An Asymptotic Expansion of Forward-Backward SDEs with a Perturbed Driver

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Abstract

This paper presents a mathematical validity of an asymptotic expansion scheme for a multiscale system of forward-backward stochastic differential equations (FBSDEs) in terms of a perturbed driver in the BSDE and a small diffusion in the FSDE. This computational scheme was proposed by Fujii and Takahashi [11], which has been successfully employed to solve the derivatives and optimal portfolio problems in Fujii and Takahashi [12] [13] and Fujii et al. [10]. In particular, we represent the coefficients up to an arbitrary order expansion of the BSDE by the solution to a system of the associated BSDEs with the FSDE, and obtain the error estimate of the expansion with respect to the driver perturbation. Accordingly, we show a concrete representation for each expansion coefficient of the volatility component, that is the martingale integrand in the BSDE. Then, we apply our proposed FSDE expansion formula with its precise error estimate to the BSDE expansion coefficients to finally obtain the total residual estimate.

Keywords Multiscale Forward-Backward SDEs, Asymptotic expansion, Malliavin calculus

1 Introduction

This paper investigates the mathematical foundation of an asymptotic expansion scheme for a multiscale system of forward-backward SDEs (FBSDEs). In particular, we concentrate on to provide a mathematical validity for the decoupled case of the scheme, which is mainly addressed in their paper.

The FBSDEs has become quite popular in finance community since El Karoui et al. [8], especially after the recent financial crises and the subsequent quite volatile markets, which leads us to recognize the importance of counter party risk management, particularly the credit value adjustments (CVA).

However, an explicit solution for a FBSDE has been known only for a simple linear or quadratic example. Although several techniques have been proposed in the last decade,

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they seem very limited in practical applications since they rely on numerical methods for non-linear PDEs or regression based Monte Carlo simulations, which are generally very difficult to implement or quite time-consuming especially for high-dimensional and long-horizon problems.

Recently, [11] has developed a simple analytical approximation scheme for the non-linear FBSDEs. They have introduced a perturbation parameter to the driver of a BSDE to expand recursively the non-linear terms around a relevant linear FBSDE. In the computation of each order, we explicitly represent the backward elements as the functions of the forward components and take those expectations. Hence, except the cases that the distributions of the forward process are explicitly known, we apply some approximations of the distributions such as an asymptotic expansion technique, which is widely applied to the analytical approximations for pricing European contingent claims and computing optimal portfolios. (For example, see [11] [12], Takahashi and Yamada [24] [26] and references therein for the details.)

They also provided two numerical examples, where the second-order analytic approximations work quite well compared to numerical techniques such as the finite difference method and the regression-based Monte Carlo simulation.

Moreover, their subsequent work [12] has applied this scheme to the optimal portfolio problem in an incomplete market with stochastic volatility, and demonstrated the accurate approximations even for long maturities such as 10 years, as opposed to the regression based Monte Carlo simulation that works well only up to short maturities such as one year. We also note that the method has the great advantage of deriving explicit expressions of the optimal portfolios and hedging strategies, that is very important in practice. Further, we can use the method for the general multi-dimensional cases, which is not true of the well-known Cole-Hopf transformation. As for the recent development of this scheme with interacting particle method, see [13] and [10].

In a different stream, Takahashi and Yamada [25] has proposed a new closed-form approximation for the solutions of FBSDEs. In particular, applying Malliavin calculus approach of Kusuoka [16] and [24] [26] to the forward SDEs with the Picard-iteration scheme for the BSDEs, they have obtained an error estimate for the approximation. Moreover, they have demonstrated the effectiveness of the method through numerical examples for pricing options with counter party risk under the local and stochastic volatility models, where the credit value adjustment (CVA) is taken into account.

This paper provides a mathematical foundation for the original scheme for a multiscale FBSDE. (The justification for the coupled case will be one of our next research topics.) It mainly consists of two parts. That is, for the BSDE expansion with a perturbed driver we obtain the coefficients up to an arbitrary order as the solution to a system of the associated BSDEs with the base FSDE, and present the error estimate of the expansion. Accordingly, we show a concrete representation for each expansion coefficient of the volatility component, that is the martingale integrand in the BSDE. For the FSDE expansion, we derive an expansion formula with its sharp error estimate for the expectation of the solution to the base FSDE in terms of a small diffusion. Then, we combine the both results, particularly applying our FSDE expansion formula to the BSDE expansion coefficients to obtain our main result, that is an asymptotic expansion of FBSDEs with a perturbed driver. In the proofs, we effectively apply the representation results in Ma and Zhang [19] for the BSDE expansion and the properties of the Kusuoka-Stroock functions in [16] for the FSDE expansion.

The organization of the paper is as follows: after the next section describes the basic setup, Section 3 provides the result for the expansion of the BSDE with respect to a perturbation parameter in the driver. Section 4 shows an expansion for the FSDE in terms of a small diffusion, which is combined with the asymptotic expansion for the BSDE in Section 3 to present our main result in Section 5.

2 Multiscale FBSDE

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which a d-dimensional Brownian motion W is defined. Let $\mathbf{F} = \{\mathcal{F}_t\}$ be the natural filtration generated by W, augmented by the P-null sets of $\mathcal{F}$. We first consider the following d-dimensional forward stochastic differential equation with parameter ε, $(X_t^\varepsilon)_t$ with $X_t^\varepsilon = (X_t^{\varepsilon,1}, \cdots, X_t^{\varepsilon,d})$:

$$\begin{aligned} dX_t^{\varepsilon,i} &= b^i(t, X_t^\varepsilon)dt + \varepsilon \sum_{j=1}^d \sigma_j^i(t, X_t^\varepsilon)dW_t^j, \quad i = 1, \cdots, d, \qquad (2.1) \\ X_0^{\varepsilon,i} &= x_0^i \in \mathbf{R}, \ i = 1, \cdots, d, \end{aligned}$$

where $b : [0, T] \times \mathbf{R}^d \mapsto \mathbf{R}^d$, $\sigma : [0, T] \times \mathbf{R}^d \mapsto \mathbf{R}^{d\times d}$ and $\varepsilon \in (0, 1]$.

Next, given FSDE (2.1), we introduce $(Y^{\alpha,\varepsilon}, Z^{\alpha,\varepsilon})$ with a perturbation parameter $\alpha \in [0, 1]$, which is the solution of the following BSDE:

$$\begin{aligned} Y_t^{\alpha,\varepsilon} &= g(X_T^\varepsilon) + \alpha \int_t^T f(s, X_s^\varepsilon, Y_s^{\alpha,\varepsilon}, Z_s^{\alpha,\varepsilon})ds \\ &\quad - \int_t^T Z_s^{\alpha,\varepsilon} \cdot dW_s, \qquad (2.2) \end{aligned}$$

or equivalently, as the differential form:

$$\begin{aligned} dY_t^{\alpha,\varepsilon} &= -\alpha\, f(t, X_t^\varepsilon, Y_t^{\alpha,\varepsilon}, Z_t^{\alpha,\varepsilon})dt + Z_t^{\alpha,\varepsilon} \cdot dW_t, \qquad (2.3) \\ Y_T^{\alpha,\varepsilon} &= g(X_T^\varepsilon), \end{aligned}$$

where $x \cdot y$ denotes the inner product of $x, y \in \mathbf{R}^d$, that is $x \cdot y = \sum_{i=1}^d x^i y^i$ for $(x^1, \cdots, x^d)$ and $y = (y^1, \cdots, y^d)$.

In the following we state the assumptions for the forward-backward SDE in this paper.

Assumption 2.1.

1. *The coefficients of the forward process, b, σ are bounded Borel functions. Moreover, $b(t, x)$ and $\sigma(t, x)$ are continuous in (t, x) and smooth in x with bounded derivatives of all orders.*
2. *There exist constants $a_i > 0$, $i = 1, 2$ such that for any vector ξ in $\mathbf{R}^d$ and any $(t, x) \in [0, T] \times \mathbf{R}^d$,*

$$a_1|\xi|^2 \le \sum_{i,j=1}^d [\sigma\sigma^T]_{i,j}(t, x)\xi_i\xi_j \le a_2|\xi|^2.$$

3. *The driver* $f : [0,T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d \mapsto \mathbf{R}$ *is continuous and bounded. Moreover,* $f(t,x,y,z)$ *is smooth in* x, y, z *with bounded derivatives of all orders.*
4. $g : \mathbf{R}^d \mapsto \mathbf{R}$ *is smooth with bounded derivatives of all orders, and* $|g(0)| \leq K$ *for a positive constant* K.

Under the assumption above, there exists the unique solution $(Y^{\alpha,\varepsilon}, Z^{\alpha,\varepsilon})$ such that for any $p > 1$, $E\left[\sup_{0\leq s\leq T} |Y_s^{\alpha,\varepsilon}|^p\right] + E\left[\left(\int_0^T |Z_s^{\alpha,\varepsilon}|^2 ds\right)^{p/2}\right] < \infty$. (e.g. See Theorem 5.1 in [8].)
Then, it also holds that

$$Y_t^{\alpha,\varepsilon} = E\left[Y_t^{\alpha,\varepsilon}|\mathcal{F}_t\right] = E\left[g(X_T^{\varepsilon})|\mathcal{F}_t\right] + \alpha E\left[\int_t^T f(s, X_s^{\varepsilon}, Y_s^{\alpha,\varepsilon}, Z_s^{\alpha,\varepsilon})ds|\mathcal{F}_t\right].$$

We note that when $\alpha = 0$, $Y_t^{0,\varepsilon}$ is the solution to the linear BSDE with $\alpha = 0$ in (2.2):

$$Y_t^{0,\varepsilon} = E\left[g(X_T^{\varepsilon})|\mathcal{F}_t\right].$$

We consider the FBSDEs (2.1) and (2.3) on the subinterval $[t,T] \subseteq [0,T]$ as follows: for $s \in [t,T]$,

$$X_s^{t,x,\varepsilon,i} = x^i + \int_t^s b^i(r, X_r^{t,x,\varepsilon})dr + \varepsilon \sum_{j=1}^d \int_t^s \sigma_j^i(r, X_r^{t,x,\varepsilon})dW_r^j \tag{2.4}$$

$$\begin{aligned} Y_s^{t,x,\alpha,\varepsilon} &= g(X_T^{t,x,\varepsilon}) + \alpha \int_s^T f(r, X_r^{t,x,\varepsilon}, Y_r^{t,x,\alpha,\varepsilon}, Z_r^{t,x,\alpha,\varepsilon})dr \\ &\quad - \int_s^T Z_r^{t,x,\alpha,\varepsilon} \cdot dW_r, \end{aligned} \tag{2.5}$$

where the subscript $\cdot^{t,x}$ shows the dependence on the initial data (t,x), and $X_t^{t,x,\varepsilon,i} = x^i$.
Hereafter, we use the notation $\partial_\alpha = \frac{\partial}{\partial\alpha}$ and $\partial_x = (\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^d})$.

Then, we recall the following well-known result (for instance, see Corollary 4.1 in [8] or Theorem 3.1 in [19]): Define $u^{\alpha,\varepsilon}(t,x)$ as

$$u^{\alpha,\varepsilon}(t,x) := Y_t^{t,x,\alpha,\varepsilon} = E\left[g(X_T^{t,x,\varepsilon}) + \alpha \int_t^T f(r, X_r^{t,x,\varepsilon}, Y_r^{t,x,\alpha,\varepsilon}, Z_r^{t,x,\alpha,\varepsilon})dr\right].$$

Then, we have

$$\partial_x u^{\alpha,\varepsilon}(t,x)\sigma(t,x) = Z_t^{t,x,\alpha,\varepsilon}. \tag{2.6}$$

We also define $\partial_x u^{\alpha,\varepsilon}\sigma : [0,T] \times \mathbf{R}^d \ni (t,x) \mapsto \partial_x u^{\alpha,\varepsilon}(t,x)\sigma(t,x)$.

