EXTREMES AND FIRST PASSAGE TIMES OF CORRELATED FRACTIONAL BROWNIAN MOTIONS

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Abstract: Let $\{X_i(t), t \ge 0\}, i = 1, 2$ be two standard fractional Brownian motions being jointly Gaussian with constant cross-correlation. In this paper we derive the exact asymptotics of the joint survival function

$$\mathbb{P}\left\{ \sup_{s\in[0,1]} X_1(s) > u, \ \sup_{t\in[0,1]} X_2(t) > u \right\}$$

as $u \to \infty$. A novel finding of this contribution is the exponential approximation of the joint conditional first passage times of X_1, X_2. As a by-product we obtain generalizations of the Borell-TIS inequality and the Piterbarg inequality for 2-dimensional Gaussian random fields.

Key Words: Extremes; first passage times; Borell-TIS inequality; Piterbarg inequality; fractional Brownian motion; Gaussian random fields.

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1. Introduction and main results

Let $\{X_i(t), t \ge 0\}, i = 1, 2$ be two standard fractional Brownian motion's (fBm's) with Hurst indexes $\alpha_i/2 \in (0, 1), i = 1, 2$, i.e., X_i is a centered Gaussian process with a.s. continuous sample paths and covariance function

$$\mathbb{C}ov(X_i(t), X_i(s)) = \frac{1}{2}(|t|^{\alpha_i} + |s|^{\alpha_i} - |\, t - s \,|^{\alpha_i}), \quad s, t \ge 0, \quad i = 1, 2.$$

Hereafter (X_1, X_2) are assumed to be jointly Gaussian with cross-correlation function $r(s, t) = \mathbb{E}\{X_1(s)X_2(t)\} / \sqrt{s^{\alpha_1} t^{\alpha_2}} \in (-1, 1)$. Calculation of the following joint survival function

$$P_r(u) := \mathbb{P}\left\{ \sup_{s\in[0,1]} X_1(s) > u, \ \sup_{t\in[0,1]} X_2(t) > u \right\}, \quad u > 0. \tag{1}$$

is important for various applications in statistics, mathematical finance and insurance mathematics. The special simple model of two correlated Brownian motions (i.e., $\alpha_1 = \alpha_2 = 1$) with $r(s, t) = r$ a constant has been well studied in the literature; see e.g., [26] and [30]. Therein an explicit expression for (1) was given through the modified Bessel function and in the form of series; recently [39] obtained some computable bounds for (1). We refer to [27] for related results. Explicit calculation of (1) is only possible for correlated Brownian motions. Since typically in applications calculation of the joint survival probability is needed for large thresholds u, one can rely on the asymptotic theory to find adequate approximations of this survival probability. In [38] logarithmic asymptotics of (1) as $u \to \infty$ for general correlated Gaussian processes X_1, X_2 was obtained; see also [13] for a general treatment of the multidimensional case. So far in the literature there are only two contributions that derive exact asymptotics of (1) for certain Gaussian processes, namely Cheng and Xiao [7] obtained an exact asymptotic expansion of (1) for two correlated smooth Gaussian processes X_1, X_2. In the aforementioned paper the result was obtained by studying the geometric properties of the processes. The second contribution is from Anshin [3] where the exact asymptotics for two correlated non-smooth Gaussian processes X_1, X_2 is derived by relying on a modified double-sum method (see [19, 32, 34, 35, 36] for details on the double-sum method). The assumptions in [3] are such that our model of two standard fBm's with $r(s, t) = r \in (-1, 1)$ is not included. Indeed, the conditions **C1-C3** therein are all invalid for our model. Due to wide applications of fBm's and their exit probabilities, we consider in this paper the exact asymptotics of $P_r(u)$ given as in (1) with X_1, X_2 being the two standard fBm's above with a constant cross-correlation function $r \in (-1,1)$. Another merit of choosing fBm's rather than other (general) Gaussian processes is that it allows for somewhat explicit formulae. In order to proceed with our analysis using a modified double-sum method, we shall extend the celebrated Borell-TIS inequality and the Piterbarg inequality for 2-dimensional Gaussian random fields in Theorem 2.1 and Theorem 2.2, respectively. These results are of independent interest given their importance in the theory of Gaussian processes and random fields; see [29] for new developments in this direction.

Before presenting our main results we recall the definition of the well-known Pickands constant

$$\mathcal{H}_\alpha := \lim_{T\to\infty} \frac{1}{T}\mathcal{H}_\alpha^0[\Lambda_T], \quad \alpha \in (0,2], \quad \Lambda_T := [0,T], \tag{2}$$

where

$$\mathcal{H}_\alpha^b[\Lambda] := \mathbb{E}\left\{\exp\left(\sup_{t\in\Lambda}\left(\sqrt{2}B_\alpha(t) - |t|^\alpha - bt\right)\right)\right\}, \quad \Lambda \subset \mathbb{R},\ b \in \mathbb{R}. \tag{3}$$

Here $\{B_\alpha(t), t\in\mathbb{R}\}$ is a standard fBm defined on $\mathbb{R}$ with Hurst index $\alpha/2 \in (0,1]$. By the symmetry about 0 of the fBm, for any $T\in(0,\infty)$ we have (with Λ_T given as in (2))

$$\mathcal{H}_\alpha^0[[-T,0]] = \mathcal{H}_\alpha^0[\Lambda_T], \quad \mathcal{H}_1^{-b}[[-T,0]] = \mathcal{H}_1^b[\Lambda_T], \quad b\in\mathbb{R}. \tag{4}$$

We refer to [2, 5, 10, 8, 9, 12, 17, 28, 36] for the basic properties of the Pickands and related constants.

Our first principle result is presented below.

Theorem 1.1. *Let $\{X_i(t), t\ge 0\}$, $i=1,2$ be two standard fBm's with Hurst indexes $\alpha_i/2 \in (0,1), i=1,2,$ respectively. If (X_1,X_2) are jointly Gaussian with a constant cross correlation function $r\in(-1,1)$, then as $u\to\infty$*

$$P_r(u) = \frac{(1+r)^{\frac{3}{2}}}{2\pi\sqrt{1-r}}\Upsilon_1(u)\Upsilon_2(u)u^{-2}\exp\left(-\frac{u^2}{1+r}\right)(1+o(1)), \tag{5}$$

where

$$\Upsilon_i(u) = \begin{cases} 2^{1-\frac{1}{\alpha_i}}(1+r)^{1-\frac{2}{\alpha_i}}\frac{1}{\alpha_i}\mathcal{H}_{\alpha_i}u^{\frac{2}{\alpha_i}-2}, & \text{if } \alpha_i\in(0,1),\\ \frac{2+r}{1+r}, & \text{if } \alpha_i = 1, \\ 1, & \text{if } \alpha_i \in (1,2), \end{cases} \qquad i=1,2.$$

Remarks: *a) The case $r=0$ can be confirmed by the fact that (cf.* [15] *or* [36]*), for a standard fBm B_α*

$$\mathbb{P}\left\{\sup_{t\in[0,1]} B_\alpha(t) > u\right\} = \frac{1}{\sqrt{2\pi}}\mathcal{F}_\alpha(u)u^{-1}\exp\left(-\frac{u^2}{2}\right)(1+o(1)), \qquad \text{as } u\to\infty, \tag{6}$$

where $\mathcal{F}_\alpha(u)$ is equal to $2^{1-1/\alpha}\alpha^{-1}\mathcal{H}_\alpha u^{2/\alpha-2}$ if $\alpha\in(0,1)$, 2 if $\alpha=1$, and 1 if $\alpha\in(1,2)$.

b) It follows from Theorem 1.1 and (6) *that $M_1 := \sup_{s\in[0,1]} X_1(s)$ and $M_2 := \sup_{t\in[0,1]} X_2(t)$ are asymptotically independent, i.e.,*

$$\lim_{u\to\infty} \frac{\mathbb{P}\{M_1>u, M_2>u\}}{\mathbb{P}\{M_2>u\}} = 0. \tag{7}$$

We refer to [16] *for the concept of the asymptotic independence of two random variables. The result in* (7) *improves that in Corollary 1.1 in* [39] *where an upper bound (1/2) for the left-hand side of* (7) *for two correlated Brownian motions was obtained.*

c) In Theorem 1.1 we considered the joint extremes of two standard fBm's on the time interval $[0,1]$. Next, we briefly discuss the case where the time interval is $[0,S]$, with S some positive constant. It follows by the self-similarity of the fBm's that, for $S^{\alpha_1-\alpha_2}\ge 1$ we have

$$\mathbb{P}\left\{\sup_{t\in[0,S]} X_1(s) > u,\ \sup_{t\in[0,S]} X_2(t) > u\right\} = \mathbb{P}\left\{M_1 > c(S^{-\frac{\alpha_2}{2}}u),\ M_2 > S^{-\frac{\alpha_2}{2}}u\right\},$$

with $c = S^{\frac{\alpha_2-\alpha_1}{2}} \in (0,1]$. *By slight modifications of the proofs of Theorem 1.1 and Lemma A we conclude that similar results can be obtained for the case* $c > r$. *However, the case* $c \le r$ *can not be dealt with similarly since we do not observe a similar result as Lemma A which is crucial for the double-sum method. It turns out that the case* $c \le r$ *may not be easily solved in general; new techniques will be explored for it elsewhere.*

In the framework of ruin theory, see, e.g., [4, 11, 18, 24, 25], [11] given that ruin happens one wants to know when it happens. With this motivation, we are interested to know when the first passages occur given that X_1, X_2 both ever pass a threshold $u > 0$ on $[0,1]$.
Define the first passage times of X_1, X_2 to the threshold u by

$$\tau_1(u) = \inf\{s \ge 0, X_1(s) > u\} \quad \text{and} \quad \tau_2(u) = \inf\{t \ge 0, X_2(t) > u\}, \tag{8}$$

respectively (here we use the common assumption that $\inf\{\emptyset\} = \infty$). Further, define $\tau_1^*(u),\ \tau_2^*(u), u > 0$ in the same probability space such that

$$(\tau_1^*(u), \tau_2^*(u)) \quad \stackrel{d}{=} \quad (\tau_1(u), \tau_2(u))\Big|(\tau_1(u) \le 1, \tau_2(u) \le 1), \tag{9}$$

where $\stackrel{d}{=}$ stands for equality of distribution functions. With motivation from the aforementioned contributions, our second principle result is concerned with the distributional approximation of the random vector $(\tau_1^*(u), \tau_2^*(u))$, as $u \to \infty$. Let $E_i, i = 1, 2$ be two independent unit exponential random variables, and denote by $\stackrel{d}{\to}$ the convergence in distribution.

