Term Structure Models
Driven by General Lévy Processes

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Nr. 33   November 1996

revised October 1998

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Abstract

As a generalization of the Gaussian Heath-Jarrow-Morton term structure model, we present a new class of bond price models that can be driven by a wide range of Lévy processes. We deduce the forward and short rate processes implied by this model and prove that, under certain assumptions, the short rate is Markovian if and only if the volatility structure has either the Vasiček or the Ho-Lee form. Finally, we compare numerically forward rates and European call option prices in a model driven by a hyperbolic Lévy motion with those in the Gaussian model.

Keywords: term structure models, martingale modelling, Lévy process, hyperbolic Lévy motion, Markov property, Vasiček model

1 Introduction

Models of the term structure of interest rates are important for many problems in economics, in particular for the valuation of contingent claims depending on interest rate sensitive assets. In the Heath-Jarrow-Morton (1992) setting, one assumes that there is a complete set of zero coupon bonds with maturities T in some time interval $[0, T^*]$. Under a risk-neutral measure, the price process of a bond maturing at T satisfies the following stochastic differential equation

$$dP(t,T) = P(t,T)(r(t)dt + \sigma(t,T)dW_t), \tag{1}$$

where W is a standard Brownian motion, which is the same for all maturities. In the corresponding models for stock prices such as the Black-Scholes [4] or better Samuelson [16] model, empirical studies revealed (see e. g. [7]) that by replacing the source of randomness W by an appropriate Lévy process L, a much better fit of return distributions is obtained. Furthermore, it is not only the fit of distributions that is improved. Purely discontinuous processes, such as e. g. the hyperbolic Lévy motion, in addition give a more realistic picture of price movements on the level of the microstructure.

Although the empirical evidence for non-Gaussian behaviour of bond prices is not as complete as in the case of stocks, it is tempting to study models driven by Lévy processes in general. It is this that we start in the following. Because one of our goals is the valuation of contingent claims, we are interested in a model in which discounted bond prices are martingales. Instead of replacing W in the stochastic differential equation above, we look at the solution of this equation, which can be given in the form

$$P(t,T) = P(0,T) \cdot \exp\left(\int_0^t r(s)ds\right) \frac{\exp\left(\int_0^t \sigma(s,T)dW_s\right)}{\mathrm{E}\left[\exp\left(\int_0^t \sigma(s,T)dW_s\right)\right]}.$$

If σ is deterministic, then $\int_0^t \sigma(s,T)dW_s$ has independent increments. Therefore, the quotient on the right-hand side—and thus the discounted price process—forms a martingale.

In section 2, the mathematical framework and the assumptions are given for the generalization where W_t is replaced by L_t. Section 3 provides a number of analytic results which lead to the following final form of the bond price process, achieved in (14):

$$P(t,T) = P(0,T) \cdot \exp\left(\int_0^t \left(r(s) - \theta(\sigma(s,T))\right) ds + \int_0^t \sigma(s,T)dL_s\right),$$

where $\theta(u)$ denotes the log moment-generating function of L_1. In section 4, we investigate the subclass of models in which the short rate $r(t)$ is a Markov process. Under further restrictions on the driving Lévy process L, for stationary volatility structures the well-known Vasiček and Ho-Lee cases reappear. In section 5, assuming a Vasiček structure we compare the forward rates resulting from a hyperbolic Lévy motion to those given in the classical Gaussian setting. Hyperbolic forward rates turn out to be slightly higher. Finally, we compare prices of European call options on bonds derived from the Gaussian and the hyperbolic model. The shape of the difference curve is the same which we know already (see [8]) from options on stocks.

2 Presentation of the model

As mentioned in the introduction, we start by modelling bond prices. A *zero coupon bond* with maturity date T, also called *bond* or T-*bond* for short, is a contract which pays an amount of one currency unit to its holder at time T. Its price at time t will be denoted by $P(t,T)$. It is assumed that for each $T \in [0,T^\star]$, $T^\star > 0$ being a fixed time horizon, a bond maturing at this time T is traded on the market. Furthermore, we shall use a numeraire process β. That is, we will express the prices of bonds in terms of another security whose value is described by the stochastic process β.

We shall denote by $L = (L_s)_{s\geq 0}$ a Lévy process, that is, a stochastic process with stationary and independent increments which is continuous in probability and satisfies $L_0 = 0$ a.s. (See [12] or [14], chap. I.4, for details.) The law of L_1, $\mathcal{L}(L_1)$, is infinitely divisible and hence by the Lévy-Khintchine formula is characterized in part by its Lévy measure F. In order to guarantee the existence of the expectations that appear in our model (3) below, we have to impose the following integrability assumption on F:
There are constants $M, \epsilon > 0$ such that

$$(2) \qquad \int_{\{|x|>1\}} \exp(vx) F(dx) < \infty \qquad \forall\ |v| \leq (1+\epsilon)M.$$

Lemma 3.1, which we will prove below, shows that this is indeed sufficient.

L is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will base our model on the completed canonical filtration associated with L, which will be denoted by $(\mathcal{F}_u)_{u\in[0,T^\star]}$. The completed canonical filtration of every Lévy process is right continuous (see e. g. [14], Theorem I.31) and thus satisfies the *usual hypotheses* of the general theory of stochastic processes. The price dynamics of all securities considered are assumed to be described by càdlàg processes adapted to this filtration. In particular, for every time $T \in [0,T^\star]$ there is a càdlàg adapted process $(P(t,T))_{0\leq t\leq T}$ giving the prices of the bond maturing at T.

The central building block of our bond price model will be a stochastic integral $X_t := \int_0^t \sigma(s) dL_s$, where σ, the *volatility*, is a deterministic, twice continuously differentiable function. For such processes, we have the following

Lemma 2.1. *Let $\sigma : [0,T] \to \mathbb{R}$ be a deterministic, continuously differentiable function. Then the following equality holds for almost all $\omega \in \Omega$:*

$$\int_0^t \sigma(s) dL_s = \sigma(t) L_t - \int_0^t L_s \sigma'(s) ds \qquad \forall\ t \in [0,T].$$

Proof: This is clear from the definition of the quadratic co-variation $[L,\sigma] := L\sigma - L_0\sigma(0) - \int L_- d\sigma - \int \sigma_- dL$ of the Lévy process L and the function σ, since σ is continuous, which implies $[L,\sigma] \equiv 0$. Note that for fixed ω, the set of s where $L_s(\omega)$ and $L_{s-}(\omega)$ can differ is an at most countable Lebesgue null set. □

In order to avoid any problems that may arise from the fact that stochastic integrals are defined up to null sets only, we will interpret all stochastic integrals with suitable deterministic integrands in the spirit of the above lemma. Note that we consider a non-denumerable set of bond price processes, hence of stochastic integrals.

As mentioned in the introduction, we start with the following form of the price process for a zero coupon bond maturing at time T:

$$P(t,T) = P(0,T) \cdot \beta(t) \cdot \frac{\exp\left(\int_0^t \sigma(s,T)dL_s\right)}{\mathrm{E}\left[\exp\left(\int_0^t \sigma(s,T)dL_s\right)\right]}, \tag{3}$$

where $\beta(t)$ denotes the value of the numeraire at time t.

Besides (2), the following standard assumptions are made throughout the paper. Recall that $T^\star$ is the fixed time horizon.

Assumption 2.1: The initial bond prices are given by a deterministic, positive, and twice continuously differentiable function $T \mapsto P(0,T)$ on the interval $[0,T^\star]$.

Assumption 2.2: $P(T,T) = 1$ for all $T \in [0,T^\star]$.

From this boundary condition, we shall derive the explicit form of the process $\beta(t)$.

