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A note on transforming PDEs to ODEs

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Abstract

In this paper, we develop a simple and general method that transforms (nonlinear) PDEs to ODEs. We apply our method to the stochastic portfolio model.

Keywords: HJB PDEs; ODEs; portfolio; stochastic factor; investment.

This note overcomes major obstacles in the area of stochastics and mathematical finance. In so doing, it transforms a cumbersome partial differential equation PDE to an ordinary differential equation ODE. We apply our method to the portfolio model (the stochastic factor model).

We consider the function $V(x, y)$; it can be expressed as $V(\beta x, y)$, where β is a shift parameter with an initial value equal to one (see Alghalith, 2008). Define $g \equiv \beta x$; differentiating $V(g, y)$ with respect to β and x, respectively, yields

$$\begin{aligned} V_\beta &= V_g x, \\ V_x &= V_g \beta. \end{aligned}$$

Thus,

$$\frac{V_x}{V_\beta} = \frac{\beta}{x} \Rightarrow V_x = \frac{\beta V_\beta}{x}. \tag{1}$$

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The second order derivatives of $V(g, y)$ with respect to β and x, respectively, are

$$V_{\beta\beta} = V_{gg}x^2,$$
$$V_{xx} = V_{gg}\beta^2.$$

Therefore,

$$\frac{V_{xx}}{V_{\beta\beta}} = \frac{\beta^2}{x^2} \Rightarrow V_{xx} = \frac{\beta^2 V_{\beta\beta}}{x^2}. \tag{2}$$

Similarly, we can rewrite $V(x, y)$ as $V(x, \beta y)$. Define $f \equiv \beta y$; differentiating $V(x,f)$ with respect to β and y, respectively, yields

$$V_{\beta} = V_f y,$$
$$V_y = V_f \beta.$$

Thus,

$$V_y = \frac{\beta V_\beta}{y}. \tag{3}$$

The second order derivatives of $V(x,f)$ with respect to β and y, respectively, are

$$V_{\beta\beta} = V_{ff}y^2,$$
$$V_{yy} = V_{ff}\beta^2.$$

Therefore,

$$\frac{V_{yy}}{V_{\beta\beta}} = \frac{\beta^2}{y^2} \Rightarrow V_{yy} = \frac{\beta^2 V_{\beta\beta}}{y^2}. \tag{4}$$

Differentiating (1) with respect to y yields

$$V_{xy} = \frac{\beta V_{\beta y}}{x} \Rightarrow V_{\beta y} = \frac{x V_{xy}}{\beta}. \tag{5}$$

Differentiating (3) with respect to y yields

$$V_{yy} = \frac{\beta[\beta y V_{\beta y} - V_\beta]}{y^2}. \tag{6}$$

Substituting (5) into (6), we obtain

$$V_{xy} = \frac{(y^2/\beta)V_{yy} + \beta V_\beta}{xy} = \frac{\beta(V_{\beta\beta} + V_\beta)}{xy}. \tag{7}$$

Practical Example: The Portfolio Model

We provide a brief description of the portfolio model (Detemple, 2013; Alghalith, 2009; Castaneda-Leyva and Hernandez-Hernandez, 2006). Thus, we have a two-dimensional Brownian motion $\{(W_s^1, W_s^{(2)}), \mathcal{F}_s\}_{t \le s \le T}$ defined on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_s, P)$, where $\{\mathcal{F}_s\}_{t \le s \le T}$ is the augmentation of filtration. The risk-free asset price process is $S_0 = e^{\int_t^T r(Y_s)ds}$, where $r(Y_s) \in C_b^2(R)$ is the rate of return and Y_s is the economic factor.

