Interest Rate, Currency and Equity Derivatives Valuation Using the Potential Approach*

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Abstract

Based on the potential approach to interest rate modeling, we introduce a simple tractable model for the unified valuation of interest rate, currency and equity derivatives. Our model is able to accommodate the initial term structure of zero-coupon bond prices, generate positive and bounded interest rates, and handle cross products such as differential swaps, quanto options, and equity swaps. As our model is specified under the actual probability measure, it can be directly used for portfolio risk management and the computation of Value at Risk. Furthermore, our model yields simple analytical formulas that are easy to calibrate and implement.

Currently there are several different methodologies for the valuation of interest rate derivatives. The first approach is based on a specification of the spot rate and the associated prices of risk. Well-known examples of this approach are Vasicek (1977) and Cox, Ingersoll and Ross (1985). More recent work in this area specifies the spot rate as an affine function of Markovian state variables [see Duffie and Kan (1996), Dai and Singleton (2000), and Duffie, Pan and Singleton (2001)] and is capable of generating analytical pricing formulas. A second approach specifies the arbitrage-free evolution of the term structure of instantaneous forward rates. The advantage of this approach is that there is no need to specify the market prices of risk, and the only inputs to the model are the initial term structure and the forward rate volatilities. In addition, many classic short-rate models are merely special cases of this framework. This approach originates from Ho and Lee (1986) and reaches its full generality in Heath, Jarrow and Morton (1992, HJM hereafter).

A third approach has appeared recently, which is based on a specification of the state price density (SPD). The SPD is usually derived from a representative-agent asset pricing model. However, one can choose to bypass the complexity of an asset pricing model by specifying the randomness of the SPD directly. An early example of this approach can be found in Constantinides (1992). More recently, Rogers (1997a, 1997b) characterizes the SPD as the mathematical construct of a potential and lays out the general groundwork for deriving the SPD in a Markovian setting.[1] This provides an interesting alternative method for generating a rich set of term structure models.

Rogers points out that the major advantage of modeling the SPD is its efficiency and internal consistency in handling yield curves in multiple currencies. Specifically, since the exchange rate process is uniquely determined by the SPDs in the two respective currencies, there is no need to specify a separate randomness for the exchange rate process, as is often the case with other approaches [see Amin and Jarrow (1991) for example].[2] Once we specify an SPD for each currency and their correlations, domestic interest rate and foreign exchange products can be valued with no ad hoc additions to the model. Consequently, the structure of a global model is also significantly simplified.

The potential approach has other attractive features. For instance, since the model is set up under the actual probability measure, the dynamics of asset prices can be specified with real (as opposed to risk-

[1] A potential is a positive supermartingale which asymptotes to zero as $t \to \infty$. It is possible to model the SPD as a potential because of restrictions on the behavior of bond prices. First, bond prices (with a face value of one dollar) are between zero and one. Second, the price of a zero-coupon bond approaches zero as its maturity goes to infinity. Details about the potential approach can be found in Section I.A.

[2] To implement the continuous-time model of Amin and Jarrow (1991), Amin and Bodurtha (1995) specify a discrete-time one-factor HJM model for the domestic term structure, another discrete-time one-factor HJM model for the foreign term structure, and a binomial model for the exchange rate process. The product lattice has eight distinct nodes for each step and generally does not recombine. On the other hand, the potential approach is a better structure for currency derivatives because it permits arbitrage-free models that are significantly simpler – in the above case, only two potentials and their correlation need to be specified. Although possible in theory, it is not clear how one should specify an exchange rate process based on the randomness driving the two HJM forward rate curves.

neutral) probabilities.[3] This implies that the model can be easily integrated into portfolio risk management. Specifically, it could be used to compute Value at Risk measures. The potential approach also guarantees positive interest rates. This is in fact the focus of Flesaker and Hughston (1996, 1997), who express bond prices in terms of a family of positive martingales under the actual measure. Their approach is also able to accommodate the initial term structure of zero-coupon bond prices. Surprisingly, Jin and Glasserman (2001) show that the Rogers potential approach and the Flesaker-Hughston approach are in fact equivalent ways of generating positive interest rates in the arbitrage-free HJM framework.

Despite its attractiveness, we note that currently there is a lack of effort in examining the pricing implications of the potential approach.[4] This is perhaps due to the completely general nature of the framework. Flesaker and Hughston (1996) suggest a class of models that they call the "rational lognormal models," which promise to yield Black-Scholes type formulas for interest rate caps and swaptions.[5] However, even this restricted class still contains a continuum of possible specifications. Instead, we further specialize the rational lognormal class by concentrating on a two-parameter family. The two parameters in our model control the upper and lower bounds of the spot rate process. Our further restriction of the rational lognormal class can be seen as an effort to derive tractable pricing formulas from which pricing implications can be easily examined. Indeed, we are able to price interest rate caps, swaptions, as well as currency options all under a unifying framework. This is a modest first step in the right direction.

Apart from pricing currency and interest rate derivatives, an important methodological contribution of the paper is that we show how to price equity derivatives using the potential approach. Traditionally, interest rate derivatives and equity derivatives valuation are handled by very different models (for instance, HJM model and the stochastic volatility model). However, as the market for new product develops, there appears to be a need to combine the features of these different models.[6] By modeling the stock price (rescaled in our case) as a potential, we ensure that previously developed tools for the general potential approach to interest rate and currency derivatives are readily applicable to equity derivatives as well. From the perspective of an investment professional who is building a valuation or risk management tool box, it is now possible to handle a large set of securities, encompassing interest rate, currency and equity derivatives, under a common umbrella called the potential approach. This is highly

[3] The SPD, or the pricing kernel, allows a transformation between the risk-neutral and the actual probability measure. Consequently, one can pursue risk-neutral pricing using the risk-neutral dynamics of asset prices, or alternatively, work with the real probability assessments along with a specified SPD under the actual measure. The potential approach belongs to the latter category.

[4] Rogers and Zane (1996) test the potential model on interest rate and foreign exchange rate data. However, they do not use the prices of interest rate and currency derivatives.

[5] In addition, see Musiela and Rutkowshi (1997) and Rutkowski (1997).

[6] For example, a stochastic volatility model with random interest rates may be important for the pricing of long-term equity anticipation securities (LEAPS). Such a model is also needed to value hybrid products such as equity swaps.

efficient from a practical point of view and also reduces the scope for internal inconsistency when models are specified and implemented on a security-by-security basis.

The rest of this paper is organized as follows. In Section I we present the general potential approach as well as the specialized assumptions of our model. In Section II we introduce the valuation of caps and swaptions as examples of interest rate derivatives, along with the valuation of currency options. In Section III we show how to extend the potential approach to include equity options, and derive the price of equity options without compromising the initial forward rate curve. In Section IV we use a two-factor model to price caps and swaptions, which generalizes the analysis in Section II. We conclude in Section V.

I. Model

This section contains the construction of our model. For the sake of completeness, we develop an abstract version of Rogers' potential approach in the first subsection. In the second subsection we provide further assumptions required to generate our tractable framework.

A. The Potential Approach

The building block of the potential approach is the state price density (SPD). To illustrate this, we start with a world in which there are multiple countries and currencies. For each currency i, there is a money market account whose value at time t is designated by B_t^i. We also assume that there is an instantaneous spot interest rate process r_t^i in currency i, so that $B_t^i = \exp\left(\int_0^t r_s^i ds\right)$. The physical probabilities are assessed according to the measure P and ϕ_t is the information set (σ-field) of all available information up to and including time t.

Let X_t^i be the price at time t of a traded asset in currency i. We assume completeness and the absence of arbitrage for the markets in each currency, so that there exists a unique (currency-specific) martingale measure Q^i equivalent to P, such that the discounted price of the traded asset is a martingale:

$$\frac{X_t^i}{B_t^i} = \tilde{E}^i\left[\frac{X_T^i}{B_T^i}|\phi_t\right], \tag{1.1}$$

where $\tilde{E}^i[\ ]$ denotes the expectation under Q^i.

The SPD in currency i is defined as $\zeta_t^i = \frac{\eta_t^i}{B_t^i}$ where η_t^i is the Radon-Nikodym density martingale transforming the actual measure P into the equivalent martingale measure Q^i. For $T < \infty$, we observe that

$$\eta_t^i = \left.\frac{dQ^i}{dP}\right|_{\phi_t} = E\left[\eta_T^i \middle| \phi_t\right], \qquad \forall t \in [0, T], \tag{1.2}$$

where $E[\ ]$ is with respect to the actual measure *P*.

By an application of the Bayes' rule to equation (1.1), we can show that the product $\zeta_t^i X_t^i$ is a martingale under the actual measure. As a result,

$$\zeta_t^i X_t^i = E\left[\zeta_T^i X_T^i \middle| \phi_t\right]. \tag{1.3}$$

An immediate consequence is that the price $P_{t,T}^i$ of a default-free discount bond in currency i with maturity T is:

$$P_{t,T}^i = E\left[\zeta_T^i \middle| \phi_t\right] \frac{1}{\zeta_t^i}. \tag{1.4}$$

Therefore, asset pricing can be treated under the actual measure if we model the SPD directly.

