HITTING LINES WITH TWO-DIMENSIONAL BROWNIAN MOTION

BY

SATISH IYENGAR

TECHNICAL REPORT NO. 429

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FOR THE OFFICE OF NAVAL RESEARCH

DEPARTMENT OF STATISTICS

STANFORD UNIVERSITY

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1. Introduction

This paper consists of the computation of several hitting time and hitting place distributions for two-dimensional Brownian motion. The motivation for this study is two-fold: first, to get a diffusion model for the firing behavior of a simple network of neurons, and second, to get an interesting two-dimensional version of the inverse Gaussian distribution.

Fienberg (1974) has reviewed various models for the firing of single neurons. A classical model of Gerstein and Mandelbrot (1964) says that if the electrical state (or potential) of the neural membrane is specified by a single number, which moves towards or away from the firing potential as the neuron receives excitatory or inhibitory input, resp., then the time to firing can be approximated by the first hitting time of a certain level for a Brownian motion with drift. The authors showed that this model could be used to provide a satisfactory fit to some data that they observed; more importantly, they showed by Monte Carlo methods that neural activity in the presence of stimuli could also be well duplicated by a modification of the above random walk model.

Next, the review by Folks and Chhikara (1978) shows that the inverse Gaussian distribution has many nice statistical properties which, to a large extent mirror those of the Gaussian distribution. It is natural, then, to ask whether there is a multivariate inverse Gaussian whose statistical properties are similar to those of the multivariate Gaussian.

Several proposals for a bivariate inverse Gaussian have already appeared in the literature. Barndorff-Nielsen and Blaesild (1983) define reproductive exponential families and propose a bivariate inverse Gaussian model; they claim that their generalization has nice statistical properties (i.e., affords tractable estimation and analysis of variance), but their proposal does not have inverse Gaussian marginals. Wasan (1969, 1972) proposes several bivariate inverse Gaussians but does not develop their properties.

2. A Simple Neural Network

Consider the three neurons of Figure 1. Neuron A sends predominantly exci-

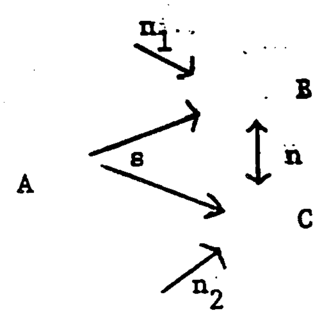

Figure 1

tatory signals, s, to B and C. B and C share a common noise, n, and they also have independent sources of noise, n_1 and n_2, resp. If the electrical states of B and C are encoded by single numbers, $X_1(t)$ and $X_2(t)$, resp., then $X_i(t)$ has three components; the common noise n, the particular noise n_i, and the signal s. Let the noise variances be σ^2 for n and σ_i^2 for n_i; then it is easy to see that

$$\operatorname{corr}(X_1(t),X_2(t)) = [(1 + \frac{\sigma_1^2}{\sigma^2})(1 + \frac{\sigma_2^2}{\sigma^2})]^{-1/2}$$

which is a function of the noise ratios. Also, we may allow the drifts of $X_1(t)$ and $X_2(t)$ (due to the signal, s) to be different since B and C may accept the same input but integrate it differently. When either Brownian reaches the firing threshold, the appropriate neuron fires, returns to its resting state, and the process starts afresh. What are of interest, then, are the firing times or the first hitting times for the Brownian motions. Alternatively, this model can be used to study a single neuron: if we postualte that the neuron has two interacting trigger zones, (Gerstein, et al. (1964)), then the components of the Brownian motion describe the electrical state of each zone. Mathematically, the two problems are the same.

