Universidad de Castilla-La Mancha

TESIS DOCTORAL

THREE ESSAYS ON STOCHASTIC STRING MODELS FOR THE TERM STRUCTURE OF INTEREST RATES

Autor:

Alberto Bueno-Guerrero

Directores:

Javier Fernández Navas y Manuel Moreno Fuentes

Facultad de Ciencias Jurídicas y Sociales

Departamento de Análisis Económico y Finanzas

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To my family

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Resumen

La Estructura Temporal de los Tipos de Interés (ETTI), es decir, la relación funcional existente entre el rendimiento de los bonos cupón-cero y su vencimiento, es uno de los conceptos más relevantes en el campo de las finanzas. De su dinámica dependen aspectos tan importantes como el propio precio de los bonos cupón-cero (y, por tanto, el de los bonos con cupón y las carteras de bonos), el de sus derivados, como los futuros y opciones sobre bonos, la cobertura de dichos derivados o la immunización de carteras de bonos.

El objetivo general de esta tesis doctoral es presentar un marco de trabajo para la dinámica en tiempo continuo de la ETTI que permita abordar los distintos aspectos enunciados anteriormente, resolver diversos problemas relacionados con los modelos actuales y obtener resultados nuevos.

El punto de partida será el modelo presentado en Santa-Clara y Sornette (2001) y conocido como modelo de cuerda estocástica. El punto esencial de este modelo consiste en la introducción de una dinámica en tiempo continuo para el tipo forward instantáneo guiada por un nuevo proceso, llamado *proceso de cuerda estocástica*. Este proceso incorpora una dependencia en el tiempo hasta el vencimiento no incluida en modelos anteriores del tipo forward, como el de Heath, Jarrow y Morton (1992) (HJM) y que da lugar a la aparición en el modelo de la covarianza entre shocks a la curva forward. Este hecho da lugar a una riquísima modelización apenas desarrollada por Santa-Clara y Sornette en su trabajo.

En esta tesis se reobtiene el modelo de cuerda estocástica con unas herramientas matemáticas más potentes que nos permitirán desarrollar sustancialmente lo realizado en el trabajo original convirtiendo este tipo de modelización en una verdadera teoría que explica y generaliza, de forma coherente, gran parte de los aspectos más importantes relacionados con la modelización en tiempo continuo de la ETTI.

En el primer capítulo se reformula el modelo de Santa-Clara y Sornette utilizando el cálculo estocástico con semimartingalas continuas. Esta reformulación potencia el modelo original permitiendo clarificar la naturaleza del proceso de cuerda estocástica y presentar nuevos resultados. Una vez presentados los resultados preliminares se redefine el proceso de cuerda estocástica, mostrando que consiste en un continuo de movimientos brownianos, uno para cada tiempo hasta el vencimiento.

A continuación se estudian dos ejemplos de procesos: la lámina de Ornstein-Uhlenbeck y su versión integrada, demostrándose que éste último es un proceso de cuerda estocástica. Tras la introducción de los supuestos necesarios se obtiene la dinámica de no arbitraje del tipo forward instantáneo y del tipo a corto plazo. Como aplicaciones nuevas del modelo se obtienen una ecuación en derivadas parciales (EDP) para el precio de los bonos en el caso markoviano y una fórmula cerrada para el precio de una opción call europea en el caso gaussiano. El capítulo primero acaba con un estudio detallado de la relación entre el modelo propuesto y otros modelos similares previamente propuestos en la literatura.

En el segundo capítulo se presenta el modelo de cuerda estocástica como una teoría unificadora de distintos aspectos relacionados con la ETTI como son los modelos HJM multifactoriales e infinito-dimensionales, el análisis de componentes principales de la curva forward, el problema de la consistencia con familias paramétricas y la valoración de opciones sobre bonos. El capítulo comienza con una clasificación de los modelos infinito-dimensionales de la ETTI y una breve descripción de un ejemplo de cada uno de ellos: el modelo HJM infinito-dimensional de Chu (1996) y el modelo de cuerda estocástica presentado en el primer capítulo. Como conclusión se obtiene que los dos modelos son equivalentes en la valoración de opciones call europeas en el caso gaussiano. La principal herramienta matemática del capítulo, el Teorema de Mercer, es enunciado en su forma válida para conjuntos no compactos. Mediante este teorema hallamos que, para un proceso de cuerda estocástica concreto, el modelo de cuerda estocástica se reduce a un modelo HJM infinito-dimensional, siendo los modelos HJM multifactoriales aproximaciones a éste. Como ejemplo se introduce una función de covarianza entre shocks que da lugar a una generalización infinito-dimensional del modelo de Hull y White (1990). Como otro resultado del análisis se obtiene una reinterpretación del modelo de cuerda estocástica como una generalización del análisis de componentes principales a un continuo de puntos muestrales. El siguiente aspecto del capítulo es una reformulación del estudio de la consistencia en nuestro marco de trabajo. El principal resultado está basado en la función de covarianza introducida anteriormente y se refiere a la consistencia de una sucesión de modelos de cuerda estocástica aproximados con una sucesión de familias paramétricas de curvas forward, extensión de la de Nelson y Siegel (1987) (NS). El último apartado de este capítulo se dedica a la obtención de un teorema para la valoración de opciones europeas con función de pago homogénea de grado uno en el caso gaussiano. Este resultado generaliza lo obtenido para modelos HJM gaussianos.

El tercer capítulo se centra en la valoración de caps y swaptions. Para ello se proponen dos aproximaciones distintas: un modelo de cuerda estocástica gaussiano y un modelo de mercado de cuerda estocástica. Adicionalmente se propone una solución a dos problemas relacionados con el estudio de caps y swaptions: el de su valoración relativa y el de la equivalencia observacional. Se empieza obteniendo sendas fórmulas para la valoración de opciones europeas de compra y venta

sobre una cartera de bonos, dentro del marco de trabajo de cuerda estocástica. Dichas fórmulas generalizan las de Brace y Musiela (1994a). A continuación se demuestra que, con la versión finito-dimensional de la función de covarianza entre shocks propuesta en el segundo capítulo, se obtiene la dinámica de Hull y White (1990) para el tipo de interés a corto plazo. Utilizando el resultado de Jamshidian (1989), obtenemos fórmulas cerradas para el precio de opciones de compra y venta sobre carteras de bonos con la covarianza anterior. En el siguiente apartado se desarrolla un modelo de cuerda estocástica gaussiano que permite obtener expresiones para el precio de caps y de swaptions, que se reducen a las fórmulas de Black (1976) bajo ciertas aproximaciones. A continuación se obtiene un modelo de cuerda estocástica para el tipo LIBOR. A partir del mismo surgen dos modelos LIBOR de mercado alternativos que permiten recuperar la fórmula de Black para caps. La diferencia entre estos dos modelos consiste en que uno de ellos es exacto, generaliza el modelo de Brace, Gatarek y Musiela (1997) (BGM) y es, por tanto, incompatible con los modelos gaussianos. El otro modelo es aproximado y compatible con ellos. Esta sección termina proporcionando la dinámica de cuerda estocástica para el tipo swap forward. Este capítulo acaba con el estudio de dos importantes problemas. El primero es el de la valoración relativa de caps y swaptions, es decir, la mala valoración de caps con la información extraída del mercado de swaptions (Longstaff, Santa-Clara y Schwartz, (2001a)) (LSS). Se analizan todos los supuestos de LSS bajo el enfoque de nuestro marco de trabajo. El supuesto de que los factores que generan la matrices de covarianzas histórica e implícita son iguales no se verifica, siendo ésta una posible explicación al problema. En cuanto al segundo problema, el de la equivalencia observacional, nuestro análisis corrobora el resultado de Kerkhoff y Pelsser (2002), es decir, LSS y BGM requieren la estimación del mismo número de parámetros. Sin embargo mostramos que dicho número resulta ser menor que el obtenido por Kerkhoff y Pelsser en la mayoría de los casos.