Assumption 2.3: $\sigma(s,T)$ is defined on the triangle $\Delta := \{(s,T) : 0 \le s \le T \le T^\star\}$. This function is twice continuously differentiable in both variables, and $\sigma(s,T) \le M$ for all $(s,T) \in \Delta$, where M is the constant from (2). Furthermore, $\sigma(s,T) > 0$ for all $(s,T) \in \Delta$, $s \neq T$, and $\sigma(T,T) = 0$ for all $T \in [0,T^\star]$.

It is quite natural to assume that the volatility structure is bounded. Also, the volatility of a just-maturing bond should be zero, since its value is known for sure.

3 Analysis of the model

Our model is designed such that bond prices, when expressed in units of the numeraire, are martingales. If the expectation appearing in the denominator of (3) is finite, this is clear since for fixed T the exponent $X_t = \int_0^t \sigma(s,T)dL_s$ is a process with independent increments:

$$\begin{aligned}\mathrm{E}\left[\exp(X_t) \mid \mathcal{F}_s\right] &= \mathrm{E}\left[\exp(X_t - X_s)\exp(X_s) \mid \mathcal{F}_s\right] \\ &= \frac{\mathrm{E}\left[\exp(X_t)\right]}{\mathrm{E}\left[\exp(X_s)\right]}\exp(X_s).\end{aligned}$$

(2) is the proper assumption to guarantee finiteness of the expectations appearing here and in (3). The following lemma at the same time provides an explicit form of these expectations in terms of the function θ, the log of the *moment generating function* of L_1, that is,

$$\theta(v) = \log E\Big[\exp(vL_1)\Big].$$

According to [17], condition (2) implies that the moment generating function $v \mapsto E[\exp(vL_1)]$ exists at least on the interval $[-(1+\epsilon)M, (1+\epsilon)M]$. Due to [13], θ is continuously differentiable (in fact even analytic) and has the following representation:

$$\theta(z) = bz + \frac{c}{2}z^2 + \int_{\mathbb{R}} (e^{zx} - 1 - zx) F(dx),$$

which is valid for $z \in \mathbb{C}, \mathrm{Re}(z) \in [-(1+\epsilon)M, (1+\epsilon)M]$.

Lemma 3.1. *Let L be a Lévy process satisfying (2). If $f : \mathbb{R}_+ \to \mathbb{C}$ is a complex-valued, left-continuous function with limits from the right, such that $|Re(f)| \leq M$, then*

$$E\Big[\exp\Big(\int_0^t f(s)dL_s\Big)\Big] = \exp\Big(\int_0^t \theta(f(s))ds\Big). \tag{4}$$

(The integrals in (4) are to be understood componentwise for real and imaginary part.)

Proof: For any partition $0 = t_0 < \cdots < t_{N+1} = t$ of the interval $[0, t]$, we get, by independence and stationarity of the increments of L:

$$\begin{aligned}
\mathrm{E}\Big[\exp\Big(\sum_{k=0}^{N} f(t_k)(L_{t_{k+1}} - L_{t_k})\Big)\Big] &= \prod_{k=0}^{N} \mathrm{E}\Big[\exp\Big(f(t_k)(L_{t_{k+1}} - L_{t_k})\Big)\Big] \\
&= \prod_{k=0}^{N} \exp\Big(\theta(f(t_k))(t_{k+1} - t_k)\Big) \\
&= \exp\Big(\sum_{k=0}^{N} \theta(f(t_k))(t_{k+1} - t_k)\Big).
\end{aligned}$$

If the mesh of the partition goes to zero, the r.h.s. converges to $\exp\Big(\int_0^t \theta(f(s))ds\Big)$, while the exponent on the l.h.s. converges in measure to $\int_0^t f(s)dL_s$ (see [12], Proposition I.4.44.) Convergence in measure is preserved under continuous transformations. Hence

$$\exp\Big(\sum_{k=0}^{N} f(t_k)(L_{t_{k+1}} - L_{t_k})\Big) \longrightarrow \exp\Big(\int_0^t f(s)dL_s\Big) \quad \text{in measure,} \tag{5}$$

Here, the approximating sequence is uniformly integrable, since it is bounded in $L^{1+\epsilon}$. (See e. g. [6], Theorem 22.) Therefore, convergence in measure implies integrability of the limit as well as convergence in L^1. $\square$

As an immediate consequence, taking $f(s) = \sigma(s,T)$ for fixed $T \in [0,T^\star]$ we get

$$
\mathrm{E}\Big[\exp\Big(\int_0^t \sigma(s,T)dL_s\Big)\Big] = \exp\Big(\int_0^t \theta(\sigma(s,T))ds\Big), \tag{6}
$$

while by choosing $f(s) = iu\sigma(s,T)$ for fixed $T \in [0,T^\star]$ and $u \in \mathbb{R}$ we get the characteristic function of the random variable $X_t = \int_0^t \sigma(s,T)dL_s$:

$$
\begin{aligned}
\mathrm{E}\Big[\exp\Big(iuX_t\Big)\Big] = \exp\Big(iub\int_0^t \sigma(s,T)ds - \frac{u^2}{2}c\int_0^t \sigma(s,T)^2 ds \\
+ \int_0^t\int_{\mathbb{R}}\Big[e^{iu\sigma(s,T)x} - 1 - iu\sigma(s,T)x\Big]F(dx)ds\Big).
\end{aligned} \tag{7}
$$

Remark: In the same way, one can derive the joint characteristic function of two stochastic integrals $\int f^1 dL$ and $\int f^2 dL$. We will need this in sections 4 and 5 in order to compute joint densities via Fourier inversion.

As a starting point for a deeper analysis of the model from the point of view of the theory of stochastic processes (e. g. for the study of path properties), let us just give the canonical representation (Jacod and Shiryaev, II.2.34) of $X = \int \sigma(s,T)dL_s$.

$$
X_t = c^{1/2}\int_0^t \sigma(s,T)dW_s + \{\sigma(s,T)x\} * \Big(\mu^L - \nu^L\Big)_t + b\int_0^t \sigma(s,T)ds, \tag{8}
$$

where W is a standard Brownian motion, μ^L the random measure of jumps of L and ν^L its compensator. The compensator ν^X of μ^X is given by

$$
\nu^X(]a,b]\times C) = \int_a^b\int_{\mathbb{R}} 1\!\!1_C(\sigma(s,T)x)F(dx)\,ds \qquad (C \in \mathcal{B}^1).
$$

The characteristic function (7) could be derived from this as well.

For fixed $t \in [0,T^\star]$, we introduce the *forward rate* with maturity T, contracted at time t, $f(t,T)$, and the *short rate* $r(t)$ as $\mathcal{F}_t$-measurable random variables

$$
f(t,T) := -\frac{\partial}{\partial T}\log P(t,T), \quad \text{and} \quad r(t) := f(t,t).
$$

Lemma 3.2. *For all $0 \le T \le T^\star$, the forward rate process $f(\cdot,T)$ exists and has the form*

$$
f(t,T) = f(0,T) + \int_0^t \theta'(\sigma(s,T))\sigma_2(s,T)ds - \int_0^t \sigma_2(s,T)dL_s \qquad (t \in [0,T]), \tag{9}
$$

where $f(0,s) := -\frac{\partial}{\partial s}\log P(0,s)$ is determined by the initial bond price structure, and where σ_2 denotes the partial derivative of σ with respect to the second variable, that is,

$$
\sigma_2(s,T) := \frac{\partial}{\partial T}\sigma(s,T).
$$

Remark: In what follows, a subscript "1" (resp. "2") on a function will always denote the partial derivative with respect to the first (resp. second) variable.

Proof of Lemma 3.2: Making use of (6), we can write the logarithm of the bond price as

$$\log P(t,T) = \log P(0,T) + \log \beta(t) - \int_0^t \theta(\sigma(s,T))ds + \int_0^t \sigma(s,T)dL_s. \tag{10}$$

This expression is continuously differentiable w.r.t. T, which is immediately clear for the first three summands. Recall the differentiability assumptions 2.1 and 2.3. The differentiability of the fourth summand in (10) becomes clear when we consider the representation

$$\int_0^t \sigma(s,T)dL_s = \sigma(t,T)L_t - \int_0^t L_s\sigma_1(s,T)ds, \tag{11}$$

which has been proved in Lemma 2.1. Calculating the derivatives of the four summands in (10) poses no problems, and by adding up we arrive at representation (9). □

We will now determine the form of the numeraire process β.

