Fractional Hida Malliavin Derivatives and Series Representations of Fractional Conditional Expectations

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Abstract

We represent fractional conditional expectations of a functional of fractional Brownian motion as a convergent series in $L^2(\mathbb{P}^H)$ space. When the target random variable is some function of a discrete trajectory of fractional Brownian motion, we obtain a backward Taylor series representation; when the target functional is generated by a continuous fractional filtration, the series representation is obtained by applying a "frozen path" operator and an exponential operator to the functional. Three examples are provided to show that our representation gives useful series expansions of ordinary expectations of target random variables.

Keywords: Fractional Brownian motion ; Malliavin calculus ; fractional Hida Malliavin derivative ; fractional Clark-Hausmann-Ocone formula
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1 Introduction

Fractional Brownian motion (fBm) ([13, 17]), can be divided into three very different classes according to the values of its Hurst index $H > 1/2$, $H = 1/2$ and $H < 1/2$. When $H > 1/2$, a huge number of phenomena can be modeled in terms of this class of processes, such as the level of the optimum dam sizing, the logarithm of the stock return and financial turbulence [17]. Unlike Brownian motion (Bm), when $H > 1/2$, the fBm has positively correlated increments. This complex covariance structure of increments results that evaluating conditional expectations of functionals of fBm $\{B_t^H\}_{t\in\mathbb{R}}$ is a notoriously difficult problem. Grippenberg and Norros [7] provided a technical and difficult approach to calculate the conditional expectation of fBm. Fink et al. [5] also addressed this problem when studying the price of a zero-coupon bond in a fractional bond market. Since fBm is generally not a Markov process, both authors restricted themselves to calculate conditional expectations given the current value of B_t^H, and not given the whole path of B_t^H preceding t. For different types of processes, there are other types of Malliavin calculus based approaches to calculate conditional expectations. We refer to Kuna and Streit [14] and all references therein.

This paper presents a different notion – fractional conditional expectation, which helps us to propose an original way to evaluate ordinary expectations of functionals of fBm with

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$H > 1/2$. The fractional conditional expectation, which shouldn't be confused with the ordinary conditional expectation, is introduced in [4]. This functional characterizes the fractional Clark-Hausmann-Ocone formula. It is based on the Malliavin calculus with respect to fBm as presented in Biagini et al. [4], Chapter 2 and Chapter 3. Since the 70's, Malliavin calculus plays an important role in analysis on the Wiener space as well as in the study on stochastic differential equations. The main advantage of Malliavin calculus is that, it allows to give sufficient conditions for the distribution of a random variable to have a smooth (differentiable) density with respect to Lebesgue measure and to give bounds for this density and its derivatives. By using the so-called fractional Clark-Hausmann-Ocone formula, our first main result (see Section 3) represents the fractional conditional expectations of functionals of fBm's discrete trajectory as a convergent series in $L^2(\mathbb{P}^H)$. Our second main result is more general under a different sufficient condition for the convergence. It is obtained from the fact that the fBm has the "martingale" property under fractional conditional expectation. This property leads to an exponential expansion of the fractional conditional expectations. The latter result partially extends our previous [11], where we proved this representation for conditional expectations of a functional of Bm. In Section 4, we show 3 examples, where our exponential formula is used to evaluate ordinary expectations of functionals of fBm. The first example is on the fractional Merton model of interest rates, where the series expansion for the bond price (including the following, all our results are limited to the bond price at time 0.) can be simplified into a regular exponential, whereas the latter is a well-known result. Our next two examples are original results. In the second example, we show how to calculate the first terms of the series of a bond price in a fractional interest rates model with time-dependent volatility and are able to approximate the bond price numerically. In the third example, we provide a new series representation of the characteristic function of geometric fBm, which leads to having solved a Fourier transform problem using power series representation. The proofs of main results are given in the appendix.

2 Preliminaries

2.1 Fractional Brownian Motion

A real-valued standard fBm can be defined independently and equivalently using a moving average representation [17] and a harmonizable representation [20]. In fact, up to a multiplicative scaling factor, fBm is the unique Gaussian, self-similar, with stationary increments process. Therefore, a standard fBm can be defined from the uniqueness of its covariance structure:

Definition 2.1 *A standard fBm $\{B_t^H\}_{t\in\mathbb{R}}$ with Hurst index $H \in (0,1)$ is the unique zero mean Gaussian process with almost surely continuous non-differentiable sample path and with covariance function: for any $s, t \in \mathbb{R}$,*

$$E\left[B_s^H B_t^H\right] = \frac{1}{2}\Big(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\Big).$$

Without any loss of generality, we restrict the fBm to nonnegative-time process and denote the corresponding probability space by $\left(\Omega, (\mathcal{F}_t^H)_{t\in\mathbb{R}_+}, \mathbb{P}^H\right)$, where $(\mathcal{F}_t^H)_{t\in\mathbb{R}_+}$ is the natural filtration generated by fractional Wiener chaos (see e.g. [4], Page 49) and $\mathbb{P}^H$ is the corresponding probability measure. It is also worth noting that, as in [4], Chapter 3, we only consider the case where $H > 1/2$ in this work. A similar study can be done for classes of fBm with $H < 1/2$ for the future.

2.2 Fractional Hida Malliavin Derivative

Let $L^2(\mathbb{P}^H) := L^2(\Omega, (\mathcal{F}_t^H)_{t\in\mathbb{R}_+}, \mathbb{P}^H)$ denote a Hilbert space of random variables F equipped with the norm

$$\|F\|_{L^2(\mathbb{P}^H)} := \sqrt{E|F|^2} < +\infty.$$

We indicate by $L^2_{\varphi_H}(\mathbb{R}_+)$ the separable Hilbert space of deterministic functions, equipped with the inner product: for any $f, g \in L^2_{\varphi_H}(\mathbb{R}_+)$,

$$\langle f, g\rangle_H := \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(s)\, g(t)\, \varphi_H(s,t)\, \mathrm{d}s\, \mathrm{d}t,$$

where $\varphi_H(s,t) := H(2H-1)|s-t|^{2H-2}$ for any $s, t \in \mathbb{R}_+$. We also denote the norm by $\|f\|_{H,\mathbb{R}_+} := \sqrt{\langle f, f\rangle_H}$.

Note that fBm is an isonormal process and, for two functions $f, g \in L^2_{\varphi_H}(\mathbb{R}_+)$, their stochastic integrals with respect to fBm $\int_{\mathbb{R}_+} f(s)\, \mathrm{d}B_s^H$ and $\int_{\mathbb{R}_+} g(s)\, \mathrm{d}B_s^H$ are well-defined, zero-mean, Gaussian random variables with covariance

$$Cov\Big(\int_{\mathbb{R}_+} f(s)\, \mathrm{d}B_s^H, \int_{\mathbb{R}_+} g(s)\, \mathrm{d}B_s^H\Big) = \langle f, g\rangle_H.$$

The dual space of $L^2_{\varphi_H}(\mathbb{R}_+)$ is denoted by $L^2_{\varphi_H}{}'(\mathbb{R}_+)$, and the pairing between $\nu \in L^2_{\varphi_H}{}'(\mathbb{R}_+)$ and $f \in L^2_{\varphi_H}(\mathbb{R}_+)$ is given as the usual inner product $< \nu, f >$. We allow this inner product to return a random variable, because by using the Bochner-Minlos theorem (see e.g. [4, 8, 15]), the probability measure $\mathbb{P}^H$ is the one which allows

$$< B^H, \chi_{[0,t]} >= \int_0^t 1\, \mathrm{d}B_u^H = B_t^H$$

to be an element in $L^2(\mathbb{P}^H)$.

For presenting an element $F \in L^2(\mathbb{P}^H)$ using backward Taylor expansion, we first introduce the notion of fractional Hida Malliavin derivative (see Definition 3.3.1 in [4]).

Definition 2.2 (Fractional Hida Malliavin Derivative) *Given an operator $G : L^2_{\varphi_H}{}'(\mathbb{R}_+) \to \mathbb{R}$ and some $\gamma \in L^2_{\varphi_H}{}'(\mathbb{R}_+)$. G is said to have a directional derivative in the direction γ if, for all $\nu \in L^2_{\varphi_H}{}'(\mathbb{R}_+)$, there exists an element $X_{\nu,\gamma}$ in $L^2(\mathbb{P}^H)$, such that*

$$\frac{G(\nu+\epsilon\gamma) - G(\nu)}{\epsilon} \xrightarrow[\epsilon\to 0]{L^2(\mathbb{P}^H)} X_{\nu,\gamma}.$$

We say G is fractional Hida Malliavin differentiable if there exists a map $\Psi : \mathbb{R}_+ \times L^2_{\varphi_H}{}'(\mathbb{R}_+) \to L^2(\mathbb{P}^H)$ such that for all $\nu \in L^2_{\varphi_H}{}'(\mathbb{R}_+)$, $\Psi(\cdot, \nu)\gamma(\cdot)$ is $L^2(\mathbb{P}^H)$-integrable and

$$X_{\nu,\gamma} = \int_{\mathbb{R}_+} \Psi(t,\nu)\,\gamma(t)\, \mathrm{d}t \text{ for all } \gamma \in L^2_{\varphi_H}{}'(\mathbb{R}_+).$$

Then we set, for all $t \in \mathbb{R}_+$,

$$D_t^H G(\nu) := \Psi(t,\nu)$$

and we call $D_t^H G(\nu)$ the fractional Hida Malliavin derivative with order H of G on ν at t.

Remark that the fractional Hida Malliavin derivative with respect to fBm extends the classical one (see [19]) with respect to Bm, and it possesses some nice properties similar to the classical derivatives. For example, the chain rule is still valid: if $G(\nu) = \Phi\left(B^H_{t_1}, ..., B^H_{t_n}\right)$, with $\Phi:\ \mathbb{R}^n \to \mathbb{R}$ being some deterministic differentiable function, then for $t \geq 0$,

$$D^H_t G(\nu) = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}\left(B^H_{t_1}, ..., B^H_{t_n}\right)\chi_{[0,t_i]}(t). \tag{2.1}$$

Now we introduce some other important features of fractional Hida Malliavin derivative that we will need to construct the convergent series of fractional conditional expectations. The most interesting one among them is the fractional Clark-Hausmann-Ocone formula.

Theorem 2.3 (Itô decomposition, see [4], Page 82) *Let $F \in L^2(\mathbb{P}^H)$, then F has the following representation via fractional Wick Itô Skorohod integral (FWISI): there exists a sequence of deterministic functions $f_n \in \hat{L}^2_{\varphi_H}\left(\mathbb{R}^n_+\right)$ such that*

$$F = \sum_{n=0}^{+\infty} I_n\left(f_n\right) \ \textit{in } L^2(\mathbb{P}^H) \textit{ with } \|F\|^2_{L^2(\mathbb{P}^H)} = \sum_{n=0}^{+\infty} n!\, \|f_n\|^2_{H,\mathbb{R}^n_+},$$

where

- *the sequence of multiple FWISI $(I_n(f_n))_{n\in\mathbb{N}}$ is defined in [4], Page 81;*
- *$\hat{L}^2_{\varphi_H}\left(\mathbb{R}^n_+\right)$ denotes the subspace of symmetric functions in $L^2_{\varphi_H}\left(\mathbb{R}^n_+\right)$;*
- *$\|f_n\|_{H,\mathbb{R}^n_+}$ is the norm defined by*

$$\|f_n\|^2_{H,\mathbb{R}^n_+} := n! \int_{\mathbb{R}^{2n}_+} f_n(u_1, \ldots, u_n) f_n(v_1, \ldots, v_n) \prod_{i=1}^{n} \varphi_H(u_i, v_i)(\,\mathrm{d}u)^{\otimes n}(\,\mathrm{d}v)^{\otimes n},$$

with

$$(\,\mathrm{d}u)^{\otimes n} := \,\mathrm{d}u_n \,\mathrm{d}u_{n-1} \ldots \,\mathrm{d}u_1. \tag{2.2}$$

Definition 2.4 *We define the sequence of Hermite polynomials $\{h_n\}_{n\geq 0}$, as solutions of the following equation, valid for all $t, x \in \mathbb{R}$:*

$$\exp\left(tx - \frac{t^2}{2}\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} h_n(x). \tag{2.3}$$

Proposition 2.5 *Let $f \in \hat{L}^2_{\varphi_H}\left(\mathbb{R}^n_+\right)$, then the multiple FWISI $I_n(f)$ exists and is given as*

$$I_n\left(f\right) = n! \int_{0\leq s_1\leq \ldots \leq s_n < +\infty} f\left(s_1, \ldots, s_n\right) \,\mathrm{d}\left(B^H_s\right)^{\otimes n},$$

where we denote by

$$\mathrm{d}\left(B^H_s\right)^{\otimes n} := \,\mathrm{d}B^H_{s_n} \,\mathrm{d}B^H_{s_{n-1}} \ldots \,\mathrm{d}B^H_{s_1}. \tag{2.4}$$

In particular, if there exists $g \in L^2_{\varphi_H}(\mathbb{R}_+)$ such that $f\left(s_1, \ldots, s_n\right) = g(s_1) \ldots g(s_n)$ for all $(s_1, \ldots, s_n) \in \mathbb{R}^n_+$, then

$$I_n\left(f\right) = \|g\|^n_{H,\mathbb{R}_+} h_n\left(\frac{\int_0^{+\infty} g\left(s\right) \,\mathrm{d}B^H_s}{\|g\|_{H,\mathbb{R}_+}}\right).$$

As a special case, when taking $f = \chi_{[t,T]^n}$ with $0 \le t < T$ in Proposition 2.5, we obtain

$$\int_{t \le s_1 \le \ldots \le s_n \le T} 1 \, \mathrm{d}\left(B_s^H\right)^{\otimes n} = \frac{(T-t)^{nH}}{n!} h_n\left(\frac{B_T^H - B_t^H}{(T-t)^H}\right). \tag{2.5}$$

From now on, the FWISI of a continuous-time stochastic process $\{X(s)\}_{s \ge 0}$ over any time interval $[a, b]$, is denoted by $\int_a^b X(s)\, \mathrm{d}B_s^H$.