The risky asset price process is given by

$$dS_s = S_s\{\mu(Y_s)ds + \sigma(Y_s)dW_s^1\}, \tag{8}$$

where $\mu(Y_s)$ and $\sigma(Y_s)$ are the rate of return and the volatility, respectively. The economic factor process is given by

$$dY_s = b(Y_s)ds + \rho dW_s^1 + \sqrt{1-\rho^2}dW_s^{(2)}, \quad Y_t = y, \tag{9}$$

where $|\rho| < 1$ is the correlation factor between the two Brownian motions and $b(Y_s) \in C^1(R)$ with a bounded derivative.

The wealth process is given by

$$X_T^\pi = x + \int_t^T \{r(Y_s)X_s^\pi + [\mu(Y_s) - r(Y_s)]\pi_s\}ds + \int_t^T \pi_s\sigma(Y_s)dW_s^1, \tag{10}$$

where x is the initial wealth, $\{\pi_s, \mathcal{F}_s\}_{t \le s \le T}$ is the portfolio process with $E\int_t^T \pi_s^2 ds < \infty$. The trading strategy $\pi_s \in \mathcal{A}(x, y)$ is admissible.

The investor's objective is to maximize the expected utility of terminal wealth

$$V(t, x, y) = \operatorname*{Sup}_{\pi} E[u(X_T) \mid \mathcal{F}_t], \tag{11}$$

where $V(.)$ is the value function and $u(.)$ is a continuous, bounded and strictly concave utility function.

The corresponding Hamilton–Jacobi–Bellman PDE is (suppressing the notations)

$$\begin{gathered} V_t + rxV_x + bV_y + \frac{1}{2}V_{yy} + \operatorname*{Sup}_{\pi_t}\left\{\frac{1}{2}\pi_t^2\sigma^2V_{xx} + \pi_t(\mu - r)V_x + \rho\sigma\pi_tV_{xy}\right\} = 0, \\ V(T, x, y) = u(x). \end{gathered} \tag{12}$$

Thus, we obtain the following well-known HJB PDE

$$V_t + rxV_x + (\mu - r)\pi_t^*V_x + \frac{1}{2}\sigma^2\pi_t^{*2}V_{xx} + bV_y + \frac{1}{2}V_{yy} + \rho\sigma\pi_t^*V_{xy} = 0, \tag{13}$$

where the asterisk denotes the optimal value. Substituting (1), (2), (4), (3) and (7) into (13) yields

$$V_t + r\beta V_\beta + (\mu - r)\pi_t^* \frac{\beta V_\beta}{x} + \frac{1}{2}\sigma^2\pi_t^{*2}\frac{\beta^2 V_{\beta\beta}}{x^2} + \frac{b\beta V_\beta}{y} + \frac{1}{2}\frac{\beta^2 V_{\beta\beta}}{y^2} + \rho\sigma\pi_t^* \frac{\beta(V_{\beta\beta} + V_\beta)}{xy} = 0.$$

Using the above procedure, we can easily show that $V_t = \beta V_\beta / t$ and thus the above equation becomes

$$\frac{\beta V_\beta}{t} + r\beta V_\beta + (\mu - r)\pi_t^* \frac{\beta V_\beta}{x} + \frac{1}{2}\sigma^2\pi_t^{*2}\frac{\beta^2 V_{\beta\beta}}{x^2} + \frac{b\beta V_\beta}{y} + \frac{1}{2}\frac{\beta^2 V_{\beta\beta}}{y^2} + \rho\sigma\pi_t^* \frac{\beta(V_{\beta\beta} + V_\beta)}{xy} = 0.$$

Setting β at its initial value, we obtain

$$\frac{V_\beta}{t} + rV_\beta + (\mu - r)\pi_t^* \frac{V_\beta}{x} + \frac{1}{2}\sigma^2\pi_t^{*2}\frac{V_{\beta\beta}}{x^2} + \frac{bV_\beta}{y} + \frac{1}{2}\frac{V_{\beta\beta}}{y^2} + \frac{\rho\sigma\pi_t^*(V_{\beta\beta} + V_\beta)}{xy} = 0.$$

The solution to the above equation is equivalent to solving an ODE.

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