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Itô-Taylor Expansions for Systems of Stochastic Differential Equations with Numerical Applications

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Abstract. Stochastic differential equations (SDEs) are playing a growing role in financial mathematics, actuarial sciences, physics, biology and engineering. In this paper, we focus on a numerical simulation of systems of SDEs based on the stochastic Taylor series expansions. At first, we apply the vector-valued Itô formula to the systems of SDEs, then, the stochastic Taylor formula is used to get the numerical schemes. In the case of higher dimensional stochastic processes and equations, the numerical schemes may be expensive and take more time to compute. Iterated integrals with several n-dimensional Brownian motions can be quite complicated. Sometimes, such integrals may be approximated. In this paper, we deal with both the systems with standard n-dimensional Brownian motions and the systems of SDEs having correlated Brownian motions. The main issue is to transform the systems of SDEs with correlated Brownian motions to the ones having independent Wiener processes, and then, to apply the Itô formula to the transformed systems. Moreover, explicit computations of multiple Itô integrals with several Brownian motions are given, supported by MATLAB. The paper ends with conclusion and outlook to future studies.

Keywords. Systems of SDEs, Itô-Taylor expansions, Correlated Brownian motions, Vector-valued Itô formula, Numerical schemes for SDEs.

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1 Introduction

There has been a great interest in the simulation methods of SDEs in the fields of, e.g., finance, insurance, medicine and the modern technologies [10, 11, 16, 20, 21]. As the need to take into account of uncertainty is more and more accepted in science and the applications, SDEs are an emerging subject of interest. Stochastic Taylor expansion provides a source for the discrete-time approximation methods. One of the simplest ways to discretize the process is *Euler* method, which approximates the integrals by using the left-point rule. The *Milstein* scheme (1974), which has the order 1.0 of strong convergence, is stronger than *Euler* method. By adding further stochastic integrals, with the equations and using the stochastic Taylor expansion, more accurate schemes can be obtained.

P.E. Kloeden and E. Platen [10] have given a methodical means of deriving the Taylor series for both Stratonovich and Itô form of a SDE. The application of Itô Taylor formula to the 1-dimensional SDEs is given explicitly in [10]. In this case, $I_{i_1,i_2,\ldots,i_k}$ represent the Itô integral, where integration is with respect to ds if $i_k = 0$, or dW_s if $i_k = 1$ [22]. For example,

$$I_{010} = \int_0^t \int_0^{s_1} \int_0^{s_2} ds_3 dW_{s_2} ds_1.$$

Such a display of integrals can be considered as a part of the inner beauty of stochastic dynamics. This beauty expresses itself in terms of *digitalization*, *algebraization* and *automization* which are not only very aesthetic indeed, but also very practical.

By recursively using the Itô formula, the obtained Taylor series can be related to a tree theory. The tree expansion is given for the true solution in [3]. Runge-Kutta method has been constructed, having order 1.5, in [3, 4, 5].

In the deterministic case, the accuracy of the numerical scheme can be obtained by comparing the obtained result with the exact solution. In the stochastic differential equations, there are two ways to measure the accuracy of the solution: strong and weak convergence. A time-discrete approximation X^h is said to *converge strongly* with order $p > 0$ at time T if there is a positive constant c, independent of h, and $h_0 > 0$ such that

$$E(|X_T - X^h(T)|) \leq ch^p, \quad h \in (0, h_0).$$

On the other hand, X_h is said to *converge weakly* with order $p > 0$ at time T if there are constants $c > 0$, independent of h, and $h_0 > 0$ such that

$$|E(X_T) - E(X^h(T))| \leq ch^p, \quad h \in (0, h_0).$$

In this paper, in order to get numerical solutions, we shall consider Taylor schemes that converge strongly.

Vector-valued Itô calculus may involve more than one Wiener process. Although it seems that the extension of one-variable SDEs to the multi-valued SDEs is easy, the computations of iterated Itô integrals are very expensive in the numerical approximations. But, the systems of SDEs model the important cases having several source of randomness and correlation; these are main reasons of financial risk and instability.

There are not many packages with regards to the simulations of SDEs [7]. The Maple package *stochastic* provides a symbolic manipulation for SDEs. The package may be downloaded from *www.math.uni-frankfurt.de/∼numerik /kloeden*. Many numerical schemes may be generated by using this package. However, the computations of the iterated multi-dimensional Itô integrals are not supported. The software package *SDE lab* generated by H. Gilsing and T. Shardlow [6] is a good source especially for 1-dimensional SDEs.

In this work, we provide a routine, supported by MATLAB, to compute the iterated multi-dimensional Itô integrals with presence of the formulations given in [10]. We use the *Polar Marsaglia method* to generate random variables. This method, that is attributed to G. Marsaglia, is a variation of the Box-Muller method. It is based on choosing random points (x, y) in the square given by $-1 < x < 1, -1 < y < 1$. For $s = x^2 + y^2 < 1$, the following pair of normal random variables is obtained:

$$x\sqrt{\frac{-2\ln(s)}{s}}, \quad y\sqrt{\frac{-2\ln(s)}{s}}.$$

This work deals with both the systems with standard Brownian motions and the systems having correlated Brownian motions. In the vector-valued case, a *standard n-dimensional Brownian motion* is defined as

$$\mathbb{Z}_t = (Z_t^1, Z_t^2, \ldots, Z_t^n)^T,$$

where Z_t^i and Z_t^j, for $i \neq j$, are independent Brownian motions with the properties

$$(dZ_t^i)^2 = dt, \; dZ_t^i dt = 0, \; (dt)^2 = 0.$$

This implies the following rule for different components:

$$dZ_t^i dZ_t^j = \delta_{ij} dt, \; \text{for } i, j = 1, 2, \ldots, n,$$

where δ_{ij} is the *Kronecker delta*:

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

We shall call the process $\mathbb{W}_t := (W_t^1, W_t^2, \dots, W_t^n)^T$ as a correlated Brownian motion if

$$dW_t^i dW_t^j = \rho_{ij} dt, \text{ for } i, j = 1, 2, \dots, n,$$

for a positive symmetric matrix $\rho = (\rho_{ij})_{1 \leq i,j \leq n}$ satisfying the follows:

$$\rho_{ii} = 1, \text{ and } \rho_{ij} = \rho_{ji} \in [-1, 1] \text{ for } i \neq j.$$

The paper is organized as follows. Multivariable Itô calculus is reviewed in Section 2. In Section 3, the systems of SDEs with independent Wiener processes and their Taylor expansions are analyzed. Then, the correlated systems are covered in Section 4. After obtaining discretization schemes in Section 5, the numerical results and implementation issues are given in Section 6. We conclude and give an outlook to the future studies in Section 7.

2 Multi-dimensional Itô Calculus

We consider the process $\mathbb{X}_t$ in $\mathbb{R}^d$. Let $\mathbb{Z}_t$ be a multi-dimensional Brownian motion, defined as $\mathbb{Z}_t = (Z_t^1, Z_t^2, \dots, Z_t^n)^T$. Then, the k^{th} component of the vector-valued SDE is given by

$$dX_t^k = a_k dt + \sum_{j=1}^{n} h_{kj} dZ_t^j, \qquad k = 1, 2, \dots, d,$$

where $a_k(t, \mathbb{X}_t)$ and $h_{kj}(t, \mathbb{X}_t)$ are the drift and the diffusion coefficients, respectively.