Definition 2.6 (Fractional Conditional Expectation) *Let $F \in L^2(\mathbb{P}^H)$ be represented as (such an expansion exists, due to Definition 3.10.1 in [4])*

$$F = \sum_{n=0}^{+\infty} \int_{\mathbb{R}_+^n} g_n(s_1, \ldots, s_n)\, \mathrm{d}\left(B_s^H\right)^{\otimes n},$$

with some sequence of functions $g_n \in \hat{L}^2_{\varphi_H}\left(\mathbb{R}_+^n\right)$. Then for $t \ge 0$, we define the fractional conditional expectation of F with respect to $\mathcal{F}_t^H$ by

$$\tilde{E}\left[F|\mathcal{F}_t^H\right] := \sum_{n=0}^{+\infty} \int_{[0,t]^n} g_n(s_1, \ldots, s_n)\, \mathrm{d}\left(B_s^H\right)^{\otimes n}. \tag{2.6}$$

Remarks: There is no simple relationship between $\tilde{E}\left[F|\mathcal{F}_t^H\right]$ and $E\left[F|\mathcal{F}_t^H\right]$. Though different from the ordinary conditional expectation, the fractional conditional expectation has some properties, which are similar to those of conditional expectation (see [4], Page 84):

1. For all $t \ge 0$, $\tilde{E}\left[F|\mathcal{F}_t^H\right] = F$ is equivalent to F is $\mathcal{F}_t^H$-measurable. In particular,
$$\tilde{E}\left[F|\mathcal{F}_0^H\right] = E[F]. \tag{2.7}$$

2. For any $s, t \ge 0$,
$$\tilde{E}\left[\tilde{E}\left[F|\mathcal{F}_s^H\right]|\mathcal{F}_t^H\right] = \tilde{E}\left[F|\mathcal{F}^H_{\min\{s,t\}}\right]. \tag{2.8}$$
By taking $t = 0$ in (2.8) and using (2.7), a special case is obtained:
$$E\left[\tilde{E}\left[F|\mathcal{F}_s^H\right]\right] = E[F] \text{ for any } s \ge 0. \tag{2.9}$$

3. From (2.6), for all $t \ge 0$, $\tilde{E}\left[F|\mathcal{F}_t^H\right]$ is $\mathcal{F}_t^H$-measurable. By virtue of Lemma 3.10.5 1) in [4] and (2.14), (2.22) in [1] (by taking $r = 0$), one has,
$$\left\|\tilde{E}\left[F|\mathcal{F}_t^H\right]\right\|_{L^2(\mathbb{P}^H)} \le \|F\|_{L^2(\mathbb{P}^H)}. \tag{2.10}$$
This inequality yields that, if $F \in L^2(\mathbb{P}^H)$, then $\tilde{E}\left[F|\mathcal{F}_t^H\right] \in L^2(\mathbb{P}^H)$.

4. Under the transform $\tilde{E}[\cdot|\mathcal{F}_t^H]$, one recovers a property very similar to that of classical martingale. Although fBm is generally not a martingale, it is shown that, for $0 \le t \le T$, $\tilde{E}\left[B_T^H|\mathcal{F}_t^H\right] = B_t^H$, $\mathbb{P}^H$-a.s..

The result below, given in [4], extends the Clark-Hausmann-Ocone formula from Bm to fBm.

Theorem 2.7 (Fractional Clark-Hausmann-Ocone Formula) *Fix $T \ge 0$, let the random variable $F \in L^2(\mathbb{P}^H)$ be $\mathcal{F}_T^H$- measurable and fractional Hida Malliavin differentiable, then $\tilde{E}\left[D_t^H F|\mathcal{F}_t^H\right] \in L^2\left(\mathbb{P}^H\right)$ for all $t \in [0, T]$ and*

$$F = E[F] + \int_0^T \tilde{E}\left[D_t^H F|\mathcal{F}_t^H\right] \mathrm{d}B_t^H.$$

3 Main Results

3.1 Series Representation via Backward Taylor Expansion

Definition 3.1 *Fix $T \geq 0$. Let $\mathbb{D}^H_{\infty,T}$ denote the subspace of $L^2(\mathbb{P}^H)$ such that, for any $F \in \mathbb{D}^H_{\infty,T}$,*

$$E\Big[\sup_{s_1,\ldots,s_n\in[0,T]} \big|D^H_{s_n}\ldots D^H_{s_1}F\big|\Big]^2 < +\infty, \text{ for any integer } n \geq 1.$$

To simplify notation, we denote by $D^{H,0}_u = Id$ (identity function) and by $D^{H,k}_u = \underbrace{D^H_u \circ \ldots \circ D^H_u}_{k-tuple}$, the k-th composition of the fractional Hida Malliavin derivative.

Definition 3.2 *Let $F \in \mathbb{D}^H_{\infty,T}$. Assume $F = \Phi(B^H_{t_1}, \ldots, B^H_{t_J})$ for some integer $J \geq 1$ and $0 = t_0 < t_1 < t_2 < \ldots < t_{J-1} < t_J = T$. $\Phi : \mathbb{R}^J \to \mathbb{R}$ is an infinitely differentiable deterministic function. For $j \in \{1, \ldots, J\}$ and $r \in [t_{j-1}, t_j]$, let $\{\psi^{(r,t_j)}_k(F)\}_{k\in\mathbb{N}}$ be the sequence given as:*

$$\psi^{(r,t_j)}_0(F) = F; \tag{3.1}$$

and for $k \geq 1$, $\psi^{(r,t_j)}_k(F)$ equals

$$2^{-k}\sum_{\sum_{l=1}^J q_l=k}\prod_{i=1}^J\left(\frac{\left(|t_j-t_{i-1}|^{2H}-|t_j-t_i|^{2H}+|t_i-r|^{2H}-|r-t_{i-1}|^{2H}\right)^{q_i}}{q_i!}\right)D^{H,q_1}_{t_1}\ldots D^{H,q_J}_{t_J}F. \tag{3.2}$$

Our first main result is given by the following theorem, where a sufficient condition is provided, such that the fractional backward Taylor expansion (see (3.4)) of the fractional conditional expectation of F converges in $L^2(\mathbb{P}^H)$:

Assumption $(\mathcal{A})$**:** Let $F \in \mathbb{D}^H_{\infty,T}$. For some $r \in [0,T]$, assume F satisfies the following:

$$\begin{aligned}&\sum_{i=0}^N \left\|\sup_{\sum_{j=1}^J w_j=2N-i}\left|D^{H,w_1}_{t_1}\ldots D^{H,w_J}_{t_J}F\right|\right\|_{L^2(\mathbb{P}^H)}\binom{N}{i}^2\frac{(i!)^{1/2}(N+J-1)!}{2^{N-i}(N!)^2}\\ &\times\left(T^{2H}-r^{2H}+(T-r)^{2H}\right)^{N-i}(T-r)^{iH}\xrightarrow[N\to+\infty]{}0.\end{aligned}$$

Contrary to appearance, this condition is not difficult to check in practice. For example, it can be shown that any F verifying

$$\left\|\sup_{\sum_{j=1}^J w_j=N}\left|D^{H,w_1}_{t_1}\ldots D^{H,w_J}_{t_J}F\right|\right\|_{L^2(\mathbb{P}^H)} \leq cN^{(1/4-\epsilon)N}$$

for some $c > 0$, some $\epsilon \in (0, 1/4]$ and for all $N \in \mathbb{N}$ satisfies Assumption $(\mathcal{A})$.

Theorem 3.3 (Fractional Backward Taylor Expansion) *Let F satisfy Assumption $(\mathcal{A})$. Define*

$$I_r := \begin{cases} 1, & \text{if } r \in [0,t_1] \\ i, & \text{if } r \in (t_{i-1},t_i], \text{ for } 2 \leq i \leq J \end{cases} \tag{3.3}$$

and $(\tilde t_{I_r-1}, \tilde t_{I_r}, \dots, \tilde t_J) := (r, t_{I_r}, \dots, t_J)$. *Then the following series is convergent in* $L^2(\mathbb{P}^H)$*:*

$$\tilde E\big[F|\mathcal{F}_r^H\big] = \sum_{l=0}^{+\infty} (-1)^l \sum_{q_{I_r}+\ldots+q_J=l} \sum_{i_{I_r}=0}^{q_{I_r}} \cdots \sum_{i_J=0}^{q_J} \prod_{k=I_r}^{J} \frac{(-1)^{i_k} (\tilde t_k - \tilde t_{k-1})^{(q_k-i_k)H}}{(q_k-i_k)!}$$
$$\times \left(h_{q_{I_r}-i_{I_r}} \left(\frac{B^H_{\tilde t_{I_r}} - B^H_{\tilde t_{I_r-1}}}{(\tilde t_{I_r} - \tilde t_{I_r-1})^H} \right) \psi_{i_{I_r}}^{(\tilde t_{I_r-1}, \tilde t_{I_r})} \right) \circ \ldots \circ \left(h_{q_J-i_J} \left(\frac{B^H_{\tilde t_J} - B^H_{\tilde t_{J-1}}}{(\tilde t_J - \tilde t_{J-1})^H} \right) \psi_{i_J}^{(\tilde t_{J-1}, \tilde t_J)} \right)$$
$$(D^{H,q_{I_r}}_{t_{I_r}} \ldots D^{H,q_J}_{t_J} F). \tag{3.4}$$

In particular, when $F = \Phi(B^H_T)$*,*

$$\tilde E\big[F|\mathcal{F}_r^H\big] = \sum_{l=0}^{+\infty} (-1)^l \sum_{k=0}^{l} \frac{(-1)^k (T-r)^{(l-k)H}}{(l-k)!} h_{l-k} \left(\frac{B^H_T - B^H_r}{(T-r)^H} \right) \psi_k^{(r,T)} (D^{H,l}_T F). \tag{3.5}$$

The proof is provided in the appendix. To see in a concrete way how to present this convergent series for $\tilde E\big[F|\mathcal{F}_r^H\big]$, we take the following example.

Example 3.4 *Consider the random variable* $e^{\sigma B^H_T}$ *with* T*,* $\sigma > 0$*. One can easily check that it verifies Assumption* $(\mathcal{A})$*. We determine* $\tilde E\big[e^{\sigma B^H_T}|\mathcal{F}^H_r\big]$ *for* $r \in [0,T]$*.*

From (3.2), we notice that for all $l \geq 0$ and all $0 \leq k \leq l$,

$$\psi_k^{(r,T)} (D^{H,l}_T e^{\sigma B^H_T}) = \frac{(T^{2H} - r^{2H})^k}{k!} \sigma^{k+l} e^{\sigma B^H_T}. \tag{3.6}$$

It follows by (3.5), (3.6) and (2.3) that

$$\tilde E\big[e^{\sigma B^H_T}|\mathcal{F}^H_r\big] = e^{\sigma B^H_T} \sum_{l=0}^{+\infty} \frac{(-\sigma(T-r)^H)^l}{l!} \sum_{k=0}^{l} \binom{l}{k} \left(-\frac{\sigma(T^{2H} - r^{2H})}{(T-r)^H} \right)^k h_{l-k} \left(\frac{B^H_T - B^H_r}{(T-r)^H} \right). \tag{3.7}$$

Recall the following property of Hermite polynomials (due to a Taylor expansion): for all $l \in \mathbb{N}$, and all $x, y \in \mathbb{R}$,

$$\sum_{k=0}^{l} \binom{l}{k} x^k h_{l-k}(y) = h_l(x+y). \tag{3.8}$$

Finally, it results from (3.7), (3.8) and (2.8) that

$$\begin{aligned} \tilde E\big[e^{\sigma B^H_T}|\mathcal{F}^H_r\big] &= e^{\sigma B^H_T} \sum_{l=0}^{+\infty} \frac{(-\sigma(T-r)^H)^l}{l!} h_l \left(\frac{-\sigma(T^{2H} - r^{2H}) + B^H_T - B^H_r}{(T-r)^H} \right) \\ &= e^{\sigma B^H_r + \frac{\sigma^2(T^{2H} - r^{2H})}{2}}. \end{aligned} \tag{3.9}$$

It is interesting to observe that, in (3.9), the function $H \longmapsto \tilde E[F|\mathcal{F}^H_t]$ is continuous over $(1/2, 1)$. And then $\tilde E[F|\mathcal{F}^H_t] \to E[F|\mathcal{F}^{1/2}_t]$ a.s., as $H \downarrow 1/2$. The latter one is the classical conditional expectation. This fact shows, in some cases, the backward Taylor expansion in Theorem 3.3 tends (as $H \downarrow 1/2$) to the one for classical conditional expectation with respect to Bm (see [11]).

