Euler time discretization of Backward Doubly SDEs and Application to Semilinear SPDEs

Achref BACHOUCH

Humboldt University of Berlin
Institute of Mathematics
Stochastic Group
e-mail: achref.bachouch@hotmail.com

Mohamed Anis BEN LASMAR

University of Tunis El Manar
Laboratoire de Modélisation Mathématique et Numérique
dans les Sciences de l'Ingénieur, ENIT
e-mail: mohamedanis.benlasmar@lamsin.rnu.tn

Anis MATOUSSI

Université du Maine
Institut du Risque et de l'Assurance
Laboratoire Manceau de Mathématiques
e-mail: anis.matoussi@univ-lemans.fr

Mohamed MNIF

University of Tunis El Manar
Laboratoire de Modélisation Mathématique et Numérique
dans les Sciences de l'Ingénieur, ENIT
e-mail: mohamed.mnif@enit.rnu.tn

Key words: Backward Doubly Stochastic Differential Equations, Forward-Backward System, Stochastic PDE, Malliavin calculus, Euler scheme, Monte Carlo method.
MSC Classification (2000): 60H10, 65C30.

Abstract: This paper investigates a numerical probabilistic method for the solution of some semilinear stochastic partial differential equations (SPDEs in short). The numerical scheme is based on discrete time approximation for solutions of systems of decoupled forward-backward doubly stochastic differential equations. Under standard assumptions on the parameters, the convergence and the rate of convergence of the numerical scheme is proven. The proof is based on a generalization of the result on the path regularity of the backward equation.

*Research partly supported by the Chair *Financial Risks* of the *Risk Foundation* sponsored by Société Générale, the Chair *Derivatives of the Future* sponsored by the Fédération Bancaire Française, and the Chair *Finance and Sustainable Development* sponsored by EDF and Calyon

†This work was partially supported by the research project MATPYL of the Fédération de Mathématiques des Pays de la Loire

1. Introduction

Stochastic partial differential equations combine the features of partial differential equations and Itô equations. Such equations play important roles in many applied fields such as the filtering of partially observable diffusion processes, genetic population and other areas. For concrete examples, we send the reader to Pardoux [32], Krylov and Rozovskii [21] and Flandoli [12]. We study the following SPDE for a predictable random field $u_t(x) = u(t,x)$, satisfying:

$$du_t(x) + \big(Lu_t(x) + f(t,x,u_t(x),\nabla u_t\sigma(x))\big)\,dt + g(t,x,u_t(x),\nabla u_t\sigma(x))\cdot \overleftarrow{dB}_t = 0, \tag{1.1}$$

over the time interval $[0,T]$, with a given final condition $u_T = \Phi$ and non-linear deterministic coefficients f and g. $Lu = \big(Lu_1,\cdots,Lu_k\big)$ is a second order differential operator and σ is the diffusion coefficient. The differential term with $\overleftarrow{dB}_t$ refers to the backward stochastic integral with respect to a l-dimensional Brownian motion on $\big(\Omega,\mathcal{F},\mathbb{P},(B_t)_{t\geq 0}\big)$. The backward stochastic integral in the SPDE is used because we will employ the framework of Backward Doubly Stochastic Differential Equation (BDSDE in short) introduced first by Pardoux and Peng [34]. It gives a probabilistic representation for the classical solution $u_t(x)$ of the SPDE (1.1) (written in the integral form) in terms of the following class of BDSDE's:

$$Y_s^{t,x} = \Phi(X_T^{t,x}) + \int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr + \int_s^T g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\overleftarrow{dB_r} - \int_s^T Z_r^{t,x}dW_r, \tag{1.2}$$

where $(X_s^{t,x})_{t\leq s\leq T}$ is a diffusion process starting from x at time t driven by the finite dimensional Brownian motion $(W_t)_{t\geq 0}$ and with infinitesimal generator L. More precisely, under some regularity assumptions on the final condition Φ and coefficients f and g , they proved that $u_t(x) = Y_t^{t,x}$ and $\nabla u_t\sigma(x) = Z_t^{t,x}$, $\forall (t,x)\in[0,T]\times\mathbb{R}^d$. Then, Bally and Matoussi [6] (see also [29]) showed that the same representation remains true in the case when the final condition (respectively the coefficients f and g) is only measurable in x (resp. are jointly measurable in (t,x) and Lispchitz in u and ∇u). In this paper, weak Sobolev solution of the equation (1.1) was considered, and the approach was based on stochastic flow techniques (see also [23, 24]). Moreover, their results were generalized in [29] to the case of a larger class of SPDE's (1.1) driven by a Kunita-Itô non-linear noise (see [23, 24, 25] for more details). In particular, the Kunita-Itô non-linear noise covers a class of infinite dimensional time-space white-colored noise (see [16], [36], [19]). The explicit resolution of semi-linear SPDEs is not generally possible, it is then necessary to resort to numerical methods.

The first approach used to solve numerically nonlinear SPDEs is analytic. It is based on time-space discretization of the SPDEs. The discretization in space can be achieved by different methods such as finite differences, finite elements, spectral Galerkin methods. Most numerical works on SPDEs concentrated on the Euler finite-difference scheme. Gyongi and Nualart [17] proved that these schemes converge, and Gyongy [18] determined the order of convergence. Very interesting results were obtained by Gyongy and Krylov [16] considering a symmetric finite difference scheme for a class of linear SPDE driven by infinite dimensional Brownian motion. They proved that the approximation error is proportional to $\widehat{h}^2$ where $\widehat{h}$ is the discretization step in space and by the Richardson acceleration method they even got the error proportional to $\widehat{h}^4$. Walsh [37] investigated schemes based on the finite elements methods. He studied the rate of convergence of these schemes for parabolic SPDEs, including the Forward and Backward Euler and the Crank-Nicholson

schemes. He found a substantially similar rate of convergence to those found for finite difference schemes.
The spectral Galerkin approximation was used by Jentzen and Kloeden [19]. They based their method on Taylor expansions derived from the solution of the SPDE, under some regularity conditions. Lototsky, Mikulevicius and Rozovskii [26] used the spectral approach for the numerical estimation of the conditional distribution solution of a linear SPDE known as the Zakai equation. Further developments on spectral methods can be found in Lototsky [27].

The other alternative for resolving numerically SPDEs is the probabilistic approach by using Monte Carlo methods. These methods are tractable especially when the dimension of the state process is large unlike the finite difference method. Furthermore, their parallel nature provides another advantage to the probabilistic approach: each processor of a parallel computer can be assigned the task of making a random trial and doing the calculus independently. Milstein and Tretyakov [28] solved a linear Stochastic Partial Differential Equation by using the characteristics method (the averaging over the characteristic formula). They proposed a numerical scheme based on the Monte Carlo technique. Moreover, they constructed Layer methods for linear and semilinear SPDEs. Picard [35] considered a filtering problem where the observation was a diffusion function corrupted by an independent white noise. He estimated the error caused by a discretization of the time interval. He obtained some approximations of the optimal filter that can be computed with Monte-Carlo methods. Crisan [9] studied a particle approximation for a class of nonlinear stochastic partial differential equations.

Another probabilistic method to solve a semilinear SPDE is based on the associated BDSDE. It requires weaker assumptions on the SPDE's coefficients. In the deterministic PDE's case i.e. $g \equiv 0$, the numerical approximation of the BSDE has already been studied in the literature by Bally [4], Zhang [38], Bouchard and Touzi [7], Gobet, Lemor and Warin[14] and Bouchard and Elie [8] among others. Zhang [38] proposed a discrete-time numerical approximation, by step processes, for a class of decoupled FBSDEs with possible path-dependent terminal values. He proved a L^2-type regularity of the BSDE's solution, the convergence of his scheme and he derived its rate of convergence. Bouchard and Touzi [7] suggested a similar numerical scheme for decoupled FBSDEs. The conditional expectations involved in their discretization scheme were computed by using the kernel regression estimation. Therefore, they used the Malliavin approach and the Monte carlo method for its computation. Crisan, Manolarakis and Touzi [10] proposed an improvement on the Malliavin weights. Gobet, Lemor and Warin in [14] proposed an explicit numerical scheme. In the stochastic PDEs case, i.e. $g \neq 0$, Aman [1] and Aboura [2] treated the particular case when g does not depend on the control variable z. Aman [1] proposed a numerical scheme following the idea used by Bouchard and Touzi [7] and obtained a convergence of order h of the square of the L^2-error (h is the discretization step in time). Aboura [2] studied the same numerical scheme under the same kind of hypothesis, but following Gobet et al. [13]. He obtained a convergence of order h in time and used the regression Monte Carlo method to implement his scheme, as in [13].

In this paper, we extend the approach of Bouchard-Touzi-Zhang in the general case when g also depends on the control variable z. We emphasize that this generalization is not obvious because of the strong impact of the backward stochastic integral term on the numerical approximation scheme. It is known that in the associated Stochastic PDE's (1.1), the term $g(u, \nabla u)$ leads to a second order perturbation type which explains the contraction condition assumed on g with respect to the variable z (see [34], [31]). This scheme is implicit in Y and explicit in Z. The convergence of our time-discretization scheme is proven and the rate of convergence given. The square of the L^2- error has an upper bound in the order of the discretization step in time. As a consequence, a scheme for the weak solution of the associated semilinear SPDE is obtained and a rate of convergence result for the later weak solution given. Then, we propose a fully implementable numerical scheme based on iterative regression functions which are approximated by projections on vector space of functions with coefficients evaluated using Monte Carlo simulations. Finally, some numerical tests are presented. Compared to the deterministic numerical method developed by Gyongy and Krylov [16], the probabilistic approach could tackle the semilinear SPDE which could be degenerate and needs fewer regularity conditions on the coefficients than the finite difference scheme. However, the rate of convergence obtained is clearly slower than the rate obtained by finite difference and finite element schemes, but our method is available in high dimension. To simplify the numerical implementation which is based on least-squares method an example is given in the one dimensional case. For the multidimensional case, we refer to Gobet and Lemor [15] who studied the numerical resolution of BSDEs and treated numerical results up to the dimension 10.

The paper is organized as follows: in section 2, preliminaries and assumptions are introduced and the approximation scheme for the BDSDEs (1.2) is described. In section 3, an upper bound result for the time discretization error is shown. In section 4, we give a Malliavin regularity result for the solution of our Forward-Backward Doubly SDEs. Then, we show an L^2-regularity result for the Z-component of the solution of the BDSDEs (1.2) which is crucial to obtain the rate of convergence of our numerical scheme. Section 5 is devoted to the numerical scheme of the SPDE's weak solution. In section 6, the convergence of this scheme is tested statistically by using a path dependent algorithm based on the regression Monte Carlo Method. Finally, some technical results are given in the Appendix.

2. Preliminaries and notations

2.1. *Forward Backward Doubly Stochastic Differential Equation*

Let $\{W_s, 0 \leq s \leq T\}$ and $\{B_s, 0 \leq s \leq T\}$ be two mutually independent standard Brownian motion processes, with values respectively in $\mathbb{R}^d$ and in $\mathbb{R}^l$ where $T > 0$ is a fixed horizon time, defined on the probability space $(\Omega, \mathcal{F}, P)$.
We shall work in the product space $\Omega := \Omega_W \times \Omega_B$, where Ω_W is the set of continuous functions from $[0, T]$ into $\mathbb{R}^d$ and Ω_B is the set of continuous functions from $[0, T]$ into $\mathbb{R}^l$.
For $t \in [0, T]$ and $s \in [t, T]$,

$$\mathcal{F}_s^t := \mathcal{F}_{t,s}^W \vee \mathcal{F}_{s,T}^B$$

is defined, where $\mathcal{F}_{t,s}^W = \sigma\{W_r - W_t, t \leq r \leq s\}$, and $\mathcal{F}_{s,T}^B = \sigma\{B_r - B_s, s \leq r \leq T\}$. We set $\mathcal{F}^W = \mathcal{F}_{0,T}^W$, $\mathcal{F}^B = \mathcal{F}_{0,T}^B$ and $\mathcal{F} = \mathcal{F}^W \vee \mathcal{F}^B$.
We define the probability measures P_W on $(\Omega_W, \mathcal{F}^W)$ and P_B on $(\Omega_B, \mathcal{F}^B)$. Then, we define the probability measure $P := P_W \otimes P_B$ on $(\Omega, \mathcal{F}^W \times \mathcal{F}^B)$. Without loss of generality, it is assumed that $\mathcal{F}^W$ and $\mathcal{F}^B$ are complete.
Note that the collection $\{\mathcal{F}_s^t, s \in [t, T]\}$ is neither increasing nor decreasing, and it does not con-

stitute a filtration. To alleviate notations, we denote $\mathcal{F}_s := \mathcal{F}_s^0$.

The following spaces are introduced:
• $C_b^k(\mathbb{R}^p, \mathbb{R}^q)$ (respectively $C_b^\infty(\mathbb{R}^p, \mathbb{R}^q)$) denotes the set of functions of class C^k from $\mathbb{R}^p$ to $\mathbb{R}^q$ whose partial derivatives of order less or equal to k are bounded (respectively the set of functions of class C^∞ from $\mathbb{R}^p$ to $\mathbb{R}^q$ whose partial derivatives are bounded).
• $C_b^k([0,T] \times \mathbb{R}^p, \mathbb{R}^q)$ denotes the set of functions of class C^k from $[0,T] \times \mathbb{R}^p$ to $\mathbb{R}^q$ whose partial derivatives of order less or equal to k are bounded.
• $L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^k)$ denotes the set of $\mathcal{F}_T$-measurable square integrable random variables with values in $\mathbb{R}^k$.
For any $m \in \mathbb{N}$ and $t \in [0,T]$, the following notations are introduced:
• $\mathbb{H}_m^2([t,T])$ denotes the set of (classes of $dP \times dt$ a.e. equal) $\mathbb{R}^m$-valued jointly measurable processes $\{\psi_u; u \in [t,T]\}$ satisfying:
(i) $||\psi||^2_{\mathbb{H}_m^2([t,T])} := E[\int_t^T |\psi_u|^2 du] < \infty$,
(ii) ψ_u is $\mathcal{F}_u$-measurable, for a.e. $u \in [t,T]$.
• $\mathbb{S}_m^2([t,T])$ denotes similarly the set of $\mathbb{R}^m$-valued continuous processes satisfying:
(i) $||\psi||^2_{\mathbb{S}_m^2([t,T])} := E[\sup_{t \le u \le T} |\psi_u|^2] < \infty$,
(ii) ψ_u is $\mathcal{F}_u$-measurable, for any $u \in [t,T]$.
• $\mathbb{S}$ the set of random variables F of the form: $F = \hat{f}(W(h_1), \dots, W(h_{m_1}), B(k_1), \dots, B(k_{m_2}))$ with $\hat{f} \in C_b^\infty(\mathbb{R}^{m_1+m_2}, \mathbb{R})$, $h_1, \dots, h_{m_1} \in L^2([t,T], \mathbb{R}^d), k_1, \dots, k_{m_2} \in L^2([t,T], \mathbb{R}^l)$, where

$$W(h_i) := \int_t^T h_i(s) dW_s, \quad B(k_j) := \int_t^T k_j(s) \overleftarrow{dB_s}.$$

For any random variable $F \in \mathbb{S}$, its Malliavin derivative $(D_s F)_s$ is defined with respect to the Brownian motion W as follows

$$D_s F := \sum_{i=1}^{m_1} \nabla_i \hat{f}\Big(W(h_1), \dots, W(h_{m_1}); B(k_1), \dots, B(k_{m_2})\Big) h_i(s),$$

where $\nabla_i \hat{f}$ is the derivative of $\hat{f}$ with respect to its i-th argument.
We define a norm on $\mathbb{S}$ by:

$$\|F\|_{1,2} := \big\{E[F^2] + E\big[\int_t^T |D_s F|^2 ds\big]\big\}^{\frac{1}{2}}.$$

• $\mathbb{D}^{1,2} \triangleq \overline{\mathbb{S}}^{\|\cdot\|_{1,2}}$ is then a Sobolev space.
• $\mathcal{S}_k^2([t,T], \mathbb{D}^{1,2})$ is the set of processes $Y = (Y_u, t \le u \le T)$ such that $Y \in \mathbb{S}_k^2([t,T])$, $Y_u^i \in \mathbb{D}^{1,2}$, $1 \le i \le k$, $t \le u \le T$ and

$$\|Y\|_{1,2} := \{E[\int_t^T |Y_u|^2 du] + E[\int_t^T \int_t^T ||D_\theta Y_u||^2 du d\theta]\}^{\frac{1}{2}} < \infty.$$

• $\mathcal{M}_{k \times d}^2([t,T], \mathbb{D}^{1,2})$ is the set of processes $Z = (Z_u, t \le u \le T)$ such that $Z \in \mathbb{H}_{k \times d}^2([t,T])$, $Z_u^{i,j} \in \mathbb{D}^{1,2}$,$1 \le i \le k$, $1 \le j \le d$, $t \le u \le T$ and

$$\|Z\|_{1,2} := \{E[\int_t^T \|Z_u\|^2 du] + E[\int_t^T \int_t^T ||D_\theta Z_u||^2 du d\theta]\}^{\frac{1}{2}} < \infty.$$

• $\mathcal{B}^2([t,T],\mathbb{D}^{1,2}) := \mathcal{S}_k^2([t,T],\mathbb{D}^{1,2}) \times \mathcal{M}_{k\times d}^2([t,T],\mathbb{D}^{1,2})$.
We define also for a given $t \in [0,T]$:
• $L^2([t,T],\mathbb{D}^{1,2})$ is the set of $(\mathcal{F}_s^t)_{s\leq T}$-measurable processes $(v_s)_{t\leq s\leq T}$ such that:
(i) $v(s,.) \in \mathbb{D}^{1,2}$, for a.e. $s \in [t,T]$,
(ii) $(s,w) \longrightarrow Dv(s,w) \in L^2([t,T]\times\Omega)$,
(iii) $E[\int_t^T |v_s|^2 ds] + E[\int_t^T\int_t^T |D_u v_s|^2 duds] < \infty$.
• $L^2([t,T],\mathbb{D}^{1,2}\times\mathbb{D}^{1,2}) := L^2([t,T],\mathbb{D}^{1,2}) \times L^2([t,T],\mathbb{D}^{1,2})$.