Lemma 3.3. *Under the hypotheses of our bond price model, the numeraire β is given by*

$$\beta(t) = \exp\left(\int_0^t r(s)ds\right).$$

Hence β is the usual money account process.

Proof: The boundary condition $P(t,t) = 1$ implies

$$\beta(t) = \frac{1}{P(0,t)} \exp\left(\int_0^t \theta(\sigma(s,t))ds - \int_0^t \sigma(s,t)dL_s\right). \tag{12}$$

On the other hand,

$$\begin{aligned}\int_0^t r(s)ds &= \int_0^t f(0,s)ds + \int_0^t\int_0^s \theta'\big(\sigma(v,s)\big)\sigma_2(v,s)dv\ ds - \int_0^t\int_0^s \sigma_2(v,s)dL_v\ ds \\ &= -\log P(0,t) + \int_0^t \theta(\sigma(s,t))ds - \int_0^t \sigma(s,t)dL_s, \end{aligned}\tag{13}$$

where in the transformation of the two double integrals we have used Fubini's theorem. (See e. g. [14], Theorem 45, for Fubini's theorem for stochastic integrals.) In addition, we have again made use of the boundary conditions $P(0,0) = 1$ and $\sigma(t,t) = 0$. This completes the proof, since clearly (13) is the logarithm of (12). □

Putting together (6) and Lemma 3.3, we have derived the following representation for the bond price process P:

$$P(t,T) = P(0,T)\cdot \exp\left(\int_0^t \Big(r(s) - \theta(\sigma(s,T))\Big)ds + \int_0^t \sigma(s,T)dL_s\right). \tag{14}$$

By Ito's formula for semimartingales, for fixed T the process $P(\cdot, T)$ satisfies the following stochastic differential equation:

$$dP(t,T) = P(t-,T) \cdot \Bigg(r(t)dt + \Big(\frac{c}{2}\sigma^2(t,T) - \theta\big(\sigma(t,T)\big)\Big) dt \\ + \sigma(t,T)dL_t + \Big(e^{\sigma(t,T)\Delta L_t} - 1 - \sigma(t,T)\Delta L_t \Big) \Bigg) \tag{15}$$

The Gaussian model, which was studied e. g. in [9], [10] and [5], is a special case of our general model. We get it by choosing for L a standard Brownian motion W. Then the law of $L_1 = W_1$ is a standard normal distribution, whose log moment-generating function has the simple form $\theta(u) = u^2/2$. Equation (15) thus reduces to the classical stochastic differential equation (1) cited in the introduction. Compare the corresponding formulas for bond prices, forward rates and the short rate. (See e. g. [2] and [3].)

4 The Markov property of the short rate

In the previous section, we have seen that the short rate process r is implicitly defined by our bond price model. Setting $T = t$ in (9) shows how r depends on the choice of the volatility structure σ. We will now discuss the question which volatility structures lead to a short rate that is a Markov process. For *stationary* volatility structures, the answer we will get is the following: The short rate is Markovian iff the volatility structure is a Ho-Lee or a Vasiček volatility structure.

For the Gaussian term structure model, this result was proved in [5]. In order to prove it for our model, where more general Lévy processes L are admitted as driving processes, we will need the additional assumption (16) concerning the characteristic function of L_1, which will enable us to prove Lemma 4.2 by arguments of absolute continuity of measures.

Lemma 4.1. *Let L be a Lévy process such that there are real constants $C, \gamma, \eta > 0$ satisfying*

$$\Big| E[\exp(iuL_1)] \Big| \leq C \cdot \exp(-\gamma |u|^\eta) \qquad \forall\, u \in \mathbb{R}. \tag{16}$$

Suppose $t \in\,]0, \infty[$, and suppose further that $f, g : [0,t] \to \mathbb{R}$ are continuous functions such that none of them is a scalar multiple of the other.
Then the joint distribution of the random variables $X := \int_0^t f(s) dL_s$ and $Y := \int_0^t g(s) dL_s$ is continuous with respect to Lebesgue measure λ^2 on $\mathbb{R}^2$.

Remarks: Condition (16) somewhat narrows the range of admissible Lévy processes. Yet, important examples of processes satisfy this assumption: This is obvious for Brownian motion, since the characteristic function of a standard normal distribution is

$u \mapsto \exp(-u^2/2)$. In section 5, we prove that assumption (16) holds for all hyperbolic Lévy motions as well.

The condition that the functions f and g are no scalar multiples of each other is equivalent to the following: There does not exist a straight line through the origin on which all points $\binom{f(s)}{g(s)}$, $s \in [0,t]$, lie.

Proof of Lemma 4.1: We use the fact that a probability distribution on $\mathbb{R}^d$ is continuous with respect to λ^d if its characteristic function is integrable over $\mathbb{R}^d$. Thus we have to prove the λ^2-integrability of the joint characteristic function $\phi(u,v)$ of X and Y.

Let $\chi(u)$ denote the cumulant generating function of L_1, i. e. $\exp(\chi(u)) = E[\exp(iuL_1)]$. According to the remark following Lemma 3.1, we have

$$\phi(u,v) \equiv \mathrm{E}\left[\exp(iuX + ivY)\right] = \exp\left(\int_0^t \chi(uf(s) + vg(s))ds\right).$$

Using (16), we get

$$|\phi(u,v)| = \exp\left(\int_0^t \mathrm{Re}\,\chi(uf(s) + vg(s))ds\right) \le C^t \exp\left(-\gamma \int_0^t |uf(s) + vg(s)|^\eta ds\right).$$

Since $uf(s) + vg(s)$ is the Euclidean scalar product of $\binom{u}{v}, \binom{f(s)}{g(s)} \in \mathbb{R}^2$, we can, for all $\binom{u}{v} \ne \binom{0}{0}$, extract the Euclidean norm of $\binom{u}{v}$ from the integral, leaving the normed vector $\binom{u^*}{v^*}$:

$$\int_0^t |uf(s) + vg(s)|^\eta ds = \left|\begin{pmatrix} u \\ v \end{pmatrix}\right|^\eta \cdot \int_0^t \left|\begin{pmatrix} u^* \\ v^* \end{pmatrix} \cdot \begin{pmatrix} f(s) \\ g(s) \end{pmatrix}\right|^\eta ds.$$

Since the remaining integral is a continuous function of the vector $\binom{u^*}{v^*}$, it has a minimum m on the unit circle in $\mathbb{R}^2$. It is obvious that $m \ge 0$. In fact, we have $m > 0$, since $m = 0$ would imply that the integrand vanishes for all s; but this is impossible by assumption, since it would mean that all the points $\binom{f(s)}{g(s)}$ lie on a straight line through the origin. From $m > 0$ follows

$$\int_{\mathbb{R}^2} |\phi(u,v)|\, d\lambda^2(u,v) \le C^t \int_{\mathbb{R}^2} \exp\left(-\gamma m \left|\begin{pmatrix} u \\ v \end{pmatrix}\right|^\eta\right) d\lambda^2(u,v) < \infty. \qquad \square$$

We shall now apply this result to our model:

Lemma 4.2. *Suppose L is a Lévy process which satisfies the integrability condition (2) and for which there exist constants $C, \gamma, \eta > 0$ satisfying (16).*
Then the short rate process r is a Markov process iff the partial derivative $\sigma_2(s,t)$ of the volatility structure σ satisfies the following: For all fixed T, U such that $0 < T < U \le T^\star$, the function $\sigma_2(\cdot, U)$ is a scalar multiple of the function $\sigma_2(\cdot, T)$ on $[0,T]$—that is, there is a real constant ξ (which may depend on T and U) such that

$$\sigma_2(t,U) = \xi \cdot \sigma_2(t,T) \qquad \forall\, t \in [0,T].$$

Proof: We have to examine the process r given by

$$r(t) = f(0,t) + \int_0^t \theta'(\sigma(s,t))\sigma_2(s,t)ds - \int_0^t \sigma_2(s,t)dL_s \qquad (t \in [0,T^\star]). \tag{17}$$

