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Self-financing strategy expression in general shape limit order book with market impacts in continuous time*

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Abstract

In this paper, we extend the self-financing strategy expression in the linear supply curve model with market impacts by Roch (2011) and Cetin *et al.* (2004) to a non-linear case. Option hedging with liquidity costs and market impacts has been a key issue since the financial crises. We generalize the continuous time expression of a self-financing strategy in the linear supply curve model with market impacts proposed by Roch (2011) and Cetin *et al.* (2004), which is a useful result when one considers an option hedging strategy under an illiquid market, to a non-linear case. After showing an expression of the maximum price to when a hedger buys a certain amount under the non-linear supply curve, we define a non-linear market impact and show the self-financing expression under the non-linear supply curve model with market impacts. We also show examples of the strategy in non-linear supply curves observed in practice.

Keywords: Liquidity risk; self-financing trading strategy; market impacts.

JEL Classifications: D40; G13

1. Introduction

Liquidity costs and market impacts are important factors in pricing options on an illiquid asset especially since the financial crisis. For instance, illiquid markets

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caused serious derivatives losses to banks in the crisis. While bid-ask spreads of the hedging instruments widened, the banks hedged similar derivatives positions from customer deals. As a result, it caused market impacts which pushed the market further in one direction and accelerated the turmoil. As this indicates, it is critical that banks take into account liquidity costs and market impacts in the derivatives positions management.

There exist a large number of preceding works on a model, which incorporates liquidity costs and market impacts, and their applications (see Alfonsi *et al.* (2010), Bertsimas and Lo (1998), Predoiu *et al.* (2010), Almgren and Chriss (2000), Bank and Baum (2004), Obizhaeva and Wang (2013) for instance). Among others, the linear supply curve model, which was introduced by Cetin *et al.* (2004) and extended by Roch (2011) to the case with market impacts (we refer the model as the Roch-CJP model hereafter), has a continuous time expression of the self-financing strategy when the hedger trades a risky asset by a position which follows a semimartingale. The expression is useful in analysis of hedging problems.

This paper further extends the Roch-CJP model to include the non-linear supply curve, especially with discontinuity at the origin, and provides an expression of the self-financing strategy when the position of a risky asset follows a semimartingale. The discontinuity at the origin, which corresponds to the case where there is a bid-offer spread in the underlying asset, is essential when banks hedge derivatives on illiquid underlying assets. In order to obtain the expression, we first show the maximum price, when the hedger buys a certain amount, as a function of the supply curve and define a non-linear market impact using that price. Then we give the self-financing strategy expression as a continuous time limit of a discrete time self-financing strategy. We also present examples of the strategy in the cases of non-linear supply curves which are observed in practice.

The paper is organized as follows. After the next section gives the expression of the maximum price, when the hedger buys a certain amount, as a function of the supply curve, Sec. 3 shows the expression of the self-financing strategy in a non-linear supply curve model in a theorem. We also give examples of the strategies for supply curves observed in practice. Finally, Sec. 4 concludes.

2. Maximum Price of Buying as Function of Supply Curve

In this section, we show the maximum price when a hedger buys a risky asset in a limit order book as a function of a supply curve.

First, we briefly introduce a supply curve of a risky asset which is the average price of the asset when the hedger buys a certain amount of it. A supply curve of a risky asset, $S(x)$, is an average price when the hedger buys amount x of it. When x is negative, it means that the hedger sells amount $|x|$ of the asset. Let $\rho(y)$ be the density of the limit order book at price y. If the hedger buys the asset by filling the orders from the price $S(0)$ to y, the total amount he/she buys is $\int_{S(0)}^{y} \rho(\eta)d\eta$, and the total cost he/she pays is $\int_{S(0)}^{y} \eta\rho(\eta)d\eta$. As $S(x)$ is the average price, the total cost the hedger pays is also written as $xS(x)$, and the following relations hold.

$$\int_{S(0)}^{y} \rho(\eta)d\eta = x, \tag{1}$$

$$\int_{S(0)}^{y} \eta\rho(\eta)d\eta = xS(x). \tag{2}$$

Proposition 1 provides the maximum price of buying and the density of the limit order book for a given supply curve. This implies that the maximum price when one buys amount x is the sum of the average price and the buying amount multiplied by the derivative of the average price with respect to the amount. The maximum price of buying is an important variable which shall be used in defining the market impact that is proportional to the price range of the orders filled in the limit order book, in Sec. 3. We remark that this also can be deduced from the expression of a total buying cost as a function of order book density in Alfonsi *et al.* (2010). We denote by $S^{(i)}(x)$ the ith derivative of $S(x)$ with respect to x.

Proposition 1. *Let $S(x)$ be a $\mathbf{R}$-valued function defined on $\mathbf{R}$ which is of class $\mathcal{C}^2$ except for at the origin. Assume that $S(0-)$, $S(0+)$, $S^{(1)}(0-)$ and $S^{(1)}(0+)$ exist, $S(0-) < S(0) < S(0+)$, $S^{(1)}(x) > 0$, and $S^{(2)}(x)x + 2S^{(1)}(x) > 0$ for all $x \in \mathbf{R}\backslash\{0\}$.*

Define $f(x)$ and $\rho(y)$, functions on $\mathbf{R}$, as $f(x) := S(x) + S^{(1)}(x)x$, for all $x \in \mathbf{R}\backslash\{0\}$, $f(0) := S(0)$, $\rho(y) := \frac{d}{dy}f^{-1}(y)$, for all $y \in \mathbf{R}\backslash[S(0-), S(0+)]$, $\rho(y) = 0$, for all $y \in [S(0-), S(0+)]$.

Let $R_1(y)$, $R_2(y)$ be $R_1(y) := \int_{S(0)}^{y} \rho(\eta)d\eta$, $R_2(y) := \int_{S(0)}^{y} \eta\rho(\eta)d\eta$.

Then

$$R_1(f(x)) = x, \tag{3}$$

$$R_2(f(x)) = xS(x) \tag{4}$$

hold.

Proof. In the case of $x = 0$, (1) and (2) hold.

First, we note that the inverse function $f^{-1}(y)$ is defined for for all $y \in f(\mathbf{R}\backslash\{0\})$ and $\frac{d}{dy}f^{-1}(y)$ exists by the inverse function theorem. We denote the derivative by $\partial f^{-1}(y)$.

