamic demands over different days or months.

4. Optimization model reformulation for high-resolution demand input data

4.1. Reformulation for variable loading time period

In this section, we consider the problem under the minute-dependent demand (i.e., a minute-based representation of passenger demand) conditions, without loss of generality. That is, all origin–destination demand matrices are represented in terms of one-minute periods. This high-fidelity representation, which is particularly suitable for on-line scheduling, allows us to describe highly dynamic existing and predicted passenger demands in the near future. We first introduce the following binary variables for this high-resolution model.

$z_j^u(t)$	departure binary variable which is equal to 1 if train j departs from station u by time t; otherwise, it is 0.
$TL_j^{u,v}(t)$	binary loading indicator variable which is equal to 1 if passengers from station u towards station v can board train j at time t; otherwise, it is 0.

The binary vector $\{z_j^u(t)|t \in [0, T]\}$ with different times t for train j at station u take the non-decreasing form of $(0, 0, \cdots, 0, 1, 1, \cdots, 1)$. In Fig. 5, as an example, train $\bar{j}$ and train j depart at minute 3 and 7, respectively. The first "1" for vector $\{z_j^u(t)|t \in [0, T]\}$ (circled) indicates the departure timestamps of train j at station u.

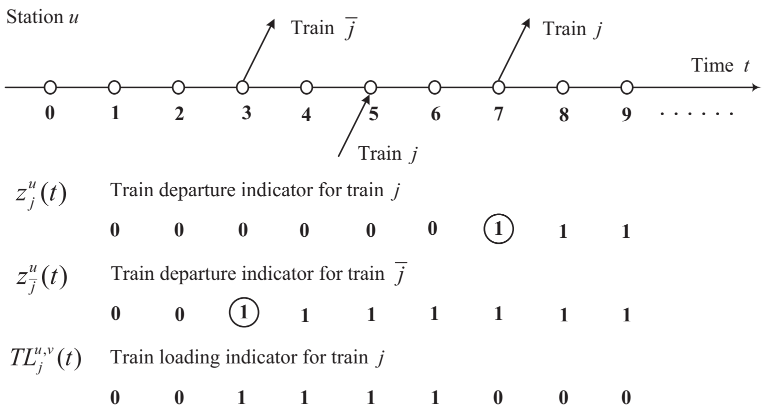

Fig. 5. Illustration of departure and loading binary variables.

Property 3. *The loading indicator variables $TL_j^{u,v}(t)$ under assumption 2 can be determined by the departure binary variables as follows.*

$$TL_j^{u,v}(t) = z_{\bar{j}}^u(t) - z_j^u(t) \tag{18}$$

Proof. In Fig. 5, the departure timestamps for train $\bar{j}$ and train j are time $t = 3$ and time $t = 7$, respectively. Under Assumption 2, the effective loading time period for train j covers $[TD_{\bar{j}}^u, TD_j^u)$. During this period, such as at times 3, 4, 5 and 6, $z_{\bar{j}}^u(t) = 1$ and $z_j^u(t) = 0$ indicate that the passengers who arrived at these times can successfully board departure train j. Except for period $[TD_{\bar{j}}^u, TD_j^u)$, the condition $z_{\bar{j}}^u(t) - z_j^u(t) = 0$ is always true. □

4.2. Additional constraints

In order to use a general purpose optimization program, such as GAMS, we need to consider how to express train indicator $\bar{j}$ when calculating the loading variables $TL_j^{u,v}(t)$ with Eq. (18). For a particular train j and station OD pair (u, v), to handle the above issue, we introduce a constant binary parameter $\beta_{j',j}^{u,v}$ with an arbitrary train indicator j' whose value equals 1 if the train index is $j' = \bar{j}$; otherwise, it is 0. According to the given train skip-stop patterns, we can easily determine the binary parameter $\beta_{j',j}^{u,v}$ for the given train indicator j and OD pair (u, v).

Based on the given binary parameter $\beta_{j',j}^{u,v}$, we can express the variables in Eq. (18) through $z_{\bar{j}}^u(t) = \sum_{j'=1}^{j} \beta_{j',j}^{u,v} \cdot z_{j'}^u(t)$. Thus, the relationship between departure binary variables $z_j^u(t)$ and binary loading indicator variables $TL_j^{u,v}(t)$ can be rewritten as below.

$$TL_j^{u,v}(t) = \sum_{j'=1}^{j} \beta_{j',j}^{u,v} \cdot z_{j'}^u(t) - z_j^u(t) = \sum_{j'=1}^{N} \beta_{j',j}^{u,v} \cdot z_{j'}^u(t) - z_j^u(t) \tag{19}$$

Using the loading time variables $TL_j^{u,v}(t)$, the number of passengers boarding train j at station u and heading towards station v can be re-expressed as $\sum_{t\in[0,T]} \tau_j^{u,v} \cdot P^{u,v}(t) \cdot TL_j^{u,v}(t)$, which is essentially a sum over a constant time index set between 0 and T.

Thus, the capacity constrain $Q_j^u \leqslant c_j$ can be updated as shown below.

$$\sum_{u'=1}^{u} \sum_{v=u+1}^{S} \sum_{t\in[0,T]} \tau_j^{u',v} \cdot P^{u',v}(t) \cdot TL_j^{u',v}(t) \leqslant c_j \tag{20}$$

For two adjacent time indicators t-1 and t, the relationship between cumulative binary variables $z_j^u(t-1)$ and $z_j^u(t)$ can be represented as follows.

$$z_j^u(t-1) \leqslant z_j^u(t) \tag{21}$$

According to the following constraint, we can also link the train departure time TD_j^u and departure binary variable $z_j^u(t)$.

$$TD_j^u = \sum_{t\in[0,T]} t \cdot [z_j^u(t) - z_j^u(t-1)] \tag{22}$$

4.3. Reformulated objective function

By using the loading time variable $TL_j^{u,v}(t)$, the total waiting times at station u between two matched trains j and $\bar{j}$ can be rewritten as follows.

$$\sum_{t\in[TD_{\bar{j}}^u, TD_j^u)} P^{u,v}(t)\cdot(TD_j^u - t) = \sum_{t\in[0,T]} P^{u,v}(t)\cdot TL_j^{u,v}(t)\cdot(TD_j^u - t) \tag{23}$$

Thus, the objective function for the real-time scheduling case can be updated as:

$$\min \sum_{j=1}^{N}\sum_{u=1}^{S-1}\sum_{v=u+1}^{S}\sum_{t\in[0,T]} \tau^{u,v}\cdot P^{u,v}(t)\cdot TL_j^{u,v}(t)\cdot(TD_j^u - t) \tag{24}$$

Now, model **M2** is reformulated to model **M3**:

Model M3 (with time-dependent high-resolution demand data):

Objective function Eq. (24).