3. Preliminaries

We start with a correlated driftless Brownian motion $X(t)$ with $EX(t) = 0$ and $\operatorname{var} X(t) = t\Sigma$. Here

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \quad \text{and} \quad \Sigma^{1/2} = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

where $\rho = \sin(2\beta)$, $|\beta| \leq \pi/4$. Thus $X(t) = \Sigma^{1/2} Z(t)$ where $Z(t)$ is a standard Brownian motion: $\operatorname{var} Z(t) = tI$. Also define the two stopping times $\tau_i = \inf\{t: X_i(t) = a_i\}$ $= \inf\{t: Z(t) \in \ell_i\}$. Here $a_i > 0$ without loss of generality, and ℓ_i is the line $\{v \in \mathbb{R}^2: v' \cdot \Sigma^{1/2} e_i = a_i\}$ and $\{e_1, e_2\}$ is the standard basis for $\mathbb{R}^2$. By the scale invariance of Brownian motion, we can take $a_1 = 1$. Finally, by elementary methods, we arrive at the following problem (see Figure 2): start a Brownian motion at x $= (x_1, x_2) = (r_0 \cos\theta_0, r_0 \sin\theta_0)$, and study the stopping times and places associated with τ_1 and τ_2.

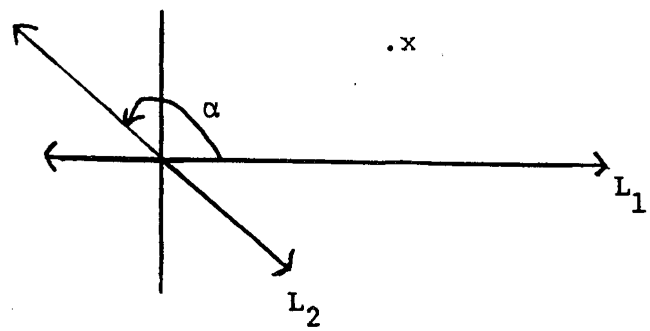

Figure 2

Here τ_i is the first hitting time of L_i, and $\alpha = \frac{\pi}{2} + \sin^{-1}\rho$. Also, let $W = \{(r,\theta): r > 0,\ 0 < \theta < \alpha\}$, and $\tau' = \tau_1 \wedge \tau_2$ be the first hitting time of ∂W. Our aim is to get the joint density of (τ_1, τ_2), and on the way we compute other quantities that are also of interest. In particular, we study the following quantities:

a) $P^x(\tau' > t,\ Z(t) \in B)$, $B \subset W$

b) $P^x(\tau' > t)$

c) $P^x(Z(\tau') \in A)$, $A \subset \partial W$

d) $P^x(\tau' \in dt,\ Z(t) \in da)$, $a \in \partial W$

e) $P^x(\tau_1 \in ds,\ \tau_2 \in dt)$

f) the above quantities in the presence of drift.

Here P^x is the measure associated with standard Brownian motion starting at x; E^x will denote the corresponding expectation. Note that the marginal distributions of τ_i are easy:

$$P^x(\tau_2 \in dt) = \frac{x_2}{t\sqrt{t}}\ \phi\left(\frac{x_2}{\sqrt{t}}\right) dt, \text{ and } P^x(\tau_1 \in dt) = \frac{A}{t\sqrt{t}}\ \phi\left(\frac{A}{\sqrt{t}}\right) dt$$

where $A = x_1\sin\alpha - x_2\cos\alpha$ and $\phi(x) = (2\pi)^{-1/2} e^{-x^2/2}$.

4. Brownian Motion in the Wedge

The main result of this section is contained in (8). Most of the subsequent results follow from it. If we have a positive bounded continuous function f defined on W, and which vanishes on ∂W, then

$$(1) \qquad u(t,x) = E^x f(Z(\tau'\wedge t)) = \int_W f(y)P^x(\tau' > t, Z(t) \in dy)$$

satisfies the heat equation with boundary and initial conditions

$$(2) \qquad u_t = \frac{1}{2}\Delta u \text{ in } W;\ u(0,x) = f(x),\ x \in W;\quad u(t,z) = 0,\quad z \in \partial W.$$

We can solve (2) when $\alpha = \pi/m$, $m = 1,2,\ldots$ by the method of images. That is, if we let $T_0 = I$, F_j be the matrix representing the reflection across the line $y = x\tan\frac{\pi j}{m}$, and $T_j = F_j \circ T_{j-1}$, and let $\tilde{f}(y) = (-1)^k f(T_k^{-1}y)$ for $y \in T_k(W)$, we have the initial value problem

$$(3) \qquad \tilde{u}_t = \frac{1}{2}\Delta\tilde{u} \text{ in } \mathbb{R}^2;\quad \tilde{u}(0,x) = \tilde{f}(x),\ x \in \mathbb{R}^2.$$