Resumiendo, en esta tesis doctoral, se han obtenido un buen número de nuevos resultados. Podemos señalar los siguientes:

- La interpretación del proceso de cuerda estocástica como un continuo de movimientos brownianos.

- La dinámica del tipo de interés a corto plazo.

- Una EDP para el precio de un bono en el caso markoviano.

- Una expresión analítica para el precio de una opción call europea.

- Una condición de ortogonalidad para las volatilidades HJM.

- La interpretación de los modelos HJM multifactoriales como aproximaciones a un modelo de cuerda estocástica infinito-dimensional.

- Un ejemplo de función de covarianza entre shocks con *buenas* propiedades.

- La reinterpretación de la modelización de cuerda estocástica como un análisis de componentes principales de la curva forward con un continuo de puntos muestrales.

- Un resultado de consistencia entre una sucesión de modelos con oportunidades de arbitraje y una sucesión de familias de NS extendidas.

- Un teorema para la valoración de opciones con función de pago homogénea de grado uno.

- Un modelo multifactorial cuya dinámica del tipo de interés a corto plazo coincide con la de Hull y White (1990).

- Una fórmula cerrada para la valoración de opciones europeas sobre carteras de bonos.

- Una fórmula cerrada y exacta para la valoración de swaptions.

- Una expresión para la dinámica del tipo swap forward.

- Una posible solución a los problemas de la valoración relativa de caps y swaptions y al de la equivalencia observacional.

Como conclusión final podemos mencionar que, a lo largo de esta tesis se generalizan o se amplían importantes modelos como los presentados en Heath, Jarrow y Morton (1992), Brace, Gatarek y Musiela (1997), Björk y Christensen (1999), Longstaff, Santa-Clara y Schwartz (2001a) y Santa-Clara y Sornette (2001).

Summary

The Term Structure of Interest Rates (TSIR), i.e., the relationship between zero-coupon bond yields and their maturities, is one of the most relevant concepts in finance. The dynamics of the term structure affects important issues as the bond price itself (and, therefore, coupon bond and bond portfolio prices); derivatives prices, such as futures and options on bonds; the hedging of these derivatives or bond portfolio immunization.

The general objective of this doctoral dissertation is to present a framework for the continuous-time dynamics of the TSIR that allows us to study the different issues mentioned before, to solve a number of problems related to existent models and to obtain new results.

The starting point will be the model presented in Santa-Clara and Sornette (2001), known as stochastic string model. The key point in this model is the introduction of a continuous-time dynamics for the instantaneous forward interest rate driven by a new process, called *stochastic string process*. This process presents a dependency in time to maturity not included in previous models of the forward rate, such as the Heath, Jarrow and Morton (1992) (HJM) model and that leads to the appearance in the model of the covariance between shocks to the forward curve. This fact brings a very rich modeling not developed in Santa-Clara and Sornette (2001).

In this thesis the stochastic string model is reobtained with more powerful tools that allow us to develop substantially what was made in the original work making this type of modeling a truly theory that explains and generalize, in a coherent way, most of the more important issues related to the continuous-time modeling of the TSIR.

In the first chapter, the Santa-Clara and Sornette model is reformulated using the stochastic calculus with continuous semimartingales. This reformulation improves the original model allowing to clarify the nature of the stochastic string process and to present new results. Once preliminary results are presented, the stochastic string process is redefined, showing that it consists of a continuum of Brownian motions, one for each time to maturity. Then, two examples of processes are studied: the Ornstein-Uhlenbeck sheet and its integrated version, proving that the last one is a stochastic string process. After the introduction of the necessary assumptions, the no-arbitrage dynamics of the instantaneous forward rate and of the short-term interest rate are obtained. As new applications

of the model, a Partial Differential Equation (PDE) for the bond price in the Markovian case and a closed formula for the price of a European call option in the Gaussian case are obtained. The first chapter ends with a detailed study of the relationship between the model proposed and other similar models previously proposed in the literature.

In the second chapter, the stochastic string model is presented as a unifying theory of different issues related with the TSIR, as the multi-factor and infinite-dimensional HJM models, the Principal Components analysis of the forward curve, the problem of consistency with parametric families, and the valuation of bond options. This chapter starts with a classification of the infinite-dimensional models of the TSIR and a brief description of one member of each class: the infinite-dimensional HJM model of Chu (1996) and the stochastic string model of the first chapter. As a conclusion, the equivalence between both models for the valuation of European call options in the Gaussian case is obtained. The main mathematical tool of the chapter, Mercer's Theorem, is stated in the form valid for non-compact sets. Using this theorem we show that, for a concrete stochastic string process, the stochastic string model reduces to an infinite-dimensional HJM one, being multi-factor HJM models approximations to it. As an example, a covariance function between shocks that leads to an infinite-dimensional generalization of Hull and White (1990) is introduced. As another result of the analysis, the stochastic string model is reinterpreted as a generalization of the Principal Components analysis to a continuum of sample points. The next issue of the chapter is a reformulation of the study of consistency in our framework. The main result is based on the covariance function introduced previously and it refers to the consistency of a sequence of approximated stochastic string models with a sequence of parametric families of forward curves, extension of that in Nelson and Siegel (1987) (NS). The last section of this chapter obtains a theorem for the valuation of European options whose pay-off function is homogeneous of degree one in the Gaussian case. This result generalizes that obtained for Gaussian HJM models.

The third chapter focuses on the valuation of caps and swaptions. To this end two different approaches are proposed: a Gaussian stochastic string model and a stochastic string market model. Additionally, a solution to the two problems of the relative valuation of caps and swaptions and to the observational equivalence is proposed. We start obtaining formulas for the valuation of European call and put options on a bond portfolio within the stochastic string framework. These formulas generalize those of Brace and Musiela (1994a). Next it is proved that, with the finite-dimensional version of the covariance function between shocks proposed in the second chapter, the dynamics of Hull and White (1990) for the short-term interest rate is obtained. Using the Jamshidian's trick and the previous covariance, we obtain closed-form expressions for pricing put and call options on bond portfolios. In the next section, a Gaussian stochastic string model is developed. This model allows us to obtain expressions for caps and swaptions prices that reduce to Black (1976) formulas

under certain approximations. Then a stochastic string model for the LIBOR rate is obtained. From this model two alternative LIBOR market models that allow to recover the Black formula for caps emerge. The difference between these two models is that one of them is exact, generalizes the Brace, Gatarek and Musiela (1997) (BGM) model and, therefore, is incompatible with Gaussian models. The other model is approximated and compatible with these models. The last result of the section is the stochastic string dynamics for the forward swap rate. This chapter ends with the study of two important problems. The first one is the relative valuation of caps and swaptions, i.e., the mispricing of caps with information extracted from the swaptions market (Longstaff, Santa-Clara and Schwartz (2001a)) (LSS). All the assumption of LSS are analyzed under our framework. The assumption that the factors that generate the historical and the implied covariance matrices are equal do not hold, being this fact a possible explanation to the problem. For the second problem, the observational equivalence, our analysis concludes the same as in Kerkhoff and Pelsser (2002), i.e., LSS and BGM require the same number of parameters to be estimated. However, in our analysis, this number results to be smaller than that in Kerkhoff and Pelsser in almost all the cases.

Summarizing, in this doctoral dissertation, a number of new results have been obtained. We can highlight the following ones:

- The interpretation of the stochastic string process as a continuum of Brownian motions.

- The dynamics of the short-term interest rate.

- A PDE for the bond price in the Markovian case.

- An analytical expression for the price of a European call option.

- An orthogonality condition for the HJM volatilities.

- The interpretation of the multi-factor HJM models as approximations to a infinite-dimensional stochastic string model.