Subject to constraints (10)–(17), (19)–(22).

Obviously, reformulated model **M3** is a mixed integer programming with a quadratic objective function and linear constraints.

5. Reformulation for low-resolution demand data and long planning horizon

5.1. Impact of long aggregation time interval and planning horizon

We now turn our attention to an off-line train timetabling problem with time-dependent matrices, which have the distinctive features of (1) a relatively long demand aggregation time interval and (2) a long planning horizon T. Without the loss of generality, we consider the aggregation time interval of the origin–destination demand matrix as one hour (i.e., 60 min), which can be generalized to 15 min or 30 min depending on the data collection accuracy and with-in-day variability of the demand changes. The notations of this newly reformatted model are listed below, and a set of binary train-to-demand period mapping variables $x_j^u(k)$ are introduced between train j and demand period k at station u.

Notation:

K	the number of time periods;
k	demand time period index; the demand for the kth period covers time interval $[60(k\text{-}1), 60k]$;
$P^{u,v}(k)$	given demand flow rate for the station OD pair (u, v) during one-hour period k;
$x_j^u(k)$	binary variable that indicates if a train departure time TD_j^u belongs to a specific period k:

$$x_j^u(k) = \begin{cases} 1 & \text{if } 60(k-1) \leqslant TD_j^u \leqslant 60k \\ 0 & \text{otherwise} \end{cases}$$

In general, the reformulated model **M3** can apply to this medium-term operational use case, by assuming that the demand values remain the same during each one-hour period. During a given period k, for example, we assume that the minute-dependent demand $P^{u,v}(t)$ is the same as $P^{u,v}(k)$ for $t\in[60(k-1), 60k]$. However, the size of the resulting model will increase dramatically due to the huge number of binary variables $z_j^u(t)$ and $TL_j^{u,v}(t)$. For a high-speed rail line with 10 stations and 70 trains, for instance, the number of variables $TL_j^{u,v}(t)$ is $10\times10\times70\times15\times60$ = 6,300,000 when we take the operational duration of a day as T = 15 h. Based on such considerations, we need to construct a new formulation for the daily operational requirements by directly using hour-dependent demand (i.e., an hour-based representation of passenger demand) matrix $P^{u,v}(k)$.

5.2. Computing passenger waiting times and the number of in-train passengers over time intervals

Using the train departure times $TD_{\bar{j}}^u$ and TD_j^u, as shown in the previous section, we obtain the total waiting time associated with train j which depends on the minute-dependent demand record $P^{u,v}(t)$. To construct a mathematically tractable form for the medium-term operational application, we need to update the formulation for counting the passenger waiting time at a station by the hour-dependent demand $P^{u,v}(k)$.

Property 3. *Under the hour-dependent demand, the total waiting time at station u, $W_j^{u,v}$, for the departure train j and station OD pair (u, v) can be represented as below.*

$$W_j^{u,v} = 0.5 \cdot \tau_j^{u,v} \cdot \sum_{k=1}^{K} P^{u,v}(k) \cdot \left\{ TD_j^u \cdot x_j^u(k) - \sum_{j'=1}^{N} \beta_{j',j}^{u,v} \cdot TD_{j'}^u \cdot x_{j'}^u(k) - \left[x_j^u(k) - \sum_{j'=1}^{N} \beta_{j',j}^{u,v} \cdot x_{j'}^u(k) \right] \cdot \sum_{k=1}^{K} x_j^u(k) \cdot 60(k-1) \right\}^2 \tag{25}$$

Proof. In order to count the total waiting time at a station, we need to distinguish the following two cases.

Case I. The departure times $TD_{\bar{j}}^u$ and TD_j^u are located within the same period.

As shown in Fig. 6, if two matched trains $\bar{j}$ and j depart from station u in the same period, then there always exists a particular period k that ensures $x_j^u(k) = x_{\bar{j}}^u(k) = 1$. Considering that equation $\sum_{k=1}^{K} x_j^u(k) = 1$ always holds, according to Property 2, the total waiting time between train departure times $TD_{\bar{j}}^u$ and TD_j^u is thus represented as below.

$$W_j^{u,v} = 0.5 \cdot \tau_j^{u,v} \cdot (TD_j^u - TD_{\bar{j}}^u)^2 \cdot \sum_{k=1}^{K} x_j^u(k) \cdot P^{u,v}(k) \tag{26}$$

Case II. The departure times $TD_{\bar{j}}^u$ and TD_j^u are located in two different but adjacent periods.

As the aggregation time interval is sufficiently long, without the loss of generality, we assume that two matched trains $\bar{j}$ and j are located in two adjacent periods $k-1$ and k, respectively. In this case, as shown in Fig. 7, condition $x_j^u(k) = x_{\bar{j}}^u(k-1) = 1$ is satisfied for a particular period k.