3 Expansion of BSDE

This section shows our main result for the expansion of $(Y^{\alpha,\varepsilon}, Z^{\alpha,\varepsilon})$ around $\alpha = 0$. Generally speaking, as above BSDE is nonlinear, solving it analytically seems not possible. In fact, for computation of $u^{\alpha,\varepsilon}(t,x)$ and $\partial_x u^{\alpha,\varepsilon}(t,x)\sigma(t,x)$, there is an unavoidable complexity mainly due to the "non-linearity" of the driver f in the BSDE. To overcome the difficulty, we expand the BSDE with respect to a driver parameter α around a linear BSDE with $\alpha = 0$, which is able to take the "non-linearity" effects into account as the expansion coefficients. Hence, we pursue to obtain an approximate solution by an asymptotic expansion around a linear BSDE.

Firstly, in the case of $\alpha = 0$ with $s = t$ in (2.5), $(Y^{t,x,0,\varepsilon}, Z^{t,x,0,\varepsilon})$ becomes the solution to the following linear BSDE:

$$Y_t^{t,x,0,\varepsilon} = g(X_T^{t,x,\varepsilon}) - \int_t^T Z_s^{t,x,0,\varepsilon} \cdot dW_s.$$

Then, we also have

$$u^{0,\varepsilon}(t,x) = Y_t^{t,x,0,\varepsilon} = E[g(X_T^{t,x,\varepsilon})],$$

and

$$\partial_x u^{0,\varepsilon}(t,x)\sigma(t,x) = Z_t^{t,x,0,\varepsilon} = \{\partial_x E[g(X_T^{t,x,\varepsilon})]\}\sigma(t,x).$$

In the mathematical finance, $u^{0,\varepsilon}(t,x)$ may be regarded as a value of a European derivative with the payoff $g(X_T^{t,x,\varepsilon})$ and $\partial_x u^{0,\varepsilon}(t,x)$ as its Delta, that is the sensitivity of the value with respect to the change in the initial value of the underlying variable x. Then, it is well known that $u^{0,\varepsilon}(t,x)$ and $\partial_x u^{0,\varepsilon}(t,x)\sigma(t,x)$ can not be obtained as closed forms, due to a generally unknown density function of $X_T^{t,x,\varepsilon}$.

Although the Monte Carlo simulation may be applied to computing those quantities, applying the simulation method becomes infeasible for computation of the higher order expansions around the linear BSDE with reasonable accuracy and computational time. Hence, it is a key element for analytical approximations of BSDE to obtain a closed form approximation of the density function of $X_s^{t,x,\varepsilon}$, $s \in (t,T]$.

One tractable and powerful approach is an asymptotic expansion by Watanabe [28] because the density of $X_s^{t,x,\varepsilon}$ is expanded with respect to the small volatility parameter ε in a unified manner and its concrete and automatic computational scheme has been developed. (e.g. Li [18], Takahashi et al. [23])

Consequently, this paper considers an expansion of the FBSDE with respect to both parameters α and ε and provides a concrete approximation method of the FBSDE and its error estimate.

3.1 Notations and Basic Result

For the preparation, we list up the notations and a lemma following [19], which will be frequently used in the next subsection. Firstly, let E (or E_1) be a generic Euclidean space.

- $C(\mathbf{F}, [0,T] \times \mathrm{E}; \mathrm{E}_1)$: the space of all E_1-valued, continuous random fields, $\varphi : \Omega \times [0,T] \times \mathrm{E} \mapsto \mathrm{E}_1$, such that for fixed $e \in \mathrm{E}$, $\varphi(\cdot,\cdot,e)$ is an $\mathbf{F}$-adapted process.
- $W^{1,\infty}(\mathrm{E}; \mathrm{E}_1)$: the space of all measurable functions $\psi : \mathrm{E} \mapsto \mathrm{E}_1$, such that for some constant $K > 0$ it holds that

$$\|\psi(x) - \psi(y)\|_{\mathrm{E}_1} \leq K\|x - y\|_{\mathrm{E}}, \ \forall x, y \in \mathrm{E}. \tag{3.1}$$

- $L^0([t,T]; W^{1,\infty}(\mathrm{E}; \mathrm{E}_1))$: for $t \in [0,T]$, the space of all measurable functions φ : $[t,T] \mapsto W^{1,\infty}(\mathrm{E}; \mathrm{E}_1)$.

- $L^p(\mathcal{G};\mathrm{E})$: for any sub-$\sigma$-field $\mathcal{G} \subseteq \mathcal{F}_T$ and $1 \le p < \infty$, the space of all E-valued, $\mathcal{G}$-measurable random variables ξ such that $E\left[|\xi|^p\right] < \infty$.

  $L^\infty(\mathcal{G};\mathrm{E})$: for any sub-$\sigma$-field $\mathcal{G} \subseteq \mathcal{F}_T$, the space of all E-valued, $\mathcal{G}$- measurable and bounded random variables.
- $C_b^\infty(\mathrm{E};\mathrm{E}_1)$: the space of all infinitely differentiable functions $\varphi : \mathrm{E} \mapsto \mathrm{E}_1$ such that the all of its derivatives are bounded. We write $C_b^\infty(\mathrm{E})$ for $C_b^\infty(\mathrm{E};\mathbf{R})$.

We also prepare the basic notations and definitions of Malliavin calculus.

- H: the Cameron-Martin space of all absolutely continuous functions $h : [0,T] \to \mathbf{R}^d$ with a square integrable derivative, *i.e.*, $h' \in L^2([0,T];\mathbf{R}^d)$, $h'(t) = \dfrac{d}{dt}h(t)$. Here, $L^2([0,T];\mathbf{R}^d)$ is the space of all $\mathbf{R}^d$-measurable functions φ on $[0,T]$ such that $\left(\int_0^T |\varphi(s)|^2 ds\right)^{1/2} < \infty$.
- $L^2(\Omega;H)$: the space of all random variables $F : \Omega \to H$ such that $\|F\|_2^2 := E[|F|^2] < \infty$.
- $\mathcal{S}$: the set of random variables F of the form

$$F = \varphi\left(\int_0^T h_1'(s)\cdot dW_s, \cdots, \int_0^T h_d'(s)\cdot dW_s\right)$$

  where $\varphi \in C_b^\infty(\mathbf{R}^d)$, $h_1, \cdots, h_d \in H$.
- *Malliavin derivative* D: If $F \in \mathcal{S}$ is of the above form, we define its derivative as follows

$$DF = \sum_{i=1}^d \frac{\partial\varphi}{\partial x^i}\left(\int_0^T h_1'(s)\cdot dW_s, \cdots, \int_0^T h_d'(s)\cdot dW_s\right) h_i,$$

  The derivative DF will be a stochastic process $(D_\tau F)_{\tau\in[0,T]}$ as follows;

$$D_\tau F = \sum_{i=1}^d \frac{\partial\varphi}{\partial x^i}\left(\int_0^T h_1'(s)\cdot dW_s, \cdots, \int_0^T h_d'(s)\cdot dW_s\right) h_i'(\tau), \quad \tau \in [0,T].$$

- $\mathbf{D}^{k,p}$: the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p} = \left(E[|F|^p] + \sum_{j=1}^k E[\|D^j F\|_{H^{\otimes j}}^p]\right)^{1/p}, \ 1 \le p, \ k \in \mathbf{N}.$$

- $\mathbf{D}^\infty$: $\mathbf{D}^\infty = \cap_{p\ge1}\cap_{k\ge1}\mathbf{D}^{k,p}$.
- *Skorohod integral* δ: We define δ as the adjoint operator of the derivative operator D, that is an unbounded operator from $L^2(\Omega;H)$ into $L^2(\Omega)$ such that the domain of δ, denoted by $Dom(\delta)$, is the set of H-valued square integrable random variables u such that $\left|E\left[\int_0^T D_\tau F u_\tau d\tau\right]\right| \le C\|F\|_2$, for all $F \in \mathbf{D}^{1,2}$, where C is some constant depending on u. For $u \in Dom(\delta)$, $\delta(u)$ is characterized by the duality relationship:

$$E[F\delta(u)] = E\left[\int_0^T D_\tau F u_\tau d\tau\right], \ \textit{for any}\ F \in \mathbf{D}^{1,2}.$$

  $\delta(u)$ is called Skorohod integral of the process u.

The next lemma is taken from Lemma 2.2. in [19] and a slight modification of Proposition 5.1 in [8], which is frequently used in the proof of Theorem 3.1.

Lemma 3.1. *1. Suppose that $\tilde{b} \in C(\mathbf{F}, [0,T] \times \mathbf{R}^d; \mathbf{R}^d) \cap L^0([0,T]; W^{1,\infty}(\mathbf{R}^d; \mathbf{R}^d))$, $\tilde{\sigma} \in C(\mathbf{F}, [0,T] \times \mathbf{R}^d; \times\mathbf{R}^{d\times d}) \cap L^0([0,T]; W^{1,\infty}(\mathbf{R}^d; \mathbf{R}^{d\times d}))$, with a common Lipschitz constant $K > 0$. Suppose also that $\tilde{b}(t,0) = 0$ and $\tilde{\sigma}(t,0) = 0$ P-a.s. For any $h^0 \in L^2(\mathbf{F}, [0,T]; \mathbf{R}^d)$ and $h^1 \in L^2(\mathbf{F}, [0,T]; \mathbf{R}^{d\times d})$, let X be the solution of the following SDE:*

$$X_t = x + \int_0^t \left[\tilde{b}(s, X_s) + h_s^0\right] ds + \int_0^t \left[\tilde{\sigma}(s, X_s) + h_s^1\right] dW_s.$$

Then, for any $p \geq 2$, there exists a constant $C > 0$ depending only on p, T and K, such that

$$E\left[|X|_{t,T}^{*,p}\right] \leq C\left\{|x|^p + E\left[\int_0^T \left[||h_t^0|^p + |h_t^1|^p\right] dt\right]\right\},$$

where $|X|_{t,T}^{*,p} := \sup_{t\leq s\leq T} \|X_s\|^p$.

2. Assume that $\tilde{f} \in C(\mathbf{F}, [0,T] \times \mathbf{R} \times \mathbf{R}^d; \mathbf{R}) \cap L^0([0,T]; W^{1,\infty}(\mathbf{R} \times \mathbf{R}^d))$ with a uniform Lipschitz constant $K > 0$, and $\tilde{f}(\omega, s, 0, 0) = 0$ P-a.e. $\omega \in \Omega$. For any $\xi \in L^p(\mathcal{F}_T; \mathbf{R})$, $p > 1$ and a $\mathbf{R}$-valued, $\mathbf{F}$-adapted process h such that $\mathrm{E}\left[\left(\int_0^T |h_t|^2 dt\right)^{p/2}\right] < \infty$ for $p > 1$. let (Y, Z) be the adapted solution to the BSDE:

$$Y_t = \xi + \int_t^T [\tilde{f}(s, Y_s, Z_s) + h_s] ds - \int_t^T Z_s \cdot dW_s.$$

Then there exists a constant $C > 0$ depending only on T, p and the Lipschitz constant of $\tilde{f}$, such that

$$E\left[|Y|_{t,T}^{*,p}\right] + E\left[\left(\int_0^T |Z_t|^2 dt\right)^{p/2}\right] \leq CE\left[|\xi|^p + \left(\int_0^T |h_t|^2 dt\right)^{p/2}\right],$$

where $|Y|_{t,T}^{*,p} := \sup_{t\leq s\leq T} \|Y_s\|^p$.

Also, in order to estimate the expansion error we define a space as in [25]. For any $\beta, \mu > 0$, let $H_{\beta,\mu,T}$ be the space of functions $v : [0,T] \times \mathbf{R}^d \to \mathbf{R}^n$ such that

$$\|v\|_{H_{\beta,\mu,T}}^2 = \int_0^T \int_{\mathbf{R}^d} e^{\beta s} |v(s,x)|^2 e^{-\mu|x|} dx ds < \infty.$$

3.2 Asymptotic Expansion for BSDE and its Representation

Hereafter, we often suppress the subscript ε for the notational simplicity. Also we frequently use abbreviated notations such as Y_t^α, Z_t^α, u^α and $\partial_x u^\alpha \sigma$ in stead of $Y_t^{t,x,\alpha,\varepsilon}$, $Z_t^{t,x,\alpha,\varepsilon}$, $u^{\alpha,\varepsilon}$ and $\partial_x u^{\alpha,\varepsilon}\sigma$, respectively.

