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Does model misspecification matter for hedging? A computational finance experiment based approach

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Abstract

To assess whether the model misspecification matters for hedging accuracy, we carefully select six increasingly complicated asset models, i.e., the Black–Scholes (BS) model, the Merton (M) model, the Heston (H) model, the Heston jump-diffusion (HJ) model, the double Heston (dbH) model and the double Heston jump-diffusion (dbHJ) model, and then impartially evaluate their performances in mitigating the risk of an option, under a controllable experimental market. In experiments, the $\mathbb{P}$ measure asset paths are piecewisely simulated by a hybrid-model (including the Black–Scholes-type and the (double) Heston-type, with or without jump-diffusion term) with randomly given properly defined parameters. We access the hedging accuracy of six models within the operational dynamic hedging framework proposed by Sun (2015), and apply the Fourier-COS-expansion method (i.e., the COS formula, Fang and Oosterlee (2008) to price options and to calculate the Greeks). Extensive numerical results indicate that the model

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misspecification shows no significant impact on hedging accuracy, but the market fit does matter critically for hedging.

Keywords: Option pricing; double Heston model; COS method; delta hedging; dynamic hedging; implied volatility surface.

1. Introduction

Hedging aims at protecting the future Profit and Loss of a given position from a set of risk factors. What does attribute to a good hedging accuracy? Having a lucky guess of market model, or having a good fit hedging model?

There are remarkable amount of empirical literature trying to answer this question, but most of them are quarrelling with each other, which makes this question highly interesting. In academic, econometric research on the data generating process of equity indices tends to favor models with jumps (see, for example, Seeger *et al.* (2015)), but the empirical hedging literature finds little evidence that including jumps improves the hedging performance (see, for example, Bakshi *et al.* (1997) and Branger *et al.* (2012)), although Green and Figlewski (1999) point out that, an inadequate (including the bad specification, incorrect implementation, false estimation for model parameters, and so on) model may lead to a bad hedging strategy. To gauge the impact of model misspecification, Carr and Wu (2014) perform the hedging exercise assuming that the hedger does not know the true underlying price dynamics, but simply computes the delta and the static hedge portfolio weight using the Black and Scholes formula with the observed option implied volatility on the target option as the volatility input. They report the hedging accuracy shows no visible deterioration (compared with performances of other complex hedging models). However, Branger and Schlag (2004) and Kim and Kim (2005) suggest that the Heston model outperforms other stochastic volatility models in hedging, especially when jumps are added to the price process. Interestingly, Broadie *et al.* (2007) among others, claim that the Heston dynamics are grossly misspecified. Though Bakshi *et al.* (1997) argue that adding a jump feature does not improve the hedging accuracy much, He *et al.* (2006) and Kennedy *et al.* (2009) achieve good hedging accuracy using Merton model to mitigate the jump-risk of an option. Moreover, as for the effect of market fit on the hedging performance of a stochastic volatility model, Dumas *et al.* (1998) find that over-fitting of a model increases the variability of hedging errors, and frequent recalibration to option data is not even consistent with most stochastic volatility models; yet Alexander and Kaeck (2012) argue that daily recalibration of the Heston model to option prices dramatically improves its hedging performance. At this moment, there seems no ends to this quarrel in near future.

In this paper, we try to settle down the dispute by computational finance experiments, rather than by empirical study, like the above mentioned literature. We carefully select six increasingly comprehensive asset models, i.e., the Black–Scholes (BS) model, the Merton (M) model, the Heston (H) model, the Heston jump-diffusion (HJ) model, the double Heston (dbH) model and the double Heston jump-diffusion (dbHJ) model, and then evaluate impartially their performances in mitigating the risk of an option, under the following market assumptions (Sun, 2015):

- All hedgers face the same $\mathbb{P}$-measure market where nontrivial jumps may happen. Without losing generality, given the $\mathbb{P}$ measure asset price paths are piecewisely simulated by randomly chosen asset models (such as the BS-type and the (double) Heston-type, with or without jump-diffusion term), equipped with randomly given well-defined parameters.
- No one knows exactly the objective market at any trading time. All hedgers must rely their own models to hedge the risk of an option to be traded.
- All Hedgers are required to hedge the risk of the same option. They share the basic hedging settings, such as the rebalancing time, the extra hedging instruments. Besides, they use the same definition of relative P&L value, and the dynamic hedging algorithm.

These assumptions are quite important for a fair comparison of a model's hedging performance, and also critical to rightly understand the seemingly abusive generalization (Sun, 2015) of dynamic hedging strategy (proposed by He *et al.*, 2006) for various kinds of $\mathbb{Q}$-measure models in real markets.

The reasons for such a group of increasingly comprehensive models that four are stochastic volatility models and half are with jump features, are two-folded: (i) It is known that the stochastic volatility model, providing additional flexibility for describing the stochastic variance, fits better the volatility smiles than the BS model does; and the dbH model is even better than the Heston model in dealing with stiff skews. (ii) In experiments, the models without jump act as the references. By this arrangement, we can easily investigate the impacts of model misspecification on hedging accuracy.

On the methodological side, our research has three advantages over the empirical studies. First, a large number of asset price paths (in this paper $N_{\text{paths}} = 10^4$) can be investigated simultaneously, while an empirical study usually chooses carefully one piece of path in the history data of an asset. From a viewpoint of statistics, a statement drawn from a large number examples at one time would be much safer than those made by one example only. Second, we can easily determine the extent of a model's misspecification in the computational finance experiments, however, it is almost impossible to judge the model error in an empirical study, since no one knows exactly the real market model. Thirdly, probably the most importantly, the impacts on the hedging accuracy of model misspecification can be evaluated impartially, precisely and repeatedly, under a controllable market; whereas in an empirical study, it seems arbitrary to conclude that it is the model misspecification that contributes to the bad hedging performance of a model, since too many other factors, such as liquidity of instruments and uncertainty in option trading, may affect it in financial practice.

We access the hedging accuracy of six models by extensive numerical experiments within the operational dynamic hedging framework proposed by Sun (2015). We use the Fourier-COS-expansion method (i.e., the COS formula, see Fang and Oosterlee (2008)), to price options and to calculate the Greeks.

2. Financial Models

We briefly introduce a rather general asset model, i.e., the double dbHJ model, mainly for notational purpose.

As an ordinary extension of the Heston model, the dbHJ model, under the risk-neutral $\mathbb{Q}$-measure, is defined by two independent variance processes and at least one jump feature attached to the price or (and) variance process.