The passenger waiting times between departure times $TD_{\bar{j}}^u$ and TD_j^u have two components, namely, $0.5 \cdot [TD_j^u - 60(k-1)]^2 \cdot P^{u,v}(k)$ and $0.5 \cdot [60(k-1) - TD_{\bar{j}}^u]^2 \cdot P^{u,v}(k-1)$. Considering different possible periods, the waiting times $W_j^{u,v}$ can be calculated by the following formula.

$$W_j^{u,v} = 0.5 \cdot \tau_j^{u,v} \cdot \left[TD_j^u - \sum_{k=1}^{K} x_j^u(k) \cdot 60(k-1) \right]^2 \cdot \sum_{k=1}^{K} x_j^u(k) \cdot P^{u,v}(k) + 0.5 \cdot \tau_j^{u,v} \cdot \left[\sum_{k=1}^{K} x_{\bar{j}}^u(k) \cdot 60(k-1) - TD_{\bar{j}}^u \right]^2 \cdot \sum_{k=1}^{K} x_{\bar{j}}^u(k) \cdot P^{u,v}(k) \tag{27}$$

To integrate both conditions in cases I and II into one objective function, we transform Eqs. (26) and (27) into a unified formula for calculating the total waiting time between two matched trains.

$$W_j^{u,v} = 0.5 \cdot \tau_j^{u,v} \cdot \sum_{k=1}^{K} P^{u,v}(k) \cdot \left\{ TD_j^u \cdot x_j^u(k) - TD_{\bar{j}}^u \cdot x_{\bar{j}}^u(k) - [x_j^u(k) - x_{\bar{j}}^u(k)] \cdot \sum_{k=1}^{K} x_j^u \cdot 60(k-1) \right\}^2 \tag{28}$$

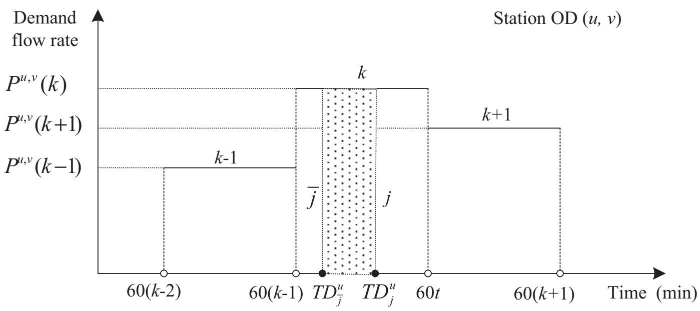

Fig. 6. Illustration of the demand flow rate for the first case.

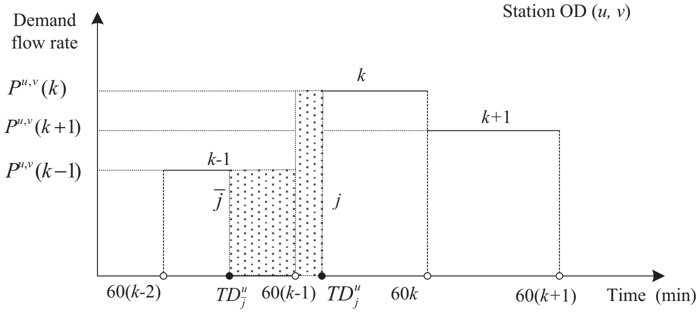

Fig. 7. Illustration of the demand flow rate for the second case.

For example, Eq. (28) is simplified as $W_j^{u,v} = 0.5 \cdot \tau_j^{u,v} \cdot [TD_j^u - TD_{\bar{j}}^u]^2 \cdot P^{u,v}(4)$ if $\{x_j^u(k)\} = \{x_{\bar{j}}^u(k)\} = \{0,0,0,1,0\}$; however, the equation is changed to $W_j^{u,v} = 0.5 \cdot \tau_j^{u,v} \cdot [120 - TD_{\bar{j}}^u]^2 \cdot \pi^{u,v}(2) + 0.5 \cdot \tau_j^{u,v} \cdot [TD_j^u - 120]^2 \cdot \pi^{u,v}(3)$ if $\{x_{\bar{j}}^u(k)\} = \{0,1,0,0,0\}$ and $\{x_j^u(k)\} = \{0,0,1,0,0\}$, where $K = 5$.

By representing the variables for the matched train $\bar{j}$ as $x_{\bar{j}}^u(k) = \sum_{j'=1}^N \beta_{j',j}^{u,v} \cdot x_{j'}^u(k)$ and $TD_{\bar{j}}^u \cdot x_{\bar{j}}^u(k) = \sum_{j'=1}^N \beta_{j',j}^{u,v} \cdot TD_{j'}^u \cdot x_{j'}^u(k)$, it is proved that property 3 holds. □

Remark 1. The waiting time $W_j^{u,v}$ from Eq. (25) is a quasi-quadratic function of train departure times, by substituting values for the binary train-to-demand period mapping variables $x_j^u(k)$.

Clearly, the waiting time $W_j^{u,v}$ in Eq. (25) is a quartic function of two sets of variables, namely TD_j^u and $x_j^u(k)$. According to the proof for Property 3, however, the value of $W_j^{u,v}$ can be obtained by counting two quadratic entries (25) and (26) when $x_j^u(k)$ has pre-set values. Considering that all but one of the 0–1 variables $x_j^u(k)$ equal zero ($\sum_{k=1}^K x_j^u(k) = 1$), the objective function $W_j^{u,v}$ can be reduced to a quadratic function through properly substituting or enumerating values for the binary variables $x_j^u(k)$, e.g., through a branch and bound solution method. Thus, we view objective function in Eq. (25) as a quasi-quadratic function.

Property 4. *Under the hour-dependent demand, the number of passengers remaining in train j, $Q_j^{u,v}$, after the train departs from station u for the station OD pair (u, v) can be represented as below.*

$$Q_j^{u,v} = \tau_j^{u,v} \cdot \sum_{k=1}^K \left\{ TD_j^u \cdot x_j^u(k) - \sum_{j'=1}^N \beta_{j',j}^{u,v} \cdot TD_{j'}^u \cdot x_{j'}^u(k) - \left[x_j^u(k) - \sum_{j'=1}^N \beta_{j',j}^{u,v} \cdot x_{j'}^u(k) \right] \cdot \sum_{k=1}^K x_j^u(k) \cdot 60(k-1) \right\} \cdot P^{u,v}(k) \tag{29}$$

Proof. Similar to Property 3, we can conclude that the number of passengers remaining in train j after the train departs from station u for the station OD pair $(u,\ v)$ is,

$$Q_j^{u,v} = \tau_j^{u,v} \cdot \sum_{k=1}^K \left\{ TD_j^u \cdot x_j^u(k) - TD_{\bar{j}}^u \cdot x_{\bar{j}}^u(k) - [x_j^u(k) - x_{\bar{j}}^u(k)] \cdot \sum_{k=1}^K x_j^u(k) \cdot 60(k-1) \right\} \cdot P^{u,v}(k) \tag{30}$$

By substituting the associated variables for train $\bar{j}$, we find that property 4 holds. □

Similar to the discussion of Remark 1, we also describe the train capacity constraint as follows.

