Preliminary Exam for the Ph.D. in Financial Engineering

Date: April 2010

Note: unless stated otherwise, all parameters are constant.

Mathematical Finance: 3 hours

Problem 1 (15 points)

Let W be Brownian motion, and α be a continuous process adapted to the Brownian filtration.

a) Find the strong solution of the equation:

$$\begin{aligned} dX(t) &= \alpha(t)dt + X(t)dW(t) \\ X(0) &= X_0 > 0 \end{aligned}$$

Hint: replace W by a deterministic differentiable function of time g, and "divide by dt". This equation becomes now a linear ordinary equation with non-constant coefficients. The impulse response for that equation is a (deterministic) exponential function. When solving the original equation above, you should replace the deterministic exponential by what is often called the "stochastic exponential".

b) Suppose that α is a function of $X(t)$, say $\alpha(t) = f(t, X(t))$.Give a condition for the solution to remain positive for $t > 0$.

c) Suppose we choose a different initial condition than $X(0) = 0$. Discuss the advantages and advantages of this process to model risk-free interest rates and price bonds, compared to the Vasicek and Cox-Ingersoll-Ross models.

Solution

a) The inhomogeneous equation

$$\begin{aligned} \frac{dx}{dt} - \frac{dg}{dt}x &= \alpha(t) \\ x(0) &= 0 \end{aligned}$$

has solution:

$$x(t) = \exp(-g(t)) \int_0^t \exp(g(s))\alpha(s)ds$$

Thus:

$$X(t) = \exp(W(t) - \frac{1}{2}t) \int_0^t \exp(-W(s) + \frac{1}{2}s)\alpha(s)ds$$

Verification:

$$
\begin{aligned}
f(t, W(t), z(t)) &= \exp(-W(t) + \frac{1}{2}t)z(t) \\
f_t &= -\frac{1}{2}f \\
f_W &= -f \\
f_Z &= \alpha \\
f_{WW} &= f
\end{aligned}
$$

Thus

$$df = -\frac{1}{2}f - fdW + \alpha dt + \frac{1}{2}f$$

The final solution is:

$$X(t) = X_0 \exp(-W(t) + \frac{1}{2}t) + \exp(W(t) - \frac{1}{2}t) \int_0^t \exp(-W(s) + \frac{1}{2}s)\alpha(s)ds$$

b) $f(t, 0) > 0$

c) Advantage: there is an analytical solution (unlike CIR). The solution is positive (unlike Vasicek).

Disadvantages: the homogeneous solution is nasty, because its effect does not decrease with probability one. The process is not Gaussian. There does not seem to be an analytical solution for bond prices.

Problem 2 (20 points)

Let B_1 and B_2 be independent Brownian motions in the physical measure. The prices of asset 1 and 2 are given by:

$$
\begin{aligned}
\frac{dS_1}{S_1} &= \mu_1 dt + \sigma_1 dB_1(t) \\
\frac{dS_1}{S_1} &= \mu_2 dt + \sigma_2(\rho dB_1(t) + \sqrt{1-\rho^2}dB_2(t))
\end{aligned}
$$

Consider an option to exchange asset 1 against asset.2 at an expiration time T. The payoff at time t of this option is given by:

$$V(T) = \max(S_1(T) - S_2(T), 0)$$

a) Write the partial differential equation that the price $V(t, S_1(t), S_2(t))$ satisfies

b) Verify that the following formula solves the partial differential equation derived earlier

$$V(t, S_1(t), S_2(t)) = S_1(t)N(d_1) - S_2(t)N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_1(t)) - \ln(S_2(t)) + \frac{1}{2}v^2(T-t)}{v\sqrt{T-t}} \\ d_2 &= d_1 - v\sqrt{T-t} \\ v &= \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \end{aligned}$$

c) How does V vary with the correlation ρ between asset returns? No proof is necessary.

