

The Dynamics of the Forward Interest Rate Curve with Stochastic String Shocks

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This article offers a new class of models for the term structure of forward interest rates. We allow each instantaneous forward rate to be driven by a different stochastic shock, but constrain the shocks so that the forward rate curve is kept continuous. We term the shocks to the forward curve “stochastic string shocks,” and construct then as solutions to stochastic partial differential equations. This article offers a variety of parameterizations that can produce, with parsimony, any correlation pattern among forward rates of different maturities. We derive the no-arbitrage condition on the drift of forward rates shocked by stochastic strings and show how to price interest rate derivatives. Although derivatives can be easily priced, they can be perfectly hedged only by trading in an infinite number of bonds of all maturities. We show that the strings model is consistent with any panel dataset of bond prices, and does not require the addition of error terms in econometric models. Finally, we empirically calibrate some versions of the model and price the delivery option embedded in long bond futures. We show that the delivery option is much more valuable in string models than in a similar one-factor model. This is due to the greater variety of shapes of the term structure that string models can produce, which induces more changes in the cheapest-to-deliver bond.

In this article we develop a new class of bond pricing models. Our model is as parsimonious and tractable as the traditional Heath, Jarrow, and Morton (1992; hereafter HJM) model, but is capable of generating a much richer class of dynamics and shapes of the term structure of interest rates. Our main innovation consists in having each instantaneous forward rate driven by its own shock, while constraining these shocks in such a way as to keep the forward

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curve continuous.¹ This shock to the curve is termed a “stochastic string,” following the physical analogy of a string whose shape changes stochastically through time.

We provide a detailed treatment of stochastic strings and their stochastic calculus. The framework we use is that of stochastic partial differential equations (SPDEs) and our main tool is the calculus of Dirac distributions. We attempt to present the results and their derivations in the simplest and most intuitive way, rather than emphasize mathematical rigor. We offer several parametric examples of stochastic strings that produce varied correlation structures of forward rates of different maturities in a parsimonious manner.

We show that there is a no-arbitrage constraint on the drift of forward rates that generalizes the no-arbitrage constraint of HJM. This condition fully specifies the drifts of forward rates under the risk-neutral probability measure. Derivatives prices can be expressed as expectations of their discounted payoffs under this measure.

Kennedy (1994, 1997) and, in recent independent work, Goldstein (2000), propose a similar approach to modeling forward rates. Kennedy (1997) simply models the forward rate curve as a Gaussian random field. Goldstein (2000) uses a model similar to ours, letting the forward rate curve be shocked by two of the strings that we analyze as special examples of our framework.

Using stochastic strings as the noise source for the dynamics of the forward curve has important economic advantages over previous approaches. In existing term structure models, the same set of shocks affects all forward rates. This feature constrains the correlations between bond prices and therefore the set of admissible shapes and dynamics of the yield curve. In term structure models with state variables [e.g., Vasicek (1977), Cox, Ingersoll, and Ross, (1985; hereafter CIR)], obtaining complex shapes of the forward curve and realistic correlations between bonds of different maturities requires the introduction of a large number of state variables. This makes the models very unparsimonious and virtually impossible to estimate. Similar problems affect the traditional HJM model, where fitting the initial term structure imposes strong constraints on the shapes of the forward rate curve in the future, which are in general not verified as the forward curve evolves through time.

In contrast to these models, in our approach, any (finite) set of bond prices is imperfectly correlated. The model is thus fully compatible with any given panel dataset of bond prices. Or to put it differently, any (finite) set of bond prices observed at some (finite) sampling frequency is consistent with the model. There is thus no need to add observation noise when estimating the model.

¹ We present our model in terms of the dynamics of instantaneous forward rates, but could as well use strings to shock bond prices, futures prices or coupon bond rates. Indeed, it is often convenient to model these instead of instantaneous forward rates. For example, Longstaff, Santa-Clara and Schwartz (1999 and 2000) offer string models of forward swap rates and forward Libor rates and show how they can be easily calibrated to fit cap and swaption volatilities.

Another feature of our approach is that, in general, it is necessary to use a portfolio with an infinite number of bonds to replicate interest rate contingent claims. However, the pricing of derivatives remains simple: interest rate options can, in general, be priced by simulation,² and, in some cases, in closed form. As with HJM, our approach does not allow the formulation of a partial differential equation to price derivatives.

Finally, we use our model to price the delivery option embedded in the long bond futures contract. We start by calibrating different parameterizations of the strings model, as well as a simple one-factor HJM model, to volatilities and correlations of bond prices. We then price the long bond futures contract and compare the value of the delivery option obtained in the different parameterizations. We find that string specifications imply a much greater value for the delivery option than the one-factor model, due to the greater variability in the choice of the cheapest-to-deliver bond at the maturity of the futures contract. This increased variability arises from the richer variety of shapes of the term structure of forward rates that the string models can produce.

The article is organized as follows. In the next section we define the primitives of our approach and solve a simple model where forward rates are driven by Brownian motion. We use this model as a benchmark to compare the models driven by stochastic strings. Section 2 defines string shocks and their properties, and offers the general no-arbitrage condition for forward rate dynamics driven by stochastic strings. Section 3 discusses the construction of stochastic strings as solutions of stochastic partial differential equations (SPDEs). In Section 4 we present a collection of examples of stochastic strings with interesting properties. Section 5 contains an empirical application of string models to the pricing of the long bond futures contract and its embedded delivery option. Section 6 concludes.

1. The Traditional Model: Brownian Motion as Noise Source

In this section we introduce a simple model of forward rate dynamics driven by Brownian motion. We solve this model both for completeness and as a benchmark against which to check the string models presented in later sections.

We postulate the existence of a stochastic discount factor (SDF) that prices all assets in this economy, and denote it by M .³ This SDF can be thought of as the nominal, intertemporal, marginal rate of substitution for consumption of a representative agent in an exchange economy. Under an adequate definition

² See, for example, Longstaff and Schwartz (2001) and Longstaff, Santa-Clara and Schwartz (1999), who use string models to price American swaptions by simulation.

³ This process is also termed the pricing kernel, the pricing operator, or the state price density. We use these terms interchangeably. See Duffie (1996) for the theory behind SDFs.

of the space of admissible trading strategies, for no arbitrage opportunities to exist, the product of the SDF with the value process of any admissible self-financing trading strategy V must be a martingale. Then

$$V(t) = E_t \left[V(s) \frac{M(s)}{M(t)} \right], \quad (1)$$

where s is a future date and $E_t[x]$ denotes the mathematical expectation of x taken at time t . In particular, we require that the prices of a bank account and of zero-coupon discount bonds of all maturities satisfy this condition.

We assume that a bank account exists in the economy, such that the value at time t of an initial investment of $B(0)$ that is continuously reinvested is given by

$$B(t) = B(0) \exp \left\{ \int_0^t ds r(s) \right\}, \quad (2)$$

where $r(t)$ is the instantaneous nominal interest rate.

We further assume that, at any time t , riskless discount bonds of all maturity dates s trade in this economy and let $P(t, s)$ denote the time t price of the s maturity bond. We require that $P(s, s) = 1$, that $P(t, s) > 0$, and that $\partial P(t, s)/\partial s$ exists.

Instantaneous forward rates at time t for all times to maturity $x > 0$, $f(t, x)$, are defined by

$$f(t, x) = - \frac{\partial \log P(t, t+x)}{\partial x}, \quad (3)$$

which is the rate that can be contracted at time t for instantaneous borrowing or lending at time $t+x$. We require that the initial forward curve $f(0, x)$, for all x is continuous.

Equivalently, from the knowledge of the instantaneous forward rates for all times to maturity between 0 and time $s-t$, the price at time t of a bond with maturity s can be obtained by

$$P(t, s) = \exp \left\{ - \int_0^{s-t} dx f(t, x) \right\}. \quad (4)$$

Forward rates thus fully represent the information in the prices of all zero-coupon bonds.

The spot interest rate at time t , $r(t)$, is the instantaneous forward rate at time t with time to maturity 0,

$$r(t) = f(t, 0). \quad (5)$$

We model forward rates with a fixed time to maturity rather than a fixed maturity date.⁴ Modeling forward rates with a fixed time to maturity is more natural for describing the dynamics of the entire forward curve as the shape of a string evolving in time. In contrast, in HJM, forward rate processes disappear as time reaches their maturities. Note, however, that we still impose the martingale condition on bonds with a fixed maturity date, since these are the financial instruments that are actually traded.

We model the dynamics of all forward rates as Itô processes,⁵

$$df(t, x) = \alpha(t, x)dt + \sigma(t, x)dW(t), \quad (6)$$

or, in integral form,

$$f(t, x) = f(0, x) + \int_0^t dv \alpha(v, x) + \int_0^t dW(v) \sigma(v, x), \quad (7)$$

where W is a Brownian motion.

From the definition of bond prices, we obtain for fixed s

$$d \log P(t, s) = f(t, s - t) dt - \int_0^{s-t} dy df(t, y). \quad (8)$$

Using Itô's lemma, we calculate the dynamics of bond prices as

$$\begin{aligned} \frac{dP(t, s)}{P(t, s)} = & \left[f(t, s - t) - \int_0^{s-t} dy \alpha(t, y) \right. \\ & \left. + \frac{1}{2} \left(\int_0^{s-t} dy \sigma(t, y) \right)^2 \right] dt - \left(\int_0^{s-t} dy \sigma(t, y) \right) dW(t). \end{aligned} \quad (9)$$

Finally, we assume the following dynamics for the SDF

$$\frac{dM(t)}{M(t)} = -r(t) dt - \phi(t) dW(t). \quad (10)$$

The drift of M is justified by the well-known martingale condition on the product of the bank account and the SDF. The process ϕ denotes the market price of risk, as measured by the covariance of asset returns with the SDF.

⁴ In contrast, HJM model forward rates with a fixed maturity date. Using a "hat" to denote the forward rates modeled by HJM,

$$\hat{f}(t, s) = f(t, s - t),$$

or equivalently,

$$f(t, x) = \hat{f}(t, t + x)$$

for fixed s or x . Musiela (1993) and Brace and Musiela (1994) define forward rates in the same fashion. Miltersen, Sandmann, and Sondermann (1997), and Brace, Gatarek, and Musiela (1997) use definitions of forward rates similar to ours, albeit for noninstantaneous forward rates.

⁵ We assume any required regularity conditions for the existence of the processes [see Duffie (1996)]. The extension to several Brownian motions is trivial along the lines of HJM.

To put in another way, ϕ is the excess return over the spot interest rate that assets must earn per unit of covariance with W .