- An example of a covariance function between shocks with *good* properties.

- The reinterpretation of the stochastic string modeling as a Principal Component analysis with a continuum of sample points.

- A consistency result between a sequence of models with arbitrage opportunities and a sequence of extended NS families.

- A theorem for the valuation of options with degree-one homogeneous payoff function.

- A multi-factor model whose short-term interest rate dynamics coincide with that of Hull and White (1990).

- A closed formula for the valuation of European options on bond portfolios.

- An exact and closed-form expression for swaptions valuation.

- An expression for the dynamics of the forward swap rate.

- A possible solution to the problems of the relative valuation of caps and swaptions and of the observational equivalence.

As final conclusion we can mention that, along this dissertation, we have extended or generalized important models as those introduced in Heath, Jarrow, and Morton (1992), Brace, Gatarek and Musiela (1997), Björk and Christensen (1999), Longstaff, Santa-Clara and Schwartz (2001a) and Santa-Clara and Sornette (2001).

Chapter 1

Stochastic String Models with Continuous Semimartingales

1.1 Introduction

Among the continuous-time models of the term structure of interest rates (TSIR), the model of Heath, Jarrow, and Morton (1992) (HJM, hereafter) is considered by many as one of the main references. In this model, the initial forward curve and the volatility structure of forward rates are taken as inputs so that we can determine the evolution of the forward curve over time from the dynamics of the instantaneous forward rate and the no arbitrage condition. The main advantages of this model are (Schmidt (2011)): a) a perfect fitting (by construction) to the initial TSIR, b) its flexibility, that allows to obtain different models for different volatility functions, and c) its generality as it nests several models.[1] Additionally, the one-factor version of this model is simple, parsimonious, and consistent with some empirical evidence (Bühler *et al.* (1999)).

However, a drawback of this model is that, for a given volatility structure, the one-factor version imposes a perfect correlation between the innovations of instantaneous forward rates for different maturities. This restriction limits the dynamics of the forward curve and contradicts the empirical evidence (Brown and Schaefer (1994)), that shows that the correlation decays exponentially as a function of the difference between maturities. Merton (1973) shows a no-arbitrage relationship between the price of a portfolio of options and the price of an option on a portfolio, determined by the correlation between the underlying assets. Then, to perform the relative pricing of caps and swaptions, the TSIR model must reflect the correlation between forward rates more appropriately than one-factor models, which do not allow a fast enough decorrelation (Rebonato and Cooper (1997)) nor an independent description of volatilities and correlations (Collin-Dufresne and Goldstein (2001, 2003)). A solution could be increasing the number of factors to consider imperfect correlation but the theoretical volatilities corresponding to each factor must be parameterized to fit simultaneously

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[1]See, for instance, Vasicek (1977), Brennan and Schwartz (1979), Cox, Ingersoll, and Ross (1985), Ho and Lee (1986), Hull and White (1990), and Black and Karasinski (1991).

empirical volatilities and correlations, a really difficult goal (Santa-Clara and Sornette (2001)).

A new type of TSIR models, known as *stochastic string* or *random fields* models, has proposed a correlation structure between forward rates. Seminal papers of this string approach are Kennedy (1994, 1997), Goldstein (2000), and Santa-Clara and Sornette (2001) which are followed by Sornette (1998), Baaquie (2001), Longstaff and Schwartz (2001), Longstaff *et al.* (2001a, b), Collin-Dufresne and Goldstein (2001), Kerkhof and Pelsser (2002), De Donno (2004) and Kimmel (2004), among many others.[2]

All these papers consider that the source of randomness generating the dynamics of the forward curve, the *string stochastic process*, is not constant along maturities but it varies point by point along the whole TSIR. The only condition is that the shocks with different maturities must be imperfectly correlated but maintaining the continuity in the forward curve. As a consequence, the string models provide several advantages with respect to the traditional HJM ones, as pointed out by Goldstein (2000), Santa-Clara and Sornette (2001), McDonald and Beard (2002), Bester (2004), Kimmel (2004), among others.

Considering these advantages, it seems reasonable to look for a TSIR framework as general as possible. Santa-Clara and Sornette (2001) is the paper closest to this goal. However, from our point of view, that paper lacks of rigorous mathematical formulation to be the benchmark we are looking for. This lack is recognized by these authors, who state: "We attempt to present the results and their derivation in the simplest and most intuitive way, rather than emphasize mathematical rigor" (see Santa-Clara and Sornette (2001, p. 150)).

Our main objective is to provide a framework, based on stochastic calculus, that can incorporate existing formulations of string models of the TSIR and that, at the same time, is able to accommodate future extensions of these models. To this aim, we rebuild the model of Santa-Clara and Sornette (2001) and provide some new results not included in that paper.

The advantage of our approach with respect to other approximations to the TSIR infinite-dimensional modeling (see, for example, Carmona and Tehranchi (2006) or Filipovic (2001)) is that we maintain the time to maturity as a fundamental variable and not included implicitly in the (infinite-dimensional) Wiener processes. This allows us to work with the correlation between stochastic shocks, a very important function for modeling the TSIR.

Our first contribution is that we characterize clearly the type of stochastic integration to be used, an issue omitted in Santa-Clara and Sornette (2001). We will use semimartingales, the more general processes for which we can define reasonably a stochastic calculus (Bichteler (1981)). Moreover, in arbitrage-free models, the price of any financial asset follows a semimartingale (Harrison and Pliska

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[2]See also Bouchaud *et al.* (1999), Collin-Dufresne and Goldstein (2003), McDonald and Beard (2002), Furrer (2003), Bester (2004), Cont (2005), Hamza *et al.* (2005), Korezlioglu (2005) and Altay (2007).

(1981), Delbaen and Schachermayer (2004)).

The second contribution of this chapter is related to the methodology. Santa-Clara and Sornette (2001) consider a certain *stochastic discount factor* and impose that the bond price (discounted using this factor) follows a martingale. Our approach follows the more standard methodology in Finance of discounting prices by means of a banking account.

Our final contribution is that we obtain certain original results with respect to Santa-Clara and Sornette (2001) such as: a) an expression for the dynamics of the short-term interest rate, b) a Markovian framework in which we will obtain a PDE to price bonds, and c) a closed-form expression for the price of European bond options in the Gaussian case.

This chapter is organized as follows. Section 1.2 details the probabilistic framework and introduces the financial concepts and mathematical results to be used later. Section 1.3 generalizes the stochastic string process of Santa-Clara and Sornette (2001) to a framework based on calculus with semimartingales. An alternative way of building examples of stochastic string processes using the random field theory is reported. Section 1.4 provides the dynamics of the variables that characterize the TSIR and the no-arbitrage condition of the model and shows that the one-factor HJM model is nested in the stochastic string one. Section 1.5 includes the conditions under which the short-term interest rate follows a Markovian process and, as an application, we obtain a bond pricing PDE that recovers some of the classical TSIR models.

To illustrate the usefulness of the model, Section 1.6 prices analytically European bond options in the Gaussian case, generalizing the formulas obtained for the Gaussian HJM models. Section 1.7 analyzes the relationship between our model and another ones proposed in the literature and provides some interesting properties on the robustness of the string stochastic models. Finally, Section 1.8 summarizes the main conclusions. Mathematical proofs are deferred to the appendix.

1.2   Notation and Preliminary Results

We consider a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual hypotheses (Protter (2004, p. 3)). We also assume that $\mathcal{F} = \mathcal{F}_T$ where T denotes the finite time horizon for trading risk-free zero-coupon bonds. All the equalities and properties related to stochastic processes will be assumed to hold almost sure.

