Correlated Defaults in Intensity-Based Models

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Abstract

This paper presents an intensity-based model of correlated defaults with application to the valuation of defaultable securities. The model assumes that the intensities of the default times are driven by common factors as well as other defaults in the system. A recursive procedure called the "total hazard construction" is used to generate default times with a broad class of correlation structures. This approach is compared to standard reduced-form models based on conditional independence as well as alternative approaches involving copula functions. Examples are given for the pricing of defaultable bonds and credit default swaps of the regular and basket type.

1 Introduction

In "standard" reduced-form models such as Lando (1998), it is customary to define the time to default τ as

$$\tau = \inf \left\{ t : \int_0^t \lambda_s ds \geq E \right\}, \tag{1.1}$$

where λ is a nonnegative process adapted to the filtration generated by a stochastic process X, and E is a unit exponential random variable independent of X. This definition ensures that

$$N_t - \int_0^t \lambda_s 1_{\{s<\tau\}} ds$$

is a martingale in the filtration generated by X and N, where $N_t = 1_{\{t \geq \tau\}}$ is the single-jump process associated with the default time. In this setting, the process λ is usually referred to as the intensity of τ.

When multiple defaults are involved, the above definition generates default times from a collection of independent unit exponential random variables $\left(E^i\right)_{i=1}^I$. This implies that the default times $\left(\tau^i\right)_{i=1}^I$ as given in (1.1) are independent given the whole history of X, and default correlation arises because of correlation of the intensities induced by their common dependence on X.[1]

In the current literature, there are three simple ways to extend this definition. First, Schönbucher and Schubert (2001) allow the heretofore independent $\left(E^i\right)_{i=1}^I$ to follow a joint distribution $C\left(U^1, \cdots, U^I\right)$, where $U^i = \exp\left(-E^i\right)$, and C denotes a copula function. Second, Schönbucher (2003) and Gouriéroux and Gagliardini (2003) assume that the stochastic process X can include "frailties," or unobserved factors; conditional on only the observed factors, default times are no longer independent. Third, Jarrow and Yu (2001) allow the intensities to depend directly on past defaults in addition to X.

This paper focuses on Jarrow and Yu's intensity-based approach. Jarrow and Yu note that as λ is allowed to depend on other defaults, (1.1) ceases to be a useful definition of default times because of the recursive structure it creates. Various authors, including Kusuoka (1999), Bielecki and Rutkowski (2002), Gouriéroux and Gagliardini (2003), and Collin-Dufresne, Goldstein, and Hugonnier (2004), have addressed this issue. However, in most cases one is limited to analytical solutions in the setting of two firms.

In this paper, I present a general algorithm for constructing an arbitrary number of default times with intensities dependent on observed defaults as well as a common

[1] Hull and White (2001) and Schönbucher and Schubert (2001) suggest that this type of models generate a default correlation that is too low relative to empirical estimates. On the other hand, Yu (2005) seems to offer some support for the assumption of conditionally independent defaults. Das, Duffie, Kapadia, and Saita (2004) reject the joint hypothesis of conditionally independent defaults and well-specified default intensities.

stochastic process X. This algorithm is based on the so-called "total hazard construction" from Norros (1987) and Shaked and Shanthikumar (1987), which traces its root to Meyer (1971). While the original prescription calls for stopping times with respect to their internal histories, it is straightforward to extend this construction to cases of broader interest, where the intensities also depend on a common stochastic process X.

The rest of this paper is organized as follows. Section 2 outlines an intensity-based model of multiple defaults. Section 3 presents the total hazard construction and shows that it indeed gives rise to default times with the desired intensities. Section 4 compares the intensity-based and copula-based approaches. Section 5 illustrates the model with applications to the pricing of defaultable bonds and credit derivatives. Section 6 concludes.

2 A Framework of Multiple Defaults

Consider a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, P\right)$, satisfying the usual conditions of right continuity and completeness with respect to P-null sets.[2] On this probability space there is a process X and I stopping times, $\left(\tau^i\right)_{i=1}^I$, representing the default times of I individual obligors. Let $N_t^i = 1_{\{t\geq\tau^i\}}$ denote the single-jump process associated with τ^i. For simplicity, it is assumed that

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^N,$$

where $\mathcal{F}_t^X = \sigma\left(X_s, 0 \leq s \leq t\right)$ and $\mathcal{F}_t^N = \sigma\left(\left(N_s^i\right)_{i=1}^I, 0 \leq s \leq t\right)$. In other words, $(\mathcal{F}_t)_{t\geq 0}$ is the smallest filtration containing the internal histories of X and the default processes $\left(N^i\right)_{i=1}^I$.

Definition 2.1 *The default process N^i has an $\mathcal{F}_t$-intensity λ^i if there exists a non-negative and $\mathcal{F}_t$-adapted process λ^i, satisfying an integrability condition*

$$\int_0^t \lambda_s^i ds < \infty,\ t \geq 0,$$

such that

$$M_t^i = N_t^i - \int_0^t \lambda_s^i ds \tag{2.1}$$

is an $\mathcal{F}_t$-martingale, or equivalently,

$$E\left(N_{t+h}^i - N_t^i | \mathcal{F}_t\right) = E\left(\int_t^{t+h} \lambda_s^i ds | \mathcal{F}_t\right),\ h > 0. \tag{2.2}$$

[2]Depending on the specific application (risk management or pricing), the probability measure P can be the physical measure or the risk-neutral measure. Of course, the risk-neutral measure is unique only if there are a sufficient number of traded assets to "complete" the market.

Because the default process N^i features a single jump at τ^i, Definition 2.1 implies that the intensity λ^i vanishes after τ^i, or $\lambda_t^i = 0$, for $t \geq \tau^i$.

Remark 2.1 For a more intuitive interpretation of the intensity, note that if the intensity is right-continuous and bounded, it follows from (2.2) that

$$\lim_{h\to 0+} \frac{1}{h} E\left(N_{t+h}^i - N_t^i | \mathcal{F}_t\right) = \lambda_t^i. \tag{2.3}$$

This gives λ_t^i the interpretation as the $\mathcal{F}_t$-conditional hazard rate of the default time τ^i. In fact, (2.3) offers a practical way to identify the intensity of an increasing process: Let $(h_n)_{n=1}^{\infty}$ be a sequence decreasing to zero and assume that the following limit exists:

$$\lim_{n\to\infty} \frac{1}{h_n} E\left(N_{t+h_n}^i - N_t^i | \mathcal{F}_t\right) = \lambda_t^i.$$

If $E\left(N_{t+h_n}^i - N_t^i | \mathcal{F}_t\right)/h_n - \lambda_t^i$ satisfies a certain technical condition, given in Aven (1985), then λ_t^i is the $\mathcal{F}_t$-intensity of N_t^i.