The no-arbitrage condition for buying and holding bonds requires that PM is a martingale in time, for any bond price P . Technically this amounts to imposing that the drift of PM be zero,

$$\begin{aligned} -r(t) + f(t, s-t) - \int_0^{s-t} dy \alpha(t, y) + \frac{1}{2} \left(\int_0^{s-t} dy \sigma(t, y) \right)^2 \\ + \phi(t) \left(\int_0^{s-t} dy \sigma(t, y) \right) = 0. \end{aligned} \quad (11)$$

Equivalently, for all t and s , with $x \equiv s - t$,

$$\begin{aligned} f(t, x) = r(t) + \int_0^x dy \alpha(t, y) - \frac{1}{2} \left(\int_0^x dy \sigma(t, y) \right)^2 \\ - \phi(t) \left(\int_0^x dy \sigma(t, y) \right). \end{aligned} \quad (12)$$

We can differentiate this no-arbitrage condition with respect to x and obtain

$$\begin{aligned} \alpha(t, x) = \frac{\partial}{\partial x} \left[f(t, x) + \frac{1}{2} \left(\int_0^x dy \sigma(t, y) \right)^2 \right. \\ \left. + \phi(t) \left(\int_0^x dy \sigma(t, y) \right) \right], \end{aligned} \quad (13)$$

or

$$\alpha(t, x) = \frac{\partial f(t, x)}{\partial x} + \sigma(t, x) \left(\int_0^x dy \sigma(t, y) + \phi(t) \right), \quad (14)$$

where $\partial f(t, x)/\partial x$ is the slope of the forward curve at time t for time to maturity x . This no-arbitrage condition clearly shows that the diffusion function σ must go to zero when the maturity date goes to infinity, in order to ensure the finiteness of the drift of forward rates [See Dybvig, Ingersoll, and Ross (1996) and Jeffrey (1997)]. The term $\partial f(t, x)/\partial x$ comes from the parameterization in terms of time to maturity, and is absent in the usual HJM formulation of forward rates with fixed maturity date. However, we stress that the two formulations are equivalent.

Replacing Equation (14) in Equation (6), we obtain the following arbitrage-free model of forward rate dynamics,

$$df(t, x) = \left(\frac{\partial f(t, x)}{\partial x} + a(t, x) \right) dt + \sigma(t, x) dW(t), \quad (15)$$

where

$$a(t, x) \equiv \sigma(t, x) \left(\int_0^x dy \sigma(t, y) + \phi(t) \right). \quad (16)$$

This condition can also be found in HJM, and with the time-to-maturity formulation in Musiela (1993)⁶ and Brace and Musiela (1994).

We see that estimating the drift and volatility functions of forward rates, along with the slope of the forward rate curve at some date t , recovers the market price of risk $\phi(t)$ on that date. This shows that the use of the SDF in the derivation of the no-arbitrage condition is just a useful presentation device. The no-arbitrage condition for a single forward rate allows us to recover the drifts of all other forward rates from the knowledge of their volatilities.

In Appendix A, we solve Equation (15) by first rewriting it as

$$\frac{\partial f(t, x)}{\partial t} - \frac{\partial f(t, x)}{\partial x} = a(t, x) + \sigma(t, x) \eta(t), \quad (17)$$

where η is a white noise process such that, when integrated over time from 0 to t , we obtain $\int_0^t dv \eta(v) = W(t)$.⁷ We show that the solution of Equation (17) is

$$\begin{aligned} f(t, x) &= f(0, t+x) + \int_0^t dv a(v, t+x-v) \\ &\quad + \int_0^t dW(v) \sigma(v, t+x-v). \end{aligned} \quad (18)$$

Notice that if $\sigma(t, x) = 0$, for all t and x , we recover the deterministic no-arbitrage condition $f(t, x) = f(0, t+x)$ which states that the instantaneous forward rate is the same at time 0 and at time t for the same maturity date $t+x$.

2. A Better Model: Stochastic String Shocks

In this section we develop the general model of the forward rate curve driven by stochastic strings. Instead of the traditional model of Equation (6), we now model the dynamics of the forward rates by

$$d_t f(t, x) = \alpha(t, x) dt + \sigma(t, x) d_t Z(t, x), \quad (19)$$

⁶ Musiela (1993) terms this formulation an SPDE for the term structure since the formulation in terms of time to maturity introduces the partial derivative of the forward curve with respect to x . Our use of SPDEs in the next two sections is very different.

⁷ Section 3 below elaborates on this construction of Brownian motion.

or, in integral form, by

$$f(t, x) = f(0, x) + \int_0^t dv \alpha(v, x) + \int_0^t d_v Z(v, x) \sigma(v, x), \quad (20)$$

where the stochastic process $Z(t, x)$ generalizes to two dimensions the previous one-dimensional Brownian motion $W(t)$. The important innovation in Equation (19) is that the stochastic process depends not only on time t but also on time to maturity x . The notation $d_t Z(t, x)$ denotes a stochastic perturbation to the forward rate curve at time t , with different magnitudes for forward rates with different times to maturity. The subscript t in the differential operator means that the increment is taken with respect to time. It is straightforward to extend the model to include more than one string shock or to combine string shocks with Brownian motion shocks.

We stress that Equation (19) is not the infinite-dimensional generalization of the multifactor HJM model, in which all forward rates are subject to the same (possibly infinite) set of Brownian motion shocks. In our model, there is one different stochastic shock for each time to maturity.

We impose several requirements on Z to qualify as a string shock to the forward rate curve:

1. $Z(t, x)$ is continuous in x at all times t .
2. $Z(t, x)$ is continuous in t for all x .
3. The string is a martingale in time t , $E[d_t Z(t, x)] = 0$, for all x .
4. The variance of the increments is equal to the time change, $\text{var}[d_t Z(t, x)] = dt$, for all x .
5. The correlation of the increments, $\text{corr}[d_t Z(t, x), d_t Z(t, y)]$, does not depend on t .

We will see that the first two conditions are automatically satisfied by taking Z to be the solution of an SPDE with at least one partial derivative in x and t . Condition 3 captures the unforecastable aspect of shocks. Condition 4 makes all shocks affecting the forward curve have the same intensity. This intensity can then be changed, or “modulated,” by the diffusion coefficient $\sigma(t, x)$ taking different values at different times to maturity x . The final condition ensures that the correlation between shocks to forward rates of different maturities depends only on these maturities. Conditions 3, 4, and 5 together make the strings Markovian. We will see that all string shocks produced as solutions of SPDEs have Gaussian distributions, which are completely characterized by the first two moments.

To have each forward rate for each time to maturity be driven by its own noise process, we could simply use infinite-dimensional Brownian motion, using a simple Brownian motion as a shock to each forward rate, independent from the Brownian motion used to perturb forward rates with different

times to maturity. However, such a model makes the forward rate curve discontinuous as a function of time to maturity so that, over time, any two forward rates can become arbitrarily distant from each other. Although a discontinuous forward curve does not violate any arbitrage conditions, it is intuitively unlikely.

We develop the model of forward rate dynamics assuming the existence of a string shock that satisfies our requirements. In the next sections we show how such string shocks can be constructed and examine several parametric examples.

We now take the process for the SDF as given by

$$\frac{dM(t)}{M(t)} = -r(t) dt - \int_0^\infty dy \phi(t, y) d_t Z(t, y), \quad (21)$$

where the pricing kernel is driven by a “weighted sum” of the shocks that affect the forward rate curve, and the market prices of risk can in principle be different for each shock to the curve.

To derive the no-arbitrage condition on the drift of forward rates, we follow the steps in Section 1. We have

$$\begin{aligned} d_t y(t, s) \equiv d_t \log P(t, s) &= \left[f(t, s-t) - \int_0^{s-t} dy \alpha(t, y) \right] dt \\ &\quad - \int_0^{s-t} dy \sigma(t, y) d_t Z(t, y). \end{aligned} \quad (22)$$

We need the dynamics for bond prices, which is obtained from Equation (22) using Itô's calculus. In order to get Itô's term in the drift, recall that it results from the fact that, with y stochastic,

$$d_t F(y(t, s)) = \frac{\partial F(y(t, s))}{\partial y(t, s)} d_t y(t, s) + \frac{1}{2} \frac{\partial^2 F(y(t, s))}{\partial y(t, s)^2} \text{var}[d_t y(t, s)], \quad (23)$$

so that the differential $d_t F(y(t, s))$ up to order dt contains contributions from all the covariances of the form $\text{cov}[d_t Z(t, x), d_t Z(t, y)]$ appearing in $\text{var}[d_t y(t, s)]$.

Taking the expectation of the square of the stochastic term in the right-hand side of Equation (22), the bond price dynamics can be written as

$$\begin{aligned} \frac{dP(t, s)}{P(t, s)} &= \left[f(t, s-t) - \int_0^{s-t} dy \alpha(t, y) + \frac{1}{2} \int_0^{s-t} dy \int_0^{s-t} dx c(x, y) \sigma(t, x) \sigma(t, y) \right] dt \\ &\quad - \int_0^{s-t} dy \sigma(t, y) d_t Z(t, y), \end{aligned} \quad (24)$$

where

$$c(x, y) \equiv \text{corr} [d_t Z(t, x), d_t Z(t, y)], \quad (25)$$

and we make use of $\text{var}[d_t Z(t, x)] = dt$. In the next sections we offer several examples of stochastic strings and the correlation functions of their time increments.

The no-arbitrage condition becomes

$$\begin{aligned} -r(t) + f(t, x) - \int_0^x dy \alpha(t, y) + \frac{1}{2} \int_0^x dy \int_0^x dz c(y, z) \sigma(t, y) \sigma(t, z) \\ + \int_0^\infty dy \int_0^x dz c(y, z) \sigma(t, z) \phi(t, y) = 0, \end{aligned} \quad (26)$$

which we can differentiate with respect to x ⁸ to obtain

$$\begin{aligned} \alpha(t, x) = \frac{\partial f(t, x)}{\partial x} + \sigma(t, x) \left(\int_0^x dy c(x, y) \sigma(t, y) \right. \\ \left. + \int_0^\infty dy c(x, y) \phi(t, y) \right). \end{aligned} \quad (27)$$

If $c(x, y)$ in Equation (27) is unity, we recover the traditional HJM model of a single Brownian motion in Equation (15).