Since $f(0,\cdot)$, $\theta(\cdot)$, and $\sigma(\cdot,\cdot)$ are deterministic functions, r is a Markov process iff the process $Z \equiv (Z(T))_{T\in[0,T^\star]}$ is Markovian, where

$$Z(T) := \int_0^T \sigma_2(s,T)dL_s. \tag{18}$$

By definition, Z is a Markov process iff for all $T, U \in \mathbb{R}$ with $0 \le T < U \le T^\star$ the following is true:

$$\mathbb{P}\left[Z(U) \in B \mid \mathcal{F}_T\right] = \mathbb{P}\left[Z(U) \in B \mid Z(T)\right] \qquad \forall\, B \in \mathcal{B}^1.$$

Proof of the necessity: Assume r is a Markov process. Then, according to the preliminary consideration above, the process Z defined by (18) is Markovian. This implies

$$\mathrm{E}\left[Z(U)|\mathcal{F}_T\right] = \mathrm{E}\left[Z(U) \mid Z(T)\right]. \tag{19}$$

Using the definition of Z, equation (19) becomes

$$\begin{aligned}\mathrm{E}\left[\int_0^T \sigma_2(s,U)dL_s\middle|\mathcal{F}_T\right] + \mathrm{E}\left[\int_T^U \sigma_2(s,U)dL_s\middle|\mathcal{F}_T\right]& \\ = \mathrm{E}\left[\int_0^T \sigma_2(s,U)dL_s\middle|Z(T)\right] &+ \mathrm{E}\left[\int_T^U \sigma_2(s,U)dL_s\middle|Z(T)\right].\end{aligned}$$

Since the integrand $\sigma_2(\cdot,U)$ is deterministic and since L is a process with independent increments, $\int_T^U \sigma_2(s,U)dL_s$ is independent of the σ-field $\mathcal{F}_T$ and, in particular, of $Z(T)$. This implies that the second summands on both sides are equal, since one can replace the conditional expectation by the expectation in these terms. In addition, $\int_0^T \sigma_2(s,U)dL_s$ is measurable with respect to $\mathcal{F}_T$. Thus, condition (19) is equivalent to

$$\int_0^T \sigma_2(s,U)dL_s = \mathrm{E}\left[\int_0^T \sigma_2(s,U)dL_s\middle|\int_0^T \sigma_2(s,T)dL_s\right].$$

But this means that the integral $\int_0^T \sigma_2(s,U)dL_s$ can be expressed as some measurable function G applied to the integral $\int_0^T \sigma_2(s,T)dL_s$. Hence, the joint distribution of these two random variables is concentrated on the Lebesgue null set

$$\{(x, G(x)) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$$

and thus cannot be continuous with respect to λ^2.

If $\sigma_2(\cdot,T) \not\equiv 0$, by Lemma 4.1 $\sigma_2(\cdot,U)$, restricted to $[0,T]$, is a scalar multiple of $\sigma_2(\cdot,T)$.

If, on the other hand, $\sigma_2(\cdot, T) \equiv 0$, then $\int_0^T \sigma_2(s, U) dL_s = G(\int_0^T \sigma_2(s, T) dL_s) = G(0)$ is deterministic, and by simple arguments we get $\sigma_2(\cdot, U) \equiv 0$ on $[0, T]$.

Proof of the sufficiency: According to our preliminary consideration, we have to show that the process Z defined in (18) is Markovian.

Suppose that T and U satisfy $0 < T < U \leq T^\star$ — the case $T = 0$ is trivial. Then we have

$$Z(U) = \int_0^T \sigma_2(s, U) dL_s + \int_T^U \sigma_2(s, U) dL_s.$$

By assumption, the first term on the r.h.s. is equal to $\int_0^T \xi \cdot \sigma_2(s, T) dL_s = \xi Z(T)$ for some constant ξ. Hence it is measurable w.r.t. $\mathcal{F}_T$. The second term is independent of $\mathcal{F}_T$. These two facts yield

$$\mathbb{P}\left[Z(U) \in B | \mathcal{F}_T\right] = \mathbb{P}\left[Z(U) \in B | \xi \cdot Z(T)\right] \qquad \forall\, B \in \mathcal{B}^1,$$

as is easy to show using elementary properties of conditional expectations. In view of the obvious relation $\sigma(\xi \cdot Z(T)) \subset \sigma(Z(T)) \subset \mathcal{F}_T$, this implies

$$\mathbb{P}\left[Z(U) \in B | \mathcal{F}_T\right] = \mathbb{P}\left[Z(U) \in B | Z(T)\right]. \qquad \square$$

Theorem 4.3. *Suppose that the conditions of Lemma 4.2 are satisfied and that $\sigma_2(\cdot, T) \not\equiv 0$ for all $T \in [0, T^\star]$. Then the short rate r is a Markov process iff the partial derivative σ_2 has a representation*

$$\sigma_2(t, T) = \tau(t) \cdot \zeta(T) \qquad \forall\, (t, T) \in \Delta,$$

where $\tau: \; [0, T^\star] \to \mathbb{R}$ and $\zeta: \; [0, T^\star] \to]0, \infty[$ are continuously differentiable functions.

Proof: Assume that r is Markovian. Then Lemma 4.2 implies that for all $0 < T < U \leq T^\star$, the restriction of $\sigma_2(\cdot, U)$ to $[0, T]$ is a scalar multiple of $\sigma_2(\cdot, T)$.

From this it is easy to prove that

$$\sigma_2(t, S) \neq 0 \qquad \forall\, S \in [t, T^\star]$$

if there is a T such that $\sigma_2(t, T) \neq 0$. (Note that by Assumption 2.3, σ_2 is a continuous function and that there is no $T \in [0, T^\star]$ such that the function $t \mapsto \sigma_2(t, T)$ is identically zero.)

In particular, we have $\sigma_2(t, T^\star) \neq 0$ if $\sigma_2(t, T) \neq 0$ for some T. Hence the constant ξ that exists by Lemma 4.2 cannot be zero. This means $\sigma_2(t, T) = \sigma_2(t, T^\star)/\xi$ on $[0, T]$, and the following definitions for $(t, T) \in \Delta$ make sense:

$$\tau(t) := \sigma_2(t, T^\star)$$

and

$$\zeta(T) := \frac{\sigma_2(s,T)}{\sigma_2(s,T^\star)},$$

for any $s \in [0,T]$ satisfying $\sigma_2(s,T) \neq 0$ (and hence $\sigma_2(s,T^\star) \neq 0$). By assumption, for every $T \in [0,T^\star]$ at least one such s exists, and by the above considerations, this definition of ζ does not depend on the choice of s.

The continuous differentiability of the functions τ and ζ defined above is clear from the continuous differentiability of σ_2. $\square$

Theorem 4.3 enables us to characterize the class of *stationary* volatility structures which lead to Markovian short rate processes:

Theorem 4.4. *Under the hypotheses of Lemma 4.2, the following is true:*
If r is a Markov process and if, in addition, the volatility structure is stationary—that is, if there exists a twice continuously differentiable function $\tilde{\sigma} : [0,T^\star] \to [0,\infty[$ such that $\sigma(t,T) = \tilde{\sigma}(T-t)$ for all $(t,T) \in \Delta$—then σ has one of the following forms:

$$\begin{aligned} &\sigma(t,T) = \frac{\widehat{\sigma}}{a} \cdot \left(1 - e^{-a\cdot(T-t)}\right) && \textit{(Vasiček volatility structure)} \\ \textit{or}\quad &\sigma(t,T) = \widehat{\sigma} \cdot (T-t) && \textit{(Ho-Lee volatility structure),} \end{aligned}$$

with real constants $\widehat{\sigma} > 0$ and $a \neq 0$.