For convenience of analysis, in this paper we only consider a popular form of the dbHJ model, that is, the double Heston with return jump as follows,

$$\begin{cases} dS(t)/S(t^-) = (r - q - \lambda_J m)dt + \sqrt{v_1(t)}dW_1^S(t) \\ \qquad\qquad + \sqrt{v_2(t)}dW_2^S(t) + (e^J - 1)d\pi(t), \\ dv_1(t) = \kappa_1(\theta_1 - v_1(t))dt + \xi_1\sqrt{v_1(t)}dW_1^v(t), \\ dv_2(t) = \kappa_2(\theta_2 - v_2(t))dt + \xi_2\sqrt{v_2(t)}dW_2^v(t), \end{cases} \tag{1}$$

where

- r is the risk-free rate of return; q the yield flow rate; $\theta_.$ the long variance, $\kappa_.$ the rate at which $v_.(t)$ reverts to $\theta_.$; $\xi_.$ the volatility of the volatility which governs the variance of $v_.(t)$.
- $W_.^S(t)$, $W_.^v(t)$ are Wiener processes, with correlation coefficient $\rho_.$. In this paper, we follow the model structure by Gauthier and Possamaï (2011) with $\langle dW_1^S(t), dW_1^v(t)\rangle = \rho_1 dt$, $\langle dW_2^S(t), dW_2^v(t)\rangle = \rho_2 dt$, $\langle dW_1^S(t), dW_2^S(t)\rangle = 0$ and $\langle dW_1^v(t), dW_2^v(t)\rangle = 0$.
- $\pi(t)$ is a Poisson process with an intensity λ_J, J is a random jump size in the logarithm of the asset price with the probability density function $\varpi(J)$. We assume that $\mathbb{E}(e^J) < \infty$ for a smooth function f. We set $m = \mathbb{E}(e^J - 1)$ to make the discounted asset process a martingale.

By carefully parameterizing, the dynamics (1) can easily turn into different models: the BS model (1991); the Merton model; the Heston stochastic volatility model; the Bates' model and the Heston stochastic volatility jump-diffusion model (with variance jump); the dbHJ model; etc.

According to Eq. (1), the return over a time interval Δt is given by

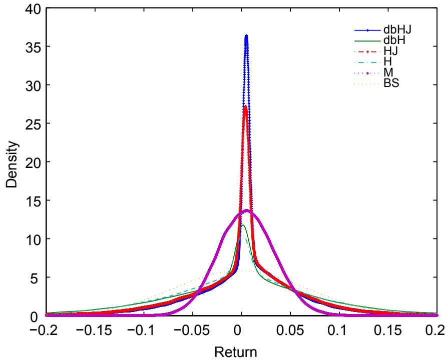
$$\frac{\Delta S(t)}{S(t)} = \exp\left\{ (r - q - \lambda_J m)\Delta t + \sqrt{v_1(t)}[W_1^S(t+\Delta t) - W_1^S(t)] \right.$$
$$\left. + \sqrt{v_2(t)}[W_2^S(t+\Delta t) - W_2^S(t)] + \sum_{i=\pi^s(t)+1}^{\pi^s(t+\Delta t)} J_s^i \right\} - 1, \tag{2}$$

where the summation over an empty set is set to be zero and the variance $v(t)$ is governed by a stochastic volatility jump-diffusion model.

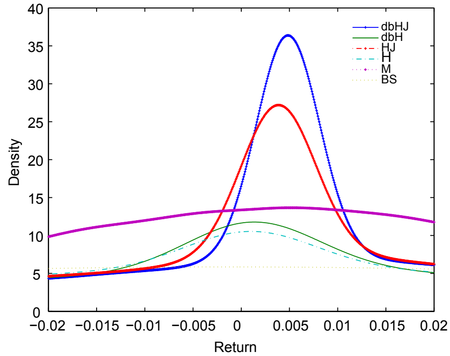
It is well known that the return distribution of BS model follows the normal distribution. However, for the stochastic volatility models, it is extremely complicate to derive an explicit form of return distribution, given the jump amplitudes distributions in return. Therefore, we approximate numerically the return distributions for stochastic volatility models by Monte Carlo method.

To gain an intuitive understanding to the leptokurtic feature of return distribution driven by the stochastic volatility and jump feature, we plot the numerical densities of return from six models in Fig. 1. For fair comparisons, parameters of six models are calibrated to the same set of option data, for example, the "V" column of options in Table 1 in Sec. 5.1. In simulation, the number of simulation paths is assigned to 10^4; the duration time is one year and $\Delta t = 1/32$ year.

The first panel of Fig. 1 compares the overall shapes of the numerical densities with the normal density driven by the BS model. The leptokurtic feature of log-return density is quite evident. The peaks of densities simulated by the dbHJ, HJ, H and M models are: 36.3695, 11.7717, 27.1889, 10.5356 and 13.6606, respectively; whereas that of the normal density is 5.8587. Evidently, the density peaks from the jump models are generally higher than those from models without jumps.



(a) The overall shape of the densities



(b) The shape around the peak area

Fig. 1. The densities of log-return simulated by the dbHJ, HJ, H, M and BS model.

Further, the peaks from stochastic volatility jump-diffusion models are higher than that from the constant volatility model, i.e., the Merton. Besides, Fig. 1 also reveals that the jump feature may have relatively more significant impact on the leptokurtic feature than the stochastic volatility feature.

This example demonstrates that when calibrated to the same option data, the dbHJ model can reproduce the most pronounced leptokurtic feature of return distribution among six models.

As for the impact on the volatility surface of the second variance and return jump, as well as of the jump feature, here we only briefly summarize it as follows. For details, please refer to Sun (2015). By adding the second variance process with a small initial variance, the dbH model is able to better deal with stiff volatility skews than the plain Heston model. Besides, the dbHJ model is further better in dealing with stiff volatility skews and smiles than the already pretty good dbH model.

3. The Operational Dynamic Hedging Framework

Following He *et al.* (2006) and Kennedy *et al.* (2009), Sun (2015) sets up an operational framework for examining impartially the dynamic hedging performance of Heston-type models in mitigating the risks of an option traded in real markets. Here, we briefly introduce this framework for notational purpose.

Denote an overall hedged position value by Π and define $\Pi := -V + eS + \mathbf{\Phi} \cdot \mathbf{I} + B$, where the explicit dependence on time t and asset price S has been dropped to ease notation. In the value formula, V is an option to be traded, with its underlying risk to be dynamically hedged by portfolio: (i) an amount B in cash; (ii) e units of the underlying asset S, and (iii) additional hedging instruments $\mathbf{I} = [\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_N]$ (written on the underlying) with weights $\mathbf{\Phi} = [\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_N]$.

First, Sun (2015) derives a mathematical representation of jump risk under the dbHJ model (1), by applying Itô lemma to Π under the $\mathbb{P}$-measure and the arbitrage-free principle, imposing the delta and pseudo-Vega neutrality condition, and assuming hedgers know exactly the $\mathbb{P}$-measure variance processes. That is,

$$d\Pi = r\Pi dt - \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}(\Delta H)dt + \Delta H d\pi^{\mathbb{P}}, \tag{3}$$

where $\Delta H = -\Delta V + e \cdot \Delta S + \mathbf{\Phi} \cdot \Delta\mathbf{I}$ in which $\Delta V = V(e^J S) - V(S), \Delta S = S(e^J - 1), \Delta\mathbf{I} = \mathbf{I}(e^J S) - \mathbf{I}(S)$; and e and $\mathbf{\Phi}$ are regarded as constant over dt.