Moreover, we use the following notations and the abbreviations especially in the next theorem:

$$
\begin{aligned}
\sum_{\mathbf{n}_\beta,\mathbf{d}^{(\beta)}}^{(n)} &:= \sum_{\beta=1}^{n}\sum_{\mathbf{n}_\beta\in L_{n,\beta}}\sum_{\mathbf{d}^{(\beta)}\in\{1,\cdots,d+1\}^\beta}\frac{1}{\beta!},\\
\sum_{\mathbf{n}_\beta,\mathbf{d}^{(\beta)},\beta=2}^{(n)} &:= \sum_{\beta=2}^{n}\sum_{\mathbf{n}_\beta\in L_{n,\beta}}\sum_{\mathbf{d}^{(\beta)}\in\{1,\cdots,d+1\}^\beta}\frac{1}{\beta!},\\
L_{n,\beta} &:= \left\{\mathbf{n}_\beta=(n_1,\cdots,n_\beta);\ \sum_{k=1}^{\beta}n_k=n;\ (n,n_k,\beta\in\mathbf{N})\right\},
\end{aligned}
$$

$$
\begin{aligned}
& Z^{t,x,\alpha,\varepsilon}=\left(Z^{t,x,\alpha,\varepsilon,1},\cdots,Z^{t,x,\alpha,\varepsilon,d}\right),\\
& \partial_\alpha Z^\alpha\equiv\partial_\alpha Z^{t,x,\alpha,\varepsilon}=\left(\partial_\alpha Z^{t,x,\alpha,\varepsilon,1},\cdots,\partial_\alpha Z^{t,x,\alpha,\varepsilon,d}\right),\\
& \Xi^\alpha\equiv\Xi^{t,x,\alpha,\varepsilon}:=\left(Y^{t,x,\alpha,\varepsilon},Z^{t,x,\alpha,\varepsilon}\right)\in\mathbf{R}\times\mathbf{R}^d,\\
& \Xi^{\alpha,i}\equiv\Xi^{t,x,\alpha,\varepsilon,i},\ i\in\{1,\cdots,d+1\},\\
& \Theta_r^\alpha\equiv\Theta_r^{t,x,\alpha,\varepsilon}:=\left(r,X_r^{t,x,\varepsilon},\Xi_r^{t,x,\alpha,\varepsilon}\right)=\left(r,X_r^{t,x,\varepsilon},Y_r^{t,x,\alpha,\varepsilon},Z^{t,x,\alpha,\varepsilon}\right)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^d,\\
& \partial_{\mathbf{d}^{(\beta)}}f(\Theta_r^\alpha):=\frac{\partial^\beta}{\partial\xi_{d_1}\cdots\partial\xi_{d_\beta}}f\left(\cdot,\cdot,\Xi_r^{t,x,\alpha,\varepsilon}\right)=\frac{\partial^\beta}{\partial\xi_{d_1}\cdots\partial\xi_{d_\beta}}f\left(\cdot,\cdot,Y_r^{t,x,\alpha,\varepsilon},Z^{t,x,\alpha,\varepsilon}\right),\\
& \left(\mathbf{d}^{(\beta)}:=(d_1,\cdots,d_\beta)\in\{1,\cdots,d+1\}^\beta,\beta\geq 1\right),\\
& \partial_y f(\Theta_r^\alpha):=\frac{\partial}{\partial y}f\left(\cdot,\cdot,Y_r^{t,x,\alpha,\varepsilon},\cdot\right),\\
& \nabla_z f(\Theta_r^\alpha):=\left(\frac{\partial f(\cdot,\cdot,\cdot,Z_r^{t,x,\alpha,\varepsilon})}{\partial z_1},\cdots,\frac{\partial f(\cdot,\cdot,\cdot,Z_r^{t,x,\alpha,\varepsilon})}{\partial z_d}\right).
\end{aligned}
$$

Section 2.4 of [8] discuss the first-order differentiation of the function $\alpha\mapsto(Y^\alpha,Z^\alpha)$. In the following theorem, we provide a representation of $\partial_\alpha^n Y_s^{t,x,\alpha}:=\frac{\partial^n}{\partial\alpha^n}Y_s^{t,x,\alpha}$ and $\partial_\alpha^n Z_s^{t,x,\alpha}:=\frac{\partial^n}{\partial\alpha^n}Z_s^{t,x,\alpha}$ for any $n\in\mathbf{N}$ and derive an asymptotic expansion of (Y^α,Z^α) with respect to the parameter α around $\alpha=0$.

Theorem 3.1. *Given the forward SDE (2.4) and $Y^{t,x,0}$ in (3.1), for $s\in[t,T]$, the derivatives $\partial_\alpha^n Y_s^{t,x,\alpha}=\frac{\partial^n}{\partial\alpha^n}Y_s^{t,x,\alpha}$ and $\partial_\alpha^n Z_s^{t,x,\alpha}=\frac{\partial^n}{\partial\alpha^n}Z_s^{t,x,\alpha}$ satisfy:*
when $n=1$,

$$
\begin{aligned}
\partial_\alpha Y_s^{t,x,\alpha} &= \int_s^T\left[f(\Theta_r^\alpha)+\alpha\partial_y f(\Theta_r^\alpha)(\partial_\alpha Y_r^\alpha)+\alpha\nabla_z f(\Theta_r^\alpha)\cdot(\partial_\alpha Z_r^\alpha)\right]dr\\
&\quad -\int_s^T(\partial_\alpha Z_r^\alpha)\cdot dW_r, \qquad (3.2)
\end{aligned}
$$

when $n\geq 2$,

$$
\begin{aligned}
\partial_\alpha^n Y_s^{t,x,\alpha} &= \int_s^T\left[H^n(r,t,x,\alpha)+\alpha\left\{\partial_y f(\Theta_r^\alpha)\partial_\alpha^n Y_r^\alpha+\nabla_z f(\Theta_r^\alpha)\cdot\partial_\alpha^n Z_r^\alpha\right\}\right]dr\\
&\quad -\int_s^T\partial_\alpha^n Z_r^\alpha\cdot dW_r, \qquad (3.3)
\end{aligned}
$$

where

$$\begin{aligned} H^n(r,t,x,\alpha) &:= n! \sum_{\mathbf{n}_\beta, \mathbf{d}^{(\beta)}}^{(n-1)} \partial_{\mathbf{d}^{(\beta)}} f(\Theta_r^\alpha) \prod_{k=1}^{\beta} \frac{1}{n_k!} \partial_\alpha^{n_k} \Xi^{\alpha, d_k} \\ &\quad + \alpha\ n! \sum_{\mathbf{n}_\beta, \mathbf{d}^{(\beta)}, \beta=2}^{(n)} \partial_{\mathbf{d}^{(\beta)}} f(\Theta_r^\alpha) \prod_{k=1}^{\beta} \frac{1}{n_k!} \partial_\alpha^{n_k} \Xi^{\alpha, d_k}. \end{aligned}$$

Moreover, for any $M \in \mathbf{N}$, there exists a constant $C(M,T) > 0$ such that

$$\begin{aligned} &\left\| u^{\alpha,\varepsilon} - \left\{ u^{0,\varepsilon} + \sum_{i=1}^{M} \alpha^i\ u_i^{0,\varepsilon} \right\} \right\|_{H_{\beta,\mu,T}}^2 \\ &+ \left\| \partial_x u^{\alpha,\varepsilon}\sigma - \left\{ \partial_x u^{0,\varepsilon}\sigma + \sum_{i=1}^{M} \alpha^i\ \partial_x u_i^{0,\varepsilon}\sigma \right\} \right\|_{H_{\beta,\mu,T}}^2 \\ &\leq \alpha^{2(M+1)} C(M,T), \end{aligned} \tag{3.4}$$

where

$$u^{0,\varepsilon}(t,x) = Y_t^{t,x,0,\varepsilon} = E\left[g(X_T^{t,x,\varepsilon}) \right], \tag{3.5}$$

$$\partial_x u^{0,\varepsilon}\sigma(t,x) = Z_t^{t,x,0,\varepsilon} = E\left[g(X_T^{t,x,\varepsilon}) N_T^{t,x,\varepsilon} \right] \sigma(t,x), \tag{3.6}$$

and

$$\begin{aligned} u_{n+1}^{0,\varepsilon}(t,x) &= \frac{1}{(n+1)!} \partial_\alpha^{n+1} Y_t^{t,x,\alpha,\varepsilon}|_{\alpha=0} \\ &= E\left[\int_t^T F^{n+1}(r, X_r^{t,x,\varepsilon}) dr \right], \quad \textit{for } n = 0,1,\cdots, \\ \partial_x u_{n+1}^{0,\varepsilon}\sigma(t,x) &= \frac{1}{(n+1)!} \partial_\alpha^{n+1} Z_t^{t,x,\alpha,\varepsilon}|_{\alpha=0} \\ &= E\left[\int_t^T [F^{n+1}(r, X_r^{t,x,\varepsilon})] N_r^{t,x,\varepsilon} dr \right] \sigma(t,x), \\ &\qquad \textit{for } n = 0,1,\cdots, \end{aligned}$$

where $N_r^{t,x,\varepsilon}$ stands for the Malliavin Delta weight:

$$N_r^{t,x,\varepsilon} = \frac{1}{(r-t)} \int_t^r \sigma(X_\tau^{t,x,\varepsilon})^{-1} \nabla X_\tau^{t,x,\varepsilon} dW_\tau. \tag{3.7}$$

Here, F^{n+1}, $n \geq 0$, is recursively given by

$$F^1(t,x) = f\left(t,x,u^{0,\varepsilon}(t,x), \partial_x u^{0,\varepsilon}\sigma(t,x)\right), \quad \textit{for } n = 0, \tag{3.8}$$

$$\begin{aligned} &F^{n+1}(t,x) \\ &= \sum_{\mathbf{n}_\beta, \mathbf{d}^{(\beta)}}^{(n)} \partial_{\mathbf{d}^{(\beta)}} f\left(t,x,u^{0,\varepsilon}(t,x), \partial_x u^{0,\varepsilon}\sigma(t,x)\right) \prod_{k=1}^{\beta} \frac{1}{n_k!} \partial_\alpha^{n_k} \hat{\Xi}^{0,d_k}, \\ &\qquad \textit{for } n \geq 1, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
&\partial_x u^{0,\varepsilon}\sigma(t,x) = \left(\partial_x u^{0,\varepsilon}\sigma(t,x))^1, \cdots, (\partial_x u^{0,\varepsilon}\sigma(t,x))^d\right), \\
&\hat{\Xi}^0 \equiv \hat{\Xi}^{t,x,0,\varepsilon} := \left(u^{0,\varepsilon}(t,x), \partial_x u^{0,\varepsilon}\sigma(t,x)\right) \\
&= \left(u^{0,\varepsilon}(t,x), (\partial_x u^{0,\varepsilon}\sigma(t,x))^1, \cdots, (\partial_x u^{0,\varepsilon}\sigma(t,x))^d\right), \\
&\hat{\Xi}^{0,d_k} \equiv \hat{\Xi}^{t,x,0,\varepsilon,d_k}.
\end{aligned}$$

Remark 3.1. *In the case of $d=1$, (3.2) and (3.3) is reduced to the following equations:*

$$\begin{aligned}
d(\partial_\alpha Y_r^\alpha) &= -\left[f(\Theta_r^\alpha) + \alpha \partial_y f(\Theta_r^\alpha)(\partial_\alpha Y_r^\alpha) + \alpha \partial_z f(\Theta_r^\alpha)(\partial_\alpha Z_r^\alpha)\right] dr \\
&\quad + \partial_\alpha Z_r^\alpha dW_r, \quad for\ n=1, \\
d\left(\partial_\alpha^n Y_r^\alpha\right) &= -\left[H^n(r,t,x,\alpha) + \alpha\left\{\partial_y f(\Theta_r^\alpha)\partial_\alpha^n Y_r^\alpha + \partial_z f(\Theta_r^\alpha)\partial_\alpha^n Z_r^\alpha\right\}\right] dr \\
&\quad + \partial_\alpha^n Z_r^\alpha dW_r, \quad for\ n \geq 2, \\
\partial_\alpha^n Y_T^\alpha &= 0,
\end{aligned}$$

where

$$\begin{aligned}
H^n(r,t,x,\alpha) &= n! \sum_{k=1}^{n-1} \sum_{\beta_1+\cdots+\beta_k=n-1, \beta_i \geq 1} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} \\
&\quad \partial_y^{k-i}\partial_z^i f(\Theta_r^\alpha) \prod_{j=1}^{k-i} \frac{1}{\beta_j!}\partial_\alpha^{\beta_j} Y_r^\alpha \prod_{j=k-i+1}^{k} \frac{1}{\beta_j!}\partial_\alpha^{\beta_j} Z_r^\alpha \\
&\quad + \alpha\ n! \sum_{k=2}^{n} \sum_{\beta_1+\cdots+\beta_k=n, \beta_i \geq 1} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} \\
&\quad \partial_y^{k-i}\partial_z^i f(\Theta_r^\alpha) \prod_{j=1}^{k-i} \frac{1}{\beta_j!}\partial_\alpha^{\beta_j} Y_r^\alpha \prod_{j=k-i+1}^{k} \frac{1}{\beta_j!}\partial_\alpha^{\beta_j} Z_r^\alpha,
\end{aligned}$$

and $\prod_j^i \equiv 1$ *when* $i<j$.