In light of Remark 2.1, Aven (1985)'s condition is assumed below, so that the intensities take the interpretation of (2.3):

Assumption 2.1 *Let $(h_n)_{n=1}^{\infty}$ be a sequence decreasing to zero and*

$$Y_n^i(t) = \frac{1}{h_n} E\left(N_{t+h_n}^i - N_t^i | \mathcal{F}_t\right).$$

There are non-negative and $\mathcal{F}_t$-adapted processes $\lambda^i(t)$ and $y^i(t)$ such that

1) For each t and i,

$$\lim_{n\to\infty} Y_n^i(t) = \lambda^i(t).$$

2) For each t and i there exists for each $\omega \in \Omega$ an $n_0^i = n_0^i(t,\omega)$ such that

$$\left|Y_n^i(s,\omega) - \lambda^i(s,\omega)\right| \leq y^i(s,\omega), \ \forall s \leq t, n \geq n_0^i.$$

3) For each t and i,

$$\int_0^t y^i(s)\, ds < \infty.$$

Consistent with Definition 2.1, when N^i consists of a single jump at τ^i, Assumption 2.1 implies that the process λ^i vanishes after τ^i.

We also make a standard assumption in the setting of default modeling. This assumption rules out simultaneous defaults and ensures that default will always occur in finite time.

Assumption 2.2 *1) For each i, $P\left(0 < \tau^i < \infty\right) = 1$. 2) For each pair of $i \neq j$, $P\left(\tau^i = \tau^j\right) = 0$.*

Lastly, it is assumed that X represents an "exogenous" process in the following sense:

Assumption 2.3 *For any t, the σ-fields $\mathcal{F}_\infty^X$ and $\mathcal{F}_t^N$ are conditionally independent given $\mathcal{F}_t^X$.*

This assumption is taken directly from Elliott, Jeanblanc, and Yor (2000). Lando (1998)'s construction of Cox processes in (1.1) clearly satisfies this requirement, as the unit exponential random variable E is assumed to be independent of X.

Remark 2.2 In the present setting, Assumption 2.3 implies that for any $\mathcal{F}_\infty^X$-measurable random variable Y, we have

$$E\left(Y|\mathcal{F}_t^X \vee \mathcal{F}_t^N\right) = E\left(Y|\mathcal{F}_t^X\right). \tag{2.4}$$

Consequently, a "transition probability," useful in simulating the sample paths of a Markovian process X, can be computed independently of the jump processes N. This is why it makes sense to call X an "exogenous" process. Note that (2.4) is linked with the invariance property of $\mathcal{F}_t^X$-martingales [see Elliott, Jeanblanc, and Yor (2000)].

Remark 2.3 Another useful implication of Assumption 2.3 is that for any $\mathcal{F}_t^N$-measurable random variable Z,

$$E\left(Z|\mathcal{F}_t^X\right) = E\left(Z|\mathcal{F}_\infty^X\right).$$

Applying this result to (2.3),

$$\begin{aligned}
&\lim_{h\to 0+}\frac{1}{h}E\left(N_{t+h}^i - N_t^i|\mathcal{F}_\infty^X \vee \mathcal{F}_t^N\right)\\
=\ &\lim_{h\to 0+}\frac{1}{h}E\left(N_{t+h}^i - N_t^i|\mathcal{F}_{t+h}^X \vee \mathcal{F}_t^N\right)\\
=\ &\lim_{h\to 0+}\frac{1}{h}E\left(N_{t+h}^i - N_t^i|\mathcal{F}_t^X \vee \mathcal{F}_t^N\right).
\end{aligned}$$

The last equality is a consequence of the right continuity of $\mathcal{F}_t^X$. Moreover it is easy to verify that the rest of Aven's condition is also satisfied with $\mathcal{F}_t$ replaced by $\mathcal{G}_t$, where

$$\mathcal{G}_t = \mathcal{F}_\infty^X \vee \mathcal{F}_t^N.$$

Therefore, under Assumption 2.3 the $\mathcal{F}_t$-intensity λ also happens to be the $\mathcal{G}_t$-intensity of the default time.

3 The Total Hazard Construction of Default Times

The previous section assumes the *existence* of default times having a certain type of stochastic intensities. This section, on the other hand, emphasizes the *construction* of default times having such intensities. The methodology follows Norros (1987) and Shaked and Shanthikumar (1987).

First, consider a special case of the model in Section 2, in which $\mathcal{F}_t = \mathcal{F}_t^N$, with $N_t^i = 1_{\{t \geq \tau^i\}}$, $1 \leq i \leq I$. In other words, the information comes solely from the internal history of the default processes. In this case, instead of merely stating that λ^i is the $\mathcal{F}_t$-intensity of the default time τ^i, we can express the information set more clearly by writing $\omega \in \mathcal{F}_t^N$ as $\omega = (I_n, T_n)$, where $I_n = (k_1, \ldots, k_n)$ and $T_n = (t_1, \ldots, t_n)$. With this shorthand, it is understood that at time t, n defaults have already occurred at $0 = t_0 < t_1 < \cdots < t_n < t$, where the jth defaulter $(1 \leq j \leq n)$ is obligor k_j. The intensity is then written as $\lambda_t^i = \lambda_i (t | I_n, T_n)$, which is a deterministic function of t and the default history. Note that this function is well-defined only on the interval between the nth default time t_n and the $(n+1)$th default time t_{n+1}.

Having specified the desired form of the intensities, we next define the total hazard accumulated by obligor i by time t as

$$\begin{aligned} \psi_i (t | I_n, T_n) &= \sum_{m=1}^{n} \Lambda_i (t_m - t_{m-1} | I_m, T_m) \\ &\quad + \Lambda_i (t - t_n | I_n, T_n), \end{aligned} \tag{3.1}$$

where

$$\Lambda_i (s | I_m, T_m) = \int_{t_m}^{t_m + s} \lambda_i (u | I_m, T_m) \, du \tag{3.2}$$

is the total hazard accumulated by obligor i for a period of s following the mth default. It is assumed that there is no default between t_n and t.

A result in Norros (1987) states that under Assumption 2.2, the total hazards accumulated by $\tau = (\tau^1, \ldots, \tau^I)$ by the time they occur are independent unit exponential random variables.[3] Using this result, Shaked and Shanthikumar (1987) construct an inverse mapping which generates a set of random times from a collection of independent unit exponential random variables.

Define

$$\Lambda_i^{-1} (x | I_n, T_n) = \inf \{ s : \Lambda_i (s | I_n, T_n) \geq x \}, \ x \geq 0. \tag{3.3}$$

The following recursive procedure constructs a new collection of random variables $\widehat{\tau} = \left(\widehat{\tau}^1, \ldots, \widehat{\tau}^I \right)$:

[3] The compensators of the default times also have to be continuous. However, this is satisfied by the integrability condition on the intensities in Definition 2.1.