Remark: The converse statement is true by Lemma 4.2, so Theorem 4.4 proves the *equivalence* of the short rate being Markovian and the volatility structure being of one of the above-mentioned forms.

Proof of the theorem: Since σ is stationary by assumption, the partial derivative $\sigma_2(t,T)$ is given by

$$\frac{\partial}{\partial T}\tilde{\sigma}(T-t) = \tilde{\sigma}'(T-t) \tag{20}$$

and thus is stationary too.

If $\sigma_2 \equiv 0$, then $\sigma \equiv 0$ as well—keep in mind the boundary condition from Assumption 2.3, namely, $\sigma(s,s) = 0$ for all $s \in [0,T^\star]$. But $\sigma \equiv 0$ is not of interest and is precluded by Assumption 2.3.

So $\sigma_2 \not\equiv 0$, and we have $\sigma_2(t,T) \neq 0$ for all $(t,T) \in \Delta$. To see this, we use the fact that for models with Markovian short rates, $\sigma_2(t,T) \neq 0$ implies $\sigma_2(t,S) \neq 0$ for all $S \in [t,T^\star]$. (This property has already been used in the proof of Theorem 4.3.) In particular, if $(t,T) \in \Delta$ satisfies $\sigma_2(t,T) \neq 0$, then this implies $\sigma_2(t,t) \neq 0$. By stationarity of σ_2, $\sigma_2(s,s) \neq 0$ for all $s \in [0,T^\star]$. Again making use of the above-mentioned property, we get the desired result.

In fact, we have $\sigma_2 > 0$, because the continuous function σ_2 has no zeros on the connected set Δ and thus is either strictly positive or strictly negative on Δ. The case $\sigma_2 < 0$ is precluded by Assumption 2.3.

$\sigma_2 > 0$ implies that, in particular, the condition "$\sigma_2(\cdot, T) \not\equiv 0$ for all $T \in [0, T^\star]$" of Theorem 4.3 is satisfied. Hence there exists a multiplicative decomposition of σ_2:

$$\sigma_2(t, T) = \tau(t)\zeta(T),$$

$\zeta > 0$ and τ being continuously differentiable functions. $\sigma_2 > 0$ now implies $\tau > 0$. Differentiation with respect to t and T, respectively, gives

$$\tau'(t)\zeta(T) = -\tilde{\sigma}''(T - t) = -\tau(t)\zeta'(T),$$

where we have used (20). Hence

$$(\log \tau)'(t) \equiv \frac{\tau'(t)}{\tau(t)} = -\frac{\zeta'(T)}{\zeta(T)} \equiv -(\log \zeta)'(T) \qquad \forall\, (t, T) \in \Delta.$$

Since t and T are independent variables, neither side of this equation can actually depend on t or T. Hence, both sides are constant. Denoting their common value by a, we thus have $\tau(t) = \exp(at + C_1)$ and $\zeta(T) = \exp(-aT + C_2)$ with two real constants C_1 and C_2, and hence

$$\sigma_2(t, T) = \tau(t)\zeta(T) = e^{C_1 + C_2} \cdot e^{-a\cdot(T-t)}. \tag{21}$$

We define $\widehat{\sigma} = e^{C_1 + C_2}$. Integrating (21) with respect to T and using $\sigma(s, s) = 0$, $s \in [0, T^\star]$, we get the desired representation. □

Remarks: Assuming the volatility structure of a bond price model to be stationary is quite natural if one works with deterministic volatilities. In fact, it is analogous to assuming the volatility parameter of a stock price model to be constant over time.
Provided the stationarity of σ, we have shown that the class of volatility structures leading to a Markovian short rate process r is parametrized by two real numbers, $\widehat{\sigma}$ and a.

Lemma 4.2 enables us to establish a stochastic differential equation that is satisfied by the short rate process r. The following result is analogous to that obtained in [5] for the Gaussian case:

Corollary 4.5. *Suppose that the driving Lévy process L satisfies the hypotheses of Lemma 4.2. Suppose further that r is a Markov process and that there is no $T \in [0, T^\star]$ such that the function $\sigma_2(\cdot, T) : [0, T] \to \mathbb{R}$ vanishes identically. Then r satisfies the stochastic differential equation*

$$\begin{aligned} dr(t) = &\left[\frac{\partial}{\partial t} f(0, t) + \theta'(0) \cdot \sigma_2(t, t) + \int_0^t \frac{\partial^2}{\partial t^2} \left(\theta(\sigma(s, t))\right) ds\right] dt \\ &- \frac{\zeta'(t)}{\zeta(t)} \left[\int_0^t \frac{\partial}{\partial t} \left(\theta(\sigma(s, t))\right) ds + f(0, t) - r(t)\right] dt - \sigma_2(t, t) dL_t, \end{aligned} \tag{22}$$

where the continuously differentiable function ζ stems from a factorization $\sigma_2(t,T) = \tau(t)\zeta(T)$ according to Theorem 4.3.

If, in addition, the volatility structure σ is stationary, we have

$$(23)\quad dr(t) = \Big[f_2(0,t) + \theta'(\sigma(0,t))\sigma_2(0,t)\Big]dt \\ - \frac{\zeta'(t)}{\zeta(t)}\Big[\theta(\sigma(0,t)) + f(0,t) - r(t)\Big]dt - \sigma_2(t,t)dL_t.$$

Proof: Consider representation (17):

$$(24)\qquad r(t) = f(0,t) + \int_0^t \frac{\partial}{\partial t}\theta(\sigma(s,t))ds - \int_0^t \sigma_2(s,t)dL_s.$$

Clearly, the first two summands in this representation are differentiable w.r.t. t. Keep in mind Assumptions 2.1 and 2.3, which guarantee the differentiability of the initial bond price structure $t \mapsto P(0,t)$ and of the volatility structure σ, as well as the fact that θ is an analytic function. For the third term, we use the decomposition $\sigma_2(t,T) = \tau(t)\zeta(T)$, which allows us to write

$$\int_0^t \sigma_2(s,t)dL_s = \int_0^t \tau(s)dL_s \cdot \zeta(t).$$

Since $X_t := \int_0^t \tau(s)dL_s$ and $Y_t := \zeta(t)$ are semimartingales, where Y has continuous paths of bounded variation, we have $d(XY) = XdY + YdX$. Using (24) to express $\int_0^t \sigma_2(s,t)dL_s$ as a function of the short rate $r(t)$, we then arrive at (22).

For stationary σ, (23) follows because $\sigma_2(t,T) = -\sigma_1(t,T)$ and the integrals appearing in the drift coefficient can be computed explicitly. $\square$

In particular, using a Vasiček volatility structure in (23), we get

$$(25)\quad dr(t) = a\Big[\frac{f_2(0,t)}{a} + \theta'\Big(\frac{\widehat{\sigma}}{a}\Big(1-e^{-at}\Big)\Big)\frac{\widehat{\sigma}}{a}e^{-at} + \theta\Big(\frac{\widehat{\sigma}}{a}\Big(1-e^{-at}\Big)\Big) \\ + f(0,t) - r(t)\Big]dt - \widehat{\sigma}dL_t$$

This means that the short rate process r is a mean-reverting process satisfying a SDE of the form

$$(26)\qquad dr(t) = a(\rho(t) - r(t))dt - \widehat{\sigma}\cdot dL_s,$$

where the mean ρ is a deterministic process and where $\widehat{\sigma}$ and a are the parameters of the Vasiček structure. If the driving Lévy process L is a Brownian motion, then (26) turns out to be a generalized Vasiček model for the short rate (see e. g. [2], p. 155). Even with a flat initial forward rate term structure we have a time-dependent mean $\rho(t)$ here. This is

