Problem 1

Consider a discrete-time model, where $R_X(t)$ is the expected return at time t of a stock with price $X(t)$, and $R_M(t)$ is the expected return at time t of the market portfolio, with value $M(t)$. In other terms:

$$\begin{aligned} R_X(t) &= E_t[\frac{X(t+1)-X(t)}{X(t)}] \\ R_M(t) &= E_t[\frac{M(t+1)-M(t)}{M(t)}] \end{aligned}$$

where E_t denotes expectation under the physical measure, conditional on all the information available at time t. We write β for the beta of the stock, and r for the risk-free rate of return. The Capital Asset Pricing Model (CAPM) shows that:

$$R_X(t) - r = \beta(R_M(t) - r)$$

a) Suppose now that stock prices change continuously, and that $r = 0$. Assuming the CAPM holds, derive a system of two stochastic differential equations (SDE) for $X(t)$ and $M(t)$.
b) Show that in this model, beta is equal to the covariance of the stock return with the market portfolio divided by the variance of the return of the market portfolio.
c) Suppose that the volatility of both $X(t)$ and $M(t)$ can be estimated. Can we also estimate β?
d) Suppose that both the drift and volatility of $X(t)$ and $M(t)$ can be estimated. Can we also estimate β?
For cases (c) and (d) you need to justify your answer.

Problem 2

Let $P(t,T)$ be the price at time t of a discount bond with maturity T. The time axis is divided in equal intervals of size δ. We define $N(t)$ as :

$$N(t) = P(t, k\delta) \quad \text{for } (k-1)\delta < t \leq k\delta$$

The rolling forward measure is the measure under which the ratio of any asset price divided by the numeraire N is a martingale. Let $F_k(t)$ be the value at time t of the simply compounded forward rate on a loan with start date $k\delta$ and maturity $(k+1)\delta$. In other terms:

$$1 + F_k(t)\delta = \frac{P(t, k\delta)}{P(t, (k+1)\delta)}$$

The forward rate is modelled as:

$$\frac{dF_k(t)}{F_k(t)} = \sum_{q=1}^{p} \zeta_{k,q} dW_q^{k\delta}(t)$$

where $W_q^{k\delta}(t)$ are independent Brownian motions in the forward measure with tenor $k\delta$. Let $W_q(t)$ be independent Brownian motions in the rolling forward measure.

a) Show that

$$\frac{dF_k(t)}{F_k(t)} = \sum_{i=m(t)}^{k} \frac{\sum_{q=1}^{p} \zeta_{i,q}\zeta_{k,q}\delta}{1+F_i(t)\delta} dt + \sum_{q=1}^{p} \zeta_{k,q} dW_q(t)$$

In this expression $m(t) = \{\min n\delta |\ n\delta > t\}$

b) How would you calculate the price of an asset in this model?

Problem 3

The original two-factor Vasicek model for the risk-free interest rate $r(t)$ is:

$$\begin{aligned} dX_1(t) &= (a_1 - b_{11}X_1(t) - b_{12}X_2(t))dt + \sigma_1 dB_1(t) \\ dX_2(t) &= (a_2 - b_{21}X_1(t) - b_{22}X_2(t))dt + \sigma_2 dB_2(t) \\ r(t) &= \varepsilon_0 + \varepsilon_1 X_1(t) + \varepsilon_2 X_2(t) \end{aligned}$$

where $B_1(t)$ and $B_2(t)$ are Brownian motions with $dB_1(t)dB_2(t) = \rho dt$. This model has 12 parameters, but it has been showed that it can be rewritten in a canonical format with only 6 parameters, thus facilitating estimation. In a first step towards the canonical model, we define the semi-canonical Vasicek model:

$$\begin{aligned} d\bar{X}_1(t) &= (\theta_1 - \lambda_1\bar{X}_1(t))dt + p_{11}\sigma_1 dB_1(t) \\ d\bar{X}_2(t) &= (\theta_1 - \kappa\bar{X}_1(t) - \lambda_2\bar{X}_2(t))dt + p_{21}\sigma_1 dB_1(t) + p_{22}\sigma_2 dB_2(t) \\ R(t) &= \varepsilon_0 + \gamma_1\bar{X}_1(t) + \gamma_2\bar{X}_2(t) \end{aligned}$$

a) Determine relations between the parameters of the original model and the semi-canonical model which ensure that $r(t)$ and $R(t)$ are the same processes.

b) Define uncorrelated Brownian motions W_1 and W_2 , and express the semi-canonical model as a function of W_1 and W_2.

Problem 4

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ that supports as many Brownian motions as you need. In each subproblem (a) to (d) you should construct a market model. In other terms, you should write equations for asset prices $X_1(t)$ and $X_2(t)$, linking them (through coefficients that are not all zero) to Brownian motions.

a) Construct a market model where there exists an arbitrage strategy.

b) Construct a complete market model, where there does not exist an arbitrage strategy.

c) Construct an incomplete market model with 3 Brownian motions.

d) Construct an incomplete model with 2 Brownian motions.

Problem 5
A security gives the right to all the dividends issued by a firm between period t and T. The price of this security at time t is denoted $X(t)$, and the value of the cumulated dividends between 0 and t is $D(t)$. Let r be the (constant) risk-free rate of return.
a) Find a formula for $X(t)$, and justify it.
b) Let $V(t)$ be the value at time t of the cumulated dividends between zero and t

$$V(t) = \exp(rt)\int_0^t e^{-r(t-s)}dD(s)$$

Show by mathematical means only and your result in (a) that $\exp(-rt)(X(t)+V(t))$ is a martingale in the risk-neutral measure.

Problem 6
Let $W'(t)$ be white noise. Solve the stochastic differential equation:

$$\begin{aligned} X''(t)+156X'(t)+169 &= W'(t) \\ X(0) &= 1 \\ X'(0) &= 0 \end{aligned}$$

What is $E[X^2(t)]$?

Problem 7
Let T and f be two differentiable functions mapping $\mathbb{R}$ to $\mathbb{R}$. Let W be Brownian motion. We would like to define a stochastic integral, which we write

$$\int_0^b f(t)dW(T(t))$$

which would be a limit in some sense (when n goes to infinity) of

$$I(f) = \sum_{i=0}^{n-1} f(\frac{ib}{n})[W(T(\frac{(i+1)b}{n})) - W(T(\frac{ib}{n}))]$$

a) How would you define such an integral ? What restriction should you impose on T?
b) Let $f(t) = t^4$ and $T(t) = t^2$. Calculate $(\int_0^b f(t)dW(T(t)))^2$ as a function of b.