Solution

a)

$$\frac{\partial V}{\partial t} + rS_1\frac{\partial V}{\partial S_1} + rS_2\frac{\partial V}{\partial S_2} + \frac{1}{2}\sigma_1^2\frac{\partial^2 V}{\partial {S_1}^2} + \sigma_1\sigma_2\rho\frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial {S_2}^2} = rV$$

b) This is going to be gory...

c) It decreases with correlation.

Problem 3 (20 points)

The price at time t of a discount bond with maturity T is denoted by $P(t,T)$. At time zero, we observe the following yield curve:

$$P(0,t) = \exp(-r(0)t - \beta t^2)$$

where α and β are positive constants. An inverse floater returns a payoff per unit of time equal to:

$$\pi(t)dt \equiv \begin{cases} (c - r(t))dt & t \le T \\ 0 & t > T \end{cases}$$

where c is a constant. Let $\tilde{W}$ be Brownian motion in the risk-neutral measure. The dynamics of the interest rate are:

$$dr(t) = (\theta(t) - ar(t))\,dt + \sigma d\tilde{W}(t)$$

a) Find a formula for the price $V(r(0),0)$ of a floater at time 0

b) Suppose you hedge this floater with a discount bond with maturity τ. Give a formula for the hedge $\Delta(0)$, i.e., the number of discount bonds needed.at time zero to hedge the inverse floater.

Solution

a)

$$
\begin{aligned}
V(r(0),0) &= \int_0^T cP(0,t)dt - E[\int_0^T r(t)\exp(-\int_0^t r(s)ds)dt] \\
&= \int_0^T cP(0,t)dt + E[\int_0^T \frac{d}{dt}\exp(-\int_0^t r(s)ds)dt] \\
&= \int_0^T cP(0,t)dt - P(0,T) + 1 \\
&= \int_0^T c\exp(-r(0)t - \beta t^2)dt - \exp(-r(0)T - \beta T^2) + 1 \\
&= \frac{\sqrt{2\pi\frac{1}{2\beta}}}{\sqrt{2\pi\frac{1}{2\beta}}}\int_0^T c\exp(-\frac{1}{2}\frac{(t+r(0)/2\beta)^2}{1/2\beta})\exp(-\beta\left(\frac{r(0)}{2\beta}\right)^2)dt - \exp(-r(0)T - \beta T^2) + 1 \\
&= \sqrt{2\pi\frac{1}{2\beta}}(N(-r(0)/2\beta, 1/2\beta, T) - N(0, 1/2\beta, T)) - \exp(-r(0)T - \beta T^2) + 1
\end{aligned}
$$

b) Differentiate the expression above w.r.t $r(0)$

Problem 4 (10 points)

Suppose interest rates are constant. Prove that early exercise is never optimal for a call on a non-dividend paying stock.

Problem 5 (20 points)

Let $\tilde{\mathbb{P}}_{US}$ be the US risk-neutral measure, and $\tilde{\mathbb{P}}_{EU}$ the Euro risk-neutral measure. The process $\tilde{W}_{EU}$ is Brownian motion under $\tilde{\mathbb{P}}_{EU}$, while the process $\tilde{W}_{US}$ and $\tilde{W}_Q$ are Brownian motions under $\tilde{\mathbb{P}}_{US}$. We write ρdt for $d\tilde{W}_{EU}d\tilde{W}_Q$. For simplicity, we suppose that there is no correlation between $\tilde{W}_{US}$ and $\tilde{W}_Q$. The Euro/US interest rate are denoted by r_{EU} and r_{US} respectively. They follow the equations:

$$
\begin{aligned}
dr_{EU} &= a_{EU}(b_{EU} - r_{EU}(t))dt + \sigma_{EU}d\tilde{W}_{EU}(t) \\
dr_{US} &= a_{US}(b_{US} - r_{US}(t))dt + \sigma_{US}d\tilde{W}_{US}(t)
\end{aligned}
$$