1. Draw a collection of i.i.d. unit exponential random variables $E = \left(E^1, \ldots, E^I\right)$.

2. Let
$$k_1 = \arg\min_{1 \le i \le I} \left\{ \Lambda_i^{-1}\left(E^i\right) \right\},$$
and let
$$\widehat{\tau}^{k_1} = \Lambda_{k_1}^{-1}\left(E^{k_1}\right).$$

3. Assume that the values of $\left(\widehat{\tau}^{k_1}, \ldots, \widehat{\tau}^{k_{m-1}}\right)$ are already determined as $T_{m-1} = (t_1, \ldots, t_{m-1})$, where $m \ge 2$. Define the defaulted set $I_{m-1} = (k_1, \ldots, k_{m-1})$ and the remaining set $\overline{I}_{m-1} = (1, \ldots, n) \setminus I_{m-1}$. Recall that $\psi_i\left(t | I_{m-1}, T_{m-1}\right)$ is the total hazard accumulated by firm i to time t given the first $m-1$ defaults, let
$$k_m = \arg\min_{i \in \overline{I}_{m-1}} \left\{ \Lambda_i^{-1}\left(E^i - \psi_i\left(t_{m-1} | I_{m-1}, T_{m-1}\right) | I_{m-1}, T_{m-1}\right) \right\},$$
and let
$$\widehat{\tau}^{k_m} = t_{m-1} + \Lambda_{k_m}^{-1}\left(E^{k_m} - \psi_{k_m}\left(t_{m-1} | I_{m-1}, T_{m-1}\right) | I_{m-1}, T_{m-1}\right).$$

4. If $m = I$, then stop. Otherwise, increase m by 1 and go to Step 3.

It is shown in Norros (1987) and Shaked and Shanthikumar (1987) that $\widehat{\tau} \overset{\text{st}}{=} \tau$ where $\overset{\text{st}}{=}$ denotes equality in law. In other words, the $\mathcal{F}_t^N$-intensities alone are sufficient to recover the law of τ.

Given the newly constructed random variables $\widehat{\tau}$ and their law P', it is straightforward to specify a new filtered probability space $\left(\Omega', \left(\mathcal{F}'_t\right)_{t \ge 0}, P'\right)$, where $\mathcal{F}'_t$ is the internal history of the processes $\widehat{N}_t^i = 1_{\left\{t \ge \widehat{\tau}^i\right\}}$, $1 \le i \le I$. It then remains to be shown that the process $\widehat{N}_t^i$ has an $\mathcal{F}'_t$-intensity of the previously specified form: $\widehat{\lambda}_t^i = \lambda_i\left(t | I_n, T_n\right)$. However, this is self-evident due to $\widehat{\tau} \overset{\text{st}}{=} \tau$ and the interpretation of the intensities as conditional hazard rates. To see this, using the more detailed notation, we have

$$\begin{aligned}
\widehat{\lambda}_t^i &= \lim_{h \to 0+} \frac{1}{h} E'\left(\widehat{N}_{t+h}^i - \widehat{N}_t^i | \mathcal{F}'_t\right) \\
&= \lim_{h \to 0+} \frac{1}{h} P'\left(t < \widehat{\tau}^i \le t + h | \widehat{\tau}^{k_1} = t_1, \ldots, \widehat{\tau}^{k_n} = t_n\right) \\
&= \lim_{h \to 0+} \frac{1}{h} P\left(t < \tau^i \le t + h | \tau^{k_1} = t_1, \ldots, \tau^{k_n} = t_n\right) \\
&= \lambda_i\left(t | I_n, T_n\right). \qquad (3.4)
\end{aligned}$$

The third equality of (3.4) uses the fact that $\widehat{\tau} \overset{\text{st}}{=} \tau$, and the last equality comes from the assumption that Aven (1985)'s condition is satisfied by the original default times τ (see Remark 2.1 and Assumption 2.1). These two observations together also imply that Aven's condition applies to $\widehat{N}_t^i$ as well. Consequently, the first equality holds. This yields the following result:

Proposition 3.1 *Let τ be default times whose associated single-jump processes satisfy Assumptions 2.1-2.2, with intensities $\lambda_t^i = \lambda_i\left(t|I_n, T_n\right)$, $1 \leq i \leq I$. Construct $\widehat{\tau}$ according to Steps 1-4 with the intensity functions $\lambda_i\left(t|I_n, T_n\right)$ in (3.1)-(3.3). Let $\mathcal{F}'_t$ be the internal history of the single jump processes associated with $\widehat{\tau}$ and P' be the law of $\widehat{\tau}$. Then $\widehat{\tau}^i$ has $(P', \mathcal{F}'_t)$-intensity of the form $\lambda_i\left(t|I_n, T_n\right)$, $1 \leq i \leq I$.*

Next, we extend Proposition 3.1 to the more general case of Section 2. That is, where $\mathcal{F}_t = \mathcal{F}_t^N \vee \mathcal{F}_t^X$ and Assumption 2.2 is satisfied. In this case, Remark 2.3 shows that the $\mathcal{F}_t$-intensity λ^i is also the $\mathcal{G}_t$-intensity where the enlarged filtration includes $\mathcal{F}_\infty^X$, the entire history of X. The strategy, then, is to follow the total hazard construction, with the understanding that the procedure is conditional on a particular element $\omega \in \mathcal{F}_\infty^X$. Note that in this case, the intensity of τ^i can be written as $\lambda_t^i = \lambda_i\left(t|I_n, T_n, \omega\right)$ to take into account the new information set.

Replace Step 1 of the total hazard construction procedure above with the following:[4]

1A. Draw a complete sample path of X, denoted by $\omega \in \mathcal{F}_\infty^X$. Draw i.i.d. unit exponential random variables $E = \left(E^1, \ldots, E^I\right)$ independent of $\mathcal{F}_\infty^X$.

Following the revised total hazard construction, Steps 1A and 2-4, one arrives at a set of random variables $\widehat{\tau}$, which, in addition to their obvious dependence on the unit exponential random variables, also depend on $\mathcal{F}_\infty^X$ through the pre-specified intensity functions. It is straightforward to specify a new filtered probability space $\left(\Omega', (\mathcal{F}'_t)_{t\geq 0}, P'\right)$ where $\mathcal{F}'_t$ is the smallest filtration containing the internal filtration of X and the single-jump processes associated with $\widehat{\tau}$:

$$\mathcal{F}'_t = \mathcal{F}_t^X \vee \mathcal{F}_t^{\widehat{N}},$$

and P' is the probability measure generated by the law of X and the law of $\widehat{\tau}$ conditional on $\omega \in \mathcal{F}_\infty^X$. Moreover, let

$$\mathcal{G}'_t = \mathcal{F}_\infty^X \vee \mathcal{F}_t^{\widehat{N}}.$$

By Shaked and Shanthikumar (1987)'s result, $\widehat{\tau}$ and the original default times τ are

[4]Step 1A is feasible because the sample paths of X can be simulated independently from the jump processes (see Remark 2.2).

equal in law given $\omega \in \mathcal{F}_\infty^X$. Therefore, the $\mathcal{G}_t'$-intensity of $\widehat{\tau}$ is given by

$$\begin{aligned}
\widehat{\lambda}_t^i &= \lim_{h\to 0+} \frac{1}{h} E'\left(\widehat{N}_{t+h}^i - \widehat{N}_t^i | \mathcal{G}_t'\right) \\
&= \lim_{h\to 0+} \frac{1}{h} P'\left(t < \widehat{\tau}^i \leq t+h | \widehat{\tau}^{k_1} = t_1, \ldots, \widehat{\tau}^{k_n} = t_n, \omega\right) \\
&= \lim_{h\to 0+} \frac{1}{h} P\left(t < \tau^i \leq t+h | \tau^{k_1} = t_1, \ldots, \tau^{k_n} = t_n, \omega\right) \\
&= \lambda_i\left(t|I_n, T_n, \omega\right). \qquad (3.5)
\end{aligned}$$