Remark 2. The number of in-train passengers $Q_j^{u,v}$ is a quasi-linear function, by substituting values for the binary train-to-demand period mapping variables $x_j^u(k)$.

5.3. Revised formulation using a piecewise linear relationship

Using the number of in-train passengers $Q_j^{u,v}$, the train capacity constraint can be updated as below.

$$Q_j^u = \sum_{u'=1}^u \sum_{v=u+1}^N Q_j^{u',v} \leqslant c_j \tag{31}$$

For a set of train departure times TD_j^u, we can use Eqs (32)–(34) to determine binary values of $x_j^u(k)$.

$$60(k-1) - TD_j^u \leqslant M \cdot [1 - x_j^u(k)] \tag{32}$$

$$TD_j^u - 60k \leqslant M \cdot [1 - x_j^u(k)] \tag{33}$$

$$\sum_{k=1}^K x_j^u(k) = 1 \tag{34}$$

Eqs. (32)–(34) provide the linking constraints between the binary variables and train departure times, where M is a very large positive number. Similar to an integer programming formulation for the piecewise linear function, inequality (32) ensures $x_j^u(k) = 0$ if $TD_j^u < 60(k-1)$; inequality (33) ensures $x_j^u(k) = 0$ if $TD_j^u > 60k$; and Eq. (34) ensures only one $x_j^u(k)$ equals 1 for all one-hour periods of k. From the computational point of view, additionally, the applied parameter M aims to accept an infeasible solution during the algorithm implementation (Klotz and Newman, 2013).

Finally, the objective function can be revised as follows.

$$\min \sum_{j=1}^{N} \sum_{u=1}^{S-1} \sum_{v=u+1}^{S} W_j^{u,v} \tag{35}$$

Thus, we reformulated model **M2** to the following model **M4** for the daily operational applications.

Model M4 (with a long demand aggregation interval):

Objective function (35).

Subject to constraints (10)–(17) and (30)–(34).

It is clear that the above reformulation leads to a quasi-quadratic mixed integer programming model. Compared to **M3**, this new optimization model **M4** has a relatively complex objective function (35), but the corresponding number of binary variables in **M4** is evidently smaller than that in **M3**. Without the binary variable $TL_j^{u,v}(t)$, for example, the numbers of parameters $z_j^u(t)$ and $P^{u,v}(t)$ in **M3** are 60 times the numbers of parameters $x_j^u(k)$ and $P^{u,v}(k)$ in this section, respectively. This reduced variable size could allow the model **M4** to be more effective and applicable for meeting daily operational requirements.

6. Numerical examples

The special formulation structure proposed in this paper, i.e., quadratic integer programming model with linear constraints, implies that high-level optimization solvers can be directly utilized for solving the problem under consideration. This paper uses GAMS (Brooke et al., 2006) to implement and solve the proposed train scheduling models. Relaxing binary variables obviously results in a loose lower bound estimate, so it is necessary to take $z_j^u(t)$, $TL_j^{u,v}(t)$ and $x_j^u(k)$ as binary integer variables. In model **M3**, Eq. (22) ensures that TD_j^u (and related TA_j^u) should be also integers when $z_j^u(t)$ are considered as binary variables. As a result, model **M3** can be taken as a nonlinear integer programming problem. In order to improve the computation efficiency and ensure the accurate results, for model **M4**, this study relaxes the integral requirements for time variables. For example, train arrival and departure times, TA_j^u and TD_j^u are relaxed as positive continuous variables. The departure time TD_j^u are linked with demand interval variables $x_j^u(k)$, particularly, TD_j^u in the final solution are still real values within a certain one-hour interval.

This study uses the Shanghai–Hangzhou High-speed Rail Line of China, which comprises 9 stations, as a test case. The free-flow running times between two adjacent stations in the considered direction are shown in Table 1. Parameters ε,

Table 1
Train free-flow running time between two adjacent stations (min).

Adjacent station fair	Free-flow running time	Adjacent station fair	Free-flow running time
1–2	5	5–6	5
2–3	6	6–7	7
3–4	5	7–8	6
4–5	6	8–9	5

Table 2
The earliest and latest departure times for each train (min).

Train number	1	2	3	4	5	6	7	8
The earliest departure time	8:01	8:07	8:14	8:21	8:28	8:35	8:42	9:00
The latest departure time	8:25	8:30	8:35	8:40	8:45	8:50	9:55	9:00

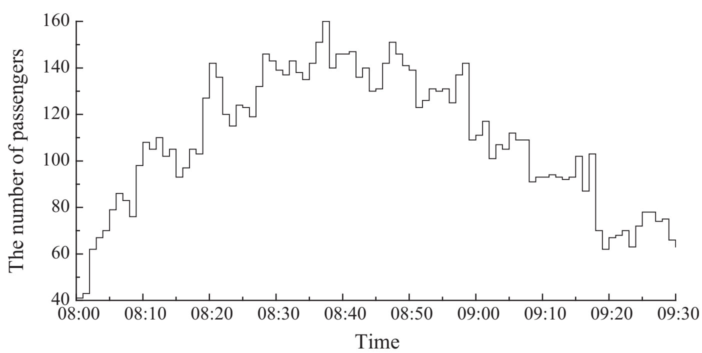

Fig. 8. The total number of arrival passengers per one minute over the study period.

$\theta_{\min}^{\text{section}}$, $\theta_{\min}^{\text{station}}$, $\tau_{\min}$, $\tau_{\max}$ and $\Delta_{\max}$ are taken as 1 min, 5 min, 5 min, 2 min, 4 min and 12 min, respectively. For simplicity, the train capacities are treated as the same value of 600 passengers per train.