Then, Sun (2015) continues to apply the local minimum-variance principle to further deal with the jump risk shown in (3), like He *et al.* (2006) and Kennedy *et al.* (2009). That is,

$$\arg\min_{e,\mathbf{\Phi}} F(e, \mathbf{\Phi}) = \int_{-\infty}^{\infty} \frac{1}{2}[-\Delta V + (e\Delta S + \mathbf{\Phi} \cdot \Delta\mathbf{I})]^2 \tilde{\varpi}(J)dJ, \tag{4}$$

where $F(e, \mathbf{\Phi})$ is a metric for the overall exposure to jump risk, and $\tilde{\varpi}(J)$ has the properties of a probability density, i.e., $\tilde{\varpi}(J) \geq 0$, $\int_{-\infty}^{\infty} \tilde{\varpi}(J)dJ = 1$.

It is worth noting that Eq. (4) definitely cannot be applied to hedge the risk of a traded option, if the $\mathbb{P}$ and $\mathbb{Q}$-measure market models are of different types, since Eq. (3) does not holds in this case. Unfortunately, this situation is quite popular in real financial markets.

However, market participators generally do not pay much attention to the heterogeneity of the $\mathbb{P}$ and $\mathbb{Q}$-measure market models. On the contrary, they may even think they are using the "right" model to hedge the risks of financial market, thus taking it for granted that Eq. (4) is used to dynamically hedge the risks. Though strictly, those hedgers are abusing Eq. (4) and confronted with the risk of model misspecification, we cannot simply say they are making wrong with the dynamic hedging strategy. Actually, they are judged by the market, i.e., the final hedging P&Ls. Thus, interestingly, Eq. (4) becomes one of unified frameworks that evaluate impartially the impacts of model misspecification on the hedging performance of various models, when facing with the same real-world market.

The algorithm of dynamic hedging strategy can be described as follows:

1) A weighting function $\tilde{\varpi}(J)$ must first be set.[a]
2) Initialize the weights of asset and instruments(e_0 and $\mathbf{\Phi}_0$) according to Eq. (4).
3) Calculate the initial bank account given by $B(0) = V(S_0, 0) - e_0 S_0 - \mathbf{\Phi}_0 \cdot \mathbf{I}(S_0, 0)$; hence, $\Pi(0) = 0$.
4) For $n = 1$ to M where M denotes as the total times of rebalancing
   - Solve the constrained optimization given by Eq. (4)
   - Update cash given by

$$\begin{aligned} B(t_n) &= \exp(r(t_n - t_{n-1}))B(t_{n-1}) - [e(t_n) - e(t_{n-1})]S_{t_n} \\ &\quad -[\mathbf{\Phi}(t_n) - \mathbf{\Phi}(t_{n-1})] \cdot \mathbf{I}(S_{t_n}, t_n); \end{aligned}$$

5) Calculate the overall hedged position at expiry T^* given by

$$\Pi(T^*) = -V(S_{T^*}, T^*) + B(t') \exp(r(T^* - t')) + e(t')S_{T^*} + \mathbf{\Phi}' \cdot \mathbf{I}(S_{T^*}, T^*);$$

and the relative

$$P\&L = \exp(-rT^*)\frac{\Pi(T^*)}{V(S_0, 0)}.$$

[a]According to Kennedy *et al.* (2009), a uniform-like weighting function generally performs well, and is much better than a poor guess.

4. Option and Greeks Pricing

4.1. The COS formula for option pricing

Given the interest rate r, $\tau = T - t_0$ and $x = \log(S(0)/K)$, $y = \log(S(T)/K)$, the COS formula (Fang and Oosterlee, 2008, under the plain Heston model) for plain vanilla European options under the HSVJJ model, reads:

$$\hat{V}(t_0, x, v) = e^{-r\tau} {\sum_{k=0}^{N-1}}' \Re\left(\phi\left(\frac{k\pi}{b-a}; x\right) \exp\left(-i\frac{ak\pi}{b-a}\right)\right) U_k, \tag{5}$$

where $\sum'$ indicates that the first term in the Summation is weighted by one-half, $\hat{V}(t_0, x, v)$ indicates the *approximate option value*, and

$$U_k = \frac{2}{b-a}\int_a^b V(T, y, v) \cos\left(k\pi\frac{y-a}{b-a}\right) dy, \tag{6}$$

are the Fourier cosine coefficients of $V(T, y, v)$, available in closed form for several payoff functions; notation $\phi(\omega; x)$ (where $\omega = k\pi/(b-a)$) is a short form of $\phi_X(\omega, T; x, v_1, v_2, t_0, J)$, i.e., the conditional characteristic function of y under the dbHJ model; the integration interval a, b is a truncated domain associated with a Fourier cosine series expansion of transitional density function $\varpi(y \mid x)$; the prime at the sum-symbol indicates that the first term in the expansion is multiplied by one-half; and $\Re$ means taking the real part of the argument.

For the dbHJ model (1), the characteristic function of $X_t = \log(\frac{S_t}{K})$ is of the following form:

$$\begin{aligned}&\phi_X(\omega, T; x, v_1, v_2, t_0, J_s)\\ &\quad = \exp(i\omega x + Q_0(\tau,\omega) + Q_1(\tau,\omega)v_1(t_0) + Q_2(\tau,\omega)v_2(t_0)),\end{aligned} \tag{7}$$

and we have (for details, please refer to Sun (2015)):

$$\begin{aligned}&\phi_X(\omega, T; x, v_1, v_2, t_0, J_s)\\ &\quad = \exp(i\omega x)\exp(i\omega r\tau + \lambda_J \Lambda(i\omega; J_s)\tau)\cdot\\ &\qquad \exp\left\{\sum_{j=1,2}\frac{\kappa_j\theta_j}{\xi_j^2}\left[-(D_j + B_j)\tau - 2\log\left(\frac{1 - G_j e^{-D_j\tau}}{1 - G_j}\right)\right]\right\}\cdot\\ &\qquad \exp\left\{\sum_{j=1,2}\frac{v_j(t_0)}{\xi_j^2}\left[-(D_j + B_j)\left(\frac{1 - e^{-D_j\tau}}{1 - G_j e^{-D_j\tau}}\right)\right]\right\},\end{aligned} \tag{8}$$

where $\tau = T - t_0$, and $A_j = \frac{1}{2}\xi_j^2, B_j = i\xi_j\rho_j\omega - \kappa_j, C_j = -\frac{1}{2}\omega(i+\omega), D_j = \sqrt{B_j^2 - 4A_jC_j}$ and $G_j = \frac{B_j + D_j}{B_j - D_j}$; and $\Lambda(\Phi; J_s) = \int_{-\infty}^{\infty} e^{J_s\Phi}\varpi_s(J_s)dJ_s - 1 - m\Phi$.