This approach also provides a convenient specification for the exchange rate process. To see this, let S_t^{ij} be the price of a unit of currency i in units of currency j at time t. If there exists a foreign exchange market across the two currencies, then $S_t^{ij} X_t^i$ is the price of a traded claim in currency j. Hence the product $\zeta_t^j S_t^{ij} X_t^i$ is a martingale under the actual measure. This implies that

$$\zeta_t^j S_t^{ij} X_t^i = E\left[\zeta_T^j S_T^{ij} X_T^i \middle| \phi_t\right]. \tag{1.5}$$

On the other hand, equation (1.3) implies that

$$\zeta_t^i S_t^{ij} X_t^i = S_t^{ij} E\left[\zeta_T^i X_T^i \middle| \phi_t\right]. \tag{1.6}$$

As the initial values of the SPD are $\zeta_0^i = \zeta_0^j = 1$, we can specialize the above two equations to a case with $t = 0$ and $T = t$:

$$\begin{aligned} S_0^{ij} X_0^i &= E\left[\zeta_t^j S_t^{ij} X_t^i\right], \\ S_0^{ij} X_0^i &= S_0^{ij} E\left[\zeta_t^i X_t^i\right]. \end{aligned} \tag{1.7}$$

Market completeness then implies that $S_t^{ij} = S_0^{ij} \frac{\zeta_t^i}{\zeta_t^j}$. This shows that the exchange rate process can be specified in terms of the SPDs in the two currencies.

The potential approach is based on the conditions that $0 < P_{t,T}^i \leq 1$ for $t \in [0, T]$ and that $\lim_{T\to\infty} P_{0,T}^i = 0$. The first condition implies that the SPD is a positive supermartingale as equation (1.4) indicates that

$$\zeta_t^i \geq \zeta_t^i P_{t,T}^i = E\left[\zeta_T^i \middle| \phi_t\right]. \tag{1.8}$$

The second condition ensures that $\lim_{T\to\infty} E\left[\zeta_T^i\right] = 0$. A positive supermartingale that asymptotes to zero is a potential [see Protter (1990)]. Therefore, the SPD admits the following representation

$$\zeta_t^i = E\left[A_\infty^i \middle| \phi_t\right] - A_t^i, \tag{1.9}$$

where A_t^i is an increasing adapted process with $A_0^i = 0$ and $E\left[A_\infty^i\right] = 1$.[7]

B. Our Tractable Framework

We define A_t^i as the following increasing process in order to construct a tractable model:

$$A_t^i \equiv \int_0^t \left(g_1^i(s) M_t^i + g_2^i(s)\right) ds, \tag{1.10}$$

where $g_1^i(s)$ and $g_2^i(s)$ are non-negative deterministic functions of time, and M_t^i is a strictly positive continuous martingale with $M_0^i = 1$ under the actual measure.

By substituting equation (1.9) into equation (1.4), it follows from $M_0^i = 1$ that

$$P_{0,t}^i = E\left[\zeta_t^i\right] = \int_t^\infty \left(g_1^i(s) + g_2^i(s)\right) ds. \tag{1.11}$$

By differentiating the above with respect to t, we have

$$-\frac{\partial P_{0,t}^i}{\partial t} = g_1^i(t) + g_2^i(t), \tag{1.12}$$

which can be used to fit any initial term structures.

Now, by setting

[7] This is the Doob decomposition of a potential. See Protter (1990) for details.

$$g_1^i(t) = -\alpha^i \frac{\partial P_{0,t}^i}{\partial t}\left(P_{0,t}^i\right)^{\gamma^i},$$
$$g_2^i(t) = -\frac{\partial P_{0,t}^i}{\partial t}\left[1-\alpha^i\left(P_{0,t}^i\right)^{\gamma^i}\right], \tag{1.13}$$

with constants $\gamma^i > 0$ and $0 < \alpha^i \le 1$, we ensure that equation (1.12) is satisfied and that the initial term structure is matched exactly. The SPD in equation (1.9) then becomes

$$\zeta_t^i = G_1^i(t)M_t^i + G_2^i(t), \tag{1.14}$$

where

$$G_1^i(t) = \int_t^\infty g_1^i(s)ds = \frac{\alpha^i}{\gamma^i+1}\left(P_{0,t}^i\right)^{\gamma^i+1},$$
$$G_2^i(t) = \int_t^\infty g_2^i(s)ds = P_{0,t}^i - G_1^i(t). \tag{1.15}$$

Furthermore, the default-free discount bond price $P_{t,T}^i$ is given by

$$P_{t,T}^i = \frac{G_1^i(T)M_t^i + G_2^i(T)}{G_1^i(t)M_t^i + G_2^i(t)} = \frac{P_{0,T}^i - \frac{\alpha^i}{\gamma^i+1}\left(P_{0,T}^i\right)^{\gamma^i+1}\left(1-M_t^i\right)}{P_{0,t}^i - \frac{\alpha^i}{\gamma^i+1}\left(P_{0,t}^i\right)^{\gamma^i+1}\left(1-M_t^i\right)}, \tag{1.16}$$

and the exchange rate S_t^{ij} is given by

$$S_t^{ij} = S_0^{ij}\frac{G_1^i(t)M_t^i + G_2^i(t)}{G_1^j(t)M_t^j + G_2^j(t)} = S_0^{ij}\frac{P_{0,t}^i - \frac{\alpha^i}{\gamma^i+1}\left(P_{0,t}^i\right)^{\gamma^i+1}\left(1-M_t^i\right)}{P_{0,t}^j - \frac{\alpha^j}{\gamma^j+1}\left(P_{0,t}^j\right)^{\gamma^j+1}\left(1-M_t^j\right)}. \tag{1.17}$$

Based on the form of the above expressions, our model can be interpreted as a two-parameter sub-family of the rational lognormal family of models suggested by Flesaker and Hughston (1996).

The two parameters of our model have the following intuitive interpretations. Since the spot rate can be derived from discount bond prices by $r_t^i = -\left.\frac{\partial \ln P_{t,T}^i}{\partial T}\right|_{T=t} = -\frac{1}{P_{t,T}^i}\left.\frac{\partial P_{t,T}^i}{\partial T}\right|_{T=t}$, we have

$$r_t^i = \frac{g_1^i(t)M_t^i + g_2^i(t)}{G_1^i(t)M_t^i + G_2^i(t)}. \tag{1.18}$$

Furthermore, equation (1.16) yields the following inequality:

$$\frac{G_1^i(T)}{G_1^i(t)} \le P_{t,T}^i \le \frac{G_2^i(T)}{G_2^i(t)}, \tag{1.19}$$

since $G_2^i(T)G_1^i(t) - G_1^i(T)G_2^i(t) \geq 0$. It is clear that the above inequality will give rise to upper and lower bounds for the spot rate as follows:

$$\frac{g_2^i(t)}{G_2^i(t)} = f^i(0,t)\frac{1-\alpha^i\left(P_{0,t}^i\right)^{\gamma^i}}{1-\frac{\alpha^i}{\gamma^i+1}\left(P_{0,t}^i\right)^{\gamma^i}} \leq r_t^i \leq f^i(0,t)(\gamma^i+1) = \frac{g_1^i(t)}{G_1^i(t)}, \tag{1.20}$$

where $f^i(0,t)$ represents the initial instantaneous forward rate with maturity t in currency i. Hence the level of the future spot rate is determined by the initial forward rate, along with the parameter γ^i which fixes the upper bound and the parameter α^i which dictates the lower bound.

For numerical illustrations, take $\gamma^i = 1.0$, $\alpha^i = 0.5$ and $P_{0,t}^i = 0.8$ for an example. In this case we have an upper bound of $2.0 \times f^i(0,t)$ and a lower bound of $0.75 \times f^i(0,t)$. While it is easy to see that γ^i completely determines the upper bound, the dependence of the lower bound on α^i is not so obvious. In Figure 1, we plot the lower bound as a function of α^i for various values of γ^i. We can see that the lower bound falls (close to zero) as α^i becomes higher (up to one), and that this relationship is not significantly affected by the choice of γ^i. We also note that the lower bound is positive and both bounds are independent of the martingale process.

In this paper, we postulate that the positive martingale M_t^i solves the stochastic differential equation

$$\frac{dM_t^i}{M_t^i} = \sigma^i(t)dw_t^i, \tag{1.21}$$

where $\sigma^i(t)$ is the positive deterministic function of time, and dw_t^i is a Wiener process under the actual measure. We also assume that across currencies, $dw^i \cdot dw^j = \rho^{ij}(t)dt$ and the correlation coefficient $\rho^{ij}(t)$ is a deterministic function of time. It is not difficult to check, using Ito's Lemma, that the stochastic differential equation for the discount bond can be expressed as

$$\begin{aligned}\frac{dP_{t,T}^i}{P_{t,T}^i} = &\left[r_t^i - \frac{G_1^i(t)\sigma^i(t)M_t^i}{G_1^i(t)M_t^i + G_2^i(t)}\left(\frac{G_1^i(T)\sigma^i(t)M_t^i}{G_1^i(T)M_t^i + G_2^i(T)} - \frac{G_1^i(t)\sigma^i(t)M_t^i}{G_1^i(t)M_t^i + G_2^i(t)}\right)\right]dt \\ &+\left(\frac{G_1^i(T)\sigma^i(t)M_t^i}{G_1^i(T)M_t^i + G_2^i(T)} - \frac{G_1^i(t)\sigma^i(t)M_t^i}{G_1^i(t)M_t^i + G_2^i(t)}\right)dw_t^i,\end{aligned} \tag{1.22}$$

and the stochastic differential equation for the exchange rate can be expressed as

$$\frac{dS_t^{ij}}{S_t^{ij}} = \left[r_t^j - r_t^i - \frac{G_1^j(t)\sigma^j(t)M_t^j}{G_1^j(t)M_t^j + G_2^j(t)} \left(\frac{G_1^i(t)\sigma^i(t)M_t^i}{G_1^i(t)M_t^i + G_2^i(t)} \rho^{ij}(t) - \frac{G_1^j(t)\sigma^j(t)M_t^j}{G_1^j(t)M_t^j + G_2^j(t)} \right) \right] dt$$
$$+ \frac{G_1^i(t)\sigma^i(t)M_t^i}{G_1^i(t)M_t^i + G_2^i(t)} dw_t^i - \frac{G_1^j(t)\sigma^j(t)M_t^j}{G_1^j(t)M_t^j + G_2^j(t)} dw_t^j . \tag{1.23}$$

The above stochastic equations are specified under the actual measure. Let λ_t^i be the market risk premium at time t for currency i. From equation (1.22), we see that the market risk premium λ_t^i is given by

$$\lambda_t^i = -\frac{G_1^i(t)\sigma^i(t)M_t^i}{G_1^i(t)M_t^i + G_2^i(t)} . \tag{1.24}$$

Therefore, we find that the volatility of the discount bond price contains the market risk premium and that the volatility of the exchange rate is equal to the distance between the market risk premium vectors for the currency pair.

