Claremont Graduate University
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Preliminary Examination on Stochastic Processes

Problem 1 (25 points)

Consider a Markov chain $\{X_n; n = 0, 1, ...\}$ having state space $\{0, 1, .., 4\}$ and transition matrix:

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.3 & \overset{0}{\cancel{0.1}} & 0.4 \\ 0 & 0.2 & 0 & 0 & 0.8 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 \\ 0 & 0.3 & 0 & 0 & 0.7 \end{bmatrix}$$

a) Find all sets of closed irreducible states.

b) For each set, determine the stationary distribution concentrated on these sets.

c) Calculate:

$$\lim_{n \to \infty} E[\exp(X_n^2) | X_0 = 0]$$

Problem 2 (25 points)

A rental car facility has N cars. Each car breaks independently, at a time distributed according to an exponential distribution with parameter λ. When a car breaks, it goes to a repair shop. The time to repair of a single car is exponentially distributed with parameter μ. Each repair time is independent. Let $X(t)$ be the number of cars in the repair shop at time t. Let

$$P_{xy}(t) = P(X(t) = y | X(0) = x)$$

a) Calculate $q_{xy} = \frac{d}{dt} P_{xy}|_{t=0}$

b) Calculate $\lim_{t \to \infty} E[X(t)]$

Hint: Let $\lambda_x = q_{x,x+1}$ and $\mu_x = q_{x,x-1}$. The unique stationary distribution $\pi(x)$ of a birth and death chain with states $\{0, .., N\}$ is given by:

$$\pi(x) = \frac{\pi_x}{\sum_{y=0}^{N} \pi_y}$$

where

$$\pi_x = \begin{cases} 1 & \text{if } x = 0 \\ \frac{\lambda_0 \ldots \lambda_{x-1}}{\mu_1 \ldots \mu_x} & \text{if } x \geq 1 \end{cases}$$

Problem 3 (25 points)
Let $U_1, .., U_n$ be independent random variables, each uniformly distributed on $(0, 1)$. Let $1[x \leq t]$ be defined by:

$$1[x \leq t] = \begin{cases} 1 & \text{if} \quad x \leq t \\ 0 & \quad\quad x > t \end{cases}$$

Then

$$X(t) = \frac{1}{n} \sum_{k=1}^{n} 1[U_k \leq t]$$

is the empirical distribution of $U_1, .., U_n$. Compute the mean and covariance functions of the $X(t)$ process.

Question 4 (25 points)
Solve the stochastic differential equation:

$$\begin{aligned} X''(t) + 4X'(t) - 21 &= W'(t) \\ X(0) &= X'(0) = 0 \end{aligned}$$

What is $E[X^2(t)]$?