The backward Taylor expansion can be useful when the random variable F is in terms of a discretized fBm trajectory. However, it fails to represent $\tilde E[F|\mathcal{F}^H_t]$ when F is an arbitrary continuous functional of fBm, such as $F = \int_0^t f(u)\,\mathrm{d}B^H_u$.

Our second main result is more general, but under a different sufficient condition for convergence. Unlike the backward taylor expansion, it allows to give series representation of F when it is functional of continuous trajectories of fBm. From the fact that $\tilde{E}[F|\mathcal{F}_t^H]$ is $\mathcal{F}_t^H$-measurable, one can imagine that the latter value only depends on the trajectories $\{B_s^H\}_{s\in[0,t]}$, no matter how ill-behaved are the remaining trajectories $\{B_s^H\}_{s\in(t,+\infty)}$. This fact inspires us to originally introduce the "frozen path" operator. More precisely, the series representation given in Theorem 3.8 below can be used to numerically evaluate a fractional conditional expectation by following a single typical path backward. It is thus an economical alternative to Monte Carlo simulation, in case where the fractional Hida Malliavin derivatives are not expensive to calculate numerically.

3.2 Series Representation via Exponential Formula

We first introduce the concept of "frozen path" operator, which plays a key role in constructing the exponential formula of fractional conditional expectations.

Definition 3.5 *Recall that for $t \geq 0$, the action of the element $B^H\chi_{[0,t]} \in {L^2_{\varphi_H}}'(\mathbb{R}_+)$ on $f \in L^2_{\varphi_H}(\mathbb{R}_+)$ is defined as*

$$< B^H\chi_{[0,t]}, f >:= \int_0^t f(s)\,\mathrm{d}B_s^H.$$

For any $F \in L^2(\mathbb{P}^H)$, let $F = G(B^H)$, where G is the operator defined in Definition 2.2. The "frozen path" operator $\gamma^t : L^2(\mathbb{P}^H) \to L^2(\mathbb{P}^H)$ is defined as:

$$F(\gamma^t) := G(B^H\chi_{[0,t]}).$$

Note that for $F \in L^2(\mathbb{P}^H)$, the operator G such that $F = G(B^H)$ always exists, as shown in Theorem 3.1.8 in [4]. Now we explain why the "frozen path" operator is well-defined. Consider the polynomial cylindrical random variables of the form

$$F = p\left(\int_0^{+\infty} f_1(s)\,\mathrm{d}B_s^H, \ldots, \int_0^{+\infty} f_n(s)\,\mathrm{d}B_s^H\right), \tag{3.10}$$

with $p \in \mathcal{P}$ (ensemble of polynomials) and $f_i \in L^2_{\varphi_H}(\mathbb{R}_+)$ for all $i = 1, \ldots, n$. The "frozen path" operator γ^t of F is thus well-defined as, for all $t \geq 0$,

$$F(\gamma^t) = p\left(\int_0^t f_1(s)\,\mathrm{d}B_s^H, \ldots, \int_0^t f_n(s)\,\mathrm{d}B_s^H\right).$$

Here it is obvious that γ^t is a linear operator and the function $t \longmapsto F(\gamma^t)$ is continuous on $[0, +\infty)$. Since the set of all random variables of the form in (3.10) is dense in $L^2(\mathbb{P}^H)$ (see e.g. Page 27 in [19] and Page 37 in [4]) and the following proposition shows the "frozen path" operator γ^t is continuous from $L^2(\mathbb{P}^H)$ to $L^2(\mathbb{P}^H)$, therefore one can naturally extend the domain of "frozen path" operator from space of polynomial cylindrical random variables to $L^2(\mathbb{P}^H)$, by preserving the above linearity and continuity. The proof of the key proposition below is given in the appendix.

Proposition 3.6 *Let $(F_M)_{M\geq 1}$ and F belong to $L^2(\mathbb{P}^H)$. If $F_M \to F$ in $L^2(\mathbb{P}^H)$ as $M \to +\infty$, then for any $t \geq 0$,*

$$F_M(\gamma^t) \xrightarrow[M\to+\infty]{L^2(\mathbb{P}^H)} F(\gamma^t).$$

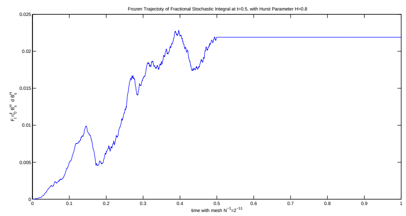

Figure 1: Simulation of the frozen path of $X_1(s) = \int_0^s B_u^H \, \mathrm{d}B_u^H$, with $H = 0.8$, frozen at time t=0.5.

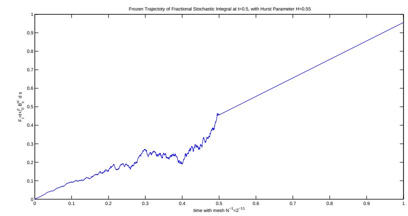

Figure 2: Simulation of the frozen path of $X_2(s) = s + \int_0^s B_u^H \, \mathrm{d}u$, with $H = 0.55$, frozen at time t=0.5.

Remark 3.7 *Assume that F is $\mathcal{F}_T^H$-measurable, it is clear that $F = G(B^H) = G(B^H \chi_{[0,T]})$ and $F(\gamma^t) = G(B^H \chi_{[0,\min\{T,t\}]})$ is $\mathcal{F}_t^H$-measurable.*

In view of Proposition 3.6, it is not difficult to explicitly compute $F(\gamma^t)$ in most cases. Here we present 2 examples to show how to obtain "frozen paths" of $L^2(\mathbb{P}^H)$ random variables. More examples are provided in a long version of this paper [12].

1. Denote by $p(B_{t_1}^H, B_{t_2}^H, \ldots, B_{t_n}^H)$ a polynomial of fBm. Set $T \geq \max\{t_1, \ldots, t_n\}$, define

$$G(B^H) = p\left(\int_0^T \chi_{[0,t_1]}(s) \, \mathrm{d}B_s^H, \ldots, \int_0^T \chi_{[0,t_n]}(s) \, \mathrm{d}B_s^H\right),$$

then for all $t \geq 0$,

$$\left(p(B_{t_1}^H, \ldots, B_{t_n}^H)\right)(\gamma^t) = p(B_{\min\{t_1,t\}}^H, \ldots, B_{\min\{t_n,t\}}^H).$$

2. For a general element $F \in L^2(\mathbb{P}^H)$, $F(\gamma^t)$ can be approximated by some sequence of polynomial cylindrical functions

$$\left\{p_n\left(\int_0^t f_1^{(n)}(s) \, \mathrm{d}B_s^H, \ldots, \int_0^t f_n^{(n)}(s) \, \mathrm{d}B_s^H\right)\right\}_{n \geq 1}$$

in $L^2(\mathbb{P}^H)$, where $p_n \in \mathcal{P}$ and $f_k^{(n)} \in L^2_{\varphi_H}(\mathbb{R}_+)$ for all $n \geq 1$, $1 \leq k \leq n$. This is thanks to Proposition 3.6 and the fact that the linear span of polynomial cylindrical functions of fBms forms a total subset of $L^2(\mathbb{P}^H)$.

In the illustrations Figure 1 and Figure 2, we show how the "frozen path" operator transforms the trajectories of stochastic processes.

Assumption ($\mathcal{B}$): Let $F \in \mathbb{D}_{\infty,T}^H$. Assume that for some $r \in [0,T]$, the following condition holds:

$$\sum_{i=0}^{+\infty} \frac{(T^{2H} - r^{2H})^i}{2^i i!} \left\| \sup_{u_{2i},\ldots,u_1 \in [0,T]} \left| \left(D_{u_{2i}}^H \ldots D_{u_1}^H F\right)(\gamma^r) \right| \right\|_{L^2(\mathbb{P}^H)} < +\infty,$$

where by convention, the first term in the above series is $\|F(\gamma^r)\|_{L^2(\mathbb{P}^H)}$.

Theorem 3.8 (Exponential Formula) *For $r \in [0,T]$ and $v \in [r,T]$, define the operator $\mathcal{A}_{v,r}: \ L^2(\mathbb{P}^H) \to L^2(\mathbb{P}^H)$ by:*

$$\mathcal{A}_{v,r}(F) := \frac{1}{2}\left(\int_0^T D_u^H D_v^H F \varphi_H(u,v) \, \mathrm{d}u + \int_0^r D_u^H D_v^H F \varphi_H(u,v) \, \mathrm{d}u\right). \tag{3.11}$$

Then under Assumption ($\mathcal{B}$), the following series converges in $L^2(\mathbb{P}^H)$:

$$\tilde{E}\left[F|\mathcal{F}_r^H\right] = \sum_{i=0}^{+\infty} \int_{r\leq v_1\leq\ldots\leq v_i\leq T} \left(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F\right)(\gamma^r)\,(\,\mathrm{d}v)^{\otimes i}, \tag{3.12}$$

where by convention, the first item in the series is $F(\gamma^r)$.

We emphasize that the integrand of the right-hand side of (3.12) must be evaluated along the path where its driving fBm is frozen at r. Below is a quick application of Theorem 3.8.

Example 3.9 *Consider $F = (B_t^H)^2 B_T^H$ for some fixed $T > 0$ and $t \in [0,T]$.*

By definition, F is $\mathcal{F}_T^H$-measurable and for $r \in [0,t]$, $F(\gamma^r) = (B_r^H)^3$. Then by means of Theorem 3.8, we get

$$\begin{aligned}\tilde{E}\left[F|\mathcal{F}_r^H\right] &= F(\gamma^r) + \frac{1}{2}\int_r^T \left(\int_0^T + \int_0^r\right)\left(D_u^H D_v^H F\right)(\gamma^r)\varphi_H(u,v)\,\mathrm{d}u\,\mathrm{d}v \\ &= (B_r^H)^3 + B_r^H\left(T^{2H} + 2t^{2H} - 3r^{2H} - (T-t)^{2H}\right).\end{aligned}$$

Instead of the backward Taylor expansion, the fractional conditional expectation of the form $F = \Phi(B_{t_1}^H, \ldots, B_{t_J}^H)$ can be also represented via exponential formula, however sometimes it is less obvious to use exponential formula than to use backward Taylor expansion to find the explicit form of $\tilde{E}[F|\mathcal{F}_t^H]$. Moreover, backward Taylor expansion is more convenient to be used for numerical approximation purpose since it is series of polynomials, while exponential formula contains integrals.

4 Applications

Unlike the ordinary conditional expectations, it might not be easy to explain the financial meaning of $\tilde{E}[F|\mathcal{F}_t^H]$ in financial modeling problems. However, since $\tilde{E}[F|\mathcal{F}_0^H] = E[F]$ (see (2.7)), then one advantage of our series expansions is that it sometimes allows to evaluate expectation of F in an extremely simple way, provided the fractional Malliavin derivatives of F are explicitly known. In this section, we motivate the exponential formula by giving some real world's applications. Note that although all the following examples are related to financial models, this non-trivial approximation approach can be used to evaluate ordinary expectations of other complicated exponential functionals of fBm, without reference to interest rates nor bond prices.