We define $\mathbb{A}(t, \mathbb{X}_t) = (a_1(t, \mathbb{X}_t), \dots, a_d(t, \mathbb{X}_t))^T$, $\mathbb{X}_t = (X_t^1, \dots, X_t^d)^T$, and

$$\mathbb{H}(t, \mathbb{X}_t) = \begin{pmatrix} h_{11}(t, \mathbb{X}_t) & \dots & h_{1n}(t, \mathbb{X}_t) \\ \vdots & \ddots & \vdots \\ h_{d1}(t, \mathbb{X}_t) & \dots & h_{dn}(t, \mathbb{X}_t) \end{pmatrix},$$

to get the following matrix formulation:

$$d\mathbb{X}_t = \mathbb{A} dt + \mathbb{H} d\mathbb{Z}_t. \tag{1}$$

3 Itô-Taylor approximation for standard Brownian Motions

In this section, we will assume that the Brownian motions are independent. Eqn. (1) can be written in integral form as

$$\mathbb{X}_t = \mathbb{X}_{t_0} + \int_{t_0}^{t} \mathbb{A}(s, \mathbb{X}_s)ds + \int_{t_0}^{t} \mathbb{H}(s, \mathbb{X}_s)d\mathbb{Z}_s. \tag{2}$$

The second integral is called *Itô stochastic integral*, which is defined by K. Itô in 1940 [9]. This integral can be approximated by stochastic Taylor method. Before going into the numerical schemes, we recall Itô Lemma in several dimensions.

Theorem 3.1 *(Itô Lemma in several dimensions,* [18]*)*
Let $\mathbb{X}_t = (X_t^1, \ldots, X_t^d)^T$ be a vector-valued Itô process satisfying Eqn. (1). Let $g : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^p$ be a given bounded function in $C^2([0, \infty) \times \mathbb{R}^d)$. Then,

$$dg(t, \mathbb{X}_t) = \frac{\partial g}{\partial t}dt + \sum_{i=1}^{d} \frac{\partial g}{\partial x_i} dX_t^i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 g}{\partial x_i, x_j} dX_t^i dX_t^j. \tag{3}$$

We consider now the k^{th} component of the system of SDEs (1):

$$dX_t^k = a_k(t, \mathbb{X}_t)dt + \sum_{j=1}^{n} h_{kj}(t, \mathbb{X}_t)dZ_t^j,$$

where $a_k(t, \mathbb{X}_t) = a_k(t, X_t^1, \ldots, X_t^d)$ and $h_{kj}(t, \mathbb{X}_t) = h_{kj}(t, X_t^1, \ldots, X_t^d)$.

For applying the Itô formula, we let $g(t, \mathbb{X}_t) = (g_1(t, \mathbb{X}_t)), \ldots, g_p(t, \mathbb{X}_t))^T$ and $Y_t = g(t, \mathbb{X}_t)$.

Then, the component Y_t^k is given by

$$dY_t^k = \frac{\partial g^k}{\partial t}dt + \sum_{i=1}^{d} \frac{\partial g^k}{\partial x^i} dX_t^i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 g^k}{\partial x^i x^j} dX_t^i dX_t^j.$$

Equivalently,

$$dY_t^k = \left(\frac{\partial g^k}{\partial t} + \sum_{i=1}^{d} a_i \frac{\partial g^k}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{p=1}^{n} h_{jp}h_{ip}\frac{\partial^2 g^k}{\partial x^i x^j} \right)dt + \sum_{j=1}^{d}\sum_{p=1}^{n} h_{jp}\frac{\partial g^k}{\partial x^j}dZ_t^p,$$

where all derivatives of g^k are to be evaluated in $(t, \mathbb{X}_t)$ and Brownian motions are independent.

The system of SDEs can be classified with respect to the shared states and Brownian motions. We give explicit formulations of each classification in the following subsections.

3.1 Completely decoupled systems

Now, we apply the theorem to

$$dX_t^k = a_k(t, X_t^k)dt + h_{kk}(t, X_t^k)dZ_t^k, \quad k = 1, 2, \ldots, d,$$

where all equations have their own independent Brownian motions and states that implies $d = n$. Then,

$$dY_t^k = (\frac{\partial g^k}{\partial t} + \sum_{i=1}^{d} a_i \frac{\partial g^k}{\partial x^i} + \frac{1}{2}\sum_{i=1}^{d} h_{ii}h_{ii}\frac{\partial^2 g^k}{\partial x^i x^i})dt + \sum_{i=1}^{d} h_{ii}\frac{\partial g^k}{\partial x^i}dZ_t^i,$$

where $g^k(t, \mathbb{X}_t) = (X_t^1, \ldots, X_t^d)^T$. Herewith,

$$dX_t^k = a_k(t, X_t^k)dt + h_{kk}(t, X_t^k)dZ_t^k. \tag{4}$$

The integral form of Eqn. (4) is

$$X_t^k = X_{t_0}^k + \int_{t_0}^{t} a_k(s, X_s^k)ds + \int_{t_0}^{t} h_{kk}(s, X_s^k)dZ_s^k. \tag{5}$$

Before obtaining the Taylor series expansions, we define the following operators:

$$L^0 := \frac{\partial}{\partial t} + \sum_{i=1}^{d} a_i \frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{p=1}^{n} h_{jp}h_{ip}\frac{\partial^2}{\partial x^i x^j},$$

and

$$L^j := \sum_{p=1}^{d} h_{pj}\frac{\partial}{\partial x^p}, \quad \text{for } j = 1, 2, \ldots, n.$$

Application of Theorem 3.1 to Eqn. (5) gives

$$Y_t^k = Y_{t_0}^k + \int_{t_0}^{t} L^0 g^k ds + \sum_{j=1}^{n}\int_{t_0}^{t} L^j g^k dZ_s^j. \tag{6}$$

In Eqn. (5), firstly, g^k is chosen as $a_k(t, X_t^k)$ and then, the Itô formula (6) is applied. Secondly, we let $g^k = h_{kk}(t, X_t^k)$ and use the Itô formula to get

$$
\begin{aligned}
X_t^k = X_{t_0}^k &+ \int_{t_0}^{t} \big[a_k(t_0, X_{t_0}^k) + \int_{t_0}^{s} \mathcal{L}^0 a_k(\tau, X_\tau^k) d\tau \\
&+ \sum_{j=1}^{n} \int_{t_0}^{s} \mathcal{L}^j a_k(\tau, X_\tau^k) dZ_\tau^j\big] ds + \int_{t_0}^{t} \big[h_{kk}(t_0, X_{t_0}^k) \\
&+ \int_{t_0}^{s} \mathcal{L}^0 h_{kk}(\tau, X_\tau^k) d\tau + \sum_{j=1}^{n} \int_{t_0}^{s} \mathcal{L}^j h_{kk}(\tau, X_\tau^k) dZ_\tau^j\big] dZ_s^k, \\
= X_{t_0}^k &+ a_k(t_0, X_{t_0}^k) I_0 + h_{kk}(t_0, X_{t_0}^k) I_k \\
&+ \int_{t_0}^{t} [\int_{t_0}^{s} \mathcal{L}^0 a_k(\tau, X_\tau^k) d\tau + \sum_{j=1}^{n} \int_{t_0}^{s} \mathcal{L}^j a_k(\tau, X_\tau^k) dZ_\tau^j] ds \qquad (7) \\
&+ \int_{t_0}^{t} [\int_{t_0}^{s} \mathcal{L}^0 h_{kk}(\tau, X_\tau^k) d\tau + \sum_{j=1}^{n} \int_{t_0}^{s} \mathcal{L}^j h_{kk}(\tau, X_\tau^k) dZ_\tau^j] dZ_s^k.
\end{aligned}
$$