For all $(t,x) \in [0,T]\times\mathbb{R}^d$, let $(X_s^{t,x})_{0\leq s\leq T}$ be the unique strong solution of the following stochastic differential equation:

$$dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dW_s, \quad s\in[t,T], \qquad X_s^{t,x} = x, \quad 0\leq s\leq t, \tag{2.1}$$

where b and σ are two measurable functions on $\mathbb{R}^d$ with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$. We will omit the dependance of the forward process X in the initial condition if it starts at time $t=0$.
We consider the following BDSDE: For all $t\leq s\leq T$,

$$\begin{cases} dY_s^{t,x} & = -f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds - g(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})\overleftarrow{dB_s} + Z_s^{t,x}dW_s, \\ Y_T^{t,x} & = \Phi(X_T^{t,x}), \end{cases} \tag{2.2}$$

where f and Φ are two measurable functions respectively on $[0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$ and $\mathbb{R}^d$ with values in $\mathbb{R}^k$ and g is a measurable function on $[0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$ with values in $\mathbb{R}^{k\times l}$.
Note that the integral with respect to $(B_s, t\leq s\leq T)$ is a "backward Itô integral" (see Kunita [25] and Nualart and Pardoux [31] for the definition) and the integral with respect to $(W_s, t\leq s\leq T)$ is a standard forward Itô integral.
Finally, for each real matrix A, we denote by $\|A\|$ its Frobenius norm defined by $\|A\| = (\sum_{i,j} a_{i,j}^2)^{1/2}$.
For a vector x, $|x|$ stands for its Euclidean norm defined by $|x| = (\sum_i |x_i|^2)^{1/2}$.
The following assumptions will be needed in our work:

Assumption (H1) There exists a positive constant K such that $\forall x, x' \in \mathbb{R}^d$

$$|b(x) - b(x')| + \|\sigma(x) - \sigma(x')\| \leq K|x - x'|.$$

Assumption (H2) There exist two constants $K > 0$ and $0\leq\alpha<1$ such that
for any $(t_1,x_1,y_1,z_1),(t_2,x_2,y_2,z_2) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$,

(i) $|f(t_1,x_1,y_1,z_1) - f(t_2,x_2,y_2,z_2)| \leq K\big(\sqrt{|t_1-t_2|} + |x_1-x_2| + |y_1-y_2| + \|z_1-z_2\|\big)$,

(ii) $\|g(t_1,x_1,y_1,z_1) - g(t_2,x_2,y_2,z_2)\|^2 \leq K^2\big(|t_1-t_2| + |x_1-x_2|^2 + |y_1-y_2|^2\big) + \alpha^2\|z_1-z_2\|^2$,

(iii) $|\Phi(x_1) - \Phi(x_2)| \leq K|x_1-x_2|$,

(iv) $\sup_{0\leq t\leq T}(|f(t,0,0,0)| + ||g(t,0,0,0)||) \leq K$.

Assumption (H3)

(i) $b \in C_b^2(\mathbb{R}^d,\mathbb{R}^d)$ and $\sigma \in C_b^2(\mathbb{R}^d,\mathbb{R}^{d\times d})$

(ii) $\Phi \in C_b^2(\mathbb{R}^d,\mathbb{R}^k)$, $f \in C_b^2([0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{d\times k},\mathbb{R}^k)$
and $g \in C_b^2([0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{d\times k},\mathbb{R}^{k\times l})$.

We state the following result proved in [34] (Theorem 1.1. p.212)

Theorem 2.1 *Under Assumptions* ***(H1)*** *and* ***(H2)****, there exists a unique solution* (Y, Z) *of the BDSDE* (2.2) *which belongs to* $\mathbb{S}_k^2([t,T]) \times \mathbb{H}_{k\times d}^2([t,T])$.

Remark 2.1 *The regularity conditions on the time-space variable* (t, x) *of* f*,* g *and* Φ *are needed for the estimates for the time discretization error of the solution* (Y, Z) *in section 3.*

From [11], [34] (Theorem 1.4 p. 217) and [22], the standard estimates for the solution of the Forward-Backward Doubly SDE (2.1)-(2.2) hold and we remind the following theorem:

Theorem 2.2 *Under Assumptions* ***(H1)*** *and* ***(H2)*** *and for some* $p \geq 2$*, there exist two positive constants* C *and* C_p *independent of* x *and an integer* q *such that:*

$$E[\sup_{t\leq s\leq T} |X_s^{t,x}|^2] \leq C(1 + |x|^2), \tag{2.3}$$

$$E\Big[\sup_{t\leq s\leq T} |Y_s^{t,x}|^p + \Big(\int_t^T \|Z_s^{t,x}\|^2 ds\Big)^{p/2}\Big] \leq C_p(1 + |x|^q). \tag{2.4}$$

Remark 2.2 *The superscript* (t, x) *indicates the dependence of the solution* (X, Y, Z) *on the initial date* (t, x)*. When it is clear, we omit the dependence of* $(Y^{t,x}, Z^{t,x})$ *on* (t, x) *.*
It should also be noted that in the next computations, the constant C *denotes a generic constant that may change from line to line. It depends on* K*,* T*,* α*,* $|b(0)|$*,* $||\sigma(0)||$*,* $|f(t, 0, 0, 0)|$ *and* $||g(t, 0, 0, 0)||$*.*

2.2. Numerical Scheme for decoupled Forward-BDSDE

In order to approximate the solution of the BDSDE (2.2), the following discretized version is introduced. Let

$$\pi : t_0 = 0 < t_1 < \ldots < t_N = T, \tag{2.5}$$

be a partition of the time interval $[0, T]$. For simplicity we take an equidistant partition of $[0, T]$ i.e. $h = \frac{T}{N}$ and $t_n = nh$, $0 \leq n \leq N$. Throughout the rest, the notations $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ and $\Delta B_n = B_{t_{n+1}} - B_{t_n}$, for $n = 1, \ldots, N$ will be used.
The forward component X will be approximated by the classical Euler scheme:

$$\begin{cases} X_{t_0}^N &= X_{t_0}, \\ X_{t_n}^N &= X_{t_{n-1}}^N + b(X_{t_{n-1}}^N)(t_n - t_{n-1}) + \sigma(X_{t_{n-1}}^N)(W_{t_n} - W_{t_{n-1}}), \text{ for } n = 1, \ldots, N. \end{cases} \tag{2.6}$$

Note the following lemma (see [20]):

Lemma 2.1 *Under Assumption* ***(H1)****, there exists a positive constant* C *independent of* x *and depending on* K*,*T*,* $|b(0)|$ *and* $\|\sigma(0)\|$ *such that for all* $s \in [t_n, t_{n+1})$ *and for all* $n = 0, \ldots, N-1$ *we have:*

$$E\Big[|X_s - X_{t_n}^N|^2 + |X_s - X_{t_{n+1}}^N|^2\Big] \leq Ch(1 + |x|^2). \tag{2.7}$$

The solution (Y, Z) of (2.2) is approximated by (Y^N, Z^N) defined by:

$$Y_{t_N}^N = \Phi(X_T^N) \text{ and } Z_{t_N}^N = 0, \tag{2.8}$$

and for $n = N-1, \dots, 0$, we set

$$Y_{t_n}^N = E_{t_n}[Y_{t_{n+1}}^N + g(t_{n+1}, \Theta_{n+1}^N)\Delta B_n] + hf(t_n, \Theta_n^N), \tag{2.9}$$

$$hZ_{t_n}^N = E_{t_n}\left[Y_{t_{n+1}}^N \Delta W_n^* + g(t_{n+1}, \Theta_{n+1}^N)\Delta B_n \Delta W_n^*\right], \tag{2.10}$$

where

$$\Theta_n^N := (X_{t_n}^N, Y_{t_n}^N, Z_{t_n}^N), \text{for all } n = 0, \dots, N.$$

$*$ denotes the transposition operator and E_{t_n} denotes the conditional expectation over the σ-algebra $\mathcal{F}_{t_n}$.

Remark 2.3 *By construction, $(Y_{t_n}^N, Z_{t_n}^N)_{n\geq 0}$ are square integrable. For the approximation of $Y_{t_n}^N$, (2.9) is well-defined, indeed $Y_{t_n}^N(\omega)$ is a fixed point of*

$$\varphi(x) = hf(t_n, X_{t_n}^N(\omega), x, Z_{t_n}^N(\omega)) + E_{t_n}[Y_{t_{n+1}}^N + g(t_{n+1}, X_{t_{n+1}}^N, Y_{t_{n+1}}^N, Z_{t_{n+1}}^N)\Delta B_n](\omega),$$

which exists and is unique as soon as $Kh < 1$. Such a condition holds when h is small enough.

For later use, a continuous approximation of the solution of BDSDE (2.2) must be introduced. We define:

$$Y_t^N := Y_{t_{n+1}}^N + \int_t^{t_{n+1}} f(t_n, \Theta_n^N)ds + \int_t^{t_{n+1}} g(t_{n+1}, \Theta_{n+1}^N)\overleftarrow{dB_s} - \int_t^{t_{n+1}} Z_s^N dW_s, \ t_n \leq t < t_{n+1}. \tag{2.11}$$

where

$$\Theta_n^N := (X_{t_n}^N, Y_{t_n}^N, Z_{t_n}^N), \text{for all } n = 0, \dots, N.$$

The following property of Z^N is needed later.

Lemma 2.2 *For all $n = 0, \dots, N-1$, we have*

$$Z_{t_n}^N = \frac{1}{h} E_{t_n}[\int_{t_n}^{t_{n+1}} Z_s^N ds] \quad P-a.s. \tag{2.12}$$

Proof. From (2.11) we have

$$\begin{aligned}\int_{t_n}^{t_{n+1}} Z_s^N dW_s \Delta W_n &= Y_{t_{n+1}}^N \Delta W_n + \int_{t_n}^{t_{n+1}} f(t_n, \Theta_n^N)ds\Delta W_n \\ &+ \int_{t_n}^{t_{n+1}} g(t_{n+1}, \Theta_{n+1}^N)\overleftarrow{dB_s}\Delta W_n - Y_{t_n}^N \Delta W_n.\end{aligned}$$

Taking the conditional expectation we get

$$\begin{aligned}E_{t_n}[\int_{t_n}^{t_{n+1}} Z_s^N dW_s \Delta W_n] &= E_{t_n}[Y_{t_{n+1}}^N \Delta W_n] + \mathbb{E}_{t_n}[\int_{t_n}^{t_{n+1}} f(t_n, \Theta_n^N)ds\Delta W_n] \\ &+ E_{t_n}[\int_{t_n}^{t_{n+1}} g(t_{n+1}, \Theta_{n+1}^N)\overleftarrow{dB_s}\Delta W_n] - E_{t_n}[Y_{t_n}^N \Delta W_n] \\ &= E_{t_n}[Y_{t_{n+1}}^N \Delta W_n] + hE_{t_n}[f(t_n, \Theta_n^N)\Delta W_n] \\ &+ E_{t_n}[g(t_{n+1}, \Theta_{n+1}^N)\Delta B_n \Delta W_n] - E_{t_n}[Y_{t_n}^N \Delta W_n].\end{aligned}$$

Using the fact that $Y^N_{t_n}$ and $f(t_n,\Theta^N_n)$ are $\mathcal{F}_{t_n}$-measurable, we obtain

$$E_{t_n}[\int_{t_n}^{t_{n+1}} Z^N_s dW_s \Delta W_n] = E_{t_n}[Y^N_{t_{n+1}}\Delta W_n] + E_{t_n}[g(t_{n+1},\Theta^N_{n+1})\Delta B_n \Delta W^*_n]. \tag{2.13}$$

By using the integration by parts formula, we have

$$\begin{aligned} E_{t_n}[\int_{t_n}^{t_{n+1}} Z^N_s dW_s \Delta W_n] &= E_{t_n}[\int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dW_u Z^N_s dW_s] \\ &+ E_{t_n}[\int_{t_n}^{t_{n+1}}\int_{t_n}^{s} Z^N_u dW_u dW_s] + E_{t_n}[\int_{t_n}^{t_{n+1}} Z^N_s ds]. \end{aligned}$$

Then

$$E_{t_n}[\int_{t_n}^{t_{n+1}} Z^N_s dW_s \Delta W_n] = E_{t_n}[\int_{t_n}^{t_{n+1}} Z^N_s ds]. \tag{2.14}$$

Equations (2.13) and (2.14) together with (2.10) give that

$$hZ^N_{t_n} = E_{t_n}[\int_{t_n}^{t_{n+1}} Z^N_s ds].$$

□

3. The discrete time approximation error

Fisrt, the step process $\bar{Z}$ is defined by

$$\begin{cases} \bar{Z}_t = \frac{1}{h}E_{t_n}[\int_{t_n}^{t_{n+1}} Z_s ds], \text{ for all } t\in[t_n,t_{n+1}), \text{ for all } n\in\{0,\dots,N-1\}, \\ \bar{Z}_{t_N} = 0. \end{cases} \tag{3.1}$$

The following theorem states an upper bound result regarding the time discretization error.

Theorem 3.1 *Define the square error by*

$$Error_N(Y,Z) := \sup_{0\le s\le T} E[|Y_s - Y^N_s|^2] + \sum_{n=0}^{N-1} E[\int_{t_n}^{t_{n+1}} ||Z_s - Z^N_s||^2 ds], \tag{3.2}$$

*where Y^N and Z^N are given by (2.11). Under Assumptions **(H1)** and **(H2)** we have*

$$\begin{aligned} Error_N(Y,Z) &\le Ch(1+|x|^2) + C\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds \\ &+ C\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds + C\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[|Y_s - Y_{t_n}|^2]ds \\ &+ C\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[|Y_s - Y_{t_{n+1}}|^2]ds. \end{aligned} \tag{3.3}$$

Before proving Theorem 3.1, we need the following lemma whose proof is given in the Appendix. For all $t \in [t_n, t_{n+1})$, $n = 0, \dots, N-1$, the following quantities are defined:

$$\left\{\begin{array}{l} \theta_t := (X_t, Y_t, Z_t)\ , \delta Y_t^N := Y_t - Y_t^N,\ \delta Z_t^N := Z_t - Z_t^N, \\ \delta f_t := f(t, \theta_t) - f(t_n, \Theta_n^N), \\ \delta g_t := g(t, \theta_t) - g(t_{n+1}, \Theta_{n+1}^N). \end{array}\right. \tag{3.4}$$

Introduce the following term: for $n \leq N-1$

$$R_n := Ch^2(1+|x|^2) + C\int_{t_n}^{t_{n+1}} E\big[|Y_s - Y_{t_n}|^2 + |Y_s - Y_{t_{n+1}}|^2 + ||Z_s - \bar{Z}_{t_n}||^2 + ||Z_s - \bar{Z}_{t_{n+1}}||^2\big] ds \tag{3.5}$$

Lemma 3.1 *Under Assumptions **(H1)** and **(H2)**, there exists a constant $\alpha' \in (0,1)$ such that for a constant $C > 0$*

$$\begin{aligned} &\frac{1}{C}\sup_{t\in[t_n,t_{n+1}]} E[|\delta Y_t^N|^2] + E\Big[|\delta Y_{t_n}^N|^2 + \frac{1+\alpha'}{2}\int_{t_n}^{t_{n+1}} \|\delta Z_s^N\|^2 ds\Big] \leq \\ &(1+Ch)\bigg\{E\Big[|\delta Y_{t_{n+1}}^N|^2 + \alpha' \mathbb{1}_{\{n<N-1\}}\int_{t_{n+1}}^{t_{n+2}} \|\delta Z_s^N\|^2 ds\Big] + R_n\bigg\}. \end{aligned} \tag{3.6}$$

Proof of Theorem 3.1. To alleviate the presentation, we introduce $y_n := E[|\delta Y_{t_n}^N|^2]$ and $z_n := E\Big[\int_{t_n}^{t_{n+1}} \|\delta Z_s^N\|^2 ds\Big]$. From Lemma 3.1, we have for all $n = 0, ..., N-1$

$$y_n + \frac{1+\alpha'}{2} z_n \leq (1+Ch)\Big(y_{n+1} + \alpha' \mathbb{1}_{\{n<N-1\}} z_{n+1} + R_n\Big). \tag{3.7}$$