6.1. Case with high-resolution minute-dependent demand

In this case, we consider the planning horizon from 8:00 AM to 9:00 AM with 8 trains. The earliest and latest departure times for each train are listed in Table 2. The time-dependent demand inputs according to the total number of arrival passengers per one minute over the study period are shown in Fig. 8.

The train skip-stop patterns for every station except the first and last stations are given in Table 3, where number "0" indicates that the corresponding train in the column skips the corresponding station in the row.

According to the special feature of **M3**, we select GAMS/AlphaECP to solve the problem for the minute-dependent demand case and set a relative solution optimization gap as zero (i.e., OPTCR = 0 in GAMS's configuration). Without any further post-processing, we directly obtain an optimal solution for problem **M3** as shown in Tables 3–5. The computational time is

Table 3
The predetermined skip-stop pattern and the optimized train dwell times.

Station number	Train number							
	1	2	3	4	5	6	7	8
2	2	**0**	3	**0**	2	**0**	4	**0**
3	2	2	4	4	2	3	4	3
4	2	2	**0**	4	4	4	**0**	3
5	3	2	4	4	4	4	4	4
6	3	2	**0**	2	2	4	**0**	4
7	4	2	4	4	4	4	4	3
8	2	**0**	4	**0**	2	**0**	3	**0**

Table 4
The optimized train buffer times at each track segment for each train.

Adjacent station fair	Train number							
	1	2	3	4	5	6	7	8
1–2	0	0	0	1	0	1	1	0
2–3	0	3	0	1	0	1	0	0
3–4	0	2	0	4	4	5	5	0
4–5	0	5	1	0	0	0	0	1
5–6	0	0	8	0	0	0	4	2
6–7	3	0	0	4	4	3	2	0
7–8	0	0	3	1	1	1	0	9
8–9	0	1	0	0	0	1	0	0

Table 5
The departure times at the first station for each train.

Train number	1	2	3	4	5	6	7	8
Departure time	08:01	08:10	08:15	08:23	08:28	08:35	08:43	09:00

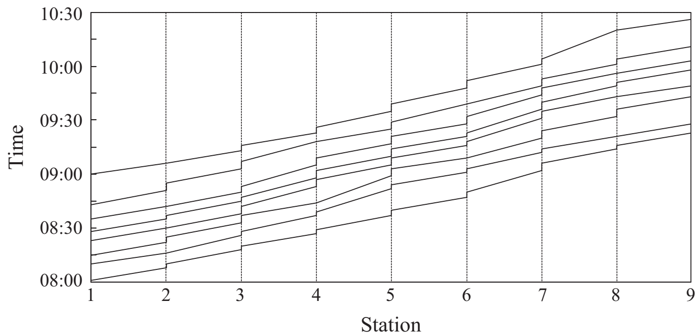

Fig. 9. The optimized train timetable from 8:00 to 9:00 over the real-time scheduling horizon.

2.7 min using the NEOS (Network-Enabled Optimization System) Server, and the objective function is 65,906 min. Table 3 shows the predetermined skip-stop pattern and the optimized dwell time at each intermediate station for each train. In Table 4, we list the optimized buffer time at every segment for each train, which corresponds to the optimized segment travel time. Table 5 shows the departure times at the first station for each train. Fig. 9 plots the corresponding scheduling results.

Table 6
The earliest and latest departure times at the first station.

Train	Earliest time	Latest time	Train	Earliest time	Latest time	Train	Earliest time	Latest time	Train	Earliest time	Latest time	Train	Earliest time	Latest time
1	06:01	06:11	16	09:00	09:10	31	11:48	11:58	46	15:24	15:34	61	18:10	18:20
2	06:15	06:25	17	09:10	09:20	32	12:00	12:10	47	15:36	15:46	62	18:20	18:30
3	06:30	06:40	18	09:20	09:30	33	12:15	12:25	48	15:48	15:58	63	18:30	18:40
4	06:45	06:55	19	09:30	09:40	34	12:30	12:40	49	16:00	16:10	64	18:40	18:50
5	07:00	07:10	20	09:40	09:50	35	12:45	12:55	50	16:12	16:22	65	18:50	19:00
6	07:12	07:22	21	09:50	10:00	36	13:00	13:10	51	16:24	16:34	66	19:00	19:10
7	07:24	07:34	22	10:00	10:10	37	13:15	13:25	52	16:36	16:46	67	19:15	19:25
8	07:36	07:46	23	10:12	10:22	38	13:30	13:40	53	16:48	16:58	68	19:30	19:40
9	07:48	07:58	24	10:24	10:34	39	13:45	13:55	54	17:00	17:10	69	19:45	19:55
10	08:00	08:10	25	10:36	10:46	40	14:00	14:10	55	17:10	17:20	70	20:00	20:15
11	08:10	08:20	26	10:48	10:58	41	14:15	14:25	56	17:20	17:30	71	20:20	20:35
12	08:20	08:30	27	11:00	11:10	42	14:30	14:40	57	17:30	17:40	72	20:40	20:55
13	08:30	08:40	28	11:12	11:22	43	14:45	14:55	58	17:40	17:50	73	21:00	21:00
14	08:40	08:50	29	11:24	11:34	44	15:00	15:10	59	17:50	18:00			
15	08:50	09:00	30	11:36	11:46	45	15:12	15:22	60	18:00	18:10			

Table 7
The demand flow rates for station OD pairs during each one-hour period (PAX/min).