Again, this result is due to the equality in law between τ and $\widehat{\tau}$ given $\mathcal{F}_\infty^X$, and the fact that Aven's condition holds for τ with respect to $\mathcal{G}_t$ and for $\widehat{\tau}$ with respect to $\mathcal{G}_t'$ (see Remark 2.3). Since $\lambda_t^i = \lambda_i\left(t|I_n, T_n, \omega\right)$ is, in fact, assumed to be $\mathcal{F}_t$-adapted, the $\mathcal{F}_t'$-intensity of $\widehat{\tau}$ is also $\lambda_i\left(t|I_n, T_n, \omega\right)$. We therefore have the following result:

Proposition 3.2 *Let τ be default times whose associated single-jump processes satisfy Assumptions 2.1-2.3, with intensities $\lambda_t^i = \lambda_i\left(t|I_n, T_n, \omega\right)$, $\omega \in \mathcal{F}_t^X$, $1 \leq i \leq I$. Construct $\widehat{\tau}$ according to Steps 1A and 2-4 with the intensity functions $\lambda_i\left(t|I_n, T_n, \omega\right)$ in (3.1)-(3.3). Let $\mathcal{F}_t'$ be the minimal filtration containing $\mathcal{F}_t^X$ and the internal history of the single-jump processes associated with $\widehat{\tau}$ and P' be the law of $(X, \widehat{\tau})$. Then $\widehat{\tau}^i$ has $(P', \mathcal{F}_t')$-intensity of the form $\lambda_i\left(t|I_n, T_n, \omega\right)$, $1 \leq i \leq I$.*

Assumption 2.3 appears to rule out cases where the stochastic process X depends on past observations of the default process N. One such well known scenario is called "flight to quality," where the default-free short rate contains downward jumps when one or more obligors default.[5] For example, the default-free short rate X_t can be specified as

$$X_t = Y_t + \sum_{i=1}^{I} \gamma_i N_t^i, \qquad (3.6)$$

where

$$dY_t = \kappa\left(\theta - Y_t\right) dt + \sigma\sqrt{Y_t} dW_t, \qquad (3.7)$$

with $\gamma_I < \cdots < \gamma_1 < 0$ and Y is a square-root diffusion with mean-reversion parameter κ, long-run mean θ, and volatility parameter σ. The default intensities are

$$\lambda_t^i = \left(\alpha_i + \beta_i X_t\right)\left(1 - N_t^i\right), \qquad (3.8)$$

where α_i and β_i are constants. We then have

$$\lim_{h\to 0+} \frac{1}{h} E\left(N_{t+h}^i - N_t^i | \mathcal{G}_t\right) = \lim_{h\to 0+} \frac{1}{h}\left(N_{t+h}^i - N_t^i\right) = 0 \neq \lambda_t^i$$

[5] When major defaults occur (such as LTCM and Enron), investors tend to flock to US Treasury securities for their safety, driving prices up and interest rates down.

due to the right-continuity of N^i and the fact that the knowledge of $(X_t)_{t=0}^{\infty}$ amounts to a complete knowledge of $\left(N_t^i\right)_{t=0}^{\infty}$, $1 \leq i \leq I$. Therefore, the construction outlined above fails.

Yet there is a simple way to circumvent this problem. In this particular case, we can substitute (3.6) into (3.8), yielding

$$\lambda_t^i = \left(\alpha_i + \sum_{i=1}^{I} \beta_i \gamma_i N_t^i + \beta_i Y_t\right)\left(1 - N_t^i\right). \tag{3.9}$$

Then, let Assumption 2.3 apply to the "new" factor process Y. In other words, the σ-fields $\mathcal{F}_\infty^Y$ and $\mathcal{F}_t^N$ are assumed to be conditionally independent given $\mathcal{F}_t^Y$, with the dynamics of Y specified as in (3.7). Consequently, the above construction of the default times can still be used—we simply simulate the sample path of Y according to (3.7) and apply the total hazard construction with the intensity functions given in (3.9).

This example shows that by rewriting X_t as one part that is measurable with respect to $\mathcal{F}_t^N$ and another part that satisfies Assumption 2.3, we have significantly broadened the scope of the model described in Section 2 without changing the procedure for generating the default times.

4 Comparison with the Copula Approach

This section compares the aforementioned intensity-based model with the copula-dependent intensity model of Schönbucher and Schubert (2001) (called the "copula approach" below for short).

As in Lando (1998), the copula approach assumes that τ^i is defined as

$$\tau^i = \inf\left\{t : \int_0^t h_s^i ds \geq E^i\right\},$$

where h^i is an $\mathcal{F}_t^X$-adapted process. However, instead of maintaining that the unit exponential random variables $\left(E^1, \ldots, E^I\right)$ are independently distributed, this approach assumes that they are governed by a joint distribution function $C\left(U^1, \cdots, U^I\right)$, where $U^i = \exp\left(-E^i\right)$ and C is a copula function. Note, in particular, that h^i is the intensity of τ^i with respect to the filtration $\mathcal{F}_t^X \vee \mathcal{F}_t^{N^i}$, and not the intensity in the more general setting (with respect to the full filtration $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^N$).

To understand the relationship between the copula approach and the intensity approach based on the total hazard construction, we again start with a simple case with only the internal histories of the default processes. It is assumed that a collection of default times $\tau = \left(\tau^1, \ldots, \tau^I\right)$ have a sufficiently smooth joint distribution function $F\left(t_1, \ldots, t_I\right) = F\left(\tau^1 \leq t_1, \ldots, \tau^I \leq t_I\right)$ and marginal distribution functions $G_i\left(t_i\right) = G_i\left(\tau^i \leq t_i\right)$. By Sklar's Theorem [see Nelson (1999)], there exists a copula function C such that

$$F(t_1, \ldots, t_I) = C(G_1(t_1), \ldots, G_I(t_I)).$$

Then, let

$$h_t^i = \frac{1}{1 - G_i(t)} \frac{dG_i(t)}{dt}. \tag{4.1}$$

This is the probability per unit time of τ^i occurring in the next instant, conditional on it not having occurred prior to t. In other words, it is the hazard rate of τ^i when other defaults are ignored.

When we apply the copula approach using the above hazard rate h^i and copula function C, it is clear that the resulting default times will have a "reverse-engineered" joint distribution function F. On the other hand, the total hazard construction starts from the conditional hazard rates given the internal history of defaults:

$$\lambda_t^i = \lim_{h \to 0+} \frac{1}{h} P\left(t < \tau^i \leq t + h | \mathcal{F}_t^N\right). \tag{4.2}$$

As shown by Shaked and Shanthikumar (1987), this procedure also yields default times with the joint distribution F. We therefore have the following result:

Proposition 4.1 *Assume default times $\tau = \left(\tau^1, \ldots, \tau^I\right)$ have a sufficiently smooth joint distribution function $F(t_1, \ldots, t_I)$ such that h_t^i in (4.1) and λ_t^i in (4.2) exist. Then both the copula-based approach and the intensity-based approach can be used to generate default times with the joint distribution function F.*

Proposition 4.1 therefore provides a sense in which the two methods constitute "equivalent" constructions of default times.

Remark 4.1 The situation is slightly different with the addition of the common stochastic process X. In this case, the construction described in Steps 1A and 2-4 gives default times with joint distribution F_ω conditional on $\omega \in \mathcal{F}_\infty^X$. If the dependence structure of F_ω is summarized by a copula function, then the choice of this copula function will in general depend on ω. In contrast, in Schönbucher and Schubert's approach the copula function is invariant to the history of X. It is in this sense that the copula-based approach is "static," whereas the intensity-based approach truly incorporates a dynamic factor X.