Given $J \sim N(\mu_J, \sigma_J^2)$, a simple calculation yields

$$\Lambda(\Phi) = e^{\mu_J\Phi+\frac{1}{2}\sigma_J^2\Phi^2} - 1 - \Phi(e^{\mu_J+\frac{1}{2}\sigma_J^2} - 1), \tag{9}$$

Integration domain $[a, b]$ is determined so that

$$\int_{\Re} f(y\,|\,x)dy - \int_a^b f(y|x)dy < \text{TOL (Tolerance)}. \tag{10}$$

By following Fang and Oosterlee (2008), a and b are given by

$$a = c_1 - L\sqrt{c_2 + \sqrt{c_4 + \cdots}}; \quad b = c_1 + L\sqrt{c_2 + \sqrt{c_4 + \cdots}}, \tag{11}$$

where c_1, c_2 and c_4 are the 1st, 2nd and 4th-order cumulant, respectively. Thus, given a parameter L, integration interval $[a, b]$ in(11) can be easily determined, though tediously, as in Carr and Wu (2014).

4.2. The COS formula for Greeks

Given the explicit formulas of characteristic function $\phi_X(\omega, T; x, v_1, v_2, t_0, J_s)$ by Eq. (8), we can easily derive the delta and pseudo-Vega $\left(\frac{\partial \mathbf{V}}{\partial S}, \frac{\partial \mathbf{V}}{\partial \upsilon_j}, \frac{\partial \mathbf{I}}{\partial S}, \frac{\partial \mathbf{I}}{\partial \upsilon_j}\right.$ where $j = 1, 2$).

$$\frac{\partial V}{\partial S} \approx \frac{e^{-r\tau}}{S} \sum_{k=0}^{N-1}{}' \Re\left(\imath \frac{k\pi}{b-a} \phi\left(\frac{k\pi}{b-a}; x, \upsilon_1, \upsilon_2, t_0, J_s\right) \exp\left(-\imath \frac{k\pi a}{b-a}\right)\right) V_k, \tag{12}$$

$$\begin{aligned}\frac{\partial V}{\partial \upsilon_j} \approx e^{-r\tau} \sum_{k=0}^{N-1}{}' \Re\Bigg\{\frac{1}{\xi^2}\left[-(D_j + B_j)\frac{1 - e^{-D_j\tau}}{1 - G_j e^{-D_j\tau}}\right]. \\ \phi\left(\frac{k\pi}{b-a}; x, \upsilon_1, \upsilon_2, t_0, J_s\right) \exp\left(-\imath \frac{k\pi a}{b-a}\right)\Bigg\} V_k.\end{aligned} \tag{13}$$

Other Greeks such as $\frac{\partial \mathbf{I}}{\partial S}$ and $\frac{\partial \mathbf{I}}{\partial \upsilon_j}$ can be obtained similarly.

Simple analysis to delta and pseudo-Vega yields

$$\lim_{t\to T_V} \frac{\partial V}{\partial S} = \begin{cases} 1, & \text{if } S(T_V) > K, \\ 0, & \text{if } S(T_V) < K, \end{cases} \tag{14}$$

$$\lim_{t\to T_{\mathbf{I}}} \frac{\partial \mathbf{I}}{\partial S} = \begin{cases} 1, & \text{if } S(T_{\mathbf{I}}) > K, \\ 0, & \text{if } S(T_{\mathbf{I}}) < K, \end{cases} \tag{15}$$

$$\lim_{t \to T_V} \frac{\partial V}{\partial \upsilon_j} = 0, \quad \text{and} \quad \lim_{t \to T_{\mathbf{I}}} \frac{\partial \mathbf{I}}{\partial \upsilon_j} = 0. \tag{16}$$

This means, the delta always finishes at 1 for options that expire in-the-money and 0 for options that expire out-of-the-money. The pseudo-Vega always come to 0 for options at the expiry.

The limits analysis above indicates that the linear system that is made of the neutrality equation and the necessary condition of the local minimum-variance problem (4) may become ill-conditioned, when the rebalancing time approximates the expiry of hedging instruments.

In order to accommodate the possibility of ill-conditioned system, a singular value decomposition (SVD) approach is adopted in the implementation of dynamic hedging strategy, at cost of losing accuracy in calculating the portfolio weights.

5. Numerical Experiments on Hedging

In this section, we evaluate and compare the hedging accuracy of the carefully selected six models.

5.1. The market data

For $\mathbb{P}$-measure market, the asset price paths are piecewise-simulated by hybrid models (including the Black–Scholes-type and the (double) Heston-type, with or without jump-diffusion term) with any given properly defined parameters. Figure 2 shows $N_{\text{paths}} = 10^4$ simulated $\mathbb{P}$-measure asset price paths.

In financial practice, the $\mathbb{Q}$-measure market models are usually calibrated to observed option market data. Since there is no such data in this experimental research, we need to generate option data. There are two ways: one is using the simulated $\mathbb{P}$-measure asset price paths to generate artificially the option data; the other is applying directly the $\mathbb{P}$-measure asset market models to simulate the option market. For convenience, we use the second way to create the option data (recorded in Table 1), then calibrate the BS, Merton, plain Heston, HJ, dbH and dbHJ model to the artificial option data, individually.

The fit errors (defined in Eq. (17)) are shown in Fig. 3. The parameters of periodically calibrated hedging models are plotted in Fig. 4. Since the $\mathbb{P}$-measure paths are randomly simulated by hybrid models, actually any single one of six hedging models is believed to be definitely misspecified (Kou, 2008).

Table 1. The simulated option data. For simplicity, we only focus on European Calls with one-year expiry and spot price 100. Assume the frequency of recalibration is every three rebalancing times. Accordingly, we need to create $\frac{\text{The total rebalancing times}}{\text{Recalibration frequency}}$ sets of option data. In this example, we need recalibrate all hedging models independently for $\frac{21}{3} = 7$ times during the whole hedging period. The recalibration time (also the rebalancing time), is labeled as I, II, ... and VII in table.