Theorem 1.2. *Under the assumptions of Theorem 1.1 we have as* $u \to \infty$

$$\big(u^2(1-\tau_1^*(u)), u^2(1-\tau_2^*(u))\big) \quad \stackrel{d}{\to} \quad \left(\frac{2(1+r)}{\alpha_1}E_1, \frac{2(1+r)}{\alpha_2}E_2\right). \tag{10}$$

Remark: *Let* M_1, M_2 *be given as in Remark b) above. By the self-similarity of the fBm's we have for any* $x_1, x_2 \ge 0, u > 0$

$$\mathbb{P}\left\{M_1 > u + \frac{x_1}{u}, M_2 > u + \frac{x_2}{u}\Big|M_1 > u, M_2 > u\right\} = \frac{\mathbb{P}\left\{\sup_{s\in[0,1]} X_1(S_{1,u}s) > u, \sup_{t\in[0,1]} X_2(S_{2,u}t) > u\right\}}{\mathbb{P}\left\{M_1 > u, M_2 > u\right\}},$$

where $S_{i,u} = (1 + x_i u^{-2})^{-2/\alpha_i}, i = 1, 2, u > 0$. *Therefore, by a similar argument as in the proof of Theorem 1.2 we conclude that*

$$\lim_{u\to\infty} \mathbb{P}\left\{M_1 > u + \frac{x_1}{u}, M_2 > u + \frac{x_2}{u}\Big|M_1 > u, M_2 > u\right\} \quad = \quad \exp\left(-\frac{x_1 + x_2}{1 + r}\right). \tag{11}$$

In view of Theorem 4.1 in [20] *(see also Section 4.1 in* [21]*)*

$$\lim_{u\to\infty} \mathbb{P}\left\{X_1(1) > u + \frac{x_1}{u}, X_2(1) > u + \frac{x_2}{u}\Big|X_1(1) > u, X_2(1) > u\right\} \quad = \quad \exp\left(-\frac{x_1 + x_2}{1 + r}\right)$$

holds for any $x_1, x_2 \in [0, \infty)$, *from which we see that* (11) *is not surprising since the minimum of the function* $h(s,t), (s,t) \in (0,1]^2$ *given in* (19) *is attained at the unique point* $(1,1)$, *at which the processes usually contribute most to the asymptotics.*

Organization of the rest of the paper: In Section 2 we present some preliminary results including the Borell-TIS inequality and the Piterbarg inequality for 2-dimensional Gaussian random fields. The proofs of Theorems 1.1 and 1.2 are given in Section 3, while proofs of other results are relegated to Appendix.

2. Preliminaries

In the asymptotic theory of Gaussian processes, two of the important inequalities are the Borell-TIS inequality (cf. [1, 36]) and the Piterbarg inequality (cf. [36]). Let $\{Z(t), t \in \mathcal{K}\}$ be a centered Gaussian process with a.s. continuous sample paths, and let $\mathcal{K} \subset \mathbb{R}$ be a compact set with Lebesgue measure $\mathrm{mes}(\mathcal{K}) > 0$. The Borell-TIS inequality, which was proved by [6] and [41] independently, states that

$$\mathbb{P}\left\{\sup_{t\in\mathcal{K}} Z(t) > u\right\} \le \exp\left(-\frac{(u-\mu)^2}{2}\tau_m^2\right) \tag{12}$$

holds for any $u \ge \mu := \mathbb{E}\{\sup_{t\in\mathcal{K}} Z(t)\}$, with $\tau_m^2 := \inf_{t\in\mathcal{K}} (\mathbb{V}ar Z(t))^{-1} \in (0,\infty)$.
The upper bound in (12) might not be precise enough for various applications due to the appearance of the constant μ. V.I. Piterbarg obtained an upper bound under a global Hölder condition on the Gaussian process, which eliminates the constant μ; see e.g., Theorem 8.1 in [36] or Theorem 8.1 in [37]. Specifically, if there are some positive constants γ and G such that $\mathbb{E}\left\{(Z(t)-Z(t'))^2\right\} \le G|t-t'|^\gamma$ for all $t, t' \in \mathcal{K}$, then

$$\mathbb{P}\left\{\sup_{t\in\mathcal{K}} Z(t) > u\right\} \le C\ \mathrm{mes}(\mathcal{K})\ u^{\frac{2}{\gamma}-1}\exp\left(-\frac{u^2}{2}\tau_m^2\right) \tag{13}$$

holds for any u large enough, with some positive constant C not depending on u. The last inequality is commonly referred to as the Piterbarg inequality; see e.g., Proposition 3.2 in [40] for the case of chi-processes.
Next, let $\mathcal{V} \subset \mathbb{R}^2$ be a compact set, and let $\{(Z_1(t), Z_2(t)), t \ge 0\}$ be a 2-dimensional centered vector Gaussian process with components which have a.s. continuous sample paths. Motivated by the findings of [13, 38], we present in Theorem 2.1 and Theorem 2.2 generalizations of the Borell-TIS and Piterbarg inequalities for 2-dimensional Gaussian random fields $\{(Z_1(s), Z_2(t)), (s,t) \in \mathcal{V}\}$. As it will be seen from the proof of Theorem 1.1, the generalized Borell-TIS and Piterbarg inequalities are very powerful tools.

Theorem 2.1. *Let $\{Z_i(t), t \ge 0\}, i = 1, 2$ be two centered Gaussian processes with a.s. continuous sample paths, variance functions $\sigma_i(t), i = 1, 2$ being further jointly Gaussian with cross-correlation function $r(s,t) \in (-1,1)$. Then there exists a constant μ such that for $u \ge \mu$*

$$\mathbb{P}\left\{\bigcup_{(s,t)\in\mathcal{V}} \{Z_1(s) > u, Z_2(t) > u\}\right\} \le \exp\left(-\frac{(u-\mu)^2}{2}\tau_m^2\right), \tag{14}$$

where $\tau_m^2 = \inf_{(s,t)\in\mathcal{V}} \sigma^2(s,t) > 0$ with (below $I(\cdot)$ stands for the indicator function)

$$\sigma^2(s,t) = \frac{1}{\min(\sigma_1^2(s), \sigma_2^2(t))}\left(1 + \frac{(c(s,t)-r(s,t))^2}{1-r^2(s,t)} I(r(s,t) < c(s,t))\right),\ c(s,t) = \min\left(\frac{\sigma_1(s)}{\sigma_2(t)}, \frac{\sigma_2(t)}{\sigma_1(s)}\right). \tag{15}$$

In particular, if $r(s,t) < c(s,t)$ for all $(s,t) \in \mathcal{V}$, then (14) holds with

$$\tau_m^2 = \inf_{(s,t)\in\mathcal{V}} \frac{\sigma_1^2(s) + \sigma_2^2(t) - 2\sigma_1(s)\sigma_2(t)r(s,t)}{\sigma_1^2(s)\sigma_2^2(t)(1-r^2(s,t))} \tag{16}$$

and further, if $r(s,t) \ge c(s,t)$ for all $(s,t) \in \mathcal{V}$, then (14) holds with

$$\tau_m^2 = \inf_{(s,t)\in\mathcal{V}} \frac{1}{\min(\sigma_1^2(s), \sigma_2^2(t))}.$$

Theorem 2.2. *Let $\{Z_i(t), t \ge 0\}, i = 1, 2$ be as in Theorem 2.1. Assume that $\sigma_1(s), \sigma_2(t), r(s,t), (s,t) \in \mathcal{V}$ are all twice continuously differentiable with respect to their arguments. If there exist some positive constants γ and L such that the following global Hölder condition*

$$\mathbb{E}\left\{(Z_i(v_i) - Z_i(w_i))^2\right\} \le L|v_i - w_i|^\gamma, \quad i = 1, 2 \tag{17}$$

holds for all $(v_1, v_2), (w_1, w_2) \in \mathcal{V}$, *then for all* u *large*

$$\mathbb{P}\left\{ \bigcup_{(s,t)\in\mathcal{V}} \{Z_1(s) > u, Z_2(t) > u\} \right\} \le C\ mes(\mathcal{V})\ u^{\frac{4}{\gamma}-1} \exp\left(-\frac{u^2}{2}\tau_m^2\right), \tag{18}$$

where τ_m^2 *is given as in Theorem 2.1, and* C *is some positive constant not depending on* u*.*

Remark 2.3. *Assume that* $\mathcal{G} = \{(s,t) \in \mathcal{V} : (s,t) = arg\inf \sigma(s,t)\}$ *is a finite set. Define* $\mathcal{G}_\varepsilon = \bigcup_{(s,t)\in\mathcal{G}} ([s-\varepsilon, s+\varepsilon] \times [t-\varepsilon, t+\varepsilon] \cap \mathcal{V})$ *for any small positive* ε*. In view of the proof of Theorem 8.1 in* [36]*, the claim* (18) *still holds if* (17) *is valid for all* $(v_1, v_2), (w_1, w_2) \in \mathcal{G}_\varepsilon$ *for some small positive* ε*.*

Now, we come back to our principle problem of finding the exact asymptotics of $P_r(u)$ as $u \to \infty$. In view of the findings of [3, 13, 38] we deduce that the constant τ_m^2 given in (16) (restricted to fBm's case) should play a crucial role in the exact asymptotics of $P_r(u)$. Thus, we need to analyze the following function

$$h(s,t) = \frac{t^{\alpha_2} + s^{\alpha_1} - 2rs^{\frac{\alpha_1}{2}} t^{\frac{\alpha_2}{2}}}{s^{\alpha_1} t^{\alpha_2}(1-r^2)}, \quad s,t \in (0,1]. \tag{19}$$

The function $h(s,t), s,t \in (0,1]$ attains its minimum at the unique point $(s_0, t_0) = (1,1)$ and further $h(1,1) = \frac{2}{1+r}$.