Let P_t^τ be the price at time t of a bond maturing at time τ, where $0 \leq t \leq T$ and $t \leq \tau$. For each fixed $\tau \geq 0$, we have a real stochastic process

$$
\begin{aligned}
P^\tau : \quad [0, \tau] \times \Omega \quad &\rightarrow \quad \mathbb{R} \\
(t, \omega) \quad &\longmapsto \quad P_t^\tau(\omega)
\end{aligned}
$$

verifying $P_\tau^\tau = 1$, $P_t^\tau > 0$, and that $\dfrac{\partial \ln P_t^\tau}{\partial \tau}$ exists for all t.

We define now the variables that characterize the TSIR.

Definition 1.1 *The **instantaneous forward interest rate** at time t with time to maturity $x = \tau - t$ is defined as $f_t^x = -\frac{\partial}{\partial x} \ln P_t^{t+x}$.*

*The **spot interest rate** (or **short-term interest rate**), at time t, r_t, is defined as*

$$r_t = f_t^0 \tag{1.1}$$

∎

It is important to highlight we are using different parameterizations for bond prices and forward rates. Bond prices are based on the index τ that indicates the maturity day because the no-arbitrage condition will be applied to tradable instruments with fixed maturity day τ (Santa-Clara and Sornette (2001)). Additionally, for forward rates, we use the Musiela (1993) parameterization based on the time to maturity $x = \tau - t$. The reason for this is that the stochastic string model considers the forward curve as a continuous curve, subject at each time t to stochastic shocks along its length, parameterized by the distance to the origin, x.

Finally, we assume the existence of a *banking account process* whose rate of return is the short-term risk-free interest rate, r_t, continuously compounded. The value at time t of a initial balance B_0 invested in this account is given by the solution of the differential equation

$$\frac{dB_t}{B_t} = r_t dt \tag{1.2}$$

We present now some results, related to martingale representation and to the correlation theory of random fields, that will be used later. These results were presented and proved in Protter (2004) and Adler (1981),

Definition 1.2 *Let X be a local martingale and let H be a predictable process. An equation of the type*

$$[X, X]_t - t = \int_0^t H_s dX_s$$

*is known as the **Emery's structure equation** and can be written in differential form as*

$$d[X, X]_t = dt + H_t dX_t$$

∎

Theorem 1.1 *(Existence of solutions of the structure equation)*. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be Lipschiz continuous. Then, the Emery's structure equation*

$$[X, X]_t - t = \int_0^t \phi(X_{s-}) dX_s \tag{1.3}$$

has a weak solution where both $(X_t)_{t \geq 0}$ and $\left(\int_0^t \phi(X_{s-}) dX_s \right)_{t \geq 0}$ are local martingales. ∎

Theorem 1.2 *Let X be a (weak) solution of the equation (1.3). Then, the following holds:*

a) *$E\left\{X_t^2\right\} = E\left\{[X, X]_t\right\} = t$ and X is a square integrable martingale on compact time sets.*

b) *All the jumps of X are of the form $\Delta X_t = \phi(X_{t-})$.*

c) *X has continuous paths if and only if ϕ is identically 0, in which case, X is an standard Brownian motion.*

d) *If a stopping time T is a jump time of X, then, it is totally inaccessible.* ■

Theorem 1.3 (Emery's Uniqueness Theorem). *Let X be a local martingale solution of the structure equation $d[X, X]_t = dt + (\alpha + \beta X_{t-})\, dX_t$, $X_0 = x_0$. Then, X is unique in law. Moreover, X is a strong Markov process.* ■

Theorem 1.4 *Let $X(\mathbf{t})$ be a Gaussian random field with a continuous correlation function $\rho(\mathbf{t}, \mathbf{s}) = \rho(\alpha)$, $\alpha = \mathbf{t} - \mathbf{s}$. If, for some finite $c > 0$ and some $\epsilon > 0$, we have*

$$1 - \rho(\alpha) \leq \frac{c}{\mid \log \|\alpha\| \mid^{1+\epsilon}}, \ \forall \ \alpha \ with \ \|\alpha\| < 1 \tag{1.4}$$

then, the random field $X(\mathbf{t})$ will have continuous sample paths with probability one. ■

1.3 The Stochastic String Process

The source of randomness for our model is the infinite-dimensional stochastic process or random field Z_t^x (the *stochastic string process*), consisting of a continuum of stochastic processes Z^x indexed by time to maturity. Concretely,

$$
\begin{aligned}
Z: \quad \Delta^2 \times \Omega &\rightarrow \mathbb{R} \\
(t, x, \omega) &\mapsto Z_t^x(\omega)
\end{aligned}
$$

where $\Delta^2 = \left\{(t, x) \in \mathbb{R}^2 : 0 \leq t \leq T; \ x \geq 0\right\}$. The process Z verifies that the application Z^x defined for each fixed $x \geq 0$ by

$$
\begin{aligned}
Z^x: \quad [0, T] \times \Omega &\rightarrow \mathbb{R} \\
(t, \omega) &\mapsto Z_t^x(\omega)
\end{aligned} \tag{1.5}
$$

and the application Z_t defined for each fixed $t \in [0, T]$ by

$$
\begin{aligned}
Z_t: \quad [0, \infty) \times \Omega &\rightarrow \mathbb{R} \\
(x, \omega) &\mapsto Z_t^x(\omega)
\end{aligned} \tag{1.6}
$$

are stochastic processes. In addition, we assume that $Z^x = (Z_t^x)_{0 \leq t \leq T}$ and $Z_t = (Z_t^x)_{x \geq 0}$ are *adapted* processes.

Santa-Clara and Sornette (2001, p. 156)) impose five conditions on the stochastic string process that extend the properties of the Brownian motion to the infinite dimensional framework. Using our notation, these properties are the following:

1. Z_t^x is continuous in x for all t.

2. Z_t^x is continuous in t for all x.

3. Martingale property: $\mathbb{E}\left[dZ_t^x\right] = 0$, $\forall\, x$.

4. The variance of the increments is equal to the time change, $var\left[dZ_t^x\right] = dt$, $\forall\, x$.

5. The correlation of the increments, $corr\left[dZ_t^x, dZ_t^y\right]$, does not depend on t.

In the previous expressions and in the rest of the chapter, we will assume that the stochastic differential is taken with respect to the subindex of the processes.

To rewrite these conditions in terms of semimartingales, we follow Back (1991) and interpret the conditional covariance of the increments of the stochastic shock, $cov\left[dZ_t^x, dZ_t^y\right]$, as the stochastic differential of the *bracket process* or *quadratic covariation process*, $d\left[Z_\cdot^x, Z_\cdot^y\right]_t$. In a similar way, $var\left[dZ_t^x\right]$ will be rewritten as $d\left[Z_\cdot^x, Z_\cdot^x\right]_t$. To see that these interpretations are coherent with the usual notation in stochastic calculus, we consider the Itô processes

$$G_t = G_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

$$H_t = H_0 + \int_0^t \alpha_s ds + \int_0^t \theta_s dW_s$$

where W is an standard Brownian motion. Applying the Lévy's Theorem and differentiating, we have that $d\left[G_\cdot, H_\cdot\right]_t = \sigma_t \theta_t dt$, equating the value of $cov\left[dG_t, dH_t\right]$ obtained with Itô calculus. In a similar way, we obtain $d\left[G_\cdot, G_\cdot\right]_t = \sigma_t^2 dt = var\left(dG_t\right)$.

With the previous interpretation, we can write now the model conditions that will be the minimum ones to recover the results from Santa-Clara and Sornette (2001) in this more general framework.

