Qualifying Examination for the Ph.D. in Financial Engineering 2011

Courses covered: Credit Risk (MATH 361B), and Optimal Portfolio Theory (MATH 458B and 458C)

PART 1: Take-home Project

Wachter (2002) applied the martingale method to find the optimal consumption and portfolio in a market where asset returns are mean-reverting. You should recover Wachter's results by applying the stochastic maximum principle.

PART 2: Credit Risk

Conditions: closed book, closed notes, no computer allowed.
Time allowed: 3 hours.

Question 1:

A 2-dimensional differentiable copula is a function $C : [0,1]^2 \to [0,1]$ which satisfies the following properties:

1) For every x, y between 0 and one:

$$\begin{aligned} C(1,y) &= y \qquad C(x,1) = x \\ C(0,y) &= 0 \qquad C(x,0) = 0 \end{aligned}$$

2) For all $a, b \in [0,1]^2$ with $a \le b$:

$$\int_{y=a_2}^{b_2} \int_{x=a_1}^{b_1} \frac{\partial^2 C}{\partial x \partial y}|_{x,y} dx dy \ge 0$$

Let $\phi(t) = (-\ln t)^\theta$ for $\theta \ge 1$. Show that the so-called Archimean copula $C(x,y) = \phi^{-1}(\phi(x) + \phi(y))$ is a copula. Do you recognize another copula as the special case of the Archimedean copula (i.e., for a special value of the parameter θ)?

Question 2:

Financial contagion is an important phenomenon. Several authors suggest to model it with more sophisticated models than sheer multi-factor extensions of single-firm credit risk models. Write a short essay describing several contagion models. Typical questions to be answered are:

- is this a pricing or risk management model?
- how does the model define default and/or describe how to simulate default?
- does the model result in fat joint tails?
- what is the main contribution of the model?

Your essay should cover but does not need to be limited to the models described in the following papers[1]:

- Modeling Default Correlation in Bond Portfolios (Davis and Lo 2001)
- Correlated Default with Incomplete Information (Giesecke 2004)
- Cyclical Correlations, Credit Contagion, and Portfolio Losses (Giesecke and Weber, 2004)
- Counterparty Risk and the Pricing of Defaultable Securities (Jarrow and Yu 2001)
- Correlated Defaults in Intensity-Based Models (Yu 2007)

[1] The exact references are in your reading list.

- Default Risk and Diversification: Theory and Empirical Implications (Jarrow, Lando, and Yu 2005)

Be as mathematically rigorous as possible while staying concise.

Question 3:
In the Merton model of credit risk, the liabilities of a firm consist of (i) one zero-coupon bond with maturity T and principal equal to D, and (ii) equity. The value of the assets of the firm V follows a geometric Brownian motion with expected return μ and relative volatility σ. The interest rate is r. Suppose that equityholders can default only at T. Show that the value of equity is:

$$S(0) = V(0)N(d_1) - D\exp(-rT)N(d_2)$$

where:

$$\begin{aligned} d_1 &= \frac{\log(V(0)/D) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

Would the answer be different in this model if the shareholders were allowed to default before T? No mathematical justification is needed.

Question 4:
A swap between counterparty A and B pays a cash flow $dD(t)$ per unit of time to counterparty A, up till maturity T. in case neither A nor B defaults in that period. However the swap is subject to counterparty risk. Counterparty X (where X$\in$ {A,B}) defaults at time τ_X, which is the first jump time of a Cox process with intensity λ_X. Call $S(t)$ the market value of the swap for counterparty A. Suppose that counterparty X defaults at time τ_X . There are two cases:

- if the swap has negative value for counterparty X , then X pays at time τ_X a lump sum equal to $S(t)\varphi^X$, where the recovery rate φ^X is between zero and one
- if the swap has positive value for counterparty X , then the contract is cancelled, and X receives the full value of the swap.

a) Let $X(t)$ be the cash flow accumulated to counterparty A up till time t. Show that:

$$\begin{aligned} dX(t) = 1\{\tau > t\}dD(t) + \delta(t - \tau_A)S(t)[1\{S(t) < 0\}\varphi^A + 1\{S(t) \geq 0\}]dt \\ +\delta(t - \tau_B)S(t)[1\{S(t) > 0\}\varphi^B + 1\{S(t) \leq 0\}] \end{aligned}$$

where $\tau = \min(\tau^A, \tau^B)$ and $\delta(t)$ is the Dirac delta function.

b) Denote the credit spread of counterparty X by $s^X = (1 - \varphi^X)\lambda^X$, the risk-free interest rate by r, and the risky rate by R, where:

$$R = r + s^A 1[S < 0] + s^B 1[S > 0]$$

Show that the price of the swap satisfies the following equation:

$$dS(t) = (R(t) - \lambda_A(t) - \lambda_B(t))S(t)dt - 1\{t < \tau\}dD(t) + dM(t) \tag{1}$$

for some martingale $M(t)$.

c) Use (b) to show that

$$S(t) = E_t^Q[\int_t^T \exp(-\int_0^s R(u)du)dD(s)]1\{\tau > t\}$$

In other words, instead of discounting the "risky" cash flow with the risk-free rate, one can discount the "riskless" cash flow with the risky rate.