5 Applications

5.1 Pricing Defaultable Bonds

For simplicity, this section treats the pricing of zero-coupon bonds with zero recovery under a constant default-free short rate r. The defaultable bond price is then proportional to the conditional survival probability, $P\left(\tau^i > T|\mathcal{F}_t\right)$, where T denotes the maturity of the bond. Therefore, we focus on this entity under various assumptions of the default correlation structure.

5.1.1 The Case of Two Firms

Jarrow and Yu (2001) present a "looping default" case with two firms, where the default of one firm affects the default intensity of the other firm. In other words, they assume two default times, τ^A and τ^B, with the following intensities with respect to the internal history of the default processes:

$$\begin{aligned}\lambda_t^A &= \left(a_1 + a_2 1_{\{t \geq \tau^B\}}\right) 1_{\{t<\tau^A\}},\\ \lambda_t^B &= \left(b_1 + b_2 1_{\{t \geq \tau^A\}}\right) 1_{\{t<\tau^B\}}.\end{aligned}$$

Unable to provide an explicit solution for the distribution of the default times, they turn to a simpler case with

$$\begin{aligned}\lambda_t^A &= a 1_{\{t<\tau^A\}},\\ \lambda_t^B &= \left(b_1 + b_2 1_{\{t \geq \tau^A\}}\right) 1_{\{t<\tau^B\}}.\end{aligned} \tag{5.1}$$

Namely, they assume that firm A is a "primary" firm not affected by counterparty default and firm B is a "secondary" firm whose default intensity jumps at the default of firm A. Below, a solution to the more general case is given using the total hazard construction.

Following the notations of Section 3, with no default up to t the total hazards accumulated from 0 to t are $\Lambda_A(t) = a_1 t$ and $\Lambda_B(t) = b_1 t$. If firm A has defaulted at t_1, the total hazard accumulated by firm B from t_1 to $t_1 + t$ is $\Lambda_B(t|A, t_1) = (b_1 + b_2)\, t$. Similarly, if firm B has defaulted at t_2, one can set $\Lambda_A(t|B, t_2) = (a_1 + a_2)\, t$. Using the inverse of these functions, two default times $\widehat{\tau}^A$ and $\widehat{\tau}^B$ can be constructed from two independent unit exponential random variables E^A and E^B as:

$$\begin{aligned}\widehat{\tau}^A &= \begin{cases} \frac{E^A}{a_1}, & \frac{E^A}{a_1} \leq \frac{E^B}{b_1},\\ \frac{E^B}{b_1} + \frac{1}{a_1+a_2}\left(E^A - \frac{a_1}{b_1}E^B\right), & \frac{E^A}{a_1} > \frac{E^B}{b_1},\end{cases}\\ \widehat{\tau}^B &= \begin{cases} \frac{E^A}{a_1} + \frac{1}{b_1+b_2}\left(E^B - \frac{b_1}{a_1}E^A\right), & \frac{E^A}{a_1} \leq \frac{E^B}{b_1},\\ \frac{E^B}{b_1}, & \frac{E^A}{a_1} > \frac{E^B}{b_1}.\end{cases}\end{aligned}$$

From this, the joint density of $\widehat{\tau}^A$ and $\widehat{\tau}^B$, which is also shared by the original default times τ^A and τ^B, can be derived:

$$f(t_1, t_2) = \begin{cases} a_1(b_1+b_2)\, e^{-(a_1-b_2)t_1-(b_1+b_2)t_2}, & t_1 \leq t_2,\\ b_1(a_1+a_2)\, e^{-(b_1-a_2)t_2-(a_1+a_2)t_1}, & t_1 > t_2.\end{cases}$$

It follows that the marginal densities of τ^A and τ^B are

$$\begin{aligned} g_A(t_1) &= \frac{(a_1+a_2)\,b_1}{b_1-a_2}\left(e^{-(a_1+a_2)t_1}-e^{-(a_1+b_1)t_1}\right)+a_1e^{-(a_1+b_1)t_1}, \\ g_B(t_2) &= \frac{(b_1+b_2)\,a_1}{a_1-b_2}\left(e^{-(b_1+b_2)t_2}-e^{-(a_1+b_1)t_2}\right)+b_1e^{-(a_1+b_1)t_2}, \end{aligned}$$

and their marginal distributions are

$$\begin{aligned} G_A(t_1) &= 1-\frac{b_1e^{-(a_1+a_2)t_1}-a_2e^{-(a_1+b_1)t_1}}{b_1-a_2}, \\ G_B(t_2) &= 1-\frac{a_1e^{-(b_1+b_2)t_1}-b_2e^{-(a_1+b_1)t_1}}{a_1-b_2}. \end{aligned}$$

These expressions show that neither G_A depends on b_2, nor G_B on a_2. In fact, the expression above for G_B coincides with Jarrow and Yu's solution for default times with intensities specified by (5.1). Intuitively, what occurs to firm B after the default of firm A should have no impact on the distribution of firm A's default time. This is why in computing G_A, one can effectively ignore the default dependency for firm B.[6]

5.1.2 A Simple Specification with Multiple Firms

The type of default dependency modeled in this paper is in fact based on solid empirical evidence. For example, Lang and Stulz (1992) find that there are significant industry-wide abnormal stock returns in response to a bankruptcy filing. Newman and Rierson (2003) show that European telecom credit spreads widened by 10 basis points in June 2000 in response to a large issue by Deutsche Telekom. Evidence that a single "credit event," defined as a large monthly change in an individual corporate bond yield spread, can affect the entire corporate bond market index is presented by Collin-Dufresne, Goldstein and Helwege (2003).

To model such contagion effects, we consider the following simple specification:

$$\lambda_t^i = \left(a_1 + a_2 1_{\{t\geq\tau_F\}}\right) 1_{\{t<\tau^i\}}, \tag{5.2}$$

where $\tau_F = \min\left(\tau^1,\ldots,\tau^I\right)$ is the first-to-default time. The coefficient a_2 can be estimated from the average price reaction of a large cross-section of bonds to "typical" credit events. The use of the first-to-default, rather than a more general default dependency relationship, can be justified in a population of high quality credits where defaults are relatively infrequent.

Using the total hazard construction, the first-to-default time $\widehat{\tau}_F$ can be written as

$$\widehat{\tau}_F = \min\left(\frac{E^1}{a_1},\frac{E^2}{a_1},\ldots,\frac{E^I}{a_1}\right),$$

[6] This interesting observation and its interpretation are first given in Bielecki and Rutkowski (2002, Chapter 9).

and the individual default times are given through their respective unit exponential random variables as

$$\widehat{\tau}^i = \frac{E^i - a_1 \widehat{\tau}_F}{a_1 + a_2} + \widehat{\tau}_F.$$

Using the fact that $\widehat{\tau}_F = \min\left(E^1/a, \widehat{\tau}'_F\right)$ where $\widehat{\tau}'_F = \min\left(E^2, \ldots, E^I\right)/a$, and $\widehat{\tau}'_F$ is an exponential random variable with rate $(I-1)\,a_1$ independent of E^1, the marginal density of $\widehat{\tau}^1$ can be derived as:

$$g_1(t_1) = \frac{(I-1)\,a_1\,(a_1+a_2)\,e^{-(a_1+a_2)t_1} - I a_1 a_2 e^{-I a_1 t_1}}{(I-1)\,a_1 - a_2}. \tag{5.3}$$

Remark 5.1 Taking the limit of $I \to \infty$ in (5.3), τ^1 converges to a default time with intensity $a_1 + a_2$ in the sense that the survival probability converges to $\exp\left(-(a_1+a_2)\,t_1\right)$. Intuitively, as I increases, the first-to-default occurs sooner. In the limit of $I \to \infty$, τ_F is equal to 0 and the intensity of τ^1 is equal to $a_1 + a_2$ almost surely.