Adjacent station OD pair	One-hour period number													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1–2	3	2	3	2	2	2	3	2	2	2	1	1	2	2	1
1–3	2	3	4	3	4	3	4	3	3	3	3	3	4	3	2
1–4	1	1	3	2	1	2	2	2	2	2	2	2	3	2	1
1–5	1	2	1	2	1	1	1	2	1	3	2	3	2	1	1
1–6	2	1	1	1	1	2	2	1	1	3	1	2	3	1	1
1–7	1	1	1	2	2	1	2	2	2	2	2	2	3	1	1
1–8	1	1	1	1	2	1	1	1	1	1	1	1	2	1	1
1–9	1	1	1	1	1	1	2	1	2	1	1	1	2	1	1
2–3	2	3	3	2	2	2	1	1	2	2	2	2	3	2	1
2–4	1	1	3	2	2	2	3	2	2	2	1	2	3	2	2
2–5	2	2	2	1	2	2	2	2	1	1	1	2	2	1	1
2–6	1	1	1	1	1	1	1	1	2	2	1	1	1	1	1
2–7	1	2	1	2	1	1	1	2	1	1	2	1	2	2	1
2–8	1	1	1	1	1	1	1	1	1	1	1	2	1	1	1
2–9	1	1	2	1	2	1	2	1	1	1	1	1	2	1	1
3–4	2	2	3	3	4	4	4	3	2	2	2	4	4	3	1
3–5	1	2	2	1	1	1	1	2	2	2	1	2	2	1	1
3–6	1	1	2	2	1	1	1	1	1	1	2	2	2	1	2
3–7	1	1	2	1	2	2	2	1	1	1	2	1	2	1	1
3–8	1	1	1	1	1	1	1	1	1	2	1	2	1	1	2
3–9	1	1	1	2	1	2	1	2	1	2	1	1	1	1	1
4–5	2	2	4	3	2	3	3	2	1	2	4	4	3	3	2
4–6	1	1	2	1	1	1	1	2	1	1	1	2	2	1	1
4–7	1	2	1	2	1	2	1	2	2	2	2	1	2	1	1
4–8	1	1	1	1	2	1	1	1	1	1	1	2	1	1	1
4–9	1	1	1	1	2	1	2	1	1	1	2	1	1	1	1
5–6	2	3	3	3	2	4	2	3	3	2	4	2	3	2	2
5–7	1	2	3	2	2	3	2	1	2	1	2	3	3	1	1
5–8	1	1	1	1	1	2	1	1	1	1	1	2	2	1	1
5–9	1	1	1	1	1	1	2	1	2	1	1	1	1	2	1
6–7	3	2	2	2	2	2	2	2	3	2	3	4	4	4	2
6–8	1	1	1	1	2	1	1	2	1	2	1	2	3	2	1
6–9	1	1	1	1	1	1	1	1	1	2	1	1	1	1	1
7–8	2	3	2	2	3	2	2	2	3	3	2	2	3	2	1
7–9	2	2	2	2	2	2	1	2	2	1	2	2	2	2	2
8–9	2	3	5	4	4	4	3	4	4	4	5	4	4	3	2

Table 8
The predetermined skip-stop pattern and the optimized train dwell times.

Train	Station							Train	Station						
	2	3	4	5	6	7	8		2	3	4	5	6	7	8
1	2	2	2	2	2	2	4	38	**0**	2	2	4	2	2	**0**
2	**0**	3	2	2	4	2	**0**	39	2	4	2	2	**0**	**0**	2
3	2	2	2	4	**0**	**0**	2	40	2	**0**	2	**0**	2	2	2
4	2	**0**	4	**0**	2	2	2	41	**0**	2	2	2	2	4	**0**
5	**0**	2	2	2	2	2	**0**	42	2	2	2	3	**0**	**0**	2
6	2	2	2	2	**0**	**0**	4	43	2	**0**	4	**0**	2	2	2
7	2	**0**	2	**0**	4	2	3	44	**0**	2	2	2	2	2	**0**
8	**0**	2	4	2	2	2	**0**	45	2	2	2	2	**0**	**0**	4
9	2	2	2	2	**0**	**0**	2	46	3	**0**	2	**0**	4	2	3
10	2	**0**	2	**0**	2	2	2	47	0	2	4	2	2	2	0
11	**0**	2	2	2	2	4	**0**	48	2	2	2	2	**0**	**0**	2
12	2	2	2	2	**0**	**0**	2	49	2	**0**	2	**0**	2	2	2
13	2	**0**	2	**0**	4	2	4	50	**0**	2	2	2	2	4	**0**
14	**0**	2	4	2	2	2	**0**	51	2	2	2	2	**0**	**0**	2
15	2	4	2	2	**0**	**0**	2	52	2	**0**	2	**0**	2	2	3
16	2	**0**	2	**0**	2	2	4	53	**0**	4	2	2	2	2	**0**
17	**0**	2	2	2	2	4	**0**	54	2	2	2	2	**0**	**0**	4
18	2	2	2	2	**0**	**0**	2	55	2	**0**	3	**0**	2	4	2
19	2	**0**	2	**0**	2	2	4	56	**0**	2	2	2	4	3	**0**
20	**0**	2	4	2	2	2	**0**	57	2	2	2	2	**0**	**0**	2
21	2	4	2	2	**0**	**0**	2	58	2	**0**	4	**0**	2	2	3
22	2	**0**	2	**0**	2	2	4	59	**0**	4	2	2	3	2	**0**
23	**0**	2	2	2	2	4	**0**	60	2	2	2	2	**0**	**0**	4
24	2	2	2	4	**0**	**0**	2	61	2	**0**	2	**0**	2	4	2
25	2	**0**	4	**0**	2	2	2	62	**0**	2	2	2	4	2	**0**
26	**0**	3	2	2	2	2	**0**	63	2	2	2	4	**0**	**0**	2
27	2	2	2	2	**0**	**0**	4	64	2	**0**	4	**0**	2	2	2
28	2	**0**	2	**0**	2	4	2	65	**0**	4	2	2	2	2	**0**
29	**0**	2	2	2	4	4	**0**	66	2	2	2	2	**0**	**0**	4
30	2	2	4	2	**0**	**0**	2	67	2	**0**	2	**0**	2	4	4
31	4	**0**	2	**0**	2	2	2	68	0	2	2	4	2	4	0
32	**0**	2	2	2	2	2	**0**	69	2	4	2	2	**0**	**0**	2
33	2	2	2	2	**0**	**0**	4	70	2	**0**	2	**0**	2	2	2
34	2	**0**	2	**0**	4	2	2	71	**0**	2	2	2	2	2	**0**
35	**0**	3	2	2	2	2	**0**	72	2	2	2	2	**0**	**0**	4
36	2	2	2	2	**0**	**0**	4	73	2	**0**	2	**0**	2	2	4
37	2	**0**	2	**0**	2	4	3								

For this calculation, the total number of binary variables $z_j^u(t)$ and $TL_j^{u,v}(t)$ is 37800. The simulation tests (Table 11) show that the computation time is intolerable if the study horizon exceeds 3 h.