4.1 Fractional Merton Model of Interest Rates

Suppose that the interest rate follows a particular simple version of the Merton model: assume $H > 1/2$, for $s \in [0,T]$, $r(s) = B_s^H$. Let $F = \exp\left(\int_0^T r(s)\,\mathrm{d}s\right)$, we are going to compute the bond price $P(0,T) = E[F]$. It is easy to check that F is $\mathcal{F}_T^H$-measurable and satisfies Assumption ($\mathcal{B}$). Observe that

$$\left(D_u^H D_v^H F\right)(\gamma^0) = (T-u)(T-v),$$

and then for all integer $i \geq 1$,

$$\begin{aligned}
&\int_{0\leq v_1\leq\ldots\leq v_i\leq T} (\mathcal{A}_{v_i,0}\ldots\mathcal{A}_{v_1,0}F)\left(\gamma^0\right)(\mathrm{d}v)^{\otimes i} \\
&= \frac{1}{2^i}\int_{0\leq v_1\leq\ldots\leq v_i\leq T}\int_{[0,T]^i}\prod_{k=1}^{i}(T-u_k)(T-v_k)\varphi_H(u_k,v_k)\,(\mathrm{d}u)^{\otimes i}\,(\mathrm{d}v)^{\otimes i} \\
&= \frac{1}{2^i i!}\left(\int_0^T\int_0^T (T-u)(T-v)\varphi_H(u,v)\,\mathrm{d}u\,\mathrm{d}v\right)^i \\
&= \frac{1}{2^i i!}\left(\frac{T^{2H+2}}{2H+2}\right)^i.
\end{aligned}$$

Then by applying Theorem 3.8, we obtain:

$$P(0,T) = \sum_{i=0}^{+\infty}\frac{1}{2^i i!}\left(\frac{T^{2H+2}}{2H+2}\right)^i = \exp\left(\frac{T^{2H+2}}{4H+4}\right).$$

4.2 Fractional CIR Model of Interest Rates: a Special Case

Consider a particular case of fractional Cox-Ingersoll-Ross model (CIR model) of interest rates: for $H > 1/2$ and $s \in [0,T]$,

$$r(s) = (B_s^H)^2.$$

Let $F = \exp(-\int_0^T r(s)\,\mathrm{d}s)$, our major goal is to compute $E[F]$. Since $r(0)=0$, then imposed to the "frozen path" operator γ^0, one has,

$$F(\gamma^0) = e^{-\left(\int_0^T (B_s^H)^2\,\mathrm{d}s\right)(\gamma^0)} = 1;\ (D_u^H D_v^H F)(\gamma^0) = -2\big(T-\max(u,v)\big).$$

Similarly one shows

$$\begin{aligned}
&\left(D_{u_2}^H D_{v_2}^H D_{u_1}^H D_{v_1}^H F\right)(\gamma^0) = 4\Big(\big(T-\max(u_1,v_2)\big)\big(T-\max(u_2,v_1)\big) \\
&\quad + \big(T-\max(u_1,u_2)\big)\big(T-\max(v_1,v_2)\big) + \big(T-\max(u_1,v_1)\big)\big(T-\max(u_2,v_2)\big)\Big).
\end{aligned}$$

Then applying Theorem 3.8,

$$\begin{aligned}
E[F] &= 1 + \frac{1}{2}\int_0^T\int_0^T \left(D_u^H D_v^H F\right)(\gamma^0)\varphi_H(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&\quad + \frac{1}{4}\int_{[0,T]^3\times[v_1,T]}\left(D_{u_2}^H D_{v_2}^H D_{u_1}^H D_{v_1}^H F\right)(\gamma^0)\varphi_H(u_1,v_1)\varphi_H(u_2,v_2)\,\mathrm{d}v_2\,\mathrm{d}u_2\,\mathrm{d}v_1\,\mathrm{d}u_1 \\
&\quad + o(T^{4H+2}) \\
&= 1 - \frac{1}{2H+1}T^{2H+1} + \left(\frac{8H^2+18H+5}{4(2H+1)^2(4H+1)} - \frac{\mathcal{B}(2H+1,2H+2)}{2H+1}\right)T^{4H+2} \\
&\quad + o(T^{4H+2}),
\end{aligned} \tag{4.1}$$

where $\mathcal{B}$ denotes the beta function, i.e. $\mathcal{B}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$, for all $x,y>0$. This result provides a numerical approximation of $E[F]$, as $T \downarrow 0$. The computations of the first two items on the right hand-side of (4.1) are straightforward. However, the third item needs more efforts to obtain. The technical computation of the third item can be found in [12].

4.3 Series Representation of Characteristic Function of Geometric FBM

In this example, we show an application of the exponential formula without satisfying Assumption ($\mathcal{B}$). For $T > 0$ and $H > 1/2$, let the geometric fBm

$$X_T = e^{\sigma B_T^H} \text{ and } F = e^{izX_T}, \text{ for } z \in \mathbb{C}.$$

Recall that X_T is a lognormal random variable, i.e. $\log X_T \sim \mathcal{N}(0, T^{2H}\sigma^2)$. Then $E[F]$ is the characteristic function of some lognormal random variable X_T evaluated at z. It is known in the literature that there is no convergent Taylor series representation of $E[F]$, due to the fact that the lognormal distribution does not only depend on its probability moments. For example, Holgate [9] showed there is no unique determination of the lognormal distribution only by its moments. Thus, there has been a number of attempts to present the lognormal characteristic functions by divergent power series, which are sufficient for generating moments, e.g. we refer to [9, 3, 2, 16]. Now we provide a new divergent power series representation of $E[F]$ using Theorem 3.8. Notice that Assumption ($\mathcal{B}$) is no longer satisfied, however the series representation by Theorem 3.8 is still of many interests.

By induction, we show that, for all $0 \le u_1, v_1, \ldots, u_n, v_n \le T$:

$$D_{u_n}^H D_{v_n}^H \ldots D_{u_1}^H D_{v_1}^H F = F\sigma^{2n} \sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} (iz)^k e^{k\sigma B_T^H},$$

where the sequence $\left(\left\{ {j \atop k} \right\} \right)_{j,k \in \mathbb{N}}$ is the sequence of Stirling numbers of the second kind (see for instance [16] for its definition and features), with convention that $b_{j,k} = \left\{ {j \atop k} \right\} = 0$ for all $0 \le j < k$. Therefore by Theorem 3.8 and the fact that $\int_{[0,T]^n} \prod_{l=1}^n \varphi_H(u_l, v_l)(\,\mathrm{d}u)^{\otimes n}$ is symmetric with respect to $v_1, \ldots, v_n$,

$$\begin{aligned}
&\frac{1}{2^n} \int_{0 \le v_1 \le \ldots \le v_n \le T} \int_{[0,T]^n} (D_{u_n}^H D_{v_n}^H \ldots D_{u_1}^H D_{v_1}^H F)(\gamma^0) \prod_{l=1}^n \varphi_H(u_l, v_l)(\,\mathrm{d}u)^{\otimes n}(\,\mathrm{d}v)^{\otimes n} \\
&= \frac{T^{2nH}}{2^n n!} \sigma^{2n} \sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} e^{iz} (iz)^k,
\end{aligned}$$

and the power series representation of $E[F]$ is given by

$$\sum_{n=0}^{+\infty} \sum_{k=0}^{2n} \frac{(\frac{T^{2H}\sigma^2}{2})^n}{n!} \left\{ {2n \atop k} \right\} e^{iz} (iz)^k. \tag{4.2}$$

The series (4.2) is divergent for all $z \in \mathbb{C}$ due to the large order of the Stirling number of second kind. However, this series representation can be used to evaluate all the moments of F, more precisely, by the following relation: for $p \in \mathbb{N}$,

$$E[X_T^p] = (-i)^p \sum_{n=0}^{+\infty} \sum_{k=0}^{2n} \frac{(\frac{T^{2H}\sigma^2}{2})^n}{n!} i^k \left\{ {2n \atop k} \right\} \frac{\mathrm{d}^p(e^{iz}z^k)}{\mathrm{d}z^p}\Big|_{z=0}. \tag{4.3}$$

This interesting result provides a new divergent representation of the lognormal distribution's characteristic function using power series.

5 Conclusion and Future Work

We provide two series representations of $\tilde{E}[F|\mathcal{F}_t^H]$, the fractional conditional expectations of functionals of fBm: backward Taylor expansion and exponential formula. Remark that we have not yet addressed the problem of calculating $E[F|\mathcal{F}_t^H]$, the ordinary conditional expectation of a functional of fBm. This problem is quite interesting and meaningful. We note that the methodology developed by Fournié et al. [6] for Bm reduces the problem to evaluating two expectations, and is also applicable to fBm with $H > 1/2$. We do not provide explicit results in that case, and leave this work for future research.

6 Appendix

6.1 Proof of Theorem 3.3

One needs the following lemmas to prove Theorem 3.3. The following lemma extends Proposition 1.2.4 in [19] from Bm to fBm.

Lemma 6.1 *Let $F \in L^2\left(\mathbb{P}^H\right)$ be fractional Hida Malliavin differentiable. For any $t, u \geq 0$, one has*

$$D_u^H \tilde{E}\left[F|\mathcal{F}_t^H\right] = \tilde{E}\left[D_u^H F|\mathcal{F}_t^H\right] \chi_{[0,t]}(u). \tag{6.1}$$

Proof of Lemma 6.1. For $f \in L^2_{\varphi_H}(\mathbb{R}_+)$, define the exponential function

$$\varepsilon(f) := \exp\left(\int_{\mathbb{R}_+} f(t)\, \mathrm{d}B_t^H - \frac{1}{2}\|f\|^2_{H,\mathbb{R}_+}\right). \tag{6.2}$$

According to Theorem 3.1.4 in [4], the set of linear span of $\varepsilon(f)$'s is dense in $L^2\left(\mathbb{P}^H\right)$. Thus it suffices to prove Lemma 6.1 for $F = \varepsilon(f)$ then argue with $L^2(\mathbb{P}^H)$-convergence.

Now we show (6.1) holds for any $\varepsilon(f)$ defined in (6.2). Observe that $\varepsilon(f)$ has the following series representation (we refer to the following statements in [4]: Corollary 3.9.3, Lemma 3.9.2 and Theorem 3.9.7):

$$\varepsilon(f) = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\mathbb{R}_+^n} f(s_1)\dots f(s_n)\, \mathrm{d}\left(B_s^H\right)^{\otimes n}.$$

Thus by definition, for any $t \geq 0$,

$$\tilde{E}\left[\varepsilon(f)|\mathcal{F}_t^H\right] = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{[0,t]^n} f(s_1)\dots f(s_n)\, \mathrm{d}\left(B_s^H\right)^{\otimes n} = \varepsilon\left(f\chi_{[0,t]}\right). \tag{6.3}$$

On the one hand, it follows by (6.3) and the chain rule that

$$\begin{aligned} D_u^H \tilde{E}\left[\varepsilon(f)|\mathcal{F}_t\right] &= e^{\int_0^t f(s)\,\mathrm{d}B_s^H - \frac{1}{2}\left\|f\chi_{[0,t]}\right\|^2_{H,\mathbb{R}_+}} D_u^H\left(\int_0^t f(s)\,\mathrm{d}B_s^H\right) \\ &= \varepsilon(f\chi_{[0,t]})f(u)\chi_{[0,t]}(u). \end{aligned} \tag{6.4}$$

On the other hand, again by the chain rule and (6.3), one gets

$$\tilde{E}\left[D_u^H \varepsilon(f)|\mathcal{F}_t\right] = \tilde{E}\left[\varepsilon(f)f(u)\big|\mathcal{F}_t^H\right] = \varepsilon(f\chi_{[0,t]})f(u). \tag{6.5}$$

Then (6.1) holds for $F = \varepsilon(f)$, thanks to (6.4) and (6.5). By the denseness of the linear span of $\varepsilon(f)$ in $L^2(\mathbb{P}^H)$, one can show (6.1) also holds for all fractional Hida Malliavin differentiable $F \in L^2(\mathbb{P}^H)$. □

Definition 6.2 *Let $F \in \mathbb{D}^H_{\infty,T}$. For $0 \le r < b \le T$, define the sequence $\left\{ \left(\int_r^b\right)^{H,k} F \,\mathrm{d}u\,\mathrm{d}v \right\}_{k\in\mathbb{N}}$ by:*

$$\left(\int_r^b\right)^{H,0} F \,\mathrm{d}u\,\mathrm{d}v := F; \tag{6.6}$$

and for $k \ge 1$,

$$\left(\int_r^b\right)^{H,k} F \,\mathrm{d}u\,\mathrm{d}v := \int_{[r\le v_k\le\ldots\le v_1\le b]\times[0,T]^k} D^H_{u_k}\ldots D^H_{u_1} F \left(\prod_{i=1}^k \varphi_H(u_i,v_i)\right)(\mathrm{d}u)^{\otimes k}(\mathrm{d}v)^{\otimes k}. \tag{6.7}$$

Lemma 6.3 (Iterated Fractional Integration) *Let $F \in \mathbb{D}^H_{\infty,T}$. For any $0 \le r < b \le T$ and any integer $n \ge 1$:*

$$\int_{r\le s_1\le\ldots\le s_n\le b} F\,\mathrm{d}\left(B^H_s\right)^{\otimes n} = \sum_{i=0}^n \frac{(-1)^i (b-r)^{(n-i)H}}{(n-i)!} h_{n-i}\left(\frac{B^H_b - B^H_r}{(b-r)^H}\right)\left(\int_r^b\right)^{H,i} F\,\mathrm{d}u\,\mathrm{d}v, \tag{6.8}$$

where $h_{n-i}(x)$ is the Hermite polynomial of degree $n-i$ defined in (2.3).

Proof of Lemma 6.3.
By repeatedly using integration by parts of the type for FWISI as below (see e.g. (2.33) in [4]):

$$\int_r^b F\,\mathrm{d}B^H_s = F\int_r^b \mathrm{d}B^H_s - \int_r^b\int_0^T D^H_u F\varphi_H(u,v)\,\mathrm{d}u\,\mathrm{d}v$$

and definitions (6.6), (6.7), we get

$$\int_{r\le s_1\le\ldots\le s_n\le b} F\,\mathrm{d}\left(B^H_s\right)^{\otimes n} = \sum_{i=0}^n (-1)^i \int_{r\le s_1\le\ldots\le s_{n-i}\le b} \mathrm{d}(B^H_s)^{\otimes(n-i)} \left(\int_r^b\right)^{H,i} F\,\mathrm{d}u\,\mathrm{d}v. \tag{6.9}$$

Then Lemma 6.3 holds thanks to (6.9) and (2.5). $\square$

Lemma 6.4 *Let F be given in Definition 3.2. For $k \in \mathbb{N}$, $r \in [t_{j-1}, t_j)$,*

$$\left(\int_r^{t_j}\right)^{H,k} F\,\mathrm{d}u\,\mathrm{d}v = \psi_k^{(r,t_j)}(F), \tag{6.10}$$

where $\psi_k^{(r,t_j)}(F)$ is defined in (3.1) and (3.2).