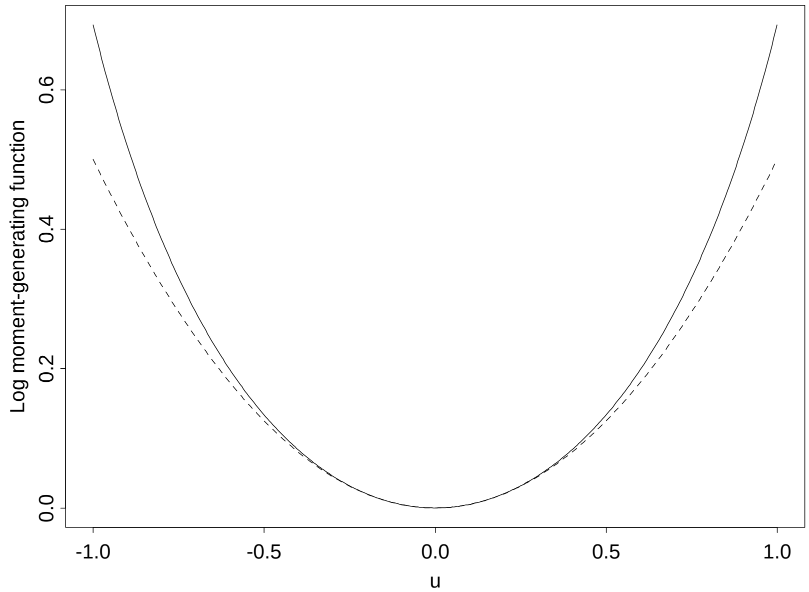

Figure 1: Log moment-generating function θ of a centered and symmetric hyperbolic distribution (parameters $\zeta = 0.01$, $\delta = 0.00707$) vs. log mgf of the standard normal (dashed line).

due to the fact that the classical Vasiček model cannot produce flat forward rate curves: Thus imposing the initial condition of a flat term structure means that we have to choose one of the *generalized* Vasiček models.

For the generalized Vasiček model, even in the case of an initially flat term structure of forward rates the mean to which the short rate reverts differs from the level of the forward rates. In our class of models, which includes the Vasiček model, we observe a similar phenomenon. Letting $t \to \infty$ in (25) and assuming $f(0,t) = \mu$ $(t \geq 0)$ and $a > 0$, we get the long-term mean

$$m := \lim_{t\to\infty} \rho(t) = \mu + \theta\Big(\frac{\widehat{\sigma}}{a}\Big).$$

We observe that $\widehat{\sigma}/a$ is the limit of the bond volatility as the time to maturity goes to infinity. Since $\theta(x) > 0$ for $x \neq 0$, invariably $m > \mu$. The size of the deviation $\theta(\widehat{\sigma}/a)$ depends on the kind of Lévy process used. In the generalized Vasiček model, $\theta_{\text{Gauss}}(u) = u^2/2$, while in a model driven by a centered and symmetric hyperbolic Lévy process (see Section 5), $\theta_{\text{Lévy}}(u) > u^2/2$ $(u \neq 0)$. See Figure 1. However, the two functions are asymptotically equal near $u = 0$, and with reasonable parameter values, e. g. $\widehat{\sigma} = 0.015$, $a = 0.5$, the difference is very small: $\theta_{\text{Lévy}}(0.03) = 0.4501 \cdot 10^{-3}$ vs. $\theta_{\text{Gauss}}(0.03) = 0.45 \cdot 10^{-3}$.

In Lemma 3.2, we have shown that the forward rates are given by

$$f(t,T) = f(0,T) + \int_0^t \theta'(\sigma(s,T))\sigma_2(s,T)ds - \int_0^t \sigma_2(s,T)dL_s \qquad (t \in [0,T]).$$

Suppose that the short rate r is a Markov process. Using the factorization $\sigma_2(s,T) = \tau(s) \cdot \zeta(T)$ from Theorem 4.3, it is then easy to show that, for a fixed time t, the forward rates $f(t,T)$, $T \in [t, T^\star]$, are deterministic functions of the current short rate $r(t)$:

$$\begin{aligned} f(t,T) = f(0,T) &+ \int_0^t \Big(\theta'(\sigma(s,T)) - \theta'(\sigma(s,t))\Big) \cdot \sigma_2(s,T)\, ds \\ &+ \frac{\zeta(T)}{\zeta(t)}\left(r(t) - f(0,t)\right). \end{aligned} \tag{27}$$

If the volatility structure σ is stationary, then we can use $\sigma_2(s,T) = -\sigma_1(s,T)$, as in the proof of (23). Using $\sigma_2(s,T) = \sigma_2(s,t)\zeta(T)/\zeta(t)$, the integral can then be computed explicitly, which yields the formula

$$\begin{aligned} f(t,T) = f(0,T) + \theta(\sigma(0,T)) - \theta(\sigma(t,T)) &- \frac{\zeta(T)}{\zeta(t)}\theta(\sigma(0,t)) \\ &+ \frac{\zeta(T)}{\zeta(t)}\left(r(t) - f(0,t)\right). \end{aligned} \tag{28}$$

Hence for the same value of the short rate and the same initial forward rate curve, the difference of the forward rates in the general Lévy model and the Gaussian model is

$$\begin{aligned} f_{\text{Lévy}}(t,T) - f_{\text{Gauss}}(t,T) = \Big(\theta(\sigma(0,T)) - \frac{\sigma(0,T)^2}{2}\Big) &- \Big(\theta(\sigma(t,T)) - \frac{\sigma(t,T)^2}{2}\Big) \\ &- \frac{\zeta(T)}{\zeta(t)}\Big(\theta(\sigma(0,t)) - \frac{\sigma(0,t)^2}{2}\Big). \end{aligned}$$

5 A hyperbolic term structure model

In this section, we shall use a hyperbolic Lévy motion as the driving Lévy process L. We will compute forward rate curves and option prices and compare them with those obtained in a Gaussian setting. As we have seen in the previous section, the choice of the Vasiček or Ho-Lee volatility structure is—under certain assumptions—equivalent to the short rate being Markovian. Therefore, it is natural to concentrate on these two types of volatility structures. We will choose the *Vasiček structure* in our computations.

The class of Lévy processes we shall use in the following is that of hyperbolic Lévy motions. These processes were introduced in finance by [7] to model stock price dynamics.

A *hyperbolic Lévy motion* L is characterized by the variable L_1 having a *hyperbolic distribution*, which is given by its Lebesgue density

$$\mathrm{hyp}_{\alpha,\beta,\delta,\mu}(x) = \frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2-\beta^2})}\exp\left(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta\cdot(x-\mu)\right),$$

where K_1 is the modified Bessel function of the third kind with index 1 (see [1], Sec. 9.6), and where the four real parameters $\alpha, \beta, \delta, \mu$ satisfy the relations $|\beta| < \alpha$ and $\delta > 0$. The characteristic function of this distribution can be easily computed:

$$(29) \qquad \phi_{\alpha,\beta,\delta,\mu}(u) = e^{iu\mu}\cdot\frac{\sqrt{\alpha^2-\beta^2}}{K_1(\delta\sqrt{\alpha^2-\beta^2})}\cdot\frac{K_1(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{\sqrt{\alpha^2-(\beta+iu)^2}} \qquad (u\in\mathbb{R}).$$

Using the asymptotic expansion

$$K_1(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}\{1+\frac{3}{8z}-\frac{15}{128z^2}-\cdots\}$$

of the function K_1 (see [1], 9.7.2), which is valid for $|\arg(z)| < 3\pi/2$, one can see from (29) that the assumptions of Theorem 4.4 are satisfied. Hence, in a model driven by a hyperbolic Lévy motion, the volatility structures of the Vasiček and of the Ho-Lee type are the only stationary volatility structures leading to a Markovian short rate process.