6.2. Hour-dependent demand case

In this case, we consider the planning period from 6:00 to 21:00 and the number of trains is 73. The earliest and latest departure times for each train are listed in Table 6. The demand flow rate inputs for each station OD pair during one hour over the study period are shown in Table 7.

The train skip-stop patterns for every station except the first and last stations, are given in Table 8, where number "0" indicates that the corresponding train in the column skips the corresponding station in the row.

Another optimization solver GAMS/DICOPT, which is particularly suitable to the case in which the part of integer variables can be relaxed, is selected to solve problem **M4.** Using the NEOS Server with 1.3 h computational time and the optimization gap as zero (i.e., OPTCR = 0), we can obtain the optimal values of the binary variables $x_j^u(k)$ and the relaxed real variables TD_j^u for problem **M4,** which corresponds to an objective function of 557,496 min. For the obtained real values of TD_j^u, we round them to the nearest integer values by a post-processing procedure. The final results are shown in Tables 8–10. Specifically, Table 8 provides the predetermined skip-stop pattern and the optimized dwell time at each intermediate station for each train. In Table 9, we show the optimized buffer time at every segment for each train, and Table 10 shows the departure times at the first station for each train. The results of Tables 8–10 correspond to the train timetable depicted in Fig. 10.

In optimization model **M4**, the number of binary variables $x_j^u(k)$ is 10056. As shown in Table 11, a number of test runs indicate that the computation time is generally acceptable for this daily operational application from 5 h to 15 h planning horizon, even though more complicated constraints (32) and (33) are included in the model. The simulation results also reveal that the binary variables cannot be relaxed in the algorithm implementation.

Table 9
The optimized train buffer times at each segment for each train.

Train	Adjacent station OD pair								Train	Adjacent station OD pair							
	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9		1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9
1	0	0	0	0	0	0	7	0	38	0	0	0	10	0	0	0	0
2	0	0	0	0	8	0	0	0	39	0	1	0	0	0	2	0	0
3	0	0	0	5	0	0	0	0	40	0	0	0	0	0	0	0	0
4	0	1	1	0	0	0	0	0	41	0	0	0	0	0	6	0	0
5	0	0	0	0	0	0	0	0	42	0	0	0	0	0	0	0	0
6	0	0	0	0	1	0	0	0	43	0	0	0	0	0	0	0	0
7	0	0	0	1	6	0	0	0	44	0	0	0	0	0	0	0	0
8	0	0	6	0	0	0	0	0	45	0	0	0	0	1	0	0	0
9	0	0	0	0	0	4	0	0	46	0	0	0	2	5	0	0	0
10	0	0	0	0	0	0	0	0	47	0	0	6	0	0	0	0	0
11	0	0	0	0	0	2	0	0	48	0	0	0	0	0	4	0	0
12	0	0	0	0	0	2	0	0	49	0	0	0	0	0	0	0	0
13	0	0	0	4	0	0	0	0	50	0	0	0	0	0	9	0	0
14	0	0	4	0	0	0	0	0	51	0	0	0	7	0	0	0	0
15	0	0	0	0	0	2	0	0	52	0	8	0	0	0	0	0	0
16	0	0	0	0	0	0	8	0	53	6	0	0	0	0	0	0	0
17	0	0	0	0	0	6	0	0	54	0	0	0	0	0	1	7	0
18	0	0	0	0	6	2	0	0	55	0	0	0	2	0	10	0	0
19	0	0	0	2	6	0	1	0	56	0	0	0	0	9	0	0	0
20	0	0	5	0	0	0	0	0	57	0	0	0	7	0	3	0	0
21	0	2	0	0	0	1	0	0	58	0	4	1	2	0	0	0	0
22	0	0	0	0	0	0	9	0	59	4	1	0	0	0	0	0	0
23	0	0	0	0	0	8	0	4	60	0	0	0	0	0	2	5	0
24	0	0	0	8	0	0	0	0	61	0	0	0	3	0	9	0	0
25	0	4	4	0	0	0	1	0	62	0	0	0	0	9	0	0	0
26	0	8	0	0	0	0	0	0	63	0	0	0	6	0	2	0	0
27	0	0	0	0	0	3	4	0	64	0	1	6	1	0	0	0	0
28	0	0	0	2	0	10	0	0	65	4	3	0	0	0	0	0	0
29	0	0	0	0	9	0	0	3	66	0	0	0	0	0	0	7	0
30	0	0	7	0	0	0	0	0	67	0	0	0	0	0	10	2	0
31	8	0	0	0	0	0	0	0	68	0	0	0	12	0	0	0	0
32	0	0	0	0	0	0	0	0	69	0	9	0	0	0	0	0	0
33	0	0	0	0	0	1	1	0	70	0	0	0	0	0	0	0	0
34	0	0	0	3	4	0	0	0	71	0	0	0	0	0	0	0	0
35	8	0	0	0	0	0	5	0	72	0	0	0	0	0	12	0	0
36	0	0	0	0	0	2	6	0	73	0	0	0	0	0	0	0	0
37	0	0	0	0	0	11	0	0									

Table 10
The optimized departure times at the first station for each train.