Summing (3.7) from $n = i$ to $n = N-1$, $i \leq N-1$, we obtain

$$\sum_{n=i}^{N-1} y_n + \frac{1+\alpha'}{2}\sum_{n=i}^{N-1} z_n \leq (1+Ch)\Big(\sum_{n=i}^{N-1} y_{n+1} + \alpha' \sum_{n=i+1}^{N-1} z_n + \sum_{n=i}^{N-1} R_n\Big).$$

Then, we have

$$y_i + \frac{1+\alpha'}{2}\sum_{n=i}^{N-1} z_n \leq y_N + Ch\sum_{n=i+1}^{N} y_n + (1+Ch)\alpha' \sum_{n=i+1}^{N-1} z_n + (1+Ch)\sum_{n=i}^{N-1} R_n.$$

This leads, for h small enough and since $\alpha' \in (0,1)$ to

$$\sum_{n=i}^{N-1} z_n \leq C\Big(y_N + h\sum_{n=i+1}^{N} y_n + \sum_{n=i}^{N-1} R_n\Big). \tag{3.8}$$

Iterating (3.7) from $n = i$ to $n = N-1$, $i \leq N-1$, we obtain

$$y_i + \frac{1+\alpha'}{2} z_i \leq C\Big(y_N + \sum_{n=i+1}^{N-1} z_n + \sum_{n=i}^{N-1} R_n\Big).$$

Combining (3.8) with the last inequality, we obtain

$$y_i \leq C\Big(y_N + h\sum_{n=i+1}^{N} y_n + \sum_{n=0}^{N-1} R_n\Big).$$

Using the discrete version of Gronwall's lemma, we get

$$\max_{0\leq i\leq N-1} y_i \leq C\Big(y_N + \sum_{n=0}^{N-1} R_n\Big).$$

From Assumption **(H2)-(iii)** we obtain

$$y_i \leq Ch(1+|x|^2) + C\sum_{n=0}^{N-1} R_n. \tag{3.9}$$

Therefore

$$h\sum_{n=i+1}^{N} y_n \leq Ch(1+|x|^2) + C\sum_{n=0}^{N-1} R_n.$$

Inserting the last inequality into (3.8) and taking $i=0$ we obtain

$$\sum_{n=i}^{N-1} z_n \leq Ch(1+|x|^2) + C\sum_{n=0}^{N-1} R_n. \tag{3.10}$$

Combining (3.6) with (3.9) and (3.10) the result is obtained. □

4. Path regularity of the process Z

The purpose of this section is to prove the L^2-regularity of the Z component of the BDSDE's solution (1.2). Such a result is crucial to obtain the convergence and the rate of convergence of this numerical scheme. To this end, the Malliavin derivatives of the solution must be introduced . This will allow us to provide a representation and regularity results for Y and Z that will immediately imply the rate of convergence of the scheme.
We recall the tools on the Malliavin calculus in the context of BDSDEs introduced by Pardoux and Peng [34]. Pardoux and Peng have skipped details of this part considering that it is just a natural extension of the work on standard BSDEs [33]. For the sake of completeness, we give some details which are crucial to obtaining regularity result of the process Z and we give some technical proofs in the Appendix.

4.1. Malliavin calculus on the Forward SDE's

In this section, we recall some properties on the differentiability in the Malliavin sense of the forward process $(X_s^{t,x})$. Under **(H3(i))**, Nualart [30] stated that $X_s^{t,x}\in\mathbb{D}^{1,2}$ for any $s\in[t,T]$ and for $l\leq k$ the derivative $D_r^l X_s^{t,x}$ is given by:
(i) $D_r^l X_s^{t,x} = 0, \quad for \quad s<r\leq T,$
(ii) For any $t<r\leq T$, a version of $\{D_r^l X_s^{t,x}, r\leq s\leq T\}$ is the unique solution of the following linear SDE

$$D_r^l X_s^{t,x} = \sigma^l(X_r^{t,x}) + \int_r^s \nabla b(X_u^{t,x}) D_r^l X_u^{t,x} du + \sum_{i=1}^d \int_r^s \nabla\sigma^i(X_u^{t,x}) D_r^l X_u^{t,x} dW_u^i,$$

where $(\sigma^i)_{i=1,\dots,d}$ denotes the i-th column of the matrix σ.
Moreover, $D_r^l X_s^{t,x} \in \mathbb{D}^{1,2}$ for all $r, s \le T$. For all $v \le T$ and $l' \le k$, we have

$$D_v^{l'} D_r^l X_s^{t,x} = 0 \text{ if } s < v \vee r,$$

and for all $s \ge v \vee r$ a version of $D_v^{l'} D_r^l X_s^{t,x}$ is the unique solution of the following SDE:

$$\begin{aligned}
D_v^{l'} D_r^l X_s^{t,x} &= \nabla\sigma^l(X_r^{t,x}) D_v^{l'} X_r^{t,x} + \sum_{i=1}^d \nabla\sigma^i(X_v^{t,x}) D_r^l X_v^{t,x} \mathbb{1}_{\{t \le v \le s\}} \\
&+ \int_r^s \Big[\sum_{j=1}^k \nabla((\nabla b)^j (X_u^{t,x})) D_v^{l'} X_u^{t,x} (D_r^l X_u^{t,x})^j + \nabla b(X_u^{t,x}) D_v^{l'} D_r^l X_u^{t,x} \Big] du \\
&+ \sum_{i=1}^d \int_r^s \Big[\sum_{j=1}^k \nabla(\nabla\sigma^i(X_u^{t,x}))^j D_v^{l'} X_u^{t,x} (D_r^l X_u^{t,x})^j + \nabla\sigma^i(X_u^{t,x}) D_v^{l'} D_r^l X_u^{t,x} \Big] dW_u^i,
\end{aligned}$$

where $((\nabla b)^j)_{j=1,\dots,k}$ (resp.$((\nabla\sigma^i(X_u^{t,x}))^j)_{j=1,\dots,k}$) denotes the j-th column of the matrix (∇b) (resp. $(\nabla\sigma^i(X_u^{t,x})))$ and $((D_r^l X_u^{t,x})^j)_{j=1,\dots,k}$ denotes the j-th component of the vector $(D_r^l X_u^{t,x})$. The following inequalities will be useful later. For the proofs, we refer to Nualart [30]. From Lemma 2.7 in [30] applied to X and $D_s X$ and any $0 \le r \le s \le T$, there exists a constant C which depends on p such that we have the following inequalities

$$E\Big[\sup_{0 \le u \le T} ||D_s X_u||^p \Big] \le C(1 + |x|^p), \tag{4.1}$$

$$E\Big[\sup_{s \vee r \le u \le T} ||D_s X_u - D_r X_u||^p \Big] \le C|s - r|^{p/2}(1 + |x|^p). \tag{4.2}$$

The same argument applied for $D_r D_s X$ shows that there exists a constant C which depends on p such that

$$E\Big[\sup_{0 \le u \le T} ||D_r D_s X_u||^p \Big] \le C(1 + |x|^{2p}). \tag{4.3}$$

4.2. Malliavin calculus for the solution of BDSDE's

Now, our aim is to study the differentiability in the Malliavin sense of the solution of the BDSDE (2.2). We start with the following lemma which shows that a backward Itô integral is differentiable in the Malliavin sense if and only if its integrand is so. We recall that Pardoux and Peng [33] proved that the result holds for the classical Itô integral.

Lemma 4.1 *Let $U \in \mathbb{H}_1^2([t, T])$ and $I_i(U) = \int_t^T U_r dW_r^i, i = 1, \dots, d$. Then, for each $\theta \in [0, T]$ we have $U_\theta \in \mathbb{D}^{1,2}$ if and only if $I_i(U) \in \mathbb{D}^{1,2}, i = 1, \dots, d$ and for all $\theta \in [0, T]$, we have*

$$\begin{aligned}
D_\theta I_i(U) &= \int_\theta^T D_\theta U_r dW_r^i + U_\theta, \ \theta > t, \\
D_\theta I_i(U) &= \int_t^T D_\theta U_r dW_r^i, \ \theta \le t.
\end{aligned}$$

For backward Itô integral, and since the Malliavin derivative is with respect to the brownian motion W, we have the following result :

Lemma 4.2 *Let $U \in \mathbb{H}_1^2([t,T])$ and $I_i(U) = \int_t^T U_r \overleftarrow{dB_r^i}, i = 1, \ldots, l$. Then for each $\theta \in [0,T]$ we have $U_\theta \in \mathbb{D}^{1,2}$ if and only if $I_i(U) \in \mathbb{D}^{1,2}, i = 1, \ldots, l$ and for all $\theta \in [0,T]$, we have*

$$\begin{aligned} D_\theta I_i(U) &= \int_\theta^T D_\theta U_r \overleftarrow{dB_r^i},\ \theta > t, \\ D_\theta I_i(U) &= \int_t^T D_\theta U_r \overleftarrow{dB_r^i},\ \theta \leq t. \end{aligned}$$

For later use, using the same argument as in the classical BSDEs setting, we can prove the a priori estimates for the solution of the BDSDE (see El Karoui et al. [11]).

Proposition 4.1 *Let (ϕ^1, f^1, g^1) and (ϕ^2, f^2, g^2) be two standard parameters of the BDSDE (2.2) and (Y^1, Z^1) and (Y^2, Z^2) the associated solutions. Let Assumption* **(H2)** *holds. For $s \in [t,T]$, set $\delta Y_s := Y_s^1 - Y_s^2$, $\delta_2 f_s := f^1(s, X_s, Y_s^2, Z_s^2) - f^2(s, X_s, Y_s^2, Z_s^2)$ and $\delta_2 g_s := g^1(s, X_s, Y_s^2, Z_s^2) - g^2(s, X_s, Y_s^2, Z_s^2)$. Then, we have*

$$||\delta Y||^2_{\mathbb{S}_d^2([t,T])} + ||\delta Z||^2_{\mathbb{H}^2_{k\times d}([t,T])} \leq CE[|\delta Y_T|^2 + \int_t^T |\delta_2 f_s|^2 ds + \int_t^T \|\delta_2 g_s\|^2 ds], \tag{4.4}$$

where C is a positive constant depending only on K, T and α.

We need also the following estimates which are deduced from the last proposition by using the Lipschitz condition for f and g and Assumption **(H2-iv)**.

Lemma 4.3 *Let $(X^{t,x}, Y^{t,x}, Z^{t,x})$ be the solution of the FBDSDE (2.1)-(2.2). Then, under Assumptions* **(H1)** *and* **(H2)**, *we have*

$$||Y^{t,x}||_{\mathbb{S}_d^2} + ||Z^{t,x}||_{\mathbb{H}^2_{k\times d}} \leq C(1 + |x|^2), \tag{4.5}$$

and for all $s', s \in [t,T], s' \leq s$, we have

$$E\Big[\sup_{s' \leq u \leq s} |Y_u^{t,x} - Y_{s'}^{t,x}|^2\Big] \leq C\Big((1+|x|^2)|s - s'| + ||Z^{t,x}||_{\mathbb{H}^2_{k\times d}[s',s]}\Big). \tag{4.6}$$

Now, we study the differentiability in the Malliavin sense of the solution of the BDSDE which is technical. To our knowledge, it does not exist in the literature. We have to precise that Pardoux and Peng [34] have skipped details considering that it was just an easy extension of the work on standard BSDEs [33]. We show in the following proposition that the derivative is a solution of a linear BDSDE (see Peng and Pardoux [33] for the standard BSDE's and also El Karoui, Peng and Quenez ([11], Proposition 5.3)). The proof is postponed to the appendix.

Proposition 4.2 *Assume that* **(H1)**-**(H3)** *hold. For any $t \in [0,T]$ and $x \in \mathbb{R}^d$, let $\{(Y_s, Z_s), t \leq s \leq T\}$ denotes the unique solution of the following BDSDE:*

$$Y_s = \Phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r, Z_r)dr + \int_s^T g(r, X_r^{t,x}, Y_r, Z_r)\overleftarrow{dB_r} - \int_s^T Z_r dW_r,\ t \leq s \leq T.$$

Then, $(Y,Z) \in \mathcal{B}^2([t,T], \mathbb{D}^{1,2})$ and $\{D_\theta Y_s, D_\theta Z_s; t \leq s, \theta \leq T\}$ is given by:
(i) $D_\theta Y_s = 0, D_\theta Z_s = 0$ for all $t \leq s < \theta \leq T$
(ii) for any fixed $\theta \in [t,T]$, $\theta \leq s \leq T$ and $1 \leq i \leq d$, a version of $(D_\theta^i Y_s, D_\theta^i Z_s)$ is the unique

solution of the following BDSDE:

$$\begin{aligned} D_\theta^i Y_s &= \nabla\Phi(X_T^{t,x}) D_\theta^i X_T^{t,x} + \int_s^T \Big(\nabla_x f(r, X_r^{t,x}, Y_r, Z_r) D_\theta^i X_r^{t,x}\Big) dr \\ &+ \int_s^T \Big(\nabla_y f(r, X_r^{t,x}, Y_r, Z_r) D_\theta^i Y_r + \sum_{j=1}^d \nabla_{z^j} f(r, X_r^{t,x}, Y_r, Z_r) D_\theta^i Z_r^j\Big) dr \\ &+ \sum_{n=1}^l \int_s^T \Big(\nabla_x g^n(r, X_r^{t,x}, Y_r, Z_r) D_\theta^i X_r^{t,x} + \nabla_y g^n(r, X_r^{t,x}, Y_r, Z_r) D_\theta^i Y_r\Big) \overleftarrow{dB_r^n} \\ &+ \sum_{n=1}^l \int_s^T \sum_{j=1}^d \Big(\nabla_{z^j} g^n(r, X_r^{t,x}, Y_r, Z_r) D_\theta^i Z_r^j\Big) \overleftarrow{dB_r^n} - \int_s^T \sum_{j=1}^d D_\theta^i Z_r^j dW_r^j, \end{aligned} \tag{4.7}$$

where $(z^j)_{1\le j\le d}$ denotes the j-th column of the matrix z, $(g^n)_{1\le n\le l}$ denotes the n-th column of the matrix g and $B = (B^1, \ldots, B^l)$.

The second order differentiability in the Malliavin sense of the solution of the BDSDE will be given in Appendix.

4.3. Representation results for BDSDEs

In this subsection, we will prove a representation result of (Z, DZ) which will be useful to prove the rate of convergence of our numerical scheme.

Proposition 4.3 *Assume that* **(H1)-(H3)** *hold. Then, for $t \le s \le T$, we have*

$$D_s Y_s^{t,x} = Z_s^{t,x}, \quad P-a.s., \qquad (4.8) \qquad \text{and} \qquad \|Z^{t,x}\|^2_{\mathbb{S}^2_{k\times d}([t,T])} \le C(1+|x|^2). \qquad (4.9)$$

Proof. To simplify the notations, we restrict ourselves to the case $k = d = 1$.
Notice that for $t \le s$, we have

$$Y_s^{t,x} = Y_t^{t,x} - \int_t^s f(r, \Sigma_r^{t,x}) dr - \int_t^s g(r, \Sigma_r^{t,x}) \overleftarrow{dB_r} + \int_t^s Z_r^{t,x} dW_r,$$

where $\Sigma_r^{t,x} := (X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})$.
It follows from Lemma 4.1 and Lemma 4.2 that, for $t < \theta \le s$

$$\begin{aligned} D_\theta Y_s^{t,x} &= Z_\theta^{t,x} - \int_\theta^s \Big(\nabla_x f(r, \Sigma_r^{t,x}) D_\theta X_r^{t,x} + \nabla_y f(r, \Sigma_r^{t,x}) D_\theta Y_r^{t,x} + \nabla_z f(r, \Sigma_r^{t,x}) D_\theta Z_r^{t,x}\Big) dr \\ &- \int_\theta^s \Big(\nabla_x g(r, \Sigma_r^{t,x}) D_\theta X_r^{t,x} + \nabla_y g(r, \Sigma_r^{t,x}) D_\theta Y_r^{t,x} + \nabla_z g(r, \Sigma_r^{t,x}) D_\theta Z_r^{t,x}\Big) \overleftarrow{dB_r} + \int_\theta^s D_\theta Z_r^{t,x} dW_r. \end{aligned}$$

Then by taking $\theta = s$, it follows that equality (4.8) holds.
From Proposition 4.2 and inequalities (2.4) and (4.1), we deduce that for each $\theta \le T$

$$E[\sup_{t\le s\le T} |D_\theta Y_s^{t,x}|^2] + E[\int_t^T |D_\theta Z_s^{t,x}|^2 ds] \le C(1+|x|^2). \tag{4.10}$$

Then, by taking $\theta = s$, we deduce that (4.9) holds. □

4.4. Path regularity

In this subsection, we extend the result of Zhang [38] which concerns the L^2-regularity of the martingale integrand Z. Such result is crucial to derive the rate of convergence of our numerical scheme. We start with the following proposition which gives an upper bound for

$$E\Big[\sup_{r\in[s,u]}|Y_r^{t,x}-Y_s^{t,x}|^2\Big] \quad \text{and} \quad E\Big[||Z_u^{t,x}-Z_s^{t,x}||^2\Big], \quad t\le s\le u\le T.$$