The price (in Euro) of a Euro discount bond with maturity T is denoted $P_{EU}(t,T)$, and the price (in \$) of a US discount bond is denoted $P_{US}(t,T)$.The exchange rate at time t is denoted $Q(t)$. It denotes the number of \$ per Euro. Its dynamics are:

$$
\frac{dQ(t)}{Q(t)} = (r_{US}(t) - r_{EU}(t))dt + \sigma_Q d\tilde{W}_Q(t)
$$

1) Write the dynamics of $P_{EU}(t,T)$ in the Euro forward measure
2) Write the dynamics of the Euro interest rate in the Euro forward measure
3) Write the dynamics of the US interest rate in the US forward measure

4) Construct an arbitrage strategy in order to write the dynamics of $P_{EU}(t,T)$ in the US forward measure.

5) Why is it important to have a single measure where to carry calculations?

Solution

1 and 2)

$$P_{EU}(t,T) = \exp(-r(t)A(T-t) + B(t,T))$$

where $A_{EU}(T-t) = \frac{1-\exp(-a_{EU}(T-t))}{a_{EU}}$. Thus:

$$\begin{aligned} \frac{dP_{EU}(t,T)}{P_{EU}(t,T)} &= r_{EU}dt - \sigma A(T-t)d\tilde{W}_{EU}(t) \\ &= -\sigma A(T-t)dW^T_{EU}(t) \end{aligned}$$

where:

$$dW^T_{EU}(t) = d\tilde{W}_{EU}(t) - \frac{r_{EU}}{\sigma A_{EU}(T-t)}dt$$

3)

$$dW^T_{US}(t) = d\tilde{W}_{US}(t) - \frac{r_{US}}{\sigma A_{US}(T-t)}dt$$

4) Since:

$$\begin{aligned} \frac{P_{EU}(T,T)}{P_{US}(T,T)} &= 1/Q(T) \\ \frac{P_{EU}(t,T)}{P_{US}(t,T)} &= E^T_{US}[\frac{1}{Q(T)}] \end{aligned}$$

Since there is no correlation between $\tilde{W}_Q$ and $\tilde{W}_{US}$, then $\tilde{W}_Q$ is also BM in the US forward measure. So:

$$\begin{aligned} d\frac{1}{Q} &= \frac{1}{Q}\{r_{EU}(t) - r_{US}(t))dt + \sigma_Q d\tilde{W}_Q(t) \\ &= (r_{US}(t) - r_{EU}(t) - \sigma_Q^2)dt + \sigma_Q d\tilde{W}_Q(t) \end{aligned}$$

Thus

$$\begin{aligned} E^T_{US}[\frac{1}{Q(T)}] &= E^T_{US}[\frac{1}{Q(t)}\exp((r_{US}(t) - r_{EU}(t) - \frac{1}{2}\sigma_Q^2)(T-t) + \sigma_Q d\tilde{W}_Q(t))] \\ &= \frac{1}{Q(t)}\exp((r_{US}(t) - r_{EU}(t))(T-t)) \end{aligned}$$

Thus

$$dP_{EU}(t,T) = d(P_{US}(t,T)\frac{1}{Q(t)}\exp((r_{US}(t) - r_{EU}(t))(T-t)))$$

5) For Monte Carlo simulation.

Problem 6 (15 points)

Consider a futures contract on a commodity, with expiration T. The price $S(t)$ of the commodity is a geometric Brownian motion with volatility σ. Denote the price of the futures contract by $Fut(t,T)$, and its volatility by σ.

a) Write the stochastic differential equation that $Fut(t,T)$ satisfies as a function of Brownian motion in the risk-neutral measure

b) Consider a call option on this future, with expiration $\tau < T$ and strike price K. Give a representation of the price $V(t)$ at time t of this option as a conditional expectation.

c) Suppose interest rates are stochastic. Write the partial differential equation that $V(t)$ satisfies.

d) Give a formula for $V(t)$ in the case where interest rates are constant. Define what you need.

