A Sharp Approximation for ATM-Forward Option Prices and Implied Volatilites*

Dan Stefanica[†] Radoš Radoičić[‡]

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Abstract

In this paper, we provide an approximation formula for at the money forward options based on a Pólya approximation of the cumulative density function of the standard normal distribution, and prove that the relative error of this approximation is uniformly bounded for options with arbitrarily large (or small) maturities and implied volatilities.

This approximation is viable in practice: for options with implied volatility less than 95% and maturity less than three years, which includes the large majority of traded options, the values given by the approximation formula fall within the tightest typical implied vol bid–ask spreads.

The relative errors of the corresponding approximate option values are also uniformly bounded for all maturities and implied volatilities.

The error bounds established here are the first results in the literature holding for all integrated volatilities, and are vastly superior to those of two other approximation formulas analyzed in this paper, including the Brenner–Subrahmanyam formula.

1 Introduction

The Black-Scholes theory of option pricing initiated by Black and Scholes [3] and Merton [18] is one of the most influential theories in finance. Its ultimate result is the celebrated Black-Scholes formula which is used daily by traders around the world. According to Lee and Li [14], it is the most frequently used equation by human beings, ahead of both Newton's laws of motion in classical mechanics and Schrödinger's equation in quantum mechanics. Besides being used to price options on stocks, it is also used to value options on futures, options on foreign currencies, and interest rates options.

In practice, the Black-Scholes formula is often used in the opposite direction; see, for example, Gatheral [9]: we observe the market option price C_m and solve for a volatility value, called the implied volatility $\sigma_{imp,BS}$, that makes the Black–Scholes value C_{BS} of the option equal to its market price C_m. Originally suggested by Latané and Rendleman [16], the use of implied volatility is so pervasive in the financial markets nowadays that it is common practice for the implied volatility

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[†]Baruch College, City University of New York, email: dan.stefanica@baruch.cuny.edu; corresponding author

[‡]Baruch College, City University of New York, email: rados.radoicic@baruch.cuny.edu

rather than the option price to be quoted. Besides being a succinct way to talk about option prices, it is constantly compared to that of other market participants, and it serves as a forward-looking measure of expectation about future market movements.

In this paper, we focus solely on European at-the-money (ATM) forward options, that is, on options with strike equal to the forward price at maturity of the underlying asset. The Black–Scholes values of ATM-forward call and put options are

$$C_{BS} \;=\; P_{BS} \;=\; 2S_0 e^{-qT}\left(N\left(\frac{\sigma\sqrt{T}}{2}\right)-\frac{1}{2}\right). \tag{1}$$

Brenner and Subrahmanyam [4, 5] used a Taylor approximation of the term $N\left(\frac{\sigma\sqrt{T}}{2}\right)$ from (1) to obtain the following approximation formula for ATM-forward options

$$C_{approx,1} = P_{approx,1} \;=\; \sigma S_0 e^{-qT}\sqrt{\frac{T}{2\pi}} \tag{2}$$

and the following corresponding approximation formula for ATM-forward options implied volatilities:

$$\sigma_{imp,approx,1} \;=\; \sqrt{\frac{2\pi}{T}}\frac{C_m}{S_0 e^{-qT}}.$$

As the performance of the approximation (2) quickly deteriorates for options which are not short dated or with underlying assets with larger volatilities, numerous approximation formulas have been derived by approximating the cumulative density function $N(\cdot)$ from the Black–Scholes formula by higher order Taylor approximations [6, 15, 17], by logistic functions [11], or by rational functions [13]. Approximation formulas for implied volatilities for generalized Black–Scholes models have also been derived; see [10]. However, all of these formulas reduce to (2) for ATM-forward options. Furthermore, no theoretical bounds on the relative errors of these formulas have been derived, as their accuracy is checked only by using limited (real and hypothetical) data. It is worth noting that only in extreme regimes (long/short expiry and fixed strike, or large strike and fixed expiry) the asymptotics of implied volatility is well understood and determined to arbitrarily high order of precision; see [2, 8].

In this paper, two approximations for the term $N\left(\frac{\sigma\sqrt{T}}{2}\right)$ from (1) are used to derive ATM-forward options approximate formulas that are simply and explicitly invertible, thus generating approximate formulas for implied volatilities of ATM-forward options; see section 2. From Pólya's approximation (10–11), see [19, 20, 21], we obtain the following approximation formulas:

$$\begin{aligned} C_{approx,2} \;=\; P_{approx,2} \;&=\; S_0 e^{-qT}\sqrt{1-e^{-\frac{\sigma^2 T}{2\pi}}}; \\ \sigma_{imp,approx,2} \;&=\; \sqrt{-\frac{2\pi}{T}\ln\left[1-\left(\frac{C_m}{S_0 e^{-qT}}\right)^2\right]}. \end{aligned}$$

From the Aludaat and Alodat approximation (14–15), see [1], we obtain the approximation formulas

$$\begin{aligned} C_{approx,3} \;=\; P_{approx,3} \;&=\; S_0 e^{-qT}\sqrt{1-e^{-\frac{\sigma^2 T}{4}\sqrt{\frac{\pi}{8}}}}; \\ \sigma_{imp,approx,3} \;&=\; \sqrt{-\frac{4}{T}\sqrt{\frac{8}{\pi}}\ln\left[1-\left(\frac{C_m}{S_0 e^{-qT}}\right)^2\right]}. \end{aligned}$$

While the approximation $\sigma_{imp,approx,1}$ deteriorates quickly for long dated options and large implied volatilities and the approximation $\sigma_{imp,approx,3}$ is not precise for short dated options and low implied volatilities, the implied volatility approximation $\sigma_{imp,approx,2}$ is accurate for any integrated volatility, including for arbitrarily large or small option maturities and volatilities. More precisely, in section 3, we prove that the approximate implied volatility $\sigma_{imp,approx,2}$ is within 12% of the implied volatility of the underlying for options with arbitrarily large maturities and implied volatilities, i.e.,

$$0 \ < \ \frac{\sigma_{imp,BS} - \sigma_{imp,approx,2}}{\sigma_{imp,BS}} \ < \ 1 - \frac{\sqrt{\pi}}{2} \approx 0.1138, \tag{3}$$

for any $\sigma_{imp,BS}$ and for any maturity T.

The relative error bound (3) holding for any maturities and volatility levels is the first such bound derived in literature.

From a practical viewpoint, the bound (3) is, in fact, much tighter. Options traders use bid–ask spreads for the implied volatility of liquid options, including near at the money options, that are roughly one basis point for each vol point; for example, for 20% implied volatility, the bid–ask spread could be 20 basis points, e.g., 19.90%–20.10%. Thus, the approximation $\sigma_{imp,approx,2}$ to the implied volatility $\sigma_{imp,BS}$ falls within the bid–ask spread around $\sigma_{imp,BS}$ for a corresponding relative error of 0.5%, In section 3, we show that, for options with implied volatility less than 95% and maturity less than three years, which covers the vast majority of traded options, the relative error of approximating $\sigma_{imp,BS}$ by $\sigma_{imp,approx,2}$ is less than 0.5%, i.e.,

$$0 \ < \ \frac{\sigma_{imp,BS} - \sigma_{imp,approx,2}}{\sigma_{imp,BS}} \ < \ 0.005, \quad \forall \ \sigma_{imp,BS}\sqrt{T} \leq 1.65,$$

for any $\sigma_{imp,BS}$ and for any maturity T. This makes the approximation $\sigma_{imp,approx,2}$ viable in practice. Also, the values given by $\sigma_{imp,approx,2}$ can be used as initial guesses for iterative computational methods including Newton's method and rational functions approximations; see [12].