We thus get the following arbitrage-free model of forward rate dynamics:

$$\begin{aligned} d_t f(t, x) = \left[\frac{\partial f(t, x)}{\partial x} + \sigma(t, x) \left(\int_0^x dy c(x, y) \sigma(t, y) \right. \right. \\ \left. \left. + \int_0^\infty dy c(x, y) \phi(t, y) \right) \right] dt + \sigma(t, x) d_t Z(t, x). \end{aligned} \quad (28)$$

Equation (28) can be solved in a similar way as in Section 1. We denote

$$A(t, x) \equiv \sigma(t, x) \left(\int_0^x dy c(x, y) \sigma(t, y) + \int_0^\infty dy c(x, y) \phi(t, y) \right) \quad (29)$$

⁸ In principle, differentiating Equation (26) is only warranted if $f(t, x)$ is differentiable in x . This is the case for some strings constructed below, such as the integrated O-U sheet, but is not true for others, such as the O-U sheet. In these cases $\partial f(t, x)/\partial x$ is not a function, but is similar to an increment of a Brownian motion. However, the results below still hold, by reading $\partial f(t, x)/\partial x$ as a convenient notation for the formal integral calculus used.

to finally obtain

$$f(t, x) = f(0, t + x) + \int_0^t dv A(v, t + x - v) + \int_0^t d_v Z(v, t + x - v) \sigma(v, t + x - v). \quad (30)$$

We can still “invert” the forward curve to obtain the market prices of risk needed for pricing derivatives. However, we now need to use the entire forward rate curve rather than a small number of forward rates (equal to the number of Brownian motions driving the curve in the traditional model). Note that if we assume that $\phi(t, x) = \phi(t)$, thus constraining the SDF of Equation (21), the “inversion” of the forward rate curve to extract the market prices of risk becomes much easier.

2.1 Empirical implications

Traditional factor models of the term structure are only compatible with samples that include, at most, as many bonds as there are factors in the model. In order to be able to use data on more bond prices to estimate these models it is necessary to add error terms to their econometric specification. For econometric tractability, these errors must be assumed to be independent from the factors [See Chen and Scott (1993) or Pearson and Sun (1994)]. This approach is correct if the incompatibility of the data with the model is indeed due to observation error, possibly due to bid-ask spreads or nonsynchronous observations. However, if the incompatibility is due to a misspecified model, the error terms will not be independent of the factors and the econometric model will be inappropriate. Models with stochastic string shocks can be estimated without any such econometric error terms. The models are compatible with any sample of forward rates (or bond prices) of finitely spaced maturities taken at some finite sampling interval. This is so because, for any parametric specification, there is always a possible path for the string shock over a finite interval that can lead from the forward curve at the beginning of the interval to the forward curve at the end of the interval. This realization may be highly unlikely, but it is always possible. The estimation exercise is thus one of finding the most likely parameters, given a set of movements of the forward curve over time.

In addition, we note that our model is as parsimonious as the traditional model. In Section 4 we will show a number of parametric examples of string shocks with interesting properties that require a single parameter to specify the correlation function.

Our model can (at least approximately) fit the entire correlation structure of forward rate increments, which is very difficult to do in the traditional

model. Note that if we extend the simple HJM model of Equation (6) to include N independent Brownian motions,

$$df(t, x) = \alpha(t, x)dt + \sum_{i=1}^N \sigma_i(t, x)dW_i(t), \quad (31)$$

we obtain the correlation function of forward rate increments equal to

$$c(x, y) = \frac{\sum_{i=1}^N \sigma_i(t, x)\sigma_i(t, y)}{\sqrt{\sum_{i=1}^N \sigma_i(t, x)^2} \sqrt{\sum_{i=1}^N \sigma_i(t, y)^2}}. \quad (32)$$

In this model it is very hard to parameterize the functions σ_i to fit both the volatilities and the correlations of forward rates, even with many Brownian motion shocks. In contrast, this is extremely easy in our model since we have separate functions σ and c to fit volatilities and the correlations of forward rates. The ability to separately fit volatilities and correlations greatly simplifies the calibration of the model to option prices, as shown by Longstaff, Santa-Clara, and Schwartz (1999, 2000).

2.2 Option pricing and replication

In our model, derivatives can in general be priced by simulation. We just need to simulate the dynamics of the forward curve under the risk-adjusted probability measure. Define a new string Z^* with dynamics

$$d_t Z^*(t, y) = d_t Z(t, y) + dt \int_0^\infty dy \, c(x, y)\phi(t, y). \quad (33)$$

Under the risk-adjusted measure, this string is a martingale, with the same correlation function as Z . The dynamics of forward rates under this probability measure do not depend on the market prices of risk,

$$\begin{aligned} d_t f(t, x) = & \left[\frac{\partial f(t, x)}{\partial x} + \sigma(t, x) \int_0^x dy \, c(x, y)\sigma(t, y) \right] dt \\ & + \sigma(t, x)d_t Z^*(t, x). \end{aligned} \quad (34)$$

Asset prices discounted at the bank account are also martingales under this probability measure. Thus the price at time t of a European contingent claim with payoff $\Phi(s)$ at time $s > t$ is

$$\Phi(t) = E_t^* \left[\Phi(s) e^{-\int_t^s du \, r(u)} \right]. \quad (35)$$

This expectation can be easily approximated with simulation methods, as illustrated in Section 5.

American options can also be priced by simulation using the approach of Longstaff and Schwartz (2001). This approach works well even with a high number of state variables, which makes it ideal to use for pricing American options in string models. Indeed, Longstaff and Schwartz (1998) and Longstaff, Santa-Clara, and Schwartz (1999) use this simulation approach to price American swaptions in a discretized string model with 20 correlated interest rates.

Unfortunately we cannot use the Feynman-Kac theorem to obtain a PDE for pricing contingent claims. In our model, as in HJM, there is no finite set of state variables that determine the value of interest rate derivatives. Derivatives' prices will in general depend on the full history of (string) shocks to the forward rate curve. This shows that to hedge them, it is necessary to use the full set of bonds of all maturities.

3. Construction of Stochastic String Shocks

This section shows how to construct stochastic string shocks that can be used to perturb the forward rate curve. We start by reviewing the construction of Brownian motion from a probabilistic point of view and then as the solution to a stochastic ordinary differential equation (SODE). This allows us to obtain a natural generalization of Brownian motion to strings as the solution of SPDEs. We then formulate the solutions to these SPDEs in terms of the Green function of the equation and find constraints on this function to impose our requirements for string shocks.

3.1 Brownian motion as solution of an SODE

Brownian motion W is typically defined in the probability literature [See Karatzas and Shreve (1991)] by

$$W(t) = \int_0^t dW(u), \quad (36)$$

with $W(0) = 0$, assumed normally distributed with

$$E[dW(t)] = 0, \quad (37)$$

and

$$\text{var}[dW(t)] = dt. \quad (38)$$

Alternatively, W can be defined as the solution of the SODE

$$\frac{dW(t)}{dt} = \eta(t), \quad (39)$$

with $W(0) = 0$, where η is white noise, characterized by the covariance

$$\text{cov}[\eta(t), \eta(s)] = \delta(s - t), \quad (40)$$

and δ designates the Dirac function.⁹

The solution of Equation (39) is formally

$$W(t) = \int_0^t dv \eta(v), \quad (41)$$

which shows that $dt \eta(t)$ is simply a notation for $dW(t)$, and, as usual, has rigorous mathematical meaning only under the integral representation. From Equation (41), we see by the central limit theorem that W is Gaussian, with mean zero, and covariance

$$\begin{aligned} \text{cov}[W(t), W(s)] &= \int_0^t dv \int_0^s du \text{cov}[\eta(u), \eta(v)] \\ &= \int_0^t dv \int_0^s du \delta(v - u) = (t \wedge s), \end{aligned} \quad (42)$$

where $(t \wedge s)$ stands for $\min(t, s)$.

Finally, we define $dW(t)$ as the limit of $[W(t + \Delta t) - W(t)]$ when the small but finite increment of time Δt becomes the infinitesimal dt . Using Equation (42), we get

$$\text{var}[W(t + \Delta t) - W(t)] = \Delta t, \quad (43)$$

which recovers Equation (38) in the infinitesimal time increment limit.

The definition of the Brownian motion as the solution of an SODE is very useful as a starting point to generate other processes. Maybe the simplest extension is the well-known Ornstein-Uhlenbeck (O-U) process $U(t)$, which can be defined as the solution of the following SODE,

$$\frac{dU(t)}{dt} = -\kappa U(t) + \eta(t), \quad (44)$$

where κ is a positive constant. In addition to the random shock η , the process is now subject to a restoring force, which tends to bring it back to the origin, so that there is mean reversion. The formal solution reads

$$U(t) = \int_0^t dv \eta(v) e^{-\kappa(t-v)}. \quad (45)$$

⁹ The Dirac function is a distribution in the sense of Schwartz's theory of distributions, or generalized functions. It is such that, for arbitrary $f(x)$, $\int_{x_1}^{x_2} f(x) \delta(x - x_0) = f(x_0)$ if $x_1 < x_0 < x_2$, $\frac{1}{2} f(x_0)$ if $x_1 = x_0$, or $x_2 = x_0$, and zero otherwise. $\delta(x)$ is obtained as the limit of functions when their width goes to zero and their height goes to infinity in such a way that their integral remains one. An example is the Gaussian density limit

$$\delta(x) = \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}.$$

Its covariance is

$$\text{cov}[U(t), U(s)] = \frac{1}{2\kappa} (e^{-\kappa|s-t|} - e^{-\kappa(t+s)}), \quad (46)$$

which goes to $\frac{1}{2\kappa}e^{-\kappa|s-t|}$ for large time intervals.¹⁰ By adding more terms in Equation (44), more complex stochastic processes can be easily generated.

3.2 Stochastic strings as solutions of SPDEs

We now extend the class of stochastic processes driving the uncertainty in the forward rate curve from solutions of SODEs to solutions of SPDEs. The introduction of partial differential equations is called for in order to account for the continuity condition and to deal with the two variables, time and time to maturity, and their interplay.

Brownian motion depicts, in physical terms, the motion of a particle subjected to random velocity variations. The analogous physical system described by SPDEs is a stochastic string. In our application of stochastic strings as the noise source in the dynamics of the forward rate curve, the length along the string is the time to maturity, and the string configuration gives the amplitude of the shocks at a given time for each time to maturity.

In the present article, we restrict our attention to linear SPDEs, in which the highest derivative is, in most cases, second order. This second-order derivative has a simple physical interpretation: the string is subjected to a tension, like a piano chord, that tends to bring it back to zero deformation. This tension forces the “coupling” among different times to maturity so that the forward rate curve remains continuous.