Solution

a)

$$dFut(t,T) = \sigma dW(t)$$

b)

$$V(t) = E[\exp(-\int_t^\tau r(s)ds \max(Fut(\tau,T) - K, 0)]$$

c)$V(t) = V(t, F_{ut}(t,T), r(t))$

d)

$$\begin{aligned} V(t) &= P(t,\tau)E[\max(Fut(\tau,T) - K, 0)] \\ &= P(t,\tau)\left(F_{ut}(t,T)N(d_1) - KN(d_2)\right) \end{aligned}$$

where

$$d_1 = \frac{\ln(Fut(t,T)/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

Stochastic Processes and Probability: 90 minutes

Problem 1 (20 points)

The main paradigm in portfolio theory is mean-variance optimization. An investor desires to maximize the mean return on his portfolio, subject to a constraint on the variance of the return of the portfolio. Consider a 2-asset problem, where asset prices are denoted by S_1 and S_2. Let N_1 and N_2 be independent Poisson processes, with mean equal to $\lambda_1 t$ and $\lambda_2 t$. The dynamics of the asset prices are:

$$
\begin{aligned}
\log(S_1(t)) &= \mu_1 t + \sigma_1 (N_1(t) - \lambda_1 t) \\
\log(S_2(t)) &= \mu_2 t + \sigma_2 \rho (N_1(t) - \lambda_1 t) + \sigma_2 \sqrt{1-\rho^2} (N_2(t) - \lambda_2(t))
\end{aligned}
$$

The number of shares of assets 1 and 2 are denoted Δ_1 and Δ_2. State the mean-variance optimization problem, and calculate the mean and variance of the value of the portfolio. You are not asked to solve the portfolio problem.

Hint: there is no probabilistic trick involved. When you carry out the calculation, remember that $\exp(i)(\lambda t)^i = (e\lambda t)^i$.

Solution

$$
\begin{aligned}
E[\exp(N(t))] &= \exp(-\lambda t) \sum_i \exp(i) \frac{(\lambda t)^i}{i!} \\
&= \exp(-\lambda t) \sum_i \frac{(e\lambda t)^i}{i!} \\
&= \exp(-\lambda t) \exp(e\lambda t) \\
&= \exp(\lambda t(e-1))
\end{aligned}
$$

Problem 2 (15 points)

Let W be Brownian motion.

a) Let $a > 0$. Solve the equation

$$
\begin{aligned}
dX(t) &= -aX(t) + \sigma dW(t) \\
X(0) &= X_0
\end{aligned}
$$

b) Calculate the correlation between $X(t)$ and $W(t)$.

Solution

a)

$$
\begin{aligned}
X(t) &= X_0 \exp(-at) + \sigma \exp(-at) \int_0^t \exp(au) \frac{dW(u)}{du} du \\
&= X_0 \exp(-at) + \sigma \exp(-at)[W(t)\exp(at) - a \int_0^t \exp(au) W(u) du]
\end{aligned}
$$

b)

$$
\begin{aligned}
E[W(t)X(t)] &= \sigma(t - a\exp(-at)\int_0^t \exp(au)E[W(u)W(t)]du) \\
&= \sigma(t - a\exp(-at)\int_0^t \exp(au)udu) \\
&= \sigma(t - a\exp(-at)(\frac{\exp(at)}{a}t - \int_0^t \exp(au)du) \\
&= \sigma(t - a\exp(-at)(\frac{\exp(at)}{a}t - \frac{\exp(at)-1}{a})
\end{aligned}
$$

Problem 3 (15 points)

Consider a birth and death process on the non-negative integers with birth parameter λ_x and death parameter μ_x for each state x.

a) Derive the equation that the stationary distribution of this process (if it exists) satisfies.

b) Solve this equation

As a particular application, you will study a N server queue. Customers arrive to a supermarket according to a Poisson process with mean λt. They are served by N servers. The service times are exponentially distributed with parameter μ. Whenever there are more than N customers waiting for service the excess customers form a queue and wait until their turn at one of the N servers.

c) Find the stationary distribution of the total number of persons in the queue (i.e., waiting to be served, and being served) whenever there is a stationary distribution.

Solution

Hoel Port Stone p. 106.