Assumption 1.1 *The infinite-dimensional process Z verifies the following properties:*

a) *The stochastic processes Z^x and Z_t, defined in (1.5) and (1.6), are continuous for each $x \geq 0$ and for each $t \in [0, T]$, respectively.*

b) *The process Z^x is a martingale for each $x \geq 0$.*

c) *The process Z_t is differentiable for each $t \in [0, T]$.*

d) For each $x, y \geq 0$, it is verified that

$$d\left[Z_\cdot^x, Z_\cdot^y\right]_t = c_t^{x,y} dt \tag{1.7}$$

where $c_t^{x,y}$ is an admissible, continuous, and differentiable correlation function for each t. ∎

The property a) incorporates conditions 1 and 2 in Santa-Clara and Sornette (2001) and, jointly with Assumption 1.2 (to be stated later), allows us to work with continuous forward curves evolving continuously over time. The continuity is not a necessary condition for the absence of arbitrage but it is intuitively desirable and leads to a model with a higher analytical tractability. The property b) corresponds to property 3 in Santa-Clara and Sornette, representing the unpredictable aspect of the stochastic shocks and will allow us to consider, in Assumption 1.2, the instantaneous forward interest rate as a semimartingale. Property c) is a novelty with respect to Santa-Clara and Sornette and responds to the need of introducing in the model the different behavior of instantaneous forward rates with respect to each parameter: regular in time to maturity (forward curve) and irregular in calendar time (Cont (2005)).

Finally, condition d) replaces conditions 4 and 5 in Santa-Clara and Sornette as taking $x = y$ and using $c_t^{x,x} = 1$, $\forall\, t$ in such condition, we get

$$d\left[Z_\cdot^x, Z_\cdot^x\right]_t = dt \tag{1.8}$$

which is condition 4 in Santa-Clara and Sornette. However, unlike these authors, we allow $c_t^{x,y}$ to depend on t, implying that our model incorporates the correlation structures of Goldstein (2000) and Collin-Dufresne and Goldstein (2001, 2003).

Condition d) is the most important one in this Assumption because it introduces the differential feature of string models: the correlation function between shocks, $c_t^{x,y}$. To see this, taking into account the previous interpretation of the differential of the quadratic covariation process and working formally with condition d), we have

$$c_t^{x,y} = \frac{d\left[Z_\cdot^x, Z_\cdot^y\right]_t}{dt} = \frac{d\left[Z_\cdot^x, Z_\cdot^y\right]_t}{\sqrt{d\left[Z_\cdot^x, Z_\cdot^x\right]_t}\sqrt{d\left[Z_\cdot^y, Z_\cdot^y\right]_t}} = \frac{cov\left[dZ_t^x, dZ_t^y\right]}{\sqrt{var\left[dZ_t^x\right]}\sqrt{var\left[dZ_t^y\right]}} = corr\left[dZ_t^x, dZ_t^y\right]$$

Then, at time t, $c_t^{x,y}$ is the correlation function between the stochastic string shocks corresponding to times to maturity x and y.

Other important properties of the stochastic string shock arise from some of the preliminary results. Looking at definition 1.2, it is clear that (1.8) is an Emery's structure equation. Theorem 1.1 guarantees, for each x, the existence of the stochastic string process as (weak) solution of (1.8). Property c) in Theorem 1.2 says that, for each x, the process Z^x is a standard Brownian motion and Theorem 1.3 says that this process is a) the (unique in law) solution of (1.8) and b) a strong

Markov process, a result we already knew as it is a Brownian motion. Moreover, Z^x is a Gaussian process, a desirable property for simulations purposes (Santa-Clara and Sornette (2001), McDonald and Beard (2002), and Bester (2004)).

It is also interesting to highlight that, in Santa-Clara and Sornette (2001), the Markovian feature of the stochastic string process is a consequence of its conditions 3 to 5, while, in our model, this property is due to conditions b) and d) in Assumption 1.1 and it does not require the assumption of a time-independent correlation. Moreover, in their work, the normality of the process is a consequence of considering the stochastic string shocks as solutions of SPDE's. On the contrary, now, this property is consequence of the stochastic string process consisting of a continuum of correlated Brownian motions.

Another important consequence of the previous results for modeling the stochastic string is that any TSIR model whose correlation structure is given by (1.7) must have a stochastic string shock that is a Brownian motion in the direction t. Then, at least in the framework proposed in our Assumption 1.1, we can not use discontinuous stochastic processes in time for modeling stochastic strings.

1.3.1 Construction of Stochastic String Processes. Examples

Santa-Clara and Sornette (2001) propose to build examples of stochastic string processes as solutions to SPDE's. Their conditions on the shocks become restrictions on the Green function associated to the SPDE. This analysis leads to expressions for the shock dZ_t^x and for the correlation function $c_t^{x,y}$ in terms of the restricted Green function. The drawback of this approach is that the calculation of the correlation function is based on Z_t^x, that is unobservable.

These authors also show that it is possible to do the opposite: we can depart from an admissible correlation function and find always an associated stochastic string process such that the correlation function verifies the previously imposed conditions. This alternative has the advantage of departing from the correlation function between shocks, a magnitude with real economic meaning.

As mentioned earlier, we will follow this last procedure emphasizing the role of the correlation function and propose a scheme based on the correlation theory of random fields as alternative to the SPDE's theory. To this aim, we will restrict in some way the conditions imposed in the model although maintaining enough generality to nest previous models.

Let $c^{x,y} = corr\left[dZ_t^x, dZ_t^y\right]$ be an admissible, continuous and differentiable correlation function that is independent of t, isotropic $c^{x,y} = \rho\left(\alpha\right)$, $\alpha = |x - y|$, and verifying that $\frac{d^2}{d\alpha^2}\rho\left(\alpha\right)$ exists and is finite in $\alpha = 0$. In the random field theory, given any function $m : \mathbb{R}^d \to \mathbb{R}$ and any positive semidefinite function $R : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, it is always possible to find a (unique) d-dimensional Gaussian random field for which m and R are, respectively, its mean and covariance function.

Consider the two-dimensional Gaussian random field Z_t^x with $\mathbb{E}[Z_t^x] = 0$, $cov[Z_t^x, Z_s^y] = (t \wedge s) c^{x,y}$, and verifying additionally the condition (1.4). With the previous considerations, the random field Z_t^x verifies the conditions of Assumption 1.1 and is therefore a stochastic string process.

To gain some intuition, we illustrate the previous way of building stochastic string processes with two important examples previously analyzed in the literature:

1. **Ornstein-Uhlenbeck Sheet**

   Consider the random field U_t^x determined by the conditions $\mathbb{E}[U_t^x] = 0$ and $cov[U_t^x, U_s^y] = (t \wedge s) e^{-\kappa|x-y|}$, $\kappa > 0$. Then, the correlation structure between shocks is given by $c^{x,y} = e^{-\kappa|x-y|}$. This random field can be expressed formally as (Santa-Clara and Sornette (2001, p. 168))

   $$U_t^x = e^{-\kappa x} \int_{y=0}^{e^{2\kappa x}} \int_{v=0}^{t} dy\, dv\, \eta(v, y)$$

   where η is a two-dimensional Gaussian white noise. This process has two fundamental features so that it is used by Kennedy (1994), Goldstein (2000), Longstaff and Schwartz (2001), Santa-Clara and Sornette (2001), and Bester (2004), among others:

   (a) The correlation between shocks is perfect for a given time to maturity, and decays exponentially for different times to maturity, a property consistent with the empirical evidence.

   (b) The parameter κ allows a smooth transition between the extreme cases of a unique Brownian motion for all the times to maturity ($\kappa \to 0$) and a continuum of independent Brownian motions ($\kappa \to \infty$).

   The main drawback of this random field is that, although $c^{x,y} = e^{-\kappa|x-y|}$ is admissible, isotropic, and verifies the condition (1.4), it does not generate a differentiable process (Abrahamsen (1997)) and, then, it does not verify condition c) in Assumption 1.1.