Let $(\hat{s}_0, \hat{t}_0) := (\hat{s}_0(u), \hat{t}_0(u)), u > 0$ be a family of points in $[0,1]^2$ satisfying $1 - \hat{s}_0 \le (\ln u)^2/u^2$ and $1 - \hat{t}_0 \le (\ln u)^2/u^2$. For the use of the double-sum method, we need to deal with the asymptotics of the following joint survival function

$$R_{\Lambda_1,\Lambda_2}(u) := \mathbb{P}\left\{ \bigcup_{(s,t)\in K_u} \{X_1(s) > u, X_2(t) > u\} \right\}, \qquad \text{as } u \to \infty,$$

where $K_u = (\hat{s}_0, \hat{t}_0) + (u^{-2/\alpha_1}\Lambda_1, u^{-2/\alpha_2}\Lambda_2)$ with $\Lambda_i, i = 1,2$ two compact sets in $\mathbb{R}$. Here in our notation, for any $\Lambda \in \mathbb{R}$ $a\Lambda := \{ax : x \in \Lambda\}$, and for any $\Lambda \in \mathbb{R}^2$ $(x_1, x_2) + \Lambda := \{(x_1, x_2) + (y_1, y_2) : (y_1, y_2) \in \Lambda\}$.
The following lemma can be seen as a generalization of Pickands and Piterbarg lemmas (cf. [22, 33, 35, 36]) for 2-dimensional Gaussian random fields. Its proof is presented in Appendix.
Lemma A. *Let* $\{X_i(t), t \ge 0\}, i = 1,2$ *be two standard fBm's with Hurst indexes* $\alpha_i/2 \in (0, 1/2], i = 1,2,$ *respectively. Assume further that the joint correlation function of them is a constant* $r \in (-1,1)$*. Then as* $u \to \infty$

$$R_{\Lambda_1,\Lambda_2}(u) = \mathcal{Q}_{\alpha_1}[\Lambda_1]\mathcal{Q}_{\alpha_2}[\Lambda_2] \frac{(1+r)^{\frac{3}{2}}}{2\pi\sqrt{1-r}} u^{-2} \exp\left(-\frac{u^2}{2} h(\hat{s}_0, \hat{t}_0)\right)(1 + o(1)), \tag{20}$$

where $h(\cdot,\cdot)$ *is given as in* (19)*, and*

$$\mathcal{Q}_{\alpha_i}[\Lambda_i] = \begin{cases} \mathcal{H}^0_{\alpha_i}\left[\Lambda_i \left(\frac{1}{\sqrt{2}(1+r)}\right)^{\frac{2}{\alpha_i}}\right], & \text{if } \alpha_i \in (0,1), \\ \mathcal{H}_1^{-(1+r)}\left[\Lambda_i \left(\frac{1}{\sqrt{2}(1+r)}\right)^2\right], & \text{if } \alpha_i = 1, \end{cases} \qquad i = 1,2.$$

3. Proofs of Theorems 1.1 and 1.2

In this section, we first present the proof of Theorem 1.1 which is based on a tailored double-sum method as in [3]; see the classical monograph [36] for a deep explanation on the double-sum method. Then we present the proof of Theorem 1.2.
Proof of Theorem 1.1: Let $\delta(u) = (\ln u)^2/u^2$, and set $D_u = \{(s,t) \in [0,1]^2 : 1 - s \le \delta(u), 1 - t \le \delta(u)\}$. With these notation we have

$$P_{1,r}(u) := \mathbb{P}\left\{ \bigcup_{(s,t)\in D_u} \{X_1(s) > u, X_2(t) > u\} \right\} \quad \le \quad P_r(u)$$

$$
\begin{aligned}
&\leq P_{1,r}(u) + \mathbb{P}\left\{\bigcup_{(s,t)\in[0,1]^2/D_u} \{X_1(s) > u, X_2(t) > u\}\right\}\\
&=: P_{1,r}(u) + P_{2,r}(u).
\end{aligned}
$$

Next, we shall derive the exact asymptotics of $P_{1,r}(u)$ as $u \to \infty$, and show that

$$
P_{2,r}(u) = o(P_{1,r}(u)), \qquad u \to \infty \tag{21}
$$

implying thus

$$
P_r(u) = P_{1,r}(u)(1 + o(1)) \qquad u \to \infty.
$$

Next, we derive an upper bound for $P_{2,r}(u)$ by utilising the generalized Borell-TIS and Piterbarg inequalities. Choose some small $\varepsilon \in (0,1)$ such that

$$
\hat{c}(s,t) := \min\left(\frac{t^{\alpha_2/2}}{s^{\alpha_1/2}}, \frac{s^{\alpha_1/2}}{t^{\alpha_2/2}}\right) > r, \quad \forall (s,t) \in [1-\varepsilon, 1]^2. \tag{22}
$$

Clearly, for any u positive

$$
P_{2,r}(u) \quad \leq \quad \mathbb{P}\left\{\bigcup_{(s,t)\in[0,1]^2/[1-\varepsilon,1]^2} \{X_1(s) > u, X_2(t) > u\}\right\} + \mathbb{P}\left\{\bigcup_{(s,t)\in[1-\varepsilon,1]^2/D_u} \{X_1(s) > u, X_2(t) > u\}\right\}.
$$

It follows from the Borell-TIS inequality in Theorem 2.1 that for all u large

$$
\mathbb{P}\left\{\bigcup_{(s,t)\in[0,1]^2/[1-\varepsilon,1]^2} \{X_1(s) > u, X_2(t) > u\}\right\} \leq \exp\left(-\frac{(u-\mu)^2}{2} \inf_{(s,t)\in(0,1]^2/[1-\varepsilon,1]^2} f(s,t)\right), \tag{23}
$$

where $\mu \in (0, \infty)$ is some constant and

$$
f(s,t) = \frac{1}{\min\left(s^{\alpha_1}, t^{\alpha_2}\right)}\left(1 + \frac{(\hat{c}(s,t) - r)^2}{1 - r^2} I(r < \hat{c}(s,t))\right), \quad (s,t) \in (0,1]^2/[1-\varepsilon,1]^2.
$$

Further, straightforward calculations yield that (recall (19) for the expression of $h(\cdot,\cdot)$)

$$
\inf_{(s,t)\in(0,1]^2/[1-\varepsilon,1]^2} f(s,t) > h(1,1) = \frac{2}{1+r}.
$$

Moreover, in view of (22) we have from the Piterbarg inequality in Theorem 2.2 and its remark that, for all u large

$$
\mathbb{P}\left\{\bigcup_{(s,t)\in[1-\varepsilon,1]^2/D_u} \{X_1(s) > u, X_2(t) > u\}\right\} \leq C u^{\frac{4}{\min(\alpha_1,\alpha_2)} - 1} \exp\left(-\frac{u^2}{2} \inf_{(s,t)\in[1-\varepsilon,1]^2/D_u} h(s,t)\right),
$$

with $C > 0$ not depending on u. In addition from the Taylor expansion of $h(s,t)$ around the point $(1,1)$ we have

$$
h(s,t) = h(1,1) + \frac{1}{1+r}\left(\alpha_1(1-s) + \alpha_2(1-t)\right)(1 + o(1)).
$$

Hence, for the chosen small enough $\varepsilon > 0$ there exists some positive constant C_1 such that

$$
h(s,t) \geq h(1,1) + C_1 \delta(u)
$$

for any $(s,t) \in [1-\varepsilon, 1]^2/D_u$, implying thus, for all u large

$$
\mathbb{P}\left\{\bigcup_{(s,t)\in[1-\varepsilon,1]^2/D_u} \{X_1(s) > u, X_2(t) > u\}\right\} \leq C u^{\frac{4}{\min(\alpha_1,\alpha_2)} - 1} \psi_r(u) \exp\left(-\frac{C_1}{2}(\ln u)^2\right), \tag{24}
$$

where we set

$$
\psi_r(u) := \exp\left(-\frac{u^2}{2} h(1,1)\right) = \exp\left(-\frac{u^2}{1+r}\right).
$$

Consequently, from (23) and (24) we obtain the following upper bound for $P_{2,r}(u)$ when u is large

$$P_{2,r}(u) \le \exp\left(-\frac{(u-\mu)^2}{2}\inf_{(s,t)\in[0,1]^2/[1-\varepsilon,1]^2} f(s,t)\right) + Cu^{\frac{4}{\min(\alpha_1,\alpha_2)}-1}\psi_r(u)\exp\left(-\frac{C_1}{2}(\ln u)^2\right). \tag{25}$$