Remark 5.2 Ignoring the default dependency in this example amounts to computing the survival probability as

$$P^{\text{incorrect}}\left(\tau^1 > t_1\right) = E\left(\exp\left(-\int_0^{t_1}\left(a_1 + a_2 1_{\{s \geq \tau_F\}}\right) ds\right)\right),$$

treating τ_F as exponentially distributed with rate $I a_1$. This is the same as replacing I in (5.3) with $I+1$. The two are clearly different and the difference is larger for smaller I, disappearing as $I \to \infty$.

Figure 1 presents a numerical illustration of this case. Specifically, we assume a constant interest rate, $a_1 = 0.01$, $a_2 = 0.001$, and price a defaultable zero-coupon bond. The figure plots the yield spread on such a bond as a function of maturity for various values of I, assuming that no default has occurred in the system. As a reference, the thin dotted line represents the case where the first-to-default has just occurred (or, as mentioned in Remark 5.1, the case of $I \to \infty$). The gap between other lines and the thin dotted line represents the jump in yield spread at the first-to-default, which can be used to identify the coefficient a_2. For example, for an industry that consists of 10 firms, if one estimates that the spreads on 5-year zero-coupon corporate bonds are equal to 150 bps, and that a default in the industry adds another 10 bps to these spreads, the default intensity parameters can be implied out as:

$$a_1 = 0.01464,\; a_2 = 0.00136.$$

5.2 Pricing Credit Default Swaps

This section examines the pricing of credit default swaps (CDS), an insurance-like security which helps to transfer credit risk. For simplicity, it is assumed that the buyer of the CDS agrees to make continuous premium payments to the seller at a fixed rate until the expiration of the contract. The seller agrees to pay \$1 to the buyer at the expiration of the contract if a reference credit defaults prior to expiration.[7]

The CDS buyer has default time τ^A with intensity λ^A, the seller, τ^B with λ^B, and the reference credit, τ^C with λ^C. The intensities are specified with respect to the internal history of the default processes as:

$$
\begin{aligned}
\lambda_t^A &= \left(a_{10} + a_{12} 1_{\{t \geq \tau^B\}} + a_{13} 1_{\{t \geq \tau^C\}}\right) 1_{\{t < \tau^A\}}, \\
\lambda_t^B &= \left(a_{20} + a_{21} 1_{\{t \geq \tau^A\}} + a_{23} 1_{\{t \geq \tau^C\}}\right) 1_{\{t < \tau^B\}}, \\
\lambda_t^C &= \left(a_{30} + a_{31} 1_{\{t \geq \tau^A\}} + a_{32} 1_{\{t \geq \tau^B\}}\right) 1_{\{t < \tau^C\}}.
\end{aligned}
\tag{5.4}
$$

A constant short rate r is used for discounting cash flows.

The present value of the premium payment from the buyer is then

$$
E\left(\int_0^T \exp\left(-\int_0^s r_u du\right) y 1_{\{s < \tau^A\}} ds\right),
$$

where y denotes the CDS premium rate and T the time to expiration of the contract. The indicator function reflects the fact that the premium payment stops given a default by the CDS buyer.

The present value of the seller's payment at time T is

$$
E\left(\exp\left(-\int_0^T r_u du\right) 1_{\{\tau^C \leq T, \tau^B > T\}}\right).
$$

The indicator here shows that the default protection is paid if the reference credit defaults prior to the CDS expiration, and then only if the seller is still alive to honor its obligations. Setting the two sides of the contract equal in value, the CDS premium y is

$$
y = \frac{E\left(\exp\left(-\int_0^T r_u du\right) 1_{\{\tau^C \leq T, \tau^B > T\}}\right)}{E\left(\int_0^T \exp\left(-\int_0^s r_u du\right) 1_{\{s < \tau^A\}} ds\right)}.
\tag{5.5}
$$

While analytical solutions using the total hazard construction are possible for the intensities specified in (5.4), the formula for the CDS premium is likely to be

[7] In a typical CDS contract, the premium payment stops when the reference credit defaults, upon which the default protection is paid. With the above assumptions, CDS pricing is more sensitive to the assumed default dependency structure. We also abstract away from realistic features such as recovery rate, physical or cash settlement, cheapest-to-deliver options, accrued interest, and considerations for debt restructuring. For an excellent introduction to the conventions of the CDS market, see Berndt et al. (2004).

inconveniently long. Therefore, we use Monte-Carlo simulation to numerically evaluate the expectations in (5.5). Specifically, we assume a base case with the following parameters: $r = 0.05$, $T = 5$, $a_{10} = a_{20} = a_{30} = 0.05$, and $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0.01$, where the coefficients a_{ij} are defined in reference to (5.4). We then vary individual coefficients, keeping others fixed at their base case values, and examine the corresponding change in the CDS premium y.

The results of this exercise are easy to understand. Generally, anything that elevates the credit risk of the buyer or reference asset, or reduces the credit risk of the seller, will increase the CDS premium. This can be a direct effect, as illustrated in the top three panels in Figure 2, or an indirect effect, as shown in the other six panels. Notably, an increase in a_{23} causes the default correlation between the seller and the reference credit to go up, reducing the CDS premium. This is the essence of an example on CDS pricing in Jarrow and Yu (2001). Here we see that default contagion can also propagate between the buyer and the seller (a_{12} and a_{21}), and between the reference and the buyer (a_{31} and a_{13}).

The absence of an effect due to changes in a_{32} can be understood as follows. Since a protection payment requires that the seller do not default until after the reference credit event, this coefficient has no effect on the numerator of (5.5). Instead, it enters CDS valuation by altering the distribution of the buyer's default time. If changing a_{10} constitutes a direct effect on the distribution of τ^A, changing a_{31} a first order effect, then this would be a second order effect, too small to be picked up by numerical simulations presented here.

5.3 Pricing Basket Credit Default Swaps

In this section we apply the simulation-based approach to the valuation of basket credit default swaps. An example of a basket CDS is a contract that pays \$1 if the first-to-default out of a portfolio of reference credits occurs prior to expiration. For simplicity, we assume that the payment (if any) occurs at expiration, and that the buyer pays a premium at the initiation of the swap. With a constant interest rate r, the premium on the nth-to-default CDS can be written as

$$y_n = \exp(-rT)\, P\left(\tau^{(n)} \leq T\right),$$

where T is the maturity of the swap and $\tau^{(n)}$ denotes the nth-to-default time.