The solution is

$$\tilde{u}(t,x) = E^x\tilde{f}(Z(t)) = \int_W f(y)\sum_{k=0}^{2m-1}(-1)^k\phi_2\left(\frac{x-T_k y}{\sqrt{t}}\right)\frac{dy}{t}$$

where $\phi_2(x) = (2\pi)^{-1}\exp(-x'x/2)$. It is easy to see that

$$(4) \qquad P^x(\tau' > t, Z(t) \in dy) = \frac{1}{t}\sum_{k=0}^{2m-1}(-1)^k\phi_2\left(\frac{x-T_k y}{\sqrt{t}}\right)dy.$$

While (4) is appropriate only for the special angles $\theta_0 = \pi/m$, the following argument gives us the result in general. To facilitate this, we use polar coordinates: let $x = (r_0\cos\theta_0, r_0\sin\theta_0)$, $y = (r\cos\theta, r\sin\theta)$ to get

$$(5)\qquad P^x(\tau' > t, Z(t) \in dy) = \frac{1}{2\pi t} e^{-\frac{r^2+r_0^2}{2t}} \sum_{k=0}^{2m-1} (-1)^k e^{\frac{rr_0}{t}\cos(\theta-\theta_k)}$$

where θ_k is the argument of $T_k y$. Note that (Magnus, et al. (1966))

$$(6)\qquad e^{\gamma z} = 2 \sum_{n=0}^{\infty} T_n(\gamma) I_n(z)$$

where T_n is the n^{th} Chebyshev polynomial and I_n is the modified Bessel function of order n. Recalling that $T_n(\cos\theta) = \cos(n\theta)$, and using the fact that

$$(7)\qquad \sum_{k=0}^{2m-1} (-1)^k \cos n(\theta-\theta_k) = 2m \sin(n\theta)\sin(n\theta_0)$$

if m divides n and zero otherwise, we get

$$(8)\quad P^x(\tau' > t, Z(t) \in dy) = \frac{2r}{t\alpha} e^{-\frac{r^2+r_0^2}{2t}} \sum_{n=0}^{\infty} \sin\frac{n\pi\theta}{\alpha} \sin\frac{n\pi\,\theta_0}{a} I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right) dr d\theta$$

whenever $\alpha = \pi/m$. But formula (8) is valid for all α, and it is easy to see that it solves problem (2).

Expression (8) has also been essentially derived by Sommerfeld (1894) by an extension of the method of images. See also Carslaw and Jaeger (1959) and Buckholtz and Wasan (1979).

We next compute $P^x(\tau' > t)$. If we integrate out θ and r in (8) and use the following identities:

$$(9)\qquad 2\, I'_\nu(x) = I_{\nu-1}(x) + I_{\nu+1}(x)$$

$$\int_0^\infty e^{-\beta t^2} I_\nu(\alpha t) dt = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\; e^{-\alpha^2/8\beta}\, I_{\nu/2}\left(\frac{\alpha^2}{8\beta}\right)$$

(Magnus, et al. (1966)) we get*

$$(10)\quad P^x(\tau' > t) = \frac{2r_0}{\sqrt{2\pi t}} e^{-\frac{3r_0^2}{4t}} \sum_{\substack{n=1\\ n\ \text{odd}}}^{\infty} \frac{1}{n} \sin\frac{n\pi\theta_0}{\alpha} \{I_{\frac{\nu-1}{2}}(\frac{r_0^2}{4t}) + I_{\frac{\nu+1}{2}}(\frac{r_0^2}{4t})\}$$

where $\nu = n\pi/\alpha$.

When the wedge angle is special - $\alpha = \pi/m$ - we have the alternate expression

$$(11)\quad P^x(\tau' > t) = 2 \sum_{k=0}^{2m-1} (-1)^{k+1} F(\frac{k\pi}{m}).$$

where

$$(12)\quad F(u) = \int_0^u \frac{r_0}{\sqrt{t}} \cos(\theta-\theta_0) R(-\frac{r_0}{\sqrt{t}} \cos(\theta-\theta_0)) d\theta.$$

Here, R is Mills' ratio: $R(x) = \Phi(-x)/\phi(x)$, where Φ and ϕ are the normal distribution and density functions, resp. We omit the details of this computation.