From now on we focus on the asymptotics of $P_{1,r}(u)$ as $u\to\infty$. Let T_1, T_2 be two positive constants. For $\alpha_i \le 1, i = 1,2$, we can split the rectangle D_u into several sub-rectangles of side lengths T_1u^{-2/α_1} and T_2u^{-2/α_2}. Specifically, let

$$\triangle_{k,l} = \triangle_k^1 \times \triangle_l^2 = [s_{k+1}, s_k]\times[t_{l+1}, t_l], \quad k,l\in\mathbb{N}\bigcup\{0\},$$

with $s_k = 1 - kT_1u^{-2/\alpha_1}$ and $t_l = 1 - lT_2u^{-2/\alpha_2}$, and further set

$$N_i(u) = \left\lfloor T_i^{-1}(\ln u)^2 u^{\frac{2}{\alpha_i}-2}\right\rfloor + 1, \quad i = 1,2.$$

Here $\lfloor\cdot\rfloor$ denotes the ceiling function. Thus

$$\bigcup_{k=0}^{N_1(u)-1}\bigcup_{l=0}^{N_2(u)-1}\triangle_{k,l} \subset D_u \subset \bigcup_{k=0}^{N_1(u)}\bigcup_{l=0}^{N_2(u)}\triangle_{k,l}. \tag{26}$$

In what follows, we deal with only three cases (distinguished by α_i's):

Case i) $\alpha_1\in(0,1)$ and $\alpha_2\in(0,1)$. Applying the Bonferroni inequality in Lemma B (given in Appendix) we obtain

$$P_{1,r}(u) \le \sum_{k=0}^{N_1(u)}\sum_{l=0}^{N_2(u)}\mathbb{P}\left\{\sup_{s\in\triangle_k^1}X_1(s)>u, \sup_{t\in\triangle_l^2}X_2(t)>u\right\}$$

and

$$P_{1,r}(u) \quad\ge\quad \sum_{k=0}^{N_1(u)-1}\sum_{l=0}^{N_2(u)-1}\mathbb{P}\left\{\sup_{s\in\triangle_k^1}X_1(s)>u, \sup_{t\in\triangle_l^2}X_2(t)>u\right\} - \Sigma_1(u) - \Sigma_2(u), \tag{27}$$

where

$$\Sigma_1(u) = \sum_{k=0}^{N_1(u)}\sum\sum_{0\le l_1<l_2\le N_2(u)}\mathbb{P}\left\{\sup_{s\in\triangle_k^1}X_1(s)>u, \sup_{t\in\triangle_{l_1}^2}X_2(t)>u, \sup_{t\in\triangle_{l_2}^2}X_2(t)>u\right\},$$

$$\Sigma_2(u) = \sum_{l=0}^{N_2(u)}\sum\sum_{0\le k_1<k_2\le N_1(u)}\mathbb{P}\left\{\sup_{s\in\triangle_{k_1}^1}X_1(s)>u, \sup_{s\in\triangle_{k_2}^1}X_1(s)>u, \sup_{t\in\triangle_l^2}X_2(t)>u\right\}.$$

Further, in view of Lemma A

$$\begin{aligned} P_{1,r}(u) \quad\le\quad & \mathcal{H}_{\alpha_1}^0\left[[-T_1,0]\left(\frac{1}{\sqrt{2}(1+r)}\right)^{\frac{2}{\alpha_1}}\right]\mathcal{H}_{\alpha_2}^0\left[[-T_2,0]\left(\frac{1}{\sqrt{2}(1+r)}\right)^{\frac{2}{\alpha_2}}\right] \\ & \times\frac{(1+r)^{\frac{3}{2}}}{2\pi\sqrt{1-r}}u^{-2}\sum_{k=0}^{N_1(u)}\sum_{l=0}^{N_2(u)}\exp\left(-\frac{u^2}{2}h(s_k,t_l)\right)(1+o(1)) \end{aligned}$$

as $u\to\infty$. Since by Taylor expansion

$$h(s_k,t_l) = h(1,1) + \frac{1}{1+r}\left(\alpha_1(1-s_k)+\alpha_2(1-t_l)\right)(1+o(1)), \quad u\to\infty$$

we have

$$\sum_{k=0}^{N_1(u)}\sum_{l=0}^{N_2(u)}\exp\left(-\frac{u^2}{2}h(s_k,t_l)\right) \quad=\quad \psi_r(u)\frac{u^{\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-4}}{T_1T_2}\prod_{j=1}^{2}\left(\int_0^\infty\exp\left(-\frac{\alpha_j}{2(1+r)}x\right)dx\right)(1+o(1)).$$

Therefore, as $u\to\infty$

$$P_{1,r}(u) \quad\le\quad \frac{2^{1-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}}(1+r)^{\frac{7}{2}-\frac{2}{\alpha_1}-\frac{2}{\alpha_2}}}{\pi\alpha_1\alpha_2\sqrt{1-r}}\frac{\mathcal{H}_{\alpha_1}^0[0,b_1T_1]}{b_1T_1}\frac{\mathcal{H}_{\alpha_2}^0[0,b_2T_2]}{b_2T_2}u^{\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-6}\psi_r(u)(1+o(1)), \tag{28}$$

where $b_i=\left(1/(\sqrt{2}(1+r))\right)^{2/\alpha_i}, i=1,2$. The same arguments yield that

$$\sum_{k=0}^{N_1(u)-1}\sum_{l=0}^{N_2(u)-1}\mathbb{P}\left\{\sup_{s\in\triangle_k^1}X_1(s)>u,\sup_{t\in\triangle_l^2}X_2(t)>u\right\} = \frac{2^{1-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}}(1+r)^{\frac{7}{2}-\frac{2}{\alpha_1}-\frac{2}{\alpha_2}}}{\pi\alpha_1\alpha_2\sqrt{1-r}}\frac{\mathcal{H}^0_{\alpha_1}[0,b_1T_1]}{b_1T_1}\frac{\mathcal{H}^0_{\alpha_2}[0,b_2T_2]}{b_2T_2} \times u^{\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-6}\psi_r(u)(1+o(1)) \tag{29}$$

as $u\to\infty$. Next, we consider the estimates of $\Sigma_i(u), i=1,2$. To this end, we define, for any $T,T_0\in(0,\infty)$

$$\mathcal{H}^0_\alpha([0,T],[T_0,T_0+T])=\int_{-\infty}^{\infty}\exp(x)\mathbb{P}\left\{\sup_{t\in[0,T]}\sqrt{2}B_\alpha(t)-|t|^\alpha>x,\sup_{t\in[T_0,T_0+T]}\sqrt{2}B_\alpha(t)-|t|^\alpha>x\right\}dx,\quad \alpha\in(0,2)$$

and denote, for any $n\ge1$

$$\mathcal{H}^0_\alpha(n;T)=\mathcal{H}^0_\alpha([0,T],[nT,(n+1)T]).$$

It follows from Lemma 3 in [3] or Lemmas 6 and 7 in [23] that

$$\lim_{T\to\infty}\frac{\sum_{n=1}^\infty\mathcal{H}^0_\alpha(n;T)}{T}=0. \tag{30}$$

Since

$$\begin{aligned}&\mathbb{P}\left\{\sup_{s\in\triangle_k^1}X_1(s)>u,\sup_{t\in\triangle_{l_1}^2}X_2(t)>u,\sup_{t\in\triangle_{l_2}^2}X_2(t)>u\right\}\\&=\mathbb{P}\left\{\sup_{s\in\triangle_k^1}X_1(s)>u,\sup_{t\in\triangle_{l_1}^2}X_2(t)>u\right\}+\mathbb{P}\left\{\sup_{s\in\triangle_k^1}X_1(s)>u,\sup_{t\in\triangle_{l_2}^2}X_2(t)>u\right\}\\&-\mathbb{P}\left\{\sup_{s\in\triangle_k^1}X_1(s)>u,\sup_{t\in\triangle_{l_1}^2\bigcup\triangle_{l_2}^2}X_2(t)>u\right\}\end{aligned}$$

similar arguments as in the derivation of (28) imply that

$$\Sigma_1(u) \le \frac{2^{1-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}}(1+r)^{\frac{7}{2}-\frac{2}{\alpha_1}-\frac{2}{\alpha_2}}}{\pi\alpha_1\alpha_2\sqrt{1-r}}\frac{\mathcal{H}^0_{\alpha_1}[0,b_1T_1]}{b_1T_1}\sum_{n=1}^\infty\frac{\mathcal{H}^0_{\alpha_2}[n;b_2T_2]}{b_2T_2} \times u^{\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-6}\psi_r(u)(1+o(1)). \tag{31}$$

Similarly

$$\Sigma_2(u) \le \frac{2^{1-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}}(1+r)^{\frac{7}{2}-\frac{2}{\alpha_1}-\frac{2}{\alpha_2}}}{\pi\alpha_1\alpha_2\sqrt{1-r}}\sum_{n=1}^\infty\frac{\mathcal{H}^0_{\alpha_1}[n;b_1T_1]}{b_1T_1}\frac{\mathcal{H}^0_{\alpha_2}[0,b_2T_2]}{b_2T_2} \times u^{\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-6}\psi_r(u)(1+o(1)). \tag{32}$$

Consequently, from (28-32) by letting $T_1,T_2\to\infty$ we obtain

$$P_{1,r}(u) = \frac{2^{1-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}}(1+r)^{\frac{7}{2}-\frac{2}{\alpha_1}-\frac{2}{\alpha_2}}}{\pi\alpha_1\alpha_2\sqrt{1-r}}\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}u^{\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-6}\psi_r(u)(1+o(1)) \quad \text{as } u\to\infty. \tag{33}$$