We can continue with an application of Itô formula (6) to the functions $\mathcal{L}^0 a_k$, $\sum_{j=1}^n \mathcal{L}^j a_k$, $\mathcal{L}^0 h_{kk}$ and $\sum_{j=1}^n \mathcal{L}^j h_{kk}$ in (7), and then, to the functions $\mathcal{L}^0\mathcal{L}^0 a_k$, $\sum_{j=1}^n \mathcal{L}^j\mathcal{L}^0 a_k, \sum_{j=1}^n \mathcal{L}^0\mathcal{L}^j a_k$, $\sum_{j,p=1}^n \mathcal{L}^j\mathcal{L}^p a_k, \mathcal{L}^0\mathcal{L}^0 h_{kk}$, $\sum_{j,p=1}^n \mathcal{L}^j\mathcal{L}^0 h_{kk}$, $\sum_{j,p=1}^n \mathcal{L}^0\mathcal{L}^j h_{kk}$, $\sum_{j,p=1}^n \mathcal{L}^j\mathcal{L}^p h_{kk}$, to obtain Itô-Taylor expansion:

$$
\begin{aligned}
X_t^k = X_{t_0}^k &+ a_k(t_0, X_{t_0}^k) I_0 + h_{kk}(t_0, X_{t_0}^k) I_k \\
&+ \mathcal{L}^0 a_k(t_0, X_{t_0}^k) I_{00} + \sum_{j=1}^{n} \mathcal{L}^j a_k(t_0, X_{t_0}^k) I_{j0} \\
&+ \mathcal{L}^0 h_{kk}(t_0, X_{t_0}^k) I_{0k} + \sum_{j=1}^{n} \mathcal{L}^j h_{kk}(t_0, X_{t_0}^k) I_{jk} \\
&+ \mathcal{L}^0\mathcal{L}^0 a_k(t_0, X_{t_0}^k) I_{000} + \sum_{j=1}^{n} \mathcal{L}^j\mathcal{L}^0 a_k(t_0, X_{t_0}^k) I_{j00} \\
&+ \sum_{j=1}^{n} \mathcal{L}^0\mathcal{L}^j a_k(t_0, X_{t_0}^k) I_{0j0} + \sum_{j,p=1}^{n} \mathcal{L}^j\mathcal{L}^p a_k(t_0, X_{t_0}^k) I_{jp0} \\
&+ \mathcal{L}^0\mathcal{L}^0 h_{kk}(t_0, X_{t_0}^k) I_{00k} + \sum_{j,p=1}^{n} \mathcal{L}^j\mathcal{L}^0 h_{kk}(t_0, X_{t_0}^k) I_{j0k} \\
&+ \sum_{j,p=1}^{n} \mathcal{L}^0\mathcal{L}^j h_{kk}(t_0, X_{t_0}^k) I_{0jk} + \sum_{j,p=1}^{n} \mathcal{L}^j\mathcal{L}^p h_{kk}(t_0, X_{t_0}^k) I_{jpk} + R,
\end{aligned}
$$

where R denotes the remainder term and $I_{i_1,i_2,\ldots,i_k}$ represent the multiple

Itô integral, which is defined as [10]:

$$I_\alpha = \begin{cases} 1, & \text{if } k = 0, \\ \int_{t_0}^{t} I_{\alpha^-} ds, & \text{if } k \geq 1 \text{ and } \alpha_k = 0, \\ \int_{t_0}^{t} I_{\alpha^-} dZ_s^{\alpha_k}, & \text{if } k \geq 1 \text{ and } \alpha_k \geq 1, \end{cases}$$

for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)^T \in (\mathbb{N}_0)^k$ with $k \geq 2$, where α^-, the multi index, can be obtained by deleting the last component of α, i.e., integration is with respect to ds if $i_k = 0$, or dZ_s^j if $i_k = j \neq 0,\ j = 1, \ldots, n$.

3.2 Systems with common states

Now, we consider the following system

$$dX_t^k = a_k(t, \mathbb{X}_t) + h_{kk}(t, \mathbb{X}_t) dZ_t^k, \quad k = 1, 2, \ldots, d, \tag{8}$$

where all equations may have all states in common but they are not sharing Brownian motions.
It has a similar formulation with completely decoupled systems

$$\begin{aligned} X_t^k = X_{t_0}^k \quad &+ \quad a_k(t_0, \mathbb{X}_{t_0}) I_0 + h_{kk}(t_0, \mathbb{X}_{t_0}) I_k \\ &+ \quad \mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0}) I_{00} + \sum_{j=1}^{n} \mathcal{L}^j a_k(t_0, \mathbb{X}_{t_0}) I_{j0} \\ &+ \quad \mathcal{L}^0 h_{kk}(t_0, \mathbb{X}_{t_0}) I_{0k} + \sum_{j=1}^{n} \mathcal{L}^j h_{kk}(t_0, \mathbb{X}_{t_0}) I_{jk} \\ &+ \quad \mathcal{L}^0 \mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0}) I_{000} + \sum_{j=1}^{n} \mathcal{L}^j \mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0}) I_{j00} \\ &+ \quad \sum_{j=1}^{n} \mathcal{L}^0 \mathcal{L}^j a_k(t_0, \mathbb{X}_{t_0}) I_{0j0} + \sum_{j,p=1}^{n} \mathcal{L}^j \mathcal{L}^p a_k(t_0, \mathbb{X}_{t_0}) I_{jp0} \\ &+ \quad \mathcal{L}^0 \mathcal{L}^0 h_{kk}(t_0, \mathbb{X}_{t_0}) I_{00k} + \sum_{j,p=1}^{n} \mathcal{L}^j \mathcal{L}^0 h_{kk}(t_0, \mathbb{X}_{t_0}) I_{j0k} \\ &+ \quad \sum_{j,p=1}^{n} \mathcal{L}^0 \mathcal{L}^j h_{kk}(t_0, \mathbb{X}_{t_0}) I_{0jk} + \sum_{j,p=1}^{n} \mathcal{L}^j \mathcal{L}^p h_{kk}(t_0, \mathbb{X}_{t_0}) I_{jpk} + R, \end{aligned}$$

where R denotes the remainder term.

3.3 General case

We apply the Theorem 3.1 to the following system:

$$dX_t^k = a_k(t, \mathbb{X}_t) + \sum_{j=1}^{n} h_{kj}(t, \mathbb{X}_t) dZ_t^j, \qquad k = 1, 2, \ldots, d, \tag{9}$$

where all equations may share both Brownian motions and states.

Then, similarly,

$$X_t^k = X_{t_0}^k + \int_{t_0}^{t} a_k(s, \mathbb{X}_s) ds + \sum_{j=1}^{n} \int_{t_0}^{t} h_{kj}(s, \mathbb{X}_s) dZ_s^j. \tag{10}$$

Again, we first choose $g^k := a_k(t, \mathbb{X}_t)$ to apply the Itô formula, and then, we let $g^k := h_{kj}(t, \mathbb{X}_t)$ to get

$$\begin{aligned}
X_t^k = X_{t_0}^k &+ \int_{t_0}^{t} \left[a_k(t_0, \mathbb{X}_{t_0}) + \int_{t_0}^{s} \mathcal{L}^0 a_k(\tau, \mathbb{X}_\tau) d\tau \right. \\
&+ \left. \sum_{l,j=1}^{n} \int_{t_0}^{s} \mathcal{L}^j a_k(\tau, \mathbb{X}_\tau) dZ_\tau^j \right] ds + \sum_{j=1}^{n} \int_{t_0}^{t} \left[h_{kj}(t_0, \mathbb{X}_{t_0}) \right. \\
&+ \left. \int_{t_0}^{s} \mathcal{L}^0 h_{kj}(\tau, \mathbb{X}_\tau) d\tau + \sum_{l=1}^{n} \int_{t_0}^{s} \mathcal{L}^l h_{kj}(\tau, \mathbb{X}_\tau) dZ_\tau^l \right] dZ_s^j,
\end{aligned}$$