Now, let σ_T^i be the one-day volatility of RiskMetrics™ for the zero-coupon bond with maturity T in currency i, and σ_S^{ij} be the one-day volatility for the exchange rate. From equations (1.22) and (1.23) we find

$$\begin{aligned} \sigma_T^i \sqrt{252} &= 1.65 \left| \frac{G_1^i(T)\sigma^i(0)M_0^i}{G_1^i(T)M_0^i + G_2^i(T)} - \frac{G_1^i(0)\sigma^i(0)M_0^i}{G_1^i(0)M_0^i + G_2^i(0)} \right| \\ &= 1.65\sigma^i(0)\frac{\alpha^i}{\gamma^i + 1}\left(1 - \left(P_{0,T}^i\right)^{\gamma^i}\right) \end{aligned} \tag{1.25}$$

and

$$\sigma_S^{ij}\sqrt{252} = 1.65\sqrt{\left(\frac{\sigma^i(0)\alpha^i}{\gamma^i + 1}\right)^2 - \frac{2\rho^{ij}(0)\sigma^i(0)\sigma^j(0)\alpha^i\alpha^j}{\left(\gamma^i + 1\right)\left(\gamma^j + 1\right)} + \left(\frac{\sigma^j(0)\alpha^j}{\gamma^j + 1}\right)^2} . \tag{1.26}$$

Here, 252 is the number of trading days in a year and 1.65 is the volatility coefficient of the 95 percent confidence interval over one day as defined by RiskMetrics™. Once the model is calibrated, the above relations can be used for the measurement of Value at Risk in interest rate and currency markets.

II. Pricing Caps, Swaptions, and Currency Options

Based on the specification of our model in Section I.B, in this section we show how to generate tractable pricing formulas for caps, swaptions and currency options. Since our model belongs to the rational lognormal family, we expect Black-Scholes type results for caps and swaptions. In addition, currency options can be easily priced once we make assumptions about the correlation between the randomness in each currency. This is an admitted advantage of the general potential approach and it is reflected in our specialized framework as well.

First, we consider the value of a caplet, which is just one leg of a cap. A caplet is essentially a call option on a LIBOR rate. Let us fix the expiration date T and the settlement date $T+\tau$, where $\tau > 0$ is a fixed number. Under this model, the price CL_0^i of a caplet with a cap rate k in currency i is equal to

$$CL_0^i = \begin{cases} (1+\tau k)\left(I_1^i N(d_1^i) - I_2^i N(d_2^i)\right), & I_1^i > 0, I_2^i > 0 \\ (1+\tau k)\left(I_1^i N(-d_1^i) - I_2^i N(-d_2^i)\right), & I_1^i < 0, I_2^i < 0 \\ (1+\tau k)\left(I_1^i - I_2^i\right), & I_1^i \geq 0, I_2^i \leq 0 \\ 0, & I_1^i \leq 0, I_2^i \geq 0 \end{cases} \tag{2.1}$$

where

$$\begin{aligned} I_1^i &= \frac{1}{1+\tau k} G_1^i(T) - G_1^i(T+\tau), \\ I_2^i &= G_2^i(T+\tau) - \frac{1}{1+\tau k} G_2^i(T), \\ d_1^i &= \frac{\ln \frac{I_1^i}{I_2^i} + \frac{1}{2}\int_0^T \sigma^i(s)^2 ds}{\sqrt{\int_0^T \sigma^i(s)^2 ds}}, \\ d_2^i &= d_1^i - \sqrt{\int_0^T \sigma^i(s)^2 ds}. \end{aligned} \tag{2.2}$$

The function $N(x)$ denotes the standard normal cumulative distribution. The proof of this result is in the appendix.

Second, we consider the value of a payers swaption, which is essentially a call option on a par swap rate. The buyer of a payers swaption has the right to enter a payers swap at the expiration date. Let us fix the payment date $T_l = T + l\tau$ for $l = 1, \cdots, n$. The price PS_0^i of a European payers swaption with a strike rate k and an expiration date T in currency i equals

$$PS_0^i = \begin{cases} J_1^i N(\tilde{d}_1^i) - J_2^i N(\tilde{d}_2^i), & J_1^i > 0, J_2^i > 0 \\ J_1^i N(-\tilde{d}_1^i) - J_2^i N(-\tilde{d}_2^i), & J_1^i < 0, J_2^i < 0 \\ J_1^i - J_2^i, & J_1^i \geq 0, J_2^i \leq 0 \\ 0, & J_1^i \leq 0, J_2^i \geq 0 \end{cases} \tag{2.3}$$

where

$$
\begin{aligned}
J_1^i &= G_1^i(T) - G_1^i(T_n) - k\sum_{l=1}^{n} \tau G_1^i(T_l), \\
J_2^i &= G_2^i(T_n) + k\sum_{l=1}^{n} \tau G_2^i(T_l) - G_2^i(T), \\
\tilde{d}_1^i &= \frac{\ln \dfrac{J_1^i}{J_2^i} + \dfrac{1}{2}\int_0^T \sigma^i(s)^2\, ds}{\sqrt{\int_0^T \sigma^i(s)^2\, ds}}, \\
\tilde{d}_2^i &= \tilde{d}_1^i - \sqrt{\int_0^T \sigma^i(s)^2\, ds}.
\end{aligned} \tag{2.4}
$$

The proof of this result is also contained in the appendix. We note that our pricing formulas in equations (2.1) and (2.3) resemble the Black model for European caps and swaptions.

Finally, we consider a European call option on an exchange rate of a unit j/i (currency *i* call/currency *j* put) with expiration date T. By substituting equations (1.14) and (1.17) into equation (1.3), the current value EC_0^{ij} of a European call on the exchange rate with a strike price κ in currency j is given by

$$
\begin{aligned}
EC_0^{ij} &= E\left[\max\left\{ S_T^{ij} - \kappa, 0 \right\} \zeta_T^j \middle| \phi_0 \right] \frac{1}{\zeta_0^j} \\
&= E\left[\max\left\{ K_1^{ij} M_T^i - K_2^{ij} M_T^j - K_3^{ij}, 0 \right\} \middle| \phi_0 \right],
\end{aligned} \tag{2.5}
$$

where

$$
\begin{aligned}
K_1^{ij} &= S_0^{ij} G_1^i(T), \\
K_2^{ij} &= \kappa G_1^j(T), \\
K_3^{ij} &= \kappa G_2^j(T) - S_0^{ij} G_2^i(T).
\end{aligned} \tag{2.6}
$$

Here, if $K_3^{ij} = 0$, it turns out that the price EC_0^{ij} is equal to

$$
EC_0^{ij} = K_1^{ij} N(d_1^{ij}) - K_2^{ij} N(d_2^{ij}), \tag{2.7}
$$

where

$$
\begin{aligned}
d_1^{ij} &= \frac{\ln \dfrac{K_1^{ij}}{K_2^{ij}} + \dfrac{1}{2}\left(\sigma_P^{ij}\right)^2}{\sigma_P^{ij}}, \\
d_2^{ij} &= d_1^{ij} - \sigma_P^{ij}, \\
\left(\sigma_P^{ij}\right)^2 &= \int_0^T \left(\sigma^i(s)^2 - 2\rho^{ij}(s)\sigma^i(s)\sigma^j(s) + \sigma^j(s)^2 \right) ds.
\end{aligned} \tag{2.8}
$$

If $K_3^{ij} \neq 0$, then a closed-form solution does not exist. Instead, we use the numerical procedure in Rubinstein (1991) to calculate the option price as follows:

$$EC_0^{ij} = \sum_{l=0}^{N} \sum_{m=0}^{N} p_l p_m \cdot Max\left\{K_1^{ij} e^{x_{l,m}} - K_2^{ij} e^{y_{l,m}} - K_3^{ij}, 0\right\}, \tag{2.9}$$

where

$$p_l = \frac{N!}{l!(N-l)!}\left(\frac{1}{2}\right)^N,$$
$$p_m = \frac{N!}{m!(N-m)!}\left(\frac{1}{2}\right)^N,$$
$$x_{l,m} = -\frac{1}{2}\left(\bar{\sigma}^i\right)^2 + \bar{\sigma}^i \frac{l-(N-l)}{\sqrt{N}},$$
$$y_{l,m} = -\frac{1}{2}\left(\bar{\sigma}^j\right)^2 + \bar{\sigma}^j \left(\bar{\rho}^{ij} \frac{l-(N-l)}{\sqrt{N}} + \sqrt{1-\left(\bar{\rho}^{ij}\right)^2} \frac{m-(N-m)}{\sqrt{N}}\right),$$
$$\bar{\sigma}^i = \sqrt{\int_0^T \sigma^i(s)^2 ds},$$
$$\bar{\sigma}^j = \sqrt{\int_0^T \sigma^j(s)^2 ds},$$
$$\bar{\rho}^{ij} = \frac{\int_0^T \rho^{ij}(s)\sigma^i(s)\sigma^j(s)ds}{\sqrt{\int_0^T \sigma^i(s)^2 ds}\sqrt{\int_0^T \sigma^j(s)^2 ds}}. \tag{2.10}$$

We note that the payoff function in equation (2.5) generally resembles that of a spread option. Although both M_T^i and M_T^j are assumed to have lognormal distribution, the distribution of the "spread" is determined by their correlation as well as the relative sizes of the terms in equation (2.5). Clearly, the distribution of the difference between two lognormal random variables is generally not lognormal. As Shimko (1994) shows, this could give rise to a volatility smile pattern.