<u>Case ii) $\alpha_1\in(0,1)$ and $\alpha_2=1$</u>. Applying the Bonferroni inequality we have

$$\begin{aligned}P_{1,r}(u) \le& \sum_{k=0}^{N_1(u)}\mathbb{P}\left\{\sup_{s\in\triangle_k^1}X_1(s)>u,\sup_{t\in\triangle_0^2}X_2(t)>u\right\}\\&+\sum_{k=0}^{N_1(u)}\sum_{l=1}^{N_2(u)}\mathbb{P}\left\{\sup_{s\in\triangle_k^1}X_1(s)>u,\sup_{t\in\triangle_l^2}X_2(t)>u\right\}\end{aligned}$$

and

$$P_{1,r}(u) \quad \geq \quad \sum_{k=0}^{N_1(u)-1} \mathbb{P}\left\{ \sup_{s\in\triangle_k^1} X_1(s) > u, \sup_{t\in\triangle_0^2} X_2(t) > u \right\} - \Sigma_3(u),$$

where

$$\Sigma_3(u) = \sum\sum_{0\leq k_1<k_2\leq N_1(u)} \mathbb{P}\left\{ \sup_{s\in\triangle_{k_1}^1} X_1(s) > u, \sup_{s\in\triangle_{k_2}^1} X_1(s) > u, \sup_{t\in\triangle_0^2} X_2(t) > u \right\}.$$

By Lemma A

$$\begin{aligned}
&\sum_{k=0}^{N_1(u)(\text{or } N_1(u)-1)} \mathbb{P}\left\{ \sup_{s\in\triangle_k^1} X_1(s) > u, \sup_{t\in\triangle_0^2} X_2(t) > u \right\} \\
&= \mathcal{H}_{\alpha_1}^0[0,b_1T_1]\mathcal{H}_1^{1+r}[0,b_2T_2]\frac{(1+r)^{\frac{3}{2}}}{2\pi\sqrt{1-r}}u^{-2}\sum_{k=0}^{N_1(u)} \exp\left(-\frac{u^2}{2}h(s_k,1)\right)(1+o(1)) \\
&= \frac{2^{-\frac{1}{\alpha_1}}(1+r)^{\frac{5}{2}-\frac{2}{\alpha_1}}}{\pi\alpha_1\sqrt{1-r}}\frac{\mathcal{H}_{\alpha_1}^0[0,b_1T_1]}{b_1T_1}\mathcal{H}_1^{1+r}[0,b_2T_2]u^{\frac{2}{\alpha_1}-4}\psi_r(u)(1+o(1)) \qquad (34)
\end{aligned}$$

as $u\to\infty$, where $b_i, i=1,2$ are the same as in (28). Similarly

$$\begin{aligned}
&\sum_{k=0}^{N_1(u)}\sum_{l=1}^{N_2(u)} \mathbb{P}\left\{ \sup_{s\in\triangle_k^1} X_1(s) > u, \sup_{t\in\triangle_l^2} X_2(t) > u \right\} \\
&= \frac{2^{-\frac{1}{\alpha_1}}(1+r)^{\frac{5}{2}-\frac{2}{\alpha_1}}}{\pi\alpha_1\sqrt{1-r}}\frac{\mathcal{H}_{\alpha_1}^0[0,b_1T_1]}{b_1T_1}\mathcal{H}_1^0[0,b_2T_2]\sum_{l=1}^{\infty}\exp\left(-\frac{T_2 l}{2(1+r)}\right)u^{\frac{2}{\alpha_1}-4}\psi_r(u)(1+o(1)) \qquad (35)
\end{aligned}$$

as $u\to\infty$. Moreover, it follows with similar arguments as in (31) that

$$\Sigma_3(u) \leq \frac{2^{-\frac{1}{\alpha_1}}(1+r)^{\frac{5}{2}-\frac{2}{\alpha_1}}}{\pi\alpha_1\sqrt{1-r}}\sum_{n=1}^{\infty}\frac{\mathcal{H}_{\alpha_1}^0[n;b_1T_1]}{b_1T_1}\mathcal{H}_1^{1+r}[0,b_2T_2]u^{\frac{2}{\alpha_1}-4}\psi_r(u)(1+o(1)) \qquad (36)$$

as $u\to\infty$. Consequently, letting $T_1,T_2\to\infty$ from (34-36) we have

$$P_{1,r}(u) = \frac{2^{-\frac{1}{\alpha_1}}(2+r)(1+r)^{\frac{3}{2}-\frac{2}{\alpha_1}}}{\pi\alpha_1\sqrt{1-r}}\mathcal{H}_{\alpha_1}u^{\frac{2}{\alpha_1}-4}\psi_r(u)(1+o(1)) \qquad \text{as } u\to\infty,$$

where we used the fact that $\mathcal{H}_1^{1+r} := \lim_{T\to\infty}\mathcal{H}_1^{1+r}[\Lambda_T] = (2+r)/(1+r)$; see e.g., [14] or [22].
Case iii) $\alpha_1\in(0,1)$ and $\alpha_2\in(1,2)$. Since $\alpha_2>1$, it follows that $\delta(u)\subset\triangle_0^2$. Thus

$$P_{1,r}(u) \quad \leq \quad \sum_{k=0}^{N_1(u)} \mathbb{P}\left\{ \sup_{s\in\triangle_k^1} X_1(s) > u, \sup_{t\in\triangle_0^2} X_2(t) > u \right\}$$

and further

$$P_{1,r}(u) \quad \geq \quad \sum_{k=0}^{N_1(u)-1} \mathbb{P}\left\{ \sup_{s\in\triangle_k^1} X_1(s) > u, X_2(1) > u \right\} - \Sigma_4(u),$$

where

$$\Sigma_4(u) = \sum\sum_{0\leq k_1<k_2\leq N_1(u)} \mathbb{P}\left\{ \sup_{s\in\triangle_{k_1}^1} X_1(s) > u, \sup_{s\in\triangle_{k_2}^1} X_1(s) > u, X_2(1) > u \right\}.$$

Using the same technique as in the proof of Lemma A (or let $T_2\to 0$ therein), we can show that

$$\begin{aligned}
\mathbb{P}\left\{ \sup_{s\in\triangle_k^1} X_1(s) > u, X_2(1) > u \right\} \quad &= \quad \mathcal{H}_{\alpha_1}^0\left[[-T_1,0]\left(\frac{1}{\sqrt{2}(1+r)}\right)^{\frac{2}{\alpha_1}}\right] \\
&\quad \times\frac{(1+r)^{\frac{3}{2}}}{2\pi\sqrt{1-r}}u^{-2}\exp\left(-\frac{u^2}{2}h(s_k,1)\right)(1+o(1))
\end{aligned}$$

as $u \to \infty$, implying

$$\sum_{k=0}^{N_1(u)-1} \mathbb{P}\left\{ \sup_{s \in \triangle_k^1} X_1(s) > u, X_2(1) > u \right\} = \frac{2^{-\frac{1}{\alpha_1}}(1+r)^{\frac{5}{2}-\frac{2}{\alpha_1}}}{\pi \alpha_1 \sqrt{1-r}} \frac{\mathcal{H}^0_{\alpha_1}[0, b_1 T_1]}{b_1 T_1} u^{\frac{2}{\alpha_1}-4} \psi_r(u)(1+o(1)), \quad u \to \infty. \tag{37}$$

Moreover

$$\Sigma_4(u) \le \frac{2^{-\frac{1}{\alpha_1}}(1+r)^{\frac{5}{2}-\frac{2}{\alpha_1}}}{\pi \alpha_1 \sqrt{1-r}} \sum_{n=1}^{\infty} \frac{\mathcal{H}^0_{\alpha_1}[n; b_1 T_1]}{b_1 T_1} u^{\frac{2}{\alpha_1}-4} \psi_r(u)(1+o(1)) \tag{38}$$

as $u \to \infty$. Consequently, letting $T_1 \to \infty, T_2 \to 0$ we conclude from (34), (37) and (38) that

$$P_{1,r}(u) = \frac{2^{-\frac{1}{\alpha_1}}(1+r)^{\frac{5}{2}-\frac{2}{\alpha_1}}}{\pi \alpha_1 \sqrt{1-r}} \mathcal{H}_{\alpha_1} u^{\frac{2}{\alpha_1}-4} \psi_r(u)(1+o(1)) \qquad \text{as } u \to \infty.$$

With all the techniques used in the proofs of Cases i)-iii) we see that the other cases for the possible choices of α_1 and α_2 can be shown similarly without any further difficulty, thus the detailed proofs are omitted. Moreover, it follows from (25) and the asymptotics of $P_{1,r}(u)$ in any of the remaining cases that (21) holds, and thus the proof is complete. $\square$