Evidently, the pricing of basket CDS is sensitive to default correlation among the reference credits. The case where all defaults tend to happen around the same time is clearly different from a case where all defaults are independent. To examine the effect of the default correlation structure on basket CDS pricing, we assume the following default intensity for I reference credits:

$$\lambda_t^i = \left(a + bF_t + \delta 1_{\left\{t \geq \tau^{(1)}\right\}}\right) 1_{\{t < \tau^i\}},$$

which generalizes (5.2) by including a common factor F. This specification recognizes that default correlation can arise by two mechanisms: 1) correlation of the intensities through exposure to the same factor F; 2) default contagion through a jump in the intensity of size δ at the first-to-default. With empirically motivated choices for the coefficients, one can then ask what fraction of the total default correlation is attributed to each source.

To simulate the default times, we set an overall horizon of, say, $N = 100$ years, and choose an interval of one year for discretization. As an illustration of this procedure, assume that there are 10 firms in the industry. We take F_t to be the first common factor in Driessen (2005). This is a square-root diffusion with parameters $\kappa = 0.03$, $\theta = 0.005$, and $\sigma = 0.016$. In the base case we take $a = 0.004$ and $b = 5.707$, which correspond to Driessen's Baa intensity parameters. In addition, we assume that $\delta = 0.002$. This represents roughly a five percent jump over the long-run mean of the default intensity.

To study the effect of the common factor, we set $\delta = 0$ and change b while maintaining the long-run mean of the default intensity at the base case value (which means adjusting the value of a simultaneously). The results are presented in the upper panels of Figure 3. In the case where $b = 0$, we let $a = 0.032535$. One can see that there is zero default correlation between the firms and that the marginal distribution reflects a constant intensity equal to a. As b is increased, the survival (default) probability increases (decreases) slightly due to the effect of Jensen's inequality. The elevation of default correlation, however, is much more noticeable.

The lower panels of Figure 3 verifies the impact of default contagion on the outputs of the model. Specifically, we move away from the base case by increasing δ from 0 to 0.004. As indicated by the plot, this increases default probability for all maturities. Default correlation, on the other hand, initially increases with δ but decreases with δ at longer maturities. This is because at longer maturities the contagion effect has already resulted in increased intensities, while the factor sensitivity of the intensities has not changed.

We then take this example to the pricing of basket CDS. Specifically, we assume a portfolio with 30 reference credits with the above intensity, a constant interest rate of 5%, and a time to expiration of 5 years. Figure 4 demonstrates how the correlation structure can affect the valuation of basket CDS. First, with the overall mean of the intensity fixed, an increase in the dependence on the common factor increases (decreases) the probability that a large (small) number of defaults are observed prior to the expiration of the swap. This is due to the default clustering effect introduced by the common factor exposure of the default intensities.

A second way to inject more default correlation into the system is to increase the parameter δ, which describes the extent of default contagion. As shown by the lower panel, an increase in δ raises the basket CDS premium for all n. The first-to-default premium is unchanged, however, as δ has no effect on the distribution of the first-to-default time.

6 Conclusion

This paper introduces an intensity-based model of correlated defaults that offers more flexibility than models based on the conditionally independent construction (i.e. Cox processes). Specifically, we assume that the default intensities are driven by the history of defaults in addition to common exogenous factors. We show that the "total hazard construction" of Norros (1987) and Shaked and Shanthikumar (1987) can be used to generate default times with such generalized intensities.

This approach can accommodate a wide range of default dependency structures. In fact, the Schönbucher and Schubert (2001) hybrid approach that combines a single-obligor intensity-based model with a copula function at the portfolio level can be considered as a special case of the model presented in this paper. Moreover, the simulation procedure is amenable to default-dependent common factors responsible for such phenomenon as flight-to-quality. We illustrate the general framework with simple examples of defaultable bonds and credit default swaps of the regular and basket type.

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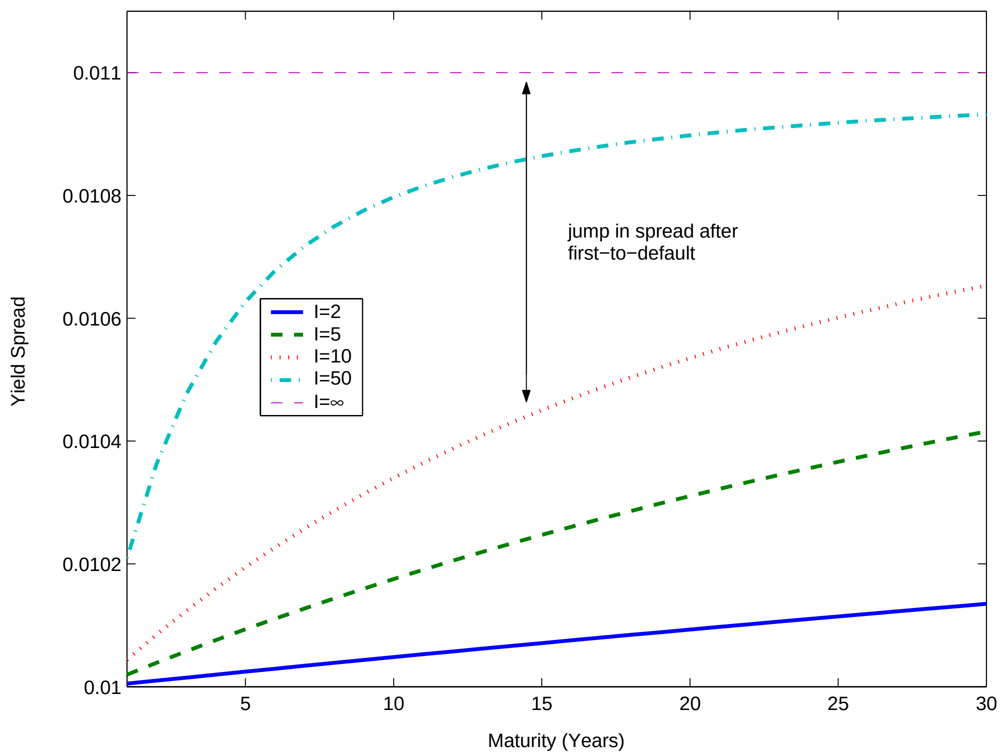

Figure 1: The term structure of yield spreads on defaultable zero-coupon bonds. It is assumed that $a_1 = 0.01$, $a_2 = 0.001$, and $I = 2$, 5, 10, 50, or ∞.