In addition, F^{n+1}, $n \geq 1$, is recursively given by

$$F^1(t,x) = f\left(t,x,u^{0,\varepsilon}(t,x), \partial_x u^{0,\varepsilon}\sigma(t,x)\right), \quad for\ n=0, \tag{3.10}$$

$$\begin{aligned}
F^{n+1}(t,x) &= \sum_{k=1}^{n} \sum_{\beta_1+\cdots+\beta_k=n, \beta_i \geq 1} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} \\
&\quad \partial_y^{k-i}\partial_z^i f\left(t,x,u^{0,\varepsilon}(t,x), \partial_x u^{0,\varepsilon}\sigma(t,x)\right) \\
&\quad \prod_{j=1}^{k-i} \frac{1}{\beta_j!} u_{\beta_j}^{0,\varepsilon}(t,x) \prod_{j=k-i+1}^{k} \frac{1}{\beta_j!} \partial_x u_{\beta_j}^{0,\varepsilon}\sigma(t,x), \quad for\ n \geq 1.
\end{aligned} \tag{3.11}$$

Proof.
We only prove the case of $d=1$ for the notational simplicity.

Firstly, as in the beginning of this section, (Y^0, Z^0) is the solution to linear BSDE:

$$Y_t^0 = g(X_T^\varepsilon) - \int_t^T Z_s^0 dW_s.$$

We have

$$u^0(t,x) = Y_t^{t,x,0}$$

and by Theorem 4.2 of [19] with null driver,

$$\partial_x u^0(t,x)\sigma(t,x) = Z_t^{t,x,0}$$

has the representation (3.6).

Next, we will apply an induction argument to the number of the times of the differentiation of (Y^α, Z^α) with respect to α, and then will prove the expansion (3.4). We also remark that we will use a generic constant $C > 0$, which is allowed to vary, depending on some constants associated with Assumption 2.1, Lemma 3.1, the time horizon, the number of the times of the differentiation and so on.

- $n = 1$ ($\partial_\alpha Y_t^\alpha$)

  In the first place, let us show the case of the first order differentiation with respect to α. For an arbitrary initial condition $(t,x) \in [0,T] \times \mathbf{R}^d$, let $(Y_{1,s}^{t,x,\alpha}, Z_{1,s}^{t,x,\alpha})_{t\le s\le T}$ be the solution to the BSDE, which is obtained by the formal differentiation of (2.5) with respect to α:

$$\begin{aligned} Y_{1,s}^{t,x,\alpha} &= \int_s^T [f(\Theta_r^{t,x,\alpha}) + \alpha\partial_y f(\Theta_r^{t,x,\alpha})Y_{1,r}^{t,x,\alpha} + \alpha\partial_z f(\Theta_r^{t,x,\alpha})Z_{1,r}^{t,x,\alpha}]dr \\ &\quad - \int_s^T Z_{1,r}^{t,x,\alpha} dW_r. \end{aligned} \tag{3.12}$$

  Applying Proposition 2.4 with its remark in p.29 of [8] or the similar argument as in the proof of Theorem 3.1 in [19], we can see $(Y_{1,s}^{t,x,\alpha}, Z_{1,s}^{t,x,\alpha})_{t\le s\le T}$ satisfies:

$$\lim_{h\to 0} E\left[\sup_{t\le s\le T}\left|\frac{Y_s^{t,x,\alpha+h} - Y_s^{t,x,\alpha}}{h} - Y_{1,s}^{t,x,\alpha}\right|^2 + \sup_{t\le s\le T}\left|Y_s^{t,x,\alpha+h} - Y_s^{t,x,\alpha}\right|^2\right] = 0,$$

  and

$$\lim_{h\to 0} E\left[\int_t^T \left|\frac{Z_s^{t,x,\alpha+h} - Z_s^{t,x,\alpha}}{h} - Z_{1,s}^{t,x,\alpha}\right|^2 ds + \int_t^T \left|Z_s^{t,x,\alpha+h} - Z_s^{t,x,\alpha}\right|^2 ds\right] = 0.$$

  Hence, hereafter we often write $Y_{1,s}^{t,x,\alpha}$ for $\partial_\alpha Y_s^{t,x,\alpha}$ and $Z_{1,s}^{t,x,\alpha}$ for $\partial_\alpha Z_s^{t,x,\alpha}$.
  Next, define

$$u_1^\alpha(t,x) := E\left[\int_t^T [f(\Theta_r^{t,x,\alpha}) + \alpha\partial_y f(\Theta_r^{t,x})Y_{1,r}^{t,x,\alpha} + \alpha\partial_z f(\Theta_r^{t,x,\alpha})Z_{1,r}^{t,x,\alpha}]dr\right], \tag{3.13}$$

  and

$$\begin{aligned} &v_1^\alpha(t,x) \\ :=\ &\frac{1}{\varepsilon}E\left[\int_t^T [f(\Theta_r^{t,x,\alpha}) + \alpha\partial_y f(\Theta_r^{t,x})Y_{1,r}^{t,x,\alpha} + \alpha\partial_z f(\Theta_r^{t,x,\alpha})Z_{1,r}^{t,x,\alpha}]N_r^{t,x}dr\right], \end{aligned} \tag{3.14}$$

where $(N_r^{t,x})_{t\le r\le T}$ is the Malliavin delta weight given by (3.7). First, it holds that $u_1^\alpha(t,x) = \partial_\alpha Y_t^{t,x,\alpha}$.
Second, since f, $\partial_y f$ and $\partial_z f$ are bounded by Assumption 2-1-3, and Lemma 3.1-2 is applied to (3.12), there exists C_1 such that for all $p>1$,

$$E\left[\left|Y_1^{t,x,\alpha}\right|_{t,T}^{*,p} + \left(\int_t^T |Z_{1,r}^{t,x,\alpha}|^2 dr\right)^{p/2}\right] \le C_1, \tag{3.15}$$

which is applied to (3.13) to obtain $|u_1^\alpha(t,x)| \le C$ for some constant C for all (t,x).
Next we consider the solution to the variational equation of the BSDE (3.12):

$$\begin{aligned} & \nabla Y_{1,s}^{t,x,\alpha} \\ = & \int_s^T \left[B^1(r,t,x,\alpha) + \ \alpha\partial_y f(\Theta_r^{t,x,\alpha})\nabla Y_{1,r}^{t,x,\alpha} + \alpha\partial_z f(\Theta_r^{t,x,\alpha})\nabla Z_{1,r}^{t,x,\alpha}\right] dr \\ & - \int_s^T \nabla Z_{1,r}^{t,x,\alpha} dW_r, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} & B^1(r,t,x,\alpha) \\ = & \partial_x f(\Theta_r^{t,x,\alpha})\nabla X_r^{t,x} + \partial_y f(\Theta_r^{t,x,\alpha})\nabla Y_r^{t,x,\alpha} + \partial_z f(\Theta_r^{t,x,\alpha})\nabla Z_r^{t,x,\alpha} \\ & +\alpha\partial_{xy} f(\Theta_r^{t,x})\nabla X_r^{t,x} Y_{1,r}^{t,x,\alpha} + \alpha\partial_{y^2} f(\Theta_r^{t,x})\nabla Y_r^{t,x,\alpha} Y_{1,r}^{t,x,\alpha} \\ & +\alpha\partial_{yz} f(\Theta_r^{t,x})\nabla Z_r^{t,x} Y_{1,r}^{t,x,\alpha} + \alpha\partial_{xz} f(\Theta_r^{t,x})\nabla X_r^{t,x} Z_{1,r}^{t,x,\alpha} \\ & +\alpha\partial_{yz} f(\Theta_r^{t,x})\nabla Y_r^{t,x,\alpha} Z_{1,r}^{t,x,\alpha} + \alpha\partial_{z^2} f(\Theta_r^{t,x})\nabla Z_r^{t,x} Z_{1,r}^{t,x,\alpha}. \end{aligned} \tag{3.17}$$

First, note that due to Lemma 3.1, we have for all $p>0$,

$$E\left[\left|\nabla X^{t,x}\right|_{t,T}^{*,p} + \left|\nabla Y^{t,x,\alpha}\right|_{t,T}^{*,p}\right] \le C_2 \text{ for some constant } C_2. \tag{3.18}$$

By Theorem 3.1-(iii) in [19] we also know that:

$$Z_s^{t,x} = \partial_x u(s, X_s^{s,x})\sigma(s, X_s^{s,x}),\ \forall s\in[t,T],\ P-a.s.$$

Thus, we have

$$\begin{aligned} \nabla_x Z_s^{t,x} \ = \ & \partial_x^2 u(s,X_s^{s,x})\nabla X_s^{t,x}\sigma(s,X_s^{s,x}) \\ & +\partial_x u(s,X_s^{s,x})\partial_x\sigma(s,X_s^{s,x})\nabla X_s^{t,x}, \\ & \qquad\qquad \forall s\in[t,T],\ P-a.s. \end{aligned}$$

Moreover, by Lemma 3.4 of Crisan and Delarue [5], $\partial_x u$ and $\partial_x^2 u$ are bounded. Hence with Assumption 2.1.1 and (3.18) we obtain for all $p>0$,

$$E\left[\left|\nabla Z^{t,x}\right|_{t,T}^{*,p}\right] \le C_3 \text{ for some constant } C_3. \tag{3.19}$$

Then, applying Assumption 2.1-.3, (3.15), (3.18) and (3.19), we obtain

$$E\left[\int_t^T \left|B^1(r,t,x,\alpha)\right|^2 dr\right] \le C_4. \text{ for some constant } C_4 \tag{3.20}$$

Here, for instance, we use the following estimate: as for the last term in $B^1(r,t,x,\alpha)$ in (3.17), by the boundedness of $\partial_{z^2} f(\Theta_r^{t,x})$ and the Hölder inequality with (3.15) and (3.19), we have for some constants $\hat{C}$ and $\bar{C}$:

$$\begin{aligned}
&E\left[\int_t^T |\alpha\partial_{z^2} f(\Theta_r^{t,x})\nabla Z_r^{t,x} Z_{1,r}^{t,x,\alpha})|^2\right] \leq \hat{C}E\left[\int_t^T |\nabla Z_r^{t,x} Z_{1,r}^{t,x,\alpha})|^2 dr\right] \\
&\leq \hat{C}E\left[|\nabla Z^{t,x}|_{t,T}^{*,2}\int_t^T |Z_{1,r}^{t,x,\alpha})|^2 dr\right] \\
&\leq \hat{C}E\left[\left(|\nabla Z^{t,x}|_{t,T}^{*,2}\right)^2\right]^{1/2} E\left[\left(\int_t^T |Z_{1,r}^{t,x,\alpha})|^2 dr\right)^2\right]^{1/2} \leq \bar{C}.
\end{aligned}$$