Proof of Theorem 1.2: First note that, for any $x_1, x_2 \ge 0, u > 0$

$$\mathbb{P}\left\{ u^2(1-\tau_1^*(u)) > x_1, u^2(1-\tau_2^*(u)) > x_2 \right\} = \frac{\mathbb{P}\left\{ \sup_{s \in [0, T_{1,u}]} X_1(s) > u,\ \sup_{t \in [0, T_{2,u}]} X_2(t) > u \right\}}{\mathbb{P}\left\{ \sup_{s \in [0,1]} X_1(s) > u,\ \sup_{t \in [0,1]} X_2(t) > u \right\}},$$

with $T_{i,u} = 1 - x_i u^{-2}, i = 1, 2$. Further, we write

$$\mathbb{P}\left\{ \sup_{s \in [0, T_{1,u}]} X_1(s) > u, \quad \sup_{t \in [0, T_{2,u}]} X_2(t) > u \right\} = \mathbb{P}\left\{ \sup_{s \in [0,1]} \widetilde{X_1}(s) > u, \ \sup_{t \in [0,1]} \widetilde{X_2}(t) > u \right\},$$

where $\widetilde{X_i}(t) := X_i(T_{i,u} t), t \in [0,1]$. Define $\widetilde{h_u}(s,t) := h(T_{1,u} s, T_{2,u} t), (s,t) \in (0,1]^2$, with $h(\cdot,\cdot)$ given as in (19). It follows from a slight modification of the proof of Lemma A that (20) holds for $\widetilde{X_1}, \widetilde{X_2}$, without any other changes apart from that $h(\cdot,\cdot)$ is replaced by $\widetilde{h_u}(\cdot,\cdot)$. With this modification of Lemma A, by a similar argument as in the proof of Theorem 1.1 we conclude that, as $u \to \infty$

$$\mathbb{P}\left\{ \sup_{s \in [0,1]} \widetilde{X_1}(s) > u, \ \sup_{t \in [0,1]} \widetilde{X_2}(t) > u \right\} = \frac{(1+r)^{\frac{3}{2}}}{2\pi\sqrt{1-r}} \Upsilon_1(u) \Upsilon_2(u) u^{-2} \exp\left(-\frac{u^2}{2} \widetilde{h_u}(1,1) \right)(1+o(1)), \tag{39}$$

where $\Upsilon_i(u), i = 1, 2$ are given as in Theorem 1.1. Consequently, from the last formula and Theorem 1.1, for any $x_1, x_2 \ge 0$

$$\begin{aligned} \mathbb{P}\left\{ u^2(1-\tau_1^*(u)) > x_1, u^2(1-\tau_2^*(u)) > x_2 \right\} &= \exp\left(-\frac{u^2}{2} (\widetilde{h_u}(1,1) - h(1,1)) \right)(1+o(1)) \\ &\to \exp\left(-\left(\frac{\alpha_1}{2(1+r)} x_1 + \frac{\alpha_2}{2(1+r)} x_2 \right) \right), \quad u \to \infty \end{aligned}$$

establishing thus the claim, and hence the proof is complete. $\square$

4. Appendix

Below we present the proofs of Theorem 2.1, Theorem 2.2 and Lemma A. We also state and prove Lemma B which is of some interest on its own.

Proof of Theorem 2.1: Denote

$$A(s,t) = \sigma_1^2(s) + \sigma_2^2(t) - 2\sigma_1(s)\sigma_2(t) r(s,t), \quad (s,t) \in \mathcal{V}.$$

Next, we introduce two nonnegative functions $a(s,t), b(s,t), (s,t) \in \mathcal{V}$ as follows

$$a(s,t) = \begin{cases} \frac{\sigma_2^2(t)-\sigma_1(s)\sigma_2(t)r(s,t)}{A(s,t)}, & \text{if } c(s,t) > r(s,t), \\ 1, & \text{if } c(s,t) \le r(s,t) \text{ and } \sigma_1(s) \le \sigma_2(t), \\ 0, & \text{otherwise,} \end{cases} \qquad (s,t) \in \mathcal{V}$$

and

$$b(s,t) = \begin{cases} \frac{\sigma_1^2(s)-\sigma_1(s)\sigma_2(t)r(s,t)}{A(s,t)}, & \text{if } c(s,t) > r(s,t), \\ 1, & \text{if } c(s,t) \le r(s,t) \text{ and } \sigma_2(t) < \sigma_1(s), \\ 0, & \text{otherwise,} \end{cases} \qquad (s,t) \in \mathcal{V}.$$

Since $a(s,t) + b(s,t) = 1, (s,t) \in \mathcal{V}$, it follows that

$$\begin{aligned} &\mathbb{P}\left\{ \bigcup_{(s,t)\in\mathcal{V}} \left\{ Z_1(s) > u, Z_2(t) > u \right\} \right\} \\ &\le \mathbb{P}\left\{ \bigcup_{(s,t)\in\mathcal{V}} \{a(s,t)Z_1(s) + b(s,t)Z_2(t) > a(s,t)u + b(s,t)u\} \right\} \\ &= \mathbb{P}\left\{ \sup_{(s,t)\in\mathcal{V}} Y(s,t;a,b) > u \right\} \end{aligned} \tag{40}$$

where

$$Y(s,t;a,b) = a(s,t)Z_1(s) + b(s,t)Z_2(t), \qquad (s,t) \in \mathcal{V}.$$

Since further

$$\begin{aligned} (\mathbb{E}\left\{(Y(s,t;a,b))^2\right\})^{-1} &= \frac{1}{a^2(s,t)\sigma_1^2(s) + b^2(s,t)\sigma_2^2(t) + 2a(s,t)b(s,t)\sigma_1(s)\sigma_2(t)r(s,t)} \\ &= \frac{\sigma_1^2(s) + \sigma_2^2(t) - 2\sigma_1(s)\sigma_2(t)r(s,t)}{\sigma_1^2(s)\sigma_2^2(t)(1-r^2(s,t))} I(c(s,t) > r(s,t)) \\ &\quad + \frac{1}{\min(\sigma_1^2(s), \sigma_2^2(t))} I(c(s,t) \le r(s,t)) \end{aligned}$$

the claim follows from the Borell-TIS inequality for one-dimensional Gaussian random fields (e.g., [1]) with

$$\mu = \mathbb{E}\left\{ \sup_{(s,t)\in\mathcal{V}} Y(s,t;a,b) \right\} < \infty$$

and thus the proof is complete. □

Proof of Theorem 2.2: We use the same notation as in the proof of Theorem 2.1. In the light of (40) and Theorem 8.1 in [36], it suffices to show that

$$\mathbb{E}\left\{(Y(s,t;a,b) - Y(s',t';a,b))^2\right\} \le L_1(|s-s'|^\gamma + |t-t'|^\gamma), \quad \forall (s,t),(s',t') \in \mathcal{V} \tag{41}$$

holds for some positive constants L_1 and γ, which can be confirmed by some straightforward calculations, and thus the claim follows. □

Proof of Lemma A: Using the classical technique, see e.g., [3, 23, 36], we have for any $u > 0$

$$\begin{aligned} R_{\Lambda_1,\Lambda_2}(u) &= \frac{1}{u^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{P}\left\{ \bigcup_{(s,t)\in K_u} \{X_1(s) > u, X_2(t) > u\} | X_1(\hat{s}_0) = u - \frac{x}{u}, X_2(\hat{t}_0) = u - \frac{y}{u} \right\} \\ &\quad \times f_{X_1(\hat{s}_0),X_2(\hat{t}_0)}\left(u - \frac{x}{u}, u - \frac{y}{u}\right) dxdy, \end{aligned} \tag{42}$$

where

$$f_{X_1(\hat{s}_0),X_2(\hat{t}_0)}\left(u-\frac{x}{u},u-\frac{y}{u}\right)=\frac{1}{2\pi\sqrt{\hat{s}_0^{\alpha_1}\hat{t}_0^{\alpha_2}(1-r^2)}}\exp\left(-\frac{1}{2\hat{s}_0^{\alpha_1}\hat{t}_0^{\alpha_2}(1-r^2)u^2}\left(\hat{t}_0^{\alpha_2}x^2+\hat{s}_0^{\alpha_1}y^2-2r\hat{s}_0^{\alpha_1/2}\hat{t}_0^{\alpha_2/2}xy\right)\right)$$

$$\times\exp\left(-\frac{1}{2\hat{s}_0^{\alpha_1}\hat{t}_0^{\alpha_2}(1-r^2)}\left(-2\hat{t}_0^{\alpha_2}x-2\hat{s}_0^{\alpha_1}y+2r\hat{s}_0^{\alpha_1/2}\hat{t}_0^{\alpha_2/2}(x+y)\right)\right)\exp\left(-\frac{u^2}{2}h(\hat{s}_0,\hat{t}_0)\right),\quad x,y\in\mathbb{R},$$

where $h(\cdot,\cdot)$ is defined as in (19). Set for $x,y\in\mathbb{R}$

$$\xi_u(s)=u(X_1(\hat{s}_0+u^{-\frac{2}{\alpha_1}}s)-u)+x,\quad \eta_u(t)=u(X_2(\hat{t}_0+u^{-\frac{2}{\alpha_2}}t)-u)+y.$$

The probability in the integrand of (42) can be rewritten as

$$p_u(x,y)=\mathbb{P}\left\{\bigcup_{(s,t)\in\Lambda_1\times\Lambda_2}\{\xi_u(s)>x,\eta_u(t)>y\}|\xi_u(0)=0,\eta_u(0)=0\right\}.$$

Next, we calculate the expectation and covariance of the conditional random vector $(\xi_u(s),\eta_u(t))|(\xi_u(0),\eta_u(0))$. We have

$$\mathbb{E}\left\{\begin{array}{c|c}\xi_u(s) & \xi_u(0)\\ \eta_u(t) & \eta_u(0)\end{array}\right\}=\mathbb{E}\left\{\begin{array}{c}\xi_u(s)\\ \eta_u(t)\end{array}\right\}+A\left(\begin{array}{c}\xi_u(0)-\mathbb{E}\{\xi_u(0)\}\\ \eta_u(0)-\mathbb{E}\{\eta_u(0)\}\end{array}\right),$$

where

$$A=\mathbb{C}ov\left(\left(\begin{array}{c}\xi_u(s)\\ \eta_u(t)\end{array}\right),\left(\begin{array}{c}\xi_u(0)\\ \eta_u(0)\end{array}\right)\right)\times\mathbb{C}ov\left(\begin{array}{c}\xi_u(0)\\ \eta_u(0)\end{array}\right)^{-1}$$

and further

$$\mathbb{C}ov\left(\begin{array}{c|c}\xi_u(t)-\xi_u(s) & \xi_u(0)\\ \eta_u(t_1)-\eta_u(s_1) & \eta_u(0)\end{array}\right)=\mathbb{C}ov\left(\begin{array}{c}\xi_u(t)-\xi_u(s)\\ \eta_u(t_1)-\eta_u(s_1)\end{array}\right)+B\ \mathbb{C}ov\left(\begin{array}{c}\xi_u(0)\\ \eta_u(0)\end{array}\right)^{-1}B^\top,$$

where

$$B=\left(\begin{array}{cc}b_{11}(u) & b_{12}(u)\\ b_{21}(u) & b_{22}(u)\end{array}\right)=\mathbb{C}ov\left(\left(\begin{array}{c}\xi_u(t)-\xi_u(s)\\ \eta_u(t_1)-\eta_u(s_1)\end{array}\right),\left(\begin{array}{c}\xi_u(0)\\ \eta_u(0)\end{array}\right)\right).$$