2. **Integrated Ornstein-Uhlenbeck Sheet**

   Goldstein (2000) and Santa-Clara and Sornette (2001) solve this problem integrating the Ornstein-Uhlenbeck sheet and building the random field

   $$V_t^x = \kappa\sqrt{2}e^{-\kappa x} \int_{y=0}^{x} dy\, e^{\kappa y} U_t^y, \quad \kappa > 0$$

   that verifies $\mathbb{E}[V_t^x] = 0$ and $cov[V_t^x, V_s^y] = (t \wedge s)(1 + \kappa|x-y|) e^{-\kappa|x-y|}$. Then, the correlation structure between shocks is given by $c^{x,y} = (1 + \kappa|x-y|) e^{-\kappa|x-y|}$ and it can be shown that the integrated Ornstein-Uhlenbeck sheet satisfies all the conditions of the stochastic string process.

1.4 The Term Structure of Interest Rates

Once we know the properties of the source of randomness in our model, the stochastic string shock, we will describe the dynamics of the different variables that characterize the term structure of interest rates and the corresponding no-arbitrage condition.

1.4.1 The Discounted Price Process

We start by modeling the evolution of the instantaneous forward interest rate.

Assumption 1.2 *For each fixed time to maturity $x \geq 0$, the dynamics of the instantaneous forward interest rate f_t^x is given by the following stochastic differential equation*

$$df_t^x = \alpha_t^x dt + \sigma_t^x dZ_t^x \tag{1.9}$$

or, in integral form,

$$f_t^x = f_0^x + A_t^x + M_t^x \tag{1.10}$$

with

$$A_t^x = \int_{v=0}^t dv \alpha_v^x, \quad M_t^x = \int_{v=0}^t dZ_v^x \sigma_v^x \tag{1.11}$$

where, for each x, α^x and $\sigma^x > 0$ are continuous adapted stochastic processes and, for each t, α_t and $\sigma_t > 0$ are continuous and differentiable adapted stochastic processes. Moreover, for each t, we get

$$\mathbb{E}\left[\int_{y=0}^\infty dy \, |M_t^y|\right] < \infty \tag{1.12}$$

∎

The previous assumption allows us to say that the process for the instantaneous forward interest rate f^x is a continuous semimartingale for each x with A^x a finite variation process and M^x a local martingale.

Note that, as Z_t^x is a Brownian motion for each fixed x, the only difference of (1.9) with the dynamics of the forward rate in a one-factor HJM model is in the dependence of x on the stochastic shock. This dependence implies that, in our model, each point in the forward curve is affected by its own shock, in contrast to the one-factor HJM models where the same shock affects all the points in this curve.

The dependence of the stochastic shock on x will also affect to the information available in the TSIR at each time, contained in the filtration $\mathbb{F}$. In a one-factor HJM model, this filtration is generated by the (only) Brownian motion that drives the dynamics of the instantaneous forward rates for any maturity. We follow Kennedy (1994) and De Donno (2004) and make the following assumption.

Assumption 1.3 *The available information is that contained in all the forward curves observed until the current time, that is,*

$$\mathcal{F}_t = \sigma\left\{Z_s^x : 0 \leq s \leq t,\, x \geq 0\right\}, \quad t \leq T \tag{1.13}$$

$\blacksquare$

The price at time t of a bond maturing at τ is $P_t^\tau = \exp\{X_t^\tau\}$ where

$$X_t^\tau = -\int_0^{\tau-t} dy\, f_t^y \tag{1.14}$$

It can be shown that A_t^x and M_t^x are continuous in x, so X_t^τ is well defined as a path-by-path Riemann integral.

Using (1.10)-(1.11), interchanging the order of integration and splitting integrals, we get

$$X_t^\tau = -\int_{y=0}^\tau dy\, f_0^y + \int_{y=\tau-t}^\tau dy\, f_0^y - \int_{v=0}^t \int_{y=0}^{\tau-v} dv\, dy\, \alpha_v^y - \int_{v=0}^t \int_{y=0}^{\tau-v} dZ_v^y dy\, \sigma_v^y$$
$$+ \int_{v=0}^t \int_{y=\tau-t}^{\tau-v} dv\, dy\, \alpha_v^y + \int_{v=0}^t \int_{y=\tau-t}^{\tau-v} dZ_v^y dy\, \sigma_v^y$$

The double integrals that include dZ_v^y should be interpreted as

$$\int_{y=0}^x \int_{v=0}^t dy\, dZ_v^y \sigma_v^y = \lim_{n\to\infty} \sum_{\substack{y_k,y_{k+1}\in\Pi_n \\ y_k \leq j_k \leq y_{k+1}}} \left[\int_{v=0}^t dZ_v^{j_k} \sigma_v^{j_k}\right] (y_{k+1} - y_k)$$

where Π_n is an increasing sequence of partitions in $[0,x]$ and

$$\int_{v=0}^t \int_{y=0}^x dZ_v^y dy\, \sigma_v^y = \lim_{m\to\infty} \sum_{T_i^m,T_{i+1}^m \in \Lambda_m} \int_{y=0}^x dy\, \sigma_{T_i^m}^y \left(Z_t^{y,T_{i+1}^m} - Z_t^{y,T_i^m}\right).$$

Here, $Z_t^{y,T} = Z_{t\wedge T}^y$ is the process Z_t^y stopped at T, Λ_n is a sequence of random partitions that tends to the identity (Protter (2004, Theorem 21, p. 64)) and the convergence is intended in ucp sense. It is not difficult to show that the two previous integrals are identical, allowing us to change the order of integration.

Using (1.11) and (1.14), applying a change of variable, reordering terms, and changing again the order of integration, we have

$$X_t^\tau = X_0^\tau + \int_{w=0}^t dw\, f_w^{\tau-w} - \int_{v=0}^t \int_{y=0}^{\tau-v} dv\, dy\, \alpha_v^y - \int_{v=0}^t \int_{y=0}^{\tau-v} dZ_v^y dy\, \sigma_v^y \tag{1.15}$$

or, in differential form,

$$d_t X_t^\tau = \left[f_t^{\tau-t} - \int_{y=0}^{\tau-t} dy\, \alpha_t^y\right] dt - \int_{y=0}^{\tau-t} dZ_t^y dy\, \sigma_t^y \tag{1.16}$$

which is the same expression as equation (22) in Santa-Clara and Sornette (2001).

Moreover, the process X^τ satisfies the following lemma.

Lemma 1.1 X^τ *is a continuous semimartingale for all* τ. ∎

Proof. See the appendix. ∎

Applying this lemma, we obtain the next theorem.

Theorem 1.5 *Under the real probability measure, the dynamics of the discounted price process is given by*

$$\frac{d_t\left(P_t^\tau B_t^{-1}\right)}{P_t^\tau B_t^{-1}} = \left[-r_t + f_t^{\tau-t} - \int_{y=0}^{\tau-t} dy\alpha_t^y + \frac{1}{2}\int_{x=0}^{\tau-t}\int_{y=0}^{\tau-t} dxdyc_t^{x,y}\sigma_t^x\sigma_t^y\right]dt - \int_{y=0}^{\tau-t} dZ_t^y dy\sigma_t^y \quad (1.17)$$

∎

Proof. See the appendix. ∎

Once obtained the dynamics of the fundamental variables, the next subsection presents the no-arbitrage condition in our model.

1.4.2 The No-Arbitrage Condition

In the HJM models, the application of the no-arbitrage principle implies the *drift condition* or *no-arbitrage condition* that says that the drift of the instantaneous forward interest rates can be expressed in terms of the volatility of these rates and the market risk premium.

In our model, we will find that there is also a no-arbitrage condition that generalizes that in the HJM model and that, as a novelty, includes the correlation function between shocks, $c_t^{x,y}$.

As in Harrison and Pliska (1981), Geman *et al.* (1995), De Donno (2004), and Hamza *et al.* (2005), the next assumption guarantees the absence of arbitrage.