Proposition 4.4 *Assume that* **(H1)**-**(H3)** *hold. Then for* $t\le s\le u\le T$, *we have*

$$E\Big[\sup_{r\in[s,u]}|Y_r^{t,x}-Y_s^{t,x}|^2\Big] \quad \le \quad C(1+|x|^2)|u-s|, \tag{4.11}$$

$$E\Big[||Z_u^{t,x}-Z_s^{t,x}||^2\Big] \quad \le \quad C(1+|x|^2)|u-s|. \tag{4.12}$$

Proof. To simplify the notations, we restrict ourselves to the case $k=d=l=1$.
(i) Plugging inequality (4.9) in the estimate (4.6), the result (4.11) holds.
(ii) From Proposition 4.3, we have

$$E\Big[|Z_u^{t,x}-Z_s^{t,x}|^2\Big] \quad \le \quad CE[|D_uY_u^{t,x}-D_sY_u^{t,x}|^2]+CE[|D_sY_u^{t,x}-D_sY_s^{t,x}|^2]. \tag{4.13}$$

From the definition of the BDSDE (4.7), we have

$$\begin{aligned}
D_uY_u^{t,x}-D_sY_u^{t,x} &= \nabla\Phi(X_T^{t,x})(D_uX_T^{t,x}-D_sX_T^{t,x})+\int_u^T\Big(\nabla_xf(r,\Sigma_r^{t,x})(D_uX_r^{t,x}-D_sX_r^{t,x})\Big)dr\\
&+ \int_u^T\Big(\nabla_yf(r,\Sigma_r^{t,x})(D_uY_r^{t,x}-D_sY_r^{t,x})+\nabla_zf(r,\Sigma_r^{t,x})(D_uZ_r^{t,x}-D_sZ_r^{t,x})\Big)dr\\
&+ \int_u^T\Big(\nabla_xg(r,\Sigma_r^{t,x})(D_uX_r^{t,x}-D_sX_r^{t,x})+\nabla_yg(r,\Sigma_r^{t,x})(D_uY_r^{t,x}-D_sY_r^{t,x})\Big)\overleftarrow{dB_r}\\
&+ \int_u^T\Big(\nabla_zg(r,\Sigma_r^{t,x})(D_uZ_r^{t,x}-D_sZ_r^{t,x})\Big)\overleftarrow{dB_r}-\int_u^T(D_uZ_r^{t,x}-D_sZ_r^{t,x})dW_r.
\end{aligned}$$

Applying the generalized Itô's formula (see [34], Lemma 1.3), we obtain

$$
\begin{aligned}
& |D_u Y_T^{t,x} - D_s Y_T^{t,x}|^2 - |D_u Y_u^{t,x} - D_s Y_u^{t,x}|^2 = \\
- & \; 2\int_u^T \nabla_x f(r,\Sigma_r^{t,x})(D_u X_r^{t,x} - D_s X_r^{t,x})(D_u Y_r^{t,x} - D_s Y_r^{t,x})dr - 2\int_u^T \nabla_y f(r,\Sigma_r^{t,x})(D_u Y_r^{t,x} - D_s Y_r^{t,x})^2 dr \\
- & \; 2\int_u^T \nabla_z f(r,\Sigma_r^{t,x})(D_u Z_r^{t,x} - D_s Z_r^{t,x})(D_u Y_r^{t,x} - D_s Y_r^{t,x})dr \\
- & \; 2\int_u^T \nabla_x g(r,\Sigma_r^{t,x})(D_u X_r^{t,x} - D_s X_r^{t,x})(D_u Y_r^{t,x} - D_s Y_r^{t,x})\overleftarrow{dB_r} \\
- & \; 2\int_u^T \nabla_y g(r,\Sigma_r^{t,x})(D_u Y_r^{t,x} - D_s Y_r^{t,x})^2\overleftarrow{dB_r} \\
- & \; 2\int_u^T \nabla_z g(r,\Sigma_r^{t,x})(D_u Z_r^{t,x} - D_s Z_r^{t,x})(D_u Y_r^{t,x} - D_s Y_r^{t,x})\overleftarrow{dB_r} \\
+ & \; 2\int_u^T (D_u Z_r^{t,x} - D_s Z_r^{t,x})(D_u Y_r^{t,x} - D_s Y_r^{t,x})dW_r \\
- & \int_u^T \left|\nabla_x g(r,\Sigma_r^{t,x})(D_u X_r^{t,x} - D_s X_r^{t,x}) + \nabla_y g(r,\Sigma_r^{t,x})(D_u Y_r^{t,x} - D_s Y_r^{t,x}) + \nabla_z g(r,\Sigma_r^{t,x})(D_u Z_r^{t,x} - D_s Z_r^{t,x}\right|^2 dr \\
+ & \int_u^T |D_u Z_r^{t,x} - D_s Z_r^{t,x}|^2 dr.
\end{aligned}
$$

From inequalities (4.10) and (4.1), using the Burkholder-Davis-Gundy's inequality and Assumption **(H2)**, the stochastic integrals which appear in the last equation disappear when we take the expectation.

By Young inequality, we obtain, for $\epsilon' > 0$

$$
\begin{aligned}
& E[|D_u Y_u^{t,x} - D_s Y_u^{t,x}|^2] + E[\int_u^T |D_u Z_r^{t,x} - D_s Z_r^{t,x}|^2]dr \leq E[|\nabla\Phi(X_T^{t,x})(D_u X_T^{t,x} - D_s X_T^{t,x})|^2] \\
+ & \; 2E[\int_u^T \nabla_x f(r,\Sigma_r^{t,x})(D_u X_r^{t,x} - D_s X_r^{t,x})(D_u Y_r^{t,x} - D_s Y_r^{t,x})dr] \\
+ & \; 2E[\int_u^T \nabla_y f(r,\Sigma_r^{t,x})(D_u Y_r^{t,x} - D_s Y_r^{t,x})^2 dr] \\
+ & \; 2E[\int_u^T \nabla_z f(r,\Sigma_r^{t,x})(D_u Z_r^{t,x} - D_s Z_r^{t,x})(D_u Y_r^{t,x} - D_s Y_r^{t,x})dr] \\
+ & \; C(1+\frac{1}{\epsilon'})E[\int_u^T \nabla_x g(r,\Sigma_r^{t,x})^2 |D_u X_r^{t,x} - D_s X_r^{t,x}|^2 dr] \\
+ & \; C(1+\frac{1}{\epsilon'})E[\int_u^T \nabla_y g(r,\Sigma_r^{t,x})^2 |D_u Y_r^{t,x} - D_s Y_r^{t,x}|^2 dr] \\
+ & \; (1+\epsilon')E[\int_u^T \nabla_z g(r,\Sigma_r^{t,x})^2 |D_u Z_r^{t,x} - D_s Z_r^{t,x}|^2 dr].
\end{aligned}
$$

Hence by using Assumption **(H2)** and Young inequality, we have for $\epsilon, \epsilon' > 0$ and $C > 0$,

$$\begin{aligned} & E[|D_u Y_u^{t,x} - D_s Y_u^{t,x}|^2] + E[\int_u^T |D_u Z_r^{t,x} - D_s Z_r^{t,x}|^2 dr] \leq K^2 E[|D_u X_T^{t,x} - D_s X_T^{t,x}|^2] \\ + & 2KE[\int_u^T |D_u X_r^{t,x} - D_s X_r^{t,x}|^2 dr] + 4KE[\int_u^T |D_u Y_r^{t,x} - D_s Y_r^{t,x}|^2 dr] \\ + & K\epsilon E[\int_u^T |D_u Y_r^{t,x} - D_s Y_r^{t,x}|^2 dr] + \frac{K}{\epsilon} E[\int_u^T |D_u Z_r^{t,x} - D_s Z_r^{t,x}|^2 dr] \\ + & CK^2(1 + \frac{1}{\epsilon'}) E[\int_u^T |D_u X_r^{t,x} - D_s X_r^{t,x}|^2 dr] + CK^2(1 + \frac{1}{\epsilon'}) E[\int_u^T |D_u Y_r^{t,x} - D_s Y_r^{t,x}|^2 dr] \\ + & (1 + \epsilon')\alpha^2 E[\int_u^T |D_u Z_r^{t,x} - D_s Z_r^{t,x}|^2 dr]. \end{aligned}$$

Then, we obtain

$$\begin{aligned} & E[|D_u Y_u^{t,x} - D_s Y_u^{t,x}|^2] + E[\int_u^T |D_u Z_r^{t,x} - D_s Z_r^{t,x}|^2 dr] \leq K^2 E[|D_u X_T^{t,x} - D_s X_T^{t,x}|^2] \\ + & K(2 + KC(1 + \frac{1}{\epsilon'})) E[\int_u^T |D_u X_r^{t,x} - D_s X_r^{t,x}|^2 dr] \\ + & (K^2 C(1 + \frac{1}{\epsilon'}) + (4 + \epsilon)K) E[\int_u^T |D_u Y_r^{t,x} - D_s Y_r^{t,x}|^2 dr] \\ + & ((1 + \epsilon')\alpha^2 + \frac{K}{\epsilon}) E[\int_u^T |D_u Z_r^{t,x} - D_s Z_r^{t,x}|^2 dr]. \end{aligned}$$

For ϵ large enough and ϵ' small enough, we have $(1 + \epsilon')\alpha^2 + \frac{K}{\epsilon} < 1$. From inequality (4.2), we deduce that

$$E[|D_u Y_u^{t,x} - D_s Y_u^{t,x}|^2] \leq C\Big((1 + |x|^2)|u - s| + E[\int_u^T |D_u Y_r^{t,x} - D_s Y_r^{t,x}|^2 dr]\Big),$$

where C is a positive constant.
From Gronwall's lemma we have

$$E[|D_u Y_u^{t,x} - D_s Y_u^{t,x}|^2] \leq C(1 + |x|^2)|u - s|. \tag{4.14}$$

Since $(D_s Y_u^{t,x})_{s \leq u \leq T}$ satisfies the BDSDE (4.7), inequalities (4.6)-(4.9) hold for $(D_s Y_u^{t,x}, D_s Z_u^{t,x})_{s \leq u \leq T}$ and yield

$$E[|D_s Y_u^{t,x} - D_s Y_s^{t,x}|^2] \leq C(1 + |x|^2)|u - s|. \tag{4.15}$$

Plugging (4.14) and (4.15) into (4.13), we obtain (4.12). □

4.5. Application to the scheme's convergence

The following theorem states the rate of convergence of our numerical scheme.

Theorem 4.1 *Under Assumptions* **(H1)**-**(H3)***, there exists a positive constant C (depending only on T, K, α, $|b(0)|$, $||\sigma(0)||$, $|f(t,0,0,0)|$ and $||g(t,0,0,0)||$) such that*

$$Error_N(Y, Z) \leq Ch(1 + |x|^2). \tag{4.16}$$

Proof. We recall that from Theorem 3.1, we have

$$\begin{aligned} Error_N(Y,Z) &\leq Ch(1+|x|^2) + C\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds \\ &+ C\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds + C\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[|Y_s - Y_{t_n}|^2]ds \\ &+ C\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[|Y_s - Y_{t_{n+1}}|^2]ds. \end{aligned}$$

First step: We deal with the Y part. We have

$$\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[|Y_s - Y_{t_n}|^2]ds \leq \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[\sup_{t_n\leq s\leq t_{n+1}} |Y_s - Y_{t_n}|^2]ds.$$

From inequality (4.11) (see Proposition 4.4), we obtain

$$\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[|Y_s - Y_{t_n}|^2]ds \leq Ch(1+|x|^2). \tag{4.17}$$

Similarly, we get

$$\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[|Y_s - Y_{t_{n+1}}|^2]ds \leq Ch(1+|x|^2). \tag{4.18}$$

Second step: From the definition (3.1), $\bar{Z}_{t_n}$ is the best approximation of $(Z_t)_{t_n\leq t<t_{n+1}}$ by $\mathcal{F}_{t_n}$-measurable random variable in the following sense

$$E\Big[\int_{t_n}^{t_{n+1}} \|Z_s - \bar{Z}_{t_n}\|^2 ds\Big] = \inf_{Z_n\in L^2(\Omega,\mathcal{F}_{t_n})} E\Big[\int_{t_n}^{t_{n+1}} \|Z_s - Z_n\|^2 ds\Big]$$

From the estimation (4.12) (see Proposition 4.4), we have

$$E\Big[||Z_s - Z_{t_n}||^2\Big] \leq C(1+|x|^2)|s-t_n| \leq Ch(1+|x|^2), \tag{4.19}$$

for all $s \in [t_n, t_{n+1}]$ and $0 \leq n \leq N-1$ where C depends only on T, K, $b(0)$, $\sigma(0)$, $f(t,0,0,0)$ and $g(t,0,0,0)$. Then

$$\sum_{n=0}^{N-1} E\Big[\int_{t_n}^{t_{n+1}} ||Z_s - \bar{Z}_{t_n}||^2 ds\Big] \leq Ch(1+|x|^2).$$

On the other hand, we have

$$E\Big[\int_{t_n}^{t_{n+1}} ||Z_s - \bar{Z}_{t_{n+1}}||^2 ds\Big] \leq 2E\Big[\int_{t_n}^{t_{n+1}} ||Z_s - Z_{t_{n+1}}||^2 ds\Big] + 2E\Big[\int_{t_n}^{t_{n+1}} ||Z_{t_{n+1}} - \bar{Z}_{t_{n+1}}||^2 ds\Big]. \tag{4.20}$$

From the definition of $\bar{Z}_{t_{n+1}}$ and the Jensen's inequality, we have

$$\begin{aligned} E\Big[||Z_{t_{n+1}} - \bar{Z}_{t_{n+1}}||^2\Big] &= E\Big[||Z_{t_{n+1}} - \frac{1}{h}E_{t_{n+1}}\Big[\int_{t_{n+1}}^{t_{n+2}} Z_s ds\Big]||^2\Big] \\ &= E\Big[||\frac{1}{h}E_{t_{n+1}}\Big[\int_{t_{n+1}}^{t_{n+2}} (Z_{t_{n+1}} - Z_s) ds\Big]||^2\Big] \\ &\leq \frac{1}{h^2}E\Big[||\int_{t_{n+1}}^{t_{n+2}} (Z_{t_{n+1}} - Z_s) ds||^2\Big]. \end{aligned}$$

By using Cauchy Schwartz inequality, we obtain

$$\begin{aligned} E\Big[||Z_{t_{n+1}} - \bar{Z}_{t_{n+1}}||^2\Big] &\leq \frac{1}{h^2} E\Big[h \int_{t_{n+1}}^{t_{n+2}} ||Z_{t_{n+1}} - Z_s||^2 ds\Big] \\ &\leq \frac{1}{h} \int_{t_{n+1}}^{t_{n+2}} E\Big[||Z_{t_{n+1}} - Z_s||^2\Big] ds \end{aligned}$$

Using the estimation (4.12), we get

$$\begin{aligned} E\Big[||Z_{t_{n+1}} - \bar{Z}_{t_{n+1}}||^2\Big] &\leq \frac{1}{h} \int_{t_{n+1}}^{t_{n+2}} C(1+|x|^2)|s - t_{n+1}| ds \\ &\leq Ch(1+|x|^2). \end{aligned}$$

Inserting the last inequality in (4.20) and using again the estimate (4.12), we obtain

$$\sum_{n=0}^{N-2} E\Big[\int_{t_n}^{t_{n+1}} ||Z_s - \bar{Z}_{t_{n+1}}||^2 ds\Big] \leq Ch(1+|x|^2).$$

Using the estimation (4.9), we obtain

$$E\Big[\int_{t_{N-1}}^{t_N} ||Z_s||^2 ds\Big] \leq Ch(1+|x|^2).$$

Then

$$\sum_{n=0}^{N-1} E\Big[\int_{t_n}^{t_{n+1}} ||Z_s - \bar{Z}_{t_{n+1}}||^2 ds\Big] \leq Ch(1+|x|^2). \tag{4.21}$$

Finally, plugging (4.17), (4.18), (4.19) and (4.21) in (3.3) in Theorem 3.1, we get

$$Error_N(Y, Z) \leq Ch(1+|x|^2).$$

□

5. Numerical scheme for the weak solution of the SPDE

Most numerical works on SPDEs are concentrated on the Euler finite-difference scheme (see [17], [18] , [16]), on finite element method (see [37]) and also on spectral Galerkin methods (see [19] and the references therein). Here, we follow a probabilistic method based on the Feynman-Kac's formula for the weak solution of the semilinear SPDE (1.1) based on BSDE approach (see [6], [29]). We consider a weak Sobolev solution of such SPDE in the sense that u shall be considered as a predictable process in some first order Sobolev space. Therefore, we improve the convergence and the rate of convergence of the L^2-norm error of such solution by using the convergence results on BDSDEs proved in section 4.