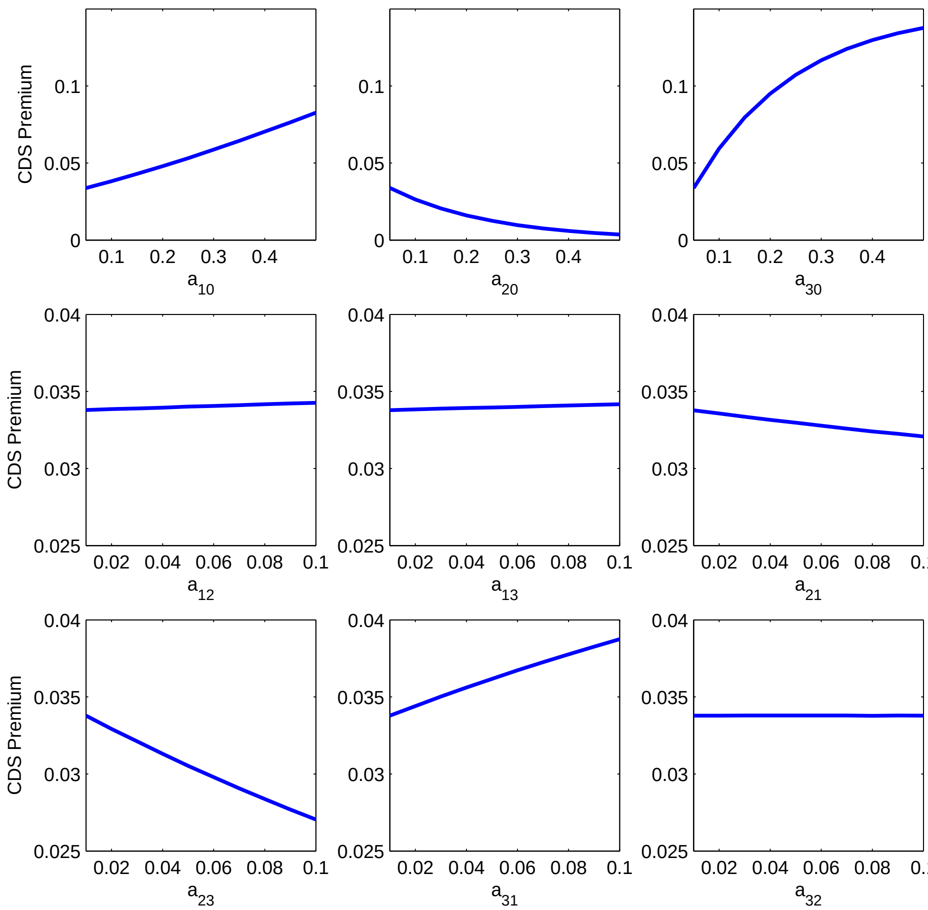

Figure 2: Changes in the CDS premium with respect to the default intensity parameters. All parameters are at their base case values except for the one being plotted.

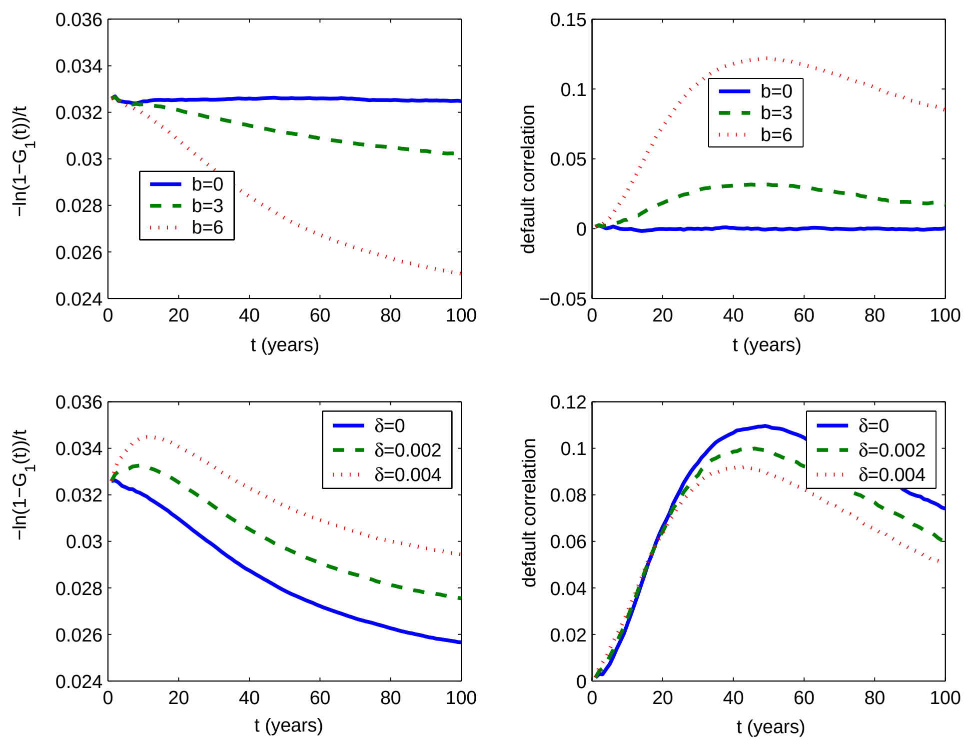

Figure 3: The effect of the common factor and default contagion on the marginal distribution and default correlation functions.

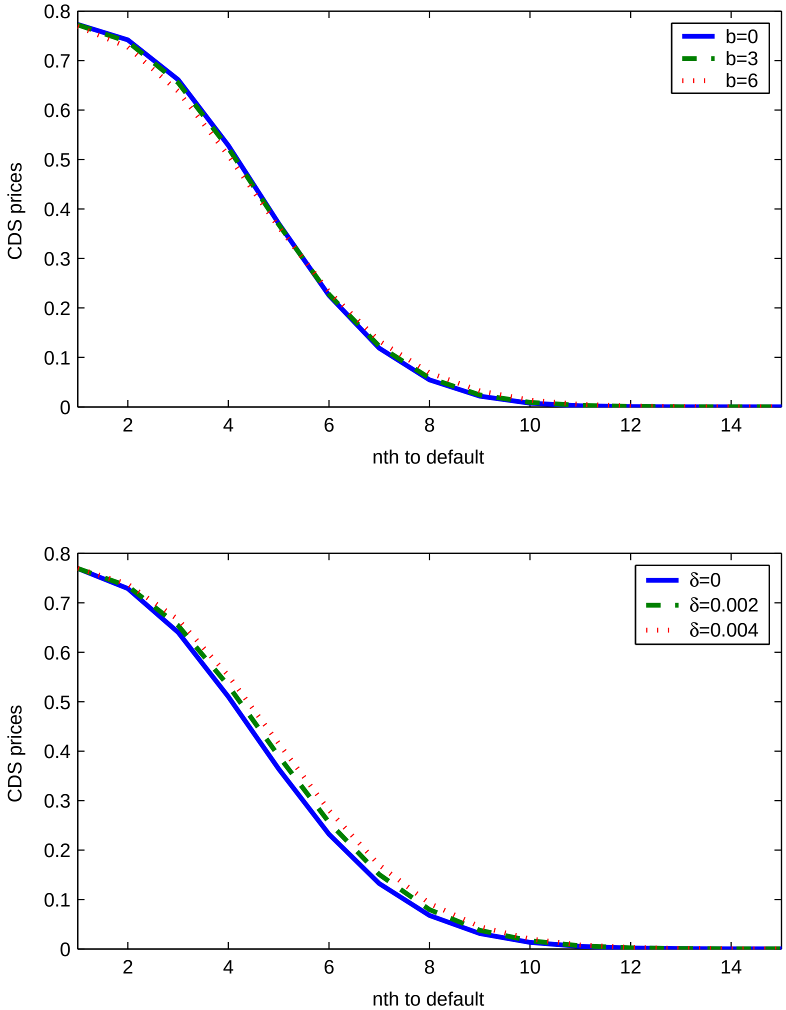

Figure 4: The effect of the common factor and default contagion on basket CDS premiums.