	Option data						
Strikes	I	II	III	IV	V	VI	VII
50	56.3137	56.0954	56.3473	56.3192	56.3761	56.0953	56.2056
60	47.805	47.3182	47.8653	47.7722	47.8628	47.3169	47.6357
70	39.5178	38.5799	39.6154	39.3862	39.5119	38.5717	39.3179
80	31.5747	30.0412	31.7234	31.2451	31.4009	30.0121	31.4268
90	24.1478	22.0801	24.3694	23.4834	23.663	22.0138	24.179
100	17.4663	15.203	17.7995	16.3274	16.545	15.0943	17.8019
110	11.8019	9.7907	12.3048	10.1616	10.5087	9.6516	12.483
120	7.4007	5.9186	8.116	5.5403	6.2221	5.7711	8.3161
130	4.3463	3.3817	5.2316	2.7861	3.7759	3.2464	5.2683
140	2.4577	1.8417	3.3885	1.4372	2.4451	1.7309	3.1891
150	1.3818	0.9646	2.2436	0.7975	1.6743	0.8815	1.86
160	0.7894	0.49	1.5274	0.4742	1.1979	0.4319	1.0561
170	0.4629	0.2434	1.0687	0.298	0.887	0.2049	0.5899
180	0.2795	0.1192	0.7666	0.1955	0.6748	0.0947	0.3271
190	0.1738	0.0579	0.5621	0.1328	0.5247	0.0428	0.1814
200	0.111	0.028	0.4202	0.0927	0.4153	0.0191	0.1012
210	0.0727	0.0136	0.3194	0.0663	0.3335	0.0084	0.057
220	0.0487	0.0067	0.2465	0.0483	0.2713	0.0036	0.0325
230	0.0333	0.0033	0.1928	0.0358	0.223	0.0016	0.0188
240	0.0232	0.0017	0.1526	0.027	0.1851	0.0007	0.011
250	0.0164	0.0008	0.122	0.0206	0.155	0.0003	0.0065

Definition 1 (The fit error).

$$\text{The fit error} = \frac{1}{N} \frac{\sum_{i=1}^{N} e^{-2\ln^2\left(\frac{\text{Strike}_i}{K_0}\right)} \left(\text{Option}_i^{cali,Q} - \text{Option}_i^{mkt,P}\right)^2}{\sum_{i=1}^{N} e^{-2\ln^2\left(\frac{\text{Strike}_i}{K_0}\right)}}, \tag{17}$$

where $\text{Option}_i^{mkt,P}$ and $\text{Option}_i^{cali,Q}$ denote the ith observed $\mathbb{P}$-measure and the calibrated $\mathbb{Q}$-measure option market values, respectively; N means the number of options to be calibrated; $Strike_i$ the strike of the ith option; and K_0 the strike of an option to be hedged.

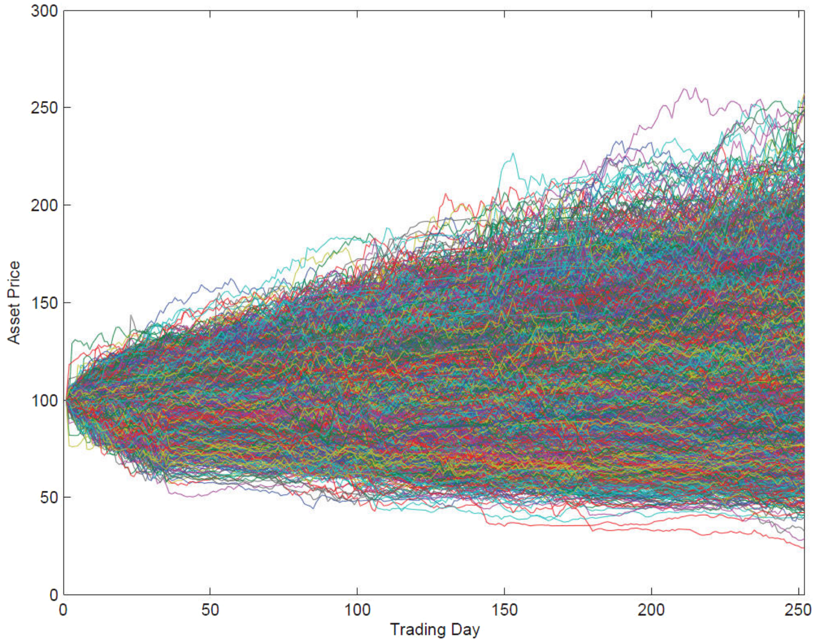


Fig. 2. $N_{\text{paths}} = 10^4$ $\mathbb{P}$-measure price paths ($S_0 = 100$, $T = 1y$).

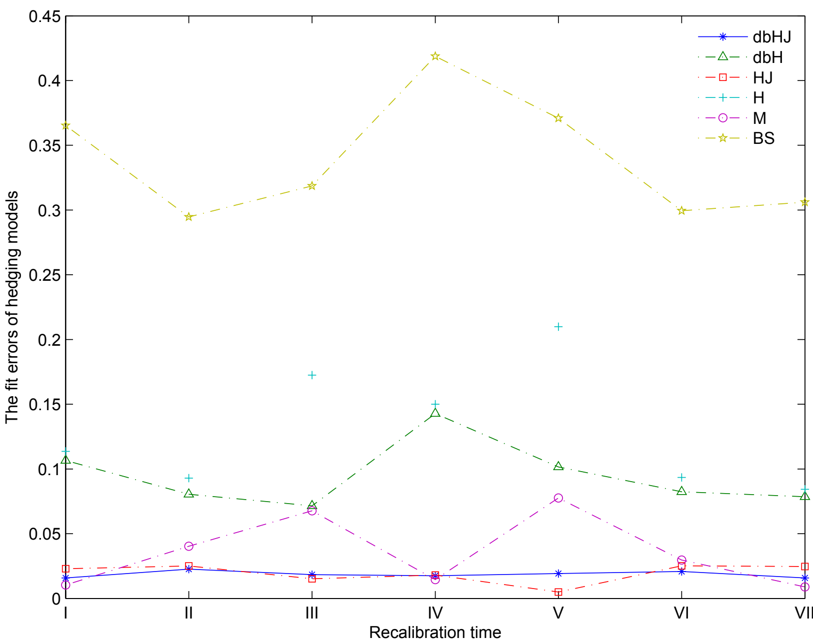
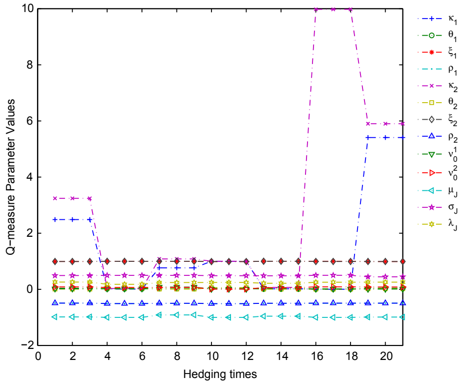
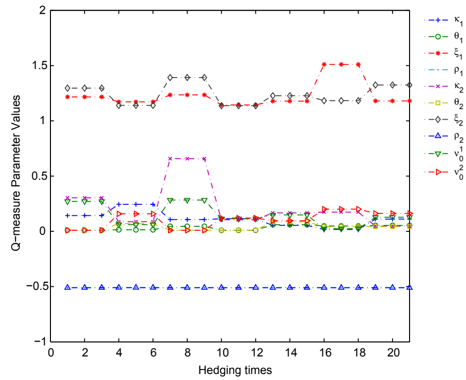


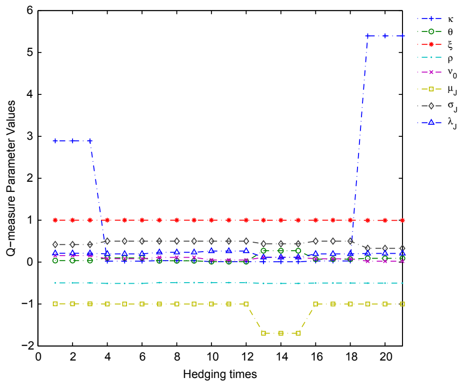
Fig. 3. The fit errors of calibrated $\mathbb{Q}$-measure market models during the hedging period.