Figure 2 illustrates the Black-Scholes implied volatility for 6-month call option on the dollar/yen exchange rate when their prices are computed using our potential approach. Specifically, currency option prices in our model are obtained through the numerical procedure in equations (2.9) and (2.10) with 500 time steps. We then plot the Black-Scholes implied volatility (in percentages) as a function of the strike price for different values of α^i, α^j, and $\bar{\rho}^{ij}$ (i-US, j-Japan). The common parameters are taken as $\gamma^i = 3.0$, $\gamma^j = 1.5$, $\bar{\sigma}^i = 0.8$, $\bar{\sigma}^j = 0.4$, $P_{0,T}^i = 0.974374$, $P_{0,T}^j = 0.997531$, $T = 187/365$ and $S_0^{ij} = 102.4$. When $\kappa = 119.6697$, $\alpha^i = \alpha^j = 0.8$ and $\bar{\rho}^{ij} = 0.2$ (dotted line in Panel B), K_3^{ij} is close to zero. This case allows us to verify the accuracy of our approximation using the closed-form solution in equation (2.7).

Clearly, increasing α^i and α^j has the effect of decreasing and increasing spot exchange rates in the future [see equation (1.17)], respectively. Figure 2 shows an important additional effect, that is, α^i mostly affects the right tail of the distribution of future spot exchange rates, whereas α^j mostly affects the left tail. As a result, an increase in α^i (keeping α^j fixed at 1.0) generates a heavier right tail compared to a lognormal distribution, which leads to a higher implied volatility for OTM calls and ITM puts (Panels A and C). Alternatively, an increase in α^j (keeping α^i fixed at 0.8) generates a heavier left tail which leads to a higher implied volatility for OTM puts and ITM calls (Panels B and D). This is generally consistent with the behavior of implied volatilities in the currency options market. The smile pattern implied by our model also depends on $\bar{\rho}_{ij}$, the "average" correlation between innovations in the SPDs. Notably, a comparison between Panels A and B and Panels C and D shows that the correlation impacts the overall level of the smile as well as its skewness.

We have thus demonstrated in this section that caps, swaptions, and currency options can be readily priced using our simple tractable model.

III. Pricing Equity Options

The potential approach is distinguished for its simplicity in dealing with cross-currency products. This is reflected in our derivation of a simple pricing formula for currency options in the preceding section. In this section, we extend the potential approach to the pricing of equity derivatives. We also derive a pricing formula for equity call options as a specific example. The main advantage of this extension is that we can easily treat the effect of random interest rates on equity option pricing, and furthermore, we can deal with cross products such as equity swaps, which periodically exchange the return on an index with a floating LIBOR rate.

We start with Y_t^i, which denotes the spot price of a stock at time t in currency i. We let δ_t^i denote the dividend yield of the stock at time t. Based on equation (1.3), the following can be obtained:

$$\zeta_t^i Y_t^i = E\left[\zeta_T^i \left(Y_T^i e^{\int_t^T \delta_s^i ds}\right) \middle| \phi_t\right] = E\left[\zeta_T^i Y_T^i \middle| \phi_t\right] + E\left[\zeta_T^i Y_T^i \left(e^{\int_t^T \delta_s^i ds} - 1\right) \middle| \phi_t\right]. \tag{3.1}$$

Define a normalized stock price $\xi_t^i \equiv \zeta_t^i Y_t^i$. Assuming that the stock price and the dividend yield are non-negative, we have

$$\xi_t^i \geq E\left[\xi_T^i \middle| \phi_t\right]. \tag{3.2}$$

This also holds in the case of non-dividend-paying stocks, i.e. $\delta_t^i = 0$ for all *t*. Equation (3.2) shows that the normalized stock price is a positive supermartingale under the actual measure.

Next, we examine the asymptotic behavior of ξ_t^i. Let $F_{t,T}^i$ be the forward stock price with maturity T at time $t \in [0,T]$ in currency i. Since a forward contract has no daily settlement and only has a payoff at maturity, from equation (1.3) we know that

$$E\left[\zeta_T^i\left(Y_T^i - F_{t,T}^i\right)\middle|\phi_t\right]\frac{1}{\zeta_t^i} = 0\,. \tag{3.3}$$

Using equation (1.4) this can be rewritten as

$$E\left[\xi_T^i\middle|\phi_t\right] = \zeta_t^i F_{t,T}^i P_{t,T}^i\,. \tag{3.4}$$

Motivated by the empirical observation that LIBOR spreads are negative in the equity swap market, we assume that

$$\frac{\partial \ln F_{t,T}^i}{\partial T} \le f^i(t,T),\ \forall t \in [0,T]\,. \tag{3.5}$$

A consequence of this is that

$$\frac{\partial\left(F_{t,T}^i P_{t,T}^i\right)}{\partial T} \le 0\,. \tag{3.6}$$

Now we define a martingale

$$N_t^i = E\left[\xi_\infty^i\middle|\phi_t\right] = \lim_{T\to\infty}\zeta_t^i F_{t,T}^i P_{t,T}^i\,, \tag{3.7}$$

assuming, of course, that the limit is well-defined. Using equation (3.6), we can show that

$$N_t^i \le \zeta_t^i F_{t,t}^i P_{t,t}^i = \zeta_t^i Y_t^i = \xi_t^i\,. \tag{3.8}$$

As a result, the normalized stock price can be decomposed into a martingale N_t^i and a potential ν_t^i such that $\xi_t^i = N_t^i + \nu_t^i$. Similar to our modeling of the SPD in Section I.B, the potential in this case can be expressed as

$$\nu_t^i = E\left[\int_0^\infty\left(h_1^i(s)Z_s^i + h_2^i(s)\right)ds\middle|\phi_t\right] - \int_0^t\left(h_1^i(s)Z_s^i + h_2^i(s)\right)ds, \tag{3.9}$$

where $h_1^i(t)$ and $h_2^i(t)$ are non-negative deterministic functions of time, and Z_t^i is a positive martingale with $Z_0^i = 1$. By substituting equations (3.9) and (3.7) into equation (3.4), it follows that

$$F_{0,t}^i P_{0,t}^i = E\left[\xi_t^i\right] = E\left[N_t^i + \nu_t^i\right] = F_{0,\infty}^i P_{0,\infty}^i + \int_t^\infty\left(h_1^i(s) + h_2^i(s)\right)ds\,. \tag{3.10}$$

Thus, we obtain

$$-\frac{\partial\left(F_{0,t}^{i}P_{0,t}^{i}\right)}{\partial t}=h_{1}^{i}(t)+h_{2}^{i}(t)\,, \tag{3.11}$$

which allows for the matching of an initial term structure of forward stock prices.

Again, we use a two-parameter specification as follows:

$$\begin{aligned} h_{1}^{i}(t) &= -\beta^{i}\frac{\partial\left(F_{0,t}^{i}P_{0,t}^{i}\right)}{\partial t}\left(\frac{F_{0,t}^{i}P_{0,t}^{i}}{Y_{0}^{i}}\right)^{\gamma_{Y}^{i}}, \\ h_{2}^{i}(t) &= -\frac{\partial\left(F_{0,t}^{i}P_{0,t}^{i}\right)}{\partial t}\left[1-\beta^{i}\left(\frac{F_{0,t}^{i}P_{0,t}^{i}}{Y_{0}^{i}}\right)^{\gamma_{Y}^{i}}\right], \end{aligned} \tag{3.12}$$

with $\gamma_{Y}^{i}>0$ and $0<\beta^{i}\leq 1$. Then ν_{t}^{i} is expressed as

$$\nu_{t}^{i}=H_{1}^{i}(t)Z_{t}^{i}+H_{2}^{i}(t), \tag{3.13}$$

where

$$\begin{aligned} H_{1}^{i}(t) &= \frac{\beta^{i}}{\gamma_{Y}^{i}+1}\frac{\left(F_{0,t}^{i}P_{0,t}^{i}\right)^{\gamma_{Y}^{i}+1}-\left(F_{0,\infty}^{i}P_{0,\infty}^{i}\right)^{\gamma_{Y}^{i}+1}}{\left(Y_{0}^{i}\right)^{\gamma_{Y}^{i}}}, \\ H_{2}^{i}(t) &= F_{0,t}^{i}P_{0,t}^{i}-F_{0,\infty}^{i}P_{0,\infty}^{i}-H_{1}^{i}(t). \end{aligned} \tag{3.14}$$

To complete the model, we need to specify the form of the martingale part of the normalized stock price. So far we have two sources of uncertainty within the model. First, M_{t}^{i} drives the stochastic evolution of the state price density. Second, Z_{t}^{i} drives the potential part of the normalized stock price. For simplicity, we assume that $N_{t}^{i}=F_{0,\infty}^{i}P_{0,\infty}^{i}\left(qZ_{t}^{i}+(1-q)M_{t}^{i}\right)$, a linear combination of the two uncertainties. Hence

$$\xi_{t}^{i}=\left(H_{1}^{i}(t)+qc^{i}\right)Z_{t}^{i}+(1-q)c^{i}M_{t}^{i}+H_{2}^{i}(t)\,, \tag{3.15}$$

where q is a constant ($0\leq q\leq 1$) and the parameter $c^{i}=F_{0,\infty}^{i}P_{0,\infty}^{i}$.