Furthermore, in section 4, we prove that the option value approximation $C_{approx,2}$ is a remarkably sharp approximation (within 2% accurate) of the Black–Scholes values for ATM-forward options with arbitrarily large maturities and implied volatilities, including for long dated options on high volatility underlying assets, i.e.,

$$0 \ < \ \frac{C_{approx,2} - C_{BS}}{C_{BS}} \ = \ \frac{P_{approx,2} - P_{BS}}{P_{BS}} \ < \ 0.02. \tag{4}$$

In contrast to the estimate on the relative error of the Brenner–Subrahmanyam approximation formula (2) provided in [22], the bound (4) is universally applicable over the whole range of volatility parameters σ and option maturities T. As was the case with (3), the relative error bound (4) for ATM-forward option prices is the first such bound established in the literature.

2 Several approximation formulas for ATM-forward options and ATM implied volatilities

The Black-Scholes values of plain vanilla European call and put options with maturity T and strike K on an asset with spot price S_0 following a lognormal distribution with volatility σ and paying dividends continuously at rate q, assuming constant risk–free interest rates equal to r, are, respectively,

$$C_{BS} \ = \ S_0 e^{-qT} N(d_1) \ - \ K e^{-rT} N(d_2); \tag{5}$$

$$P_{BS} \ = \ K e^{-rT} N(-d_2) \ - \ S_0 e^{-qT} N(-d_1), \tag{6}$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$ is the cumulative distribution function of the standard normal random variable and

$$d_1 \; = \; \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \; = \; \frac{\ln\left(\frac{S_0 e^{(r-q)T}}{K}\right)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}; \quad d_2 \; = \; d_1 - \sigma\sqrt{T}. \tag{7}$$

At-the-money (ATM) forward options are options with strike equal to the forward price at maturity of the underlying asset, i.e., with $K = S_0 e^{(r-q)T}$. Thus, for ATM-forward options, it follows from (7) that $d_1 = \frac{\sigma\sqrt{T}}{2}$ and $d_2 = -\frac{\sigma\sqrt{T}}{2}$. Moreover, if $K = S_0 e^{(r-q)T}$, then $Ke^{-rT} = S_0 e^{-qT}$, and, from (5–6), and using the fact that $N\left(-\frac{\sigma\sqrt{T}}{2}\right) = 1 - N\left(\frac{\sigma\sqrt{T}}{2}\right)$, we obtain that

$$C_{BS} \; = \; P_{BS} \; = \; S_0 e^{-qT} \left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - N\left(-\frac{\sigma\sqrt{T}}{2}\right) \right) \; = \; S_0 e^{-qT} \left(2N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1 \right).$$

In other words, the Black-Scholes values of European ATM-forward options are

$$C_{BS} \; = \; P_{BS} \; = \; 2S_0 e^{-qT} \left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - \frac{1}{2} \right). \tag{8}$$

Using the linear Taylor approximation $N(x) \approx \frac{1}{2} + \frac{x}{\sqrt{2\pi}}$, as $x \to 0$, it follows that $N\left(\frac{\sigma\sqrt{T}}{2}\right) \approx \frac{1}{2} + \frac{\sigma}{2}\sqrt{\frac{T}{2\pi}}$ and, from (8), we obtain the following approximation formula for the values of ATM-forward call and put options:

$$C_{approx,1} = P_{approx,1} \; = \; \sigma S_0 e^{-qT} \sqrt{\frac{T}{2\pi}}. \tag{9}$$

For a non–dividend–paying underlying asset, the approximation formula (9) corresponds to the classical ATM approximation of Brenner and Subrahmanyam [4, 5].

A more accurate approximation for the cumulative density function $N(x)$ can be found in Pólya [19], where it is established that

$$0 < A(x) - N(x) < 0.003, \; \forall \; x > 0; \quad 0 < N(x) - A(x) < 0.003, \; \forall \; x < 0, \tag{10}$$

with the function $A(x)$ given by

$$A(x) \; = \; \frac{1}{2} + \frac{\operatorname{sgn}(x)}{2}\sqrt{1 - e^{-\frac{2x^2}{\pi}}} \; = \; \begin{cases} \frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{-\frac{2x^2}{\pi}}}, & \text{if } \; x \; \geq \; 0; \\ \frac{1}{2} - \frac{1}{2}\sqrt{1 - e^{-\frac{2x^2}{\pi}}}, & \text{if } \; x \; < \; 0. \end{cases} \tag{11}$$

Using the function $A(x)$ instead of $N(x)$ in (8), the following approximation formula for the value of ATM-forward call and put options is obtained:

$$\begin{aligned} C_{approx,2} \; = \; P_{approx,2} \; &= \; 2S_0 e^{-qT} \left(A\left(\frac{\sigma\sqrt{T}}{2}\right) - \frac{1}{2} \right) \\ &= \; S_0 e^{-qT} \sqrt{1 - e^{-\frac{\sigma^2 T}{2\pi}}}. \end{aligned} \tag{12}$$

In other words,

$$C_{approx,2} \;=\; P_{approx,2} \;=\; S_0 e^{-qT}\sqrt{1-e^{-\frac{\sigma^2 T}{2\pi}}}. \tag{13}$$

An approximation for $N(x)$ which is even sharper than (10–11) was introduced by Aludaat and Alodat [1] as follows: Let

$$B(x) \;=\; \frac{1}{2}+\frac{\operatorname{sgn}(x)}{2}\sqrt{1-e^{-\sqrt{\frac{\pi}{8}}x^2}}, \quad \forall\, x\in\mathbf{R}. \tag{14}$$

Then,

$$|N(x)-B(x)| \;<\; 0.002, \quad \forall\, x\in\mathbf{R}. \tag{15}$$

Using the function $B(x)$ given by (14) instead of $N(x)$ in (8), the following approximation formula for the value of ATM-forward call and put options is obtained:

$$C_{approx,3} \;=\; P_{approx,3} \;=\; 2S_0 e^{-qT}\left(B\left(\frac{\sigma\sqrt{T}}{2}\right)-\frac{1}{2}\right) \;=\; S_0 e^{-qT}\sqrt{1-e^{-\frac{\sigma^2 T}{4}\sqrt{\frac{\pi}{8}}}}.$$

In other words,

$$C_{approx,3} \;=\; P_{approx,3} \;=\; S_0 e^{-qT}\sqrt{1-e^{-\frac{\sigma^2 T}{4}\sqrt{\frac{\pi}{8}}}}. \tag{16}$$

The approximation formulas (9), (13), and (16) for ATM-forward options can be used to derive approximation formulas for implied volatilities.

Note that the Black-Scholes values and the approximate values above are the same for ATM-forward call options and for ATM-forward put options. Throughout the rest of this paper, we only refer to implied volatilities corresponding to ATM-forward call options; similar results hold for implied volatilities corresponding to ATM-forward put options.