We do the same things as in the completely decoupled case, but only difference is that a_k and h_{kj} depend on $(t, \mathbb{X}_t)$. Then,

$$\begin{aligned}
X_t^k = X_{t_0}^k &+ a_k(t_0, \mathbb{X}_{t_0}) I_0 + \sum_{j=1}^{n} h_{kj}(t_0, \mathbb{X}_{t_0}) I_j \\
&+ \mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0}) I_{00} + \sum_{j=1}^{n} \mathcal{L}^j a_k(t_0, \mathbb{X}_{t_0}) I_{j0} \\
&+ \sum_{j=1}^{n} \mathcal{L}^0 h_{kj}(t_0, \mathbb{X}_{t_0}) I_{0j} + \sum_{l,j=1}^{n} \mathcal{L}^l h_{kj}(t_0, \mathbb{X}_{t_0}) I_{lj} \\
&+ \mathcal{L}^0 \mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0}) I_{000} + \sum_{j=1}^{n} \mathcal{L}^j \mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0}) I_{j00} \\
&+ \sum_{j=1}^{n} \mathcal{L}^0 \mathcal{L}^j a_k(t_0, \mathbb{X}_{t_0}) I_{0j0} + \sum_{j,p=1}^{n} \mathcal{L}^j \mathcal{L}^p a_k(t_0, \mathbb{X}_{t_0}) I_{jp0} \\
&+ \sum_{j=1}^{n} \mathcal{L}^0 \mathcal{L}^0 h_{kj}(t_0, \mathbb{X}_{t_0}) I_{00j} + \sum_{j,l=1}^{n} \mathcal{L}^l \mathcal{L}^0 h_{kj}(t_0, \mathbb{X}_{t_0}) I_{l0j} \\
&+ \sum_{j,p,l=1}^{n} \mathcal{L}^0 \mathcal{L}^l h_{kj}(t_0, \mathbb{X}_{t_0}) I_{0lj} + \sum_{j,p,l=1}^{n} \mathcal{L}^l \mathcal{L}^p h_{kj}(t_0, \mathbb{X}_{t_0}) I_{lpj} + R.
\end{aligned} \tag{11}$$

4 Itô-Taylor approximation for Correlated Brownian Motions

In this section, we assume that Brownian motions are correlated, so we use $\mathbb{W}$ instead of $\mathbb{Z}$ in order to point out the difference from the standard Brownian motions. Now, we consider the system

$$dX_t^k = a_k(t, \mathbb{X}_t) + \sum_1^n h_{kj}(t, \mathbb{X}_t) dW_t^j, \quad k = 1, 2, \ldots, d = n,$$

where $dW_t^i dW_t^j = \rho_{ij} dt$.

Then, we write the correlation matrix ρ as:

$$\rho = \begin{pmatrix} 1 & \rho_{12} & \ldots & \rho_{1n} \\ \rho_{21} & 1 & \ldots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \ldots & 1 \end{pmatrix}, \quad \rho_{ij} = \rho_{ji} \in [-1, 1].$$

Here, ρ is positive symmetric matrix which means that $\rho = \rho^T$ and

$$\sum_{i,j=1}^n \rho_{ij} x_i x_j \geq 0,$$

for all $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$.

By using standard Linear Algebra, one can find an $n \times n$ matrix $\mathbb{B} = (b_{ij})_{1 \leq i,j \leq n}$ such that

$$\rho = \mathbb{B}\mathbb{B}^T.$$

Moreover, using Cholesky Decomposion, we can take $\mathbb{B}$ as an upper (or lower) triangular matrix.

Correlated Brownian motions can be interpreted as

$$\mathbb{W}_t = \mathbb{B}\mathbb{Z}_t,$$

where $\mathbb{W}_t = (W_t^1, \ldots, W_t^n)^T$ is a standard n-dimensional Brownian motion and the Brownian motions, W_t^i for $i = 1, 2, \ldots, n$, are correlated.

In componentwise, notation,

$$W_t^i = \sum_{j=1}^n b_{ij} Z_t^j, \quad i = 1, 2, \ldots, n.$$

Example 1. We consider 2-dimensional version of *weakly-coupled Ornstein-Uhlenbeck model*:

$$
\begin{aligned}
dX_t^1 &= \alpha_1(\theta_1 - X_t^1)dt + \sigma_1 dW_t^1, \\
dX_t^2 &= \alpha_2(\theta_2 - X_t^2)dt + \sigma_2 dW_t^2,
\end{aligned} \tag{12}
$$

where $dW_t^1 dW_t^2 = \rho dt$, all coefficients are real and positive, and $\rho \in (-1, 1)$.

Then, the correlation matrix ρ can be written as:

$$
\rho = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix},
$$

by Cholesky Decomposition [2].

Thus,

$$
\mathbb{W}_t = \mathbb{B}\mathbb{Z}_t = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} Z_t^1 \\ Z_t^2 \end{pmatrix}.
$$

So, Eqn. (12) becomes

$$
\begin{aligned}
dX_t^1 &= \alpha_1(\theta_1 - X_t^1)dt + \sigma_1 dZ_t^1, \\
dX_t^2 &= \alpha_2(\theta_2 - X_t^2)dt + \sigma_2 \rho dZ_t^1 + \sigma_2\sqrt{1-\rho^2} dZ_t^2.
\end{aligned} \tag{13}
$$

In integral form,

$$
\begin{aligned}
X_t^1 &= X_{t_0}^1 + \alpha_1\theta_1 \int_{t_0}^t ds - \alpha_1 \int_{t_0}^t X_s^1 ds + \sigma_1 \int_{t_0}^t dZ_s^1, \\
X_t^2 &= X_{t_0}^2 + \alpha_2\theta_2 \int_{t_0}^t ds - \alpha_2 \int_{t_0}^t X_s^2 ds + \sigma_2\rho \int_{t_0}^t dZ_s^1 + \sigma_2\sqrt{1-\rho^2} \int_{t_0}^t dZ_s^2.
\end{aligned}
$$

Applying the procedure in the previous sections, we get

$$
\begin{aligned}
X_t^1 = X_{t_0}^1 &+ \alpha_1(\theta_1 - X_{t_0}^1)I_0 + \sigma_1 I_1 - \alpha_1^2(\theta_1 - X_{t_0}^1)I_{00} \\
&- \alpha_1\sigma_1 I_{10} + \alpha_1^3(\theta_1 - X_{t_0}^1)I_{000} + \alpha_1^2\sigma_1 I_{100} + R, \\
X_t^2 = X_{t_0}^2 &+ \alpha_2(\theta_2 - X_{t_0}^2)I_0 + \sigma_2\rho I_1 + \sigma_2\sqrt{1-\rho^2} I_2 \\
&- \alpha_2^2(\theta_2 - X_{t_0}^2)I_{00} - \rho\alpha_2\sigma_2 I_{10} - \alpha_2\sigma_2\sqrt{1-\rho^2} I_{20} \\
&+ \alpha_2^3(\theta_2 - X_{t_0}^2)I_{000} + \rho\alpha_2^2\sigma_2 I_{100} + \sqrt{1-\rho^2}\alpha_2^2\sigma_2 I_{200} + R.
\end{aligned}
$$

Example 2. We state the 2-dimensional version of *strongly-coupled Ornstein-Uhlenbeck model*

$$
\begin{aligned}
dX_t^1 &= (-\alpha_{11}X_t^1 - \alpha_{12}X_t^2)dt + \sigma_1 dW_t^1, \\
dX_t^2 &= (-\alpha_{21}X_t^1 - \alpha_{22}X_t^2)dt + \sigma_2 dW_t^2,
\end{aligned} \tag{14}
$$

where $dW_t^1 dW_t^2 = \rho dt$ and $\theta_1, \theta_2 = 0$.