Train	Departure time	Train	Departure time	Train	Departure time	Train	Departure time	Train	Departure time	Train	Departure time	Train	Departure time
1	06:01	12	08:20	23	10:14	34	12:32	45	15:14	56	17:20	67	19:21
2	06:18	13	08:30	24	10:24	35	12:45	46	15:24	57	17:30	68	19:30
3	06:30	14	08:40	25	10:36	36	13:07	47	15:36	58	17:40	69	19:45
4	06:45	15	08:50	26	10:48	37	13:19	48	15:48	59	17:50	70	20:09
5	07:03	16	09:04	27	11:06	38	13:30	49	16:00	60	18:04	71	20:20
6	07:14	17	09:12	28	11:16	39	13:45	50	16:12	61	18:13	72	20:40
7	07:25	18	09:20	29	11:24	40	14:00	51	16:24	62	18:21	73	21:00
8	07:36	19	09:30	30	11:36	41	14:15	52	16:36	63	18:30		
9	07:48	20	09:40	31	11:48	42	14:30	53	16:48	64	18:40		
10	08:00	21	09:50	32	12:06	43	14:45	54	17:03	65	18:50		
11	08:10	22	10:06	33	12:16	44	15:04	55	17:12	66	19:09		

Relaxing binary variables leads to a large deviation from the previous values, as discussed above, the binary variables $z_j^u(t)$, $TL_j^{u,v}(t)$ and $x_j^u(k)$ cannot be relaxed to continuous real variables for both models **M3** and **M4**. Our additional and repeated numerical results also confirmed this conclusion.

We have also obtained the train timetables by other simple strategies, such as constant-headway regular schedules, in order to demonstrate the advantage of the models and algorithms developed in this paper. The comparison from our experiment results indicates that the proposed approach outperforms the constant-based regular scheduling methods, which is similar to our findings in our early study (Niu and Zhou, 2013). We also found, additionally, that the clustering and dispersion of train trajectories in the resulting timetables is closely synchronized with the time-varying passenger arrival and

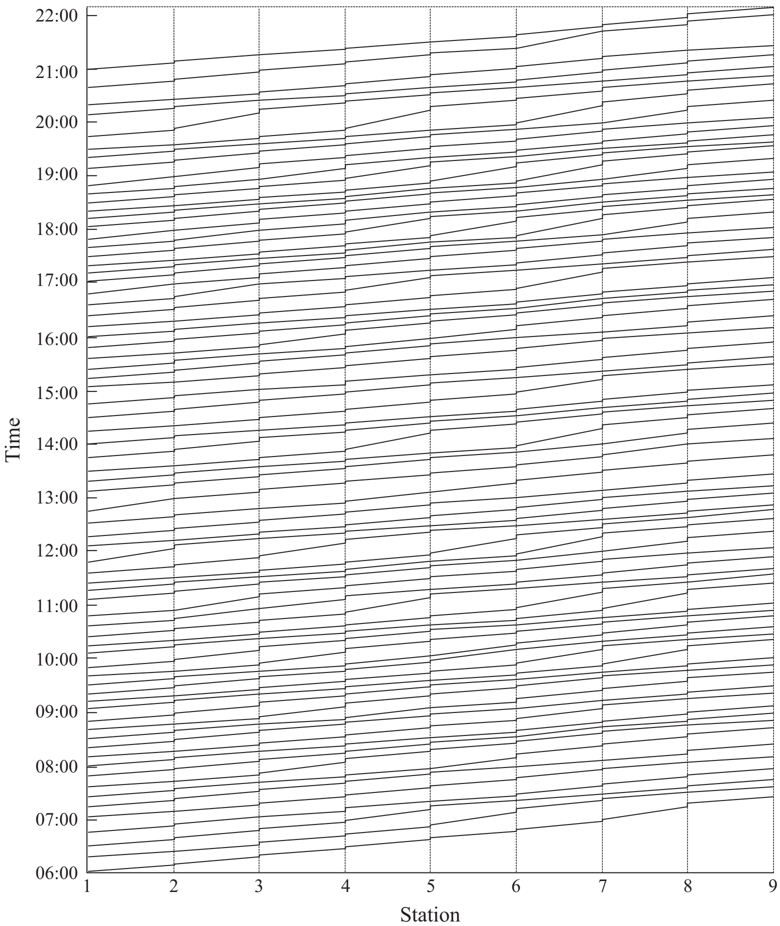

Fig. 10. The optimized train timetable from 6:00 to 21:00 over the daily operational horizon.

Table 11
Comparison of implemented models **M3** and **M4** with different planning horizons.

Model	**M3** (with high-resolution demand data)	**M4** (with long demand aggregation interval)				
Planning horizon	1 h	2 h	3 h	5 h	10 h	15 h
Number of trains	8	16	24	44	61	73
Number of continuous variables	38,049	126,257	236,545	7669	10,576	12,336
Number of binary variables	37,800	125,760	235,800	6112	8424	10,056
Number of constraints	38,233	126,641	237,129	15,127	20,975	24,521
Computational time	2.7 min	10.7 min	138 min	16.9 min	40.6 min	78 min

departure rates. As shown in Figs. 9 and 10, for example, the high-density train distribution corresponds to the larger demand patterns around 8:30 am. This is further demonstrated that the train spatial distributions and sequence structures for an optimized timetable is in general consistent with the passenger demands and train skip-stop patterns.

7. Conclusions

This paper focuses on the demand-oriented passenger train timetable optimization for a rail corridor under time-dependent demand conditions. A unified nonlinear integer programming model with linear constraints is developed for both high-resolution and medium-resolution time-varying demand data. By connecting two matched train departure events by their effective loading time periods, our approach can effectively coordinate train activities with complex spatial and temporal demand distributions. We construct a number of mathematically tractable forms to characterize the total passenger waiting time at stations using general purpose optimization models, which are implemented in GAMS with test cases in the real-world corridor. In particular, by introducing binary loading indicator variables, we formulate a train timetable optimization model with minute-dependent passenger demand inputs for short-term scheduling applications. By using a simplified, linear piecewise function to represent time-varying demand over a long planning horizon, we also construct an optimization model for daily operational applications with hour-dependent passenger demand records. To the best of our knowledge, the proposed models are the first type of nonlinear mixed integer optimization models that can be efficiently solved by general-purpose solvers for the train timetabling problem to minimize the total passenger waiting time under time-dependent demand conditions and skip-stop patterns.

Our future research will consider variable skip-stop pattern and further exploit the opportunity to jointly optimize stop patterns and travel speeds, for both real-time scheduling and long-term planning applications. An important extension for our approach is a more general framework for a network with connected corridors. For large-scale problems, it is also useful to conduct additional tests with alternative optimization solvers.

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