Recall that, if C_m is the market price of an ATM-forward call option, the implied volatility $\sigma_{imp,BS}$ of the ATM-forward call option is the value of the volatility parameter σ which makes the Black–Scholes value C_{BS} of the option equal to its market price C_m, and is obtained by solving $C_{BS}=C_m$ for σ as follows: From (8), we obtain that

$$\begin{aligned} C_{BS}=C_m \quad &\Longleftrightarrow \quad 2S_0 e^{-qT}\left(N\left(\frac{\sigma\sqrt{T}}{2}\right)-\frac{1}{2}\right) \;=\; C_m \\ &\Longleftrightarrow \quad N\left(\frac{\sigma\sqrt{T}}{2}\right) \;=\; \frac{C_m}{2S_0 e^{-qT}}+\frac{1}{2}. \end{aligned} \tag{17}$$

By solving (17) for σ, we find that

$$\sigma_{imp,BS} \;=\; \frac{2}{\sqrt{T}}\; N^{-1}\left(\frac{C_m}{2S_0 e^{-qT}}+\frac{1}{2}\right), \tag{18}$$

where $N^{-1}(x)$ is the inverse of the cumulative normal distribution function.[1]

[1] Note that, for no–arbitrage, the market value of the option must be less than the present value of one unit of the asset, i.e., $C_m < S_0 e^{-qT}$. Thus,

$$\frac{C_m}{2S_0 e^{-qT}}+\frac{1}{2} \;<\; \frac{1}{2}+\frac{1}{2} \;<\; 1, \tag{19}$$

and therefore the term $N^{-1}\left(\frac{C_m}{2S_0 e^{-qT}}+\frac{1}{2}\right)$ from (18) is well defined.

The approximate implied volatility $\sigma_{imp,approx,1}$ corresponding to the Brenner–Subrahmanyam approximation (9) is obtained by solving $C_{approx,1} = C_m$ for σ as follows:

$$C_{approx,1} = C_m \iff \sigma S_0 e^{-qT}\sqrt{\frac{T}{2\pi}} = C_m \iff \sigma = \sqrt{\frac{2\pi}{T}}\frac{C_m}{S_0 e^{-qT}}.$$

Thus,

$$\sigma_{imp,approx,1} = \sqrt{\frac{2\pi}{T}}\frac{C_m}{S_0 e^{-qT}}. \tag{20}$$

The approximate implied volatility $\sigma_{imp,approx,2}$ corresponding to the approximate formula (13) for ATM-forward options is obtained by solving $C_{approx,2} = C_m$ for σ as follows:

$$\begin{aligned} C_{approx,2} = C_m \iff S_0 e^{-qT}\sqrt{1 - e^{-\frac{\sigma^2 T}{2\pi}}} &= C_m \\ \iff e^{-\frac{\sigma^2 T}{2\pi}} &= 1 - \left(\frac{C_m}{S_0 e^{-qT}}\right)^2. \end{aligned} \tag{21}$$

By solving (21) for σ, it follows that

$$\sigma_{imp,approx,2} = \sqrt{-\frac{2\pi}{T}\ln\left[1 - \left(\frac{C_m}{S_0 e^{-qT}}\right)^2\right]}. \tag{22}$$

Note that, for no–arbitrage, $\frac{C_m}{2S_0 e^{-qT}} + \frac{1}{2} < 1$, see (19), and therefore the log term from (22) is well defined.

The approximate implied volatility $\sigma_{imp,approx,3}$ corresponding to the approximate formula (16) for ATM-forward options is obtained by solving $C_{approx,3} = C_m$ for σ as follows:

$$\begin{aligned} C_{approx,3} = C_m \iff S_0 e^{-qT}\sqrt{1 - e^{-\frac{\sigma^2 T}{4}\sqrt{\frac{\pi}{8}}}} &= C_m \\ \iff e^{-\frac{\sigma^2 T}{4}\sqrt{\frac{\pi}{8}}} &= 1 - \left(\frac{C_m}{S_0 e^{-qT}}\right)^2. \end{aligned} \tag{23}$$

By solving (23) for σ, it follows that

$$\sigma_{imp,approx,3} = \sqrt{-\frac{4}{T}\sqrt{\frac{8}{\pi}}\ln\left[1 - \left(\frac{C_m}{S_0 e^{-qT}}\right)^2\right]}. \tag{24}$$

To analyze the accuracy of the implied volatility approximation formulas (20), (22), and (24) we plot the approximation errors $\sigma_{imp,BS}\sqrt{T} - \sigma_{imp,approx,1}\sqrt{T}$, $\sigma_{imp,BS}\sqrt{T} - \sigma_{imp,approx,2}\sqrt{T}$, and $\sigma_{imp,BS}\sqrt{T} - \sigma_{imp,approx,3}\sqrt{T}$, as functions of $\sigma_{imp,BS}\sqrt{T}$, for all the values of $\sigma_{imp,BS}$ and T where computations are possible;[2] see Figure 1.

We note that:

- The implied volatility approximation $\sigma_{imp,approx,1}$ corresponding to the Brenner–Subrahmanyam approximation deteriorates quickly for moderately high values of $\sigma_{imp,BS}\sqrt{T}$, i.e., for longer dated options and larger implied volatilities.

[2] The numerical value of the term $N\left(\frac{\sigma\sqrt{T}}{2}\right)$ is within epsilon machine of 1 when $\sigma\sqrt{T} \approx 16$.

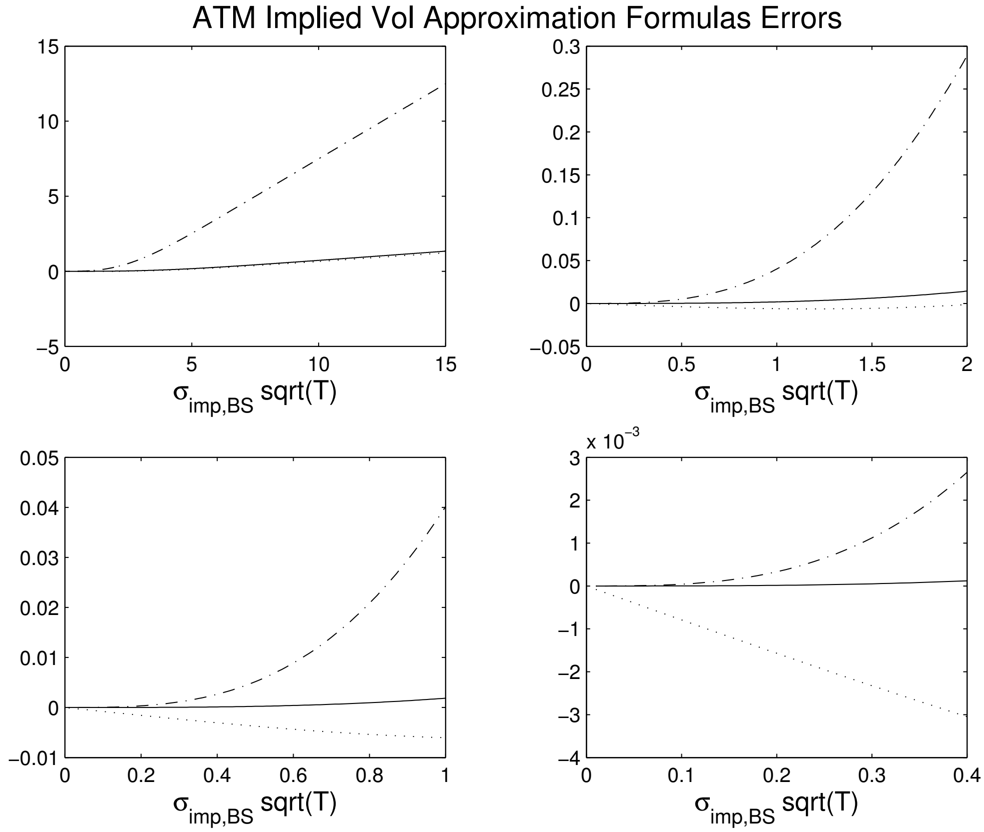

Figure 1: Approximation errors: • dash-dotted line: $\sigma_{imp,BS}\sqrt{T} - \sigma_{imp,approx,1}\sqrt{T}$; • solid line: $\sigma_{imp,BS}\sqrt{T} - \sigma_{imp,approx,2}\sqrt{T}$; • dotted line: $\sigma_{imp,BS}\sqrt{T} - \sigma_{imp,approx,3}\sqrt{T}$.