For $y \geq S(0+)$,

$$\begin{aligned} R_1(y) &= \int_{S(0)}^{y} \rho(\eta)d\eta \\ &= \int_{S(0)}^{S(0+)} \rho(\eta)d\eta + \int_{S(0+)}^{y} \rho(\eta)d\eta \end{aligned}$$

$$= \lim_{\epsilon\downarrow 0}[f^{-1}(\eta)]_{S(0+)+\epsilon}^{y}$$
$$= f^{-1}(y) - \lim_{\epsilon\downarrow 0} f^{-1}(S(0+)+\epsilon) = f^{-1}(y). \tag{5}$$

Therefore, $R_1(f(x)) = x$, for all $x > 0$.
Next, for $y \geq S(0+)$,

$$\begin{aligned} R_2(y) &= \int_{S(0)}^{y} \eta\rho(\eta)d\eta \\ &= \int_{S(0)}^{S(0+)} \eta\rho(\eta)d\eta + \int_{S(0+)}^{y} \eta\rho(\eta)d\eta \\ &= \int_{S(0+)}^{y} \eta\rho(\eta)d\eta. \end{aligned} \tag{6}$$

We show that

$$\lim_{x\to 0+} R_2(f(x)) = \lim_{x\to 0+} xS(x), \tag{7}$$
$$\frac{d}{dx}R_2(f(x)) = \frac{d}{dx}(xS(x)). \tag{8}$$

For $x > 0$,

$$\lim_{x\to 0+} R_2(f(x)) = R_2(S(0+)) = 0, \tag{9}$$
$$\lim_{x\to 0+} xS(x) = 0, \tag{10}$$
$$\begin{aligned} \frac{d}{dx}R_2(f(x)) &= f(x)\rho(f(x))\frac{d}{dx}f(x) \\ &= f(x)\partial f^{-1}(f(x))\frac{d}{dx}f(x) \\ &= f(x)\frac{d}{dx}(f^{-1}(f(x))) \\ &= f(x), \end{aligned} \tag{11}$$
$$\frac{d}{dx}(xS(x)) = S(x) + xS^{(1)}(x) = f(x). \tag{12}$$

Therefore, $R_2(f(x)) = xS(x)$, for all $x > 0$. The same is true for $x < 0$. □

2.1. Examples of general supply curves

In this subsection, we show examples of the maximum price of buying and the density of the order book corresponding to a given supply curve. Let $\mathrm{sgn}(x)$ be a function defined as $\mathrm{sgn}(x) = +1(x > 0),\ -1(x < 0),\ 0(x = 0)$. Let $M, N > 0$.

Example 1. $S(x) = S(0) + Mx, f(x) = S(0) + 2Mx, \rho(y) = \frac{1}{2M}$.

Example 1 shows the case of a linear supply curve which appears in Roch (2011). This supply curve corresponds to the constant order book density of $\frac{1}{2M}$. The maximum price $S(0) + 2Mx$ is used to define the size of the price impact in Roch (2011).

Example 2. $S(x) = S(0) + \mathrm{sgn}(x)Nx^2, \quad f(x) = S(0) + \mathrm{sgn}(x)3Nx^2, \quad \rho(y) = \frac{1}{\sqrt{12N|y-S(0)|}}$.

Example 2 shows the case of a quadratic supply curve. The corresponding density function is $\frac{1}{\sqrt{12N|y-S(0)|}}$. This is an important example with a characteristic observed in practice, where limit orders are centralizing around the mid price.

Example 3. $S(x) = S(0) + \mathrm{sgn}(x)K + Mx, f(x) = S(0) + \mathrm{sgn}(x)K + 2Mx, \rho(y) = 0(y \in [S(0) - K, S(0) + K]), \frac{1}{2M}$ (otherwise).

Example 3 shows the case where there is a bid-offer spread of $2K$. In this example, when the hedger buys/sells the underlying asset, the hedger has to pay a spread of K from the mid price $S(0)$ regardless of the size of the trade. The corresponding density function indicates that there is no order in the price range between $S(0) - K$ and $S(0) + K$, which is usually observed in practice for illiquid assets.

3. Self-Financing Strategy Expression in Non-Linear Supply Curve with Market Impacts

In this section, we show an expression of a self-financing strategy in a non-linear supply curve model with market impacts, which is an extension of the ones obtained in Cetin *et al.* (2004) and Roch (2011). First we briefly define the market impact by a trade of the hedger and a self-financing strategy in discrete time. Then, in Theorem 1, we show an expression of a self-financing strategy in continuous time as a limit of the discretized strategy when the time interval tends to zero. Moreover, we also show examples of the self-financing strategy for different supply curves, which include important cases in practice where there is a bid-offer spread or the order book density is not uniform.

We consider an economy consisting of a money market account and a risky asset, and a portfolio of the two numerated by the money market account. Let Y_t be a position of the money market account, and X_t be a position of the risky asset. Let $S(t,x)$ be a supply curve of the risky asset at time t, that is, the average price when one buys amount x of the asset at time t. Hereafter for any process Z_t, we denote by $\Delta^N Z_{t_i}$, $Z_{t_i} - Z_{t_{i-1}}$, $(1 \le i \le N)$, and by $\Delta^N Z_{t_0}$, $Z_{t_0} - Z_{t_{0-}}$.

In the setting, the self-financing condition is characterized by the following equation:

$$Y_{t_k} - Y_{t_{k-1}} + (X_{t_k} - X_{t_{k-1}})S(t_k, \Delta^N X_{t_k}) = 0.$$

This represents that the cost necessary for changing the position of the risky asset is compensated by the change in the money market position, and there is no money inflow. By summing from $k = 1$ to N,

$$Y_{t_N} = Y_{t_0} - \sum_{k=1}^{N} (X_{t_k} - X_{t_{k-1}})S(t_k, \Delta^N X_{t_k}). \tag{13}$$

Moreover, the changes in the positions at the first trading time t_0 is

$$Y_{t_0} - Y_{t_{0-}} + (X_{t_0} - X_{t_{0-}})S(t_0, \Delta X_{t_0}) = 0.$$

Then we obtain

$$Y_{t_N} = Y_{t_{0-}} - \Delta X_{t_0} S(t_0, \Delta X_{t_0}) - \sum_{k=1}^{N} (X_{t_k} - X_{t_{k-1}})S(t_k, \Delta^N X_{t_k}). \tag{14}$$

Next, we define the market impact on the risky asset. Let $G(t,x)$ be a spread of the observed price process $S(t,x)$ from the mid price process $S(t,0)$, satisfying

$$S(t,x) = S(t,0) + G(t,x). \tag{15}$$

From Proposition 1, the maximum price of buying when one buys the amount $\Delta^N X_{t_k}$ of the risky asset is given by

$$S(t_k, \Delta^N X_{t_k}) + G^{(1)}(t_k, \Delta^N X_{t_k})\Delta^N X_{t_k}, \tag{16}$$

where $G^{(1)}(t,x)$ is the first derivative of $G(t,x)$ with respect to x.

We consider the situation where there are market impacts on the observed mid price process $S(t,0)$ after trades by the hedger. We assume that the size of the market impact is proportional to the difference between the observed mid price and the maximum price of buying, which is given by

$$\lambda(G(t_{k-1}, \Delta^N X_{t_{k-1}}) + G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}})\Delta^N X_{t_{k-1}}), \tag{17}$$

where $0 \le \lambda \le 1$ is a constant.

Suppose that there exists a price process $\tilde{S}_t$ that drives the observed mid price process $S(t,0)$. We call it the unaffected mid price process. We assume that the change of the observed mid price between time t_{k-1} and t_k is the sum of the change in the unaffected mid price and the market impact by a hedge, which is given by

$$\begin{aligned} S(t_k, 0) = S(t_{k-1}, 0) + \Delta^N \tilde{S}_{t_k} + \lambda(G(t_{k-1}, \Delta^N X_{t_{k-1}}) \\ + G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}})\Delta^N X_{t_{k-1}}). \end{aligned} \tag{18}$$

By summing over k,

$$S(t_k, \Delta^N X_{t_k}) = \tilde{S}_{t_k} + \lambda \sum_{i=1}^{k} (G(t_{i-1}, \Delta^N X_{t_{i-1}}) + G^{(1)}(t_{i-1}, \Delta^N X_{t_{i-1}}) \Delta^N X_{t_{i-1}}) + G(t_k, \Delta^N X_{t_k}). \tag{19}$$

In this setting of a general supply curve with market impacts, we define a self-financing strategy in continuous time as the limit in probability of the discretized self-financing strategy. This is given by Theorem 1, the main result of the paper.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. We denote by $G_t^{(i)}(x)$ the ith derivative of $G_t(x)$ with respect to x.