Thus, applying Lemma 3.1 and the similar argument as in the proof of Theorem 3.1 in Ma and Zhang [19] to (3.16), we have

$$\lim_{h\to 0} E\left[\sup_{t\leq s\leq T}\left|\frac{Y_{1,s}^{t,x+h,\alpha}-Y_{1,s}^{t,x,\alpha}}{h}-\nabla Y_{1,s}^{t,x,\alpha}\right|^2+\sup_{t\leq s\leq T}\left|Y_{1,s}^{t,x+h,\alpha}-Y_{1,s}^{t,x,\alpha}\right|^2\right]=0,$$

$$\lim_{h\to 0} E\left[\int_t^T\left|\frac{Z_{1,s}^{t,x+h,\alpha}-Z_{1,s}^{t,x,\alpha}}{h}-\nabla Z_{1,s}^{t,x,\alpha}\right|^2 ds+\int_t^T\left|Z_{1,s}^{t,x+h,\alpha}-Z_{1,s}^{t,x,\alpha}\right|^2 ds\right]=0,$$

and

$$E\left[\left|\nabla Y_1^{t,x,\alpha}\right|_{t,T}^{*,2}+\int_t^T|\nabla Z_{1,r}^{t,x,\alpha}|^2 dr\right]\leq\int_t^T\left|B^1(r,t,x,\alpha)\right|^2 dr\leq C_4. \tag{3.21}$$

Next, let

$$\begin{aligned}
&\bar{v}_1^\alpha(t,x) \\
:=\ & E\left[\int_t^T[B^1(r,t,x,\alpha)+\alpha\partial_y f(\Theta_r^{t,x,\alpha})\nabla Y_{1,r}^{t,x,\alpha}+\alpha\partial_z f(\Theta_r^{t,x,\alpha})\nabla Z_{1,r}^{t,x,\alpha}]dr\right].
\end{aligned} \tag{3.22}$$

Then, by Assumption 2-1-3, (3.20) and (3.21), we obtain $|\bar{v}_1^\alpha(t,x)|\leq C$.
Moreover, let us show $v_1^\alpha=\bar{v}_1^\alpha=\partial_x u_1^\alpha$ in the following way.
Firstly, using basic results of Malliavin calculus, we calculate the Malliavin derivatives of $f(r,\Theta_r)$, $\{\partial_y f(\Theta_r^{t,x})Y_{1,r}^{t,x,\alpha}\}$ and $\{\partial_z f(\Theta_r^{t,x})Z_{1,r}^{t,x,\alpha}\}$:

$$\begin{aligned}
D_\tau\{f(r,\Theta_r)\} \;=\;& \{\partial_x f(\Theta_r^{t,x,\alpha})\nabla X_r^{t,x} \\
&+\partial_y f(\Theta_r^{t,x,\alpha})\nabla Y_r^{t,x,\alpha} \\
&+\partial_z f(\Theta_r^{t,x,\alpha})\nabla Z_r^{t,x,\alpha}\}(\nabla X_\tau^{t,x})^{-1}\varepsilon\sigma(\tau,X_\tau^{t,x}),
\end{aligned}$$

$$\begin{aligned}
&D_\tau\{\partial_y f(\Theta_r^{t,x})Y_{1,r}^{t,x,\alpha}\} \\
=\;&\{D_\tau\partial_y f(\Theta_r^{t,x})\}Y_{1,r}^{t,x,\alpha}+\partial_y f(\Theta_r^{t,x})\{D_\tau Y_{1,r}^{t,x,\alpha}\} \\
=\;&[\partial_{xy} f(\Theta_r^{t,x})\nabla X_r^{t,x}Y_{1,r}^{t,x,\alpha}+\partial_{y^2} f(\Theta_r^{t,x})\nabla Y_r^{t,x,\alpha}Y_{1,r}^{t,x,\alpha} \\
&+\partial_{yz} f(\Theta_r^{t,x})\nabla Z_r^{t,x}Y_{1,r}^{t,x,\alpha}+\partial_y f(\Theta_r^{t,x})\nabla Y_{1,r}^{t,x,\alpha}](\nabla X_\tau^{t,x})^{-1}\varepsilon\sigma(\tau,X_\tau^{t,x}),
\end{aligned}$$

$$
\begin{aligned}
& D_\tau\{\partial_z f(\Theta_r^{t,x}) Z_{1,r}^{t,x,\alpha}\} \\
= \;& \{D_\tau \partial_z f(\Theta_r^{t,x})\} Z_{1,r}^{t,x,\alpha} + \partial_z f(\Theta_r^{t,x})\{D_\tau Z_{1,r}^{t,x,\alpha}\} \\
= \;& [\partial_{xz} f(\Theta_r^{t,x}) \nabla X_r^{t,x} Z_{1,r}^{t,x,\alpha} + \partial_{yz} f(\Theta_r^{t,x}) \nabla Y_r^{t,x,\alpha} Z_{1,r}^{t,x,\alpha} \\
& + \partial_{z^2} f(\Theta_r^{t,x}) \nabla Z_r^{t,x} Z_{1,r}^{t,x,\alpha} + \partial_z f(\Theta_r^{t,x}) \nabla Z_{1,r}^{t,x,\alpha}](\nabla X_\tau^{t,x})^{-1} \varepsilon \sigma(\tau, X_\tau^{t,x}).
\end{aligned}
$$

Then, by applying the integration by parts on the Wiener space, we have

$$
\begin{aligned}
& E[B^1(r,t,x,\alpha) + \alpha \partial_y f(\Theta_r^{t,x,\alpha}) \nabla Y_{1,r}^{t,x,\alpha} + \alpha \partial_z f(\Theta_r^{t,x,\alpha}) \nabla Z_{1,r}^{t,x,\alpha}] \\
= \;& E\Bigg[\frac{1}{\varepsilon(r-t)} \int_t^r D_\tau \{ f(\Theta_r^{t,x,\alpha}) + \alpha \partial_y f(\Theta_r^{t,x}) Y_{1,r}^{t,x,\alpha} \\
& + \alpha \partial_z f(\Theta_r^{t,x,\alpha}) Z_{1,r}^{t,x,\alpha} \} \sigma(\tau, X_\tau^{t,x})^{-1} (\nabla X_\tau^{t,x}) d\tau \Bigg] \\
= \;& \frac{1}{\varepsilon} E\left[\{ f(\Theta_r^{t,x,\alpha}) + \alpha \partial_y f(\Theta_r^{t,x}) Y_{1,r}^{t,x,\alpha} + \alpha \partial_z f(\Theta_r^{t,x,\alpha}) Z_{1,r}^{t,x,\alpha} \} N_r^{t,x,\varepsilon} \right],
\end{aligned}
$$

where $N_r^{t,x,\varepsilon}$ is given by (3.7). Thus, we have $v_1^\alpha = \bar{v}_1^\alpha$, that is (3.14) = (3.22). Further, as $\partial_x u_1^\alpha(t,x) = \nabla Y_{1,t}^{t,x,\alpha} = \bar{v}_1^\alpha(t,x)$, we obtain that $v_1^\alpha = \bar{v}_1^\alpha = \partial_x u_1^\alpha$. Therefore, we conclude that for all $(t,x) \in [0,T] \times \mathbf{R}^d$,

$$
|\partial_x u_1^\alpha(t,x)| \leq C. \tag{3.23}
$$

Moreover, following the similar argument of Theorem 3.1-(iii) of [19], we know that

$$
Z_{1,s}^{t,x} = \partial_x u_1(s, X_s^{t,x}) \sigma(s, X_s^{t,x}) \quad \forall s \in [t,T], \; P-a.s.
$$

Thus, with (3.23) and Assumption 2.1-1., we also have for all $p > 0$,

$$
E\left[\left| Z_1^{t,x} \right|_{t,T}^{*,p} \right] \leq C.
$$

- *Induction*
  Based on the inductive argument, for an arbitrary fixed $n \in \mathbf{N}$ we assume that $(Y_{n,s}^{t,x,\alpha}, Z_{n,s}^{t,x,\alpha})_{t \leq s \leq T}$ is the solution to the following BSDE:

$$
\begin{aligned}
Y_{n,s}^{t,x,\alpha} = \;& \int_s^T [H^n(r,t,x,\alpha) + \alpha \partial_y f(\Theta_r^{t,x,\alpha}) Y_{n,r}^{t,x,\alpha} + \alpha \partial_z f(\Theta_r^{t,x,\alpha}) Z_{n,r}^{t,x,\alpha}] dr \\
& - \int_s^T Z_{n,r}^{t,x,\alpha} dW_r,
\end{aligned} \tag{3.24}
$$

where

$$
\begin{aligned}
& H^n(r,t,x,\alpha) \\
= \; & n! \sum_{k=1}^{n-1} \sum_{\beta_1+\cdots+\beta_k=n-1, \beta_l \geq 1} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} \\
& \partial_y^{k-i} \partial_z^i f(\Theta_r^{t,x,\alpha}) \prod_{j=1}^{k-i} \frac{1}{\beta_j!} Y_{\beta_j,r}^{t,x,\alpha} \prod_{j=k-i+1}^{k} \frac{1}{\beta_j!} Z_{\beta_j,r}^{t,x,\alpha} \\
& + \alpha \; n! \sum_{k=2}^{n} \sum_{\beta_1+\cdots+\beta_k=n, \beta_l \geq 1} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} \\
& \partial_y^{k-i} \partial_z^i f(\Theta_r^{t,x,\alpha}) \prod_{j=1}^{k-i} \frac{1}{\beta_j!} Y_{\beta_j,r}^{t,x,\alpha} \prod_{j=k-i+1}^{k} \frac{1}{\beta_j!} Z_{\beta_j,r}^{t,x,\alpha}
\end{aligned}
$$

Here, for some constants $\bar{C}_n$ and C_n,

$$
E\left[\int_t^T |H^n(r,t,x,\alpha,\omega)|^2 \, dr\right] \leq \bar{C}_n, \; E\left[\left|Y_n^{t,x,\alpha}\right|_{t,T}^{*,p} + \left(\int_t^T \left|Z_{n,r}^{t,x,\alpha}\right|^2 dr\right)^{p/2}\right] \leq C_n, \tag{3.25}
$$

and we also suppose that for all $p > 0$,

$$
E\left[\left|\nabla Y_{n-1}^{t,x,\alpha}\right|_{t,T}^{*,p} + \left|\nabla Z_{n-1}^{t,x,\alpha}\right|_{t,T}^{*,p}\right] \leq \hat{C}_n, \text{ for some constant } \hat{C}_n. \tag{3.26}
$$

Consequently, we assume that $u_n^\alpha(t,x) = \dfrac{1}{n!} \partial_\alpha^n Y_t^{t,x}$ and $Z_n^{t,x,\alpha}$ satisfy

$$
|u_n^\alpha(t,x)| \leq C, \; |\partial_x u_n^\alpha(t,x)| \leq C, \; E\left[\left|Z_n^{t,x,\alpha}\right|_{t,T}^{*,p}\right] \leq C, \; \forall p > 0. \tag{3.27}
$$