Proof of Lemma 6.4. When $k = 0$, (6.10) is trivial. When $k \ge 1$, by using the fact that $F = \Phi(B^H_{t_1}, \ldots, B^H_{t_J})$ and (2.1), one has, for $u_1, \ldots, u_k \in [0,T]$,

$$D^H_{u_k} D^H_{u_{k-1}} \ldots D^H_{u_1} F = D^{H,\sharp\{u_i : u_i\in[0,t_1],\ i=1,\ldots,k\}}_{t_1} \ldots D^{H,\sharp\{u_i : u_i\in(t_{J-1},t_J],\ i=1,\ldots,k\}}_{t_J} F, \tag{6.11}$$

where $\sharp\{\cdot\}$ denotes the cardinality of set. It follows from (6.7), (6.11), the multinomial theorem, (3.1) and (3.2) that

$$\begin{aligned}\left(\int_r^{t_j}\right)^{H,k} F\,\mathrm{d}u\,\mathrm{d}v &= \frac{1}{k!}\int_{[r,t_j]^k\times[0,t_J]^k} D^H_{u_k}\ldots D^H_{u_1} F \prod_{i=1}^k \varphi_H(u_i,v_i)\,(\mathrm{d}u)^{\otimes k}(\mathrm{d}v)^{\otimes k} \\ &= \frac{1}{k!}\sum_{\sum_{i=1}^J q_i = k} \binom{k}{q_1, q_2, \ldots, q_J} D^{H,q_1}_{t_1}\ldots D^{H,q_J}_{t_J} F \prod_{i=1}^J \left(\int_r^{t_j}\int_{t_{i-1}}^{t_i} \varphi_H(u,v)\,\mathrm{d}u\,\mathrm{d}v\right)^{q_i} \\ &= \psi_k^{(r,t_j)}(F),\end{aligned} \tag{6.12}$$

where $\binom{k}{q_1,q_2,\ldots,q_J} := \frac{k!}{q_1!q_2!\ldots q_J!}$ denotes multinomial coefficient. □

Before introducing the next lemma, to simplify notation, for some proper stochastic process $X := \{X(s)\}_{s\in\mathbb{R}^N}$, we indicate its multiple FWISI by

$$\delta^N(X) := \int_{\mathbb{R}^N} X(s_1,\ldots,s_N)\,\mathrm{d}\left(B_s^H\right)^{\otimes N}.$$

The following technical lemma provides an identification of $E[\delta^N(X)^2]$. It is a particular case of Equation (2.12) in [18]. More than the latter reference, we present an explicit form of the norm of tensor product of the Hilbert space $L^2_{\varphi_H}(\mathbb{R})$.

Lemma 6.5 *For any symmetric stochastic process $X := \{X(s)\}_{s\in\mathbb{R}^N}$ satisfying Assumption $(\mathcal{A})$, we have*

$$E[\delta^N(X)^2] = \sum_{i=0}^{N}\binom{N}{i}^2 i!E\left\|D^{N-i}X\right\|^2_{L^2_{\varphi_H}(\mathbb{R}_+)^{\otimes(2N-i)}},$$

where $L^2_{\varphi_H}(\mathbb{R})^{\otimes(2N-i)}$ denotes the $(2N-i)$th tensor product of $L^2_{\varphi_H}(\mathbb{R})$, i.e., $\left\|D^{N-i}X\right\|^2_{L^2_{\varphi_H}(\mathbb{R})^{\otimes(2N-i)}}$ is given as

$$\begin{aligned}
&\left\|D^{N-i}X\right\|^2_{L^2_{\varphi_H}(\mathbb{R})^{\otimes(2N-i)}}\\
&:= \int_{\mathbb{R}^{4N-2i}}\left(D^{H,N-i}_{x_1,\ldots,x_{N-i}}X(s_1,\ldots,s_N)\prod_{r=1}^{N-i}\varphi_H(s'_r,x_r)(\,\mathrm{d}x)^{\otimes(N-i)}(\,\mathrm{d}s')^{\otimes(N-i)}\right)\\
&\quad\times\left(D^{H,N-i}_{y_1,\ldots,y_{N-i}}X(s'_1,\ldots,s'_N)\prod_{r=1}^{N-i}\varphi_H(s_r,y_r)(\,\mathrm{d}y)^{\otimes(N-i)}(\,\mathrm{d}s)^{\otimes(N-i)}\right)\\
&\quad\times\prod_{r=1}^{i}\varphi_H(s_{N-i+r},s'_{N-i+r})\,\mathrm{d}s_N\ldots\,\mathrm{d}s_{N-i+1}\,\mathrm{d}s'_N\ldots\,\mathrm{d}s'_{N-i+1}.
\end{aligned}$$

Here we denote by $D^{H,N-i}_{x_1,\ldots,x_{N-i}} := D^H_{x_1}\ldots D^H_{x_{N-i}}$; $(\,\mathrm{d}x)^{\otimes(N-i)} := \,\mathrm{d}x_{N-i}\ldots\,\mathrm{d}x_1$.

Lemma 6.5 can be obtained without much effort by induction with initial step (which develops the deterministic space $L^2_{\varphi_H}(\mathbb{R})$ to space of stochastic processes, see (3.41) in [4]):

$$\begin{aligned}
\|X\|^2_{L^2_{\varphi_H}(\mathbb{R})} &= \int_{\mathbb{R}^2} X(s)X(t)\varphi_H(s,t)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad+\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}\varphi_H(s,v)D^H_vX(t)\,\mathrm{d}v\right)\left(\int_{\mathbb{R}}\varphi_H(t,u)D^H_uX(s)\,\mathrm{d}u\right)\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

Thus we omit its proof.

Proof of Theorem 3.3. Let $r\in[0,T]$ be fixed. Since $\tilde{E}\left[F|\mathcal{F}^H_r\right]$ is $\mathcal{F}^H_r$-measurable, then by the fractional Clark-Hausmann-Ocone formula given in Theorem 2.7,

$$\tilde{E}\left[F|\mathcal{F}^H_r\right] = E\left[\tilde{E}\left[F|\mathcal{F}^H_r\right]\right] + \int_0^r \tilde{E}\left[D^H_{s_1}\tilde{E}\left[F|\mathcal{F}^H_r\right]|\mathcal{F}^H_{s_1}\right]\mathrm{d}B^H_{s_1}.$$

Since $r\le T$, by applying Lemma 6.1, (2.9) and (2.8),

$$\begin{aligned}
\tilde{E}\left[F|\mathcal{F}^H_r\right] &= \left(E\left[\tilde{E}\left[F|\mathcal{F}^H_T\right]\right] + \int_0^T\tilde{E}\left[D^H_{s_1}F|\mathcal{F}^H_{s_1}\right]\mathrm{d}B^H_{s_1}\right) - \int_r^T\tilde{E}\left[D^H_{s_1}F|\mathcal{F}^H_{s_1}\right]\mathrm{d}B^H_{s_1}\\
&= \tilde{E}\left[F|\mathcal{F}^H_T\right] - \int_r^T\tilde{E}\left[D^H_{s_1}F|\mathcal{F}^H_{s_1}\right]\mathrm{d}B^H_{s_1}\\
&= F - \int_r^T\tilde{E}\left[D^H_{s_1}F|\mathcal{F}^H_{s_1}\right]\mathrm{d}B^H_{s_1}, \qquad (6.13)
\end{aligned}$$

because F is $\mathcal{F}_T^H$-measurable. Repeatedly using the fractional Clark-Hausmann-Ocone formula n times in (6.13) leads to

$$\tilde{E}\left[F|\mathcal{F}_r^H\right] = F - \int_r^T D_{s_1}^H F \,\mathrm{d}B_s^H + \sum_{l=2}^{N-1} (-1)^l \int_r^T \int_{s_1}^T \cdots \int_{s_{l-1}}^T D_{s_l}^H \ldots D_{s_1}^H F \,\mathrm{d}(B_s^H)^{\otimes l} + R^{(N)}, \tag{6.14}$$

where $R^{(N)}$ is the remainder of the series given by

$$R^{(N)} := (-1)^N \int_r^T \int_{s_1}^T \cdots \int_{s_{N-1}}^T \tilde{E}\left[D_{s_N}^H \ldots D_{s_1}^H F|\mathcal{F}_{s_N}^H\right] \mathrm{d}(B_s^H)^{\otimes N}. \tag{6.15}$$

Next, by the fact that $F = \Phi(B_{t_1}^H, \ldots, B_{t_J}^H)$ and (2.1), one has, for $s_1 \in (r, T]$,

$$D_{s_1}^H F = D_{t_{I_r}}^H F \chi_{(r,t_{I_r}]}(s_1) + D_{t_{I_r+1}}^H F \chi_{(t_{I_r},t_{I_r+1}]}(s_1) + \ldots + D_T^H F \chi_{(t_{J-1},t_J]}(s_1).$$

In consequence, for $l \geq 1$,

$$\begin{aligned}\int_{r\leq s_1\leq\ldots\leq s_l\leq T} D_{s_l}^H \ldots D_{s_1}^H F \,\mathrm{d}(B_s^H)^{\otimes l} = &\sum_{q_{I_r}+\ldots+q_J=l} \int_{r\leq s_1\leq\ldots\leq s_l\leq T} D_{t_{I_r}}^{H,q_{I_r}} \ldots D_{t_J}^{H,q_J} F \\ &\times \chi_{([r,t_{I_r}]^{q_{I_r}}\times(t_{I_r},t_{I_r+1}]^{q_{I_r+1}}\ldots\times(t_{J-1},t_J]^{q_J})}(s_1, \ldots, s_l) \,\mathrm{d}(B_s^H)^{\otimes l}.\end{aligned} \tag{6.16}$$

It results from (6.16) and Lemma 6.3 that

$$\begin{aligned}&\int_{r\leq s_1\leq\ldots\leq s_l\leq T} D_{s_l}^H \ldots D_{s_1}^H F \,\mathrm{d}(B_s^H)^{\otimes l} \\ &= \sum_{q_{I_r}+\ldots+q_J=l} \sum_{i_{I_r}=0}^{q_{I_r}} \cdots \sum_{i_J=0}^{q_J} \prod_{k=I_r}^{J} \frac{(-1)^{i_k} (\tilde{t}_k - \tilde{t}_{k-1})^{(q_k-i_k)H}}{(q_k-i_k)!} \\ &\quad \times h_{q_{I_r}-i_{I_r}}\left(\frac{B_{\tilde{t}_{I_r}}^H - B_{\tilde{t}_{I_r-1}}^H}{(\tilde{t}_{I_r} - \tilde{t}_{I_r-1})^H}\right) \left(\int_r^{t_{I_r}}\right)^{H,i_{I_r}} \ldots h_{q_J-i_J}\left(\frac{B_{\tilde{t}_J}^H - B_{\tilde{t}_{J-1}}^H}{(\tilde{t}_J - \tilde{t}_{J-1})^H}\right) \left(\int_{t_{J-1}}^{t_J}\right)^{H,i_J} \\ &\quad D_{t_{I_r}}^{H,q_{I_r}} \ldots D_{t_J}^{H,q_J} F \,(\,\mathrm{d}u\,\mathrm{d}v)^{\otimes(J-I_r+1)}.\end{aligned} \tag{6.17}$$

Therefore, by (6.17) and Lemma 6.4, Theorem 3.3 holds provided that

$$\|R^{(N)}\|_{L^2(\mathbb{P}^H)} \xrightarrow[N\to+\infty]{} 0. \tag{6.18}$$

Now we show (6.18) holds under Assumption $(\mathcal{A})$. Similar to (6.16),

$$\begin{aligned}R^{(N)} \quad = \quad (-1)^N &\sum_{q_{I_r}+\ldots+q_J=N} \int_{r\leq s_1\leq\ldots\leq s_N\leq T} \tilde{E}\left[D_{t_{I_r}}^{H,q_{I_r}} \ldots D_{t_J}^{H,q_J} F|\mathcal{F}_{s_N}^H\right] \\ &\times\chi_{([r,t_{I_r}]^{q_{I_r}}\times(t_{I_r},t_{I_r+1}]^{q_{I_r+1}}\ldots\times(t_{J-1},t_J]^{q_J})}(s_1, \ldots, s_N) \,\mathrm{d}(B_s^H)^{\otimes N}.\end{aligned}$$