Theorem 1. *Let X_t, $\tilde{S}_t$ be semimartingales. Let $\{G_t(x)\}_{x \in \mathbf{R}}$ be a family of continuous semimartingales. We assume that (i) $G_t(x)$ is of class $\mathcal{C}^3$ on x. $G_t(0) = 0$ for all $t \geq 0$, (ii) $G_t^{(1)}(x) > 0$, $xG_t^{(2)}(x) + 2G_t^{(1)}(x) > 0$ for all $x \in \mathbf{R}$, and (iii) $G_t^{(1)}(x)$, $G_t^{(2)}(x)$, $G_t^{(3)}(x)$ are continuous with respect to (t, x). Let $\{\Delta^N\}_{N \in \mathbf{N}}$ be a sequence of partition of $[t_0, t] \subset [0, \infty)$, $\Delta^N : 0 \leq t_0 < t_1 < \cdots < t_N = t$ satisfying $\lim_{N \to \infty} |\Delta^N| = 0$, where $|\Delta^N|$ is the width of the partition defined by $|\Delta^N| := \max_{1 \leq k \leq N} |t_k - t_{k-1}|$. We assume that Δ^N be a finer partition of Δ^{N-1} and $\sum_{k=1}^{N} (\Delta^N X_{t_k})^2$ converges to $[X, X]_t - [X, X]_{t_0}, \mathbf{P} - a.s.$ Let $0 \leq \lambda \leq 1$ be a constant. Define $S(t_k, \Delta^N X_{t_k}) := \tilde{S}_{t_k} + \lambda \sum_{i=1}^{k} (G(t_{i-1}, \Delta^N X_{t_{i-1}}) + G^{(1)}(t_{i-1}, \Delta^N X_{t_{i-1}}) \Delta^N X_{t_{i-1}}) + G(t_k, \Delta^N X_{t_k})$, for $1 \leq k \leq N$, $S(t_0, 0) := \tilde{S}_{t_0}$.*

Let $S(t, 0)$ be the limit of $S(t_N, 0)$ in probability as $N \to \infty$.

Then $-\sum_{k=1}^{N} \Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k})$ converges to below in probability as $N \to \infty$.

$$\begin{aligned}
&+ X_{t_0} S(t_0, 0) - X_t S(t, 0) + \int_{t_0+}^{t} X_{u-} d\tilde{S}_u - \int_{t_0+}^{t} G^{(1)}(u, 0) d[X, X]_u \\
&\quad - \sum_{t_0 < s \leq t} (G(s, \Delta X_s) \Delta X_s - G^{(1)}(s, 0)(\Delta X_s)^2) \\
&\quad + 2\lambda \int_{t_0}^{t-} X_{u-} G^{(1)}(u, 0) dX_u + 2\lambda \int_{t_0}^{t-} X_{u-} d[G^{(1)}(., 0), X]_u \\
&\quad + 2\lambda \int_{t_0}^{t-} G^{(1)}(u, 0) d[X, X]_u + \frac{3}{2}\lambda \int_{t_0}^{t-} X_{u-} G^{(2)}(u, 0) d[X, X]_u \\
&\quad + \lambda \sum_{t_0 \leq s < t} X_s \left(G(s, \Delta X_s) - G^{(1)}(s, 0) \Delta X_s - \frac{1}{2} G^{(2)}(s, 0)(\Delta X_s)^2 \right) \\
&\quad + \lambda \sum_{t_0 \leq s < t} X_s (G^{(1)}(s, \Delta X_s) \Delta X_s - G^{(1)}(s, 0) \Delta X_s - G^{(2)}(s, 0)(\Delta X_s)^2) \\
&\quad + \frac{3}{2}\lambda \sum_{t_0 \leq s < t} G^{(2)}(s, 0)(\Delta X_s)^3.
\end{aligned} \tag{20}$$

Proof. For each $\omega \in \Omega$, for all $\epsilon > 0$, let $A(\epsilon, t)$ and $B(\epsilon, t)$ be subsets of $[0, t]$, satisfying $A(\epsilon, t) \cap B(\epsilon, t) = \emptyset$, $A(\epsilon, t) \cup B(\epsilon, t) = \{s \in [0, t] | \Delta X_s \neq 0\}$, $\# A(\epsilon, t) < \infty$, $\sum_{s \in B(\epsilon, t)} (\Delta X_s)^2 \leq \epsilon^2$. We can take such subsets because the number of jump times in $[0, t]$ is countable as X_s is càdlàg, and $\sum_{s \in A(\epsilon, t) \cup B(\epsilon, t)} (\Delta X_s)^2 \leq [X, X]_t < \infty$.

Hereafter we denote $\sum_{1 \leq k \leq N,\, (t_{k-1}, t_k] \cap A(\epsilon, t) \neq \emptyset}$ by $\sum_{k,\, A(\epsilon, t)}$, $\sum_{1 \leq k \leq N,\, (t_{k-1}, t_k] \cap A(\epsilon, t) = \emptyset}$ by $\sum_{k,\, B(\epsilon, t)}$, $\sum_{1 \leq k \leq N,\, (t_{k-2}, t_{k-1}] \cap A(\epsilon, t) \neq \emptyset}$ by $\sum_{k-1,\, A(\epsilon, t)}$, and $\sum_{1 \leq k \leq N,\, (t_{k-2}, t_{k-1}] \cap A(\epsilon, t) = \emptyset}$ by $\sum_{k-1,\, B(\epsilon, t)}$.

First we note that

$$\begin{aligned}
&-\Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k}) \\
&\quad = -\Delta^N X_{t_k} S(t_k, 0) - \Delta^N X_{t_k} G(t_k, \Delta^N X_{t_k}) \\
&\quad = X_{t_{k-1}} \Delta^N S(t_k, 0) - \Delta^N X_{t_k} G(t_k, \Delta^N X_{t_k}) - \Delta^N (X_{t_k} S(t_k, 0)).
\end{aligned} \tag{21}$$

Then

$$\begin{aligned}
&-\sum_{k=1}^{N} \Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k}) \\
&\qquad = +\sum_{k=1}^{N} X_{t_{k-1}} \Delta^N \tilde{S}(t_k) + \lambda \sum_{k=1}^{N} X_{t_{k-1}} G(t_{k-1}, \Delta^N X_{t_{k-1}}) \\
&\qquad\quad + \lambda \sum_{k=1}^{N} X_{t_{k-1}} G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}}) \Delta^N X_{t_{k-1}} \\
&\qquad\quad - \sum_{k=1}^{N} \Delta^N X_{t_k} G(t_k, \Delta^N X_{t_k}) - X_{t_N} S_{t_N} + X_{t_0} S_{t_0}.
\end{aligned} \tag{22}$$