The moment generating function of a hyperbolic distribution is

$$M_{\alpha,\beta,\delta,\mu}(u) = e^{u\mu}\cdot\frac{\sqrt{\alpha^2-\beta^2}}{K_1(\delta\sqrt{\alpha^2-\beta^2})}\cdot\frac{K_1(\delta\sqrt{\alpha^2-(\beta+u)^2})}{\sqrt{\alpha^2-(\beta+u)^2}} \qquad (|\beta+u|<\alpha),$$

from which we get the function θ by taking the logarithm. Since the moment generating function exists for all u satisfying $|u| < \alpha - |\beta|$, the boundedness condition given in Assumption 2.3 reads as follows:

$$\sup\{|\sigma(t,T)| : 0 \le t \le T \le T^\star\} < \alpha - |\beta|.$$

We will limit our examinations to *centered and symmetric* hyperbolic distributions. Therefore, the moment generating function can be parametrized by only two real numbers, namely, δ from above and $\zeta := \alpha\cdot\delta$:

$$(30) \qquad M_{\zeta,\delta}(u) = \frac{\zeta}{K_1(\zeta)}\cdot\frac{K_1(\sqrt{\zeta^2-\delta^2u^2})}{\sqrt{\zeta^2-\delta^2u^2}} \qquad (|u|<\frac{\zeta}{\delta}).$$

An analogous formula is valid for the characteristic function, which in this case is real-valued.

In order to make the hyperbolic Lévy motion comparable to a standard Brownian motion, we furthermore require L_1 to have unit variance. This eliminates the parameter δ, which now can be expressed as a function of ζ:

$$\delta(\zeta) := \sqrt{\frac{\zeta K_1(\zeta)}{K_2(\zeta)}},$$

where K_2 denotes the modified Bessel function of the third kind with index 2. (Using Abramowitz and Stegun 9.6.28, this formula is easily deduced from (30) or from the analogous formula for the characteristic function, respectively.)

In our numerical examples, we examine two different values for the parameter ζ. $\zeta = 10$ corresponds to a hyperbolic distribution whose density is very close to that of a standard normal. $\zeta = 0.01$ represents a hyperbolic distribution which is far from a standard normal and puts considerably more mass in the center and in the tails. The corresponding values for δ are $\delta(10) = 2.94$ and $\delta(0.01) = 0.00707$.

Furthermore, we always choose a Vasiček volatility structure

$$\sigma(t,T) = \frac{\widehat{\sigma}}{a}\left(1 - e^{-a\cdot(T-t)}\right)$$

with parameters $\widehat{\sigma} = 0.015$ and $a = 0.5$. The term structure at time 0 is assumed to be flat, $f(0,t) = 0.05$ for all $t \in [0, T^\star]$.

We start with a numerical examination of the forward rate curves. Using Equation (28) with a Vasiček volatility structure gives

$$f(t,T) = f(0,T) + \theta(\sigma(0,T)) - \theta(\sigma(t,T)) + e^{-a\cdot(T-t)}\Big(f(0,t) - \theta(\sigma(0,t)) - r(t)\Big).$$

Evaluating this for different values of T and $r(t)$, we get the diagram given in Figure 2. Figure 3 shows the difference of hyperbolic minus Gaussian forward rates. This difference turns out to be positive and very small.

In the term structure model presented here, the martingale measure is unique. (See [15].) This means that once we have fixed the term structure model, prices of interest-rate sensitive contingent claims are uniquely determined. Thus we are in the same situation as in the Black-Scholes setting, where option prices are uniquely determined by the underlying stock prices.

Since the model is formulated under a martingale measure $\mathbb{P}$, prices of integrable contingent claims are given by the expected discounted payoff under this measure $\mathbb{P}$.

Thus, the time 0 price of a European call option on a bond maturing at T, with exercise date t and strike price K, is given by

$$C(0,t,T;K) := \mathrm{E}\left[\frac{1}{\beta(t)}\Big(P(t,T) - K\Big)^+\right]. \tag{31}$$

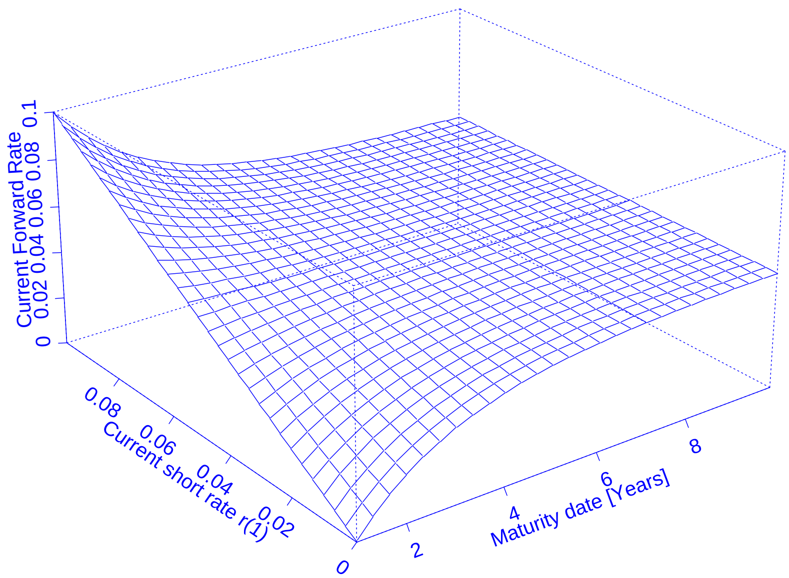

Figure 2: Forward rates with driving hyperbolic Lévy motion ($\zeta = 0.01$).

In the Gaussian setting, the option price (31) can be calculated using the change-of-numeraire approach described in [10] and [9]:

$$C(0,t,T;K) = P(0,T)\cdot\Phi(d) - K\cdot P(0,t)\cdot\Phi(d-\Sigma),$$

where Φ denotes the standard normal distribution function, and where Σ and d are given by

$$\begin{aligned}\Sigma &:= \sqrt{\int_0^t (\sigma(s,T)-\sigma(s,t))^2 ds}\\ d &:= \frac{\log P(0,T) - \log P(0,t) - \log K}{\Sigma} + \frac{1}{2}\Sigma.\end{aligned}$$

In a model driven by a hyperbolic Lévy motion, it is still possible to evaluate the option pricing formula (31) numerically: As we have mentioned before, Assumption 2.2, that is, the boundary condition $P(t,t) = 1$, implies

$$\frac{1}{\beta(t)} = P(0,t)\exp\left(-\int_0^t \theta(\sigma(s,t))ds + \int_0^t \sigma(s,t)dL_s\right).$$

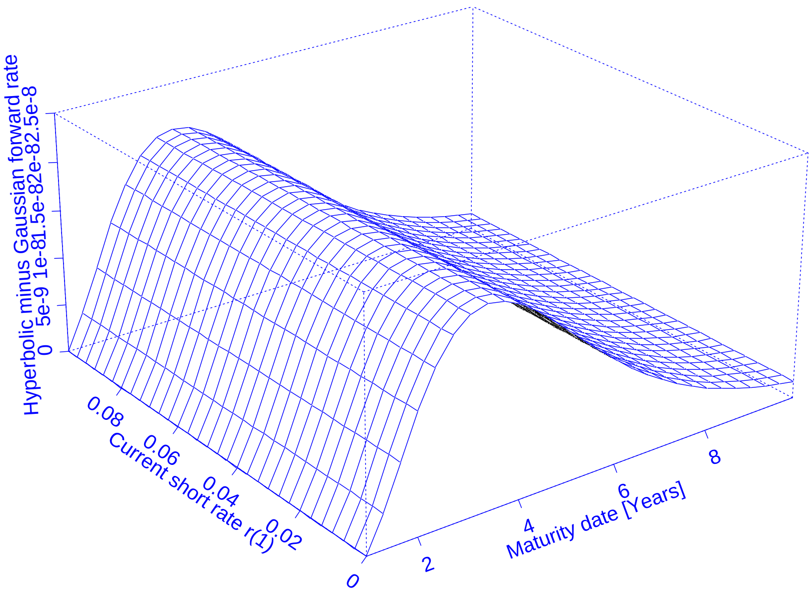

Figure 3: Difference of hyperbolic minus Gaussian forward rates.