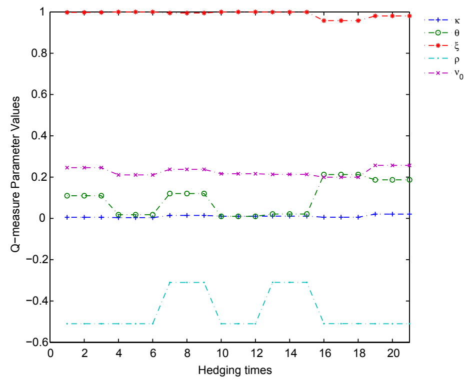
(a) The dbHJ model



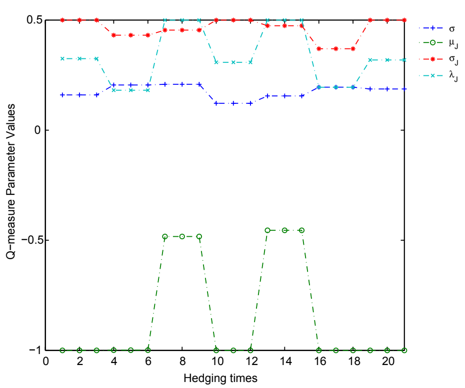
(b) The dbH model



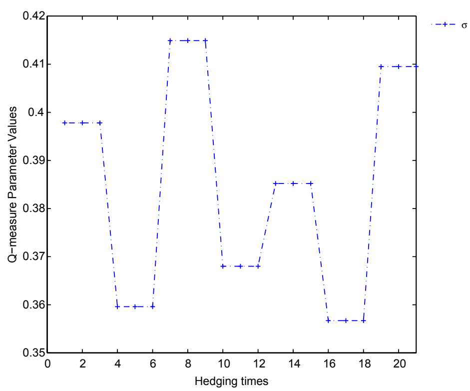
(c) The HJ model



(d) The H model



(e) The M model



(f) The BS model

Fig. 4. The recalibrated parameters of six hedging models are plotted at each rebalancing time. These curves show clearly how the calibrated parameters in various hedging models vary as the underlying market changes.

Remark 1 (On the fit error).

(i) The fit error (also so-called calibration function) can be defined in many different ways, such as mean square error (MSE), root mean square error (RMSE), average absolute error (AAE), and maximum absolute relative error (MARE), etc. In this paper, we consider the widely-used MSE as the definition of fit error.

(ii) In financial practice, in order to give more importance to the most liquid options, practitioners usually take into account the bid-ask spreads when building the calibration function. However, since there is no any trade data in this research, we don't consider the ask/bid spreads, but do weigh more on options whose strikes are closer to K_0.

(iii) Some practitioners suggest that it might be better to minimize the differences of implied volatilities instead of those of the option prices. However, this involves additional computational cost. Therefore, Lindstrom *et al.* (2008) suggest that a reasonable approximation is to minimize the square differences of option prices.

Figure 3 shows evidently that, during the whole hedging period, the models with jump feature generally have smaller fit errors than the models without. Further, among all models with jump-diffusion, the dbHJ model exceeds the HJ model, whilst both are superior to the Merton model. For models without jump feature, we have a similar observation, i.e., the dbH model generally surpasses the plain Heston model, whilst both are better than the Merton model in fitting the option data.

Figure 4 plots the periodically recalibrated parameters of six hedging models along the rebalancing time. These curves show clearly how the calibrated parameters in various hedging models vary as the underlying market progresses. For example, Fig. 4(a) shows the recalibrated parameters κ_1 and κ_2 in the dbHJ model change dramatically along the hedging time, whilst the rest wave much more gently. Figure 4 indicates that: (i) the underlying $\mathbb{P}$-world market does change randomly as time passes by; (ii) All hedgers manage their own models to hedge the risk of an option to be traded, since no one knows exactly the objective market at any trading time.

5.2. Basic settings for hedging

The option to be hedged is a one-year European call option with the underlying asset following hybrid models (including the BS types and Heston types model, with or without jump feature).

Based on the liquidity considerations, the hedging instruments are 0.25-year European calls and puts with strikes in increments of ± 10 of the underlying's current value S_0. In implementation, we take the spot price $S_0 = 100$, the $\mathbb{P}$ and $\mathbb{Q}$-measure interest rate $r^{\mathbb{P}} = 0.13$ and $r^{\mathbb{Q}} = 0.03$, and yields rate $q = 0$ for all hedging schemes. Each simulation set consists of 10^4 individual asset paths.[1] The rebalancing time is 0.05 year.

Just before each rebalancing, the hedgers recalibrate their own models (i.e., the six models) to the same option data listed in Table 1. For hedgers who use Heston-type models as the hedging model, they should retrieve the variance process. Generally, an operational $\upsilon_t^{\mathbb{Q}}$ for hedging with Heston-type models, can be constructed by using directly the implied volatility (see Nalholm and Poulsen, 2005), or fixing $\upsilon_t^{\mathbb{Q}} = \upsilon_0$, or applying the particle Kalman Filter to retrieve the variance, etc. In this paper, we simply take $\upsilon_t^{\mathbb{Q}} = \upsilon_0$ for Heston-type models.

5.3. Numerical results of dynamic hedging

According to the hedging schemes listed in Table 2, we execute dynamic hedging (with extra instruments strategy) algorithms. The relative P&L values are recorded in Table 3 below.

We have the following observations about the hedging results:

- The numerical P&L values in Table 3 suggest that, the dynamic hedging algorithm in this research is both effective and fast convergent. The convergence is also evidently shown in Fig. 5(a).

Table 2. Hedging schemes. In table, "BS" means the Black–Scholes model; "M" means the Merton model; "H" means the Heston model; "HJ" means the Heston stochastic volatility and jump-diffusion model; so as to "dbH" and "dbHJ".