Next, by Taylor's theorem and by decomposing $\sum_{k=1}^{N} = \sum_{k,\, A(\epsilon, t)} + \sum_{k,\, B(\epsilon, t)}$, $\sum_{k=1}^{N} = \sum_{k-1,\, A(\epsilon, t)} + \sum_{k-1,\, B(\epsilon, t)}$ for the terms including a remainder.

$$\begin{aligned}
&-\sum_{k=1}^{N} \Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k}) \\
&\qquad = +\sum_{k=1}^{N} X_{t_{k-1}} \Delta^N \tilde{S}(t_k) + 2\lambda \sum_{k=1}^{N} X_{t_{k-1}} G^{(1)}(t_{k-1}, 0) \Delta^N X_{t_{k-1}} \\
&\qquad\quad + \frac{3}{2} \lambda \sum_{k=1}^{N} X_{t_{k-1}} G^{(2)}(t_{k-1}, 0) (\Delta^N X_{t_{k-1}})^2 \\
&\qquad\quad + \lambda \sum_{k-1,\, B(\epsilon, t)} X_{t_{k-1}} R_1(t_{k-1}, \Delta^N X_{t_{k-1}})
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda \sum_{k-1, B(\epsilon,t)} X_{t_{k-1}} R_2(t_{k-1}, \Delta^N X_{t_{k-1}}) \Delta^N X_{t_{k-1}} \\
&+ \lambda \sum_{k-1, A(\epsilon,t)} X_{t_{k-1}} \Big(G(t_{k-1}, \Delta^N X_{t_{k-1}}) - G^{(1)}(t_{k-1}, 0) \Delta^N X_{t_{k-1}} \\
&- \frac{1}{2} G^{(2)}(t_{k-1}, 0)(\Delta^N X_{t_{k-1}})^2 \Big) \\
&+ \lambda \sum_{k-1, A(\epsilon,t)} X_{t_{k-1}} \Delta^N X_{t_{k-1}} (G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}}) \\
&- G^{(1)}(t_{k-1}, 0) - G^{(2)}(t_{k-1}, 0) \Delta^N X_{t_{k-1}}) \\
&- \sum_{k=1}^{N} G^{(1)}(t_{k-1}, 0)(\Delta^N X_{t_k})^2 - \sum_{k=1}^{N} \Delta^N G^{(1)}(t_k, 0)(\Delta^N X_{t_k})^2 \\
&- \sum_{k, B(\epsilon,t)} R_3(t_k, \Delta^N X_{t_k}) \Delta^N X_{t_k} \\
&- \sum_{k, A(\epsilon,t)} \Delta^N X_{t_k} (G(t_k, \Delta^N X_{t_k}) - G^{(1)}(t_k, 0) \Delta^N X_{t_k}) \\
&- X_{t_N} S_{t_N} + X_{t_0} S_{t_0},
\end{aligned} \tag{23}
$$

where $R_1(t_{k-1}, \Delta^N X_{t_{k-1}})$, $R_2(t_{k-1}, \Delta^N X_{t_{k-1}})$, $R_3(t_k, \Delta^N X_{t_k})$ are the remainders defined by

$$
\begin{aligned}
R_1(t_{k-1}, \Delta^N X_{t_{k-1}}) := {} & G(t_{k-1}, \Delta^N X_{t_{k-1}}) - G^{(1)}(t_{k-1}, 0) \Delta^N X_{t_{k-1}} \\
& - \frac{1}{2} G^{(2)}(t_{k-1}, 0)(\Delta^N X_{t_{k-1}})^2,
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
R_2(t_{k-1}, \Delta^N X_{t_{k-1}}) := {} & G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}}) - G^{(1)}(t_{k-1}, 0) \\
& - G^{(2)}(t_{k-1}, 0) \Delta^N X_{t_{k-1}},
\end{aligned} \tag{25}
$$

$$
R_3(t_k, \Delta^N X_{t_k}) := G(t_k, \Delta^N X_{t_k}) - G^{(1)}(t_k, 0) \Delta^N X_{t_k}, \tag{26}
$$

and satisfies the following evaluations:

$$
|R_1(t_{k-1}, \Delta^N X_{t_{k-1}})| \leq \frac{1}{6} \sup_{0 \leq |x| \leq |\Delta^N X_{t_{k-1}}|} |G^{(3)}(t_{k-1}, x)| |\Delta^N X_{t_{k-1}}|^3, \tag{27}
$$

$$
|R_2(t_{k-1}, \Delta^N X_{t_{k-1}})| \leq \frac{1}{2} \sup_{0 \leq |x| \leq |\Delta^N X_{t_{k-1}}|} |G^{(3)}(t_{k-1}, x)| |\Delta^N X_{t_{k-1}}|^2, \tag{28}
$$

$$
|R_3(t_k, \Delta^N X_{t_k})| \leq \frac{1}{2} \sup_{0 \leq |x| \leq |\Delta^N X_{t_k}|} |G^{(2)}(t_k, x)| |\Delta^N X_{t_k}|^2. \tag{29}
$$

Next we evaluate the terms including the remainders. For the term including $R_3(t_k, \Delta^N X_{t_k})$,

$$\left| \sum_{k, B(\epsilon, t)} \Delta^N X_{t_k} R_3(t_k, \Delta^N X_{t_k}) \right| \leq \frac{1}{2} \sum_{k, B(\epsilon, t)} \sup_{0 \leq |x| \leq |\Delta^N X_{t_k}|} |G^{(2)}(t_k, x)| |\Delta^N X_{t_k}|^3$$

$$\leq \frac{1}{2} K_2 \max_{k, B(\epsilon, t)} |\Delta^N X_{t_k}| \sum_{1 \leq k \leq N} |\Delta^N X_{t_k}|^2. \tag{30}$$

As X_t is càdlàg, $\limsup_{N\to\infty} \max_{k, B(\epsilon, t)} |\Delta^N X_{t_k}| \leq \sup_{s \in B(\epsilon, t)} |\Delta X_s| \leq \epsilon$. From the assumption, $\limsup_{N\to\infty} \sum_{k=1}^N |\Delta^N X_{t_k}|^2 \leq [X, X]_t$.

By taking $\limsup_{N\to\infty}$ for the both sides,

$$\limsup_{N\to\infty} \left| \sum_{k, B(\epsilon, t)} \Delta^N X_{t_k} R_3(t_k, \Delta^N X_{t_k}) \right| \leq \frac{1}{2} \epsilon K_2 [X, X]_t. \tag{31}$$

Letting $\epsilon \downarrow 0$, we have

$$\lim_{\epsilon \downarrow 0} \limsup_{N\to\infty} \left| \sum_{k, B(\epsilon, t)} \Delta^N X_{t_k} R_3(t_k, \Delta^N X_{t_k}) \right| = 0. \tag{32}$$

By noting $X_s(\omega)$ is càdlàg and bounded in $[0, t]$, for the terms including $R_1(t_{k-1}, \Delta^N X_{t_{k-1}})$, $R_2(t_{k-1}, \Delta^N X_{t_{k-1}})$, we have

$$\lim_{\epsilon \downarrow 0} \limsup_{N\to\infty} \left| \sum_{k-1, B(\epsilon, t)} X_{t_{k-1}} R_1(t_{k-1}, \Delta^N X_{t_{k-1}}) \right| = 0, \tag{33}$$

$$\lim_{\epsilon \downarrow 0} \limsup_{N\to\infty} \left| \sum_{k-1, B(\epsilon, t)} X_{t_{k-1}} R_2(t_{k-1}, \Delta^N X_{t_{k-1}}) \Delta^N X_{t_{k-1}} \right| = 0. \tag{34}$$