We obtain the following system in terms of standard Brownian motions:

$$
\begin{aligned}
dX_t^1 &= (-\alpha_{11}X_t^1 - \alpha_{12}X_t^2)dt + \sigma_1 dZ_t^1, \\
dX_t^2 &= (-\alpha_{21}X_t^1 - \alpha_{22}X_t^2)dt + \sigma_2\rho dZ_t^1 + \sigma_2\sqrt{1-\rho^2}dZ_t^2.
\end{aligned} \tag{15}
$$

In a similar way, we obtain:

$$
\begin{aligned}
X_t^1 = X_{t_0}^1 &- (\alpha_{11}X_{t_0}^1 + \alpha_{12}X_{t_0}^2)I_0 + \sigma_1 I_1 - (\sigma_1\alpha_{11} + \rho\sigma_2\alpha_{12})I_{10} \\
&+ \Big[\alpha_{11}(\alpha_{11}X_{t_0}^1 + \alpha_{12}X_{t_0}^2) + \alpha_{12}(\alpha_{21}X_{t_0}^1 + \alpha_{22}X_{t_0}^2)\Big]I_{00} \\
&- \sigma_2\alpha_{12}\sqrt{1-\rho^2}I_{20} - \Big[(\alpha_{11}^2 + \alpha_{12}\alpha_{21})(\alpha_{11}X_{t_0}^1 + \alpha_{12}X_{t_0}^2) \\
&+ (\alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{22})(\alpha_{21}X_{t_0}^1 + \alpha_{22}X_{t_0}^2)\Big]I_{000} \\
&+ \Big[\sigma_1(\alpha_{11}^2 + \alpha_{12}\alpha_{21}) + \sigma_2\rho(\alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{22})\Big]I_{100} \\
&+ \sigma_2(\alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{22})\sqrt{1-\rho^2}I_{200} + R,
\end{aligned}
$$

and

$$
\begin{aligned}
X_t^2 = X_{t_0}^2 &- (\alpha_{21}X_{t_0}^1 + \alpha_{22}X_{t_0}^2)I_0 + \sigma_2\rho I_1 + \sigma_2\sqrt{1-\rho^2}I_2 \\
&- (\sigma_1\alpha_{21} + \rho\sigma_2\alpha_{22})I_{10} + \big[\alpha_{21}(\alpha_{11}X_{t_0}^1 + \alpha_{12}X_{t_0}^2) \\
&+ \alpha_{22}(\alpha_{21}X_{t_0}^1 + \alpha_{22}X_{t_0}^2)\big]I_{00} - \sigma_2\alpha_{22}\sqrt{1-\rho^2}I_{20} \\
&- [(\alpha_{21}\alpha_{11} + \alpha_{22}\alpha_{21})(\alpha_{11}X_{t_0}^1 + \alpha_{12}X_{t_0}^2) + (\alpha_{21}\alpha_{12} \\
&+ \alpha_{22}^2)(\alpha_{21}X_{t_0}^1 + \alpha_{22}X_{t_0}^2)]I_{000} + [\sigma_1(\alpha_{11}\alpha_{21} + \alpha_{22}\alpha_{21}) \\
&+ \sigma_2\rho(\alpha_{21}\alpha_{12} + \alpha_{22}^2)]I_{100} + \sigma_2(\alpha_{12}\alpha_{21} + \alpha_{22}^2)\sqrt{1-\rho^2}I_{200} + R.
\end{aligned}
$$

5 Discretization schemes with Strong Taylor approximations

We represent some numerical approximations of Itô integrals. Let Δ denote the increments, then:

$$
\begin{aligned}
I_j &= \int_{\tau_n}^{\tau_{n+1}} dZ_s^j = \Delta Z^j = Z_{\tau_{n+1}}^j - Z_{\tau_n}^j,\\
I_{j0} &= \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s} dZ_u^j ds = \Delta \tilde{W},\\
I_{0j} &= \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s} du dZ_s^j = (\Delta Z^j)\Delta - \Delta \tilde{W},\\
I_{jj} &= \frac{1}{2}\big((\Delta Z^j)^2 - \Delta\big),\\
I_{jj0} &= I_{0jj} = I_{j0j} = \frac{1}{6}\Delta\big((\Delta Z^j)^2 - \Delta\big),\\
I_{jjj} &= \frac{1}{6}\Delta\big((\Delta Z^j)^3 - 3\Delta(\Delta Z^j)\big),\\
I_{j00} &= I_{0j0} = I_{00j} = \frac{1}{6}\Delta^2 \Delta Z_s^j,
\end{aligned}
$$

where the $\Delta\tilde{W}$ and ΔZ^j are Gaussian random variables with $\Delta Z^j \sim N(0,\Delta)$, $\Delta\tilde{W} \sim N(0, \frac{1}{3}\Delta^3)$ and $E\big(\Delta Z^j \Delta \tilde{W}\big) = \frac{1}{2}\Delta^2$.

Now, we shall use the above relations to propose some strong approximations.

5.1 The Euler-Maruyama Scheme

The simplest example of a strong Taylor approximation Y of the solution of Eqn. (9) is the Euler-Maruyama or Euler method attaining the order of strong convergence 0.5. The k^{th} component of the *Euler scheme* is of the form

$$
Y_{n+1}^k = Y_n^k + a_k\Delta + \sum_{j=1}^{n} h_{kj}\Delta Z^j, \text{ for } k = 1, 2, \ldots, d. \tag{16}
$$

In the cases where drift and difussion coefficients are nearly constant, this method generally gives us good numerical results. However, when the coefficients are nonlinear the method can provide a poor estimate of the solution. So, higher-order schemes should be used to obtain more satisfactory schemes.

Let us consider the system (15) by taking $\alpha_{11} = \alpha_{21} = \alpha_{22} = 1, \alpha_{12} = 2, \sigma_1 = \sigma_2 = 1$ and $\rho = 0.6$ to get

$$
\begin{aligned}
dX_t^1 &= (-X_t^1 - 2X_t^2)dt + dZ_t^1,\\
dX_t^2 &= (-X_t^1 - X_t^2)dt + 0.6dZ_t^1 + 0.8dZ_t^2.
\end{aligned} \tag{17}
$$

Euler scheme reads the system (17) as:

$$\begin{aligned}
Y_{n+1}^1 &= Y_n^1 + (-Y_n^2 - 2Y_n^1)\Delta + \Delta Z^1, \\
Y_{n+1}^2 &= Y_n^2 + (-Y_n^2 - Y_n^1)\Delta + 0.6\Delta Z^1 + 0.8\Delta Z^2.
\end{aligned}$$

5.2 The Milstein Scheme

Milstein scheme has the order of strong convergence 1.0. By including an additional term from Itô-Taylor expansion (11), we obtain the *Milstein scheme* whose k^{th} component is of the form

$$Y_{n+1}^k = Y_n^k + a_k\Delta + \sum_{j=1}^{n} h_{kj}\Delta Z^j + \sum_{j_1,j_2=1}^{n} L^{j_1} h_{kj_2} I_{j_1 j_2}, \text{ for } k = 1, 2, \ldots, d. \tag{18}$$

We note that Milstein scheme is identical to the Euler scheme when the diffusion term does not contain an $\mathbb{X}_t$ term.