5.1. Weak solution for SPDE

Since we work on the whole space $\mathbb{R}^d$, we introduce a weight function ρ satisfying the following conditions : ρ is a positive locally integrable function , $\frac{1}{\rho}$ is locally integrable and $\int_{\mathbb{R}^d}(1+|x|^2)\rho(x)dx < \infty$. For example, we can take $\rho(x) = e^{-\frac{x^2}{2}}$ or $\rho(x) = e^{-|x|}$. As a consequence of (**H3**), we have $\displaystyle\int_{\mathbb{R}^d} |\Phi(x)|^2\rho(x)dx < \infty$, $\int_0^T \int_{\mathbb{R}^d} |f(t,x,0,0)|^2\rho(x)dxdt < \infty$ and $\int_0^T \int_{\mathbb{R}^d} |g(t,x,0,0)|^2\rho(x)dxdt < \infty$.

We denote by $L^2(\mathbb{R}^d, \rho(x)dx)$ the weighted Hilbert space and we employ the following notation for its scalar product and its norm: $(u,v)_\rho = \int_{\mathbb{R}^d} u(x)v(x)\rho(x)dx$ and $\|u\|_\rho = (u,u)_\rho^{\frac{1}{2}}$. Then, we define by $\mathrm{H}^1_\rho(\mathbb{R}^d)$ the associated weighted first order Dirichlet space and its norm $\|u\|_{\mathrm{H}^1_\sigma(\mathbb{R}^d)} = (\|u\|_\rho^2 + \|\nabla u\sigma\|_\rho^2)^{\frac{1}{2}}$. Finally, $(.,.)$ denotes the usual scalar product in $L^2(\mathbb{R}^d, dx)$.
We also define $\mathcal{D} := \mathcal{C}_c^\infty([0,T]) \otimes \mathcal{C}_c^2(\mathbb{R}^d)$ the space of test functions where $\mathcal{C}_c^\infty([0,T])$ denotes the space of all real valued infinite differentiable functions with compact support in $[0,T]$ and $\mathcal{C}_c^2(\mathbb{R}^d)$ the set of C^2-functions with compact support in $\mathbb{R}^d$.
We introduce $\mathcal{H}_T$ the space of predictable processes $(u_t)_{t\geq 0}$ with values in $\mathrm{H}^1_\rho(\mathbb{R}^d)$ such that

$$\|u\|_T = \Big(E\Big[\sup_{0\leq t\leq T}\|u_t\|_\rho^2\Big] + E\Big[\int_0^T \|\nabla u_t\sigma\|_\rho^2 dt\Big]\Big)^{\frac{1}{2}} < \infty.$$

Définition 5.1 *We say that $u \in \mathcal{H}_T$ is a weak solution of the equation (1.1) associated with the terminal condition Φ and the coefficients (f,g), if the following relation holds almost surely, for each $\varphi \in \mathcal{D}$*

$$\begin{aligned} &\int_t^T (u(s,.), \partial_s\varphi(s,.))ds + \int_t^T \mathcal{E}(u(s,.),\varphi(s,.))ds + (u(t,.),\varphi(t,.)) - (\Phi(.),\varphi(T,.)) \qquad (5.1)\\ = \quad &\int_t^T (f(s,.,u(s,.),(\nabla u\sigma)(s,.)),\varphi(s,.))ds + \sum_{i=1}^l \int_t^T (g(s,.,u(s,.),(\nabla u\sigma)(s,.)),\varphi(s,.))\overleftarrow{dB_s^i}, \end{aligned}$$

where $\mathcal{E}(u,\varphi) = (Lu,\varphi) = \int_{\mathbb{R}^d}((\nabla u\sigma)(\nabla\varphi\sigma) + \varphi\nabla((\frac{1}{2}\sigma^*\nabla\sigma + b)u))(x)dx$ is the energy associated to the diffusion operator.

From Bally and Matoussi [6], we have the following result:

Theorem 5.1 *Under Assumptions* (**H1**) $-$ (**H3**)*, there exists a unique weak solution $u \in \mathcal{H}_T$ of the SPDE (1.1). Moreover, $u(t,x) = Y_t^{t,x}$ and $Z_t^{t,x} = \nabla u_t\sigma$, $dt\otimes dx\otimes dP$ a.e. where $(Y_s^{t,x}, Z_s^{t,x})_{t\leq s\leq T}$ is the solution of the BDSDE (1.2). Furthermore, we have for all $s \in [t,T]$, $u(s, X_s^{t,x}) = Y_s^{t,x}$ and $(\nabla u\sigma)(s, X_s^{t,x}) = Z_s^{t,x}$ $dt\otimes dx\otimes dP$ a.e.*

5.2. Rate of convergence for the weak solution of SPDEs

Our aim is to approximate the random field $(u_t(x))_{0\leq t\leq T}$ for all $x \in \mathbb{R}^d$. We recall that the continuous approximation of the solution of BDSDE (2.2) is given by:

$$Y_s^{N,t,x} := Y_{t_{n+1}}^{N,t,x} + \int_s^{t_{n+1}} f(t_n, \Theta_n^{N,t,x})du + \int_s^{t_{n+1}} g(t_{n+1}, \Theta_{n+1}^{N,t,x})\overleftarrow{dB_u} - \int_s^{t_{n+1}} Z_u^{N,t,x}dW_u,\ t_n \leq s < t_{n+1}. \qquad (5.2)$$

where

$$\Theta_n^{N,t,x} := (X_{t_n}^{N,t,x}, Y_{t_n}^{N,t,x}, Z_{t_n}^{N,t,x}), \text{for all } n = 0, \dots, N.$$

We define $n_t = \inf\{n, n = 0, ...N, \text{ such that } t \le t_n\} \wedge N$. We recall that the square error of the discrete time approximation is given by

$$Error_N(Y^{t,x}, Z^{t,x}) := \sup_{t \le s \le T} E[|Y_s^{t,x} - Y_s^{N,t,x}|^2] + \sum_{n=n_t}^{N-1} E[\int_{t_n}^{t_{n+1}} ||Z_s^{t,x} - Z_s^{N,t,x}||^2 ds],$$

We recall that $u(t,x) = Y_t^{t,x}$ and $v(t,x) = Z_t^{t,x}$ $dt \otimes dx \otimes dP$ a.e. We define the process $(u_s^N, v_s^N)_{t \le s \le T}$, the numerical approximation of the SPDE (1.1) as follows:

$$u_s^N(x) := Y_s^{N,s,x} \text{ and } v_s^N(x) := Z_s^{N,s,x}. \tag{5.3}$$

We define the square error between the solution of the SPDE and the numerical scheme as follows:

$$\begin{aligned} Error_N(u,v) &:= \sup_{0 \le s \le T} E[\int_{\mathbb{R}^d} |u_s^N(x) - u(s,x)|^2 \rho(x) dx] \\ &+ \sum_{n=0}^{N-1} E[\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \|v_s^N(x) - v(s,x)\|^2 ds \rho(x) dx]. \end{aligned} \tag{5.4}$$

Note that the error $Error_N(u,v)$ is defined by integrating over the whole domain the error $Error_N(Y^{t,x}, Z^{t,x})$ where $(Y^{t,x}, Z^{t,x})$ is the solution of the associated BDSDE.
The following theorem shows the convergence of the numerical scheme (5.3).

Theorem 5.2 *Assume that* **(H1)-(H3)** *hold. Then, there exists a positive constant C (depending only on T, K, α, $|b(0)|$, $||\sigma(0)||$, $|f(t,0,0,0)|$ and $||g(t,0,0,0)||$) such that*

$$Error_N(u,v) \le Ch. \tag{5.5}$$

Proof. We have

$$\begin{aligned} E[\int_{\mathbb{R}^d} |u_s^N(x) - u(s,x)|^2 \rho(x) dx] &= E[\int_{\mathbb{R}^d} |Y_s^{N,s,x} - Y_s^{s,x}|^2 \rho(x) dx] \\ &\le \int_{\mathbb{R}^d} \sup_{s \le u \le T} E[|Y_u^{N,s,x} - Y_u^{s,x}|^2] \rho(x) dx \end{aligned}$$

From Theorem 4.1, we get

$$\sup_{0 \le s \le T} E[\int_{\mathbb{R}^d} |u_s^N(x) - u(s,x)|^2 \rho(x) dx] \le Ch \int_{\mathbb{R}^d} (1 + |x|^2) \rho(x) dx \le Ch$$

For the Z part, we have

$$\sum_{n=0}^{N-1} E[\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \|v_s^N(x) - v(s,x)\|^2 ds \rho(x) dx] = \sum_{n=0}^{N-1} E[\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \|Z_s^{N,s,x} - Z_s^{s,x}\|^2 ds \rho(x) dx].$$

From Theorem 4.1, we get

$$\sum_{n=0}^{N-1} E[\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \|v_s^N(x) - v(s,x)\|^2 ds \rho(x) dx] \le Ch \int_{\mathbb{R}^d} (1 + |x|^2) \rho(x) dx \le Ch,$$

and then (5.5) holds. □

6. Implementation and numerical tests

In this part, we are interested in implementing our numerical scheme. Our aim is only to demonstrate empirically its convergence. We leave for future research the numerical analysis of the fully implementable algorithm.

6.1. Notations and algorithm

We use a path-dependent algorithm, for every fixed path of the brownian motion B, we approximate by a regression method the solution of the associated PDE. Then, we replace the conditional expectations which appear in (2.9) and (2.10) by $L^2(\Omega, \mathcal{P})$ projections on the function basis approximating $L^2(\Omega, \mathcal{F}_{t_n})$. We compute $Z^N_{t_n}$ in an explicit manner and $Y^N_{t_n}$ in a implicit way by using I Picard iterations where I is a natural number. Actually, we proceed as in [14], except that in our case the solutions $Y^N_{t_n}$ and $Z^N_{t_n}$ are measurable functions of $(X^N_{t_n}, (\Delta B_i)_{n\le i\le N-1})$. So, each solution given by our algorithm depends on the fixed path of B.

6.1.1. Numerical scheme

We take $k = d = 1$ i.e. W and B are one dimensional Brownian motions. For each fixed path of B, the solution of (2.1)-(2.2) is approximated by (Y^N, Z^N) defined by (2.9)-(2.10)
We stress that at each discretization time, the solution of the algorithm depends on the fixed path of the brownian motion B.

6.1.2. Vector spaces of functions

At every t_n, we select 2 deterministic functions bases $(p_{i,n}(.))_{i\in\{0,1\}}$ and we look for approximations of $Y^N_{t_n}$ and $Z^N_{t_n}$ which will be denoted respectively by y^N_n and z^N_n, in the vector space $(P_{i,n}(.))_{i\in\{0,1\}}$ spanned by the basis $p_{0,n}(.)$ and $p_{1,n}(.)$. Each basis $p_{i,n}(.)$ is considered as a vector of functions of dimension $L_{i,n}$. In other words, $P_{i,n}(.) = \{\alpha.p_{i,n}(.), \alpha \in \mathbb{R}^{L_{i,n}}\}$.
As an example, we cite the hypercube basis (**HC**) used in [14]. In this case, $p_{i,n}(.)$ does not depend nor on i neither on n and its dimension is simply denoted by L. A domain $D\subset\mathbb{R}$ centered on $X_0 = x$, that is $D = (x - a, x + a]$, can be partitioned on small hypercubes of edge δ. Then, $D=\bigcup_{i_1,\ldots,i_d} D_{i_1,\ldots,i_d}$ where $D_{i_1,\ldots,i_d} = (x-a+i_1\delta, x-a+(i_1+1)\delta]\times\ldots\times(x-a+i_d\delta, x-a+(i_d+1)\delta]$.
Finally we define $p_{i,n}(.)$ as the indicator functions of this set of hypercubes.

6.1.3. Monte Carlo simulations

To compute the projection coefficients α, we will use M independent Monte Carlo simulations of $X_{t_n}^N$ and ΔW_n which will be respectively denoted by $X^{N,m}_{t_n}$ and $\Delta W^m_n, m=1,\ldots,M$.

6.1.4. Description of the algorithm

$\to$ Initialization: For $n = N$, we set $(y^{N,m,I}_N) = (\Phi(X^{N,m}_{t_N}))$ and $(z^{N,m}_N) = 0$.
$\to$ Iteration: For $n = N - 1, \ldots, 0$:

• We approximate (2.10) by computing

$$\begin{aligned}\alpha_{1,n}^{M} &= \underset{\alpha}{\operatorname{arginf}} \frac{1}{M}\sum_{m=1}^{M}\Big|y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m})\frac{\Delta W_n^m}{h} \\ &+ g\Big(X_{t_{n+1}}^{N,m}, y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m}), z_{n+1}^{N,M}(X_{t_{n+1}}^{N,m})\Big)\frac{\Delta B_n \Delta W_n^m}{h} - \alpha.p_{1,n}(X_{t_n}^{N,M})\Big|^2.\end{aligned}$$

Then we set $z_n^{N,M}(.) = (\alpha_{1,n}^{M}.p_{1,n}(.))$.

• We use I Picard iterations to obtain an approximation of Y_{t_n} in (2.9):

· For $i = 0$: $\alpha_{0,n}^{M,0} = 0$.

· For $i = 1, \ldots, I$: We approximate (2.9) by calculating $\alpha_{0,n}^{M,i}$ as the minimizer of:

$$\begin{aligned}&\frac{1}{M}\sum_{m=1}^{M}\Big|y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m}) + hf\Big(X_{t_n}^{N,m}, y_n^{N,M,i-1}(X_{t_n}^{N,m}), z_n^{N,M}(X_{t_n}^{N,m})\Big) \\ &+g\Big(X_{t_{n+1}}^{N,m}, y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m}), z_{n+1}^{N,M}(X_{t_{n+1}}^{N,m})\Big)\Delta B_n - \alpha.p_{0,n}(X_{t_n}^{N,M})\Big|^2.\end{aligned}$$

Finally, we define $y_n^{N,M,I}(.)$ as:

$$y_n^{N,M,I}(.) = (\alpha_{0,n}^{M,I}.p_{0,n}(.)).$$

6.1.5. Function bases

We use the basis (**HC**) defined above. So we set:

$$d_1 = \min_{n,m} X_{t_n}^m, \quad d_2 = \max_{n,m} X_{t_n}^m \text{ and } L = \frac{d_2 - d_1}{\delta}$$

where δ is the edge of the hypercubes $(D_j)_{1\le j\le L}$ defined by $D_j = \Big[d + (j-1)\delta, d + j\delta\Big), \forall j$.
At each time t_n, we set

$$1_{D_j}(X_{t_n}^{N,m}) = 1_{[d+(j-1)\delta, d+j\delta)}(X_{t_n}^{N,m}), j = 1, \ldots, L$$

and

$$(p_{i,n}^m(.)) = \Big\{\sqrt{\frac{M}{card(D_j)}}1_{D_j}(X_{t_n}^{N,m}), 1 \le j \le L\Big\}, i = 0, 1,$$

where $Card(D_j)$ denotes the number of simulations of $X_{t_n}^N$ which are in the cube D_j.
This system is orthonormal with respect to the empirical scalar product defined by

$$< \psi_1, \psi_2 >_{n,M} := \frac{1}{M}\sum_{m=1}^{M}\psi_1(X_{t_n}^{N,m})\psi_2(X_{t_n}^{N,m}).$$

In this case, the solutions of our least squares problems are given by:

$$\begin{aligned}\alpha_{1,n}^{M} &= \frac{1}{M}\sum_{m=1}^{M} p_{1,n}(X_{t_n}^{N,m})\Big\{y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m})\frac{\Delta W_n^m}{h} \\ &+ g\Big(X_{t_{n+1}}^{N,m}, y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m}), z_{n+1}^{N,M,}(X_{t_{n+1}}^{N,m})\Big)\frac{\Delta B_n \Delta W_n^m}{h}\Big\}, \\ \alpha_{0,n}^{M,i} &= \frac{1}{M}\sum_{m=1}^{M} p_{0,n}(X_{t_n}^{N,m})\Big\{y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m}) + hf\Big(X_{t_n}^{N,m}, y_n^{N,M,i-1}(X_{t_n}^{N,m}), z_n^{N,M}(X_{t_n}^{N,m})\Big) \\ &+ g\Big(X_{t_{n+1}}^{N,m}, y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m}), z_{n+1}^{N,M}(X_{t_{n+1}}^{N,m})\Big)\Delta B_n\Big\}.\end{aligned}$$

Remark 6.1 *We note that for each value of M, N and δ, we launch the algorithm 50 times and we denote by $(Y^{0,x,N,M,I}_{0,m'})_{1\leq m'\leq 50}$ the set of collected values. Then we calculate the empirical mean $\overline{Y}^{0,x,N,M,I}_0$ and the empirical standard deviation $\sigma^{N,M,I}$ defined by:*

$$\overline{Y}^{0,x,N,M,I}_0 = \frac{1}{50}\sum_{m'=1}^{50} Y^{0,x,N,M,I}_{0,m'} \text{ and } \sigma^{N,M,I} = \sqrt{\frac{1}{49}\sum_{m'=1}^{50} |Y^{0,x,N,M,I}_{0,m'} - \overline{Y}^{0,x,N,M,I}_0|^2}. \tag{6.1}$$

We also note before starting the numerical examples that our algorithm converges after at most three Picard iterations. Finally, we stress that (6.1) gives us an approximation of $u(0,x)$ the solution of the SPDE (1.1) at time $t = 0$ given the path of B.

