Preliminary Ph.D. exam in Mathematical Finance

Date: November 17, 2015
Duration: 3 hours

The exam is closed book and closed notes. Access to Internet is not allowed. All problems have the same number of points.

Notation: W is Brownian motion. $\tilde{W}$ is Brownian motion in the risk-neutral measure. r is the (constant) interest rate.

Problem 1

Let W be Brownian motion, and α be a continuous process adapted to the Brownian filtration.

a) Find the strong solution of the equation:

$$\begin{aligned} dX(t) &= \alpha(t)dt + \sigma X(t)dW(t) \\ X(0) &= X_0 > 0 \end{aligned}$$

Hint: consider σW to be a deterministic differentiable function of time g, and "divide by dt". This equation becomes now a linear ordinary equation with non-constant coefficients. The impulse response for that equation is a (deterministic) exponential function. When solving the original equation above, you should replace the deterministic exponential by what is often called the "stochastic exponential".

b) Suppose that α is a function of $X(t)$, say $\alpha(t) = f(t, X(t))$.Give a condition for the solution to remain positive for $t > 0$.

c) Suppose we choose a different initial condition than $X(0) = 0$. Discuss the advantages and advantages of this process to model risk-free interest rates and price bonds, compared to the Vasicek and Cox-Ingersoll-Ross models.

Problem 2

Let T and f be two differentiable functions mapping $\mathbb{R}$ to $\mathbb{R}$. Let W be Brownian motion. We would like to define a stochastic integral, which we write

$$\int_0^b f(t)dW(T(t))$$

a) Use integration by parts to define the stochastic integral. What restriction would you impose on $T(t)$?

b) Does your definition in part a) coincide with taking an appropriate stochastic limit, as $n \to \infty$ of the expression

$$I(f) = \sum_{i=0}^{n-1} f(\frac{ib}{n})[W(T(\frac{(i+1)b}{n})) - W(T(\frac{ib}{n}))]$$

c) Let $f(t) = t^4$ and $T(t) = t^2$. Calculate $(\int_0^b f(t)dW(T(t)))^2$ as a function of b.

Problem 3

In the Merton model of credit risk, the liabilities of a firm consist of (i) one zero-coupon bond with maturity T and principal equal to D, and (ii) equity. The value of the assets of the firm V follows a geometric Brownian motion with expected return μ and relative volatility σ. Suppose that equityholders can default only at T. Show that the value of equity is:

$$S(0) = V(0)N(d_1) - D\exp(-rT)N(d_2)$$

where:

$$\begin{aligned} d_1 &= \frac{\log(V(0)/D) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

Would the answer be different in this model if the shareholders were allowed to default before T? No mathematical justification is needed.

Problem 4

Let the stock price $S(t)$ satisfy a local volatility model:

$$\frac{dS(t)}{S(t)} = rdt + \sigma(S(t), t)d\tilde{W}(t)$$

a) Show that the price $C(T, K)$ of a call option with expiration T and strike K satisfies:

$$\frac{\partial C(T,K)}{\partial T} = \frac{1}{2}\sigma^2(K,T)K^2\frac{\partial^2 C(T,K)}{\partial K^2} - rK\frac{\partial C(T,K)}{\partial K}$$

b) What is a practical application of the result above?

Hint: remember the Fokker-Planck equation for the density $\tilde{p}(T, y)$ that the stock price $S(T)$ is equal to y:

$$\frac{\partial \tilde{p}(T,y)}{\partial T} = -\frac{\partial}{\partial y}(ry\tilde{p}(T,y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(T,y)y^2\tilde{p}(T,y))$$

Problem 5

Suppose that the dividend rate $\delta(t)$ on a stock is modelled as geometric Brownian motion.

$$\frac{d\delta(t)}{\delta(t)} = \mu dt + \sigma d\tilde{W}(t)$$

with $0 < \mu < r$.

a) Show that the price of the stock is given by:

$$S(t) = \frac{\delta(t)}{r - \mu}$$

b) Find the stochastic differential equation that $S(t)$ satisfies.