Further

$$\mathbb{C}ov\left(\begin{array}{c}\xi_u(0)\\ \eta_u(0)\end{array}\right)^{-1}=\frac{u^{-2}}{\hat{t}_0^{\alpha_2}\hat{s}_0^{\alpha_1}(1-r^2)}\left(\begin{array}{cc}\hat{t}_0^{\alpha_2} & -r\hat{t}_0^{\frac{\alpha_2}{2}}\hat{s}_0^{\frac{\alpha_1}{2}}\\ -r\hat{t}_0^{\frac{\alpha_2}{2}}\hat{s}_0^{\frac{\alpha_1}{2}} & \hat{s}_0^{\alpha_1}\end{array}\right)\tag{43}$$

and

$$A=\frac{1}{\hat{t}_0^{\alpha_2}\hat{s}_0^{\alpha_1}(1-r^2)}\times$$
$$\times\left(\begin{array}{cc}\begin{array}{c}\frac{1}{2}\hat{t}_0^{\alpha_2}\left(\hat{s}_0^{\alpha_1}+(\hat{s}_0+u^{-\frac{2}{\alpha_1}}s)^{\alpha_1}-u^{-2}s^{\alpha_1}\right)-\\ -r^2\hat{t}_0^{\alpha_2}\hat{s}_0^{\frac{\alpha_2}{2}}\left(\hat{s}_0+u^{-\frac{2}{\alpha_1}}s\right)^{\frac{\alpha_1}{2}}\end{array} & \begin{array}{c}-\frac{1}{2}r\hat{t}_0^{\frac{\alpha_2}{2}}\hat{s}_0^{\frac{\alpha_1}{2}}\left(\hat{s}_0^{\alpha_1}+(\hat{s}_0+u^{-\frac{2}{\alpha_1}}s)^{\alpha_1}-u^{-2}s^{\alpha_1}\right)+\\ +r\hat{t}_0^{\frac{\alpha_2}{2}}\hat{s}_0^{\alpha_1}\left(\hat{s}_0+u^{-\frac{2}{\alpha_1}}s\right)^{\frac{\alpha_1}{2}}\end{array}\\ \begin{array}{c}-\frac{1}{2}r\hat{t}_0^{\frac{\alpha_2}{2}}\hat{s}_0^{\frac{\alpha_1}{2}}\left(\hat{t}_0^{\alpha_2}+(\hat{t}_0+u^{-\frac{2}{\alpha_2}}t)^{\alpha_2}-u^{-2}t^{\alpha_2}\right)+\\ +r\hat{s}_0^{\frac{\alpha_1}{2}}\hat{t}_0^{\alpha_2}\left(\hat{t}_0+u^{\frac{2}{\alpha_2}}t\right)^{\frac{\alpha_2}{2}}\end{array} & \begin{array}{c}\frac{1}{2}\hat{s}_0^{\alpha_1}\left(\hat{t}_0^{\alpha_2}+(\hat{t}_0+u^{-\frac{2}{\alpha_2}}t)^{\alpha_2}-u^{-2}t^{\alpha_2}\right)-\\ -r^2\hat{t}_0^{\frac{\alpha_2}{2}}\hat{s}_0^{\alpha_1}\left(\hat{t}_0+u^{-\frac{2}{\alpha_2}}t\right)^{\frac{\alpha_2}{2}}\end{array}\end{array}\right).$$

Set next

$$\left(\begin{array}{c}e_1(u)\\ e_2(u)\end{array}\right):=\mathbb{E}\left\{\begin{array}{c|c}\xi_u(s) & \xi_u(0)=0\\ \eta_u(t) & \eta_u(0)=0\end{array}\right\}=\left(\begin{array}{c}x-u^2\\ y-u^2\end{array}\right)+A\left(\begin{array}{c}u^2-x\\ u^2-y\end{array}\right).$$

It follows that

$$e_1(u)=\frac{1}{\hat{t}_0^{\alpha_2}\hat{s}_0^{\alpha_1}(1-r^2)}\left(-\left(\frac{1}{2}\hat{t}_0^{\alpha_2}-\frac{1}{2}r\hat{t}_0^{\frac{\alpha_2}{2}}\hat{s}_0^{\frac{\alpha_1}{2}}\right)|s|^{\alpha_1}+\lambda_1(u)u^2+\lambda_2(u)x+\lambda_3(u)y\right),$$

where

$$\lambda_1(u) = \frac{1}{2}\hat{t}_0^{\frac{\alpha_2}{2}} \left(\hat{t}_0^{\frac{\alpha_2}{2}} \left((\hat{s}_0 + u^{-\frac{2}{\alpha_1}} s)^{\frac{\alpha_1}{2}} + \hat{s}_0^{\frac{\alpha_1}{2}} - 2r^2 \hat{s}_0^{\frac{\alpha_1}{2}}\right) - r\hat{s}_0^{\frac{\alpha_1}{2}} \left((\hat{s}_0 + u^{-\frac{2}{\alpha_1}} s)^{\frac{\alpha_1}{2}} - \hat{s}_0^{\frac{\alpha_1}{2}}\right)\right)$$
$$\times \left((\hat{s}_0 + u^{-\frac{2}{\alpha_1}} s)^{\frac{\alpha_1}{2}} - \hat{s}_0^{\frac{\alpha_1}{2}}\right)$$
$$\lambda_2(u) = \frac{1}{2}\hat{t}_0^{\alpha_2} \left((\hat{s}_0 + u^{-\frac{2}{\alpha_1}} s)^{\frac{\alpha_1}{2}} + \hat{s}_0^{\frac{\alpha_1}{2}} - 2r^2 \hat{s}_0^{\frac{\alpha_1}{2}}\right) \left(\hat{s}_0^{\frac{\alpha_1}{2}} - (\hat{s}_0 + u^{-\frac{2}{\alpha_1}} s)^{\frac{\alpha_1}{2}}\right)$$
$$\lambda_3(u) = \frac{1}{2}r\hat{t}_0^{\frac{\alpha_2}{2}} \hat{s}_0^{\frac{\alpha_1}{2}} \left(\hat{s}_0^{\frac{\alpha_1}{2}} - (\hat{s}_0 + u^{-\frac{2}{\alpha_1}} s)^{\frac{\alpha_1}{2}}\right)^2.$$

Further

$$e_2(u) = \frac{1}{\hat{t}_0^{\alpha_2} \hat{s}_0^{\alpha_1}(1-r^2)} \left(-\left(\frac{1}{2}\hat{s}_0^{\alpha_1} - \frac{1}{2}r\hat{t}_0^{\frac{\alpha_2}{2}} \hat{s}_0^{\frac{\alpha_1}{2}}\right)|t|^{\alpha_2} + \delta_1(u)u^2 + \delta_2(u)x + \delta_3(u)y\right),$$

where

$$\delta_1(u) = \frac{1}{2}\hat{s}_0^{\frac{\alpha_1}{2}} \left(\hat{s}_0^{\frac{\alpha_1}{2}} \left((\hat{t}_0 + u^{-\frac{2}{\alpha_2}} t)^{\frac{\alpha_2}{2}} + \hat{t}_0^{\frac{\alpha_2}{2}} - 2r^2 \hat{t}_0^{\frac{\alpha_2}{2}}\right) - r\hat{t}_0^{\frac{\alpha_2}{2}} \left((\hat{t}_0 + u^{-\frac{2}{\alpha_2}} t)^{\frac{\alpha_2}{2}} - \hat{t}_0^{\frac{\alpha_2}{2}}\right)\right)$$
$$\times \left((\hat{t}_0 + u^{-\frac{2}{\alpha_2}} t)^{\frac{\alpha_2}{2}} - \hat{t}_0^{\frac{\alpha_2}{2}}\right)$$
$$\delta_2(u) = \frac{1}{2}r\hat{t}_0^{\frac{\alpha_2}{2}} \hat{s}_0^{\frac{\alpha_1}{2}} \left(\hat{t}_0^{\frac{\alpha_2}{2}} - (\hat{t}_0 + u^{-\frac{2}{\alpha_2}} t)^{\frac{\alpha_2}{2}}\right)^2$$
$$\delta_3(u) = \frac{1}{2}\hat{s}_0^{\alpha_1} \left((\hat{t}_0 + u^{-\frac{2}{\alpha_2}} t)^{\frac{\alpha_2}{2}} + \hat{t}_0^{\frac{\alpha_2}{2}} - 2r^2 \hat{t}_0^{\frac{\alpha_2}{2}}\right) \left(\hat{t}_0^{\frac{\alpha_2}{2}} - (\hat{t}_0 + u^{-\frac{2}{\alpha_2}} t)^{\frac{\alpha_2}{2}}\right).$$