Applying the scheme to the system (17), we obtain the Milstein scheme:

$$\begin{aligned}
Y_{n+1}^1 &= Y_n^1 + (-Y_n^2 - 2Y_n^1)\Delta + \Delta Z^1, \\
Y_{n+1}^2 &= Y_n^2 + (-Y_n^2 - Y_n^1)\Delta + 0.6\Delta Z^1 + 0.8\Delta Z^2.
\end{aligned}$$

5.3 The Order 1.5 Strong Taylor Scheme

We can get more accurate strong Taylor schemes by including further multiple stochastic integrals from the stochastic Taylor approximation (11). The k^{th} component of *the order 1.5 strong Taylor scheme* is given by

$$\begin{aligned}
Y_{n+1}^k = Y_n^k &+ a_k\Delta + \frac{1}{2}L^0 a_k \Delta^2 + \sum_{j=1}^{n}(h_{kj}\Delta Z^j + L^0 h_{kj} I_{0j} + L^j a_k I_{j0} \\
&+ \sum_{j_1,j_2=1}^{n} L^{j_1} h_{kj_2} I_{j_1 j_2} + \sum_{j_1,j_2,j_3=1}^{n} L^{j_1} L^{j_2} h_{kj_3} I_{j_1 j_2 j_3}
\end{aligned} \tag{19}$$

for $k = 1, 2, \ldots, d$.

Applying this method to the system (17), we get:

$$\begin{aligned}
Y_{n+1}^1 &= Y_n^1 + (-Y_n^2 - 2Y_n^1)\Delta + \frac{1}{2}(3Y_n^2 + 5Y_n^1)\Delta^2 + \Delta Z^1 - 2.6 I_{10} - 0.8 I_{20}, \\
Y_{n+1}^2 &= Y_n^2 + (-Y_n^2 - Y_n^1)\Delta + \frac{1}{2}(2Y_n^2 + 3Y_n^1)\Delta^2 + 0.6\Delta Z^1 \\
&\quad + 0.8\Delta Z^2 - 1.6 I_{10} - 0.8 I_{20}.
\end{aligned}$$

5.4 The Order 2.0 Strong Taylor Scheme

The numerical results which was given by *the order 2.0 strong Taylor scheme* is better than other three method. The k^{th} component of *the order 2.0 strong Taylor scheme* takes the form

$$\begin{aligned} Y_{n+1}^k = Y_n^k &+ a_k\Delta + \frac{1}{2}L^0 a_k \Delta^2 + \sum_{j=1}^{n}(h_{kj}\Delta Z^j + L^0 h_{kj} I_{0j} + L^j a_k I_{j0}) \\ &+ \sum_{j_1,j_2=1}^{n} \big(L^{j_1} h_{kj_2} I_{j_1 j_2} + L^0 L^{j_1} h_{kj_2} I_{0 j_1 j_2} + L^{j_1} L^0 h_{kj_2} I_{j_1 0 j_2} \\ &+ L^{j_1} L^{j_2} a_k I_{j_1 j_2 0}\big) + \sum_{j_1,j_2,j_3=1}^{n} L^{j_1} L^{j_2} h_{kj_3} I_{j_1 j_2 j_3} \\ &+ \sum_{j_1,j_2,j_3,j_4=1}^{n} L^{j_1} L^{j_2} L^{j_2} h_{kj_4} I_{j_1 j_2 j_3 j_4}, \end{aligned} \tag{20}$$

for $k = 1, 2, \ldots, d$.

Finally, for the Eqn. (17) this formulation can be reduced to:

$$\begin{aligned} Y_{n+1}^1 &= Y_n^1 + (-Y_n^2 - 2Y_n^1)\Delta + \frac{1}{2}(3Y_n^2 + 5Y_n^1)\Delta^2 + \Delta Z^1 - 2.6 I_{10} - 0.8 I_{20}, \\ Y_{n+1}^2 &= Y_n^2 + (-Y_n^2 - Y_n^1)\Delta + \frac{1}{2}(2Y_n^2 + 3Y_n^1)\Delta^2 + 0.6\Delta Z^1 \\ &\quad +0.8\Delta Z^2 - 1.6 I_{10} - 0.8 I_{20}. \end{aligned}$$

6 Numerical Results and Implementation Details

In this section, we consider numerical examples for both the systems with independent and correlated Brownian motions. In order to implement the discrete scheme, we use MATLAB. There are many documentations that describes the main features of MATLAB commands related to SDE. Some numerical interpretations can be found in the SDEs MATLAB packages [7]. However, numerical examples implemented in MATLAB are mostly in the 1-dimensional case. Some coupled SDEs are considered, but having symmetric coefficients allowing easy computations that arise from multiple Itô integrals. Our first example demonstrates the triple SDEs having independent Brownian motions.

Example Run 1. The system of SDE consisting of three equations proposed in Hofmann, Platen and Schweizer [8] is considered as:

$$\begin{cases} dX_t^1 = X_t^1 X_t^2 dZ_t^1, \\ dX_t^2 = -(X_t^2 - X_t^3)dt + 0.3 X_t^2 dZ_t^2, \\ dX_t^3 = \frac{1}{\alpha}(X_t^2 - X_t^3)dt, \end{cases}$$

where X_t^1, X_t^2 and X_t^3 represent the asset price, the instantenous volatility, and the averaged volatility, respectively. As in [7], Milstein scheme is obtained as:

$$\begin{aligned} dX_{n+1}^1 = X_n^1 &\quad + \quad X_n^1 X_n^2 \Delta Z_n^1 + \frac{1}{2} X_n^1 (X_n^2)^2 \{(\Delta Z_n^1)^2 - \Delta\} \\ &\quad + \quad 0.3 X_n^1 X_n^2 \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dZ_s^2 dZ_s^1, \\ dX_{n+1}^2 = X_n^2 &\quad - \quad (X_n^2 - X_n^3)\Delta + 0.3 X_n^2 \Delta Z_n^2 + 0.045 X_n^2 \{(\Delta Z_n^1)^2 - \Delta\}, \\ dX_{n+1}^3 = dX_n^3 &\quad + \quad \frac{1}{\alpha}(X_n^2 - X_n^3)\Delta. \end{aligned}$$

We take $\alpha = 1$, $X_0^1 = 1$, $X_0^2 = 0.1$, $X_0^2 = 0.1$ and $T = 1$; Δ is considered as 2^{-9}.

The scheme has the double integral $\int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dZ_s^2 dZ_s^1$. In [7], this integral is approximated by Euler method. Although it is a bit challenging, we compute such integrals by using the following formulations from [10] as follows:

$$\begin{aligned} &I_0^p = \Delta, \quad I_j^p = \sqrt{\Delta}\xi_j, \quad I_{00}^p = \frac{1}{2}\Delta^2, \\ &I_{j0}^p = \frac{1}{2}\Delta\big(\sqrt{\Delta}\xi_j + a_{j0}\big), \quad I_{0j}^p = \frac{1}{2}\Delta\big(\sqrt{\Delta}\xi_j - a_{j0}\big). \end{aligned}$$

Here,

$$\begin{aligned} a_{j0} &= -\frac{1}{\pi}\sqrt{2\Delta}\sum_{r=1}^{p}\frac{1}{r}\zeta_{jr} - 2\sqrt{\Delta\rho_p}\mu_{jp}, \\ I_{j_1 j_2}^p &= \frac{1}{2}\Delta\xi_{j_1}\xi_{j_2} - \frac{1}{2}\sqrt{\Delta}(a_{j_2 0}\xi_{j_1} - a_{j_1 0}\xi_{j_2}) + \Delta A_{j_1 j_2}^p, \\ A_{j_1 j_2}^p &= \frac{1}{2\pi}\sum_{r=1}^{p}\frac{1}{r}(\zeta_{j_1 r}\eta_{j_2 r} - \zeta_{j_2 r}\eta_{j_1 r}), \end{aligned}$$

with

$$\xi_j = \frac{1}{\sqrt{\Delta}} W^j, \quad \zeta_{jr} = \sqrt{\frac{2}{\Delta}} \pi r a_{jr}, \quad \eta_{jr} = \sqrt{\frac{2}{\Delta}} \pi r b_{jr},$$

$$\mu_{jp} = \frac{1}{\sqrt{\Delta \rho_p}} \sum_{r=p+1}^{\infty} a_{jr}, \quad \rho_p = \frac{1}{12} - \frac{1}{2\pi^2} \sum_{r=1}^{p} \frac{1}{r^2},$$

where $j = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, p$, for a positive number p chosen as follows:

$$p = p(\Delta) \geq \frac{K}{\Delta^2},$$

for an appropriate positive constant K to ensure the convergence order of the numerical scheme.