The general form of second-order SPDEs reads

$$\begin{aligned} & A(t, x) \frac{\partial^2 X(t, x)}{\partial t^2} + 2B(t, x) \frac{\partial^2 X(t, x)}{\partial t \partial x} + C(t, x) \frac{\partial^2 X(t, x)}{\partial x^2} \\ &= H\left(t, x, X(t, x), \frac{\partial X(t, x)}{\partial t}, \frac{\partial X(t, x)}{\partial x}\right), \end{aligned} \quad (47)$$

where X is the stochastic string shock we want to characterize. We reserve the notation Z for string shocks that satisfy the requirements of Section 2. In the present article, we restrict our discussion to *linear* equations, where H has the form

$$\begin{aligned} & H\left(t, x, X(t, x), \frac{\partial X(t, x)}{\partial t}, \frac{\partial X(t, x)}{\partial x}\right) \\ &= D(t, x) \frac{\partial X(t, x)}{\partial t} + E(t, x) \frac{\partial X(t, x)}{\partial x} \\ &+ F(t, x) X(t, x) + S(t, x) \end{aligned} \quad (48)$$

¹⁰ Using $2(t \wedge s) = t + s - |s - t|$.

and S is the random “source” term. Equation (47) together with Equation (48) is the most general linear second-order SPDE in two variables. The solution X exists and its uniqueness is assured once “boundary” conditions are given, such as, for instance, the initial value of the string $X(0, x)$, for all x [see Morse and Feshbach (1953)].

The solutions to second-order linear SPDEs have the important property of being representable as an integral of a (Green) function. In the next subsection we exploit this formulation to find the constraints on the SPDE that ensure that the properties for stochastic string shocks listed in Section 2 are verified.

3.3 Green function formulation

All solutions of the above linear SPDE can be characterized by

$$X(t, x) = X(0, x) + \int_0^t dv \int_{-\infty}^{\infty} dz G(t, x|v, z) S(v, z), \quad (49)$$

where the Green function G contains all the information in the underlying SPDE [see Morse and Feshbach (1953)]. Consider a single impulse source term $S(t, x) = \delta(t - t_0)\delta(x - x_0)$. Then,

$$X(t, x) = X(0, x) + G(t, x|t_0, x_0). \quad (50)$$

The Green function G thus describes the deterministic evolution in the future of the process X at all times to maturity due to an impulsive perturbation that occurs at time t_0 at the time to maturity x_0 .

We now take the source term to be white noise, denoted by η , and characterized by the covariance

$$\text{cov}[\eta(t, x), \eta(s, y)] = \delta(s - t) \delta(y - x), \quad (51)$$

where δ denotes, as before, the Dirac distribution. Then, from Equation (49), the increment of the string is

$$\begin{aligned} dX(t, x) = & dt \int_{-\infty}^{\infty} dz G(t, x|t, z) \eta(t, z) \\ & + dt \int_0^t dv \int_{-\infty}^{\infty} dz \frac{\partial G(t, x|v, z)}{\partial t} \eta(v, z). \end{aligned} \quad (52)$$

In order for X to be a martingale in time, we need Equation (52) to depend only on innovations at time t . We thus need the second term in Equation (52) to be zero, which corresponds to the Green function not depending on time t . We thus write the Green function as $h(v, x, z)$.

Then the increment of the string can be written as

$$dX(t, x) = dt \int_{-\infty}^{\infty} dz h(t, x, z) \eta(t, z). \quad (53)$$

The covariance of the increments is simply

$$\text{cov}[dX(t, x), dX(t, y)] = dt \int_{-\infty}^{\infty} dz h(t, x, z) h(t, y, z), \quad (54)$$

and, in particular, the variance of the string shock is

$$\text{var}[dX(t, x)] = dt \int_{-\infty}^{\infty} dz [h(t, x, z)]^2. \quad (55)$$

In order for the variance in Equation (55) not to depend on time t , we further restrict the Green function. We now require the Green function not to depend on v , and write it as $g(x, z)$.

We therefore write the string shocks that satisfy all the requirements of Section 2 as

$$Z(t, x) = Z(0, x) + \int_0^t dv \int_0^{\infty} dz g(x, z) \eta(v, z), \quad (56)$$

or, in differential notation,

$$d_t Z(t, x) = dt \int_0^{\infty} dz g(x, z) \eta(t, z), \quad (57)$$

where we set the lower limit of integration to zero because time to maturity is always positive. In order for the variance of the increments to be dt for all x , we impose that

$$\int_0^{\infty} dz [g(x, z)]^2 = 1. \quad (58)$$

We denote the string by Z instead of X since it now satisfies our requirements for an admissible string shock.

We see that the increments of the string shock have mean zero, since they are the sum of random variables η with mean zero, and are normally distributed, by the Central Limit theorem since the underlying random variables η are independent. From Equations (57) and (58), the correlation of the increments of the string shock is

$$c(x, y) = \int_0^{\infty} dz g(x, z) g(y, z). \quad (59)$$

In Appendix B we verify that the correlation defined by Equation (59) is symmetric, $c(x, y) = c(y, x)$, that it has values between minus one and one, $-1 \leq c(x, y) \leq 1$, that it has value one when y is equal to x ,

$c(x, x) = 1$, and that it is positive semidefinite. Thus string shocks constructed with Equation (56), with the constraint of Equation (58), produce admissible correlation functions.

Most importantly, we also show in Appendix B that any function $c(x, y)$ that has the above four properties, and is therefore a correlation function, can be written in the form of Equation (59). Our string formulation is therefore completely general in the sense that it can produce any admissible correlation function for the increments of the shocks. We can then parameterize the forward rate dynamics of Equation (28) by formulating directly a correlation function of the string shock and be sure of having an admissible model. However, care must be taken to ensure that the function used in the model is positive definite. Unfortunately this is not always easy to check, and we show in the appendix a simple example of a function that seems “reasonable” but is not an admissible correlation function, since it is not positive semidefinite.

A simple formulation that encompasses many of the parametric examples given in the next section is

$$Z(t, x) = Z(0, x) + \int_0^t dv \int_0^{j(x)} dz \frac{1}{\sqrt{j(x)}} \eta(v, z), \quad (60)$$

or, in differential notation,

$$d_t Z(t, x) = dt \int_0^{j(x)} dz \frac{1}{\sqrt{j(x)}} \eta(t, z), \quad (61)$$

where j is an arbitrary continuous function. We thus obtain

$$\text{var} [d_t Z(t, x)] = dt, \quad (62)$$

and the correlation of the increments is

$$c(x, y) \equiv \text{corr} [d_t Z(t, x), d_t Z(t, y)] = \sqrt{\frac{j(x) \wedge j(y)}{j(x) \vee j(y)}}. \quad (63)$$

4. Parametric Examples

In this section we examine several examples of stochastic strings that are solutions of SPDEs and consider their application as a noise source in a model of the dynamics of the forward rate curve.

4.1 The Brownian sheet

We start by discussing the simplest example of an SPDE. This example does not satisfy our criteria for admissible string shocks, but is nonetheless useful as a basis for the derivation of other string processes.

The simplest SPDE in the class given by Equations (47) and (48) is

$$\frac{\partial^2 W(t, x)}{\partial t \partial x} = \eta(t, x), \quad (64)$$

where $\eta(t, x)$ is white noise both in time and time to maturity, characterized by the covariance function of Equation (51). By inspection of Equation (64), the first-order time derivative ensures that W will be a Brownian motion in time at fixed x , while the introduction of the partial derivative with respect to x ensures the continuity of W with respect to x . The solution of Equation (64) reads

$$W(t, x) = \int_0^x dy \int_0^t dv \eta(v, y). \quad (65)$$

This process in Equation (65) is known as the Brownian sheet [see Walsh (1986)] and has the following covariance, which can be readily obtained using Equation (51) with Equation (65),

$$\text{cov}[W(t, x), W(s, y)] = (t \wedge s)(x \wedge y). \quad (66)$$

For the calculus used above in the implementation of the no-arbitrage condition on forward rate dynamics, we need the correlation function of the increments of the string. The covariance is the limit when $\Delta t \rightarrow dt$ of $\text{cov}[W(t + \Delta t, x) - W(t, x), W(t + \Delta t, y) - W(t, y)]$. Its calculation reduces to that of four terms of the form in Equation (66) that simplify to

$$\text{cov}[d_t W(t, x), d_t W(t, y)] = (x \wedge y) dt, \quad (67)$$

or stated in terms of the correlation,

$$c(x, y) \equiv \text{corr}[d_t W(t, x), d_t W(t, y)] = \frac{(x \wedge y)}{\sqrt{x} y}. \quad (68)$$

Note that the variance of $d_t W(t, x)$ is $x dt$, which makes this process unsatisfactory as a stochastic perturbation in Equation (19). In particular, the instantaneous spot rate ($x = 0$) would have no volatility in this model.

4.2 The modified Brownian sheet

A simple modification of the Brownian sheet can be obtained by making $j(x) = x$ in Equation (61), which leads to

$$d_t \widehat{W}(t, x) = dt \frac{1}{\sqrt{x}} \int_0^x dy \eta(t, y). \quad (69)$$

This process is just like the Brownian sheet, but it is rescaled by the factor $\frac{1}{\sqrt{x}}$ to ensure homogeneity in x . This form could have been guessed directly

from Equation (68) of the correlation of the Brownian sheet. The correlation function of the modified Brownian sheet is

$$c(x, y) \equiv \text{corr}[d_t \widehat{W}(t, x), d_t \widehat{W}(t, y)] = \sqrt{\frac{x \wedge y}{x \vee y}}. \quad (70)$$

4.3 The Ornstein-Uhlenbeck sheet

A natural model for a stochastic string is one similar to the Brownian sheet but in which the variance is the same for all times to maturity. We look for a string with the same variance for all x and covariance between x and y that depends only on the difference $(x - y)$.

The O-U string satisfies Equation (60) with $j(x) = e^{2\kappa x}$, which leads to¹¹

$$U(t, x) = e^{-\kappa x} \int_0^{e^{2\kappa x}} dy \int_0^t dv \eta(v, y). \quad (71)$$

We can see that the Green function that corresponds to the O-U sheet is

$$g(x, z) = \begin{cases} \sqrt{2\kappa} e^{-\kappa(x-z)} & \text{if } z \leq x \\ 0 & \text{otherwise.} \end{cases} \quad (72)$$

By inspection of Equation (71), we get the following properties,

$$\text{cov}[U(t, x), U(s, s)] = (t \wedge s) e^{-\kappa|x-y|} \quad (73)$$

and

$$c(x, y) \equiv \text{corr}[d_t U(t, x), d_t U(t, y)] = e^{-\kappa|x-y|}. \quad (74)$$

The correlations along time to maturity are strongest between close times to maturity and vanish exponentially for times to maturity distant from each other. The parameter κ allows us to span a set of strings, from a single Brownian motion to an infinity of uncorrelated Brownian motions. Indeed, for $\kappa \rightarrow 0$, all times to maturity are so strongly coupled that they all consist of the same single Brownian motion. This limit $\kappa \rightarrow 0$ thus recovers the single-factor model. Figure 1 shows an example of the increment in the O-U sheet for different values of κ . We see that as κ decreases the shape of the string shock has fewer humps.