Let $(Y_{(n+1),s}^{t,x,\alpha}, Z_{(n+1),s}^{t,x,\alpha})_{t \leq s \leq T}$ be the solution to the following BSDE which corresponds to the formal differentiation of the BSDE (3.24) with respect to α:

$$
\begin{aligned}
Y_{(n+1),s}^{t,x,\alpha} = & \int_s^T \left[H^{n+1}(r,t,x,\alpha) + \alpha \partial_y f(\Theta_r^{t,x,\alpha}) Y_{(n+1),r}^{t,x,\alpha} + f(\Theta_r^{t,x,\alpha}) \partial_\alpha^{n+1} Z_{(n+1),r}^{t,x,\alpha}\right] dr \\
& - \int_s^T Z_{(n+1),r}^{t,x,\alpha} dW_r,
\end{aligned} \tag{3.28}
$$

where

$$
\begin{aligned}
& H^{n+1}(r,t,x,\alpha) \\
= \; & \partial_\alpha H^n(r,t,x,\alpha) + \partial_y f(\Theta_r^{t,x,\alpha}) Y_{n,r}^{t,x,\alpha} + \partial_z f(\Theta_r^{t,x,\alpha}) Z_{n,r}^{t,x,\alpha} \\
& + \alpha\{\partial_\alpha \partial_y f(\Theta_r^{t,x,\alpha})\} Y_{n,r}^{t,x,\alpha} + \alpha\{\partial_\alpha \partial_z f(\Theta_r^{t,x,\alpha})\} Z_{n,r}^{t,x,\alpha} \\
= \; & (n+1)! \sum_{k=1}^{n} \sum_{\beta_1+\cdots+\beta_k=n, \beta_i \geq 1} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} \\
& \partial_y^{k-i} \partial_z^i f(\Theta_r^{t,x,\alpha}) \prod_{j=1}^{k-i} \frac{1}{\beta_j!} Y_{\beta_j,r}^{t,x,\alpha} \prod_{j=k-i+1}^{k} \frac{1}{\beta_j!} Z_{\beta_j,r}^{t,x,\alpha} \\
& + \alpha \; (n+1)! \sum_{k=2}^{n+1} \sum_{\beta_1+\cdots+\beta_k=n+1, \beta_i \geq 1} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} \\
& \partial_y^{k-i} \partial_z^i f(\Theta_r^{t,x,\alpha}) \prod_{j=1}^{k-i} \frac{1}{\beta_j!} Y_{\beta_j,r}^{t,x,\alpha} \prod_{j=k-i+1}^{k} \frac{1}{\beta_j!} Z_{\beta_j,r}^{t,x,\alpha}.
\end{aligned}
$$

Then, as in the case of $n=1$, following the similar argument as in the proof of Proposition 2.4 with its remark in p.29 of [8] or in the proof of Theorem 3.1 of [19], we are able to show

$$
\lim_{h\to 0} E\left[\sup_{t\leq s\leq T}\left|\frac{Y_{n,s}^{t,x,\alpha+h}-Y_{n,s}^{t,x,\alpha}}{h} - Y_{(n+1),s}^{t,x,\alpha}\right|^2 + \sup_{t\leq s\leq T}\left|Y_{n,s}^{t,x,\alpha+h}-Y_{n,s}^{t,x,\alpha}\right|^2\right] = 0,
$$

and

$$
\lim_{h\to 0} E\left[\int_t^T \left|\frac{Z_{n,s}^{t,x,\alpha+h}-Z_{n,s}^{t,x,\alpha}}{h} - Z_{(n+1),s}^{t,x,\alpha}\right|^2 ds + \int_t^T \left|Z_{n,s}^{t,x,\alpha+h}-Z_{n,s}^{t,x,\alpha}\right|^2 ds\right] = 0.
$$

Next, let

$$
\begin{aligned}
u_{n+1}^\alpha(t,x) \;\; = \;\; & \frac{1}{(n+1)!} E\Bigg[\int_t^T \Bigg[H^{n+1}(r,t,x,\alpha,\omega) + \alpha \partial_y f(\Theta_r^{t,x}) Y_{(n+1),r}^{t,x,\alpha} \\
& + \alpha \partial_z f(\Theta_r^{t,x,\alpha}) Z_{(n+1),r}^{t,x,\alpha}\Bigg] dr\Bigg].
\end{aligned}
$$

Then, by using Assumption 2.1-3, (3.25) and (3.27) to apply Lemma 3-2 to (3.28), we have for some constants $\bar{C}_{n+1}$ and C_{n+1},

$$
\begin{aligned}
& E\left[\int_t^T \left|H^{n+1}(r,t,x,\alpha,\omega)\right|^2 dr\right] \leq \bar{C}_{n+1}, \\
& E\left[\left|Y_{(n+1)}^{t,x,\alpha}\right|_{t,T}^{*,p} + \int_t^T \left(\left|Z_{(n+1),r}^{t,x,\alpha}\right|^2\right) dr\right] \leq C_{n+1}, \text{ for all } p>1,
\end{aligned} \tag{3.29}
$$

and hence $|u_{n+1}^\alpha(t,x)| \leq C$.

Moreover, let

$$\begin{aligned} v^{\alpha}_{n+1}(t,x) &= \frac{1}{(n+1)!}\frac{1}{\varepsilon}E\left[\int_t^T\left[H^{n+1}(r,t,x,\alpha)+\alpha\partial_y f(\Theta^{t,x}_r)Y^{t,x,\alpha}_{(n+1),r}\right.\right. \\ &\quad \left.\left.+\alpha\partial_z f(\Theta^{t,x,\alpha}_r)Z^{t,x,\alpha}_{(n+1),r}\right]N^{t,x}_r dr\right], \end{aligned}$$

where $(N^{t,x}_r)_{t\le r\le T}$ is the Malliavin Delta weight given by (3.7), again.
Then, as in the case $n=1$, $\partial_x u^{\alpha}_{n+1}(t,x)=\nabla Y^{t,x,\alpha}_{(n+1),t}$, and applying integration by parts on the Wiener space, we have $v^{\alpha}_{n+1}(t,x)=\partial_x u^{\alpha}_{n+1}(t,x)$ and $|\partial_x u^{\alpha}_{n+1}(t,x)|\le C$.

- Asymptotic expansion (3.4):
  By the Taylor expansion, we have the following formulas:

$$\begin{aligned} Y^{t,x,\alpha}_t &= Y^{t,x,0}_t+\sum_{i=1}^M\frac{\alpha^i}{i!}\frac{\partial^i}{\partial\alpha^i}Y^{t,x,\alpha}_t|_{\alpha=0} \\ &\quad +\alpha^{M+1}\int_0^1\frac{(1-u)^M}{M!}\frac{\partial^{M+1}}{\partial\nu^{M+1}}Y^{t,x,\nu}_t|_{\nu=\alpha u}du \\ &= u^0(t,x)+\sum_{i=1}^M\alpha^i u^0_i(t,x)+\alpha^{M+1}\int_0^1(1-u)^M\tilde{u}^{\alpha u}_{M+1}(t,x)du, \end{aligned}$$

$$\begin{aligned} Z^{t,x,\alpha}_t &= Z^{t,x,0}_t+\sum_{i=1}^M\frac{\alpha^i}{i!}\frac{\partial^i}{\partial\alpha^i}Z^{t,x,\alpha}_t|_{\alpha=0} \\ &\quad +\alpha^{M+1}\int_0^1\frac{(1-u)^M}{M!}\frac{\partial^{M+1}}{\partial\nu^{M+1}}Z^{t,x,\nu}_t|_{\nu=\alpha u}du \\ &= \partial_x u^{\alpha}\sigma(t,x)=\partial_x u^0\sigma(t,x)+\sum_{i=1}^M\alpha^i\partial_x u^0_i\sigma(t,x) \\ &\quad +\alpha^{M+1}\int_0^1(1-u)^M\partial_x\tilde{u}^{\alpha u}_{M+1}\sigma(t,x)du, \end{aligned}$$

where $\tilde{u}^{\alpha}_{M+1}(t,x):=(M+1)u^{\alpha}_{M+1}(t,x)$ and $\partial_x\tilde{u}^{\alpha}_{M+1}\sigma(t,x):=(M+1)\partial_x u^{\alpha}_{M+1}\sigma(t,x)$.
On the other hand, by the previous result, we have $|\tilde{u}^{\alpha}_{M+1}(t,x)|\le C$ and $|\partial_x\tilde{u}^{\alpha}_{M+1}(t,x)|\le C$ for all $(t,x)\in[0,T]\times\mathbf{R}^d$. Therefore, we finally obtain:

$$\begin{aligned} &\left\|u^{\alpha}-\left\{u^0+\sum_{i=1}^M\alpha^i u^0_i\right\}\right\|^2_{H_{\beta,\mu,T}}+\left\|\partial_x u^{\alpha}\sigma-\left\{\partial_x u^0\sigma+\sum_{i=1}^M\alpha^i\partial_x u^0_i\sigma\right\}\right\|^2_{H_{\beta,\mu,T}} \\ &\le \alpha^{2(M+1)}C(M,T). \end{aligned}$$

Then, we have the assertion. □

4 Expansion of FSDE

Before providing our main result, we state an asymptotic expansion of $E[\varphi(X_T^{t,x,\varepsilon})]$ in terms of a small diffusion parameter ε, which is a slight modification of [25] [26]. Here, $\varphi \in C_b^\infty(\mathbf{R}^d)$, $X_T^{t,x,\varepsilon} = (X_T^{t,x,\varepsilon,1}, \cdots, X_T^{t,x,\varepsilon,d})$, and $X_T^{t,x,\varepsilon,i}$, $i = 1, \cdots, d$ is the solution to the forward SDE (2.4) with $s = T$. We remark that there exist related or other works on expansions in theoretical and practical aspects such as Baudoin [1], Bayer and Laurence [2], Ben Arous and Laurence [3], Bismut [4], Fouque et. al. [9], Gatheral et al. [15], Li [18], Siopacha and Teichmann [22], Violante [27].

Firstly, let us present *the Kusuoka-Stroock Functions*, which is useful to clarify the order of a Wiener functional with respect to the time parameter t in a unified manner, and thus to evaluate the error terms in asymptotic expansions.

4.1 The Kusuoka-Stroock Functions

This subsection introduces the space of Wiener functionals $\mathcal{K}_r^T$ developed by [16] and its properties. The element of $\mathcal{K}_r^T$ is called the *Kusuoka-Stroock function.* See Nee [21], [5] and Crisan et al. [6] for more details of the notations and the proofs.

Let E be a separable Hilbert space and $\mathbf{D}^{n,\infty}(E) = \cap_{1 \le p < \infty} \mathbf{D}^{n,p}(E)$ be the space of E-valued functionals that admit the Malliavin derivatives up to the n-th order. The following definition and lemma correspond to Definition 2.1 and Lemma 2.2 of [5].

Definition 4.1. *Given $r \in \mathbf{R}$ and $n \in \mathbf{N}$, we denote by $\mathcal{K}_r^T(E, n)$ the set of functions $G : (0, T] \times \mathbf{R}^d \mapsto \mathbf{D}^{n,\infty}(E)$ satisfying the following:*

1. *$G(t, \cdot)$ is n-times continuously differentiable and $[\partial^\alpha G / \partial x^\alpha]$ is continuous in $(t, x) \in (0, T] \times \mathbf{R}^d$ a.s. for any multi-index α of the elements of $\{1, \cdots, d\}$ with length $|\alpha| \le n$.*
2. *For all $k \le n - |\alpha|$, $p \in [1, \infty)$,*

$$\sup_{t \in (0,T], x \in \mathbf{R}^d} t^{-r/2} \left\| \frac{\partial^\alpha G}{\partial x^\alpha}(t, x) \right\|_{\mathbf{D}^{k,p}} < \infty.$$

We write $\mathcal{K}_r^T(n)$ for $\mathcal{K}_r^T(\mathbf{R}, n)$ and $\mathcal{K}_r^T$ for $\mathcal{K}_r^T(\mathbf{R}, \infty)$.

The properties of the Kusuoka-Stroock functions are the following. (See Lemma 75 of [6] for the proof.)

Lemma 4.1. *[Properties of Kusuoka-Stroock functions]*

1. *The function $(s, x) \in (0, T] \times \mathbf{R}^d \mapsto X_s^{t,x,\varepsilon}$ belongs to $\mathcal{K}_0^T$.*
2. *Suppose $G \in \mathcal{K}_r^T(n)$ where $r \ge 0$. Then, for $i = 1, \cdots, d$,*

$$(a) \int_0^\cdot G(s, x) dW_s^i \in \mathcal{K}_{r+1}^T(n), \text{ and } (b) \int_0^\cdot G(s, x) ds \in \mathcal{K}_{r+2}^T(n). \tag{4.1}$$

3. *If $G_i \in \mathcal{K}_{r_i}^T(n_i)$, $i = 1, \cdots, N$, then*

$$(a) \prod_i^N G_i \in \mathcal{K}_{r_1 + \cdots + r_N}^T(\min_i n_i), \text{ and } (b) \sum_{i=1}^N G_i \in \mathcal{K}_{\min_i r_i}^T(\min_i n_i). \tag{4.2}$$

Next, we summarize the Malliavin's integration by parts formula using Kusuoka-Stroock functions. For any multi-index $\alpha^{(k)} := (\alpha_1, \cdots, \alpha_k) \in \{1, \cdots, d\}^k$, $k \geq 1$, we denote by $\partial_{\alpha^{(k)}}$ the partial derivative $\dfrac{\partial^k}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_k}}$.