6.2. Examples

6.2.1. Case when f and g are linear in y and independent of z

$$\begin{cases} dX_t = X_t(\mu dt + \sigma dW_t), \\ \Phi(x) = -x + K,\ f(y) = a_0 y,\ g(y) = b_0 y \end{cases}$$

and we set $K = 115$, $r = 0.01$, $R = 0.06$, $X_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 0.25$, $d_1 = 60$, $d_2 = 200$, a_0 and b_0 are fixed constants.
Let $Y_{explicit}$ be the solution of our BDSDE in this particular case. By the integration by parts formula, we get

$$Y^{t,x}_{t,explicit} = E[\Phi(X^{t,x}_T)e^{a_0(T-t)+b_0(B_T-B_t)-\frac{1}{2}b_0^2(T-t)}/\mathcal{F}^B_{t,T}].$$

At t=0, we have

$$\begin{aligned} Y^{0,x}_{0,explicit} &= E[\Phi(X^{0,x}_T)e^{(a_0-\frac{1}{2}b_0^2)T+b_0B_T}/\mathcal{F}^B_{0,T}] \\ &= e^{(a_0-\frac{1}{2}b_0^2)T+b_0B_T}E[\Phi(X^{0,x}_T)] \\ &= e^{(a_0-\frac{1}{2}b_0^2)T+b_0B_T}(K - xe^{\mu T}). \end{aligned}$$

Then, we define $\overline{Y}^{0,x,N,M,I}_0$ as the numerical approximation of the solution of the BDSDE in this case (computed by our algorithm) and $\sigma^{N,M,I}$ as its standard deviation.

For $a_0 = 0.5$, $b_0 = 0.5$ and $\delta = 1$

N=20, $Y^{0,x}_{explicit} = 13.724$

M	$\overline{Y}^{0,x,N,M,I}_0(\sigma^{N,M,I})$	$\frac{\lvert Y^{0,x}_{explicit} - \overline{Y}^{0,x,N,M,I}_0\rvert}{Y^{0,x}_{explicit}}$
100	13.911(1.178)	0.013
1000	13.793(0.309)	0.004
5000	13.848(0.117)	0.009
10000	13.856(0.091)	0.009

For $a_0 = 0.5$, $b_0 = 0.5$ and $\delta = 0.5$

N=30, $Y^{0,x}_{explicit} = 14.115$

M	$\overline{Y}_0^{0,x,N,M,I}(\sigma^{N,M,I})$	$\frac{\lvert Y^{0,x}_{explicit} - \overline{Y}_0^{0,x,N,M,I}\rvert}{Y^{0,x}_{explicit}}$
100	14.245(1.045)	0.009
1000	14.194(0.337)	0.005
5000	14.235(0.129)	0.008
10000	14.263(0.101)	0.01

In the linear case we have a benchmark. We see that in the maturity the numerical approximation of the BDSDE's solution is closed to the exact solution. We also note that the bias is constant depending on the number of simulation.

6.2.2. Comparison of numerical approximations of the solutions of the FBDSDE and the FBSDE: the general case

Now we set

$$\begin{cases} \Phi(x) = -x + K, \\ f(t,x,y,z) = -\theta z - ry + (y - \frac{z}{\sigma})^{-}(R-r), \\ g(t,x,y,z) = 0.1z + 0.5y + log(x) \end{cases}$$

The associated nonlinear SPDE is given by:

$$du_t(x) + \big(Lu_t(x) + f(t,x,u_t(x),\nabla u_t\sigma(x))\big)\,dt + g(t,x,u_t(x),\nabla u_t\sigma(x)) \cdot \overleftarrow{dB}_t = 0,$$

where

$$Lu_t(x) = \sigma^2 x^2 \frac{\partial^2}{\partial x^2} u_t(x) + \mu x \frac{\partial}{\partial x} u_t(x).$$

We set $\theta = (\mu - r)/\sigma$, $K = 115$, $X_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $r = 0.01$, $R = 0.06$, $\delta = 1$, $N = 20$, $T = 0.25$ and we fix $d_1 = 60$ and $d_2 = 200$ as in [13]. The function g is sufficiently regular and Lipschitz on $[60,200] \times \mathbb{R} \times \mathbb{R}$ and could be extended to regular Lipschitz function on $\mathbb{R}^3$. In this case, Assumptions (**H1**), (**H2**) and (**H3**)(**i**) are satisfied. (**H3**)(**ii**) is not satisfied because f is not differentiable.

We compare the numerical solution of our BDSDE (noted again $\overline{Y}_t^{t,x,N,M,I} = u_t(X_0)$) and the BSDE's one (noted here by $\overline{Y}^{0,x,N,M}_{t,BSDE}$), without g and B.

When t is close to maturity

M	$\overline{Y}^{0,x,N,M}_{t_{15},BSDE}(\sigma^{N,M})$	$u_{t_{15}}(X_0) = \overline{Y}^{0,x,N,M,I}_{t_{15}}(\sigma^{N,M,I})$
128	14.168(0.905)	17.894(1.096)
512	14.113(0.388)	17.774(0.429)
2048	13.988(0.226)	17.607(0.270)
8192	13.985(0.093)	17.623(0.104)
32768	13.994(0.055)	17.627(0.064)

When $t = 0$

M	$\overline{Y}^{0,x,N,M}_{0,BSDE}(\sigma^{N,M})$	$u_0(X_0) = \overline{Y}^{0,x,N,M,I}_0(\sigma^{N,M,I})$
128	15.431(1.005)	13.571(1.146)
512	15.029(0.428)	13.173(0.500)
2048	14.763(0.243)	12.885(0.280)
8192	14.718(0.098)	12.825(0.106)
32768	14.715(0.060)	12.804(0.064)

We see the convergence of the BDSDE's solution when we increase the number of simulations M.

In figure 1, we examine the convergence of our scheme for five different path of the Brownian B. We fix all the parameters ($\delta = 1$ and $M = 2000$) and we draw the map of the BDSDE's solution with respect to the number of time discretization steps N.

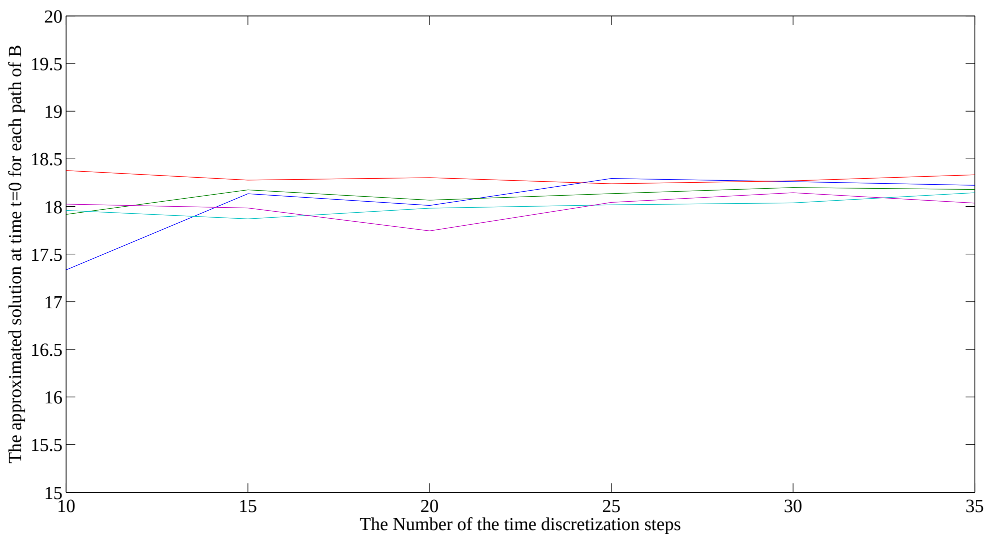


Figure 1. *The BDSDE's solution with respect to the number of time discretization steps for five different paths of B. The figure is obtained for $M = 2000$ and $\delta = 1$.*

We see on Figure 2 the impact of the function g on the solution; we variate N, M and δ as in [14], by taking these quantities as follows: First we fix $d_1 = 40$ and $d_2 = 180$ (which means that $x \in [d_1, d_2] = [40, 180]$ and in this case our assumptions (**H1**)-(**H3**) are satisfied). Let $j \in \mathbb{N}$, we take $N = 2(\sqrt{2})^{(j-1)}$, $M = 2(\sqrt{2})^{3(j-1)}$ and $\delta = 50/(\sqrt{2})^{(j-1)}$. Then, we draw the map of each solution at $t = 0$ with respect to j.

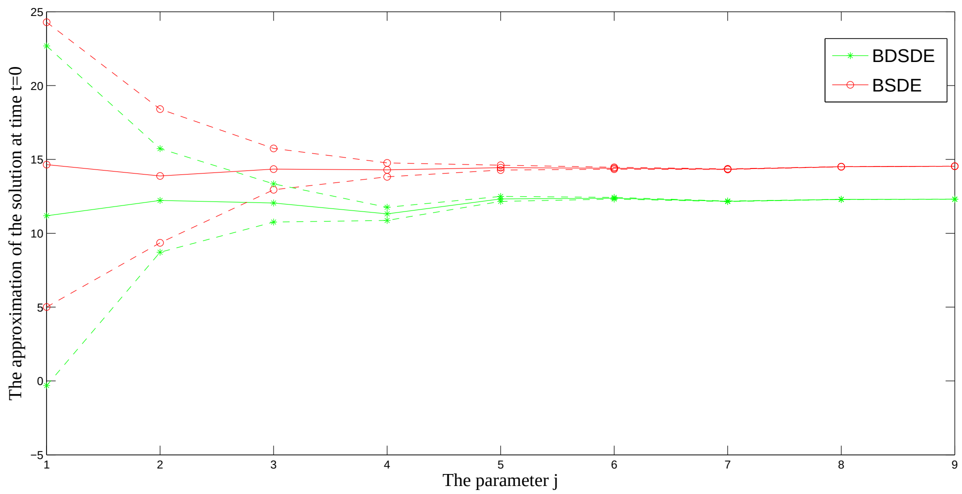


Figure 2. *Comparison of the BSDE's solution and the BDSDE's one: The solution of the BSDE is with circle markers, the solution of the BDSDE is with star markers. Confidence intervals are with dotted lines.*

7. Appendix

7.1. Proof of Lemma 3.1.

From (2.11), we have for all $t \in [t_n, t_{n+1})$

$$\delta Y_t^N = \delta Y_{t_{n+1}}^N + \int_t^{t_{n+1}} \delta f_s ds + \int_t^{t_{n+1}} \delta g_s \overleftarrow{dB_s} - \int_t^{t_{n+1}} \delta Z_s^N dW_s.$$

Using the Generalized Itô's Lemma (see Lemma 1.3, [34]), we obtain

$$\begin{aligned} |\delta Y_t^N|^2 + \int_t^{t_{n+1}} \|\delta Z_s^N\|^2 ds - |\delta Y_{t_{n+1}}^N|^2 &= 2\int_t^{t_{n+1}} (\delta Y_s^N, \delta f_s) ds + 2\int_t^{t_{n+1}} (\delta Y_s^N, \delta g_s \overleftarrow{dB_s}) \\ &+ \int_t^{t_{n+1}} \|\delta g_s\|^2 ds - 2\int_t^{t_{n+1}} (\delta Y_s^N, \delta Z_s^N dW_s), \forall t \in [t_n, t_{n+1}), \end{aligned}$$

where $(.,.)$ is the inner product associated with the euclidean norm.
Then taking the expectation, we have

$$\begin{aligned} A_t^n := E[|\delta Y_t^N|^2] + \int_t^{t_{n+1}} E[\|\delta Z_s^N\|^2] ds - E[|\delta Y_{t_{n+1}}^N|^2] &= 2\int_t^{t_{n+1}} E[(\delta Y_s^N, \delta f_s)] ds \\ &+ \int_t^{t_{n+1}} E[\|\delta g_s\|^2] ds. \end{aligned} \tag{7.1}$$

From Assumption **(H2)-(ii)**, we have

$$\begin{aligned} \int_t^{t_{n+1}} E[\|\delta g_s\|^2] ds \leq K^2 h^2 + K^2 \int_t^{t_{n+1}} E[|X_s - X_{t_{n+1}}^N|^2] ds \\ + \quad K^2 \int_t^{t_{n+1}} E[|Y_s - Y_{t_{n+1}}^N|^2] ds + \alpha^2 E\big[\int_t^{t_{n+1}} ||Z_s - Z_{t_{n+1}}^N||^2 ds\big]. \end{aligned} \tag{7.2}$$

Using the Young's inequality, for a positive constant ϵ, we obtain for all $n = 0, \dots, N-1$,

$$\begin{aligned} E\big[\int_t^{t_{n+1}} ||Z_s - Z_{t_{n+1}}^N||^2 ds\big] &\leq (1 + \frac{1}{\epsilon}) E\big[\int_t^{t_{n+1}} ||Z_s - \bar{Z}_{t_{n+1}}||^2 ds\big] \\ &+ (1+\epsilon) E\big[\int_t^{t_{n+1}} ||\bar{Z}_{t_{n+1}} - Z_{t_{n+1}}^N||^2 ds\big]. \end{aligned} \tag{7.3}$$

For all $n = 0, \dots, N-2$, we use Lemma 2.2, the definition of $\bar{Z}$ and the Jensen's inequality to get

$$\begin{aligned} E\big[||\bar{Z}_{t_{n+1}} - Z_{t_{n+1}}^N||^2\big] &= E\Big[||\frac{1}{h} E_{t_{n+1}}\big[\int_{t_{n+1}}^{t_{n+2}} \delta Z_r^N dr\big]||^2\Big]. \\ &\leq \frac{1}{h^2} E\Big[E_{t_{n+1}}\big[||\int_{t_{n+1}}^{t_{n+2}} \delta Z_r^N dr||^2\big]\Big]. \end{aligned}$$

By using Cauchy Schwartz inequality, we obtain for all $n = 0, \dots, N-2$

$$E\big[||\bar{Z}_{t_{n+1}} - Z_{t_{n+1}}^N||^2\big] \leq \frac{1}{h} E\Big[\int_{t_{n+1}}^{t_{n+2}} \|\delta Z_r^N\|^2 dr\Big]. \tag{7.4}$$

Plugging (7.4) in (7.3) then (7.3) in (7.2), we get for all $n = 0, \dots, N-1$

$$\begin{aligned} \int_t^{t_{n+1}} E[\|\delta g_s\|^2] ds \leq K^2 h^2 + K^2 \int_t^{t_{n+1}} E[|X_s - X_{t_{n+1}}^N|^2] ds + K^2 \int_t^{t_{n+1}} E[|Y_s - Y_{t_{n+1}}^N|^2] ds \\ + \quad (1 + \frac{1}{\epsilon})\alpha^2 \int_t^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2] ds + (1+\epsilon)\alpha^2 \mathbb{1}_{\{n<N-1\}} \int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_s^N\|^2] ds. \end{aligned} \tag{7.5}$$

We set $\alpha' := (1+\epsilon)\alpha^2$. We choose ϵ such that $\alpha' \in (0,1)$. This is possible since $\alpha^2 \in (0,1)$. Then, we use the inequality $2ab \leq \frac{1-\alpha'}{16K^2}a^2 + \frac{16K^2}{1-\alpha'}b^2$ and equation (7.5) to obtain for all $n = 0, \ldots, N-1$

$$\begin{aligned} A_t^n &\leq \frac{16K^2}{1-\alpha'}\int_t^{t_{n+1}} E[|\delta Y_s^N|^2]ds + \frac{1-\alpha'}{16K^2}\int_t^{t_{n+1}} E[|\delta f_s|^2]ds + K^2h^2 \\ &+ K^2\int_t^{t_{n+1}} E[|X_s - X_{t_{n+1}}^N|^2]ds + K^2\int_t^{t_{n+1}} E[|Y_s - Y_{t_{n+1}}^N|^2]ds \\ &+ (1+\frac{1}{\epsilon})\alpha^2\int_t^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds + \alpha'\mathbb{1}_{\{n<N-1\}}\int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_s^N\|^2]ds \end{aligned}$$

Now using Assumption **(H2)-(i)** in the last inequality, we get

$$\begin{aligned} A_t^n &\leq \frac{16K^2}{1-\alpha'}\int_t^{t_{n+1}} E[|\delta Y_s^N|^2]ds + \frac{1-\alpha'}{16K^2}4K^2\{h^2 + \int_t^{t_{n+1}} E[|X_s - X_{t_n}^N|^2]ds + \int_t^{t_{n+1}} E[|Y_s - Y_{t_n}^N|^2]ds \\ &+ \int_t^{t_{n+1}} E[||Z_s - Z_{t_n}^N||^2]ds\} + K^2h^2 + K^2\int_t^{t_{n+1}} E[|X_s - X_{t_{n+1}}^N|^2]ds + K^2\int_t^{t_{n+1}} E[|Y_s - Y_{t_{n+1}}^N|^2]ds \\ &+ (1+\frac{1}{\epsilon})\alpha^2\int_t^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds + \alpha'\mathbb{1}_{\{n<N-1\}}\int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_s^N\|^2]ds. \end{aligned}$$