This assumption allows both generality and tractability. We illustrate two simple but rather intuitive cases. In the first case, consider a stock that does not pay any dividends. Then the normalized stock price is a martingale and equation (3.4) shows that $F_{0,T}^{i}P_{0,T}^{i}=Y_{0}^{i}$ regardless of the value of T. Therefore, the potential part ν_{t}^{i} would vanish because $H_{1}^{i}(t)$ and $H_{2}^{i}(t)$ are both equal to zero. In the second case, consider a dividend yield that is bounded below by a positive constant almost surely. Then the only way for ξ_t^i to stay finite in equation (3.1) is for $\lim_{T\to\infty} E\left[\xi_T^i\right] = 0$. As a result, the normalized stock price is itself a potential and the martingale part N_t^i would disappear because $c^i = F_{0,\infty}^i P_{0,\infty}^i = 0$.

We are now ready to derive equity option prices. From equation (3.15), the stock price is given by

$$
\begin{aligned}
Y_t^i &= \frac{\left(H_1^i(t) + qc^i\right) Z_t^i + (1-q)c^i M_t^i + H_2^i(t)}{G_1^i(t) M_t^i + G_2^i(t)} \\
&= \frac{F_{0,t}^i P_{0,t}^i - \left\{ c^i + \dfrac{\beta^i}{\gamma_Y^i + 1} \dfrac{\left(F_{0,t}^i P_{0,t}^i\right)^{\gamma_Y^i + 1} - \left(c^i\right)^{\gamma_Y^i + 1}}{\left(Y_0^i\right)^{\gamma_Y^i}} \right\} \left(1 - Z_t^i\right) - (1-q)c^i Z_t^i + (1-q)c^i M_t^i}{P_{0,t}^i - \dfrac{\alpha^i}{\gamma^i + 1} \left(P_{0,t}^i\right)^{\gamma^i + 1} \left(1 - M_t^i\right)}.
\end{aligned}
\tag{3.16}
$$

We also postulate that the positive martingale Z_t^i follows

$$
\frac{dZ_t^i}{Z_t^i} = \sigma_Y^i(t) dw_t^{Y,i}, \tag{3.17}
$$

where $\sigma_Y^i(t)$ is a positive deterministic function of time and $dw_t^{Y,i}$ is a standard Wiener process under the actual measure. We assume that $dw^{Y,i} \cdot dw^i = \rho_Y^i(t) dt$ where the correlation coefficient $\rho_Y^i(t)$ is also a deterministic function of time. Using equation (3.17) and Ito's Lemma, the stochastic differential equation for the stock price can be expressed as

$$
\begin{aligned}
\frac{dY_t^i}{Y_t^i} = &\left[r_t^i - y_t^i - \frac{G_1^i(t)\sigma^i(t) M_t^i}{G_1^i(t) M_t^i + G_2^i(t)} \times \left(\begin{aligned} &\frac{\left(H_1^i(t) + qc^i\right)\sigma_Y^i(t) Z_t^i}{\left(H_1^i(t) + qc^i\right) Z_t^i + (1-q)c^i M_t^i + H_2^i(t)} \rho_Y^i(t) \\ &+ \frac{(1-q)c^i \sigma^i(t) M_t^i}{\left(H_1^i(t) + qc^i\right) Z_t^i + (1-q)c^i M_t^i + H_2^i(t)} \\ &- \frac{G_1^i(t)\sigma^i(t) M_t^i}{G_1^i(t) M_t^i + G_2^i(t)} \end{aligned} \right) \right] dt \\
&+ \frac{\left(H_1^i(t) + qc^i\right)\sigma_Y^i(t) Z_t^i}{\left(H_1^i(t) + qc^i\right) Z_t^i + (1-q)c^i M_t^i + H_2^i(t)} dw_t^{Y,i} \\
&+ \left(\frac{(1-q)c^i \sigma^i(t) M_t^i}{\left(H_1^i(t) + qc^i\right) Z_t^i + (1-q)c^i M_t^i + H_2^i(t)} - \frac{G_1^i(t)\sigma^i(t) M_t^i}{G_1^i(t) M_t^i + G_2^i(t)} \right) dw_t^i,
\end{aligned}
\tag{3.18}
$$

where

$$y_t^i = \frac{h_1^i(t)Z_t^i + h_2^i(t)}{\left(H_1^i(t) + qc^i\right)Z_t^i + (1-q)c^i M_t^i + H_2^i(t)}. \tag{3.19}$$

Let $\sigma_{Y_0}^i$ denote the one-day volatility of RiskMetrics™ for the stock price in currency i. From equation (3.18) we see that

$$\sigma_{Y_0}^i \sqrt{252} = 1.65 \sqrt{\begin{aligned} &\sigma_Y^i(0)^2 \left(\frac{\beta^i}{\gamma_Y^i + 1} \left(1 - \frac{\left(c^i\right)^{\gamma_Y^i+1}}{\left(Y_0^i\right)^{\gamma_Y^i+1}} \right) + \frac{qc^i}{Y_0^i} \right)^2 \\ &+2\rho_Y^i(0)\sigma_Y^i(0)\sigma^i(0) \left(\frac{\beta^i}{\gamma_Y^i + 1} \left(1 - \frac{\left(c^i\right)^{\gamma_Y^i+1}}{\left(Y_0^i\right)^{\gamma_Y^i+1}} \right) + \frac{qc^i}{Y_0^i} \right) \left(\frac{(1-q)c^i}{Y_0^i} - \frac{\alpha^i}{\gamma^i + 1} \right) \\ &+\sigma^i(0)^2 \left(\frac{(1-q)c^i}{Y_0^i} - \frac{\alpha^i}{\gamma^i + 1} \right)^2 \end{aligned}} \tag{3.20}$$

is an approximation to $\sigma_{Y_0}^i$.

Finally, we consider a European call option with expiration date T. By substituting equations (3.16) and (1.14) into equation (1.3), the current value EC_0^i of a European call with strike κ is given by

$$\begin{aligned} EC_0^i &= E\left[\max\left\{ Y_T^i - \kappa, 0 \right\} \zeta_T^i \middle| \phi_0 \right] \frac{1}{\zeta_0^i} \\ &= E\left[\max\left\{ \xi_T^i - \kappa \zeta_T^i \right\} \middle| \phi_0 \right] \\ &= E\left[\max\left\{ O_1^i Z_T^i - O_2^i M_T^i - O_3^i, 0 \right\} \middle| \phi_0 \right], \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} O_1^i &= H_1^i(T) + qc^i \\ O_2^i &= \kappa G_1^i(T) - (1-q)c^i \\ O_3^i &= \kappa G_2^i(T) - H_2^i(T). \end{aligned} \tag{3.22}$$

This formula is identical in form to the one used to price currency options in Section II and can be approximated similarly. Compared to existing technologies for pricing equity options, our method respects the initial term structure of default-free interest rates as well as the initial term structure of forward stock prices. Its simple structure allows easy calibration and empirical implementation. Although random interest rates are shown to affect only long-term (LEAPS) options in a significant way [see Bakshi, Cao and Chen (2000)], our framework can also handle hybrid products such as equity swaps. We leave the exploration of these issues to future research.

IV. A Two-Factor Interest Rate Model

In this section, we extend the single-factor formulation of the potential model in Section I.B to two factors. A one-factor model implies that points on the term structure of interest rates are perfectly correlated. Instead, a two-factor model provides the flexibility to calibrate the correlation between any two points on the term structure to its empirically estimated value. This could be useful for the hedging of fixed-income instruments, for example.

The extension is straightforward within a potential framework. Instead of using equation (1.10), we specify the increasing adapted process A_t^i with $A_0^i = 0$ as:

$$A_t^i \equiv \int_0^t \left(g_1^i(s) M_s^{1,i} + g_2^i(s) M_s^{2,i} + g_3^i(s) \right) ds , \tag{4.1}$$

where $g_1^i(t)$, $g_2^i(t)$ and $g_3^i(t)$ are non-negative deterministic functions of time, and $M_t^{1,i}$ and $M_t^{2,i}$ are positive martingales with $M_0^{1,i} = M_0^{2,i} = 1$ under the actual measure. Using the same procedure in Section I.B, we obtain

$$-\frac{\partial P_{0,t}^i}{\partial t} = g_1^i(t) + g_2^i(t) + g_3^i(t) . \tag{4.2}$$

Therefore, we set

$$\begin{aligned} g_1^i(t) &= -\alpha_1^i \frac{\partial P_{0,t}^i}{\partial t} \left(P_{0,t}^i \right)^{\gamma^i} , \\ g_2^i(t) &= -\alpha_2^i \left(P_{0,t}^i \right)^{\gamma^i} \frac{\partial P_{0,t}^i}{\partial t} \left[1 - \alpha_1^i \left(P_{0,t}^i \right)^{\gamma^i} \right], \\ g_3^i(t) &= -\left[1 - \alpha_2^i \left(P_{0,t}^i \right)^{\gamma^i} \right] \frac{\partial P_{0,t}^i}{\partial t} \left[1 - \alpha_1^i \left(P_{0,t}^i \right)^{\gamma^i} \right], \end{aligned} \tag{4.3}$$

with $\gamma^i > 0$, $0 \le \alpha_1^i \le 1$, and $0 \le \alpha_2^i \le 1$ as three constant parameters. If $\alpha_1^i = 0$ or $\alpha_2^i = 0$, then this two-factor model reduces to the one-factor model illustrated in Section I.B.