The quantity $P^x(\tau > t)$ was also studied by Spitzer (1958), who computed its transform. Checking the asymptotics of modified Bessel functions, it is easy to see that $E^x(\tau')^\beta = \int_0^\infty \tau^{\beta-1} P^x(\tau' > t) dt < \infty$ if and only if $\alpha\beta < \pi/2$, independent of x.

The distribution of $Z(\tau')$ is also of interest. Now $u(x;A) = P^x(Z(\tau') \in A)$ satisfies Laplace's equation $\Delta u = 0$ with boundary condition $u(x,A) = I\{x \in A\}$. The Green's function for the wedge is easily computed, and we have

$$(13)\quad P^x(Z(\tau') \in da) = \frac{1}{c} \frac{(\frac{a}{r_0})^{\frac{\pi}{\alpha} - 1} \sin\frac{\pi\theta_0}{\alpha}}{\sin^2\frac{\pi\theta_0}{\alpha} + (1 \pm \cos\frac{\pi\theta_0}{\alpha})^2} \frac{da}{r_0}$$

where we use the plus sign for $\tau_2 < \tau_1$ and the minus sign otherwise. Using elementary estimates, it is easy to see that $E^x Z_i(\tau')^\beta$ exists iff $\alpha\beta < \pi$, again

*Expression (10) corrects a mistake in Wasan and Buckholtz (1979).

independent of the initial position, x.

5. Joint distribution of (τ_1,τ_2)

Using an argument similar to that of Daniels (1982)*, it can be shown that if $P^x(\tau > t,\ Z(t) \in dy) = f(t,x,y)dy$, then

$$(14) \qquad P^x(\tau' \in dt, Z(\tau) \in da) = \frac{1}{2}\left[\frac{\partial}{\partial n} f(t,x,y)\Big|_{y=a}\right]dadt$$

where $\frac{\partial}{\partial n}$ denotes the derivative in the inward normal direction. See Figure 3.

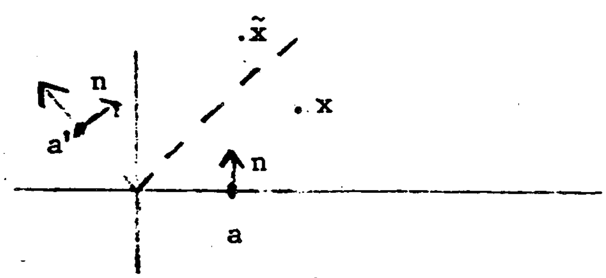

Figure 3

When $\theta = 0(\alpha)$, $\frac{\partial}{\partial n} = \frac{1}{r}\frac{\partial}{\partial\theta}\left(-\frac{1}{r}\frac{\partial}{\partial\theta}\right)$, so from (8) we get

$$(15) \quad P^x(\tau \in dt, Z(\tau') \in da) = \frac{\pi}{\alpha^2 ta} e^{-\frac{a^2+r_0^2}{2t}} \sum_{n=0}^{\infty} \delta_n\, n \sin\frac{n\pi\theta_0}{\alpha}\, I_{\frac{n\pi}{\alpha}}\left(\frac{ar_0}{t}\right)$$

where δ_n is 1 if $\theta = 0$ and $(-1)^{n+1}$ if $\theta = \alpha$. It is clear by symmetry that $P^x(\tau' \in dt, X(\tau') \in da) = P^{\tilde{x}}(\tau' \in dt, Z(\tau') \in da')$ where $\tilde{x}$ is the reflection of x across the line $y = x \tan\frac{\alpha}{2}$. And for the special angle $\alpha = \pi/m$, a simpler formula is available:

---

*I thank Professor D. Siegmund for this reference.

(16) $P^x(\tau'\epsilon\, dt, Z(\tau')\epsilon\, da) = \frac{1}{t^2}\sum_{0}^{2m-1}(-1)^k\ \phi_2(\frac{x-T_k y}{\sqrt{t}})\Big|_{y_2=0}(x' T_k e_2)$

with $P^x(\tau'\epsilon\, dt,\ Z(\tau')\epsilon\, da')$ computed by symmetry.