This process has been used by Kennedy (1997) and Goldstein (2000) to model forward rates. Our approach clarifies the special role played by this process and its relationship with SPDEs.

¹¹ Alternatively, we can construct the O-U sheet from the Brownian sheet

$$U(t, x) = e^{-\kappa x} W(t, e^{2\kappa x}).$$

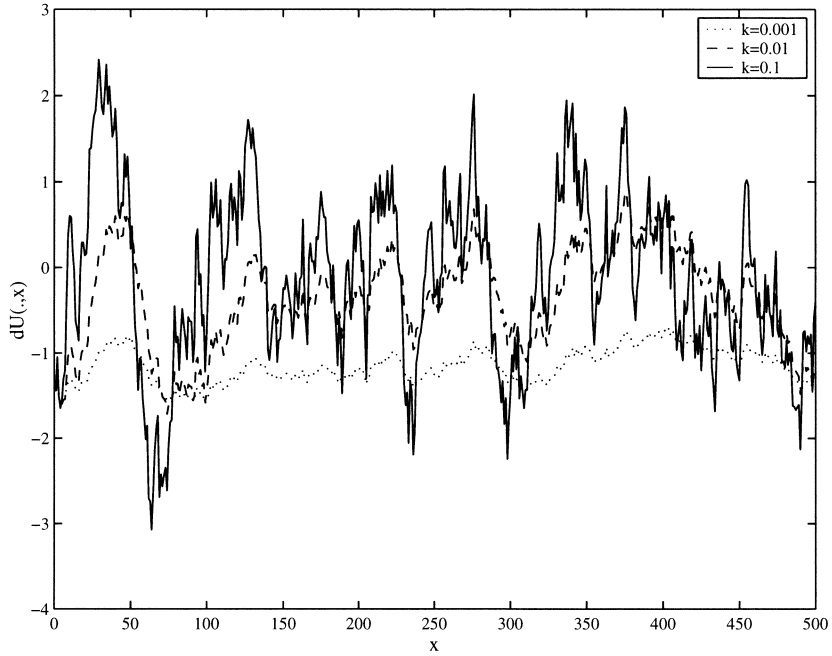


Figure 1

Sample increment to the O-U sheet, with correlation function of Equation (74) for different values of the parameter κ . The time increment is taken to be 1, measured in the same units of time to maturity.

4.4 String shocks differentiable in time-to-maturity

The processes studied above are continuous in t and x and are not differentiable in either t or x . Of special interest is the nondifferentiability in x , and we would now like to discuss a class of processes which are nondifferentiable in time but differentiable in time to maturity.¹²

Economics has little to say with respect to the differentiability of forward rate curves, and unfortunately this issue cannot be resolved empirically. Note that the differentiability of the forward curve in x relates to the smoothness of the forward curve between two very close maturities x and y . Therefore in order to determine whether the forward curve is differentiable we would need to observe instantaneous forward rates that are contiguous. Unfortunately such forward rates cannot be observed, since only discretely spaced rates can be measured with market prices. The decision to use string shocks that produce differentiable curves is thus fundamentally a matter of taste.

¹² The traditional HJM models provide obvious examples of stochastic processes that are nondifferentiable in time and differentiable in x , but they are a degenerate case as the same stochastic process drives all forward rates.

An intuitive strategy to obtain a shock that is differentiable in time to maturity is to integrate any of the previously defined strings in x ,¹³

$$Y(t, x) = \int_0^x dy Z(t, y). \quad (75)$$

By definition, $\partial Y(t, x)/\partial x = Z(t, x)$. Since $Z(t, x)$ is continuous in x , $\partial Y(t, x)/\partial x$ is also continuous in x and $Y(t, x)$ is thus differentiable with respect to x . Since $Z(t, y)$ satisfies an SPDE, we see that by replacing $Z(t, y)$ with $\partial Y(t, x)/\partial x$ in that SPDE, $Y(t, x)$ is also the solution of an SPDE.

As an interesting example, introduce the “integrated O-U sheet,” defined by

$$V_1(t, x) = \int_0^x dy U(t, y), \quad (76)$$

where $U(t, y)$ is the O-U sheet defined in Equation (71). By definition, $\partial V_1(t, x)/\partial x = U(t, x)$. Since $U(t, x)$ is continuous in x , $\partial V_1(t, x)/\partial x$ is also continuous in x , so that $V_1(t, x)$ is differentiable with respect to x . The correlation of the integrated O-U sheet is

$$\begin{aligned} c(x, y) &\equiv \text{corr}[d_t V_1(t, x), d_t V_1(t, y)] = \int_0^x dw \int_0^y dz \text{corr}[U(t, w), U(t, z)] \\ &= \frac{1}{\kappa} \left(2(x \wedge y) - \frac{1}{\kappa} (1 - e^{-\kappa x})(1 - e^{-\kappa y}) \right), \end{aligned} \quad (77)$$

where we have used Equation (74).

An alternative formulation of the integrated O-U sheet $V_2(t, x)$ is

$$V_2(t, x) = \kappa \sqrt{2} e^{-\kappa x} \int_0^x dy e^{\kappa y} U(t, y). \quad (78)$$

Again by construction, $\partial V_2(t, x)/\partial x$ is continuous and thus $V_2(t, x)$ is differentiable in x . The correlation of the time increments of this process looks nicer than Equation (77),

$$\begin{aligned} c(x, y) &\equiv \text{corr}[d_t V_2(t, x), d_t V_2(t, y)] \\ &= 2\kappa^2 e^{-\kappa(x+y)} \int_0^x dw \int_0^y dz e^{\kappa(w+z)} \text{corr}[d_t U(t, w), d_t U(t, z)] \\ &= (1 + \kappa|x - y|) e^{-\kappa|x - y|}. \end{aligned} \quad (79)$$

Figure 2 shows an example of the increment in the integrated O-U sheet for different values of κ . We see that the string is always smooth and, as κ

¹³ Smoother strings can be obtained by higher-order integration.

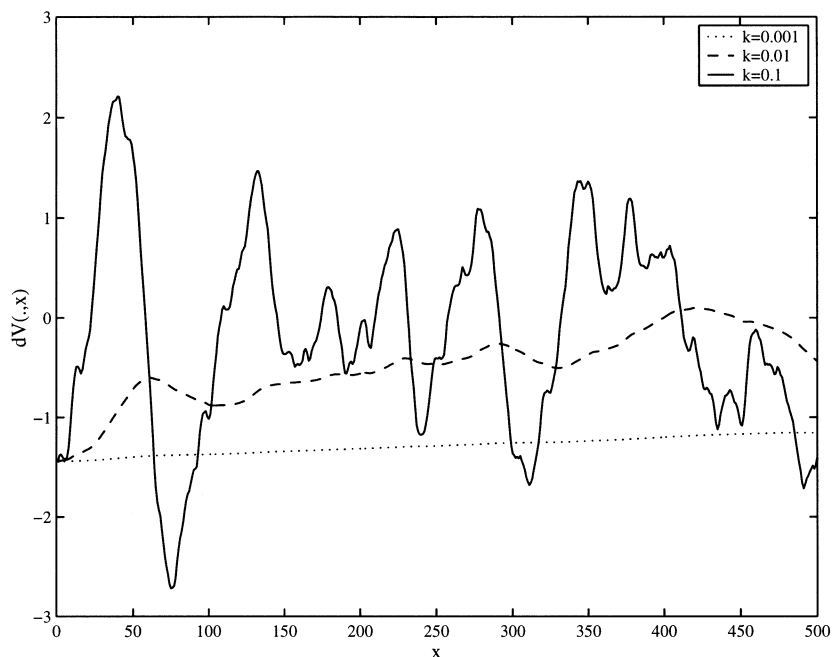


Figure 2

Sample increments to the integrated O-U sheet, with correlation function of Equation (79) for different values of the parameter κ . The time increment is taken to be 1, measured in the same units of time to maturity.

increases, the number of humps decreases. This process has also been used by Goldstein (2000) to model forward rates. Our approach shows that it is one particular example among an arbitrarily large class of processes exhibiting similar properties.

4.5 Other strings

Many parametric examples of string shocks can be obtained by noting that if $g(x, z)$ satisfies Equation (58), so does $g(f(x), z)$, for any function $f(x)$ such that $g(f(x), z)$ is well defined in the domain of x . Thus if $c(x, y)$ is an admissible correlation function, so is $c(f(x), f(y))$.

For example, we can construct examples of string shocks that produce a “term structure of correlations.” Empirical observation shows that the correlation between forward rates, with maturities separated by a given interval, increases with maturity. Take the parameterization in Equation (61), for two time to maturities x and $y = x + \Delta x$, where $\Delta x > 0$ is small compared to x . Expanding the square root in Equation (63), we get

$$c(x, y) \equiv \text{corr}[d_t Z(t, x), d_t Z(t, y)] \approx \left(1 - \frac{\Delta x}{2} \frac{d \log j(x)}{dx}\right). \quad (80)$$

We see that $c(x, y)$ increases with x if $d \log j(x)/dx$ decreases (while remaining positive) as x increases. An interesting situation is when $d \log j(x)/dx \rightarrow 0$ as x gets large. In this case, the increments become perfectly correlated for large time to maturities.

Parametrizing $j(x) = e^{i(x)}$, the above condition implies that $i(x)$ must be a function which increases slower than x , in such a way that its derivative decreases with x , that is, $i(x)$ must be concave. This is not the case for the O-U sheet or the integrated O-U sheet. However, the previous example of Equation (70), where $j(x) = x$, qualifies. In fact, any expression like $j(x) = e^{x^\alpha}$ with an exponent $0 < \alpha < 1$ produces correlations that increase with maturity. This corresponds to

$$c(x, y) \equiv \text{corr}[d_t Z(t, x), d_t Z(t, y)] = e^{-\kappa|x^\alpha - y^\alpha|}. \quad (81)$$

For instance, take $j(x) = e^{2\kappa\sqrt{x}}$, the correlation function is

$$c(x, y) \equiv \text{corr}[d_t Z(t, x), d_t Z(t, y)] = e^{-\kappa|\sqrt{x} - \sqrt{y}|}. \quad (82)$$

Figure 3 shows the correlations between forward rates at maturities spaced by different time intervals, as the maturities increase. We see that the correlation increases both with the proximity of the forward rates and with the times to maturity of the forward rates.