Therefore, to prove (6.18), it is sufficient to show the sequence below tends to 0 as $N \to +\infty$: for any $0 \leq r < T$ and $q_{I_r}, \ldots, q_J \in \{0, \ldots, N\}$,

$$\begin{aligned}&\sum_{q_{I_r}+\ldots+q_J=N} \Bigg\| \int_{r\leq s_1\leq\ldots\leq s_N\leq T} \tilde{E}\left[D_{t_{I_r}}^{H,q_{I_r}} \ldots D_{t_J}^{H,q_J} F|\mathcal{F}_{s_N}^H\right] \\ &\times \chi_{([r,t_{I_r}]^{q_{I_r}}\times(t_{I_r},t_{I_r+1}]^{q_{I_r+1}}\ldots\times(t_{J-1},t_J]^{q_J})}(s_1, \ldots, s_N) \,\mathrm{d}\left(B_s^H\right)^{\otimes N} \Bigg\|_{L^2(\mathbb{P}^H)} \\ &\xrightarrow[N\to+\infty]{} 0.\end{aligned} \tag{6.19}$$

To this end we define

$$R_{[r,T]}^{(N)}(q_{I_r},\ldots,q_J) := \int_{r\le s_1\le\ldots\le s_N\le T} \tilde{E}\left[D_{t_{I_r}}^{H,q_{I_r}}\ldots D_{t_J}^{H,q_J}F|\mathcal{F}_{s_N}^H\right]$$
$$\times\,\chi_{([r,t_{I_r}]^{q_{I_r}}\times(t_{I_r},t_{I_r+1}]^{q_{I_r+1}}\ldots\times(t_{J-1},t_J]^{q_J})}(s_1,\ldots,s_N)\,\mathrm{d}\left(B_s^H\right)^{\otimes N}.$$

By a symmetrization of the above integrand,

$$R_{[r,T]}^{(N)}(q_{I_r},\ldots,q_J) = \frac{(-1)^N}{N!}\int_{[r,T]^N}\mathcal{H}_N(s_1,\ldots,s_N)\,\mathrm{d}\left(B_s^H\right)^{\otimes N}, \tag{6.20}$$

where

$$\mathcal{H}_N(s_1,\ldots,s_N) := \sum_{\sigma\in S_N}\tilde{E}\left[D_{t_{I_r}}^{H,q_{I_r}}\ldots D_{t_J}^{H,q_J}F|\mathcal{F}_{s_{\sigma(N)}}^H\right]\chi_A(s_{\sigma(1)},\ldots,s_{\sigma(N)}) \tag{6.21}$$

is symmetric with respect to $s_1,\ldots,s_N$ with S_N being the symmetric group on $\{1,2,\ldots,N\}$ and

$$A := \{(s_1,\ldots,s_N)\in[r,t_{I_r}]^{q_{I_r}}\times\ldots\times(t_{J-1},t_J]^{q_J}\,:\, r\le s_1\le\ldots\le s_N\le t_J\}.$$

According to Lemma 6.5 (we restrict the processes to nonnegative-time) and Fubini theorem,

$$\begin{aligned}
&E\left[\delta^N(\mathcal{H}_N)^2\right] = \sum_{i=0}^N\binom{N}{i}^2 i!E\left\|D^{N-i}\mathcal{H}_N\right\|^2_{L^2_{\varphi_H}(\mathbb{R}_+)^{\otimes(2N-i)}}\\
&=\sum_{i=0}^N\binom{N}{i}^2 i!\int_{\mathbb{R}_+{}^{4N-2i}}E\left[D_{x_1,\ldots,x_{N-i}}^{H,N-i}\mathcal{H}_N(s_1,\ldots,s_N)D_{y_1,\ldots,y_{N-i}}^{H,N-i}\mathcal{H}_N(s_1',\ldots,s_N')\right]\\
&\quad\times\prod_{r=1}^{N-i}\varphi_H(s_r',x_r)\varphi_H(s_r,y_r)\prod_{r=1}^{i}\varphi_H(s_{N-i+r},s_{N-i+r}')\\
&\quad(\mathrm{d}x)^{\otimes(N-i)}(\mathrm{d}y)^{\otimes(N-i)}(\mathrm{d}s)^{\otimes N}(\mathrm{d}s')^{\otimes N}.
\end{aligned}\tag{6.22}$$

On the one hand, since $\mathcal{H}_N(s_1',\ldots,s_N')$ is symmetric, then by the following change of variable in (6.22): $(s_{N-i+1}',\ldots,s_N')\longmapsto(s_{N-i+\sigma''(1)}',\ldots,s_{N-i+\sigma''(i)}')$ for any $\sigma''\in S_i$, we get for any $i\in\{0,1,\ldots,N\}$,

$$\begin{aligned}
&\int_{\mathbb{R}_+{}^{4N-2i}}E\Big[D_{x_1,\ldots,x_{N-i}}^{H,N-i}\mathcal{H}_N(s_1,\ldots,s_N)\\
&\quad\times D_{y_1,\ldots,y_{N-i}}^{H,N-i}\mathcal{H}_N(s_1',\ldots,s_{N-i}',s_{N-i+\sigma''(1)}',\ldots,s_{N-i+\sigma''(i)}')\Big]\\
&\quad\times\prod_{r=1}^{N-i}\varphi_H(s_r',x_r)\varphi_H(s_r,y_r)\prod_{r=1}^{i}\varphi_H(s_{N-i+r},s_{N-i+\sigma''(r)}')\\
&\quad(\mathrm{d}x)^{\otimes(N-i)}(\mathrm{d}y)^{\otimes(N-i)}(\mathrm{d}s)^{\otimes N}(\mathrm{d}s')^{\otimes N}\\
&=\int_{\mathbb{R}_+{}^{4N-2i}}E\left[D_{x_1,\ldots,x_{N-i}}^{H,N-i}\mathcal{H}_N(s_1,\ldots,s_N)D_{y_1,\ldots,y_{N-i}}^{H,N-i}\mathcal{H}_N(s_1',\ldots,s_N')\right]\\
&\quad\times\prod_{r=1}^{N-i}\varphi_H(s_r',x_r)\varphi_H(s_r,y_r)\prod_{r=1}^{i}\varphi_H(s_{N-i+r},s_{N-i+\sigma''(r)}')\\
&\quad(\mathrm{d}x)^{\otimes(N-i)}(\mathrm{d}y)^{\otimes(N-i)}(\mathrm{d}s)^{\otimes N}(\mathrm{d}s')^{\otimes N}.
\end{aligned}$$

It yields

$$E\left[\delta^N(\mathcal{H}_N)^2\right] = \sum_{i=0}^{N}\binom{N}{i}^2 \int\limits_{\mathbb{R}_+{}^{4N-2i}} E\left[D^{H,N-i}_{x_1,\dots,x_{N-i}}\mathcal{H}_N(s_1,\dots,s_N)D^{H,N-i}_{y_1,\dots,y_{N-i}}\mathcal{H}_N(s'_1,\dots,s'_N)\right]$$
$$\times \prod_{r=1}^{N-i}\varphi_H(s'_r,x_r)\varphi_H(s_r,y_r)\sum_{\sigma''\in S_i}\prod_{r=1}^{i}\varphi_H(s_{N-i+r},s'_{N-i+\sigma''(r)})$$
$$(\,\mathrm{d}x)^{\otimes(N-i)}(\,\mathrm{d}y)^{\otimes(N-i)}(\,\mathrm{d}s)^{\otimes N}(\,\mathrm{d}s')^{\otimes N}. \tag{6.23}$$

On the other hand, by (6.21), the linearity of expectation, Cauchy-Schwarz inequality, (2.10) and the fact that $F=\Phi(B^H_{t_1},\dots,B^H_{t_J})$, we get

$$\begin{aligned}
&E\Big[D^{H,N-i}_{x_1,\dots,x_{N-i}}\mathcal{H}_N(s_1,\dots,s_N)D^{H,N-i}_{y_1,\dots,y_{N-i}}\mathcal{H}_N(s'_1,\dots,s'_N)\Big]\\
&=\sum_{\sigma\in S_N}\sum_{\sigma'\in S_N}E\Big[\tilde{E}\left[D^{H,N-i}_{x_1,\dots,x_{N-i}}D^{H,q_{I_r}}_{t_{I_r}}\dots D^{H,q_J}_{t_J}F|\mathcal{F}^H_{s_{\sigma(N)}}\right]\\
&\quad\times\tilde{E}\left[D^{H,N-i}_{y_1,\dots,y_{N-i}}D^{H,q_{I_r}}_{t_{I_r}}\dots D^{H,q_J}_{t_J}F|\mathcal{F}^H_{s'_{\sigma'(N)}}\right]\Big]\\
&\quad\times\chi_{[0,s_{\sigma(N)}]^{N-i}\times[0,s'_{\sigma'(N)}]^{N-i}}(x_1,\dots,x_{N-i},y_1,\dots,y_{N-i})\\
&\quad\times\chi_{A^2}(s_{\sigma(1)},\dots,s_{\sigma(N)},s'_{\sigma'(1)},\dots,s'_{\sigma'(N)})\\
&\le E\Big[\sup_{p_{I_r}+\ldots+p_J=N-i}\Big|D^{H,p_{I_r}+q_{I_r}}_{t_{I_r}}\dots D^{H,p_J+q_J}_{t_J}F\Big|\Big]^2\\
&\quad\times\sum_{\sigma\in S_N}\sum_{\sigma'\in S_N}\chi_{[0,T]^{2N-2i}}(x_1,\dots,x_{N-i},y_1,\dots,y_{N-i})\\
&\quad\times\chi_{A^2}(s_{\sigma(1)},\dots,s_{\sigma(N)},s'_{\sigma'(1)},\dots,s'_{\sigma'(N)}).
\end{aligned}\tag{6.24}$$

Plugging (6.24) into (6.23), taking $w_i=p_i+q_i$ with $i=I_r,\dots,J$ and using the fact that $q_{I_r}+\ldots+q_J=N$, we obtain

$$\begin{aligned}
E[\delta^N(\mathcal{H}_N)^2]&\le\sum_{i=0}^{N}\binom{N}{i}^2E\Big[\sup_{w_{I_r}+\ldots+w_J=2N-i}\Big|D^{H,w_{I_r}}_{t_{I_r}}\dots D^{H,w_J}_{t_J}F\Big|\Big]^2\\
&\times\int_{\mathbb{R}_+{}^{4N-2i}}\sum_{\sigma\in S_N}\sum_{\sigma'\in S_N}\chi_{[0,T]^{2N-2i}}(x_1,\dots,x_{N-i},y_1,\dots,y_{N-i})\\
&\times\chi_{A^2}(s_{\sigma(1)},\dots,s_{\sigma(N)},s'_{\sigma'(1)},\dots,s'_{\sigma'(N)})\\
&\times\prod_{r=1}^{N-i}\varphi_H(s'_r,x_r)\varphi_H(s_r,y_r)\sum_{\sigma''\in S_i}\prod_{r=1}^{i}\varphi_H(s_{N-i+r},s'_{N-i+\sigma''(r)})\\
&\quad(\,\mathrm{d}x)^{\otimes(N-i)}(\,\mathrm{d}y)^{\otimes(N-i)}(\,\mathrm{d}s)^{\otimes N}(\,\mathrm{d}s')^{\otimes N}.
\end{aligned}\tag{6.25}$$

Observe that, inside the right-hand side of (6.25), for each $(\sigma,\sigma')\in S_N^2$, the integral

$$\int\limits_{\mathbb{R}_+{}^{2N-2i}}\chi_{[0,T]^{2N-2i}}(x_1,\dots,x_{N-i},y_1,\dots,y_{N-i})\chi_{A^2}(s_{\sigma(1)},\dots,s_{\sigma(N)},s'_{\sigma'(1)},\dots,s'_{\sigma'(N)})$$
$$\times\prod_{r=1}^{N-i}\varphi_H(s'_r,x_r)\varphi_H(s_r,y_r)\sum_{\sigma''\in S_i}\prod_{r=1}^{i}\varphi_H(s_{N-i+r},s'_{N-i+\sigma''(r)})(\,\mathrm{d}x)^{\otimes(N-i)}(\,\mathrm{d}y)^{\otimes(N-i)}$$

returns the same value. Also notice that, for any positive-valued function f and $a \le b$,

$$\int_{a\le s_1\le\ldots\le s_N\le b} f(s_1,\ldots,s_N)(\,\mathrm{d}s)^{\otimes N} \le \int_{\substack{a\le s_1\le\ldots\le s_{N-i}\le b,\\ a\le s_{N-i+1}\le\ldots\le s_N\le b}} f(s_1,\ldots,s_N)(\,\mathrm{d}s)^{\otimes N}.$$