Scheme	Market model	Writer's model	Hedging strategy
1		dbHJ	Dynamic hedging (with extra options)
2		dbH	Dynamic hedging (with extra options)
3	Randomly	HJ	Dynamic hedging (with extra options)
4	Simulated by	H	Dynamic hedging (with extra options)
5	hybrid-models	M	Dynamic hedging (with extra options)
6		BS	Dynamic hedging (with extra options)

[1]Our hedging algorithm is robust. It makes only trivial difference between the hedging results for 10^4 simulation paths and much less, say 10^3.

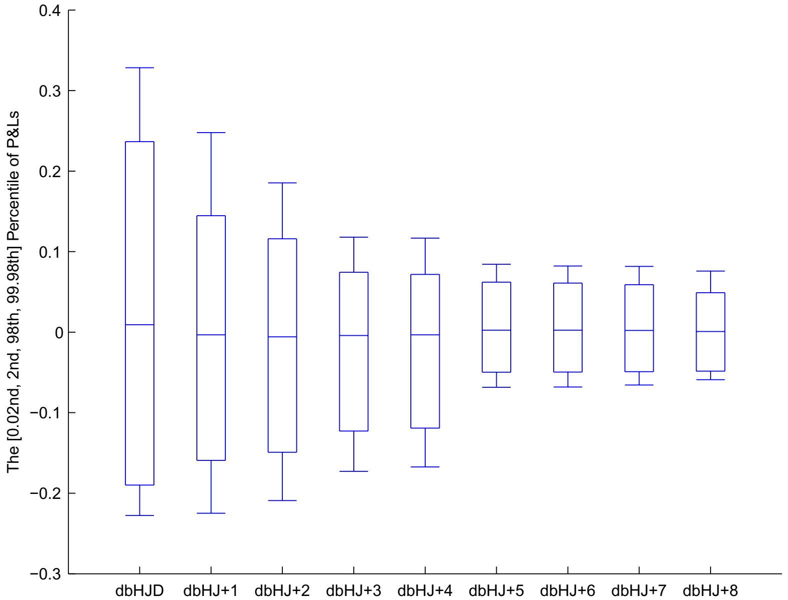
Table 3. Relative P&Ls under different hedging models: The double Heston with jump-diffusion (dbHJ) model, Heston with jump-diffusion (HJ) model, double Heston (dbH) and plain Heston (H) model, Merton (M) and Black–Scholes (BS) model. For simplicity of notation, we denote BSD as the BS based Delta hedging, HJD as the HJ based Delta hedging, so as to dHJD, dHD, HD and MD. Notation "+ number" in the first column, means that an extra number of instruments are needed for this kind of hedging strategy.

			Percentiles			
Hedging strategy	Mean	Std. Dev.	0.02%	0.2%	99.8%	99.98%
dbHJD	0.0092	0.1054	−0.2276	−0.1899	0.2366	0.3284
dbHJ+1	−0.0034	0.0691	−0.2248	−0.1592	0.1446	0.2478
dbHJ+2	−0.0058	0.0593	−0.2091	−0.1492	0.1159	0.1854
dbHJ+3	−0.0041	0.0452	−0.1729	−0.1228	0.0743	0.1179
dbHJ+4	−0.0034	0.0427	−0.1674	−0.1192	0.0717	0.1168
dbHJ+5	0.0025	0.0319	−0.0684	−0.0497	0.0621	0.0843
dbHJ+6	0.0025	0.0317	−0.0681	−0.0495	0.0610	0.0822
dbHJ+7	0.0022	0.0311	−0.0656	−0.0491	0.0589	0.0816
dbHJ+8	8.5309e−04	0.0296	−0.0591	−0.0484	0.0490	0.0759
dbHD	0.0194	0.0734	−0.2393	−0.1586	0.1696	0.2243
dbH+1	0.0164	0.0676	−0.2301	−0.1425	0.1586	0.2019
dbH+2	0.0166	0.0637	−0.2135	−0.1288	0.1561	0.2006
dbH+3	0.0173	0.0579	−0.1905	−0.1081	0.1539	0.1982
dbH+4	0.0159	0.0559	−0.1821	−0.1012	0.1529	0.1984
dbH+5	0.0091	0.0528	−0.1752	−0.1115	0.1349	0.1891
dbH+6	0.0053	0.0491	−0.1737	−0.1075	0.1237	0.1854
dbH+7	0.0050	0.0489	−0.1737	−0.1075	0.1236	0.1854
dbH+8	0.0047	0.0486	−0.1737	−0.1068	0.1236	0.1854
HJD	0.0968	0.1319	−0.3410	−0.2160	0.3279	0.3905
HJ+1	0.0198	0.1127	−0.2469	−0.1996	0.2262	0.2744
HJ+2	−0.0040	0.0767	−0.2331	−0.1729	0.1608	0.2480
HJ+3	−0.0088	0.0723	−0.2257	−0.1719	0.1435	0.2236
HJ+4	−0.0146	0.0594	−0.2150	−0.1617	0.0921	0.1778
HJ+5	−0.0140	0.0558	−0.2059	−0.1538	0.0828	0.1670
HJ+6	−0.0122	0.0410	−0.1199	−0.1015	0.0497	0.0756
HJ+7	−0.0120	0.0406	−0.1184	−0.0993	0.0496	0.0753
HJ+8	−0.0117	0.0400	−0.1174	−0.0979	0.0495	0.0732
HJ+9	−0.0037	0.0306	−0.0770	−0.0571	0.0489	0.0659
HD	0.0425	0.0855	−0.2684	−0.1703	0.2096	0.2542
H+1	0.0316	0.0752	−0.2151	−0.1489	0.1716	0.2161
H+2	0.0290	0.0672	−0.1884	−0.1212	0.1658	0.2077
H+3	0.0277	0.0670	−0.1884	−0.1210	0.1658	0.2077
H+4	0.0265	0.0659	−0.1884	−0.1201	0.1655	0.2077
H+5	0.0258	0.0655	−0.1913	−0.1194	0.1650	0.2074
H+6	0.0258	0.0654	−0.1890	−0.1194	0.1650	0.2074

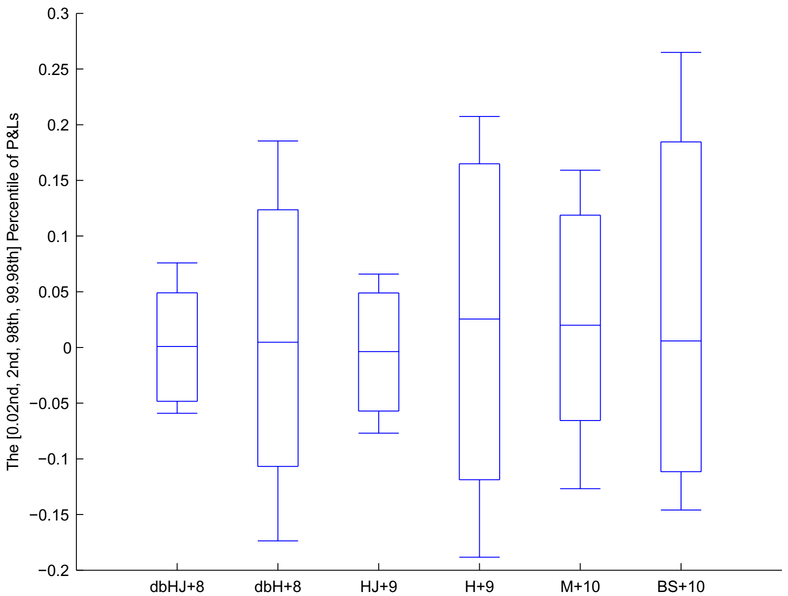
Table 3. (*Continued*)