The above leads to the following SPD:

$$\zeta_t^i = G_1^i(t) M_t^{1,i} + G_2^i(t) M_t^{2,i} + G_3^i(t), \tag{4.4}$$

where

$$
\begin{aligned}
G_1^i(t) &= \frac{\alpha_1^i}{\gamma^i+1}\left(P_{0,t}^i\right)^{\gamma^i+1}, \\
G_2^i(t) &= \alpha_2^i\left(P_{0,t}^i\right)^{\gamma^i+1}\left[\frac{1}{\gamma^i+1}-\frac{\alpha_1^i}{2\gamma^i+1}\left(P_{0,t}^i\right)^{\gamma^i}\right], \\
G_3^i(t) &= P_{0,t}^i - G_1^i(t) - G_2^i(t).
\end{aligned}
\tag{4.5}
$$

Then the default-free discount bond price $P_{t,T}^i$ is given by

$$
P_{t,T}^i = \frac{G_1^i(T)M_t^{1,i} + G_2^i(T)M_t^{2,i} + G_3^i(T)}{G_1^i(t)M_t^{1,i} + G_2^i(t)M_t^{2,i} + G_3^i(t)}. \tag{4.6}
$$

This equation does not necessarily yield explicit bounds on the spot interest rate like the one-factor model in Section I.B. Although the bounds ensure that interest rates are positive and non-explosive, it is also important to practitioners that there is otherwise minimal restriction on the variations of interest rates.[8] Our two-factor model provides more flexibility while maintaining positive interest rates.

To proceed, we postulate that the positive martingale $M_t^{j,i}$ ($j = 1,2$) solves the stochastic differential equation

$$
\frac{dM_t^{j,i}}{M_t^{j,i}} = \sigma_j^i(t)dw_t^{j,i}, \tag{4.7}
$$

where $\sigma_j^i(t)$ is a positive deterministic function of time and $dw_t^{j,i}$ is a standard Wiener process under the actual measure. We also assume that $\sigma_1^i(t) = \sigma^i(t)\cos\theta_i(t)$ and $\sigma_2^i(t) = \sigma^i(t)\sin\theta_i(t)$ where $\theta_i(t)$ is a deterministic function of time, and that the two Wiener processes are uncorrelated.

Now, the stochastic differential equation for the SPD can be expressed as

$$
\frac{d\zeta_t^i}{\zeta_t^i} = -r_t^i dt + V_t^{1,i}dw_t^{1,i} + V_t^{2,i}dw_t^{2,i}, \tag{4.8}
$$

where

$$
\begin{aligned}
V_t^{1,i} &= \frac{G_1^i(t)\sigma_1^i(t)M_t^{1,i}}{G_1^i(t)M_t^{1,i} + G_2^i(t)M_t^{2,i} + G_3^i(t)}, \\
V_t^{2,i} &= \frac{G_2^i(t)\sigma_2^i(t)M_t^{2,i}}{G_1^i(t)M_t^{1,i} + G_2^i(t)M_t^{2,i} + G_3^i(t)}.
\end{aligned}
\tag{4.9}
$$

These volatilities can be identified with market risk premiums for the two factors [see Pennacchi et.al. (1996)].

The stochastic differential equation for the discount bond price can be expressed as

---

[8] See Goldberg (1998).

$$\frac{dP_{t,T}^i}{P_{t,T}^i} = \left[r_t^i - V_{t,t}^{1,i}\left(V_{t,T}^{1,i} - V_{t,t}^{1,i}\right) - V_{t,t}^{2,i}\left(V_{t,T}^{2,i} - V_{t,t}^{2,i}\right) \right] dt \\ + \left(V_{t,T}^{1,i} - V_{t,t}^{1,i}\right) dw_t^{1,i} + \left(V_{t,T}^{2,i} - V_{t,t}^{2,i}\right) dw_t^{2,i}, \tag{4.10}$$

where

$$V_{t,T}^{1,i} = \frac{G_1^i(T)\sigma_1^i(t)M_t^{1,i}}{G_1^i(T)M_t^{1,i} + G_2^i(T)M_t^{2,i} + G_3^i(T)},\ V_{t,t}^{1,i} = V_t^{1,i}, \\ V_{t,T}^{2,i} = \frac{G_2^i(T)\sigma_2^i(t)M_t^{2,i}}{G_1^i(T)M_t^{1,i} + G_2^i(T)M_t^{2,i} + G_3^i(T)},\ V_{t,t}^{2,i} = V_t^{2,i}. \tag{4.11}$$

The above equation yields the variance and covariance of the bond return as follows:

$$Var\left(\frac{dP_{0,T}^i}{P_{0,T}^i}\right) = \left(V_{0,T}^{1,i} - V_{0,0}^{1,i}\right)^2 + \left(V_{0,T}^{2,i} - V_{0,0}^{2,i}\right)^2 \\ = \sigma^i(0)^2\left(1 - \left(P_{0,T}^i\right)^{\gamma^i}\right)^2 \begin{bmatrix} \left(\frac{\alpha_1^i}{\gamma^i+1}\cos\theta_i(0)\right)^2 \\ + \left(\frac{1}{\gamma^i+1} - \frac{\alpha_1^i}{2\gamma^i+1}\left(1+\left(P_{0,T}^i\right)^{\gamma^i}\right)\right)^2 \left(\alpha_2^i \sin\theta_i(0)\right)^2 \end{bmatrix}, \tag{4.12}$$

$$Cov\left(\frac{dP_{0,T}^i}{P_{0,T}^i}, \frac{dP_{0,s}^i}{P_{0,s}^i}\right) = \left(V_{0,T}^{1,i} - V_{0,0}^{1,i}\right)\left(V_{0,s}^{1,i} - V_{0,0}^{1,i}\right) + \left(V_{0,T}^{2,i} - V_{0,0}^{2,i}\right)\left(V_{0,s}^{2,i} - V_{0,0}^{2,i}\right) \\ = \sigma^i(0)^2\left(1 - \left(P_{0,T}^i\right)^{\gamma^i}\right)\left(1 - \left(P_{0,s}^i\right)^{\gamma^i}\right) \begin{bmatrix} \left(\frac{\alpha_1^i}{\gamma^i+1}\cos\theta_i(0)\right)^2 \\ + \left(\frac{1}{\gamma^i+1} - \frac{\alpha_1^i}{2\gamma^i+1}\left(1+\left(P_{0,T}^i\right)^{\gamma^i}\right)\right) \\ \times \left(\frac{1}{\gamma^i+1} - \frac{\alpha_1^i}{2\gamma^i+1}\left(1+\left(P_{0,s}^i\right)^{\gamma^i}\right)\right) \\ \times \left(\alpha_2^i \sin\theta_i(0)\right)^2 \end{bmatrix}. \tag{4.13}$$

This gives us the flexibility to calibrate the return covariance between two bonds of our choice.

Next, we use the two-factor model to price interest rate derivatives. First, the current value CL_0^i of a European caplet with cap rate k in currency i is given by:

$$CL_0^i = \left(1 + \tau k\right) E\left[\max\left\{ I_1^i M_T^{1,i} - I_2^i M_T^{2,i} - I_3^i, 0 \right\} \middle| \phi_0 \right], \tag{4.14}$$

where

$$
\begin{aligned}
I_1^i &= \frac{1}{1+\tau k} G_1^i(T) - G_1^i(T+\tau), \\
I_2^i &= G_2^i(T+\tau) - \frac{1}{1+\tau k} G_2^i(T), \\
I_3^i &= G_3^i(T+\tau) - \frac{1}{1+\tau k} G_3^i(T).
\end{aligned}
\tag{4.15}
$$

Second, the current value PS_0^i of a European payers swaption with strike rate k and expiration date T in currency i is given by:

$$
PS_0^i = E\left[\max\left\{ J_1^i M_T^{1,i} - J_2^i M_T^{2,i} - J_3^i, 0 \right\} \middle| \phi_0 \right], \tag{4.16}
$$

where

$$
\begin{aligned}
J_1^i &= G_1^i(T) - G_1^i(T_n) - k\sum\nolimits_{l=1}^{n} \tau G_1^i(T_l), \\
J_2^i &= G_2^i(T_n) + k\sum\nolimits_{l=1}^{n} \tau G_2^i(T_l) - G_2^i(T), \\
J_3^i &= G_3^i(T_n) + k\sum\nolimits_{l=1}^{n} \tau G_3^i(T_l) - G_3^i(T).
\end{aligned}
\tag{4.17}
$$

Similar to currency options in Section II and equity options in Section III, the formulas for caplets and swaptions in a two-factor model have the form of a spread option. Therefore our model has the ability to generate a volatility smile whose shape is controlled by the relative balance between the two constants α_1^i and α_2^i. This can again be considered as an improvement upon a single-factor formulation.