• The approximation $\sigma_{imp,approx,3}$ based on the Aludaat–Alodat approximation (14) of $N(x)$ is not precise for small values of $\sigma_{imp,BS}\sqrt{T}$, i.e., for short dated options and low implied volatilities.

• The implied volatility approximation $\sigma_{imp,approx,2}$ based on the Pólya approximation of the cumulative distribution function $N(x)$ appears to approximate the Black–Scholes implied volatility $\sigma_{imp,BS}$ very well for all regimes of small and large maturities as well as for all values of implied volatility.

We further analyze the approximation properties of $\sigma_{imp,approx,2}$ in section 3, and we prove that this approximation is indeed sharp for all integrated volatilities.

3 The sharpness of an implied volatility approximation for ATM-forward options

In this section, we show that the approximate implied volatility $\sigma_{imp,approx,2}$ for ATM-forward options is accurate for any integrated volatility, including for arbitrarily large or small option maturities and volatilities; see Theorem 3.1.

Theorem 3.1. *The approximate implied volatility $\sigma_{imp,approx,2}$ is within 12% of the Black–Scholes* *implied volatility for options with arbitrarily large maturities and implied volatilities, i.e.,*

$$0 < \frac{\sigma_{imp,BS} - \sigma_{imp,approx,2}}{\sigma_{imp,BS}} < 1 - \frac{\sqrt{\pi}}{2} \approx 0.1138, \tag{25}$$

for any $\sigma_{imp,BS}$ and for any maturity T.

Moreover, the upper bound $1 - \frac{\sqrt{\pi}}{2}$ from the inequality (25) is sharp and cannot be improved.

We provide further insights on the results of Theorem 3.1, followed by its proof.

The relative error $\frac{\sigma_{imp,BS} - \sigma_{imp,approx,2}}{\sigma_{imp,BS}}$ as a function of $\sigma_{imp,BS}\sqrt{T}$ is plotted in Figure 2.

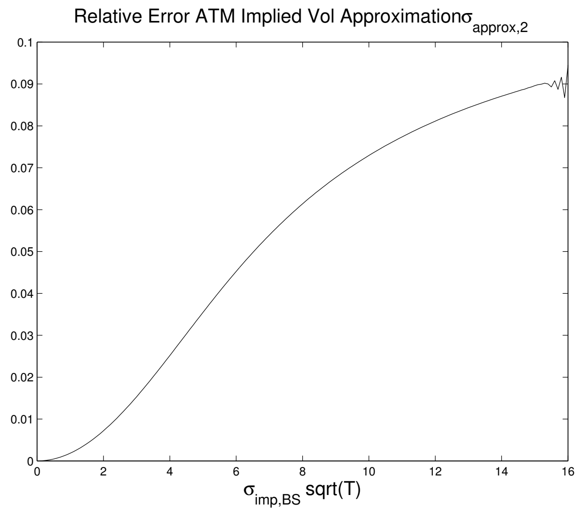

Figure 2: Relative error $\frac{\sigma_{BS} - \sigma_{approx,2}}{\sigma_{BS}}$ as a function of $\sigma_{BS}\sqrt{T}$

Note that, for options with implied volatility less than 95% and maturity less than three years, which corresponds to $\sigma_{imp,BS}\sqrt{T} \leq 1.65$ and covers the vast majority of traded options, the relative error of approximating $\sigma_{imp,BS}$ by $\sigma_{imp,approx,2}$ is less than 0.5%, i.e.,

$$0 < \frac{\sigma_{imp,BS} - \sigma_{imp,approx,2}}{\sigma_{imp,BS}} < 0.005, \quad \forall \; \sigma_{imp,BS}\sqrt{T} \leq 1.65; \tag{26}$$

see also Figure 3.

This is of practical importance, since bid–ask spreads for implied volatilities of liquid options are roughly one basis point for each vol point, which corresponds to a one percent bid–ask spread relative to the mid implied volatilivty. Thus, a 0.5% relative approximation of the implied volatility, which corresponds to the bound and range from (26), falls within the bid–ask spread for at–the–money options.

Also, note from Figure 2 that, when $\sigma_{imp,BS}\sqrt{T} \approx 16$, the numerical computations are no longer accurate, since computing $N(8)$ is already within epsilon machine of 1 and numerical accuracy is lost.

However, it turns out that, although impossible to detect computationally, it is possible to prove that the upper bound $1 - \frac{\sqrt{\pi}}{2} \approx 0.1138$ from (25) of Theorem 3.1 is, in fact, sharp and cannot be improved, as shown in the proof below.

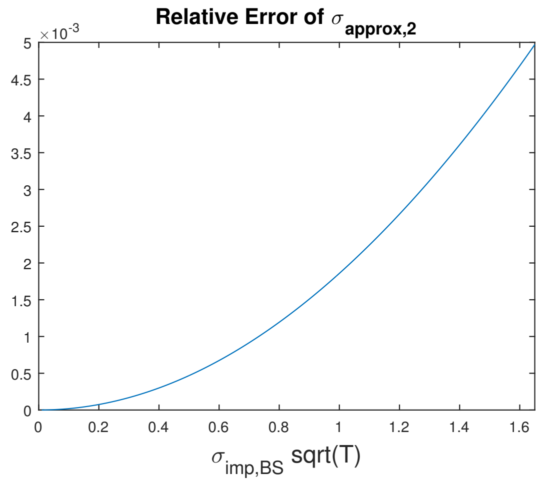

Figure 3: Relative error $\frac{\sigma_{BS}-\sigma_{approx,2}}{\sigma_{BS}}$ as a function of $\sigma_{BS}\sqrt{T}$

The bound (25) for the relative error of the approximate implied volatility $\sigma_{imp,approx,2}$ holds for all maturities and all implied volatility regimes. It is the first bound with these properties to occur in the literature.

Proof of Theorem 3.1. By using the formula (12), for $C_{approx,2}$, we find that

$$\begin{aligned} C_{approx,2} = C_m \quad \Longleftrightarrow \quad 2S_0e^{-qT}\left(A\left(\frac{\sigma\sqrt{T}}{2}\right)-\frac{1}{2}\right) &= C_m \\ \Longleftrightarrow \quad A\left(\frac{\sigma\sqrt{T}}{2}\right) &= \frac{C_m}{2S_0e^{-qT}}+\frac{1}{2}. \end{aligned} \tag{27}$$

By solving (27) for σ, we obtain the following formula for $\sigma_{imp,approx,2}$:

$$\sigma_{imp,approx,2} = \frac{2}{\sqrt{T}}\,A^{-1}\left(\frac{C_m}{2S_0e^{-qT}}+\frac{1}{2}\right), \tag{28}$$

where the inverse function $A^{-1}(x)$ is given by

$$A^{-1}(x) = \sqrt{-\frac{\pi}{2}\ln\left(4x(1-x)\right)}, \quad \forall\, x \in \left[\frac{1}{2},1\right]. \tag{29}$$

Using (18) for $\sigma_{imp,BS}$ and (28) for $\sigma_{imp,approx,2}$, it follows that

$$\frac{\sigma_{imp,BS}-\sigma_{imp,approx,2}}{\sigma_{imp,BS}} = \frac{N^{-1}\left(\frac{C_m}{2S_0e^{-qT}}+\frac{1}{2}\right)-A^{-1}\left(\frac{C_m}{2S_0e^{-qT}}+\frac{1}{2}\right)}{N^{-1}\left(\frac{C_m}{2S_0e^{-qT}}+\frac{1}{2}\right)}. \tag{30}$$