Here, we note that ζ_{jr}, η_{jr} and μ_{jp} are independent Gaussian random variables. We use the *Polar Marsaglia Method* to generate the pairs of random variables.
The following lines show the implementation of this method in MATLAB:

```
%Polar Marsaglia Method
function     [z1,z2]= Polar
l=0.5;
while l>0
u1 = rand;u2 = rand;
v1 = 2*u1 - 1;v2 = 2*u2 - 1;
V = (v1.*v1)+(v2.*v2);
if (V<=1)&&(V>0)
    break;
end
end
z1 = v1.*sqrt(-2*log(V)./V);
z2 = v2.*sqrt(-2*log(V)./V);
```

We can compute the iterated integral $\int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dZ_s^2 dZ_s^1$ as:

```
%Approximation of I_ij
function I_ij=ito_ij(p,Delta,G1,G2,mu1_j,mu2_j,ro)
a_ij=0;
for i=1:p
    [zeta1, zeta2]= Polar;[eta1 ,eta2 ]=Polar;
    a_ij=a_ij+(1/i)*(zeta1*(sqrt(2)*G2+eta2)...
```

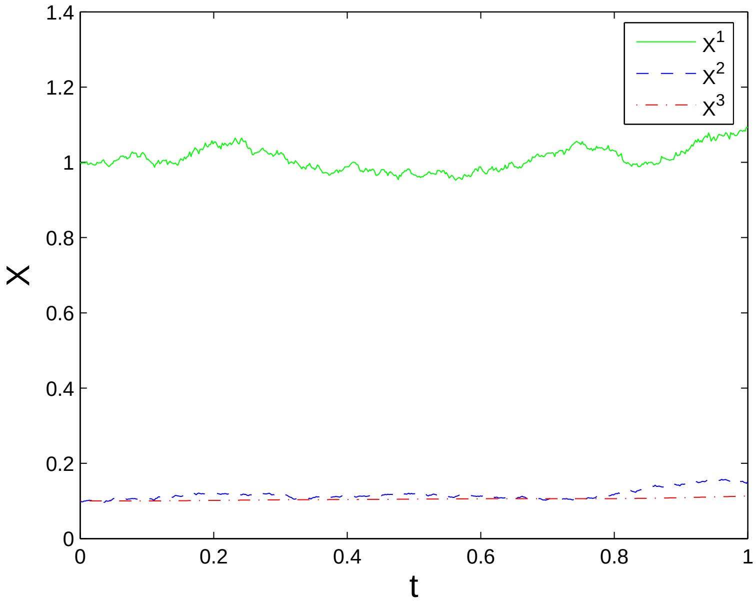

Figure 1: Numerical result of Example Run1 with Milstein approximation.

```
   -zeta2*(sqrt(2)*G1+eta1));
end
I_ij=a_ij*Delta/(pi);
I_ij=I_ij+Delta*(G1*G2/2+sqrt(ro)*(mu1_j*G2-mu2_j*G1));
```

In Figure 1, we give the numerical result of Example Run1. We have the same results as in [7].

Example Run 2. *(Correlated Brownian motions)* We recall the strongly-coupled Ornstein-Uhlenbeck process (14):

$$\begin{aligned} dX_t^1 &= (-\alpha_{11}X_t^1 - \alpha_{12}X_t^2)dt + \sigma_1 dW_t^1, \\ dX_t^2 &= (-\alpha_{21}X_t^1 - \alpha_{22}X_t^2)dt + \sigma_2 dW_t^2, \end{aligned}$$

where $dW_t^1 dW_t^2 = \rho dt$ and the transformed form (17) is

$$\begin{aligned} dX_t^1 &= (-X_t^1 - 2X_t^2)dt + dZ_t^1, \\ dX_t^2 &= (-X_t^1 - X_t^2)dt + 0.6dZ_t^1 + 0.8dZ_t^2. \end{aligned}$$

Our Taylor scheme with order 1/2 gives

$$
\begin{aligned}
Y^1_{n+1} &= Y^1_n + (-Y^2_n - 2Y^1_n)\Delta + \frac{1}{2}(3Y^2_n + 5Y^1_n)\Delta^2 + \Delta Z^1 - 2.6 I_{10} - 0.8 I_{20}, \\
Y^2_{n+1} &= Y^2_n + (-Y^2_n - Y^1_n)\Delta + \frac{1}{2}(2Y^2_n + 3Y^1_n)\Delta^2 + 0.6\Delta Z^1 \\
&\quad + 0.8\Delta Z^2 - 1.6 I_{10} - 0.8 I_{20}.
\end{aligned}
$$

We compute the integrals $I_{20},\ I_{10}$ as in the following lines:

```
%Approximation of I_j0 and I_0j
function [I_10,I_20]=ito_j0(p,Delta,G1,G2,mu1_j,mu2_j,ro)
a_10=0;a_20=0;
for i=1:p
   [eta1,eta2]=Polar;
    a_10=a_10+(1/i)*eta1;
    a_20=a_20+(1/i)*eta2;
end
I_10=a_10*(1/pi)*sqrt(Delta*2)+2*sqrt(Delta*ro)*mu1_j;
I_10=(1/2)*Delta*I_10+(1/2)*Delta*sqrt(Delta)*G1;
I_20=a_20*(1/pi)*sqrt(Delta*2)+2*sqrt(Delta*ro)*mu2_j;
I_20=(1/2)*Delta*I_20+(1/2)*Delta*sqrt(Delta)*G2;
```

The main file can be run as:

```
clf
randn('state',1)
T = 1; Delta = 2^(-9); delta = Delta^2;
L = T/Delta; K = Delta/delta;
X1 = zeros(1,L+1); X2 = zeros(1,L+1);
X1(1) = 1;X2(1) = 0.1;
p=2;ro=0;
for i=1:p
   ro=ro+1/(i*i);ro=(pi*pi)/6-ro;ro=ro/(2*pi*pi);
end
for j = 1:L
     G1 = randn;   G2 = randn;
Winc2 = sqrt(Delta)*G2;Winc1 = sqrt(Delta)*G1;
 [mu1, mu2 ]=Polar;

[I10,I20]=ito_j0(p,Delta,G1,G2,mu1,mu2,ro);
     X1(j+1) = X1(j) +(-2*X1(j)-X2(j))*Delta +...
     (0.5)*(3*X2(j)+5*X1(j))*Delta^2+Winc1-(2.6)*I10-(0.8)*I20;
    X2(j+1) = X2(j) + (-X2(j)-X1(j))*Delta+(0.5)*(2*X2(j)...
```