V. Conclusion

This paper presents tractable pricing formulas for European caps, swaptions, currency options and equity options in a positive interest framework using the potential approach. Our particular choice of the model can be thought of as a further specialization within the rational lognormal family of models postulated by Flesaker and Hughston (1996). The parameters of our model have the intuitive interpretation as upper and lower bounds that ensure both positive and non-explosive spot interest rates. Further advantages are that cross products can be handled with simplicity, that the initial term structure of discount bond prices is taken as an input, that the model yields actual probability assessments that facilitate risk management, and that volatility smiles in FX options can be fitted. We also extend the Flesaker and Hughston framework to a two-factor setting, which allows for the calibration of correlations between any two points on the term structure as well as volatility smiles in interest rate options.

Our extension of the potential approach to the pricing of equity derivatives is quite general and involves only two additional assumptions. First, the dividend yields on stocks are non-negative. Second, the instantaneous return on the forward stock price is less than the instantaneous default-free forward rate. This latter assumption is motivated by the empirical regularity that LIBOR spreads of plain vanilla equity swaps are negative. Our extension integrates equity derivatives into the family of interest rate and currency products and enriches the set of cross products that can be handled by the potential approach.

Given such tractability and flexibility, it is only natural that the next stage of research should address empirical tests of the model including the development of an efficient parameter calibration method. Moreover, it seems promising to extend the potential approach to other areas such as derivatives on commodities.

Appendix

We assume that the LIBOR term structure is default-free in order to simplify the following analysis. Consequently we have the following identity at time T because the LIBOR is a simple rate contracted for the period $(T, T+\tau)$:

$$1 + LIBOR \times \tau = \frac{P^i_{T,T}}{P^i_{T,T+\tau}}. \tag{A.1}$$

We can account for credit risk in two ways. First, we can directly model the stochastic LIBOR spread over default-free Treasury rates. Second, we can interpret $P^i_{t,T}$ as the LIBOR term structure which results from an adjusted spot rate that accounts for a generic AA credit quality. This is in the spirit of Duffie and Singleton (1999) and it preserves our analysis below.

By substituting equations (1.14) and (1.16) into equation (1.3), the current value CL^i_0 of a caplet with cap rate k in currency i is given by

$$\begin{aligned}
CL^i_0 &= E\left[\tau P^i_{T,T+\tau} \max\left\{LIBOR - k, 0\right\} \zeta^i_T \middle| \phi_0\right] \frac{1}{\zeta^i_0} \\
&= E\left[\tau P^i_{T,T+\tau} \max\left\{\left(\frac{1}{P^i_{T,T+\tau}} - 1\right)\frac{1}{\tau} - k, 0\right\} \zeta^i_T \middle| \phi_0\right] \frac{1}{\zeta^i_0} \\
&= (1+\tau k)\, E\left[\max\left\{\frac{1}{1+\tau k} - P^i_{T,T+\tau}, 0\right\} \zeta^i_T \middle| \phi_0\right] \frac{1}{\zeta^i_0} \\
&= (1+\tau k)\, E\left[\max\left\{\begin{array}{l} \frac{1}{1+\tau k}\left(G^i_1(T) M^i_T + G^i_2(T)\right) \\ \qquad -\left(G^i_1(T+\tau) M^i_T + G^i_2(T+\tau)\right), 0 \end{array}\right\} \middle| \phi_0\right] \\
&= (1+\tau k)\, E\left[\max\left\{I^i_1 M^i_T - I^i_2, 0\right\} \middle| \phi_0\right],
\end{aligned} \tag{A.2}$$

where

$$\begin{aligned}
I^i_1 &= \frac{1}{1+\tau k} G^i_1(T) - G^i_1(T+\tau) \\
&= \frac{\alpha^i}{\gamma^i + 1}\left\{\frac{1}{1+\tau k}\left(P^i_{0,T}\right)^{\gamma^i+1} - \left(P^i_{0,T+\tau}\right)^{\gamma^i+1}\right\}, \\
I^i_2 &= G^i_2(T+\tau) - \frac{1}{1+\tau k} G^i_2(T) \\
&= \left\{P^i_{0,T+\tau} - \frac{\alpha^i}{\gamma^i+1}\left(P^i_{0,T+\tau}\right)^{\gamma^i+1}\right\} - \frac{1}{1+\tau k}\left\{P^i_{0,T} - \frac{\alpha^i}{\gamma^i+1}\left(P^i_{0,T}\right)^{\gamma^i+1}\right\}.
\end{aligned} \tag{A.3}$$

Accordingly, we can apply the standard Black-Scholes analysis at this point.

Next, we have the following identity at time T because the (par) swap rate gives the swap zero initial value:

$$\left(\sum_{l=1}^{n} LIBOR_l \times \tau P^i_{T,T_l}\right) = 1 - P^i_{T,T_n} = SwapRate \times \sum_{l=1}^{n} \tau P^i_{T,T_l} \ . \tag{A.4}$$

Hence, by use of equations (1.3) and (1.16), the current value PS^i_0 of a European payers swaption with strike rate k and expiration date T in currency i is given by:

$$\begin{aligned}
PS^i_0 &= E\left[\zeta^i_T \max\left\{SwapRate - k, 0\right\} \sum\nolimits_{l=1}^{n} \tau P^i_{T,T_l} \middle| \phi_0\right] \frac{1}{\zeta^i_0} \\
&= E\left[\zeta^i_T \max\left\{\frac{1 - P^i_{T,T_n}}{\sum\nolimits_{l=1}^{n} \tau P^i_{T,T_l}} - k, 0\right\} \sum\nolimits_{l=1}^{n} \tau P^i_{T,T_l} \middle| \phi_0\right] \frac{1}{\zeta^i_0} \\
&= E\left[\max\left\{ 1 - P^i_{T,T_n} - k \sum\nolimits_{l=1}^{n} \tau P^i_{T,T_l}, 0\right\} \zeta^i_T \middle| \phi_0\right] \frac{1}{\zeta^i_0} \\
&= E\left[\max\left\{\begin{aligned} G^i_1(T) M^i_T + G^i_2(T) - \left(G^i_1(T_n) M^i_T + G^i_2(T_n)\right) \\ -k \sum\nolimits_{l=1}^{n} \tau\left(G^i_1(T_l) M^i_T + G^i_2(T_l)\right), 0 \end{aligned}\right\} \middle| \phi_0\right] \\
&= E\left[\max\left\{J^i_1 M^i_T - J^i_2, 0\right\} \middle| \phi_0\right],
\end{aligned} \tag{A.5}$$

where

$$\begin{aligned}
J^i_1 &= G^i_1(T) - G^i_1(T_n) - k \sum\nolimits_{l=1}^{n} \tau G^i_1(T_l) \\
&= \frac{\alpha^i}{\gamma^i + 1}\left\{\left(P^i_{0,T}\right)^{\gamma^i+1} - \left(P^i_{0,T_n}\right)^{\gamma^i+1} - k \sum\nolimits_{l=1}^{n} \tau \left(P^i_{0,T_l}\right)^{\gamma^i+1}\right\}, \\
J^i_2 &= G^i_2(T_n) + k \sum\nolimits_{l=1}^{n} \tau G^i_2(T_l) - G^i_2(T) \\
&= \left\{P^i_{0,T_n} - \frac{\alpha^i}{\gamma^i+1}\left(P^i_{0,T_n}\right)^{\gamma^i+1}\right\} + k \sum\nolimits_{l=1}^{n} \tau \left\{P^i_{0,T_l} - \frac{\alpha^i}{\gamma^i+1}\left(P^i_{0,T_l}\right)^{\gamma^i+1}\right\} \\
&\quad - \left\{P^i_{0,T} - \frac{\alpha^i}{\gamma^i+1}\left(P^i_{0,T}\right)^{\gamma^i+1}\right\}.
\end{aligned} \tag{A.6}$$

This calculation is similar to that for a caplet.