From these and the evaluations of the remainder terms,

$$+\lambda \sum_{k-1, B(\epsilon, t)} X_{t_{k-1}} R_1(t_{k-1}, \Delta^N X_{t_{k-1}})$$
$$+ \lambda \sum_{k-1, B(\epsilon, t)} X_{t_{k-1}} R_2(t_{k-1}, \Delta^N X_{t_{k-1}}) \Delta^N X_{t_{k-1}}$$
$$- \sum_{k, B(\epsilon, t)} R_3(t_k, \Delta^N X_{t_k}) \Delta^N X_{t_k}$$

$$
\begin{aligned}
&+\ \lambda \sum_{k-1, A(\epsilon,t)} X_{t_{k-1}} \bigg(G(t_{k-1}, \Delta^N X_{t_{k-1}}) - G^{(1)}(t_{k-1}, 0)\Delta^N X_{t_{k-1}} \\
&- \frac{1}{2} G^{(2)}(t_{k-1}, 0)(\Delta^N X_{t_{k-1}})^2 \bigg) \\
&+\ \lambda \sum_{k-1, A(\epsilon,t)} X_{t_{k-1}} \Delta^N X_{t_{k-1}} (G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}}) - G^{(1)}(t_{k-1}, 0) \\
&- G^{(2)}(t_{k-1}, 0)\Delta^N X_{t_{k-1}}) \\
&- \sum_{k, A(\epsilon,t)} \Delta^N X_{t_k} (G(t_k, \Delta^N X_{t_k}) - G^{(1)}(t_k, 0)\Delta^N X_{t_k}) \qquad (35)
\end{aligned}
$$

converges to

$$
\begin{aligned}
&-\sum_{t_0 < s \le t} (G(s, \Delta X_s)\Delta X_s - G^{(1)}(s, 0)(\Delta X_s)^2) \\
&\quad +\ \lambda \sum_{t_0 \le s < t} X_s \bigg(G(s, \Delta X_s) - G^{(1)}(s, 0)\Delta X_s - \frac{1}{2} G^{(2)}(s, 0)(\Delta X_s)^2 \bigg) \\
&\quad +\ \lambda \sum_{t_0 \le s < t} X_s (G^{(1)}(s, \Delta X_s)\Delta X_s - G^{(1)}(s, 0)\Delta X_s - G^{(2)}(s, 0)(\Delta X_s)^2) \qquad (36)
\end{aligned}
$$

in absolute convergence as $N \to \infty$.

We note that

$$
\begin{aligned}
&+2\lambda \sum_{k=1}^{N} X_{t_{k-1}} G^{(1)}(t_{k-1}, 0)\Delta^N X_{t_{k-1}} + \frac{3}{2}\lambda \sum_{k=1}^{N} X_{t_{k-1}} G^{(2)}(t_{k-1}, 0)(\Delta^N X_{t_{k-1}})^2 \\
&\quad = +2\lambda \sum_{k=0}^{N-1} X_{t_{k-1}} \Delta^N G^{(1)}(t_k, 0)\Delta^N X_{t_k} + 2\lambda \sum_{k=0}^{N-1} G^{(1)}(t_{k-1}, 0)(\Delta^N X_{t_k})^2 \\
&\quad\quad +\ 2\lambda \sum_{k=0}^{N-1} \Delta^N G^{(1)}(t_k, 0)(\Delta^N X_{t_k})^2 + 2\lambda \sum_{k=0}^{N-1} X_{t_{k-1}} G^{(1)}(t_{k-1}, 0)\Delta^N X_{t_k} \\
&\quad\quad +\ \frac{3}{2}\lambda \sum_{k=0}^{N-1} X_{t_{k-1}} \Delta^N G^{(2)}(t_k, 0)(\Delta^N X_{t_k})^2 + \frac{3}{2}\lambda \sum_{k=0}^{N-1} G^{(2)}(t_{k-1}, 0)(\Delta^N X_{t_k})^3 \\
&\quad\quad +\ \frac{3}{2}\lambda \sum_{k=0}^{N-1} \Delta^N G^{(2)}(t_k, 0)(\Delta^N X_{t_k})^3 + \frac{3}{2}\lambda \sum_{k=0}^{N-1} X_{t_{k-1}} G^{(2)}(t_{k-1}, 0)(\Delta^N X_{t_k})^2. \qquad (37)
\end{aligned}
$$

Then from the fact that $G^{(1)}(t,0), G^{(2)}(t,0)$ are uniformly continuous in $[0,t]$ and $\limsup_{N\to\infty} \sum_{k=0}^{N-1} (\Delta^N X_{t_k})^2 \leq [X,X]_t < \infty$, $\sum_{k=0}^{N-1} \Delta^N G^{(1)}(t_k,0)(\Delta^N X_{t_k})^2$, $\sum_{k=0}^{N-1} X_{t_{k-1}} \Delta^N G^{(2)}(t_k,0)(\Delta^N X_{t_k})^2$ and $\sum_{k=0}^{N-1} \Delta^N G^{(2)}(t_k,0)(\Delta^N X_{t_k})^3$ converge to 0 as $N \to \infty$.

We also note that $\sum_{k=0}^{N-1} G^{(2)}(t_{k-1},0)(\Delta^N X_{t_k})^3 \to \sum_{t_0 \leq s < t} G^{(2)}(s,0)(\Delta X_s)^3$.

Then, $-\sum_{k=1}^{N} \Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k})$ converges to

$$
\begin{aligned}
&+X_{t_0} S(t_0,0) - X_t S(t,0) + \int_{t_{0+}}^{t} X_u d\tilde{S}_u - \int_{t_{0+}}^{t} G^{(1)}(u,0) d[X,X]_u \\
&\quad - \sum_{t_0 < s \leq t} (G(s,\Delta X_s)\Delta X_s - G^{(1)}(s,0)(\Delta X_s)^2) \\
&\quad + 2\lambda \int_{t_0}^{t-} X_{u-} G^{(1)}(u,0) dX_u + 2\lambda \int_{t_0}^{t-} X_{u-} d[G^{(1)}(.,0),X]_u \\
&\quad + 2\lambda \int_{t_0}^{t-} G^{(1)}(u,0) d[X,X]_u + \frac{3}{2}\lambda \int_{t_0}^{t-} X_{u-} G^{(2)}(u,0) d[X,X]_u \\
&\quad + \lambda \sum_{t_0 \leq s < t} X_s \left(G(s,\Delta X_s) - G^{(1)}(s,0)\Delta X_s - \frac{1}{2} G^{(2)}(s,0)(\Delta X_s)^2 \right) \\
&\quad + \lambda \sum_{t_0 \leq s < t} X_s (G^{(1)}(s,\Delta X_s)\Delta X_s - G^{(1)}(s,0)\Delta X_s - G^{(2)}(s,0)(\Delta X_s)^2) \\
&\quad + \frac{3}{2}\lambda \sum_{t_0 \leq s < t} G^{(2)}(s,0)(\Delta X_s)^3
\end{aligned} \tag{38}
$$

in probability as $N \to \infty$. □

The observed mid price process $S(t,0)$ in Theorem 1 has the following expression by the unaffected price process $\tilde{S}_t$ and the supply curve.