Proposition 4.1. *Let $G : (0, T] \times \mathbf{R}^d \to \mathbf{D}^\infty = \mathbf{D}^{\infty,\infty}(\mathbf{R})$ be an element of $\mathcal{K}_r^T$ and let f be a function that belongs to the space $C_b^\infty(\mathbf{R}^d)$. Then for any multi-index $\alpha^{(k)} \in \{1, \cdots, d\}^k$, $k \geq 1$, there exists $H_{\alpha^{(k)}}(X_s^{t,x,\varepsilon}, G(s,x)) \in \mathcal{K}^T_{r-|\alpha^{(k)}|} = \mathcal{K}^T_{r-k}$ such that*

$$E\left[\partial_{\alpha^{(k)}} f(X_s^{t,x,\varepsilon}) G(s,x)\right] = E\left[f(X_s^{t,x,\varepsilon}) H_{\alpha^{(k)}}(X_s^{t,x,\varepsilon}, G(s,x))\right],$$

with

$$\sup_{x \in \mathbf{R}^d} \|H_{\alpha^{(k)}}(X_s^{t,x,\varepsilon}, G(s,x))\|_{L^p} \leq C(s-t)^{(r-k)/2},$$

where $H_{\alpha^{(k)}}(X_s^{t,x,\varepsilon}, G(s,x))$ is recursively given by

$$\begin{aligned} H_{(i)}(X_s^{t,x,\varepsilon}, G(s,x)) &= \delta\left(\sum_{j=1}^d G \gamma_{ij}^{X_s^{t,x,\varepsilon}} D X_s^{t,x,\varepsilon,j}\right), \\ H_{\alpha^{(k)}}(X_s^{t,x,\varepsilon}, G(s,x)) &= H_{(\alpha_k)}(X_s^{t,x,\varepsilon}, H_{\alpha^{(k-1)}}(X_s^{t,x,\varepsilon}, G(s,x))), \end{aligned}$$

and a positive constant C. Here, δ is the Skorohod integral and $(\gamma_{ij}^{X_s^{t,x,\varepsilon}})_{1 \leq i,j \leq d}$ is the inverse matrix of the Malliavin covariance of $X_s^{t,x,\varepsilon}$.

Proof. Apply Corollary 3.7 of Kusuoka and Stroock [17] and Lemma 8-(3) of [16] with Proposition 2.1.4 of Nualart [20]. □

4.2 Asymptotic Expansions for the Expectation of the Solution to FSDE

This subsection derives the asymptotic expansions for the expectations of the composite functionals of smooth test functions $\varphi \in C_b^\infty(\mathbf{R}^d)$ and the solution to the forward SDE (2.1). Hereafter, let us define $X_{i,T}^{t,x,\varepsilon}$ as $\dfrac{1}{i!}\dfrac{\partial^i}{\partial \varepsilon^i} X_T^{t,x,\varepsilon}$, $i \in \mathbf{N}$. In the first place, we characterize the expansion of the solution to the SDE (2.1) as a Kusuoka-Stroock function.

Lemma 4.2. *For $s \in (t, T]$,*

$$X_{i,s}^{t,x,\varepsilon} \in \mathcal{K}_i^T, \ i \in \mathbf{N}.$$

Proof. See Lemma 5.1 of [25]. □

We denote $\displaystyle\sum_{l_k, \alpha^{(k)}}^{(i)}$ by

$$\sum_{l_k, \alpha^{(k)}}^{(i)} = \sum_{k=1}^{i} \sum_{\mathbf{l}_k \in L_{i,k}} \sum_{\alpha^{(k)} \in \{1,\cdots,d\}^k} \frac{1}{k!},$$

with

$$L_{i,k} := \left\{ \mathbf{l}_k = (l_1, \cdots, l_k);\ \sum_{j=1}^{k} l_j = j;\ (i, l_j, k \in \mathbf{N}) \right\}.$$

The next proposition presents precise evaluation of the asymptotic expansions for the expectations of $E\left[\varphi(X_T^{t,x,\varepsilon})\right]$ and $E\left[\varphi(X_T^{t,x,\varepsilon})N_T^{t,x,\varepsilon}\right]\sigma(t,x)$ for a given smooth function φ.

Proposition 4.2.

1. *For $\varphi \in C_b^\infty(\mathbf{R}^d)$, there exists a constant $C(N)$ depending on N such that*

$$\begin{aligned} &\left| E[\varphi(X_T^{t,x,\varepsilon})] - \left\{ E[\varphi(\bar{X}_T^{t,x,0})] + \sum_{i=1}^{N} \varepsilon^i E[\varphi(\bar{X}_T^{t,x,0})\pi_{i,T}^{t,x}] \right\} \right| \\ &\leq \varepsilon^{N+1} C(N) \sum_{i=1}^{N+1} (T-t)^{(N+1+i)/2}, \end{aligned}$$

where $\bar{X}_T^{t,x,0} = X_T^{t,x,0} + \varepsilon X_{1,T}^{t,x,0}$ and

$$\pi_{i,T}^{t,x} = \sum_{l_k, \alpha^{(k)}}^{(i)} H_{\alpha^{(k)}}(X_{1,T}^{t,x,0}, \prod_{j=1}^{k} X_{l_j+1,T}^{t,x,0,\alpha_j}),\ i = 1, \cdots, N$$

Here, we use the following notations:

$$\begin{aligned} X_{1,T}^{t,x,0} &:= \frac{\partial}{\partial \varepsilon} X_T^{t,x,\varepsilon}|_{\varepsilon=0} \\ &= \sum_{j=1}^{d} \int_t^T \nabla X_T^{t,x,0} (\nabla X_u^{t,x,0})^{-1} \sigma_j(u, X_u^{t,x,0}) dW_u^j, \\ X_{i,T}^{t,x,0,\alpha_j} &:= \frac{1}{i!} \frac{\partial^i}{\partial \varepsilon^i} X_T^{t,x,\varepsilon,\alpha_j}|_{\varepsilon=0}. \end{aligned}$$

2. *For $\varphi \in C_b^\infty(\mathbf{R}^d)$, there exists C depending on N, T and x such that*

$$\begin{aligned} &\Big| E[\varphi(X_T^{t,x,\varepsilon})N_T^{t,x,\varepsilon}]\sigma(t,x) \\ &- \left\{ E[\varphi(\bar{X}_T^{t,x,0})N_{0,T}^{t,x}]\sigma(t,x) + \sum_{i=1}^{N} \varepsilon^i E[\varphi(\bar{X}_T^{t,x,0})N_{i,T}^{t,x}]\sigma(t,x) \right\} \Big| \\ &\leq \varepsilon^{N+1} C(N) \sum_{i=1}^{N+1} (T-t)^{(N+1+i)/2}, \end{aligned}$$

where $\bar{X}_T^{t,x,0} = X_T^{t,x,0} + \varepsilon X_{1,T}^{t,x,0}$; $N_{0,T}^{t,x} = (N_{0,T}^{t,x,1}, \cdots, N_{0,T}^{t,x,d})$ and $N_{i,T}^{t,x} = (N_{i,T}^{t,x,1}, \cdots, N_{i,T}^{t,x,d})$, $i = 1, \cdots, N$ are given respectively by

$$N_{0,T}^{t,x,k} = \sum_{j=1}^{d} H_{(j)}(\bar{X}_T^{t,x,0}, \partial_k \bar{X}_T^{t,x,0,j}),\quad 1 \leq k \leq d,$$

and

$$N^{t,x,k}_{i,T} = \sum_{j=1}^{d} H_{(j)}(\bar{X}^{t,x,0}_T, \partial_k \bar{X}^{t,x,0,j}_T \pi^{t,x}_{i,T}) + \partial_k \pi^{t,x}_{i,T}, \quad 1 \le k \le d.$$

Remark 4.1. *The result 1 has some similarity as in Lipschitz case. That is, for a Lipschitz function φ on $\mathbf{R}^d$, there exists a constant $C(N)$ depending on N such that*

$$\begin{aligned} & \left| E[\varphi(X^{t,x,\varepsilon}_T)] - \left\{ E[\varphi(\bar{X}^{t,x,0}_T)] + \sum_{i=1}^{N} \varepsilon^i E[\varphi(\bar{X}^{t,x,0}_T)\pi^{t,x}_{i,T}] \right\} \right| \\ \le & \ \varepsilon^{N+1} C(N)(T-t)^{(N+2)/2}. \end{aligned}$$

However, in Lipschitz case, the expansion error for $E[\varphi(X^{t,x,\varepsilon}_T)N^{t,x,\varepsilon}_T]\sigma(t,x)$ is given by

$$\begin{aligned} & \Bigg| E[\varphi(X^{t,x,\varepsilon}_T)N^{t,x,\varepsilon}_T]\sigma(t,x) \\ & - \left\{ E[\varphi(\bar{X}^{t,x,0}_T)N^{t,x}_{0,T}]\sigma(t,x) + \sum_{i=1}^{N} \varepsilon^i E[\varphi(\bar{X}^{t,x,0}_T)N^{t,x}_{i,T}]\sigma(t,x) \right\} \Bigg| \\ \le & \ \varepsilon^{N+1} C(N)(T-t)^{(N+1)/2}. \end{aligned} \tag{4.3}$$

We also remark that when φ is a bounded Borel function (even if it is non-smooth), we have

$$\begin{aligned} & \left| E[\varphi(X^{t,x,\varepsilon}_T)] - \left\{ E[\varphi(\bar{X}^{t,x,0}_T)] + \sum_{i=1}^{N} \varepsilon^i E[\varphi(\bar{X}^{t,x,0}_T)\pi^{t,x}_{i,T}] \right\} \right| \\ \le & \ \varepsilon^{N+1} C(N)(T-t)^{(N+1)/2}, \end{aligned}$$

$$\begin{aligned} & \Bigg| E[\varphi(X^{t,x,\varepsilon}_T)N^{t,x,\varepsilon}_T]\sigma(t,x) \\ & - \left\{ E[\varphi(\bar{X}^{t,x,0}_T)N^{t,x}_{0,T}]\sigma(t,x) + \sum_{i=1}^{N} \varepsilon^i E[\varphi(\bar{X}^{t,x,0}_T)N^{t,x}_{i,T}]\sigma(t,x) \right\} \Bigg| \\ \le & \ \varepsilon^{N+1} C(N)(T-t)^{N/2}. \end{aligned}$$

See [25] for the details.

Proof. We use the similar argument as in the proof of Proposition 5.1 and 5.2 in [25].