Let

$$z = N^{-1}\left(\frac{C_m}{2S_0e^{-qT}}+\frac{1}{2}\right). \tag{31}$$

Note that $z > 0$. From (31), it follows that

$$\frac{C_m}{2S_0e^{-qT}}+\frac{1}{2} = N(z). \tag{32}$$

Using (31) and (32), we obtain from (30) that

$$\frac{\sigma_{imp,BS} - \sigma_{imp,approx,2}}{\sigma_{imp,BS}} \;=\; \frac{z - A^{-1}(N(z))}{z}. \tag{33}$$

From (33), we find that the double inequality (25) can be written as

$$0 \;<\; \frac{z - A^{-1}(N(z))}{z} \;<\; 1 - \frac{\sqrt{\pi}}{2}, \quad \forall\ z > 0, \tag{34}$$

and therefore, in order to prove (25), we need to show that

$$z \;>\; A^{-1}(N(z)), \quad \forall\ z > 0. \tag{35}$$

$$A^{-1}(N(z)) \;>\; \frac{\sqrt{\pi}}{2} z, \quad \forall\ z > 0. \tag{36}$$

Recall from (10) that $A(z) > N(z)$ for any $z > 0$. Since $A(z)$ is an increasing function, it follows that A^{-1} is an increasing function and therefore $A^{-1}(A(z)) > A^{-1}(N(z))$ for all $z > 0$. Thus,

$$z \;=\; A^{-1}(A(z)) \;>\; A^{-1}(N(z)), \quad \forall\ z > 0,$$

which is the same as the inequality (35).

To prove the inequality (36), we use the formula (29) for $A^{-1}(x)$, and find that (36) can be written as

$$\begin{aligned}
& A^{-1}(N(z)) \;>\; \frac{\sqrt{\pi}}{2} z, \quad \forall\ z > 0 && (37)\\
\Longleftrightarrow\quad & \sqrt{-\frac{\pi}{2}\ln\left(4N(z)(1-N(z))\right)} \;\geq\; \frac{\sqrt{\pi}}{2} z, \quad \forall\ z > 0 \\
\Longleftrightarrow\quad & -\frac{\pi}{2}\ln\left(4N(z)(1-N(z))\right) \;>\; \frac{\pi}{4} z^2, \quad \forall\ z > 0 \\
\Longleftrightarrow\quad & \ln\left(4N(z)(1-N(z))\right) \;<\; -\frac{z^2}{2}, \quad \forall\ z > 0 \\
\Longleftrightarrow\quad & 4N(z) - 4(N(z))^2 \;<\; e^{-\frac{z^2}{2}}, \quad \forall\ z > 0 \\
\Longleftrightarrow\quad & 4(N(z))^2 - 4N(z) + e^{-\frac{z^2}{2}} \;>\; 0, \quad \forall\ z > 0. && (38)
\end{aligned}$$

The largest root of the quadratic equation $4t^2 - 4t + e^{-\frac{z^2}{2}} = 0$ corresponding to (38) is $\frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{-\frac{z^2}{2}}}$. Therefore, the inequality (38) is established if we show that

$$N(z) \;>\; \frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{-\frac{z^2}{2}}}, \quad \forall\ z > 0. \tag{39}$$

Note that

$$\begin{aligned}
N(x) \;&=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt \;=\; \frac{1}{2} \;+\; \frac{1}{\sqrt{2\pi}}\int_{0}^{x} e^{-\frac{t^2}{2}}\, dt \\
&=\; \frac{1}{2} \;+\; \frac{1}{2\sqrt{2\pi}}\int_{-x}^{x} e^{-\frac{t^2}{2}}\, dt. && (40)
\end{aligned}$$

Moreover,

$$\left(\int_{-x}^{x} e^{-\frac{t^2}{2}}\,dt\right)^2 = \left(\int_{-x}^{x} e^{-\frac{u^2}{2}}\,du\right)\cdot\left(\int_{-x}^{x} e^{-\frac{v^2}{2}}\,dv\right) = \int_{-x}^{x}\int_{-x}^{x} e^{-\frac{u^2+v^2}{2}}\,dudv = \iint_{[-x,x]\times[-x,x]} e^{-\frac{u^2+v^2}{2}}\,dudv. \tag{41}$$

Since the discus $D(0,x)$ centered at 0 and of radius x is inside the square $[-x,x]\times[-x,x]$, it follows that

$$\iint_{[-x,x]\times[-x,x]} e^{-\frac{u^2+v^2}{2}}\,dudv > \iint_{D(0,x)} e^{-\frac{u^2+v^2}{2}}\,dudv, \quad \forall\ x>0. \tag{42}$$

We use the polar coordinates transformation

$$(u,v)\in D(0,x) \quad\rightarrow\quad (r,\theta)\in[0,x]\times[0,2\pi) \quad \text{with} \quad u=r\cos\theta;\ v=r\sin\theta$$

to compute the integral from the right hand side of (42). Recall that $dudv = r\ d\theta dr$. Then,

$$\iint_{D(0,x)} e^{-\frac{u^2+v^2}{2}}\,dudv = \int_0^x\int_0^{2\pi} r\ e^{-\frac{r^2}{2}}\,d\theta dr = 2\pi\left(-e^{-\frac{r^2}{2}}\right)\Big|_{r=0}^{r=x} = 2\pi\left(1-e^{-\frac{x^2}{2}}\right). \tag{43}$$

From (41), (42), and (43), it follows that

$$\left(\int_{-x}^{x} e^{-\frac{t^2}{2}}\,dt\right)^2 > 2\pi\left(1-e^{-\frac{x^2}{2}}\right), \quad \forall\ x>0,$$

and therefore we conclude from (40) that

$$N(x) > \frac{1}{2} + \frac{1}{2\sqrt{2\pi}}\sqrt{2\pi\left(1-e^{-\frac{x^2}{2}}\right)} = \frac{1}{2} + \frac{1}{2}\sqrt{1-e^{-\frac{x^2}{2}}}, \quad \forall\ x>0.$$

Thus, (39) is established, and therefore the inequality (36) holds; see also (37) and (38).

From (33–36), we conclude that the double inequality (25) is established.

To show that the upper bound in the inequality (25) is sharp, recall from (33) and (34) that

$$0 < \frac{\sigma_{imp,BS}-\sigma_{imp,approx,2}}{\sigma_{imp,BS}} = \frac{z-A^{-1}(N(z))}{z} < 1-\frac{\sqrt{\pi}}{2}, \quad \forall\ z>0. \tag{44}$$

From Lemma 5.1, we find that

$$\lim_{z\to\infty}\frac{z-A^{-1}(N(z))}{z} = 1-\frac{\sqrt{\pi}}{2}. \tag{45}$$

From (18) and (31), it follows that $z = \frac{\sigma_{imp,BS}\sqrt{T}}{2}$, and, from (44) and (45), we conclude that

$$\lim_{\sigma_{imp,BS}\sqrt{T}\to\infty} \frac{\sigma_{imp,BS} - \sigma_{imp,approx}}{\sigma_{imp,BS}} \;=\; 1 - \frac{\sqrt{\pi}}{2}.$$

Thus, the upper bound in the inequality (25) is sharp and cannot be improved. □

4 The sharpness of an ATM-forward options approximation formula

Of the three approximation formulas (9), (13), and (16) for ATM-forward options, we only investigate the properties of the approximation formula $C_{approx,2}$ given by (13) which corresponds to the best approximation of ATM-forward implied volatilities.[3] In Theorem 4.1, we prove that $C_{approx,2}$ is a remarkably sharp approximation (within 2% accurate) of the Black–Scholes value for ATM-forward options, for any integrated volatilities $\sigma\sqrt{T}$, including, for example, long dated options or options on high volatility underlying assets. Moreover, we establish numerically at the end of this section that the approximation given by (13) is within 0.71% of the Black–Scholes value of the option.