Thus we have

$$\lim_{u\to\infty} e_1(u) = \begin{cases} -\frac{1}{2(1+r)}|s|^{\alpha_1}, & \text{if } \alpha_1 \in (0,1), \\ -\frac{1}{2(1+r)}|s| + \frac{1}{2}s & \text{if } \alpha_1 = 1 \end{cases} \tag{44}$$

and

$$\lim_{u\to\infty} e_2(u) = \begin{cases} -\frac{1}{2(1+r)}|t|^{\alpha_2}, & \text{if } \alpha_2 \in (0,1), \\ -\frac{1}{2(1+r)}|t| + \frac{1}{2}t & \text{if } \alpha_2 = 1. \end{cases} \tag{45}$$

Similarly

$$b_{11}(u) = \frac{1}{2}\left(s^{\alpha_1} - t^{\alpha_1} + u^2\left(\left(\hat{s}_0 + u^{-\frac{2}{\alpha_1}} t\right)^{\alpha_1} - \left(\hat{s}_0 + u^{-\frac{2}{\alpha_1}} s\right)^{\alpha_1}\right)\right),$$
$$b_{12}(u) = u^2 r\hat{t}_0^{\frac{\alpha_2}{2}} \left(\left(\hat{s}_0 + u^{-\frac{2}{\alpha_1}} t\right)^{\frac{\alpha_1}{2}} - \left(\hat{s}_0 + u^{-\frac{2}{\alpha_1}} s\right)^{\frac{\alpha_1}{2}}\right),$$
$$b_{21}(u) = u^2 r\hat{s}_0^{\frac{\alpha_1}{2}} \left(\left(\hat{t}_0 + u^{-\frac{2}{\alpha_2}} t_1\right)^{\frac{\alpha_2}{2}} - \left(\hat{t}_0 + u^{-\frac{2}{\alpha_2}} s_1\right)^{\frac{\alpha_2}{2}}\right),$$
$$b_{22}(u) = \frac{1}{2}\left(s_1^{\alpha_2} - t_1^{\alpha_2} + u^2\left(\left(\hat{t}_0 + u^{-\frac{2}{\alpha_2}} t_1\right)^{\alpha_2} - \left(\hat{t}_0 + u^{-\frac{2}{\alpha_2}} s_1\right)^{\alpha_2}\right)\right),$$

which together with (43) gives that

$$B\ \mathbb{C}ov \begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix}^{-1} B^\top = \begin{pmatrix} o(1) & o(1) \\ o(1) & o(1) \end{pmatrix}$$

as $u \to \infty$. Further

$$\mathbb{C}ov(\xi_u(t) - \xi_u(s), \xi_u(t) - \xi_u(s)) = |t-s|^{\alpha_1}, \quad \mathbb{C}ov(\eta_u(t_1) - \eta_u(s_1), \eta_u(t_1) - \eta_u(s_1)) = |t_1 - s_1|^{\alpha_2},$$
$$\mathbb{C}ov(\xi_u(t) - \xi_u(s), \eta_u(t_1) - \eta_u(s_1)) = u^2 r\left((\hat{s}_0 + u^{-\frac{2}{\alpha_1}} t)^{\frac{\alpha_1}{2}} - (\hat{s}_0 + u^{-\frac{2}{\alpha_1}} s)^{\frac{\alpha_1}{2}}\right)$$
$$\times \left((\hat{t}_0 + u^{-\frac{2}{\alpha_2}} t_1)^{\frac{\alpha_2}{2}} - (\hat{t}_0 + u^{-\frac{2}{\alpha_2}} s_1)^{\frac{\alpha_2}{2}}\right) = o(1),$$

as $u\to\infty$. Therefore,

$$\mathbb{C}ov\left(\begin{array}{c|c}\xi_u(t)-\xi_u(s) & \xi_u(0)=0\\ \eta_u(t_1)-\eta_u(s_1) & \eta_u(0)=0\end{array}\right)=\left(\begin{array}{cc}|t-s|^{\alpha_1} & o(1)\\ o(1) & |t_1-s_1|^{\alpha_2}\end{array}\right),\quad \text{as } u\to\infty.$$

Consequently, using similar arguments as in [3] (see also [10], [23] or [36]) we obtain

$$\lim_{u\to\infty} p_u(x,y)=\mathbb{P}\left\{\sup_{s\in\Lambda_1}\chi_1(s)>x\right\}\mathbb{P}\left\{\sup_{t\in\Lambda_2}\chi_2(t)>y\right\}$$

for any $x,y\in\mathbb{R}$, where χ_1 and χ_2 are two independent stochastic processes given by

$$\chi_1(s)=\widehat{B}_{\alpha_1}(s)+\begin{cases}-\frac{1}{2(1+r)}|s|^{\alpha_1}, & \text{if } \alpha_1\in(0,1),\\ -\frac{1}{2(1+r)}|s|+\frac{1}{2}s & \text{if } \alpha_1=1,\end{cases}\qquad s\in\mathbb{R}$$

and

$$\chi_2(t)=\widetilde{B}_{\alpha_2}(t)+\begin{cases}-\frac{1}{2(1+r)}|t|^{\alpha_2}, & \text{if } \alpha_2\in(0,1),\\ -\frac{1}{2(1+r)}|t|+\frac{1}{2}t & \text{if } \alpha_2=1\end{cases}\qquad t\in\mathbb{R}.$$

Here $\widehat{B}_{\alpha_1}$ and $\widetilde{B}_{\alpha_2}$ are two independent fBm's defined on $\mathbb{R}$ with Hurst indexes $\alpha_1/2$ and $\alpha_2/2\in(0,1)$, respectively. Similar arguments as in [3] and [23] show that the limit (letting $u\to\infty$) can be passed under the integral sign in (42). It follows then that

$$\begin{aligned}R_{\Lambda_1,\Lambda_2}(u)= &\quad (1+o(1))\frac{1}{2\pi\sqrt{1-r^2}u^2}\exp\left(-\frac{u^2}{2}h(\hat{s}_0,\hat{t}_0)\right)\\ &\times\prod_{i=1}^{2}\left(\int_{-\infty}^{\infty}\exp\left(\frac{x}{1+r}\right)\mathbb{P}\left\{\sup_{s\in\Lambda_i}\chi_i(s)>x\right\}dx\right),\quad u\to\infty.\end{aligned}$$

Since

$$\int_{-\infty}^{\infty}\exp\left(\frac{x}{1+r}\right)\mathbb{P}\left\{\sup_{s\in\Lambda_i}\chi_i(s)>x\right\}dx=\begin{cases}(1+r)\mathcal{H}^0_{\alpha_i}\left[\Lambda_1\left(\frac{1}{\sqrt{2}(1+r)}\right)^{\frac{2}{\alpha_i}}\right], & \text{if } \alpha_i\in(0,1),\\ (1+r)\mathcal{H}_1^{-(1+r)}\left[\Lambda_i\left(\frac{1}{\sqrt{2}(1+r)}\right)^2\right], & \text{if } \alpha_i=1\end{cases}$$

the claim follows. □

Lemma B. *Let $(\Omega,\mathfrak{F},\mathbb{P})$ be a probability space and $A_1,\cdots,A_n$ and $B_1,\cdots,B_m$ be $n+m$ events in $\mathfrak{F}$ for $n,m\geq 2$. Then*

$$\begin{aligned}\mathbb{P}\left\{\bigcup_{\substack{k=1,\cdots,n\\ l=1,\cdots,m}}(A_k\cap B_l)\right\}&\geq\sum_{k=1}^{n}\sum_{l=1}^{m}\mathbb{P}\left\{A_k\cap B_l\right\}\\ &-\sum_{k=1}^{n}\sum_{1\leq l_1<l_2\leq m}\mathbb{P}\left\{A_k\cap B_{l_1}\cap B_{l_2}\right\}-\sum_{l=1}^{m}\sum_{1\leq k_1<k_2\leq n}\mathbb{P}\left\{A_{k_1}\cap A_{k_2}\cap B_l\right\}.\end{aligned}\tag{46}$$

Proof of Lemma B: The proof relies on the following Bonferroni inequality; see e.g., Lemma 2 in [31].

$$\sum_{k=1}^{n}\mathbb{P}\left\{A_k\right\}\geq\mathbb{P}\left\{\bigcup_{k=1}^{n}A_k\right\}\geq\sum_{k=1}^{n}\mathbb{P}\left\{A_k\right\}-\sum_{1\leq k_1<k_2\leq n}\mathbb{P}\left\{A_{k_1}\cap A_{k_2}\right\}.$$

Since further

$$\begin{aligned}\mathbb{P}\left\{\bigcup_{\substack{k=1,\cdots,n\\ l=1,\cdots,m}}(A_k\cap B_l)\right\}&=\mathbb{P}\left\{\bigcup_{k=1}^{n}(A_k\cap(\bigcup_{l=1}^{m}B_l))\right\}\\ &\geq\sum_{k=1}^{n}\mathbb{P}\left\{A_k\cap(\bigcup_{l=1}^{m}B_l)\right\}-\sum_{1\leq k_1<k_2\leq n}\mathbb{P}\left\{A_{k_1}\cap A_{k_2}\cap(\bigcup_{l=1}^{m}B_l)\right\}\end{aligned}$$

$$\geq \sum_{k=1}^{n}\sum_{l=1}^{m} \mathbb{P}\left\{A_k \cap B_l\right\} - \sum_{k=1}^{n} \sum_{1\leq l_1<l_2\leq m} \mathbb{P}\left\{A_k \cap B_{l_1} \cap B_{l_2}\right\} - \sum_{l=1}^{m} \sum_{1\leq k_1<k_2\leq n} \mathbb{P}\left\{A_{k_1} \cap A_{k_2} \cap B_l\right\}$$

the proof is complete. □

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