Finally, we can compute $P^x(\tau_1\epsilon\, ds,\ \tau_2\epsilon\, dt)$, the joint density of (τ_1,τ_2). By the strong Markov property we have for $s<t$,

(17) $$P^x(\tau_1\epsilon\, ds, \tau_2\epsilon\, dt) = \int_{\partial W} P^x(\tau_1\epsilon\, ds, Z(\tau')\epsilon\, da,\ \tau_2\epsilon\, dt)$$

$$= \int_{\partial W} P^x(\tau_1 = \tau'\epsilon\, ds, Z(\tau')\epsilon\, da)\ P^{a\sin\alpha}(\tau_2\epsilon\, dt-s).$$

But the first term of the integrand is given by (15) and the second term is just

(18) $$P^{a\sin\alpha}(\tau_2\epsilon\, dt-s) = \frac{a\sin\alpha}{(t-s)^{3/2}}\ \phi(\frac{a\sin\alpha}{\sqrt{t-s}})dt,$$

since τ_2 is just a one-dimensional inverse Gaussian. After some computation and (9) we get

(19) $P^x(\tau_1\epsilon\, ds,\tau_2\epsilon\, dt) =$

$$\frac{\pi\ \sin\alpha}{2\alpha^2\sqrt{s}(t-s)\sqrt{t-s\cos^2\alpha}}\ \exp(-\frac{r_0^2}{2s}\frac{2(t-s)+(t-s\cos 2\alpha)}{(t-s)+(t-s\cos 2\alpha)})\sum_{n=0}^{\infty}(-1)^{n+1} n\sin\frac{n\pi\theta_0}{\alpha}\ I_{\frac{n\pi}{2\alpha}}(\frac{r_0(t-s)/s}{[2(t-s)+2(t-s\cos 2\alpha}$$

Finally, using the fact that (see Figure 3)

$$P^x(\tau_1\epsilon\, ds,\tau_2\epsilon\, dt) = P^{\tilde{x}}(\tau_1\epsilon\, dt,\tau_2\epsilon\, ds),\quad s<t$$

the joint density of (τ_1,τ_2) is determined for $s\neq t$. In (19) we can let $t\to s$ and use the fact that as $z \to 0$, $I_\nu(z) \sim (\frac{z}{2})^\nu / \Gamma(\nu+1)$ to get

$$
\text{(20)} \qquad P^x(\tau_1 \in ds, \tau_2 \in dt) \to \begin{cases} 0 & \text{if } 0 < \alpha < \frac{\pi}{2} \\ \infty & \text{if } \frac{\pi}{2} < \alpha < \pi \\ \frac{r_0^2 \sin 2\theta_0}{4\pi s^3} \exp(-\frac{r_0^2}{2s}) & \text{if } \alpha = \pi/2 \end{cases}
$$

Thus the joint density of (τ_1, τ_2) is discontinuous on the line $s = t$ only when the original Brownian motion $X(t)$ is positively correlated. Of course, we could have started with (16) to get a simpler expression for the joint density of (τ_1, τ_2) for the special angles $\alpha = \pi/m$; we omit the straightforward calculation.

6. Brownian Motion with Drift

Of course, the case of Brownian-motion with drift is of more interest. The analysis, however, is considerably more complicated and so it is not always possible to evaluate the integrals that arise. In this section, we extend the results of the previous sections to Brownian motion with drift.

If, in section 3, the correlated process $X(t)$ has drift $\theta = (\theta_1, \theta_2)'$ where $\theta_1 > 0$, it is easy to see that for the corresponding process $Z(t)$, in the wedge W, we have drift $\mu = (\mu_1, \mu_2)' = (\rho\theta_2 - \theta_1, -\theta_2 (1-\rho^2)^{1/2})$. Let

$$
\text{(21)} \qquad f(t; a, b) = \sqrt{\frac{b}{2\pi t^3}} \exp(-\frac{b}{2a^2} \frac{(t-a)^2}{t})
$$

be the inverse Gaussian density in its usual form [6, p. 263]. Then it is easy to see that τ_1 has density $f(t; \frac{|x_1 \sin\alpha - x_2 \cos\alpha|}{\theta_1\sqrt{1-\rho^2}}, (x_1 \sin\alpha - x_2 \cos\alpha)^2)$ and τ_2 has density $f(t; \frac{x_2}{\theta_2\sqrt{1-\rho^2}}, x_2^2)$.