Then, by plugging $\bar{Z}_{t_n}$ in the last inequality and from (7.4), we obtain

$$\begin{aligned} A_t^n &\leq \frac{16K^2}{1-\alpha'}\int_t^{t_{n+1}} E[|\delta Y_s^N|^2]ds + \frac{1-\alpha'}{4}\{h^2 + \int_t^{t_{n+1}} E[|X_s - X_{t_n}^N|^2]ds + \int_t^{t_{n+1}} E[|Y_s - Y_{t_n}^N|^2]ds \\ &+ 2\int_t^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + 2\int_{t_n}^{t_{n+1}} E[||\delta Z_s^N||^2]ds\} + K^2h^2 + K^2\int_t^{t_{n+1}} E[|X_s - X_{t_{n+1}}^N|^2] \\ &+ K^2\int_t^{t_{n+1}} E[|Y_s - Y_{t_{n+1}}^N|^2]ds + (1+\frac{1}{\epsilon})\alpha^2\int_t^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds \\ &+ \alpha'\mathbb{1}_{\{n<N-1\}}\int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_s^N\|^2]ds. \end{aligned}$$

We have

$$E[|Y_s - Y_{t_{n+1}}^N|^2] \leq C\{E[|Y_s - Y_{t_{n+1}}|^2] + E[|\delta Y_{t_{n+1}}^N|^2]\} \tag{7.6}$$

and similarly we have

$$E[|Y_s - Y_{t_n}^N|^2] \leq C\{E[|Y_s - Y_{t_n}|^2] + E[|\delta Y_{t_n}^N|^2]\}, \tag{7.7}$$

where C is a positive constant independent of x.
From Lemma 2.1, (7.6) and (7.7), we obtain

$$\begin{aligned} A_t^n &\leq C\int_t^{t_{n+1}} E[|\delta Y_s^N|^2]ds + ChE[|\delta Y_{t_{n+1}}^N|^2] + ChE[|\delta Y_{t_n}^N|^2] + Ch^2(1+|x|^2) \\ &+ C\int_t^{t_{n+1}} E[|Y_s - Y_{t_n}|^2]ds + C\int_t^{t_{n+1}} E[|Y_s - Y_{t_{n+1}}|^2]ds \\ &+ C\int_t^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + \frac{1-\alpha'}{2}\int_{t_n}^{t_{n+1}} E[||\delta Z_s^N||^2]ds \\ &+ (1+\frac{1}{\epsilon})\alpha^2\int_t^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds + \alpha'\mathbb{1}_{\{n<N-1\}}\int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_s^N\|^2]ds. \end{aligned} \tag{7.8}$$

where C is a generic positive constant depending on α' and independent of x.
Using (7.8) for $t = t_n$, we get

$$E[|\delta Y_{t_n}^N|^2] + \frac{1+\alpha'}{2}\int_{t_n}^{t_{n+1}} E[||\delta Z_s^N||^2]ds \le C\int_{t_n}^{t_{n+1}} E[|\delta Y_s^N|^2]ds + ChE[|\delta Y_{t_n}^N|^2] + B_n, \tag{7.9}$$

where we set for all $n = 0, \dots, N-1$:

$$\begin{aligned} B_n &:= E[|\delta Y_{t_{n+1}}^N|^2] + ChE[|\delta Y_{t_{n+1}}^N|^2] + Ch^2(1+|x|^2) \\ &+ C\int_{t_n}^{t_{n+1}} E[|Y_s - Y_{t_n}|^2]ds + C\int_{t_n}^{t_{n+1}} E[|Y_s - Y_{t_{n+1}}|^2]ds \\ &+ C\int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds \\ &+ (1+\frac{1}{\epsilon})\alpha^2\int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds + \alpha'\mathbb{1}_{\{n<N-1\}}\int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_s^N\|^2]ds. \end{aligned} \tag{7.10}$$

From (7.9), we obtain

$$\int_{t_n}^{t_{n+1}} E[||\delta Z_s^N||^2]ds \le C(h\sup_{t\in[t_n,t_{n+1}]} E[|\delta Y_t^N|^2]) + B_n.$$

Combining the previous inequality with (7.8), we get for h small enough

$$\sup_{t\in[t_n,t_{n+1}]} E[|\delta Y_t^N|^2] \le CB_n,$$

which proves the first part of the Lemma.
Inserting the previous inequality into (7.9), we get

$$E\Big[|\delta Y_{t_n}^N|^2 + \frac{1+\alpha'}{2}\int_{t_n}^{t_{n+1}} ||\delta Z_s^N||^2 ds\Big] \le (1+Ch)\Big\{E\Big[|\delta Y_{t_{n+1}}^N|^2 + \alpha'\mathbb{1}_{\{n<N-1\}}\int_{t_{n+1}}^{t_{n+2}} \|\delta Z_s^N\|^2 ds\Big] + R_n\Big\},$$

which proves the second part of the Lemma. □

7.2. Proof of Proposition 4.2.

To simplify the notations, we restrict ourselves to the case $k = d = l = 1$. $(D_\theta Y, D_\theta Z)$ is well defined and from inequalities (2.4) and (4.1), we deduce that for each $\theta \le T$

$$E[\sup_{t\le s\le T} |D_\theta Y_s|^2] + E[\int_t^T |D_\theta Z_s|^2 ds] \le C(1+|x|^2).$$

We define recursively the sequence (Y^m, Z^m) as follows. First we set $(Y^0, Z^0) = (0, 0)$. Then, given (Y^{m-1}, Z^{m-1}), we define (Y^m, Z^m) as the unique solution in $\mathbb{S}_k^2([t,T]) \times \mathbb{H}_{k\times d}^2([t,T])$ of

$$Y_s^m = \Phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{m-1}, Z_r^{m-1})dr + \int_s^T g(r, X_r^{t,x}, Y_r^{m-1}, Z_r^{m-1})\overleftarrow{dB_r} - \int_s^T Z_r^m dW_r.$$

We recursively show that $(Y^m, Z^m) \in \mathcal{B}^2([t,T], \mathbb{D}^{1,2})$. Suppose that $(Y^m, Z^m) \in \mathcal{B}^2([t,T], \mathbb{D}^{1,2})$ and let us show that $(Y^{m+1}, Z^{m+1}) \in \mathcal{B}^2([t,T], \mathbb{D}^{1,2})$.

From the induction assumption, we have $\Phi(X_T)+\int_s^T f(r,\Sigma_r^m)dr \in \mathbb{D}^{1,2}$.
We have $g(r,\Sigma_r^m)\in\mathbb{D}^{1,2}$ for all $r\in[t,T]$. From Lemma 4.2, we have $\int_t^T g(r,\Sigma_r^m)\overleftarrow{dB_r}\in\mathbb{D}^{1,2}$. then

$$Y_s^{m+1}=E\big[\Phi(X_T^{t,x})+\int_s^T f(r,\Sigma_r^m)dr+\int_s^T g(r,\Sigma_r^m)\overleftarrow{dB_r}|\mathcal{F}_{t,s}^W\vee\mathcal{F}_{t,T}^B\big]\in\mathbb{D}^{1,2},$$

where $\Sigma_r^m:=(X_r^{t,x},Y_r^m,Z_r^m)$.
Hence

$$\int_t^T Z_r^{m+1}dW_r=\Phi(X_T^{t,x})+\int_t^T f(r,\Sigma_r^m)dr+\int_t^T g(r,\Sigma_r^m)\overleftarrow{dB_r}-Y_t^{m+1}\in\mathbb{D}^{1,2}.$$

It follows from Lemma 4.1 that $Z^{m+1}\in\mathcal{M}^2_{k\times d}([t,T],\mathbb{D}^{1,2})$ and we have $D_\theta Y_s^{m+1}=D_\theta Z_s^{m+1}=0$ for $t\le s\le\theta$ and for $\theta\le s\le T$, we have

$$\begin{aligned}&D_\theta Y_s^{m+1}=\nabla\Phi(X_T^{t,x})D_\theta X_T^{t,x}\\&+\int_s^T\Big(\nabla_x f(r,\Sigma_r^m)D_\theta X_r+\nabla_y f(r,\Sigma_r^m)D_\theta Y_r^m+\nabla_z f(r,\Sigma_r^m)D_\theta Z_r^m\Big)dr\\&+\int_s^T\Big(\nabla_x g(r,\Sigma_r^m)D_\theta X_r+\nabla_y g(r,\Sigma_r^m)D_\theta Y_r^m+\nabla_z g(r,\Sigma_r^m)D_\theta Z_r^m\Big)\overleftarrow{dB_r}\\&-\int_s^T D_\theta Z_r^{m+1}dW_r.\end{aligned}\tag{7.11}$$

From inequality (2.4), we deduce that for each $\theta\le T$

$$E[\sup_{t\le s\le T}|D_\theta Y_s^{m+1}|^2]+E[\int_t^T|D_\theta Z_s^{m+1}|^2ds]\le C(1+|x|^2).$$

It is known that inequality (2.4) holds for (Y^{m+1},Z^{m+1}) and so we deduce that

$$\|Y^{m+1}\|_{1,2}+\|Z^{m+1}\|_{1,2}<\infty,$$

which shows that $(Y^{m+1},Z^{m+1})\in\mathcal{B}^2([t,T],\mathbb{D}^{1,2})$. Using the contraction mapping argument as in El Karoui, Peng and Quenez [11], we deduce that (Y^{m+1},Z^{m+1}) converges to (Y,Z) in $\mathbb{S}^2([t,T])\times\mathbb{H}^2([t,T])$. We will show that $(D_\theta Y^m,D_\theta Z^m)$ converges to (Y^θ,Z^θ) in $L^2(\Omega\times[t,T]\times[t,T],dP\otimes dt\otimes dt)$, where $Y_s^\theta=Z_s^\theta=0$ for all $t\le s\le\theta$ and $(Y_s^\theta,Z_s^\theta,\theta\le s\le T)$ is the solution of the following BDSDE

$$\begin{aligned}Y_s^\theta\;=\;&\nabla\Phi(X_T^{t,x})D_\theta X_T^{t,x}\\+\;&\int_s^T\Big(\nabla_x f(r,\Sigma_r)D_\theta X_r+\nabla_y f(r,\Sigma_r)Y_r^\theta+\nabla_z f(r,\Sigma_r)Z_r^\theta\Big)dr\\+\;&\int_s^T\Big(\nabla_x g(r,\Sigma_r)D_\theta X_r+\nabla_y g(r,\Sigma_r)Y_r^\theta+\nabla_z g(r,\Sigma_r)Z_r^\theta\Big)\overleftarrow{dB_r}\\-\;&\int_s^T Z_r^\theta dW_r.\end{aligned}\tag{7.12}$$

From equations (7.11) and (7.12), we have

$$D_\theta Y_s^{m+1} - Y_s^\theta = \int_s^T \Big((\nabla_x f(r,\Sigma_r^m) - \nabla_x f(r,\Sigma_r)) D_\theta X_r^{t,x}$$
$$+ \nabla_y f(r,\Sigma_r^m) D_\theta Y_r^m - \nabla_y f(r,\Sigma_r) Y_r^\theta + \nabla_z f(r,\Sigma_r^m) D_\theta Z_r^m - \nabla_z f(r,\Sigma_r) Z_r^\theta \Big) dr$$
$$+ \int_s^T \Big((\nabla_x g(r,\Sigma_r^m) - \nabla_x g(r,\Sigma_r)) D_\theta X_r^{t,x} + \nabla_y g(r,\Sigma_r^m) D_\theta Y_r^m - \nabla_y g(r,\Sigma_r) Y_r^\theta \Big) \overleftarrow{dB_r}$$
$$+ \int_s^T \Big(\nabla_z g(r,\Sigma_r^m) D_\theta Z_r^m - \nabla_z g(r,\Sigma_r) Z_r^\theta \Big) \overleftarrow{dB_r}$$
$$- \int_s^T (D_\theta Z_r^{m+1} - Z_r^\theta) dW_r.$$

From Proposition 4.1, we have

$$E[\sup_{\theta \le s \le T} |D_\theta Y_s^{m+1} - Y_s^\theta|^2] + E[\int_s^T |D_\theta Z_r^{m+1} - Z_r^\theta|^2 dr] \tag{7.13}$$
$$\le CE\Big[\int_s^T \Big| (\nabla_x f(r,\Sigma_r^m) - \nabla_x f(r,\Sigma_r)) D_\theta X_r^{t,x} + \nabla_y f(r,\Sigma_r^m) Y_r^\theta - \nabla_y f(r,\Sigma_r) Y_r^\theta$$
$$+ \nabla_z f(r,\Sigma_r^m) Z_r^\theta - \nabla_z f(r,\Sigma_r) Z_r^\theta \Big|^2 dr \Big]$$
$$+ CE\Big[\int_s^T \Big| (\nabla_x g(r,\Sigma_r^m) - \nabla_x g(r,\Sigma_r)) D_\theta X_r + \nabla_y g(r,\Sigma_r^m) Y_r^\theta - \nabla_y g(r,\Sigma_r) Y_r^\theta$$
$$+ \nabla_z g(r,\Sigma_r^m) Z_r^\theta - \nabla_z g(r,\Sigma_r) Z_r^\theta \Big|^2 dr \Big].$$

Therefore, we obtain

$$E[\int_t^T \int_t^T |D_\theta Y_s^{m+1} - Y_s^\theta|^2 ds d\theta] + E[\int_t^T \int_t^T |D_\theta Z_s^{m+1} - Z_s^\theta|^2 ds d\theta] \tag{7.14}$$
$$\le \quad CE[\int_t^T \int_t^T |\delta_{r,\theta}^m|^2 dr d\theta] + CE[\int_t^T \int_t^T |\rho_{r,\theta}^m|^2 dr d\theta],$$

where

$$\begin{aligned} \delta_{r,\theta}^m &= (\nabla_x f(r,\Sigma_r^m) - \nabla_x f(r,\Sigma_r)) D_\theta X_r^{t,x} + \nabla_y f(r,\Sigma_r^m) Y_r^\theta - \nabla_y f(r,\Sigma_r) Y_r^\theta \\ &+ \nabla_z f(r,\Sigma_r^m) Z_r^\theta - \nabla_z f(r,\Sigma_r) Z_r^\theta, \end{aligned} \tag{7.15}$$

and

$$\begin{aligned} \rho_{r,\theta}^m &= (\nabla_x g(r,\Sigma_r^m) - \nabla_x g(r,\Sigma_r)) D_\theta X_r^{t,x} + \nabla_y g(r,\Sigma_r^m) Y_r^\theta - \nabla_y g(r,\Sigma_r) Y_r^\theta \\ &+ \nabla_z g(r,\Sigma_r^m) Z_r^\theta - \nabla_z g(r,\Sigma_r) Z_r^\theta. \end{aligned} \tag{7.16}$$

From the definition of $(\delta_{r,\theta}^m)_{t \le r,\theta \le T}$, we have $E[\int_t^T \int_t^T |\delta_{r,\theta}^m|^2 dr d\theta] \le C \int_t^T (A_m(\theta,t,T) + B_m(\theta,t,T)) d\theta$, where

$$\begin{aligned} A_m(\theta,t,T) &= E\Big[\int_t^T |(\nabla_x f(r,\Sigma_r^m) - \nabla_x f(r,\Sigma_r)) D_\theta X_r^{t,x}|^2 dr \Big], \\ B_m(\theta,t,T) &= E\Big[\int_t^T \Big| (\nabla_y f(r,\Sigma_r) - \nabla_y f(r,\Sigma_r^m)) Y_r^\theta \Big|^2 dr \Big] \\ &+ E\Big[\int_t^T \Big| (\nabla_z f(r,\Sigma_r) - \nabla_z f(r,\Sigma_r^m)) Z_r^\theta \Big|^2 dr \Big] \end{aligned}$$

Moreover, since $\nabla_x f$ is bounded and continuous with respect to (x,y,z), it follows by the dominated convergence theorem and inequality (2.3) that

$$\lim_{m\to\infty}\int_t^T A_m(\theta,t,T)d\theta = 0. \tag{7.17}$$

Furthermore, since $\nabla_y f$ and $\nabla_z f$ are bounded and continuous with respect to (x,y,z), it follows, also, by the dominated convergence theorem and inequality (2.4) that

$$\lim_{m\to\infty}\int_t^T B_m(\theta,t,T)d\theta = 0. \tag{7.18}$$

From the definition of $(\rho^m_{r,\theta})_{s\le r,\theta\le T}$, we have $E[\int_t^T\int_t^T|\rho^m_{r,\theta}|^2drd\theta]\le C\int_t^T(A'_m(\theta,t,T)+B'_m(\theta,t,T))d\theta$, with

$$\begin{aligned}
A'_m(\theta,t,T) &= E\Big[\int_t^T|(\nabla_x g(r,\Sigma^m_r)-\nabla_x g(r,\Sigma_r))D_\theta X^{t,x}_r|^2dr\Big],\\
B'_m(\theta,t,T) &= E\Big[\int_t^T\big|(\nabla_y g(r,\Sigma_r)-\nabla_y g(r,\Sigma^m_r))Y^\theta_r\big|^2dr\Big]\\
&+ E\Big[\int_t^T\big|(\nabla_z g(r,\Sigma_r)-\nabla_z g(r,\Sigma^m_r))Z^\theta_r\big|^2dr\Big].
\end{aligned}$$

Similarly as shown above, since $\nabla_y g$ and $\nabla_z g$ are bounded and continuous with respect to (x,y,z) we can show that:

$$\lim_{m\to\infty}\int_t^T A'_m(\theta,t,T)d\theta = \lim_{m\to\infty}\int_t^T B'_m(\theta,t,T)d\theta = 0. \tag{7.19}$$

Plugging (7.17), (7.18) and (7.19) into inequality (7.14), we deduce that

$$\lim_{m\to\infty}E[\int_t^T\int_t^T|D_\theta Y^{m+1}_s-Y^\theta_s|^2dsd\theta]+E[\int_t^T\int_t^T|D_\theta Z^{m+1}_s-Z^\theta_s|^2dsd\theta]=0.$$

It follows that (Y^m,Z^m) converges to (Y,Z) in $L^2([t,T],\mathbb{D}^{1,2}\times\mathbb{D}^{1,2})$ and a version of $(D_\theta Y, D_\theta Z)$ is given by (Y^θ,Z^θ), which is the desired result. □

7.3. Second order Malliavin derivative of the solution of BDSDE's

We apply similar computation to get the second order Malliavin derivative representations of the solution of BDSDE 's, so we will omit the proof.