Another simple scheme to generate interesting string shocks is to take $[g(x, z)]^2$ in Equation (58) to be a probability density function in z , to make sure that it integrates to one. As an example, we can take $[g(x, z)]^2$ to be a normal density with mean x ,

$$[g(x, z)]^2 = \frac{e^{-\frac{(x-z)^2}{2}}}{\sqrt{2\pi}}. \quad (83)$$

Then the correlation function is

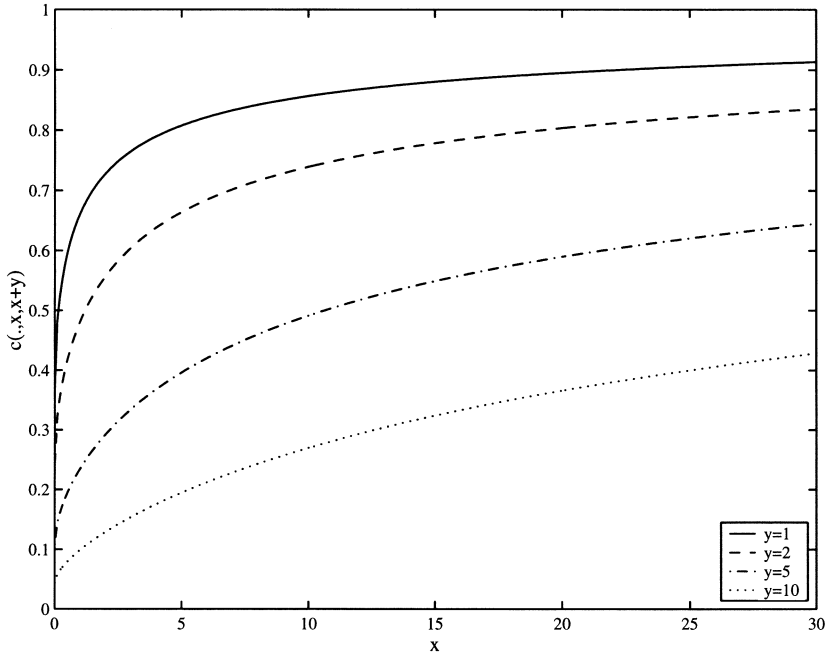
$$c(x, y) = e^{-\frac{(x-y)^2}{8}}. \quad (84)$$

We can also use the normal density with variance x^2 ,

$$[g(x, z)]^2 = \frac{e^{-\frac{z^2}{2x^2}}}{\sqrt{2\pi x}}. \quad (85)$$

The corresponding correlation is

$$c(x, y) = \sqrt{\frac{2xy}{x^2 + y^2}}. \quad (86)$$


Figure 3

Correlations between forward rates with times to maturity separated by different intervals, for the subexponential correlation string shock with correlation function of Equation (82). The parameter κ is taken to be 1.

In order to obtain negative correlations, we may define the function $g(x, z)$ as

$$g(x, z) = \begin{cases} \sqrt{\frac{-\frac{z^2}{2 \log^2 x}}{\sqrt{2\pi \log x}}} & \text{if } x > 1 \\ \sqrt{\delta(z)} & \text{if } x = 1 \\ -\sqrt{-\frac{\frac{z^2}{2 \log^2 x}}{\sqrt{2\pi \log x}}} & \text{if } 0 < x < 1 \end{cases} \quad (87)$$

Then, if $1 < x < y$ or $0 < x < y < 1$,

$$c(x, y) = \sqrt{\frac{2 \log x \log y}{\log^2 x + \log^2 y}}, \quad (88)$$

and if $0 < x < 1 < y$,

$$c(x, y) = -\sqrt{\frac{-2 \log x \log y}{\log^2 x + \log^2 y}}. \quad (89)$$

5. Pricing the Long Bond Futures Delivery Option

This section provides an application of string models to option pricing. Specifically we price the delivery option embedded in long bond futures by Monte Carlo simulation. The objective is twofold: to show the ease of implementation of the models, namely in terms of estimating the required parameters and in carrying out the simulation of the strings, and to show that the greater variety of shapes of the term structure that strings offer does matter significantly for pricing derivatives.

The long bond contract is traded on the Chicago Board of Trade. The underlying security of the contract is specified to be a 20-year Treasury bond, with 8% coupon. However, this is only a notional security, and it does not exist in the market. In reality, the seller of the contract can deliver any Treasury bond with at least 15 years to maturity or first call date. Obviously not all deliverable bonds are worth the same. Therefore the exchange publishes conversion factors that determine the number of units of each bond that the seller must deliver to settle the contract.¹⁴ The conversion factors are meant to make all bonds worth approximately the same amount at delivery, so that the sellers will be indifferent about which bond to deliver. However, this equalization of bond values for delivery is imperfect. There is usually one bond that is cheapest to deliver. The option to choose the bond to deliver at the expiration of the contract is termed the delivery option and makes the futures price lower than it would be if the cheapest-to-deliver bond was known for sure. The long bond futures contract also has an embedded timing option: the seller can choose when to deliver in any day of the delivery month, at the futures price that is fixed at the end of the last trading day before the start of the delivery month.¹⁵

We price the March 1999 long bond futures and the delivery option as of October 15, 1998. We ignore the timing option in this empirical exercise, and assume that the expiration of the contract coincides with the delivery date, and take this date to be the middle of the month, March 15, 1999. We price the contract assuming that the volatility function is always¹⁶

$$\sigma(t, x) = \sigma e^{-\gamma x}, \quad (90)$$

and using three different models for the correlation function: the simple Brownian motion

$$c(x, y) = 1; \quad (91)$$

¹⁴ A conversion factor K means that if the bond is delivered, the seller will receive K times the futures price plus the accrued interest on the bond.

¹⁵ A good reference for the characteristics of the long bond futures contract is Tuckman (1996).

¹⁶ Another simple specification of the volatility function would be proportional to some function g of the forward rate itself,

$$\sigma(t, x) = g(f(t, x))\sigma e^{-\gamma x}.$$

the O-U sheet

$$c(x, y) = e^{-\kappa|x-y|}; \quad (92)$$

and one subexponential correlation string

$$c(x, y) = e^{-\kappa|\sqrt{x}-\sqrt{y}|}. \quad (93)$$

These models are chosen for simplicity and in order to be able to contrast the strings models with the simple Brownian motion case. It is likely that we could find functions with more parameters that would fit the data better, but a study of the best formulation of string models is beyond the present article.

In order to use our model to price the long bond futures, we need several inputs: the forward rate curve on October 15, 1998; parameter estimates for the volatility and correlation functions; and the characteristics of the deliverable bonds. We obtain an estimate of the forward rate curve from Bloomberg. The data consists of 6-month forward rates for all 6-month intervals over the next 30 years. Figure 4 displays this forward curve. We use it directly as an input, ignoring the difference between discrete and continuous compounding. The estimated forward rate curve seems to be of very poor quality, and was probably obtained from simple interpolation of a few fitted points. We use it nevertheless, as our objective is to illustrate the use of the model, and show that it is possible to use readily available inputs.

Bloomberg also provides a list of all deliverable bonds, ranked by cheapness to deliver. Table 1 reproduces this list, including the maturity, coupon rate, conversion factor, and price of each bond in the first four columns. The fifth column shows the price of the bond computed from the estimated forward curve.¹⁷ Although the computed bond prices are not biased, the quality of fit is very poor.

We calibrate the parameters of the three models to fit volatilities and correlations of zero-coupon bond prices. RiskMetrics, a joint venture of J.P. Morgan and Reuters, provides daily estimates of such volatilities and correlations,¹⁸ shown in Table 2 for October 15, 1998. Again the quality of these estimates is questionable, for example, the covariance matrix has negative eigenvalues.

We estimate the parameters of the models by minimizing the Frobenius norm¹⁹ of the difference between the covariances of bond prices as given by the models

¹⁷ We price the bonds as simple coupon bonds, ignoring any option features they may have embedded.

¹⁸ See the RiskMetrics (1996) technical documentation for details of the estimated volatilities and correlations.

¹⁹ The sum of squares of all the elements of a matrix.

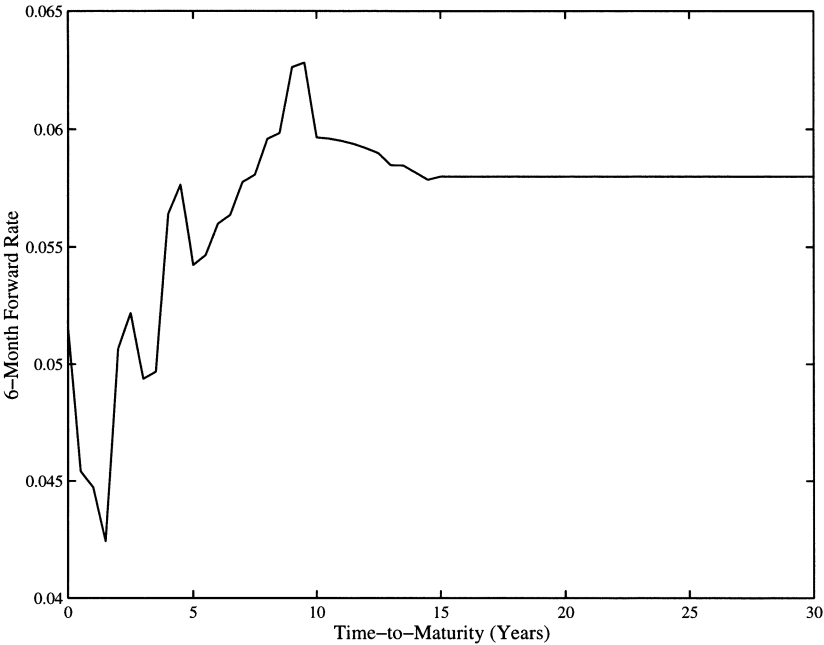


Figure 4
Six-month forward interest rates on 15 October 1998. Data obtained from Bloomberg.

$$\text{cov} \left[\frac{dP(t, s)}{P(t, s)}, \frac{dP(t, u)}{P(t, u)} \right] = dt \int_0^{s-t} dx \int_0^{u-t} dy c(x, y) \sigma(t, x) \sigma(t, y) \quad (94)$$

with the volatility function of Equation (90) and each of the correlation functions of Equation (91), (92), and (93), and the covariance of bond prices estimated by RiskMetrics. Table 3 shows the estimates, along with standard errors and the R^2 coefficients of the fitted matrices.