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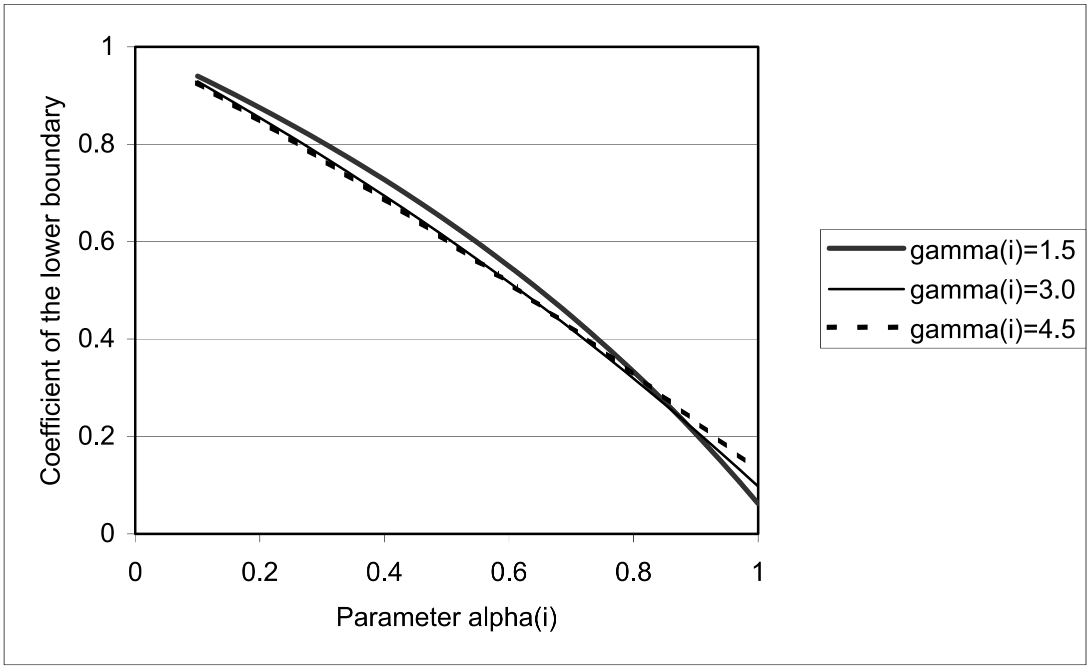
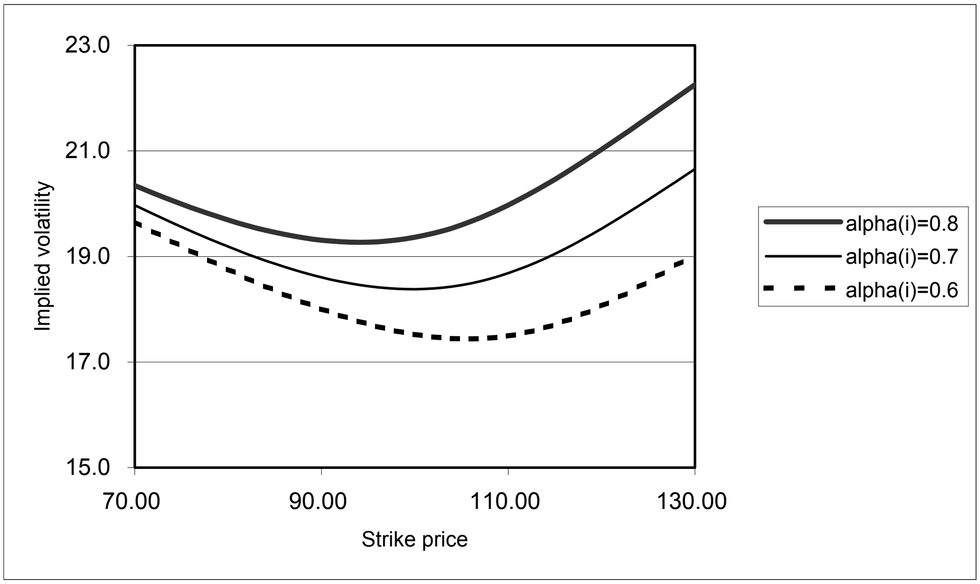
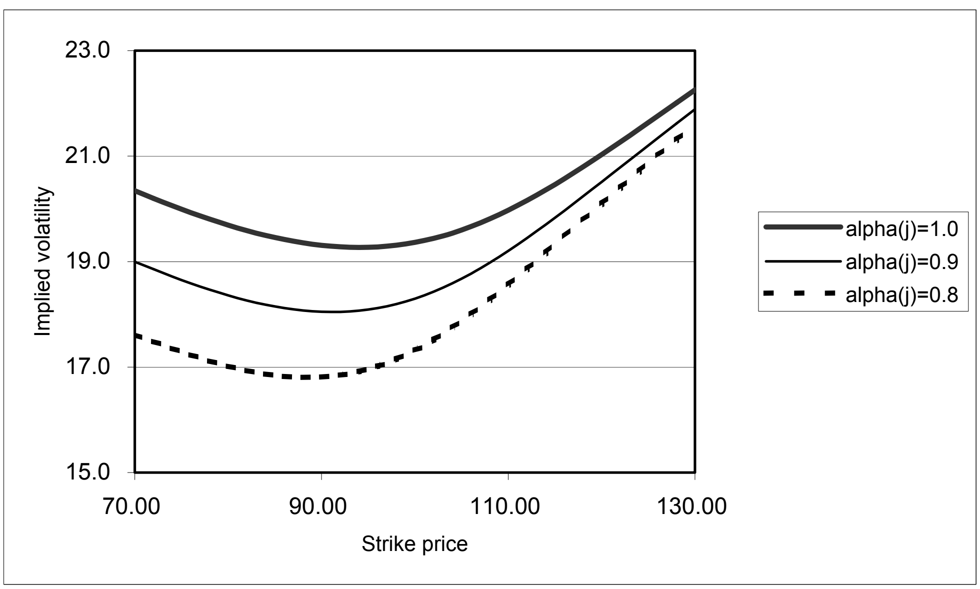


Figure 1. The coefficient of the lower bound as a function of α^i for various values of γ^i.

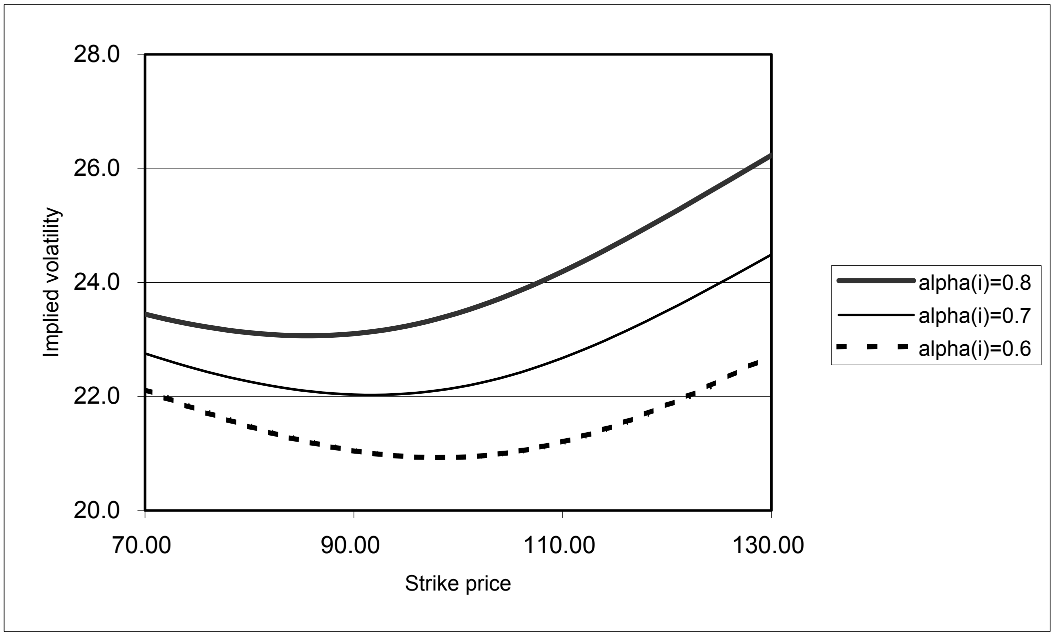
Panel A. $\alpha^j = 1.0$ and $\bar{\rho}^{ij} = 0.2$.



Panel B. $\alpha^i = 0.8$ and $\bar{\rho}^{ij} = 0.2$.



Panel C. $\alpha^j = 1.0$ and $\bar{\rho}^{ij} = -0.2$.



Panel D. $\alpha^i = 0.8$ and $\bar{\rho}^{ij} = -0.2$.

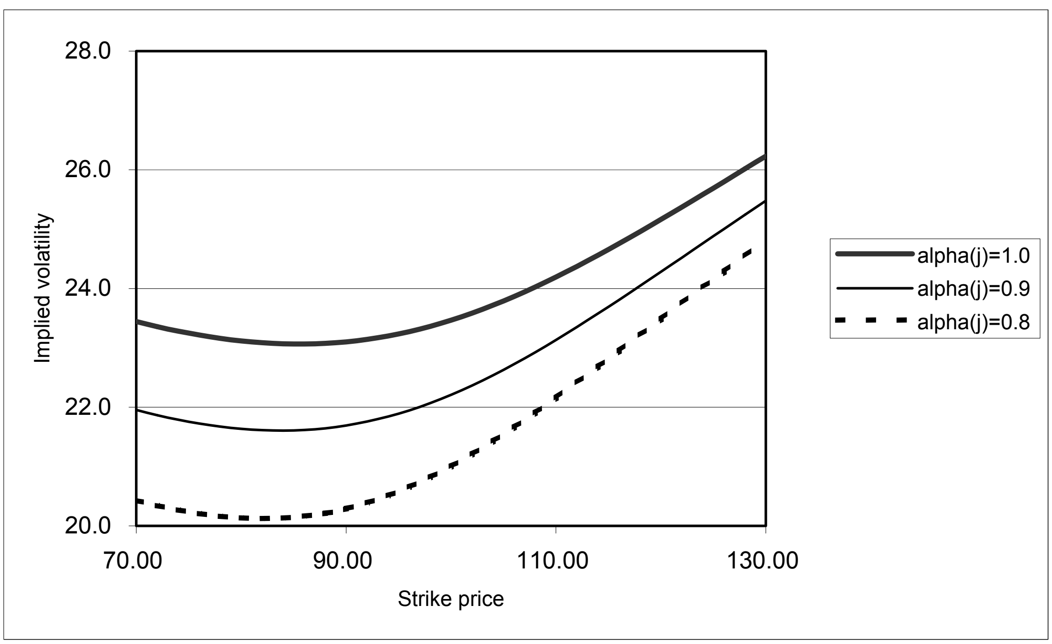


Figure 2. The Black-Scholes implied volatility of currency options as a function of strike price.