1. $X^{t,x,\varepsilon}_T$ degenerates when $\varepsilon \downarrow 0$. Then, we define $F^{t,x,\varepsilon}_T$ as follows:

$$F^{t,x,\varepsilon}_T := \frac{X^{t,x,\varepsilon}_T - X^{t,x,0}_T}{\varepsilon}.$$

$F_T^{t,x,\varepsilon} \in \mathbf{D}^\infty$ is a non-degenerate Wiener functional under Assumption 2.1. Then, the expectation $E[\varphi(F_T^{t,x,\varepsilon})]$ is calculated by the integration by parts;

$$E[\varphi(F_T^{t,x,\varepsilon})] = E[\varphi(F_T^{t,x,0})] + \sum_{i=1}^{N} \varepsilon^i E[\varphi(F_T^{t,x,0})\pi_{i,T}^{t,x}]$$
$$+\varepsilon^{N+1}\int_0^1 (1-u)^N (N+1) \sum_{l_k,\alpha^{(k)}}^{(N+1)} E[\partial_{\alpha^{(k)}}\varphi(F_T^{t,x,\varepsilon u})\prod_{j=1}^{k} X_{l_j+1,T}^{t,x,\varepsilon,\alpha_j})]du,$$

Then, by the transform $X_T^{t,x,\varepsilon} = X_T^{t,x,0} + \varepsilon F_T^{t,x,\varepsilon}$, we have

$$E[\varphi(X_T^{t,x,\varepsilon})] = E[\varphi(\bar{X}_T^{t,x,0})] + \sum_{i=1}^{N} \varepsilon^i E[\varphi(\bar{X}^{t,x,0})\pi_{i,T}^{t,x}]$$
$$+\varepsilon^{N+1}\int_0^1 (1-u)^N (N+1) \sum_{l_k,\alpha^{(k)}}^{(N+1)} E[\partial_{\alpha^{(k)}}\varphi(\tilde{X}_T^{t,x,\varepsilon u})\prod_{j=1}^{k} X_{l_j+1,T}^{t,x,\varepsilon,\alpha_j})]du,$$

where $\tilde{X}_T^{t,x,\varepsilon u} = X_T^{t,x,0} + \varepsilon F_T^{t,x,\varepsilon u}$ for $u \in [0,1]$. Therefore, by using Lemma 4.1 and 4.2, we are able to see

$$\left| \sum_{l_k,\alpha^{(k)}}^{(N+1)} E[\partial_{\alpha^{(k)}}\varphi(F_T^{t,x,\varepsilon u})\prod_{j=1}^{k} X_{l_j+1,T}^{t,x,\varepsilon,\alpha_j})] \right| \le C\sum_{i=1}^{N+1} \|\nabla^i\varphi\|_\infty (T-t)^{N+1+i}. \tag{4.4}$$

Then, we obtain the assertion.

2. Differentiating $E[\varphi(X_T^{t,x,\varepsilon})]$ with respect to x, we have

$$\begin{aligned}
& E[\varphi(X_T^{t,x,\varepsilon})N_T^{t,x,\varepsilon}] \\
= \;& E[\varphi(\bar{X}_T^{t,x,0})N_{0,T}^{t,x}] + \sum_{i=1}^{N} \varepsilon^i E[\varphi(\bar{X}_T^{t,x,0})N_{i,T}^{t,x}] \\
& +\varepsilon^{N+1}\int_0^1 (1-u)^N \sum_{k=2}^{N+2} \sum_{\alpha^{(k)}\in\{1,\cdots,d\}^k} E\left[\partial_{\alpha^{(k)}}\varphi(\tilde{X}_T^{t,x,\varepsilon u})\xi_{k,T}^{t,x,\varepsilon u}\right] du
\end{aligned}$$

where $\xi_{k,T}^{t,x,\varepsilon u} \in \mathcal{K}_{N+k}^T$, $2 \le k \le N+2$. Then, we obtain the assertion. □

5 Main result: Asymptotic Expansion of Multi-scale FBSDE

This section finally derives our main result which is asymptotic expansions of $u^{\alpha,\varepsilon}(t,x)$ in (2.6) and $\partial_x u^{\alpha,\varepsilon}(t,x)\sigma(t,x)$ in (2.6).

First, applying the Malliavin weights $\pi_{i,s}^{t,x}$ and $N_{i,s}^{t,x}$, $s \in (t,T]$, $1 \le i \le N$ in Proposition 4.2, we define an approximation sequence for $(u^{0,\varepsilon}, \partial_x u^{0,\varepsilon}\sigma)$. Let $p^0(t,s,x,y)$ be the density of $\bar{X}_s^{t,x,0}$ given by

$$\begin{aligned}
& p^0(t,s,x,y) \\
= \;& \frac{1}{(2\pi\varepsilon^2)^{d/2}\det(\Sigma(t,s))^{1/2}} e^{-\frac{(y-X_s^{t,x,0})^\top \Sigma^{-1}(t,s)(y-X_s^{t,x,0})}{2\varepsilon^2}},
\end{aligned} \tag{5.1}$$

with the covariance matrix $\Sigma(t,s) = (\Sigma(t,s)_{i,j})_{1\le i,j\le d}$.

$$
\begin{aligned}
&\Sigma(t,s)_{i,j} \\
= \ &\sum_{k=1}^{d}\int_t^s (\nabla X_s^{t,x,0}(\nabla X_u^{t,x,0})^{-1}\sigma_k(u,X_u^{t,x,0}))_i(\nabla X_s^{t,x,0}(\nabla X_u^{t,x,0})^{-1}\sigma_k(u,X_u^{t,x,0}))_j du, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1\le i,j\le d.
\end{aligned} \tag{5.2}
$$

and $(u^{0,\varepsilon,N}, \partial_x u^{0,\varepsilon,N}\sigma)$, $N\in\mathbf{N}$ be

$$
u^{0,\varepsilon,N}(t,x) := \int_{\mathbf{R}^d} g(y)\left\{1+\sum_{i=1}^{N}\varepsilon^i E\left[\pi_{i,T}^{t,x}|\bar{X}_T^{t,x,0}=y\right]\right\}p^0(t,T,x,y)dy,
$$

$$
\begin{aligned}
(\partial_x u^{0,\varepsilon,N}\sigma)(t,x) &:= (\partial_x u^{0,\varepsilon,N}(t,x))\sigma(t,x) \\
&= \int_{\mathbf{R}^d} g(y)E\left[N_{0,T}^{t,x}|\bar{X}_T^{t,x,0}=y\right]p^0(t,T,x,y)dy\sigma(t,x) \\
&\qquad +\sum_{i=1}^{N}\varepsilon^i\int_{\mathbf{R}^d} g(y)E\left[N_{i,T}^{t,x}|\bar{X}_T^{t,x,0}=y\right]p^0(t,T,x,y)dy\sigma(t,x).
\end{aligned}
$$

Also, for $n\in\mathbf{N}$ we define $(u_n^{0,\varepsilon,N}, \partial_x u_n^{0,\varepsilon,N}\sigma)$, $N\in\mathbf{N}$ as

$$
\begin{aligned}
&u_n^{0,\varepsilon,N}(t,x) \\
:= \ &E\left[\int_t^T F^n(r,t,x,0,X_r^{t,x,0})dr\right] + \sum_{i=1}^{N}\varepsilon^i E\left[\int_t^T F^n(r,t,x,0,X_r^{t,x,0})\pi_{i,r}^{t,x}dr\right] \\
= \ &\int_t^T\int_{\mathbf{R}^d} F^n(r,t,x,0,y)\left\{1+\sum_{i=1}^{N}\varepsilon^i E\left[\pi_{i,r}^{t,x}|\bar{X}_r^{t,x,0}=y\right]\right\}p^0(t,r,x,y)dyds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_x u_n^{0,\varepsilon,N}\sigma(t,x) \\
= \ &E\left[\int_t^T [F^n(r,t,x,0,X_r^{t,x,0})]N_{0,r}^{t,x}dr\right]\sigma(t,x) \\
&+\sum_{i=1}^{N}\varepsilon^i E\left[\int_t^T [F^n(r,t,x,0,X_r^{t,x,0})]N_{i,r}^{t,x}dr\right]\sigma(t,x) \\
= \ &\int_t^T\int_{\mathbf{R}^d} F^n(r,t,x,0,y) \\
&\left\{E\left[N_{0,r}^{t,x}|\bar{X}_r^{t,x,0}=y\right]+\sum_{i=1}^{N}\varepsilon^i E\left[N_{i,r}^{t,x}|\bar{X}_r^{t,x,0}=y\right]\right\}p^0(t,r,x,y)dyds\sigma(t,x),
\end{aligned}
$$

where F^n is defined as (3.8) and (3.9) in Theorem 3.1.

Then, setting each g and F^n as φ in Proposition 4.2 , we obtain the following result.

Corollary 5.1. *It holds that:*

$$
\begin{aligned}
\left\|u^{0,\varepsilon}-u^{0,\varepsilon,N}\right\|_{H_{\beta,\mu,T}}^2 &\le \varepsilon^{2(N+1)}C(N,T), \\
\left\|\partial_x u^{0,\varepsilon}\sigma-\partial_x u^{0,\varepsilon,N}\sigma\right\|_{H_{\beta,\mu,T}}^2 &\le \varepsilon^{2(N+1)}C(N,T),
\end{aligned}
$$

and that for each $n, N \in \mathbf{N}$,

$$\left\| u_n^{0,\varepsilon} - u_n^{0,\varepsilon,N} \right\|_{H_{\beta,\mu,T}}^2 \leq \varepsilon^{2(N+1)} C(N,T),$$

$$\left\| \partial_x u_n^{0,\varepsilon} \sigma - \partial_x u_n^{0,\varepsilon,N} \sigma \right\|_{H_{\beta,\mu,T}}^2 \leq \varepsilon^{2(N+1)} C(N,T),$$

where $C(N,T)$ stands for a generic constant depending on N, T.

Finally, combining Theorem 3.1. and Corollary 5.1 above, we state our main theorem, which shows expansions of $u^{\alpha,\varepsilon}(t,x)$ and $\partial_x u^{\alpha,\varepsilon}(t,x)\sigma(t,x)$ in terms of the perturbation parameters of the driver α and the forward SDE ε.

Theorem 5.1. *For any $M, N \in \mathbf{N}$, there exist generic constants $C(M,T)$ depending on M, T and $C(M,N,T)$ depending on M, N, T such that*

$$\begin{aligned} & \left\| u^{\alpha,\varepsilon} - \left\{ u^{0,\varepsilon,N} + \sum_{i=1}^M \alpha^i u_i^{0,\varepsilon,N} \right\} \right\|_{H_{\beta,\mu,T}}^2 \\ & + \left\| \partial_x u^{\alpha,\varepsilon} \sigma - \left\{ \partial_x u^{0,\varepsilon,N} \sigma + \sum_{i=1}^M \alpha^i \partial_x u_i^{0,\varepsilon,N} \sigma \right\} \right\|_{H_{\beta,\mu,T}}^2 \\ \leq \; & \alpha^{2(M+1)} C(M,T) + \varepsilon^{2(N+1)} C(M,N,T). \end{aligned}$$

Proof. We have the following inequality:

$$\begin{aligned} & \left\| u^{\alpha,\varepsilon} - \left\{ u^{0,\varepsilon,N} + \sum_{i=1}^M \alpha^i u_i^{0,\varepsilon,N} \right\} \right\|_{H_{\beta,\mu,T}}^2 \\ & + \left\| \partial_x u^{\alpha,\varepsilon} \sigma - \left\{ \partial_x u^{0,\varepsilon,N} \sigma + \sum_{i=1}^M \alpha^i \partial_x u_i^{0,\varepsilon,N} \sigma \right\} \right\|_{H_{\beta,\mu,T}}^2 \\ \leq \; & 2 \left\| u^{\alpha,\varepsilon} - \left\{ u^{0,\varepsilon} + \sum_{i=1}^M \alpha^i u_i^{0,\varepsilon} \right\} \right\|_{H_{\beta,\mu,T}}^2 \\ & +2 \left\| \left\{ u^{0,\varepsilon} + \sum_{i=1}^M \alpha^i u_i^{0,\varepsilon} \right\} - \left\{ u^{0,\varepsilon,N} + \sum_{i=1}^M \alpha^i u_i^{0,\varepsilon,N} \right\} \right\|_{H_{\beta,\mu,T}}^2 \\ & +2 \left\| \partial_x u^{\alpha,\varepsilon} \sigma - \left\{ \partial_x u^{0,\varepsilon} \sigma + \sum_{i=1}^M \alpha^i \partial_x u_i^{0,\varepsilon} \sigma \right\} \right\|_{H_{\beta,\mu,T}}^2 \\ & +2 \left\| \left\{ \partial_x u^{0,\varepsilon} \sigma + \sum_{i=1}^M \alpha^i \partial_x u_i^{0,\varepsilon} \sigma \right\} - \left\{ \partial_x u^{0,\varepsilon,N} \sigma + \sum_{i=1}^M \alpha^i \partial_x u_i^{0,\varepsilon,N} \sigma \right\} \right\|_{H_{\beta,\mu,T}}^2 . \end{aligned}$$

By Theorem 3.1 and Corollary 5.1 we have the statement. □

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