Theorem 4.1. *The approximate values $C_{approx,2}$ and $P_{approx,2}$ of ATM-forward call and put options given by (13) are within 2% of the Black–Scholes value C_{BS} and P_{BS}, i.e.,*

$$0 \;<\; \frac{C_{approx,2} - C_{BS}}{C_{BS}} \;=\; \frac{P_{approx,2} - P_{BS}}{P_{BS}} \;<\; c, \tag{46}$$

for any volatility parameter σ of the underlying asset and for any maturity T, where

$$c \;=\; \left(\frac{\pi-2}{4-\pi}\right)^{\frac{\pi-3}{2}} - 1 \;\approx\; 0.0204. \tag{47}$$

Proof. In order to prove (46), it is enough to show that

$$0 \;<\; \frac{C_{approx,2} - C_{BS}}{C_{BS}} \;<\; c, \tag{48}$$

since $P_{BS} = C_{BS}$ and $P_{approx,2} = C_{approx,2}$; see (8) and (13).

From (8) and (12), we obtain that establishing (48) is equivalent to proving that

$$\begin{aligned}
& C_{BS} \;<\; C_{approx,2} \;<\; (1+c)C_{BS} \\
\iff\; & \frac{C_{BS}}{2S_0e^{-qT}} \;<\; \frac{C_{approx,2}}{2S_0e^{-qT}} \;<\; (1+c)\frac{C_{BS}}{2S_0e^{-qT}} \\
\iff\; & N\left(\frac{\sigma\sqrt{T}}{2}\right) - \frac{1}{2} \;<\; A\left(\frac{\sigma\sqrt{T}}{2}\right) - \frac{1}{2} \;<\; (1+c)\left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - \frac{1}{2}\right), \;\; \forall\, \sigma > 0, T > 0 \\
\iff\; & N(x) - \frac{1}{2} \;<\; A(x) - \frac{1}{2} \;<\; (1+c)\left(N(x) - \frac{1}{2}\right), \;\; \forall\, x > 0.
\end{aligned} \tag{49}$$

[3] The relative error $\frac{C_{approx,1} - C_{BS}}{C_{BS}}$ of the Brenner–Subrahmanyam approximation $C_{approx,1}$ given by (9) is $O(\sigma^2 T)$; see Stefanica [22]. This result does not provide a good assessment of the quality of the approximation $C_{approx,1}$ for large volatilities and long dated options, and, in fact, the relative error $\frac{C_{approx,1} - C_{BS}}{C_{BS}}$ deteriorates quickly for long dated options or for underlying assets with moderately large volatilities.

The double inequality (49) is equivalent to

$$N(x) \ < \ A(x), \quad \forall\, x > 0; \tag{50}$$
$$(1+c)N(x) - A(x) - \frac{c}{2} \ > \ 0, \quad \forall\, x > 0. \tag{51}$$

The inequality (50) can be obtained as a direct consequence of (10).

To prove (51), it is enough to show that the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) \ = \ (1+c)N(x) - A(x) - \frac{c}{2} \tag{52}$$

is a strictly increasing function for $x > 0$. Then, since

$$f(0) \ = \ (1+c)N(0) - A(0) - \frac{c}{2} \ = \ (1+c)\frac{1}{2} - \frac{1}{2} - \frac{c}{2} \ = \ 0,$$

we obtain that

$$f(x) \ > \ f(0) \ = \ 0, \quad \forall\, x > 0,$$

which is the same as (51).

We complete the proof by showing that the function $f(x)$ given by (52) in a strictly increasing function for $x > 0$.

Recall that

$$N(x) \ = \ \frac{1}{2} \ + \ \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} \ dt. \tag{53}$$

Using (53) and (11), we find that the function $f(x)$ from (52) can be written as

$$\begin{aligned} f(x) \ &= \ (1+c)\left(\frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} \ dt\right) - \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{-\frac{2x^2}{\pi}}}\right) - \frac{c}{2} \\ &= \ \frac{1+c}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} \ dt \ - \ \frac{1}{2}\sqrt{1 - e^{-\frac{2x^2}{\pi}}}. \end{aligned}$$

We will show that $f'(x) > 0$ for all $x > 0$. Note that

$$f'(x) \ = \ \frac{1+c}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \ - \ \frac{x}{\pi\sqrt{1 - e^{-\frac{2x^2}{\pi}}}} e^{-\frac{2x^2}{\pi}}.$$

Thus,

$$\begin{aligned} & f'(x) \ > \ 0, \quad \forall\, x > 0 \\ \Longleftrightarrow \quad & \frac{1+c}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \ > \ \frac{x}{\pi\sqrt{1 - e^{-\frac{2x^2}{\pi}}}} e^{-\frac{2x^2}{\pi}}, \quad \forall\, x > 0 \\ \Longleftrightarrow \quad & (1+c) e^{\frac{2x^2}{\pi} - \frac{x^2}{2}} \ \sqrt{1 - e^{-\frac{2x^2}{\pi}}} \ > \ \sqrt{\frac{2}{\pi}}\, x, \quad \forall\, x > 0 \\ \Longleftrightarrow \quad & (1+c)^2 e^{\frac{4x^2}{\pi} - x^2} \ \left(1 - e^{-\frac{2x^2}{\pi}}\right) \ > \ \frac{2x^2}{\pi}, \quad \forall\, x > 0. \end{aligned} \tag{54}$$

Let $t = \frac{2x^2}{\pi}$. Then, $\frac{4x^2}{\pi} - x^2 = \frac{4-\pi}{\pi}x^2 = \frac{4-\pi}{2}t$, and the inequality (54) is equivalent to

$$(1+c)^2 e^{\frac{4-\pi}{2}t}\ \left(1 - e^{-t}\right)\ >\ t, \quad \forall\, t > 0. \tag{55}$$

If we define the function $g : [0, \infty) \to \mathbf{R}$ as

$$g(t)\ =\ (1+c)^2 e^{\frac{4-\pi}{2}t}\left(1 - e^{-t}\right)\ -\ t,$$

proving that (55) holds true is equivalent to showing that $g(t) > 0$ for all $t > 0$.

Note that

$$\begin{aligned} g'(t) &= (1+c)^2 e^{\frac{4-\pi}{2}t}\left(\frac{\pi}{2} - 1\right)\left(\frac{4-\pi}{\pi-2} + e^{-t}\right)\ -\ 1; \\ g''(t) &= (1+c)^2 e^{\frac{4-\pi}{2}t}\left(\frac{\pi}{2} - 1\right)^2\left(\left(\frac{4-\pi}{\pi-2}\right)^2 - e^{-t}\right). \end{aligned}$$

The only solution t_0 of $g''(t) = 0$ corresponds to $e^{-t_0} = \left(\frac{4-\pi}{\pi-2}\right)^2$, and is given by $t_0 = 2\ln\left(\frac{\pi-2}{4-\pi}\right) > 0$. Note that $g''(t) < 0$ if $0 < t < t_0$ and $g''(t) > 0$ if $t_0 < t$. In other words, the function $g'(t)$ is strictly decreasing when $0 < t < t_0$ and strictly increasing when $t_0 < t$, and therefore $g'(t) > g'(t_0)$ for all $t > 0$, $t \neq t_0$. A simple computation[4] shows that, if $c = \left(\frac{\pi-2}{4-\pi}\right)^{\frac{\pi-3}{2}} - 1$, then $g'(t_0) = 0$, and therefore $g'(t) > 0$ for all $t > 0$, $t \neq t_0$.