Now, in order to simulate the evolution of the forward curves, we need to discretize the process both in time and time to maturity. Let Δt be the length of the discrete time interval, 1 day in our simulations, and Δx be the length of the discretized time-to-maturity interval, 6 months in our simulations. We produce simulated paths of

$$f(t + \Delta t, x) = f(t, x) + \frac{f(t, x + \Delta x) - f(t, x)}{\Delta x} \Delta t + \sigma(t, x) \left(\int_0^x dy c(x, y) \sigma(t, y) \right) \Delta t + \sigma(t, x) \Delta Z(t, x) \quad (95)$$

for $x = 0, 0.5, 1, \dots, 30$, where ΔZ is a normal random vector, with variance Δt and correlation

Table 1
Deliverable bonds and cheapest to deliver

Maturity	Coupon	Conversion factor	Market price	Fitted price	Proportion of times bond is CTD		
					Brownian motion	O-U sheet	Subexponential correlation
15 Feb 15	11.250	1.2879	167.9688	168.9371	0.8333	0.7353	0.7589
15 Nov 15	9.875	1.1701	152.5000	157.1257	0.1046	0.0648	0.0678
15 Aug 15	10.625	1.2361	161.1563	162.8504	0	0.0005	0.0003
15 Feb 16	9.250	1.1140	145.2188	147.7561	0	0	0
15 May 17	8.750	1.0709	140.3125	145.4681	0	0.0077	0.0053
15 Aug 17	8.875	1.0830	141.9688	145.0675	0	0.0002	0.0001
15 May 16	7.250	0.9310	122.1563	126.3994	0.0517	0.0509	0.0561
15 Nov 16	7.500	0.9533	125.2500	129.6987	0	0.0118	0.0105
15 May 18	9.125	1.1089	145.9063	151.2916	0	0.0094	0.0075
15 Nov 18	9.000	1.0979	144.9688	150.2478	0	0.0127	0.0101
15 Feb 19	8.875	1.0859	143.8125	146.6453	0	0.0003	0.0001
15 Aug 19	8.125	1.0122	134.6875	137.3938	0	0.0001	0.0001
15 May 20	8.750	1.0757	143.3438	148.2148	0	0.0137	0.0097
15 Feb 20	8.500	1.0500	139.9063	142.5698	0	0.0001	0
15 Aug 20	8.750	1.0758	143.5938	146.2113	0	0.0014	0.0010
15 May 21	8.125	1.0128	136.1563	140.4172	0	0.0081	0.0073
15 Feb 21	7.875	0.9870	132.7813	134.9001	0	0.0001	0
15 Aug 21	8.125	1.0127	136.3750	138.5243	0	0.0004	0.0004
15 Nov 21	8.000	1.0000	135.0000	138.9749	0	0.0068	0.0056
15 Nov 22	7.625	0.9605	130.6875	134.2292	0	0.0127	0.0103
15 Aug 22	7.250	0.9212	125.6875	127.0421	0	0.0001	0.0001
15 Feb 23	7.125	0.9074	124.1563	125.4756	0	0.0001	0.0001
15 Aug 23	6.250	0.8137	111.8750	113.3530	0	0.0022	0.0014
15 Nov 24	7.500	0.9460	130.7188	133.3085	0	0.0113	0.0086
15 Feb 25	7.625	0.9592	132.5938	133.3373	0	0.0008	0.0006
15 Aug 25	6.875	0.8771	122.2188	122.6824	0	0.0001	0
15 Aug 26	6.750	0.8620	121.0625	121.1415	0	0.0001	0.0001
15 Nov 26	6.500	0.8342	117.6250	119.1435	0.0022	0.0113	0.0084
15 Feb 26	6.000	0.7805	110.2188	110.0333	0	0.0008	0.0001
15 Feb 27	6.625	0.8475	119.9063	119.4286	0	0.0004	0.0001
15 Aug 27	6.375	0.8189	116.6250	115.8254	0	0.0001	0
15 Nov 27	6.125	0.7907	115.0000	113.6663	0.0082	0.0314	0.0276
15 Aug 28	5.500	0.7189	108.1250	102.8027	0	0.0045	0.0020

The first four columns provide the prices and characteristics of the deliverable bonds. Data obtained from Bloomberg. The fifth column shows the price of the bonds computed from the forward interest rate curve shown in Figure 4. The final three columns show the frequency with which each bond is the cheapest to deliver out of 50,000 Monte Carlo simulations of the forward rate curve under the three models studied.

$$\text{corr}[\Delta Z(t, x), \Delta Z(t, y)] = c(x, y). \quad (96)$$

We take $f(t, x)$ for $x > 30$ to be constant, and equal to the longest forward rate on October 15, 1998, reflecting the fact that in our model long forward rates are constant. Note that the simulation is done under the risk-neutral probability measure, which is the one relevant for pricing.

We take 50,000 simulations of the forward rate curve daily evolution between October 15, 1998, and March 15, 1999.²⁰ For each simulation we evaluate all deliverable bonds with the final forward curve, and find the cheapest-to-deliver bond, together with the corresponding futures price at

²⁰ We use antithetic variables to lower the variance of the simulation error.

Table 2
Volatilities and correlations of zero-coupon bounds

	0.833	0.25	0.5	1	2	3	4	5	7	9	10	15	20	30
	Volatilities (Annualized)													
	Correlations													
1	0.9029	0.6874	0.6038	0.0134	-0.0007	-0.0143	-0.0265	-0.0403	-0.0541	-0.0608	-0.0832	-0.1148	-0.2052	
0.9029	1	0.7873	0.7433	0.1430	0.1416	0.1363	0.1283	0.1187	0.1069	0.1002	0.0801	0.0482	-0.0765	
0.6874	0.7873	1	0.9387	0.1337	0.1180	0.0997	0.0808	0.0736	0.0647	0.0597	0.0519	0.0383	-0.0190	
0.6038	0.7433	0.9387	1	0.2502	0.2538	0.2504	0.2418	0.2320	0.2183	0.2101	0.1977	0.1743	0.0553	
0.0134	0.1430	0.1337	0.2502	1	0.9866	0.9490	0.8946	0.8509	0.7929	0.7591	0.7319	0.6718	0.3143	
-0.0007	0.1416	0.1180	0.2538	0.9866	1	0.9878	0.9556	0.9244	0.8785	0.8503	0.8271	0.7719	0.4138	
-0.0143	0.1363	0.0997	0.2504	0.9490	0.9878	1	0.9899	0.9713	0.9381	0.9161	0.8972	0.8481	0.4986	
-0.0265	0.1283	0.0808	0.2418	0.8946	0.9556	0.9899	1	0.9933	0.9724	0.9565	0.9419	0.8995	0.5651	
-0.0403	0.1187	0.0736	0.2320	0.8509	0.9244	0.9713	0.9933	1	0.9928	0.9837	0.9726	0.9357	0.6158	
-0.0541	0.1069	0.0647	0.2183	0.7929	0.8785	0.9381	0.9724	0.9928	1	0.9982	0.9908	0.9602	0.6599	
-0.0608	0.1002	0.0597	0.2101	0.7591	0.8503	0.9161	0.9565	0.9837	0.9982	1	0.9946	0.9673	0.6785	
-0.0832	0.0801	0.0519	0.1977	0.7319	0.8271	0.8972	0.9419	0.9726	0.9908	0.9946	1	0.9884	0.7510	
-0.1148	0.0482	0.0383	0.1743	0.6718	0.7719	0.8481	0.8995	0.9357	0.9602	0.9673	0.9884	1	0.8425	
-0.2052	-0.0765	-0.0190	0.0553	0.3143	0.4138	0.4986	0.5651	0.6158	0.6599	0.6785	0.7510	0.8425	1	

Historically estimated volatilities and correlations of zero-coupon bonds of different maturities. Data obtained from RiskMetrics.

Table 3
Parameter estimates of volatility and correlation functions

Parameter	Brownian Motion	O-U Sheet	Subexponential Correlation
σ	0.0316 (0.0043)	0.0396 (0.0036)	0.0485 (0.0053)
γ	0.0359 (0.0038)	0.0046 (0.0005)	0.0228 (0.0024)
κ	— —	0.1782 (0.0150)	1.0888 (0.1012)
R^2	0.8134	0.9105	0.9097

Parameter estimates, with standard errors in parentheses, for the models specified by the volatility function of Equation (90) and each of the correlation functions of Equations (91), (92), and (93). The R^2 s of the estimated covariance matrices are also provided.

maturity. The futures price on October 15, 1998, is then simply the mean of all the simulated maturity futures prices. Note that there is no need for discounting as the futures contract is marked-to-market daily.²¹ In order to price the delivery option, we compute in the same manner the futures price of the current cheapest-to-deliver bond, the 11 1/4, with maturity in February 2015, and subtract from it the futures price that includes the delivery option.

Table 4 shows the futures price and the value of the delivery option under each of the models we examine. For comparison, the actual price observed in the market on October 15, 1998, is also presented. It is remarkable that the delivery option is found to be about four times more valuable in the string models than in the model driven by Brownian motion. The reason is simple: the string models produce a much greater variety of shapes of the final forward rate curve, which induces greater variability of the cheapest-to-deliver bond, and makes the option more valuable. Table 1 shows the frequency with which each bond ends up being the cheapest to deliver in the three models. When the forward curve is driven by a single Brownian motion, there are only five bonds out of the 33 deliverable that ever end up being cheapest to deliver. In contrast, the string models allow around 30 of the 33 bonds to be cheapest to deliver in at least some of the simulations.

6. Conclusion

This article presents a new approach to model the term structure of forward interest rates based on stochastic string shocks. In our model, each point along the term structure is a distinct random variable with its own dynamics. Each point, however, is correlated with the other points in the term structure.

²¹ The prices of continuously resettled assets are martingales under the risk-neutral measure. We assume that this is the case of the futures price, even though it is only resettled daily instead of continuously.

Table 4
Futures price and delivery option

Model	Futures price of today's CTD	Futures price	Delivery option
Brownian motion	128.3367	128.2252	0.1115
O-U sheet	128.4045	127.9595	0.4450
Subexponential correlation	128.4094	128.0536	0.3557
Market		128.3750	

Futures price according to the different models. Futures priced by Monte Carlo simulation, with 50,000 replications and antithetic variables. CTD = cheapest to deliver.

We show that our approach generalizes the model of HJM, while preserving its simplicity and appeal.

We derive conditions to generate admissible string shocks, and characterize their correlation functions. In a number of examples, we show a large diversity of correlation functions that capture stylized facts about interest rates that depend on very few parameters.

Finally, we show in a realistic example that our models are easy to estimate and simulate. More importantly we show that using string models matters significantly in pricing contingent claims.