Let P_μ^x be the measure associated with uncorrelated Brownian motion starting at x and with drift μ:

$$P_\mu^x(Z(t_1)\epsilon A_1,\ldots,Z(t_n)\epsilon A_n) = P_0^x(Z(t_1)+\mu t_1\epsilon A_1,\ldots,Z(t_n)+\mu t_n\epsilon A_n) \tag{22}$$

for all n, for all $t_1<\ldots<t_n$, and for all Borel sets A_i. Our basic tool is the exponential (likelihood ratio) martingale

$$\frac{dP_\mu^x}{dP_0^x} = \exp(\mu'(Z(t)-x)-t|\mu|^2/2) \tag{23}$$

on F_t, the sigma field generated by $\{Z(s): s\leq t\}$. Thus, we have that

$$v(t,x,y) = P_\mu^x(\tau'>t,\ Z(t)\epsilon dy) = e^{\mu'(y-x)-|\mu|^2t/2}P_0^x(\tau'>t,\ Z(t)\epsilon\ dy) \tag{24}$$

is a solution to the diffusion equation with convection or drift:

$$v_t = \frac{1}{2}\Delta v+\mu'\nabla v;\ v(0,x,y)=\delta_{x-y};\ v(t,a,y)=0,\quad a\in\partial W \tag{25}$$

where δ is the Dirac delta. Of course, in (24), $P_0^x(\tau'>t,\ Z(t)\epsilon\ dy)$ is given by (8). The expressions for $P_\mu^x(\tau'>t)$ and $P_\mu^x(Z(\tau')\epsilon A)$ do not seem to be convenient as they were for the driftless case ((10) and (13) in section 4), but the joint density of τ' and $Z(\tau')$ is available. In fact, Daniels' argument gives (see Figure 3)

$$P_\mu^x(\tau'\epsilon dt,Z(\tau')\epsilon da) = e^{-x'\mu-t|\mu|^2/2+\mu'a}\,P_0^x(\tau'\epsilon dt,Z(\tau')\epsilon da) \tag{25}$$

$$P_\mu^x(\tau'\epsilon dt,Z(\tau')\epsilon da') = e^{-x'\mu-t|\mu|^2/2+a'(\mu_1\cos\alpha+\mu_2\sin\alpha)}P_0^x(\tau'\epsilon dt,Z(\tau')\epsilon da'),$$

where we use (15) for $P_0^x(\tau'\epsilon dt,Z(\tau')\ \epsilon\ da)$.

Finally, we have for $s<t$

$$(27) \qquad P_\mu^x(\tau_1 \in ds, \tau_2 \in dt) = \int_{\partial W} P_\mu^x(\tau_1 \in ds, Z(\tau_1) \in da) P^{a \sin\alpha}(\tau_2 \in dt-s).$$

Note that $P_\mu^{a \sin\alpha}(\tau_2 \in dt-s)$ is just the inverse Gaussian density and $P_\mu^x(\tau_1 \in ds, Z(\tau_1) \in da)$ is given by (21). The integral involved, however, does not seem to be tractable.

7. Concluding Remarks

Clearly, the $\mu = 0$ case is much easier than the $\mu \neq 0$ case; however, the physical motivation demands $\mu \neq 0$. For higher dimensional problems, similar methods can be used to get the joint distribution of $\tau = (\tau_1, \ldots, \tau_p)$, where $\tau_i = \inf\{t>0: X_i(t) = a_i\}$, $a_i > 0$, and $X(t) \sim N(0, t\Sigma)$ is a driftless correlated Brownian motion. The geometry in $\mathbb{R}^p$ is quite complicated though; for certain patterened Σ (e.g., $\Sigma_{ij} \equiv \rho$), the transformation to independence can be done in closed form, and the joint distribution of τ is available. The same problem for general Σ and for $\mu \neq 0$ requires further investigation, and will be the subject of a subsequent paper.

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20. ABSTRACT

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