Assumption 1.4 *There exists a probability measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$ *such that the discounted price process for any asset is a martingale under* $\mathbb{Q}$*. The probability measure* $\mathbb{Q}$ *is known as the equivalent martingale measure.* ∎

Applying the Radon-Nikodym Theorem (Klebaner (2005, p. 271)), the existence of a probability measure $\mathbb{Q}$ that is absolutely continuous with respect to $\mathbb{P}$ guarantees the existence of a random variable $\eta > 0$ such that $\mathbb{E}_{\mathbb{P}}[\eta] = 1$ and $\mathbb{Q}(A) = \int_A \eta d\mathbb{P}$ for any measurable set A. The random variable η is known as the *Radon-Nikodym derivative* and is usually denoted as $\eta = \frac{d\mathbb{Q}}{d\mathbb{P}}$.

Assumption 1.5 *Every local martingale* M *can be expressed as*

$$M_t = M_0 + \int_{v=0}^{t}\int_{u=0}^{\infty} dZ_v^u du j_v^u$$

where, for each u*,* j_v^u *is a predictable stochastic process.* ∎

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In fact, this assumption is a martingale representation property. Similar properties are assumed in De Donno (2004) and in Björk *et al.* (2002).

Consider the martingale $\eta_t = E_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t\right]$. If we apply the Assumption 1.5 to this process, we get

$$\eta_t = 1 + \int_0^t dN_s \eta_s \tag{1.18}$$

with

$$N_t = -\int_{v=0}^t \int_{u=0}^\infty dZ_v^u du \lambda_v^u$$

and where, for each u, we have introduced a new process λ^u given by $\lambda_v^u = -j_v^u/\eta_v$.

For the following, we need some regularity conditions on the process λ^u.

Assumption 1.6 *For each $t \in [0,T]$ it is verified that*

i) $\mathbb{E}\left[\int_{u=0}^\infty du \left|\int_{v=0}^t dZ_v^u du \lambda_v^u\right|\right] < \infty$

ii) $\mathbb{E}\left[e^{\frac{1}{2}[N,N]_t}\right] < \infty$ *(Novikov condition)* ∎

Expression (1.18) is a stochastic differential equation whose unique solution can be expressed as $\eta_t = \varepsilon(N)_t$, where ε is the stochastic exponential. By condition i) in Assumption 1.6 we obtain that the process N is a martingale (the proof is similar to that of Lemma 1.1). The Novikov condition guarantees that the stochastic exponential of the process N defines a martingale as η.

The next lemma shows the relationship between the stochastic string shocks under the new measure $\mathbb{Q}$ and under the original measure $\mathbb{P}$ (see Santa-Clara and Sornette (2001)).

Lemma 1.2 *Given $d\widetilde{Z}_t^y$, the stochastic string shock under the new measure $\mathbb{Q}$, and dZ_t^y, the shock under the original measure $\mathbb{P}$, we have*

$$d\widetilde{Z}_t^y = dZ_t^y + dt \int_{u=0}^\infty du c_t^{u,y} \lambda_t^u \tag{1.19}$$

∎

Proof. See the appendix. ∎

Applying this lemma we can obtain the following result.

Theorem 1.6 *The no-arbitrage condition in the stochastic string model is given by*

$$\alpha_t^x = \frac{\partial f_t^x}{\partial x} + \sigma_t^x \left[\int_{y=0}^x dy c_t^{x,y} \sigma_t^y + \int_{y=0}^\infty dy c_t^{x,y} \lambda_t^y\right] \tag{1.20}$$

∎

Proof. See the appendix. ■

Substituting this no-arbitrage condition in the dynamics of the instantaneous forward interest rates (see (1.9)) and using (1.19), we obtain the no-arbitrage dynamics of the model under the equivalent martingale measure

$$df_t^x = \left[\frac{\partial f_t^x}{\partial x} + \sigma_t^x \int_{y=0}^x dy \tilde{c}_t^{x,y} \sigma_t^y \right] dt + \sigma_t^x d\tilde{Z}_t^x \tag{1.21}$$

This expression is the same as equation (34) in Santa-Clara and Sornette (2001) and it is used in that paper to simulate the evolution of the forward curve. With changes corresponding to the different parameterization, the drift of this equation appears as the no-arbitrage condition in Goldstein (2000) and De Donno (2004).

In a similar way, using (1.19), (1.61), and (1.63) (see the appendix), we can write the dynamics of the bond return as

$$\frac{dP_t^\tau}{P_t^\tau} = r_t dt - \int_{y=0}^{\tau-t} d\tilde{Z}_t^y dy \sigma_t^y \tag{1.22}$$

As expected, this expression reflects that, under the measure $\mathbb{Q}$, the expected bond return is equal to the risk-free interest rate.

The stochastic string model generalizes the one-factor HJM model in the Musiela parameterization. To see it, if we assume stochastic shocks that are the same for all times to maturity, we have that $Z_t^x = W_t, \forall x$. Then, we obtain a perfect correlation between shocks and we can write (1.19) as $d\tilde{W}_t = dW_t + \lambda_t dt$ where $\lambda_t \equiv \int_0^\infty du \lambda_t^u$. Moreover, by applying Girsanov Theorem, we can identify λ_t with the *market risk premium* in the HJM model and λ_t^u with a *specific market risk premium* associated to a bond with time to maturity u. In this case, the no-arbitrage condition (1.20) becomes

$$\alpha_t^x = \frac{\partial f_t^x}{\partial x} + \sigma_t^x \left[\int_0^x dy \sigma_t^y + \lambda_t \right]$$

that coincides with the drift condition in the HJM model written in terms of the Musiela parameterization.

1.4.3 The Short-Term Interest Rate

Replacing the integral version of (1.21) in the definition of the short-term interest rate (see (1.1)), we have

$$r_t = f_t^0 = r_0 + \int_{v=0}^t dv \left. \frac{\partial f_v^x}{\partial x} \right|_{x=0} + \int_{v=0}^t d\tilde{Z}_v^0 \sigma_v^0 \tag{1.23}$$

or, in differential form,

$$dr_t = \left. \frac{\partial f_t^x}{\partial x} \right|_{x=0} dt + \sigma_t^0 d\tilde{Z}_t^0 \tag{1.24}$$

where we see that, similarly to one-factor HJM models, the drift of the process for the short-term interest rate is equal to the slope of the forward curve at the origin (Jeffrey (1995, p. 626)).

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Applying the Leibnitz's rule (see Abramowitz and Stegun (1972)) to compute the drift of (1.24) and using $c_t^{x,x} = 1$, $\forall t$, we obtain the next theorem.

Theorem 1.7 *Under the measure $\mathbb{Q}$, the no-arbitrage dynamics of the short-term interest rate is given by*

$$dr_t = \left[\left. \frac{\partial f_0^x}{\partial x} \right|_{x=0} + \int_{u=0}^t du \left[\left. \frac{\partial^2 f_u^x}{\partial x^2} \right|_{x=0} + \left(\sigma_u^0 \right)^2 \right] + \int_{u=0}^t d_u \widetilde{Z}_u^0 \left. \frac{\partial \sigma_u^x}{\partial x} \right|_{x=0} \right] dt + \sigma_t^0 d\widetilde{Z}_t^0 \qquad (1.25)$$

∎

Expression (1.25) coincides with the one-factor HJM dynamics of r_t in the Musiela parameterization as, in this case, $d\widetilde{Z}_t^0 = d\widetilde{W}_t$. This is because $c_t^{x,y}$ appears in the expression of f_t^x, contributing for all $y \leq x$. As $r_t = f_t^0$, the contribution of the term that includes $c_t^{x,y}$ is null. This fact is important as it indicates that the stochastic string modeling does not introduce any change with respect to the HJM model in terms of the properties that depend only on the dynamics of the short-term interest rate.