Proposition 2.

$$
S(t_N,0) := \tilde{S}_{t_N} + \lambda \sum_{i=1}^{N} (G(t_{i-1}, \Delta^N X_{t_{i-1}}) + G^{(1)}(t_{i-1}, \Delta^N X_{t_{i-1}}) \Delta^N X_{t_{i-1}}) \tag{39}
$$

converges to

$$
\begin{aligned}
\tilde{S}_t + \lambda \Bigg(& 2\int_{t_0}^{t-} G^{(1)}(s,0) dX_s + \frac{3}{2} \int_{t_0}^{t-} G^{(2)}(s,0) d[X,X]_s + 2\int_{t_0}^{t-} d[G^{(1)}(.,0),X]_s \\
& + \sum_{t_0 \leq s < t} \left(G(s,\Delta X_s) - G^{(1)}(s,0)\Delta X_s - \frac{1}{2} G^{(2)}(s,0)(\Delta X_s)^2 \right) \\
& + \sum_{t_0 \leq s < t} (G^{(1)}(s,\Delta X_s) - G^{(1)}(s,0) - G^{(2)}(s,0)\Delta X_s)\Delta X_s \Bigg)
\end{aligned} \tag{40}
$$

in probability as $N \to \infty$.

Proof. First, $S(t_N,0)=\tilde{S}_{t_N}+\lambda\sum_{i=1}^{N}(G(t_{i-1},\Delta^N X_{t_{i-1}})+G^{(1)}(t_{i-1},\Delta^N X_{t_{i-1}})\Delta^N X_{t_{i-1}})$. By Taylor's theorem, $G(t_{i-1},\Delta^N X_{t_{i-1}})=G^{(1)}(t_{i-1},0)\Delta^N X_{t_{i-1}}+\frac{1}{2}G^{(2)}(t_{i-1},0)(\Delta^N X_{t_{i-1}})^2+R^{(1)}(t_{i-1},\Delta^N X_{t_{i-1}})$, $G^{(1)}(t_{i-1},\Delta^N X_{t_{i-1}})=G^{(1)}(t_{i-1},0)+G^{(2)}(t_{i-1},0)\Delta^N X_{t_{i-1}}+R^{(2)}(t_{i-1},\Delta^N X_{t_{i-1}})$.

$$\sum_{i=1}^{N}G(t_{i-1},\Delta^N X_{t_{i-1}})=\sum_{i=0}^{N-1}\Bigg(G^{(1)}(t_{i-1},0)\Delta^N X_{t_i}+\Delta^N G^{(1)}(t_i,0)\Delta^N X_{t_i}+\frac{1}{2}G^{(2)}(t_{i-1},0)(\Delta^N X_{t_i})^2+\frac{1}{2}\Delta^N G^{(2)}(t_i,0)(\Delta^N X_{t_i})^2+R^{(1)}(t_i,\Delta^N X_{t_i})\Bigg). \tag{41}$$

converges to $\int_{t_0}^{t-}G^{(1)}(s,0)dX_s+\frac{1}{2}\int_{t_0}^{t-}G^{(2)}(s,0)d[X,X]_s+\int_{t_0}^{t-}d[G^{(1)}(.,0),X]_s$ in probability except for the remainder terms.

$$\sum_{i=1}^{N}G^{(1)}(t_{i-1},\Delta^N X_{t_{i-1}})\Delta^N X_{t_{i-1}}=\sum_{i=0}^{N-1}(G^{(1)}(t_{i-1},0)+\Delta^N G^{(1)}(t_i,0)+G^{(2)}(t_{i-1},0)\Delta^N X_{t_i}+\Delta^N G^{(2)}(t_i,0)\Delta^N X_{t_i}+R^{(2)}(t_i,\Delta^N X_{t_i}))\Delta^N X_{t_i} \tag{42}$$

converges to $\int_{t_0}^{t-}G^{(1)}(s,0)dX_s+\int_{t_0}^{t-}G^{(2)}(s,0)d[X,X]_s+\int_{t_0}^{t-}d[G^{(1)}(.,0),X]_s$ except for the remainder terms.

For the remainder terms,

$$\begin{aligned}\sum_{i=0}^{N-1}R^{(1)}(t_i,\Delta^N X_{t_i})&=\sum_{i,A}R^{(1)}(t_i,\Delta^N X_{t_i})+\sum_{i,B}R^{(1)}(t_i,\Delta^N X_{t_i})\\&=\sum_{i,A}\Bigg(G(t_i,\Delta^N X_{t_i})-G^{(1)}(t_i,0)\Delta^N X_{t_i}\\&\quad-\frac{1}{2}G^{(2)}(t_i,0)(\Delta^N X_{t_i})^2\Bigg)+\sum_{i,B}R^{(1)}(t_i,\Delta^N X_{t_i})\end{aligned} \tag{43}$$

converges to $\sum_{t_0\le s<t}(G(s,\Delta X_s)-G^{(1)}(s,0)\Delta X_s-\frac{1}{2}G^{(2)}(s,0)(\Delta X_s)^2)$.

$$\begin{aligned}&\sum_{i=0}^{N-1}R^{(2)}(t_i,\Delta^N X_{t_i})\Delta^N X_{t_i}\\&\quad=\sum_{i,A}R^{(2)}(t_i,\Delta^N X_{t_i})\Delta^N X_{t_i}+\sum_{i,B}R^{(2)}(t_i,\Delta^N X_{t_i})\Delta^N X_{t_i}\end{aligned}$$

$$= \sum_{i,A} (G^{(1)}(t_i, \Delta^N X_{t_i}) - G^{(1)}(t_i, 0) - G^{(2)}(t_i, 0)(\Delta^N X_{t_i}))\Delta^N X_{t_i}$$
$$+ \sum_{i,B} R^{(1)}(t_i, \Delta^N X_{t_i}) \tag{44}$$

converges to $\sum_{t_0 \le s < t}(G^{(1)}(s, \Delta X_s) - G^{(1)}(s, 0) - G^{(2)}(s, 0)\Delta X_s)\Delta X_s$, which completes the proof. □

The next proposition is on the limit of the Stieltjes sum with respect to a finite variation process.

Proposition 3. *Let K_t be a non-negative continuous semimartingale and X_t be a finite variation semimartingale. Then $-\sum_{k=1}^{N} \Delta^N X_{t_k}(K_{t_k}\mathrm{sgn}(\Delta^N X_{t_k}))$ converges to $-\int_{t_{0+}}^{t} K_u|dX_u|$ in probability as $N \to \infty$.*

Proof.

$$\sum_{k=1}^{N} -\Delta^N X_{t_k}(K_{t_k}\mathrm{sgn}(\Delta^N X_{t_k})) = \sum_{k=1}^{N} -K_{t_k}|\Delta^N X_{t_k}|$$
$$= \sum_{k=1}^{N} -\Delta^N K_{t_k}|\Delta^N X_{t_k}| + \sum_{k=1}^{N} -K_{t_{k-1}}|\Delta^N X_{t_k}| \tag{45}$$

converges to $-\int_{t_{0+}}^{t} K_u|dX_u|$ in probability. □

By this proposition, we obtain the following corollary, which gives an expression of the self-financing strategy in the case where there is $2K_t$ of a bid-offer spread. Here we assume that the market impact is proportional to the difference between the maximum/minimum price of buying/selling and the lowest/highest offer/bid price when there is a bid-offer spread.