Thus, the function $g(t)$ is strictly increasing for $t \geq 0$. Since $g(0) = 0$, it follows that $g(t) > 0$ for all $t > 0$.

Recall that $f'(x) > 0$ for all $x > 0$ is equivalent to $g(t) > 0$ for all $t > 0$. We conclude that the function $f(x)$ is strictly increasing for $x > 0$, which is what we wanted to show in order to complete the proof. □

The result of Theorem 4.1 can be checked numerically as follows: In Figure 4, we plot the relative error $\frac{C_{approx,2} - C_{BS}}{C_{BS}}$ in terms of the integrated volatility $\sigma\sqrt{T}$ and note that the graph is below 0.02, as stated in (46).

---

[4]To see that $g'(t_0) = 0$, recall that $e^{-t_0} = \left(\frac{4-\pi}{\pi-2}\right)^2$ and $c = \left(\frac{\pi-2}{4-\pi}\right)^{\frac{\pi-3}{2}} - 1$. Then,

$$\begin{aligned} g'(t_0) &= (1+c)^2 e^{\frac{4-\pi}{2}t_0}\left(\frac{\pi}{2} - 1\right)\left(\frac{4-\pi}{\pi-2} + e^{-t_0}\right)\ -\ 1 \\ &= (1+c)^2\left(\frac{4-\pi}{\pi-2}\right)^{-4+\pi} \cdot \frac{\pi-2}{2} \cdot \left(\frac{4-\pi}{\pi-2} + \left(\frac{4-\pi}{\pi-2}\right)^2\right)\ -\ 1 \\ &= (1+c)^2\left(\frac{4-\pi}{\pi-2}\right)^{\pi-3}\ -\ 1 \\ &= \left(\frac{\pi-2}{4-\pi}\right)^{\pi-3} \cdot \left(\frac{4-\pi}{\pi-2}\right)^{\pi-3}\ -\ 1 \\ &= 0. \end{aligned}$$

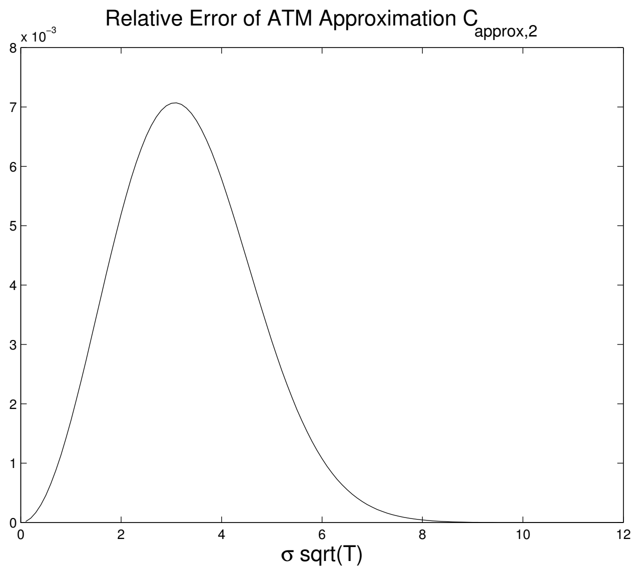

Figure 4: Relative error $\frac{C_{approx,2}-C_{BS}}{C_{BS}}$ as a function of integrated volatility $\sigma\sqrt{T}$

In fact, we find numerically an even tighter bound for the relative error of the ATM approximation formula (46) as follows:

$$0 \ < \ \frac{C_{approx,2} - C_{BS}}{C_{BS}} \ = \ \frac{P_{approx,2} - P_{BS}}{P_{BS}} \ < \ 0.0071;$$

see Figure 4.

In other words, the approximate values $C_{approx,2}$ and $P_{approx,2}$ of ATM-forward call and put options given by (13) are within 0.71% of the Black–Scholes options values C_{BS} and P_{BS} for all volatility parameters σ and for all maturities T.

5 Appendix: Technical Result

This section contains a technical result used in the proof of Theorem 3.1.

Lemma 5.1. *Let $N(x)$ be the cumulative distribution function of the standard normal random variable, and let $A(x)$ be the function given by*

$$A(x) \ = \ \frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{-\frac{2x^2}{\pi}}}, \quad \forall \ x \geq 0.$$

Then,

$$\lim_{x\to\infty} \frac{x - A^{-1}(N(x))}{x} \ = \ 1 - \frac{\sqrt{\pi}}{2}; \tag{56}$$

Proof. We will use l'Hôpital's rule repeatedly to compute the limit (56).

The inverse function $A^{-1}(x)$ is given by

$$A^{-1}(x) \ = \ \sqrt{-\frac{\pi}{2}\ln\left(4x(1-x)\right)}, \quad \forall \ x \in \left[\frac{1}{2}, 1\right], \tag{57}$$

see also (29), and its derivative is

$$
\begin{aligned}
(A^{-1})'(x) &= \sqrt{\frac{\pi}{2}} \cdot \frac{-(\ln(4x(1-x)))'}{2\sqrt{-\ln(4x(1-x))}} \\
&= \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{2x-1}{x(1-x)\sqrt{-\ln(4x(1-x))}}, \quad \forall\, x \in \left[\frac{1}{2}, 1\right]. \qquad (58)
\end{aligned}
$$

Since $N(x) \geq \frac{1}{2}$ for $x \geq 0$, it follows that

$$
(A^{-1})'(N(x)) = \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{2N(x)-1}{N(x)(1-N(x))\sqrt{-\ln(4N(x)(1-N(x)))}}, \quad \forall\, x \geq 0, \qquad (59)
$$

The derivative of $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$ is

$$
N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \forall\, x \in \mathbf{R}. \qquad (60)
$$

By using l'Hôpital's rule and (59–60), we find that

$$
\begin{aligned}
& \lim_{x\to\infty} \frac{x - A^{-1}(N(x))}{x} \qquad (61) \\
= & \lim_{x\to\infty} \frac{\left(x - A^{-1}(N(x))\right)'}{(x)'} = \lim_{x\to\infty} \left(1 - (A^{-1})'(N(x)) \cdot N'(x) \right) \\
= & \ 1 - \frac{\sqrt{\pi}}{2\sqrt{2}} \lim_{x\to\infty} \frac{2N(x)-1}{N(x)(1-N(x))\sqrt{-\ln(4N(x)(1-N(x)))}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\
= & \ 1 - \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{1}{\sqrt{2\pi}} \lim_{x\to\infty} \frac{2N(x)-1}{N(x)} \cdot \frac{e^{-\frac{x^2}{2}}}{(1-N(x))\sqrt{-\ln(1-N(x)) - \ln(4N(x))}} \\
= & \ 1 - \frac{1}{4} \lim_{x\to\infty} \frac{e^{-\frac{x^2}{2}}}{(1-N(x))\sqrt{-\ln(1-N(x)) - \ln(4)}}, \qquad (62)
\end{aligned}
$$

since $\lim_{x\to\infty} N(x) = 1$.