Thus, the expectation in (31) can be written as

$$
\text{(32)} \quad \mathrm{E}\bigg[\bigg(P(0,T)\exp\Big(-\int_0^t \theta(\sigma(s,T))ds + \int_0^t \sigma(s,T)dL_s\Big) \\
- K \cdot P(0,t)\exp\Big(-\int_0^t \theta(\sigma(s,t))ds + \int_0^t \sigma(s,t)dL_s\Big)\bigg)^+\bigg].
$$

The only random variables appearing here are the two stochastic integrals $X := \int_0^t \sigma(s,T)dL_s$ and $Y := \int_0^t \sigma(s,t)dL_s$. The remaining terms are deterministic and can be computed easily.

In order to get the expectation (32), we have to know the joint distribution of X and Y. According to the remark following Lemma 3.1, the corresponding characteristic function is

$$
\text{(33)} \qquad \mathrm{E}\left[\exp\left(iuX + ivY\right)\right] = \exp\left(\int_0^t \chi(u \cdot \sigma(s,T) + v \cdot \sigma(s,t))ds\right).
$$

Remember that $\chi(u)$ denotes the cumulant generating function of L_1.

At the beginning of this section, we have verified that every hyperbolic Lévy motion satisfies the conditions of Lemma 4.1. Since we use a Vasiček volatility structure, for $t < T$ the functions $\sigma(\cdot,t)$ and $\sigma(\cdot,T)$ are continuous, and neither of them is a scalar multiple of the other. Hence, Lemma 4.1 implies that the joint distribution of the two

Strike price	Gaussian	Hyperbolic ($\zeta = 10$)	Hyperbolic ($\zeta = 0.01$)
0.9	0.048731	0.048731	0.048731
0.91	0.039219	0.039219	0.039222
0.92	0.029707	0.029708	0.029725
0.93	0.020217	0.020227	0.020290
0.94	0.011095	0.011105	0.011170
0.95	0.004002	0.003961	0.003615
0.96	0.000741	0.000741	0.000757
0.97	0.000058	0.000074	0.000162
0.98	0.000002	0.000005	0.000035
0.99	0.000000	0.000000	0.000008
1	0.000000	0.000000	0.000002

Table 1: Prices of European call options.

random variables X and Y defined above is continuous with respect to λ^2. Therefore, the joint distribution of X and Y possesses a Lebesgue density $\rho(x,y)$ given by

(34)
$$\rho(x,y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp\left(\int_0^t \chi(u \cdot \sigma(s,T) + v \cdot \sigma(s,t)) ds \right) e^{-iux-ivy} \, d\lambda^2(u,v).$$

In order to further investigate the effect of going from a Gaussian term structure model to a model driven by a hyperbolic Lévy motion, we calculate some option prices. In Table 1, we give option prices for different exercise prices. The underlying bond matures at $T = 2$, and the exercise date is chosen to be $t = 1$. The second column shows the option prices in a model driven by a standard Brownian motion W, while the third (resp. fourth) column gives the corresponding prices in a model driven by the centered and symmetric hyperbolic Lévy motion L with parameter $\zeta = 10$ (resp. $\zeta = 0.01$).

The hyperbolic prices for $\zeta = 10$ are quite similar to those in the Gaussian model, reflecting the fact that a hyperbolic distribution with $\zeta = 10$ is close to a normal distribution. For $\zeta = 0.01$, i. e. a hyperbolic distribution which is far from the standard normal, we have larger differences. In both cases, the largest absolute deviation occurs when the strike price is about 0.95. A closer numerical examination reveals that the maximum deviation is seen if the strike price equals the time $t = 1$ *forward price* of the $T = 2$-bond, which is defined as $F(t,T) = P(0,T)/P(0,t)$. In our example, $F(t,T) = \exp(-0.05) \approx 0.951$. One could call these options "at the money". At-the-money prices for the hyperbolic model ($\zeta = 0.01$) are almost 10% below those of the Gaussian model.

If the strike is 1, both Gaussian and hyperbolic option prices are virtually zero. (To conform with reality, these prices should be exactly zero, since real zero-bond prices are

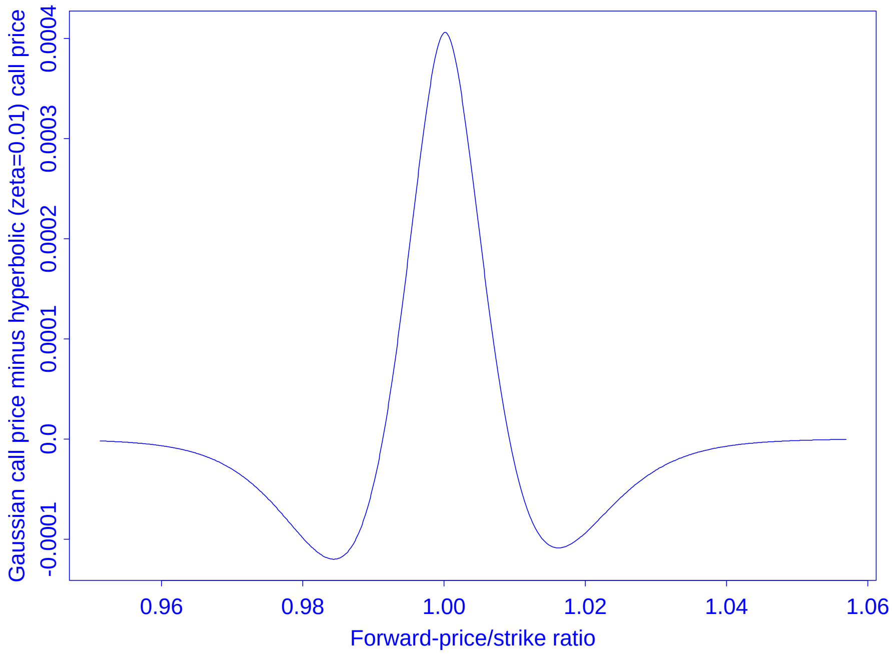

Figure 4: Difference of Gaussian minus hyperbolic ($\zeta = 0.01$) call option prices.

always below 1.)

Plotting the difference of option prices as a function of the forward-price/strike ratio, we get the W-shaped pattern shown in Figure 4: In the hyperbolic model, at-the-money prices are lower, while in-the-money and out-of-the-money prices are higher than in the Gaussian model.

6 Conclusion

As a generalization of the Gaussian term structure model, we present a model that admits as driving processes a large class of Lévy processes. The key idea is not to start with a stochastic differential equation, but with the explicit bond price formula. These two approaches turn out to be equivalent in the Gaussian case, but if one replaces the Brownian motion by a Lévy process that has a purely discontinuous component, they no longer are. We derive explicit formulas for the corresponding forward rate and short rate processes in the general model.

An interesting subclass of models is that in which the short rate process is a Markov process. Under an additional assumption concerning the characteristic function, we show that in this model the short rate process is Markovian iff the volatility structure admits a multiplicative decomposition. Using this decomposition, we see that the only *stationary* volatility structures which lead to Markovian short rate processes are those of the Vasiček and the Ho-Lee type.

For numerical examination, we choose a model driven by a hyperbolic Lévy motion and compare it to the Gaussian model. Forward rates turn out to be slightly higher in the hyperbolic model. Option prices, plotted as a function of the forward-price/strike ratio, show a W-shaped deviation from option prices in the Gaussian model. Similarly structured deviations are observed in [8], where hyperbolic stock option prices are compared to standard Black-Scholes prices.

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