Therefore (6.25) leads to

$$\begin{aligned}
E[\delta^N(\mathcal{H}_N)^2] &\le \sum_{i=0}^{N}\binom{N}{i}^2 (N!)^2 E\left[\sup_{w_{I_r}+\ldots+w_J=2N-i}\left|D^{H,w_{I_r}}_{t_{I_r}}\ldots D^{H,w_J}_{t_J}F\right|\right]^2\\
&\quad\times\int_B\prod_{r=1}^{N-i}\varphi_H(s'_r,x_r)\varphi_H(s_r,y_r)\sum_{\sigma''\in S_i}\prod_{r=1}^{i}\varphi_H(s_{N-i+r},s'_{N-i+\sigma''(r)})\\
&\quad(\,\mathrm{d}x)^{\otimes(N-i)}(\,\mathrm{d}y)^{\otimes(N-i)}(\,\mathrm{d}s)^{\otimes N}(\,\mathrm{d}s')^{\otimes N}\\
&= \sum_{i=0}^{N}\binom{N}{i}^2 (N!)^2 E\left[\sup_{w_{I_r}+\ldots+w_J=2N-i}\left|D^{H,w_{I_r}}_{t_{I_r}}\ldots D^{H,w_J}_{t_J}F\right|\right]^2\frac{i!}{((N-i)!)^2(i!)^2}\\
&\quad\times\int_{[0,T]^{2N-2i}\times[r,T]^{2N}}\prod_{r=1}^{N-i}\varphi_H(s'_r,x_r)\varphi_H(s_r,y_r)\prod_{r=1}^{i}\varphi_H(s_{N-i+r},s'_{N-i+r})\\
&\quad(\,\mathrm{d}x)^{\otimes(N-i)}(\,\mathrm{d}y)^{\otimes(N-i)}(\,\mathrm{d}s)^{\otimes N}(\,\mathrm{d}s')^{\otimes N}\\
&= \sum_{i=0}^{N}\binom{N}{i}^2\frac{(N!)^2 i!}{((N-i)!)^2(i!)^2}E\left[\sup_{w_{I_r}+\ldots+w_J=2N-i}\left|D^{H,w_{I_r}}_{t_{I_r}}\ldots D^{H,w_J}_{t_J}F\right|\right]^2\\
&\quad\times\left(\frac{T^{2H}-r^{2H}+(T-r)^{2H}}{2}\right)^{2N-2i}(T-r)^{2iH}, \qquad (6.26)
\end{aligned}$$

where $B := \{(x,y,s,s')\in[0,T]^{4N-2i}:\ x,y\in[0,T]^{N-i}, r\le s_1\le\ldots\le s_{N-i}\le T, r\le s'_1\le\ldots\le s'_{N-i}\le T, r\le s_{N-i+1}\le\ldots\le s_N\le T, r\le s'_{N-i+1}\le\ldots\le s'_N\le T\}$. Combining (6.26) and (6.20), one gets

$$\begin{aligned}
E\left[R^{(N)}_{[r,T]}(q_{I_r},\ldots,q_J)\right]^2 &\le \sum_{i=0}^{N} E\left[\sup_{w_{I_r}+\ldots+w_J=2N-i}\left|D^{H,w_{I_r}}_{t_{I_r}}\ldots D^{H,w_J}_{t_J}F\right|\right]^2\\
&\quad\times\binom{N}{i}^4\frac{i!}{(N!)^2}\left(\frac{T^{2H}-r^{2H}+(T-r)^{2H}}{2}\right)^{2N-2i}(T-r)^{2iH}. \qquad (6.27)
\end{aligned}$$

Now we upper bound $\|R^{(N)}\|_{L^2(\mathbb{P}^H)}$. Observe that

$$\sharp\{(q_{I_r},\ldots,q_J):\ q_{I_r}+\ldots+q_J=N\} = \binom{N+J-I_r}{J-I_r} \le c\frac{(N+J-1)!}{N!},$$

where $c>0$ is some constant which does not depend on r nor on N. And since $I_r\ge 1$,

$$E\left[\sup_{\sum_{j=I_r}^{J}w_j=2N-i}\left|D^{H,w_{I_r}}_{t_{I_r}}\ldots D^{H,w_J}_{t_J}F\right|\right]^2 \le E\left[\sup_{\sum_{j=1}^{J}w_j=2N-i}\left|D^{H,w_1}_{t_1}\ldots D^{H,w_J}_{t_J}F\right|\right]^2.$$

Then, it follows from (6.19), (6.20), (6.27) and the triangle inequality that

$$\begin{aligned}\left\|R^{(N)}\right\|_{L^2(\mathbb{P}^H)} &\leq \sum_{q_{I_r}+\ldots+q_J=N}\left\|R^{(N)}_{[r,T]}(q_{I_r},\ldots,q_J)\right\|_{L^2(\mathbb{P}^H)}\\ &\leq c\frac{(N+J-1)!}{N!}\sum_{i=0}^{N}\left\|\sup_{\sum_{j=1}^J w_j=2N-i}\left|D^{H,w_1}_{t_1}\ldots D^{H,w_J}_{t_J}F\right|\right\|_{L^2(\mathbb{P}^H)}\\ &\quad\times\binom{N}{i}^2\frac{(i!)^{1/2}}{2^{N-i}N!}\left(T^{2H}-r^{2H}+(T-r)^{2H}\right)^{N-i}(T-r)^{iH},\end{aligned}$$

which tends to 0 as $N\to+\infty$, thanks to Assumption $(\mathcal{A})$. $\square$

6.2 Proof of Proposition 3.6

The following lemmas are useful to the proof of Proposition 3.6.

Lemma 6.6 *Let $F\in L^2(\mathbb{P}^H)$ be $\mathcal{F}^H_T$-measurable. There exist a sequence of smooth functions $(S_n)_{n\geq 1}$ in $C^\infty(\mathbb{R}^n)$ and a sequence of functions $(f^{(n)}_k)_{n\geq1,1\leq k\leq n}$ in $L^2_{\varphi_H}(\mathbb{R}_+)$, such that*

$$\sup_{r\in[0,T]}\left\|S_n\left(\int_0^r f^{(n)}_1(s)\,\mathrm{d}B^H_s,\ldots,\int_0^r f^{(n)}_n(s)\,\mathrm{d}B^H_s\right)-G(B^H\chi_{[0,\min\{r,T\}]})\right\|_{L^2(\mathbb{P}^H)}\xrightarrow[n\to+\infty]{}0. \tag{6.28}$$

The proof of Lemma 6.6 is given in the appendix of [12].

The following preliminary result is a consequence of the fractional Wiener Itô chaos expansion theorem, it extends Theorem 3.1.8 in [4] and Theorem 2.6 in [10].

Theorem 6.7 *For $F\in L^2(\mathbb{P}^H)$, there exists a sequence of real numbers $(c_\alpha)_{\alpha\in\mathcal{J}}$, such that for all $r\geq 0$,*

$$F(\gamma^r)=\sum_{\alpha\in\mathcal{J}}c_\alpha\tilde{\mathcal{H}}_\alpha(\gamma^r),\ \text{ in } L^2(\mathbb{P}^H), \tag{6.29}$$

and

$$\|F\|^2_{L^2(\mathbb{P}^H)}=\sum_{\alpha\in\mathcal{J}}\alpha!c_\alpha^2, \tag{6.30}$$

where

- *$\mathcal{J}:=\{(\alpha_1,\ldots,\alpha_m)\in\mathbb{N}^m:\ m\geq 1\}$ is the set of all finite sequence of nonnegative integers;*
- *$\tilde{\mathcal{H}}_\alpha(\gamma^r):=\prod_{i=1}^m h_{\alpha_i}(<B^H\chi_{[0,r]},e_i>)$, with $(e_i)_{i\geq1}$ being the orthonormal basis of $L^2_{\varphi_H}(\mathbb{R}_+)$ given in (3.10) of [4] and h_{α_i} being the Hermite polynomial defined in (2.3);*
- *for $\alpha=(\alpha_1,\ldots,\alpha_m)$, $\alpha!:=\alpha_1!\alpha_2!\ldots\alpha_m!$.*

Proof of Theorem 6.7. Notice that (6.30) is straightforwardly given in Theorem 3.1.8 of [4], thus we only prove (6.29). Assume that F is $\mathcal{F}^H_T$-measurable, then for $r>T$, Theorem 6.7 holds thanks to Theorem 3.1.8 in [4]. Now assume $r\in[0,T]$. Set

$$F_n=\sum_{k=1}^n a^{(n)}_k\varepsilon(f^{(n)}_k),$$

for $(a_k^{(n)})_{1\le k\le n}\in\mathbb{R}^n$ and $f_k^{(n)}\in L^2_{\varphi_H}(\mathbb{R}_+)$. It is shown that (see (3.15) in [4]) for each $f_k^{(n)}$, there exists a sequence of constants $c_{\alpha,k}^{(n)}$ such that

$$\varepsilon(f_k^{(n)})(\gamma^r)=\sum_{\alpha\in\mathcal{J}}c_{\alpha,k}^{(n)}\tilde{\mathcal{H}}_\alpha(\gamma^r),\ \text{in } L^2(\mathbb{P}^H).$$

Therefore Theorem 6.7 holds for F_n:

$$F_n(\gamma^r)=\sum_{k=1}^n a_k^{(n)}\varepsilon(f_k^{(n)})(\gamma^r)=\sum_{\alpha\in\mathcal{J}}\left(\sum_{k=1}^n a_k^{(n)}c_{\alpha,k}^{(n)}\right)\tilde{\mathcal{H}}_\alpha(\gamma^r).$$

In a general case when $F\in L^2(\mathbb{P}^H)$ is arbitrary, we apply Lemma 6.6 to claim that there exists a sequence of functions which is uniformly convergent to $F(\gamma^r)$ in $L^2(\mathbb{P}^H)$, with respect to $r\in[0,T]$:

$$F_n(\gamma^r)=\sum_{k=1}^n a_k^{(n)}\varepsilon(f_k^{(n)})(\gamma^r)\xrightarrow[n\to+\infty]{L^2(\mathbb{P}^H)}F(\gamma^r).$$

Also observe that, by (3.15) in [4],

$$F_n=\sum_{\alpha\in\mathcal{J}}\left(\sum_{k=1}^n a_k^{(n)}c_{\alpha,k}^{(n)}\right)\tilde{\mathcal{H}}_\alpha,$$

and by Example 3.1.9 in [4], the fact that $\lim_{n\to+\infty}F_n=F$ in $L^2(\mathbb{P}^H)$, for $\alpha\in\mathcal{J}$,

$$\sum_{k=1}^n a_k^{(n)}c_{\alpha,k}^{(n)}=\frac{1}{\alpha!}E\left[F_n\tilde{\mathcal{H}}_\alpha\right]\xrightarrow[n\to+\infty]{}\frac{1}{\alpha!}E\left[F\tilde{\mathcal{H}}_\alpha\right].$$

Therefore,

$$\begin{aligned}
F(\gamma^r)&=\lim_{n\to+\infty}\sum_{\alpha\in\mathcal{J}}\left(\sum_{k=1}^n a_k^{(n)}c_{\alpha,k}^{(n)}\right)\tilde{\mathcal{H}}_\alpha(\gamma^r)\\
&=\sum_{\alpha\in\mathcal{J}}\lim_{n\to+\infty}\left(\sum_{k=1}^n a_k^{(n)}c_{\alpha,k}^{(n)}\right)\tilde{\mathcal{H}}_\alpha(\gamma^r)\\
&=\sum_{\alpha\in\mathcal{J}}\frac{1}{\alpha!}E\left[F\tilde{\mathcal{H}}_\alpha\right]\tilde{\mathcal{H}}_\alpha(\gamma^r),\ \text{in } L^2(\mathbb{P}^H).
\end{aligned}$$

It follows that Theorem 6.7 is proven, if we take $c_\alpha=\frac{1}{\alpha!}E[F\tilde{\mathcal{H}}_\alpha]$. □

Proof of Proposition 3.6. Let $(F_M)_{M\ge1}$ and F satisfy $F_M\xrightarrow[M\to+\infty]{L^2(\mathbb{P}^H)}F$. By (6.29), there exists a sequence of constants $(c_\alpha^{(M)})_{M\ge1,\alpha\in\mathcal{J}}$ such that for all $r\ge0$,

$$(F_M-F)(\gamma^r)=\sum_{\alpha\in\mathcal{J}}c_\alpha^{(M)}\tilde{\mathcal{H}}_\alpha(\gamma^r),\ \text{in } L^2(\mathbb{P}^H).\tag{6.31}$$

Notice that, by (6.30),

$$\|F_M-F\|^2_{L^2(\mathbb{P}^H)}=\sum_{\alpha\in\mathcal{J}}\alpha!(c_\alpha^{(M)})^2\xrightarrow[M\to+\infty]{}0.$$

This together with the fact that $\alpha! > 0$ yields $\lim_{M\to+\infty} c_\alpha^{(M)} = 0$ for all $\alpha \in \mathcal{J}$. From (6.31), we see that for any $M \geq 1$,

$$\|(F_M - F)(\gamma^r)\|^2_{L^2(\mathbb{P}^H)} = E\left[\sum_{\alpha\in\mathcal{J}}\sum_{\beta\in\mathcal{J}} c_\alpha^{(M)} c_\beta^{(M)} \tilde{\mathcal{H}}_\alpha(\gamma^r)\tilde{\mathcal{H}}_\beta(\gamma^r)\right] < +\infty. \tag{6.32}$$