Corollary 1. *Let K_t be a non-negative continuous semimartingale, X_t be a finite variation semimartingale, and $\tilde{S}_t$ be a semimartingale. Let $\{G_t(x)\}_{x\in\mathbf{R}}$ be a family of continuous semimartingales. We assume that (i) $G_t(x)$ is of class $\mathcal{C}^3$ on x. $G_t(0) = 0$ for all $t \ge 0$, (ii) $G_t^{(1)}(x) > 0$, $xG_t^{(2)}(x) + 2G_t^{(1)}(x) > 0$ for all $x \in \mathbf{R}$, and (iii) $G_t^{(1)}(x)$, $G_t^{(2)}(x)$, $G_t^{(3)}(x)$ are continuous with respect to (t, x). Let $\{\Delta^N\}_{N\in\mathbf{N}}$ be a sequence of partition of $[t_0, t] \subset [0, \infty)$, $\Delta^N : 0 \le t_0 < t_1 < \cdots < t_N = t$ satisfying $\lim_{N\to\infty} |\Delta^N| = 0$, where $|\Delta^N|$ is the width of the partition defined by $|\Delta^N| := \max_{1\le k\le N} |t_k - t_{k-1}|$. We assume that Δ^N be a finer partition of Δ^{N-1} and $\sum_{k=1}^{N} (\Delta^N X_{t_k})^2$ converges to $[X, X]_t - [X, X]_{t_0}, \mathbf{P} - a.s.$ Let $0 \le \lambda \le 1$ be a constant. Define $S(t_k, \Delta^N X_{t_k}) := \tilde{S}_{t_k} + \lambda \sum_{i=1}^{k}(G(t_{i-1}, \Delta^N X_{t_{i-1}}) + G^{(1)}(t_{i-1}, \Delta^N X_{t_{i-1}})\Delta^N X_{t_{i-1}}) + G(t_k, \Delta^N X_{t_k}) + K_{t_k}\mathrm{sgn}(\Delta^N X_{t_k})$, for $1 \le k \le N$,*

$S(t_0,0) := \tilde{S}_{t_0}$. Let $S(t,0)$ be the limit of $S(t_N,0)$ in probability as $N \to \infty$. Then $-\sum_{k=1}^{N} \Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k})$ converges to below in probability as $N \to \infty$.

$$
\begin{aligned}
&+X_{t_0}S(t_0,0) - X_t S(t,0) + \int_{t_{0+}}^{t} X_{u-}d\tilde{S}_u - \int_{t_{0+}}^{t} K_u|dX_u| \\
&\quad - \sum_{t_0<s\le t} G(s,\Delta X_s)\Delta X_s + 2\lambda \int_{t_0}^{t-} X_{u-}G^{(1)}(u,0)dX_u \\
&\quad + 2\lambda \sum_{t_0\le s<t} G^{(1)}(s,0)(\Delta X_s)^2 + \frac{3}{2}\lambda \sum_{t_0\le s<t} X_{s-}G^{(2)}(s,0)(\Delta X_s)^2 \\
&\quad + \lambda \sum_{t_0\le s<t} X_s\left(G(s,\Delta X_s) - G^{(1)}(s,0)\Delta X_s - \frac{1}{2}G^{(2)}(s,0)(\Delta X_s)^2\right) \\
&\quad + \lambda \sum_{t_0\le s<t} X_s(G^{(1)}(s,\Delta X_s)\Delta X_s - G^{(1)}(s,0)\Delta X_s - G^{(2)}(s,0)(\Delta X_s)^2) \\
&\quad + \frac{3}{2}\lambda \sum_{t_0\le s<t} G^{(2)}(s,0)(\Delta X_s)^3.
\end{aligned} \tag{46}
$$

Proof. Since X_t is a finite variation semimartingale,

$$-\int_{t_{0+}}^{t} G^{(1)}(u,0)d[X,X]_u = -\sum_{t_0<s\le t} G^{(1)}(s,0)(\Delta X_s)^2 \tag{47}$$

$$2\lambda \int_{t_0}^{t-} X_{u-}d[G^{(1)}(\cdot,0),X]_u = 0 \tag{48}$$

$$2\lambda \int_{t_0}^{t-} G^{(1)}(u,0)d[X,X]_u = 2\lambda \sum_{t_0<s\le t} G^{(1)}(s,0)(\Delta X_s)^2 \tag{49}$$

$$\frac{3}{2}\lambda \int_{t_0}^{t-} X_{u-}G^{(2)}(u,0)d[X,X]_u = \frac{3}{2}\lambda \sum_{t_0<s\le t} X_{s-}G^{(2)}(s,0)(\Delta X_s)^2, \tag{50}$$

and we obtain the result. □

3.1. Examples of self-financing strategy for general supply curve with market impacts

In this subsection, we show some examples of the self-financing strategy for a general supply curve with market impacts. In the first two examples, we present that Theorem 1 includes the self-financing strategy expressions discussed in Cetin *et al.* (2004) and Roch (2011). Then, in the third example, we show an important example of Corollary 1 for the case where there is a bid-offer spread in an order book and the supply curve is non-linear.

Example 1. **(Cetin *et al.*, 2004)** This shows that when there is no market impact, Theorem 1 yields the expression in Cetin *et al.* (2004). In Theorem 1, when $\lambda = 0$ (no market impact) and $X_{t_0-} = 0$,

$$\begin{aligned} Y_t &= Y_{t_0-} - X_{t_0}S(t_0, X_{t_0}) + X_{t_0}S(t_0, 0) - X_tS(t, 0) \\ &\quad + \int_{t_{0+}}^{t} X_{u-}d\tilde{S}_u - \int_{t_{0+}}^{t} G^{(1)}(u, 0)d[X, X]_u \\ &\quad - \sum_{t_0<s\leq t}(G(s, \Delta X_s)\Delta X_s - G^{(1)}(s, 0)(\Delta X_s)^2) \\ &= Y_{t_0-} - X_tS(t, 0) \\ &\quad + \int_{t_{0+}}^{t} X_{u-}d\tilde{S}_u - \int_{t_{0+}}^{t} G^{(1)}(u, 0)d[X, X]_u^c - \sum_{t_0\leq s\leq t} G(s, \Delta X_s)\Delta X_s, \end{aligned} \tag{51}$$

where $[X, X]_t^c$ is the continuous part of the quadratic variation process $[X, X]_t$.