Hedging strategy	Mean	Std. Dev.	Percentiles 0.02%	0.2%	99.8%	99.98%
H+7	0.0258	0.0654	−0.1890	−0.1194	0.1650	0.2074
H+8	0.0258	0.0654	−0.1890	−0.1190	0.1650	0.2074
H+9	0.0255	0.0650	−0.1884	−0.1187	0.1649	0.2074
MD	0.3923	0.2120	−0.4212	−0.1464	0.6570	0.7382
M+1	0.1837	0.1305	−0.2550	−0.1102	0.3853	0.4301
M+2	0.0553	0.0845	−0.1763	−0.1186	0.2181	0.2980
M+3	0.0413	0.0739	−0.1681	−0.1099	0.1793	0.2307
M+4	0.0347	0.0587	−0.1214	−0.0750	0.1618	0.1963
M+5	0.0341	0.0552	−0.1035	−0.0583	0.1610	0.1954
M+6	0.0275	0.0461	−0.0929	−0.0495	0.1368	0.1827
M+7	0.0213	0.0450	−0.1249	−0.0645	0.1207	0.1629
M+8	0.0204	0.0449	−0.1280	−0.0658	0.1194	0.1591
M+9	0.0199	0.0448	−0.1268	−0.0656	0.1189	0.1591
M+10	0.0199	0.0448	−0.1268	−0.0656	0.1188	0.1591
BSD	0.3060	0.1699	−0.3220	−0.0643	0.6456	0.7243
BS+1	0.1510	0.0942	−0.1941	−0.0647	0.3069	0.3438
BS+2	0.0491	0.1135	−0.2340	−0.1881	0.2416	0.3088
BS+3	0.0155	0.0950	−0.2028	−0.1649	0.2161	0.3080
BS+4	0.0084	0.0790	−0.1900	−0.1501	0.1914	0.2649
BS+5	0.0056	0.0759	−0.1871	−0.1480	0.1898	0.2649
BS+6	0.0042	0.0718	−0.1776	−0.1383	0.1876	0.2649
BS+7	0.0032	0.0693	−0.1703	−0.1324	0.1864	0.2649
BS+8	0.0032	0.0690	−0.1703	−0.1309	0.1854	0.2649
BS+9	0.0059	0.0635	−0.1460	−0.1116	0.1845	0.2649
BS+10	0.0059	0.0635	−0.1460	−0.1115	0.1845	0.2649

- Figure 5(b) shows clearly that the hedging accuracy from models with jump-diffusion (i.e., the dbHJ model, the HJ model and the Merton model), outperform evidently those from without considering jump feature (i.e., the dbH model, the Heston model and the BS model). In fact, Fig. 5(b) presents that none of models without jump-diffusion, can well mitigate the jump-risk of the option.
- Even for models with jump feature, the stochastic volatility models (i.e., the dbHJ and HJ model) perform better the nonstochastic volatility (i.e., the Merton model) in hedging the jump risk of the traded option. Table 3 records, the dbHJ model with extra n instruments, generally exceed (if not, at least equals to), the HJ model with extra $n + 1$ instruments, which is further better than the Merton model with extra $n + 2$ extra instruments. For example, Fig. 5(b) shows the hedging accuracy of strategy dbHJ+8 outperforms that of HJ+9, whilst both are much better than that of M+10.



(a) The convergence of dynamic hedging strategy (the case of the dbHJ model)



(b) The accuracy of dynamic hedging strategy

Fig. 5. The hedging performances of six models. Box description: On each box, the central mark is the mean of P&Ls, the edges of the box are the 2nd (below) and 98th (upper) percentiles, the whiskers extend to the most extreme (i.e., the 0.02nd and 99.98th percentiles) data points not considered outliers.

5.4. Discussions

According to Kou (2008), all models in financial practice are essentially "wrong" and only rough approximations of reality. From this viewpoint, all of six hedging models in above hedging experiments can be said "wrong" (or misspecified), since the $\mathbb{P}$-measure market are piecewise-simulated by hybrid models (blind to all hedgers) in this research.

Interestingly, our hedging results show that the models with jump feature (though misspecified) do achieve good accuracy. This indicates that the model misspecification does not shows significant impact on the hedging accuracy.

We explain this as follows: though misspecified, the jump-diffusion models are able to reproduce the leptokurtic and skew feature of the return distribution (Sun, 2015; Kou, 2008), thus leading to a good fit to option markets.

We can also interpret why the dbHJ and HJ model are better than the Merton model in mitigating the jump risk of options, though three models are of jump feature, by appealing for the good property of stochastic volatility feature, i.e., capturing well the volatility clustering effects of return.

Now that both stochastic volatility and jump features can improve significantly the market fit of an asset model, no wonder the dbHJ model has the best hedging accuracy among six models. However, we cannot attribute the good hedging performance of the dbHJ model to the lucky guess of the underlying market, since it changes randomly during the whole hedging period. Essentially, it is the model's distinguished market fit that brings forward the excellent hedging performance.

Likewise, it is the not-so-good fits of the BS model, the plain Heston model and the double Heston model that explain their unsatisfactory hedging accuracy, when the underlying market follows jump processes.

At this moment, there appears to arrive an inner logic between the hedging performance of a model and its market fit: If a model can describe the underlying asset market better, for example, capturing more precisely the empirical phenomena such as skewness, fat tails and excess kurtosis of the return distribution, then this model can have a closer fit to the option market, which thus results in a superior hedging accuracy.

Since 10^4 randomly simulated $\mathbb{P}$-measure asset paths are considered in this research, the above conclusion is robust with respect to the underlying market.

6. Concluding Remarks

This paper investigates what does help to achieve a good hedging accuracy: having a lucky guess of market model, or having a good fit hedging model. The numerical hedging results of six increasingly comprehensive models, uniting with results from calibration experiments, provide a sure answer to this question: the model misspecification shows no significant impact on hedging accuracy, but the market fit does matter critically for hedging. This paper suggests that it is advisable for a hedger to work on a well calibrated hedging model, rather than wasting time in guessing the "right" specification of $\mathbb{P}$-measure market model.

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