Since for all $\alpha, \beta \in \mathcal{J}$, $\lim_{M\to+\infty} |c_\alpha^{(M)} c_\beta^{(M)} \tilde{\mathcal{H}}_\alpha(\gamma^r)\tilde{\mathcal{H}}_\beta(\gamma^r)| = 0$. Therefore by (6.32) and the dominated convergence theorem,

$$\|(F_M - F)(\gamma^r)\|^2_{L^2(\mathbb{P}^H)} \xrightarrow[M\to+\infty]{} 0.$$

Proposition 3.6 is proven. $\square$

6.3 Proof of Theorem 3.8

Let $F \in \mathbb{D}^H_{\infty,T}$, since the linear span of $\varepsilon(f\chi_{[0,T]})$ for $f \in L^2_{\varphi_H}(\mathbb{R}_+)$ is dense in $\mathbb{D}^H_{\infty,T}$, we first show Theorem 3.8 holds for $F = \varepsilon(f\chi_{[0,T]})$. Applying (3.11) to $F = \varepsilon(f\chi_{[0,T]})$ leads to, for any $v \geq 0$, $r \in [0,T]$,

$$\begin{aligned}
\mathcal{A}_{v,r}\left(\varepsilon(f\chi_{[0,T]})\right) &= \frac{1}{2}\left(\int_0^T + \int_0^r\right) D_u^H D_v^H \varepsilon(f\chi_{[0,T]})\varphi_H(u,v)\,\mathrm{d}u \\
&= \frac{\varepsilon\left(f\chi_{[0,T]}\right)}{2}\left(\int_0^T + \int_0^r\right) f(u)f(v)\varphi_H(u,v)\,\mathrm{d}u.
\end{aligned} \tag{6.33}$$

Repeatedly using (6.33) implies that for any $v_1, \ldots, v_i \geq 0$,

$$\begin{aligned}
&\left(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}\varepsilon(f\chi_{[0,T]})\right)(\gamma^r) \\
&= \frac{\varepsilon\left(f\chi_{[0,T]}\right)(\gamma^r)}{2^i}\prod_{k=1}^i\left(\left(\int_0^T + \int_0^r\right) f(u)\, f(v_k)\,\varphi_H(u,v_k)\,\mathrm{d}u\right).
\end{aligned} \tag{6.34}$$

Observing that $(v_1,\ldots,v_i) \longmapsto \left(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}\left(\varepsilon(f\chi_{[0,T]})\right)\right)(\gamma^r)$ is symmetric, we obtain from (6.34) that, the following series converges in $L^2(\mathbb{P}^H)$:

$$\begin{aligned}
&\sum_{i=0}^N \int_{r\leq v_1\leq\ldots\leq v_i\leq T}\left(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}\varepsilon\left(f\chi_{[0,T]}\right)\right)(\gamma^r)\,(\mathrm{d}v)^{\otimes i} \\
&= \varepsilon\left(f\chi_{[0,T]}\right)(\gamma^r)\sum_{i=0}^N\frac{1}{i!}\left(\frac{1}{2}\int_r^T\left(\int_0^T + \int_0^r\right) f(u)\,f(v)\,\varphi_H(u,v)\,\mathrm{d}u\,\mathrm{d}v\right)^i \\
&\xrightarrow[N\to+\infty]{L^2(\mathbb{P}^H)} \varepsilon\left(f\chi_{[0,T]}\right)(\gamma^r)\exp\left(\frac{1}{2}\|f\chi_{[0,T]}\|^2_{H,\mathbb{R}_+} - \frac{1}{2}\|f\chi_{[0,r]}\|^2_{H,\mathbb{R}_+}\right).
\end{aligned} \tag{6.35}$$

Now we determine $\varepsilon\left(f\chi_{[0,T]}\right)(\gamma^r)$. It follows from (6.35) and the fact that $\varepsilon(f\chi_{[0,T]})(\gamma^r) = \exp(\int_0^r f(s)\,\mathrm{d}B_s^H - \|f\chi_{[0,T]}\|^2_{H,\mathbb{R}_+}/2)$ that

$$\begin{aligned}
&\sum_{i=0}^N \int_{r\leq v_1\leq\ldots\leq v_i\leq T}\left(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}\left(\varepsilon(f\chi_{[0,T]})\right)\right)(\gamma^r)\,(\mathrm{d}v)^{\otimes i} \\
&\xrightarrow[N\to+\infty]{L^2(\mathbb{P}^H)} e^{\int_0^r f(s)\,\mathrm{d}B_s^H - \frac{1}{2}\|f\chi_{[0,T]}\|^2_{H,\mathbb{R}_+}} e^{\frac{1}{2}\|f\chi_{[0,T]}\|^2_{H,\mathbb{R}_+} - \frac{1}{2}\|f\chi_{[0,r]}\|^2_{H,\mathbb{R}_+}} \\
&= \varepsilon(f\chi_{[0,r]}).
\end{aligned} \tag{6.36}$$

By (6.3) and (6.36), Theorem 3.8 holds for $F = \varepsilon(f\chi_{[0,T]})$.

We turn to the general case when $F \in \mathbb{D}^H_{\infty,T}$ is arbitrary. Since $r \longmapsto F(\gamma^r)$ is continuous, by Lemma 6.6, there exists a sequence of coefficients $(a_i^{(M)})_{1\le i\le M}$ such that: for all $r \ge 0$,

$$F_M(\gamma^r) := \sum_{i=1}^{M} a_i^{(M)} \varepsilon(f_i^{(M)}\chi_{[0,T]})(\gamma^r) \xrightarrow[M\to+\infty]{L^2(\mathbb{P}^H)} F(\gamma^r). \tag{6.37}$$

On the one hand, by (6.36) and the linearity of fractional conditional expectation, one gets

$$\sum_{i=0}^{N} \int_{r\le v_1\le\ldots\le v_i\le T} (\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F_M)(\gamma^r)\,(\,\mathrm{d}v)^{\otimes i} \xrightarrow[N\to+\infty]{L^2(\mathbb{P}^H)} \tilde{E}[F_M|\mathcal{F}^H_r]. \tag{6.38}$$

The fact that (see (2.10))

$$E\left|\tilde{E}[F_M - F|\mathcal{F}^H_r]\right|^2 \le E\,|F_M - F|^2 \xrightarrow[M\to+\infty]{} 0$$

leads to

$$\tilde{E}[F_M|\mathcal{F}^H_r] \xrightarrow[M\to+\infty]{L^2(\mathbb{P}^H)} \tilde{E}[F|\mathcal{F}^H_r]. \tag{6.39}$$

Therefore, from (6.38) and (6.39) we see the following convergence holds in $L^2(\mathbb{P}^H)$:

$$\lim_{M\to+\infty}\lim_{N\to+\infty}\sum_{i=0}^{N} \int_{r\le v_1\le\ldots\le v_i\le T} (\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F_M)(\gamma^r)\,(\,\mathrm{d}v)^{\otimes i} = \tilde{E}[F|\mathcal{F}^H_r]. \tag{6.40}$$

On the other hand, one can show that the above $\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}$ is a closable operator for almost every $v_1,\ldots,v_i \in [r,T]$. In fact, since for almost every $u \ge 0$, D^H_u is a closable operator from $L^2(\mathbb{P}^H)$ to $L^2(\mathbb{P}^H)$ (see e.g. [4], Page 38), then by the dominated convergence theorem and (6.37), for almost every $v \in [r,T]$,

$$\begin{aligned}
\mathcal{A}_{v,r}F_M &= \frac{1}{2}\left(\int_0^T D^H_u D^H_v F_M \varphi_H(u,v)\,\mathrm{d}u + \int_0^r D^H_u D^H_v F_M\varphi_H(u,v)\,\mathrm{d}u\right)\\
&\xrightarrow[M\to+\infty]{L^2(\mathbb{P}^H)} \frac{1}{2}\left(\int_0^T D^H_u D^H_v F\varphi_H(u,v)\,\mathrm{d}u + \int_0^r D^H_u D^H_v F\varphi_H(u,v)\,\mathrm{d}u\right)\\
&= \mathcal{A}_{v,r}F.
\end{aligned}$$

By induction, for almost all $v_1,\ldots,v_i \in [r,T]$,

$$\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F_M \xrightarrow[M\to+\infty]{L^2(\mathbb{P}^H)} \mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F.$$

Further, it follows from Proposition 3.6 that, for all $i \ge 1$ and almost all $v_1,\ldots,v_i \in [r,T]$,

$$(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F_M)(\gamma^r) \xrightarrow[M\to+\infty]{L^2(\mathbb{P}^H)} (\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F)(\gamma^r). \tag{6.41}$$

Now to show the exponential formula in Theorem 3.8 converges, it suffices to demonstrate that the series

$$\sum_{i=0}^{+\infty}\int_{r\le v_1\le\ldots\le v_i\le T} (\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F)(\gamma^r)(\,\mathrm{d}v)^{\otimes i} \tag{6.42}$$

is convergent in $L^2(\mathbb{P}^H)$. To this end one observes, from the binomial theorem, that for $i \geq 1$,

$$
\begin{aligned}
&\left(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F\right)(\gamma^r)\\
&= \frac{1}{2^i}\sum_{k=0}^{i}\binom{i}{k}\int\limits_{[0,T]^k\times[0,r]^{i-k}}\left(D^H_{u_i}D^H_{v_i}\ldots D^H_{u_1}D^H_{v_1}F\right)(\gamma^r)\prod_{j=1}^{i}\varphi_H(u_j,v_j)(\,\mathrm{d}u)^{\otimes i}\\
&\leq \left(\sup_{u_{2i},\ldots,u_1\in[0,T]}\left|\left(D^H_{u_{2i}}\ldots D^H_{u_1}F\right)(\gamma^r)\right|\right)\frac{1}{2^i}\sum_{k=0}^{i}\binom{i}{k}\int\limits_{[0,T]^k\times[0,r]^{i-k}}\prod_{j=1}^{i}\varphi_H(u_j,v_j)(\,\mathrm{d}u)^{\otimes i}.
\end{aligned}
$$

It yields

$$
\begin{aligned}
&\sum_{i=0}^{N}\int_{r\leq v_1\leq\ldots\leq v_i\leq T}\left(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F\right)(\gamma^r)(\,\mathrm{d}v)^{\otimes i}\\
&\leq \sum_{i=0}^{N}\frac{1}{2^i i!}\sum_{k=0}^{i}\binom{i}{k}\left(\int_r^T\int_0^T\varphi_H(u,v)\,\mathrm{d}u\,\mathrm{d}v\right)^k\left(\int_r^T\int_0^r\varphi_H(u,v)\,\mathrm{d}u\,\mathrm{d}v\right)^{i-k}\\
&\quad\times\sup_{u_{2i},\ldots,u_1\in[0,T]}\left|\left(D^H_{u_{2i}}\ldots D^H_{u_1}F\right)(\gamma^r)\right|\\
&= \sum_{i=0}^{N}\frac{(T^{2H}-r^{2H})^i}{2^i i!}\left(\sup_{u_{2i},\ldots,u_1\in[0,T]}\left|\left(D^H_{u_{2i}}\ldots D^H_{u_1}F\right)(\gamma^r)\right|\right).
\end{aligned}
\tag{6.43}
$$

Next by applying the triangle inequality to (6.43), one obtains

$$
\begin{aligned}
&\left\|\sum_{i=0}^{N}\int_{r\leq v_1\leq\ldots\leq v_i\leq T}\left(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F\right)(\gamma^r)(\,\mathrm{d}v)^{\otimes i}\right\|_{L^2(\mathbb{P}^H)}\\
&\leq \sum_{i=0}^{N}\frac{(T^{2H}-r^{2H})^i}{2^i i!}\left\|\sup_{u_{2i},\ldots,u_1\in[0,T]}\left|\left(D^H_{u_{2i}}\ldots D^H_{u_1}F\right)(\gamma^r)\right|\right\|_{L^2(\mathbb{P}^H)}.
\end{aligned}
$$

The above series is convergent as $N \to +\infty$, thanks to Assumption $(\mathcal{B})$. Finally, it follows from (6.40), (6.42), the triangle inequality and the monotone convergence theorem that the following convergence holds in $L^2(\mathbb{P}^H)$:

$$
\begin{aligned}
&\sum_{i=0}^{+\infty}\int_{r\leq v_1\leq\ldots\leq v_i\leq T}\left(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F\right)(\gamma^r)\,(\,\mathrm{d}v)^{\otimes i}\\
&= \lim_{N\to+\infty}\lim_{M\to+\infty}\sum_{i=0}^{N}\int_{r\leq v_1\leq\ldots\leq v_i\leq T}\left(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F_M\right)(\gamma^r)\,(\,\mathrm{d}v)^{\otimes i}\\
&= \lim_{M\to+\infty}\lim_{N\to+\infty}\sum_{i=0}^{N}\int_{r\leq v_1\leq\ldots\leq v_i\leq T}\left(\mathcal{A}_{v_i,r}\ldots\mathcal{A}_{v_1,r}F_M\right)(\gamma^r)\,(\,\mathrm{d}v)^{\otimes i}\\
&= \tilde{E}[F|\mathcal{F}^H_r].
\end{aligned}
$$

Therefore Theorem 3.8 has been proven. □

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