PART 3: Optimal Portfolio Theory
Conditions: the following books are allowed:

- Korn, Optimal Portfolios (1997)
- Ingersoll, Theory of Financial Decision Making (1987)
- Intriligator, Mathematical Optimization and Economic Theory (1971)
- Oksendal, An Introduction to Malliavin Calculus, with Applications to Economics (1997)

Handwritten notes are allowed, as well as a calculator. All other documentation is forbidden.

Time allowed: 3 hours.

Question 1
Let U be a differentiable concave increasing function: $\mathbb{R} \to \mathbb{R}$. The conjugate function V is defined, for $y \geq 0$ by:

$$V(y) = \sup_x U(x) - yx$$

a) Can you define U as the result of an operation on V?
b) Let $U(x) = \ln x$ for $x > 0$. Find $V(y)$.
c) Is V convex or concave? No proof is needed.
d) By Lagrange duality, under certain regularity conditions, we have:

$$\max_{x \leq x'} U(x) = \min_{y \geq 0} \max_z U(z) + y(x' - z)$$

This relation seems to contradict your result in subquestion (a). Show that it is not the case (the proof is very short).

Question 2:
Let g be a deterministic function. Calculate the Malliavin derivatives at time t of the following random variables:
a) $\exp(\int_0^T g(s)dW(s))$
b) $\int_0^T \exp(\int_0^t g(s)dW(s))dt$
c) $\int_0^T \exp(\int_0^t g(s)dW(s))dW(t)$

Question 3:
Consider a static market with 2 risky assets and a riskless asset. The latter has expected rate of return $R = 2\%$. The 2 risky assets have a variance-covariance matrix Σ and expected return vector μ given by:

$$\Sigma = \begin{bmatrix} 0.04 & 0.03 \\ 0.03 & 0.09 \end{bmatrix}$$

$$\mu = \begin{bmatrix} 0.05 \\ 0.10 \end{bmatrix}$$

Calculate the tangency portfolio w_t. This represents the proportion of the market capitalization invested in risky assets 1 and 2. Provide then a geometric characterization of the minimum-variance set in the expected return-standard deviation plane.

Question 4:
Let $X(t)$ be the value of a portfolio composed of one stock and one bond, with $X(0) = x$. There is intermediate consumption $c(t)$, but otherwise the portfolio is self-financing. The optimal portfolio problem

consists in choosing an optimal consumption stream $c^*(t)$ and an optimal fraction of wealth $\pi^*(t)$ invested in the stock. The goal is to maximize the expected value of the integral of the utility of consumption plus the utility of bequest. The optimal value of the latter is called the value function at time 0, and is written $J(x,0)$. More generally, the value function at time s given that wealth at time s is equal to $X(s)$ is denoted by $J(X(s),s)$, where:

$$J(X(s),s) = \max_{w(.),c(.)} E[\int_s^T \ln(c(t))dt + \ln X(T)|X(s)]$$

The stock is geometric Brownian motion with drift b and volatility σ. Assume for simplicity that the interest rate r is zero.

a) Use the results in Korn p. 72 (and nothing else) to show that:

$$J(x,0) = (T+1)\ln\frac{x}{T+1} + 2T + \frac{b}{2\sigma^2}(\frac{T^2}{2} + T)$$

b) Show that

$$J(X(s),s) = (T+1-s)\ln\frac{X(s)}{T+1-s} + 2(T-s) + \frac{b}{2\sigma^2}(\frac{(T-s)^2}{2} + (T-s))$$

c) Verify that the value function found in (b) satisfies the Hamilton-Jacobi-Bellman equation.

Hint: use the fact that $\frac{d}{dt}[(-1+\ln(T+1-t))(t-T-1)] = \ln(T+1-t)$.

Question 5:

Let $\delta(w(t))$ represent the Skorohod integral of a stochastic process w between 0 and T. Let the stock price be:

$$X(t) = x\exp(W(t) - t/2)$$

It can be proved that:

$$E[f(\int_0^T X(t)dt)\delta(w)] = E[f'(\int_0^T X(t)dt)\int_{t=0}^T X(t)\int_{s=0}^t w(s)dsdt]$$

An Asian option has a payoff equal to the value of a function f evaluated at the time-integral of a function of the stock price. The delta of an Asian option can be calculated using the following step:

$$\Delta \equiv \frac{\partial}{\partial x}E[f(\int_0^T X(t)dt)] = \frac{1}{x}E[f'(\int_0^T X(t)dt)\int_0^T X(t)dt]$$

a) Prove that the Greek of an Asian option satisfies:

$$\Delta = \frac{1}{x}E[f(\int_0^T X(t)dt)\delta(\frac{2X(t)}{\int_0^T X(t)dt})]$$

b) Prove that it also satisfies:

$$\Delta = \frac{1}{x}E[f(\int_0^T X(t)dt)\left(\frac{2\int_0^T X(s)dW(s)}{\int_0^T X(s)ds} + 1\right)] \tag{2}$$

c) Suppose you use Monte Carlo simulation to estimate Δ. What is the computational advantage of the representation (2)?