Proposition 7.1 *We set $t\in[0,T]$. Then, under Assumptions* **(H2)** *and* **(H3)**, *for each $t\le\theta\le T$, $(D_\theta Y, D_\theta Z)$ belongs to $\mathcal{B}^2([t,T],\mathbb{D}^{1,2})$. For each $t\le v\le T$ and $1\le i,j\le d$, we have*

$$D^j_vD^i_\theta Y_s = D^j_vD^i_\theta Z^n_s = 0,\ 1\le n\le d,\ \textit{if } s<\theta\vee v,$$

and a version of $(D^j_vD^i_\theta Y_s, D^j_vD^i_\theta Z_s)_{v\vee\theta\le s\le T}$ is the unique solution of the following equation:

$$D^j_vD^i_\theta Y_s = T_1(\Phi)+T_2(f)+T_3(g)+T_4(W),$$

where

$$T_1(\Phi)=\sum_{n_1=1}^k\nabla((\nabla\Phi)^{n_1}(X^{t,x}_T))D^j_vX^{t,x}_T(D^i_\theta X^{t,x}_T)^{n_1}+\nabla\Phi(X^{t,x}_T)D^j_vD^i_\theta X^{t,x}_T,$$

$$
\begin{aligned}
T_2(f) &= \int_s^T \sum_{n_1=1}^k \Big(\nabla_x((\nabla_x f)^{n_1}(r, X_r^{t,x}, Y_r, Z_r)) D_v^j X_r^{t,x} (D_\theta^i X_r^{t,x})^{n_1} \\
&+ \nabla_x f(r, X_r^{t,x}, Y_r, Z_r) D_v^j D_\theta^i X_r^{t,x}\Big) dr \\
&+ \int_s^T \Big(\sum_{n_1=1}^k \nabla_y((\nabla_y f)^{n_1}(r, X_r^{t,x}, Y_r, Z_r)) D_v^j Y_r (D_\theta^i Y_r)^{n_1} \\
&+ \nabla_y f(r, X_r^{t,x}, Y_r, Z_r) D_v^j D_\theta^i Y_r\Big) dr \\
&+ \sum_{n_2=1}^d \int_s^T \sum_{n_1=1}^k \nabla_{z^{n_2}}((\nabla_{z^{n_2}} f)^{n_1}(r, X_r^{t,x}, Y_r, Z_r)) D_v^j Z_r^{n_2} (D_\theta^i Z_r^{n_2})^{n_1} dr \\
&+ \sum_{n_2=1}^d \int_s^T \nabla_{z^{n_2}} f(r, X_r^{t,x}, Y_r, Z_r) D_v^j D_\theta^i Z_r^{n_2} dr,
\end{aligned}
$$

$$
\begin{aligned}
T_3(g) &= \sum_{n_3=1}^l \int_s^T \sum_{n_1=1}^k \nabla_x((\nabla_x g^{n_3})^{n_1}(r, X_r^{t,x}, Y_r, Z_r)) D_v^j X_r^{t,x} (D_\theta^i X_r^{t,x})^{n_1} \overleftarrow{dB_r^{n_3}} \\
&+ \sum_{n_3=1}^l \int_s^T \nabla_x g^{n_3}(r, X_r^{t,x}, Y_r, Z_r) D_v^j D_\theta^i X_r^{t,x} \overleftarrow{dB_r^{n_3}} \\
&+ \sum_{n_3=1}^l \int_s^T \sum_{n_1=1}^k \nabla_y((\nabla_y g^{n_3})^{n_1}(r, X_r^{t,x}, Y_r, Z_r)) D_v^j Y_r (D_\theta^i Y_r)^{n_1} \overleftarrow{dB_r^{n_3}} \\
&+ \sum_{n_3=1}^l \int_s^T \nabla_y g^{n_3}(r, X_r^{t,x}, Y_r, Z_r) D_v^j D_\theta^i Y_r \overleftarrow{dB_r^{n_3}} \\
&+ \sum_{n_3=1}^l \sum_{n_2=1}^d \int_s^T \sum_{n_1=1}^k \nabla_{z^{n_2}}((\nabla_{z^{n_2}} g^{n_3})^{n_1}(r, X_r^{t,x}, Y_r, Z_r)) D_v^j Z_r^{n_2} (D_\theta^i Z_r^{n_2})^{n_1} \overleftarrow{dB_r^{n_3}} \\
&+ \sum_{n_3=1}^l \sum_{n_2=1}^d \int_s^T \nabla_{z^{n_2}} g^{n_3}(r, X_r^{t,x}, Y_r, Z_r) D_v^j D_\theta^i Z_r^{n_2} \overleftarrow{dB_r^{n_3}},
\end{aligned}
$$

$$
T_4(W) = -\sum_{n_2=1}^d \int_s^T D_v^j D_\theta^i Z_r^{n_2} dW_r^{n_2},
$$

$(z^j)_{1 \le j \le d}$ *denotes the* j*-th column of the matrix* z*,* $(g^{n_3})_{1 \le n_3 \le l}$ *denotes the* n_3*-th column of the matrix* g*,* $B = (B^1, \ldots, B^l)$*,* $(D_\theta^i X_r^{t,x})^{n_1}$ *is the* n_1*-th component of the vector* $(D_\theta^i X_r^{t,x})$*,* $(D_\theta^i Y_r)^{n_1}$ *is the* n_1*-th component of the vector* $(D_\theta^i Y_r)$ *and* $(D_\theta^i Z_r^{n_2})^{n_1}$ *is the* n_1*-th component of the vector* $(D_\theta^i Z_r^{n_2})$*.*

References

[1] Aman, A. (2013): A numerical scheme for backward doubly stochastic differential equations. *Bernouilli*, Vol. 19, No.1, 93-114.

[2] Aboura, O. (2011): A Regression Monte-Carlo method for Backward Doubly Stochastic Differential Equations. hal-00607274, version 1 (2011).

[3] Bachouch A. (2014): Numerical Computations for Backward Doubly Stochastic Differential Equations and Non-linear Stochastic PDEs. *Ph.D. dissertation, University of Le Mans and National Engineering School of Tunis*.

[4] Bally, V. (1997): Approximation scheme for solutions of BSDE. *Pitman Research Notes in Mathematics*, 364, 177–191.

[5] Barles G., Lesigne E. (1997): SDE, BSDE and PDE. *Pitman Research Notes in Mathematics Series*, 364, 47-80.

[6] Bally V., Matoussi A. (1999): Weak solutions for SPDEs and Backward Doubly Stochastic Differential Equations. *J. Theoretical Probability*, Vol.14,No.1, 125-164.

[7] Bouchard B., Touzi N. (2004): Discrete time approximation and Monte-Carlo Simulation of Backward Stochastic differential equations. *Stochastic Processes and Their applications*, 111, 175-206.

[8] Bouchard B., Elie R.(2008) : Discrete-time approximation of decoupled forward-backward SDEs with jumps. *Stochastic Processes and Their applications*, 118(1), 53-57.

[9] Crisan D.(2006): Particle Approximations for a Class of Stochastic Partial Differential Equations. *Appl Math Optim*, 54, 293-314.

[10] Crisan D., Manolarakis K. and Touzi N.(2010): On the Monte Carlo simulation of BSDEs: An improvement on the Malliavin weights. *Stochastic Processes and Their applications*, Vol. 120, No. 7, 1133-1158.

[11] El Karoui N., Peng S.,Quenez M.C.(1997): Backward Stochastic Differential Equations in finance. Mathematical Finance, Vol.7, No. 1, 1-71.

[12] Flandoli, F.: Stochastic Problems in Fluid Dynamics. *Stochastic Partial Differential Equations and Applications*, Giuseppe Da Prato, Luciano Tubaro, 2002, Marcel Dekker, Inc.

[13] Gobet, E., Lemor, J.P, Warin X.(2005): A regression-based Monte Carlo method to solve backward stochastic differntiel equations. *The Annals of Applied Probability*, Vol.15 , No. 3, 2172-2202.

[14] Gobet, E., Lemor, J.P, Warin X.(2006): Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations. *Bernoulli* , Vol. 12 , No. 5, 889-916.

[15] Gobet, E., Lemor, J.P. (2005): Numerical simulation of BSDEs using empirical regression methods: theory and practice.*Arxiv* 0806.4447v1.

[16] Gyongy I., Krylov N. (2010): Accelerated finite difference schemes for linear stochastic partial differential equations in the whole space. *SIAM J.Math.Anal.*,42, 2275-2296.

[17] Gyongy I., Nualart D. (1995): Implicit scheme for quasi-linear parabolic partial differential equations perturbed by space-time white noise. *Stochastic Processes and Their applications*, Vol. 58, 57-72.

[18] Gyongy I. (1999): Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by a space-time white noise I. *Potential Anal.* 11 (1999), 1-37.

[19] Jentzen A., Kloeden P. (2010): Taylor Expansions of solutions of stochastic partial differential equation with additive noise. *The Annals of Probability*, Vol.38, No.2, 532-569.

[20] Kloeden P.E., Platen,E. (1992): Numerical Solution of Stochastic Differential Equations. Springer, Berlin.

[21] Krylov N.V. , Rozovskii B.L.:Stochastic evolution equations. *In Current problems in mathematics*,Vol 14 (Russian), pages 71-147, 256. Akad. Nauk SSSR, Inst. Nauki i Tekhniki, Moscow 1979.

[22] Kunita H.: Stochastic differential equations and stochastic flows of diffeomorphisms. *Ecole d'été de Probabilité de Saint-Flour XII -1982* (PL Hennequin, ed), Lecture Notes in Mathematics, vol.1097, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984, 143-303.

[23] Kunita, H. (1994) : Stochastic Flows Acting on Schwartz Distributions. *Journal of Theoretical Probability*, Vol. 7, 247-278.

[24] Kunita, H. (1994) : Generalized Solutions of a Stochastic Partial Differential Equation. *Journal of Theoretical Probability*, Vol.7, 279-308.

[25] Kunita, H. (1990) : Stochastic Flows and Stochastic Differential Equations. *Cambridge University Press*.

[26] Lototsky S., Mikulevicius R., Rozovskii B. L. (1997): Nonlinear filtering revisited: a spectral approach. *SIAM J. Control Optim.*, Vol. 35 (2): 435-461.

[27] Lototsky S. (2004): Optimal filtering of stochastic parabolic equations. *in recent developments stochastic analysis and related topics*, pages 330-353. World Sci. Publ., Hackensack, NJ.

[28] Milstein G.N. and Tretyakov M.V.(2009) : Solving Parabolic Stochastic Partial Differential Equations via Averaging Over Characteristics. *Mathematics of computation*, Vol. 78, 2075-2106.

[29] Matoussi A. and Scheutzow, M. (2002) : Semilinear Stochastic PDE's with nonlinear noise and Backward Doubly SDE's. *J. of Theoret. Probab.*, Vol. 15, 1-39.

[30] Nualart D. (2006): The Malliavin calculus and related topics. *Springer-Verlag Berlin Heidelberg*.

[31] Nualart, D. and E. Pardoux (1988) : Stochastic calculus with anticipating integrands. *Prob. Theory Relat. Fields*, Vol. 78, 535-581.

[32] Pardoux E. (1979): Stochastic Partial Differential Equations and Filtering of Diffusion Processes. *Stochastics*, Vol. 3, 127-167.

[33] Pardoux E., Peng S. (1992): Backward Stochastic Differential Equations and Quasilinear Parabolic Partial Differential Equations. *Lect. Notes Inf. Sci.*, Vol. 176, 200-217.

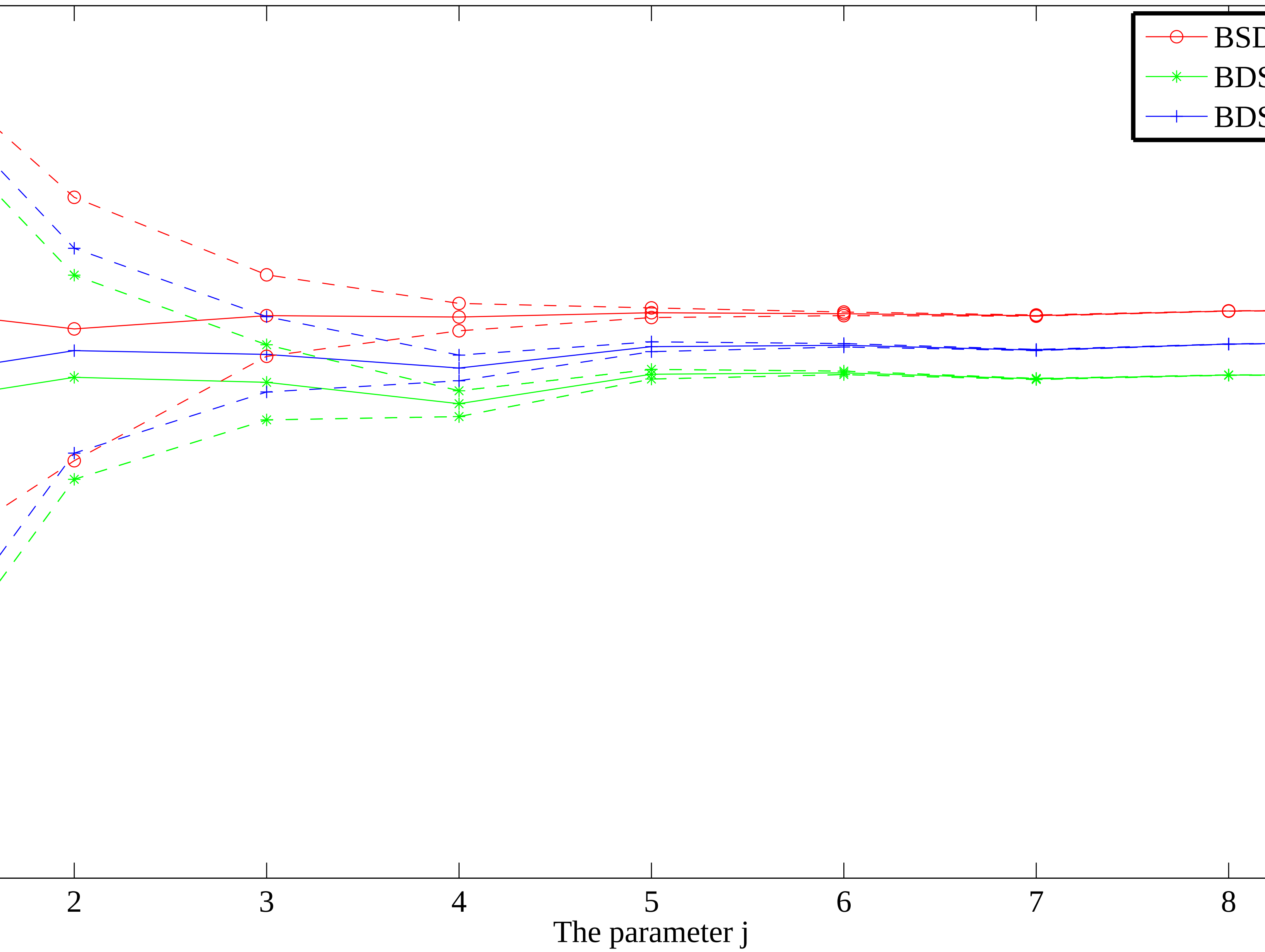
[34] Pardoux E., Peng S. (1994): Backward doubly stochastic differential equations and systems of quasilinear SPDE's. *Probab. Theory Relat. Fields*, Vol. 98, 209-227.

[35] Picard J. (1984): Approximation of nonlinear filtering problems and order of convergence. Lecture Notes in *Control and Inf. Sci.* 61, Springer (1984) 219-236.

[36] Prevot, C., Rochner, M. (2007). A concise course on stochastic partial differential equations. *Lecture Notes in Mathematics* 1905, Springer, Berlin.

[37] Walsh J. B. (2005): Finite Elements Methods for Parabolic Stochastic PDE's. *Potential Analysis*, Vol. 23, 1-43.

[38] Zhang J. (2004): A numerical scheme for BSDE's. *The Annals of Applied Probability*, Vol. 14; No. 1, 459-488.



BSD
BDS
BDS
2
3
4
5
6
7
8
The parameter j