The second line follows by the fact that:

$$\begin{aligned} &-\int_{t_{0+}}^{t} G^{(1)}(u, 0)d[X, X]_u \\ &\qquad = -\int_{t_{0+}}^{t} G^{(1)}(u, 0)d[X, X]_u^c - \sum_{t_0<s\leq t} G^{(1)}(s, 0)(\Delta X_s)^2, \end{aligned} \tag{52}$$

$$-X_{t_0}S(t_0, X_{t_0}) + X_{t_0}S(t_0, 0) = -G(t_0, \Delta X_{t_0})\Delta X_{t_0}. \tag{53}$$

Example 2. **(Roch, 2011)** This example shows that when the supply curve is linear, Theorem 1 derives the case in Roch (2011). In Theorem 1, when $G(t, x) = M_tx$ (linear supply curve), where M_t is a continuous positive semimartingale, and $\tilde{S}_t$ is a continuous semimartingale,

$$\begin{aligned} Y_t &= Y_{t_0-} - \Delta X_{t_0}S(t_0, \Delta X_{t_0}) + X_{t_0}S(t_0, 0) - X_tS(t, 0) \\ &\quad + \int_{t_0}^{t} X_{u-}d\tilde{S}_u - \int_{t_{0+}}^{t} M_u d[X, X]_u \\ &\quad + 2\lambda\int_{t_0}^{t-} M_uX_{u-}dX_u + 2\lambda\int_{t_0}^{t-} X_{u-}d[M, X]_u + 2\lambda\int_{t_0}^{t-} M_u d[X, X]_u \\ &= Y_{t_0-} + X_{t_0-}(S(t_0, 0) - \lambda X_{t_0-}M_{t_0}) - X_t(S(t+, 0) - \lambda X_tM_t) \\ &\quad + \int_{t_0}^{t} X_{u-}d\tilde{S}_u - \int_{t_0}^{t}(1-\lambda)M_u d[X, X]_u - \lambda\int_{t_0}^{t} X_{u-}^2 dM_u. \end{aligned} \tag{54}$$

The first line follows from the fact that

$$G^{(1)}(t, x) = M_t \tag{55}$$

$$G^{(2)}(t, x) = 0. \tag{56}$$

The second line is obtained as follows:

First, note that

$$-\Delta X_{t_0} S(t_0, \Delta X_{t_0}) + X_{t_0} S(t_0, 0) = -\Delta X_{t_0}(S(t_0, 0) + M_{t_0}\Delta X_{t_0}) + X_{t_0} S(t_0, 0) = +X_{t_0-} S(t_0, 0) - M_{t_0}(\Delta X_{t_0})^2. \quad (57)$$

Applying Ito's formula to $X_t^2 M_t$, we have

$$\lambda \int_{t_0+}^{t} 2X_{u-} d[X, M]_u = \lambda \Bigg(X_t^2 M_t - X_{t_0}^2 M_{t_0} - \int_{t_0+}^{t} 2X_{u-} M_u dX_u - \int_{t_0+}^{t} X_{u-}^2 dM_u - \int_{t_0+}^{t} M_u d[X, X]_u \Bigg). \quad (58)$$

Noting that

$$-\lambda X_{t_0}^2 M_{t_0} = -\lambda (X_{t_0-} + \Delta X_{t_0})^2 M_{t_0} = -\lambda X_{t_0-}^2 M_{t_0} - 2\lambda X_{t_0-} \Delta X_{t_0} M_{t_0} - \lambda (\Delta X_{t_0})^2 M_{t_0}, \quad (59)$$

we obtain

$$\begin{aligned} Y_t = {} & Y_{t_0-} + X_{t_0-} S(t_0, 0) - X_t S(t, 0) + \int_{t_0}^{t} X_{u-} d\tilde{S}_u \\ & - \int_{t_0}^{t} M_u d[X, X]_u + 2\lambda \int_{t_0}^{t-} M_u X_{u-} dX_u + 2\lambda \int_{t_0}^{t-} M_u d[X, X]_u + \lambda X_t^2 M_t \\ & - \lambda X_{t_0-}^2 M_{t_0} - 2\lambda \int_{t_0}^{t} X_{u-} M_u dX_u - \lambda \int_{t_0}^{t} X_{u-}^2 dM_u - \lambda \int_{t_0}^{t} M_u d[X, X]_u \\ = {} & Y_{t_0-} + X_{t_0-}(S(t_0, 0) - \lambda X_{t_0-} M_{t_0}) - X_t (S(t, 0) - \lambda X_t M_t) \\ & + \int_{t_0}^{t} X_{u-} d\tilde{S}_u - 2\lambda X_{t-} M_t \Delta X_t - \int_{t_0}^{t} (1 - \lambda) M_u d[X, X]_u \\ & - 2\lambda M_t (\Delta X_t)^2 - \lambda \int_{t_0}^{t} X_{u-}^2 dM_u \\ = {} & Y_{t_0-} + X_{t_0-}(S(t_0, 0) - \lambda X_{t_0-} M_{t_0}) - X_t (S(t+, 0) - \lambda X_t M_t) \\ & + \int_{t_0}^{t} X_{u-} d\tilde{S}_u - \int_{t_0}^{t} (1 - \lambda) M_u d[X, X]_u - \lambda \int_{t_0}^{t} X_{u-}^2 dM_u, \end{aligned} \quad (60)$$

where we used

$$+2\lambda M_t X_t \Delta X_t - 2\lambda M_t X_{t-} \Delta X_t - 2\lambda M_t (\Delta X_t)^2 = 0. \quad (61)$$

Example 3. Finally, this example shows an important case in practice where there is a bid-offer spread in the order book and the supply curve is non-linear. In

Corollary 1, where there is $2K_t$ of a bid-offer spread, if X_t is absolutely continuous after t_0 with an expression of $X_t = X_{t_0} + \int_{t_0}^t \dot{X}_u du (t_0 \le t), 0 (0 \le t < t_0)$, and $G(t,x) = N_t x^3$ (cubic supply curve) where N_t is a positive continuous semimartingale,

$$\begin{aligned} Y_t = Y_{t_0-} &- (N_{t_0} X_{t_0}^4 + K_{t_0}|X_{t_0}|) - X_t S(t,0) \\ &+ \int_{t_0+}^t X_u d\tilde{S}_u - \int_{t_0}^t K_u |\dot{X}_u| du + 4\lambda N_{t_0} X_{t_0}^4. \end{aligned} \tag{62}$$

In fact, noting that

$$G^{(1)}(t,x) = 3N_t x^2, \tag{63}$$

$$G^{(2)}(t,x) = 6N_t x, \tag{64}$$

we have

$$\begin{aligned} Y_t &= Y_{t_0-} - X_{t_0}(S(t_0,0) + N_{t_0} X_{t_0}^3 + \mathrm{sgn}(X_{t_0}) K_{t_0}) + X_{t_0} S(t_0,0) - X_t S(t,0) \\ &\quad + \int_{t_0+}^t X_u d\tilde{S}_u - \int_{t_0}^t K_u |\dot{X}_u| du + 4\lambda X_{t_0}^4 N_{t_0} \\ &= Y_{t_0-} - N_{t_0} X_{t_0}^4 - K_{t_0}|X_{t_0}| - X_t S(t,0) \\ &\quad + \int_{t_0+}^t X_u d\tilde{S}_u - \int_{t_0}^t K_u |\dot{X}_u| du + 4\lambda X_{t_0}^4 N_{t_0}. \end{aligned} \tag{65}$$

4. Conclusion

In this paper, we have extended the Roch-CJP linear supply curve model with market impacts to include the case of a non-linear supply curve which is discontinuous at the origin. After showing the maximum price when the hedger buys a certain amount of a risky asset as a function of the supply curve, we have presented the continuous time expression of the self-financing strategy when the hedger trades the risky asset following a semimartingale. We have also provided examples of the strategy for some supply curves, which include the cases where a bid-offer spread exists or the order book density is not uniform for the underlying asset. Such a situation is essential in practice in pricing options on an illiquid underlying asset.

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