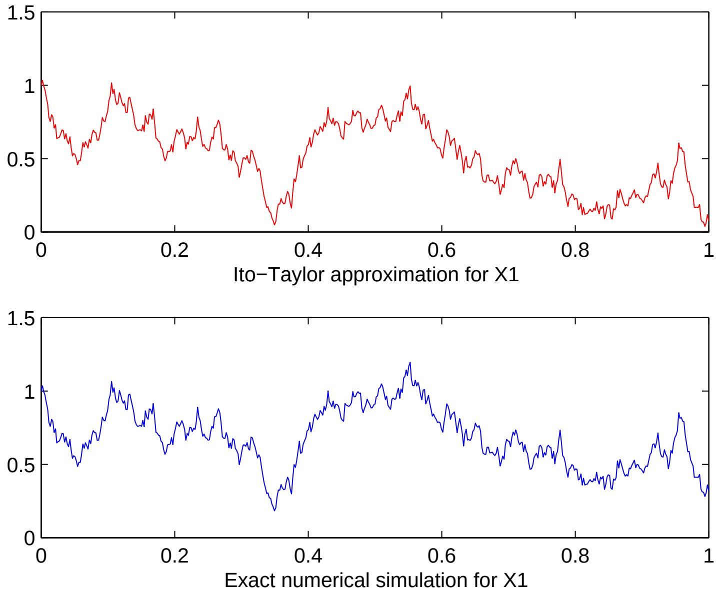

Figure 2: Comparison of exact numerical solution of Run2 with Taylor Scheme of order 1/2 for X1.

```
    +3*X1(j))*Delta^2+ (0.6)*Winc1 +(0.8)*Winc2-(1.6)*I10-(0.8)*I20;
end
plot([0:Delta:T],X1,'r-'), hold on
plot([0:Delta:T],X2,'bl--')
xlabel('t','FontSize',16), ylabel('X','FontSize',16)
legend('X^1','X^2')
```

Exact solution of the system (14) is obtained in matrix formulation as:

$$\mathbb{X}_t = \mathbb{X}_0 \exp(-t\mathbb{A}) + \int_0^t \exp((s-t)\mathbb{A})\mathbb{B}d\mathbb{Z}_t, \tag{21}$$

where $\mathbb{A} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, and $\mathbb{B} = \begin{pmatrix} 1 & 0 \\ 0.6 & 0.8 \end{pmatrix}$.

We perform the exact numerical simulation for the system (14). We first compute the matrix multiplications in the Eqn. (21), and then approximate componentwise. In Figures 2-3, we compare the obtained results. It can be seen easily that they are almost the same.

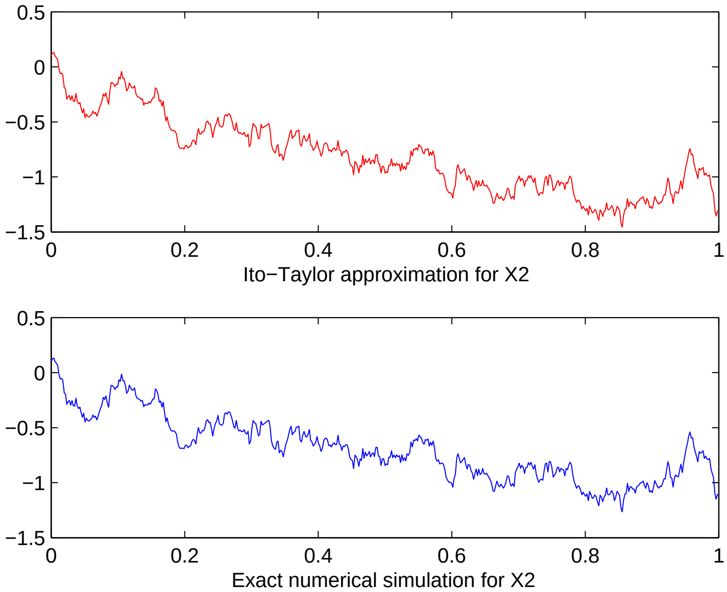

Figure 3: Comparison of exact numerical solution of Run2 with Taylor Scheme of order 1/2 for X2.

7 Conclusion and Outlook

This work have totally covered the numerical solutions of SDEs by means of Itô expansions. We obtained the stochastic Taylor series expansions of the systems os SDEs with both correlated and independent Wiener processes. In the case of correlated processes, we first transformed the system into one having independent Brownian motions. And then, we obtained the Itô Taylor series expansions. Also, we performed the computations of the iterated Itô integrals with several Wiener processes by using MATLAB.

As a future work, the relation between the Itô Taylor approximations and stochastic control problems can be considered. Furthermore, as for application to the more complicated systems of SDEs, e.g., Heston model, may be handled by means of Itô Taylor formula. A connection between the iterated Itô integrals and *Malliavin calculus* should be investigated [1, 12, 13, 14, 15, 17, 19]. A detailed package for the systems of SDEs, which has a higher-order convergence, can be generated via MATLAB.

References

[1] Bell, D., 2007, The Malliavin Calculus, Dover.

[2] Brummelhuis, R. Mathematical Methods Lecture Notes 6, Department of Economics, Mathematics and Statistics, Birkbeck, University of London, 2009.

[3] Burrage, K. and Burrage, P.M., 1999, High strong order methods for non-commutative stochastic ordinary differential equation system and the Magnus formula, Physica D 133, 34-48.

[4] Burrage, K. and Burrage, P.M., 2000, Order conditions of stochastic Runge-Kutta methods by B-series, SIAM J. Numer. Anal. 38, 1626-1646.

[5] Burrage, P.M., 1999, Runge-Kutta Methods for Stochastic Differential Equations, PhD Thesis, Department of Mathematics, University of Queensland, Australia.

[6] Gilsing, H., and Shardlow, T., SDELab: A package for solving stochastic differential equations in MATLAB, Journal of Computational and Applied Mathematics 205 (2007), no. 2, 1002-1018.

[7] Higham, D. J., and Kloeden, P. E., MAPLE and MATLAB for Stochastic Differential Equations in Finance, 2002, in Programming Languages and Systems in Computational Economics and Finance, 233-270.

[8] Hoffman, N., Platen, E., and Schweizer, M., 1992, Option pricing under incompleteness and stochastic volatility, J. Mathematical Finance, 2, 153-187.

[9] Ito K., 1944, Stochastic Integral, Proc. Imperial Acad. Tokyo, 20, 519-524

[10] Kloeden, P.E. and Platen, E., 1992, Numerical Solution of Stochastic Differential Equations, Springer-Verlag, Berlin.

[11] Kunita, H., 1990, Stochastic Flows and Stochastic Differential Equations, Vol 24 of Cambridge Studies in Advanced Mathematics, Cambridge University Press.

[12] Kusuoka, S. and Stroock, D., 1982, Applications of Malliavin Calculus I, Stochastic Analysis, Proceedings Taniguchi International Symposium Katata and Kyoto, 271-306.

[13] Kusuoka, S. and Stroock, D., 1985, Applications of Malliavin Calculus II, J. Faculty Sci. Uni. Tokyo Sect. 1A Math., 32, 1-76.

[14] Kusuoka, S. and Stroock, D., 1987, Applications of Malliavin Calculus III, J. Faculty Sci. Univ. Tokyo Sect. 1A Math., 34, 391-442.

[15] Malliavin, P., Thalmaier, A., 2005, Stochastic Calculus of Variations in Mathematical Finance, Springer.

[16] Milstein, G.N., 1995, Numerical Integration of Stochastic Differential Equations, Kluwer, Dordrecht/Boston/London.

[17] Nualart, D., 2006, The Malliavin calculus and related topics (Second edition ed.), Springer-Verlag.

[18] Øksendal, B., 2000, Stochastic Differential Equations, An Introduction with Applications, 5th edition

[19] Øksendal, B., 1997, An Introduction To Malliavin Calculus With Applications To Economics, Thesis, Dept. of Mathematics, University of Oslo.

[20] Platen, E., 1999, An introduction to numerical methods for stochastic differential equations, Acta Numerica, 8, 197-246.

[21] Rümelin, W., 1982, Numerical treatment of stochastic differential equations, SIAM J. Numer. Anal. 19, 604-613.

[22] Yılmaz, F., Öz, H., Weber, G.W., 2012, Change of Time Method and Stochastic Taylor Expansion with Computation of Expectation, Book Chapter, "*Modeling, Optimization, Dynamics and Bioeconomy*", Springer, to appear.