Appendix A: Solution of the Traditional Model

In this appendix we obtain the integral representation of the arbitrage-free process for forward rates of the traditional model of Section 1. We perform a change of variable $(t, x) \rightarrow (\tau \equiv x + t, \xi \equiv x - t)$ and denote $\hat{f}(\tau, \xi) \equiv f(t, x)$. We then have

$$\frac{\partial f(t, x)}{\partial t} = \frac{\partial \tau}{\partial t} \frac{\partial \hat{f}(\tau, \xi)}{\partial \tau} + \frac{\partial \xi}{\partial t} \frac{\partial \hat{f}(\tau, \xi)}{\partial \xi} = \frac{\partial \hat{f}(\tau, \xi)}{\partial \tau} - \frac{\partial \hat{f}(\tau, \xi)}{\partial \xi}, \quad (97)$$

and similarly

$$\frac{\partial f(t, x)}{\partial x} = \frac{\partial \tau}{\partial x} \frac{\partial \hat{f}(\tau, \xi)}{\partial \tau} + \frac{\partial \xi}{\partial x} \frac{\partial \hat{f}(\tau, \xi)}{\partial \xi} = \frac{\partial \hat{f}(\tau, \xi)}{\partial \tau} + \frac{\partial \hat{f}(\tau, \xi)}{\partial \xi}. \quad (98)$$

Replacing these in the left-hand side of Equation (17), we get

$$\frac{\partial f(t, x)}{\partial t} - \frac{\partial f(t, x)}{\partial x} = -2 \frac{\partial \hat{f}(\tau, \xi)}{\partial \xi}. \quad (99)$$

Equation (17) thus becomes

$$-\frac{\partial \hat{f}(\tau, \xi)}{\partial \xi} = -\frac{1}{2} a\left(\frac{\tau - \xi}{2}, \frac{\tau + \xi}{2}\right) - \frac{1}{2} \sigma\left(\frac{\tau - \xi}{2}, \frac{\tau + \xi}{2}\right) \eta\left(\frac{\tau - \xi}{2}\right), \quad (100)$$

where we have expressed the values of t and x in terms of the new variables τ and ξ : $t = (\tau - \xi)/2$ and $x = (\tau + \xi)/2$. This change of variable allows us to get an ODE from the initial PDE and its straightforward integration gives

$$\hat{f}(\tau, \xi) - \hat{f}(\tau, \xi_0) = -\frac{1}{2} \int_{\xi_0}^{\xi} dw a\left(\frac{\tau-w}{2}, \frac{\tau+w}{2}\right) - \frac{1}{2} \int_{\xi_0}^{\xi} dW(w) \sigma\left(\frac{\tau-w}{2}, \frac{\tau+w}{2}\right). \quad (101)$$

Now, by the definition of the change of variables, $\hat{f}(\tau, \xi) = f(t, x)$. We have a free choice for ξ_0 . Since we want to express $f(t, x)$ as a function of $f(0, \cdot)$, ξ_0 is chosen such that it corresponds to $t_0 = 0$. To see what this leads to, we use the definition $\hat{f}(\tau, \xi_0) = f(t_0 = (\tau - \xi_0)/2, (\tau + \xi_0)/2)$. For t_0 to be zero, this implies that $\xi_0 = \tau$ and thus $(\tau + \xi_0)/2 = \tau = x + t$. The left-hand side of Equation (101) thus reads

$$\hat{f}(\tau, \xi) - \hat{f}(\tau, \xi_0) = f(t, x) - f(0, x + t). \quad (102)$$

To tackle the right-hand side of Equation (101), we perform the change of variable from w to $v = (\tau - w)/2$. Then $(\tau + w)/2 = \tau - v$ and $\int_{\xi_0=\tau-x+t}^{\xi=x-t} dw = -2 \int_0^t dv$. Finally, we get

$$f(t, x) = f(0, x + t) + \int_0^t dv a(v, t + x - v) + \int_0^t dW(v) \sigma(v, t + x - v). \quad (103)$$

Let us now comment on the validity of this derivation when $f(t, x)$ is not differentiable in x . As an example, consider the SPDE

$$\frac{\partial W(t, x)}{\partial t} - \frac{\partial W(t, x)}{\partial x} = \eta(t, x). \quad (104)$$

Performing the same change of variable as above, we get

$$\frac{\partial \hat{W}(\tau, \xi)}{\partial \xi} = -\frac{1}{2} \hat{\eta}(\tau, \xi), \quad (105)$$

where

$$\hat{\eta}(\tau, \xi) \equiv \eta((\tau - \xi)/2, (\tau + \xi)/2). \quad (106)$$

The solution of Equation (105) reads

$$\hat{W}(\tau, \xi) = -\frac{1}{2} \int_0^{\xi} dy \hat{\eta}(\tau, y) = -\frac{1}{2} \int_0^{x-t} dy \eta((x + t - y)/2, (x + t + y)/2). \quad (107)$$

One can readily check that Equation (107) obeys Equation (104).

Appendix B: Correlation Functions for String Shocks

In this appendix we derive the conditions that must be satisfied by correlation functions of string shocks. We use some results of functional analysis that hold in L^2 [see Riesz and Sz.-Nagy (1990)]. In order to make sure that all the functions below are in L^2 , we assume that the times to maturity are bounded, so that $x, y < K$, for some large enough constant K .

From Equation (58), we assume that the function $g(x, z)$ is such that

$$\int_0^K dz [g(x, z)]^2 = 1, \quad (108)$$

which we use to define the correlation function $c(x, y)$ as

$$c(x, y) = \int_0^K dz g(x, z) g(y, z). \quad (109)$$

The function $c(x, y)$ is obviously symmetric, $c(x, y) = c(y, x)$. By Equation (108), $c(x, x) = 1$. The function $c(x, y)$ only takes values in the interval $[-1, 1]$, since, from the expansion of $\int_0^K [g(x, z) \pm g(y, z)]^2 dz$, together with Equation (108), we have

$$\begin{aligned} \int_0^K dz [g(x, z)]^2 + \int_0^K dz [g(y, z)]^2 + 2 \int_0^K dz g(x, z)g(y, z) \\ = 2 + 2 \int_0^K dz g(x, z)g(y, z) \geq 0, \end{aligned} \quad (110)$$

and

$$\begin{aligned} \int_0^K dz [g(x, z)]^2 + \int_0^K dz [g(y, z)]^2 - 2 \int_0^K dz g(x, z)g(y, z) \\ = 2 - 2 \int_0^K dz g(x, z)g(y, z) \geq 0. \end{aligned} \quad (111)$$

Finally, $c(x, y)$ is positive semidefinite, which means that, for all functions $h(x)$ in L^2 ,

$$\int_0^K \int_0^K dx dy h(x)c(x, y)h(y) \geq 0. \quad (112)$$

To show positive semidefiniteness, note that if we replace the correlation function of Equation (109) in Equation (112), we obtain

$$\int_0^K \int_0^K \int_0^K dz dx dy h(x)h(y)g(x, z)g(y, z) = \int_0^K dz \left(\int_0^K dx h(x)g(x, z) \right)^2, \quad (113)$$

which is nonnegative.

Conversely, we can show that for any given function $c(x, y)$ that is positive semidefinite, there exists a symmetric function $g(x, z)$ such that

$$c(x, y) = \int_0^K dz g(x, z)g(y, z) \quad (114)$$

[see Riesz and Sz.-Nagy (1990)]. Since $c(x, y)$ is symmetric, it has the following “orthogonal” decomposition

$$c(x, y) = \int_0^K du P(u, x)D(u)P(u, y), \quad (115)$$

where $P(u, x)$ obeys

$$\int_0^K du P(u, x)P(u, y) = \delta(x - y), \quad (116)$$

and $D(u)$ is nonnegative. Obtaining the orthogonal decomposition of Equation (115) is a standard procedure when we discretize the function $c(x, y)$ in matrix form. Let us assume that

the functions $g(x, z)$ are also symmetric in their arguments x and z . As a consequence, the orthogonal decomposition of Equation (115) can be applied to them as

$$g(x, z) = \int_0^K du Q(u, x) d(u) Q(u, z), \quad (117)$$

with

$$\int_0^K du Q(u, x) Q(u, y) = \delta(x - y). \quad (118)$$

Inserting Equation (117) into Equation (59) leads to

$$\begin{aligned} c(x, y) &= \int_0^K dz \int_0^K du Q(u, x) d(u) Q(u, z) \int_0^K dv Q(v, y) d(v) Q(v, z) \\ &= \int_0^K du \int_0^K dv Q(u, x) d(u) Q(v, y) d(v) \int_0^K dz Q(u, z) Q(v, z) \\ &= \int_0^K du \int dv Q(u, x) d(u) Q(v, y) d(v) \delta(u - v) \\ &= \int_0^K du Q(u, x) [d(u)]^2 Q(u, y). \end{aligned} \quad (119)$$

Comparing with Equation (115), this shows that $g(x, z)$ is determined from Equation (117) with $Q = P$ and $d = \sqrt{D}$. This completes the general construction of the string shock, given a specific correlation function of its increments in Equation (59). The problem thus boils down to obtaining the orthogonal decomposition of Equation (115). Note that this construction procedure is always possible with the parameterization of Equations (56) and (58) since $D(u)$ is nonnegative and its square root is therefore real.

Note that positive semidefiniteness is an important constraint on the generation of admissible correlation functions. As an example of how this condition may fail on a seemingly reasonable function, consider the function $c(x, y)$ defined on $0 \leq x, y \leq K$ by

$$c(x, y) = \begin{cases} 1 - \frac{|x-y|}{a} & \text{if } |x-y| < a \\ \frac{a}{|x-y|} - 1 & \text{if } |x-y| \geq a, \end{cases} \quad (120)$$

where $0 < a < K$ is a constant. It is obvious that this function satisfies the conditions: $c(x, y) = c(y, x)$, $c(x, x) = 1$, and $-1 \leq c(x, y) \leq 1$. We now show that $c(x, y)$ is *not* positive definite. Let $h(x) = 1$ be the constant function over $0 \leq x \leq K$. Clearly $h(x)$ is in L^2 . We have

$$\begin{aligned} &\int_0^K \int_0^K dx dy h(x) c(x, y) h(y) \\ &= \int_0^K \int_0^K dx dy c(x, y) \\ &= \int \int_{0 \leq |x-y| < a} dx dy c(x, y) + \int \int_{a \leq |x-y| < K} dx dy c(x, y) \\ &= 2 \int \int_{0 \leq x-y < a} dx dy c(x, y) + 2 \int \int_{a \leq x-y < K} dx dy c(x, y) \\ &= 2 \int \int_{0 \leq x-y < a} dx dy \left(1 - \frac{|x-y|}{a} \right) + 2 \int \int_{a \leq x-y < K} dx dy \left(\frac{a}{|x-y|} - 1 \right) \\ &= \left(aK - \frac{a^2}{3} \right) + \left(aK \ln K - aK \ln a - \frac{K^2 - a^2}{2} \right). \end{aligned} \quad (121)$$

For given K , the above integral converges to $-K^2/2$ as $a \rightarrow 0$. Therefore $c(x, y)$ is not positive semidefinite and is not an admissible correlation function. It is easy to construct other counter examples by modifying this simple one.

A natural recipe to construct admissible correlation functions is to use Equation (109) with $g(x, z)$ symmetric. This expression provides a necessary and sufficient condition for $c(x, y)$ to be an admissible correlation function.

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