Moreover, since $\lim_{x\to\infty} \ln(1-N(x)) = -\infty$, it follows that

$$
\lim_{x\to\infty} \frac{\sqrt{-\ln(1-N(x))}}{\sqrt{-\ln(1-N(x)) - \ln(4)}} = \lim_{x\to\infty} \frac{1}{\sqrt{1 + \frac{\ln(4)}{\ln(1-N(x))}}} = 1. \qquad (63)
$$

From (61), (62), and (63), we obtain that

$$
\begin{aligned}
& \lim_{x\to\infty} \frac{x - A^{-1}(N(x))}{x} \qquad (64) \\
= & \ 1 - \frac{1}{4} \lim_{x\to\infty} \frac{e^{-\frac{x^2}{2}}}{(1-N(x))\sqrt{-\ln(1-N(x))}} \cdot \frac{\sqrt{-\ln(1-N(x))}}{\sqrt{-\ln(1-N(x)) - \ln(4)}} \\
= & \ 1 - \frac{1}{4} \lim_{x\to\infty} \frac{e^{-\frac{x^2}{2}}}{(1-N(x))\sqrt{-\ln(1-N(x))}}. \qquad (65)
\end{aligned}
$$

Note that

$$\lim_{x\to\infty} \frac{e^{-\frac{x^2}{2}}}{(1-N(x))\sqrt{-\ln(1-N(x))}} = \left(\lim_{x\to\infty} \left(\frac{e^{-\frac{x^2}{2}}}{(1-N(x))\sqrt{-\ln(1-N(x))}} \right)^2 \right)^{\frac{1}{2}}$$
$$= \sqrt{\lim_{x\to\infty} \frac{e^{-x^2}}{-(1-N(x))^2\ln(1-N(x))}}. \tag{66}$$

Thus, from (64), (65), and (66), it follows that

$$\lim_{x\to\infty} \frac{x - A^{-1}(N(x))}{x} = 1 - \frac{1}{4}\sqrt{\lim_{x\to\infty} \frac{e^{-x^2}}{-(1-N(x))^2\ln(1-N(x))}}. \tag{67}$$

Since

$$\left(-(1-t)^2\ln(1-t)\right)' = 2(1-t)\ln(1-t) + (1-t) = (1-t)(2\ln(1-t)+1), \tag{68}$$

it follows using the Chain Rule, (68), and (60), that

$$\left(-(1-N(x))^2\ln(1-N(x))\right)' = (1-N(x))(2\ln(1-N(x))+1)\cdot N'(x)$$
$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \cdot (1-N(x)) \cdot 2\left(\ln(1-N(x)) + \frac{1}{2}\right). \tag{69}$$

By using l'Hôpital's rule and (69), we obtain that

$$\lim_{x\to\infty} \frac{e^{-x^2}}{-(1-N(x))^2\ln(1-N(x))} \tag{70}$$
$$= \lim_{x\to\infty} \frac{\left(e^{-x^2}\right)'}{\left(-(1-N(x))^2\ln(1-N(x))\right)'}$$
$$= \lim_{x\to\infty} \frac{-2xe^{-x^2}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \cdot (1-N(x)) \cdot 2\left(\ln(1-N(x)) + \frac{1}{2}\right)}$$
$$= \sqrt{2\pi}\lim_{x\to\infty} \frac{xe^{-\frac{x^2}{2}}}{-(1-N(x))\left(\ln(1-N(x)) + \frac{1}{2}\right)}$$
$$= \sqrt{2\pi}\lim_{x\to\infty} \frac{xe^{-\frac{x^2}{2}}}{-(1-N(x))\ln(1-N(x))} \cdot \frac{\ln(1-N(x))}{\ln(1-N(x)) + \frac{1}{2}}$$
$$= \sqrt{2\pi}\lim_{x\to\infty} \frac{xe^{-\frac{x^2}{2}}}{-(1-N(x))\ln(1-N(x))}, \tag{71}$$

since $\lim_{x\to\infty} \ln(1-N(x)) = -\infty$, and therefore

$$\lim_{x\to\infty} \frac{\ln(1-N(x))}{\ln(1-N(x)) + \frac{1}{2}} = \lim_{x\to\infty} \frac{1}{1 + \frac{1}{2\ln(1-N(x))}} = 1.$$

Moreover, since $(-(1-t)\ln(1-t))' = \ln(1-t)+1$, it follows using the Chain Rule and (60) that

$$\begin{aligned}(-(1-N(x))\ln(1-N(x)))' &= (\ln(1-N(x))+1)\cdot N'(x) \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot(\ln(1-N(x))+1). \qquad (72)\end{aligned}$$

By using l'Hôpital's rule and (72), we obtain that

$$\begin{aligned}&\sqrt{2\pi}\lim_{x\to\infty}\frac{xe^{-\frac{x^2}{2}}}{-(1-N(x))\ln(1-N(x))} \qquad (73)\\ =\ &\sqrt{2\pi}\lim_{x\to\infty}\frac{\left(xe^{-\frac{x^2}{2}}\right)'}{(-(1-N(x))\ln(1-N(x)))'}\\ =\ &\sqrt{2\pi}\lim_{x\to\infty}\frac{e^{-\frac{x^2}{2}}-x^2e^{-\frac{x^2}{2}}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot(\ln(1-N(x))+1)}\\ =\ &2\pi\lim_{x\to\infty}\frac{1-x^2}{\ln(1-N(x))+1}. \qquad (74)\end{aligned}$$

By using l'Hôpital's rule twice, we find that

$$\begin{aligned}&\lim_{x\to\infty}\frac{1-x^2}{\ln(1-N(x))+1} = \lim_{x\to\infty}\frac{(1-x^2)'}{(\ln(1-N(x))+1)'} = \lim_{x\to\infty}\frac{-2x}{-\frac{1}{1-N(x)}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}}\\ =\ &2\sqrt{2\pi}\lim_{x\to\infty}\frac{1-N(x)}{\frac{e^{-\frac{x^2}{2}}}{x}} = 2\sqrt{2\pi}\lim_{x\to\infty}\frac{(1-N(x))'}{\left(\frac{e^{-\frac{x^2}{2}}}{x}\right)'} = 2\sqrt{2\pi}\lim_{x\to\infty}\frac{-\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}}{\frac{-x^2e^{-\frac{x^2}{2}}-e^{-\frac{x^2}{2}}}{x^2}}\\ =\ &2\lim_{x\to\infty}\frac{-x^2e^{-\frac{x^2}{2}}}{-(x^2+1)e^{-\frac{x^2}{2}}} = 2\lim_{x\to\infty}\frac{x^2}{x^2+1} = 2.\end{aligned}$$

In other words,

$$\lim_{x\to\infty}\frac{1-x^2}{\ln(1-N(x))+1} = 2. \qquad (75)$$

Then, from (70), (71), (73), (74), and (75), it follows that

$$\lim_{x\to\infty}\frac{-e^{-x^2}}{-(1-N(x))^2\ln(1-N(x))} = 2\pi\lim_{x\to\infty}\frac{1-x^2}{\ln(1-N(x))+1} = 4\pi. \qquad (76)$$

From (67) and (76), we conclude that

$$\begin{aligned}\lim_{x\to\infty}\frac{x-A^{-1}(N(x))}{x} &= 1-\frac{1}{4}\sqrt{\lim_{x\to\infty}\frac{-e^{-x^2}}{(1-N(x))^2\ln(1-N(x))}}\\ &= 1-\frac{1}{4}\cdot\sqrt{4\pi}\\ &= 1-\frac{\sqrt{\pi}}{2},\end{aligned}$$

which is the same as (56). □

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