Thus, for example, we can see that the drift of (1.25) includes stochastic shocks in the interval $[0, t]$. This fact introduces a path-dependence in the dynamics of the short-term interest rates that implies a non-Markovian behavior, a drawback inherited from the HJM model. As a consequence, the pricing of zero-coupon bonds by using the expression

$$P_t^\tau = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^\tau r_s ds} | \mathcal{F}_t \right] \qquad (1.26)$$

is specially difficult because the fact that r_t is non-Markovian complicates its simulation and we can not apply the Feynman-Kac Theorem to obtain the corresponding pricing equation. The following section suggests a possible solution to this issue.

1.5 Bond Pricing in the Markovian Case

Following Inui and Kijima (1998), in this section we provide the sufficient conditions to obtain a Markovian model. In this case, we will be able to apply the Feynman-Kac Theorem to obtain a PDE for the bond price.

In our model, we assume that the volatility of the instantaneous forward interest rate satisfies the condition

$$\frac{\partial \sigma_t^x}{\partial x} = -\varkappa(x) \sigma_t^x \qquad (1.27)$$

or, equivalently, $\sigma_t^x = \sigma_t^0(r_t) e^{-\int_0^x \varkappa(u)du}$ where $\varkappa(x)$ is a deterministic function. Note that the short rate volatility depends only on the short rate itself. Substituting (1.27) in (1.25) and using (1.23),

we get

$$dr_t = \left[\varkappa\left(0\right) \left[\int_{u=0}^{t} du \left. \frac{\partial f_u^x}{\partial x}\right|_{x=0} - \left(r_t - r_0\right) \right] + \left. \frac{\partial f_0^x}{\partial x}\right|_{x=0} + \int_{u=0}^{t} du \left. \frac{\partial^2 f_u^x}{\partial x^2}\right|_{x=0} + \phi_t \right] dt + \sigma_t^0 \left(r_t\right) d\widetilde{Z}_t^0$$

(1.28)

where $\phi_t = \int_0^t du \left[\sigma_u^0 \left(r_u\right)\right]^2$ or, in differential form,

$$d\phi_t = \left[\sigma_t^0 \left(r_t\right)\right]^2 dt$$

(1.29)

Then, the condition (1.27) guarantees the existence of the two-dimensional Markovian process $X_t = \left(r_t, \phi_t\right)$ determined by equations (1.28)-(1.29). Expression (1.28) is formally analogous to expression (13) in Inui and Kijima (1998), except for the derivatives with respect to time to maturity that appear as a consequence of the Musiela parameterization.

Unfortunately, unlike in Inui and Kijima (1998), in general, we can not obtain an explicit expression for the bond price as a function of the state variables. This is because, in our model, the dependence on x precludes writing the stochastic integral of the integral version of (1.21) as a function of the short-term interest rate. However, as we shall see now, we can obtain a bond pricing equation.

The existence of the two-dimensional Markovian process X_t allows us to write (1.26) as

$$P_t^\tau = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^\tau r(s, X_s) ds} | \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^\tau r(s, X_s) ds} | X_t \right] = P_t^\tau \left(X_t\right) = P_t^\tau \left(r_t, \phi_t\right)$$

Assuming the necessary regularity conditions (see, for instance, Duffie (1996, Appendix E)), we can apply the Feynman-Kac Theorem to obtain the price at time t of a bond with maturity τ, $P_t^\tau \left(r, \phi\right)$, as the solution to the PDE

$$\frac{\partial P_t^\tau \left(r, \phi\right)}{\partial t} + \frac{\partial P_t^\tau \left(r, \phi\right)}{\partial r} b_t \left(r, \phi\right) + \frac{\partial P_t^\tau \left(r, \phi\right)}{\partial \phi} \left[\sigma_t^0 \left(r\right)\right]^2 + \frac{1}{2} \left[\sigma_t^0 \left(r\right)\right]^2 \frac{\partial^2 P_t^\tau \left(r, \phi\right)}{\partial r^2} - r P_t^\tau \left(r, \phi\right) = 0$$

subject to the terminal condition $P_\tau^\tau \left(r, \phi\right) = 1$, where b_t is the drift of expression (1.28).

If the volatility of the short-term interest rate is deterministic, i.e., $\sigma_t^0 \left(r\right) = \sigma_t^0$, the only state variable is the short-term interest rate, r_t, and then the PDE becomes

$$\frac{\partial P_t^\tau \left(r\right)}{\partial t} + \frac{\partial P_t^\tau \left(r\right)}{\partial r} b_t + \frac{1}{2} \left(\sigma_t^0\right)^2 \frac{\partial^2 P_t^\tau \left(r\right)}{\partial r^2} - r P_t^\tau \left(r\right) = 0$$

(1.30)

subject to the terminal condition $P_\tau^\tau \left(r\right) = 1$.

We can analyze two particular cases:

1. If we make $\sigma_t^0 = \sigma > 0$ and use (1.28), we obtain that the dynamics of the short-term interest rate is given by $dr_t = \left[\omega\left(t\right) - \varkappa r_t\right] dt + \sigma d\widetilde{Z}_t^0$ where

$$\varkappa\left(0\right) = \varkappa$$

$$\omega\left(t\right) = \varkappa \left[\int_{u=0}^{t} du \left. \frac{\partial f_u^x}{\partial x}\right|_{x=0} + r_0 \right] + \left. \frac{\partial f_0^x}{\partial x}\right|_{x=0} + \int_{u=0}^{t} du \left. \frac{\partial^2 f_u^x}{\partial x^2}\right|_{x=0} + \sigma^2 t$$

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With our parameterization, this model is analogous to that presented in Hull and White (1990). In this case, the analytic expression for the price of a zero-coupon bond is given by

$$P_t^\tau(r) = e^{A(t,\tau) - B(t,\tau)r} \tag{1.31}$$

where

$$A(t,\tau) = \frac{1}{2}\sigma^2 \int_{s=t}^{\tau} ds\, B^2(s,\tau) - \int_{s=t}^{\tau} ds\, \omega(s)\, B(s,\tau)$$

$$B(t,\tau) = \frac{1 - e^{-\varkappa(\tau - t)}}{\varkappa}$$

2. If we make $\varkappa(x) = 0$, $\forall x$ and $\sigma_t^0 = \sigma > 0$, we can write the dynamics of the short-term interest rate (1.28) in the form $dr_t = \theta(t)\, dt + \sigma d\widetilde{Z}_t^0$ with $\theta(t) = \left.\frac{\partial f_0^x}{\partial x}\right|_{x=0} + \int_{u=0}^{t} du \left.\frac{\partial^2 f_u^x}{\partial x^2}\right|_{x=0} + \sigma^2 t$ that corresponds, with the changes in the drift due to the parameterization, to the continuous-time version of the Ho and Lee (1986) model. As the previous particular case, the solution of the PDE (1.30) is given by (1.31) where we have now

$$A(t,\tau) = \frac{1}{6}\sigma^2 B^3(t,\tau) - \int_{s=t}^{\tau} ds\, \theta(s) B(s,\tau)$$

$$B(t,\tau) = \tau - t$$

1.6   Pricing European Bond Options

As indicated by Santa-Clara and Sornette, in general, in the framework we have presented we can not apply the Feynman-Kac Theorem to obtain a PDE to price bond derivatives. This is because even in the previous Markovian models, and except in some particular cases, we can not express the bond price in terms of the state variables.

However, we can price derivatives choosing adequately a numeraire to express the price in terms of the mathematical expectations with respect to other equivalent probability measures (Geman *et al.* (1995)). In our case, the numeraire will be P_t^τ, the current price of a bond maturing at time τ.

Applying the general result for changes of numeraire (see Geman *et al.* (1995) or Shreve (2004)), there exists a probability measure $\mathbb{Q}^\tau$, named τ-*forward measure*, equivalent to $\mathbb{Q}$, whose Radon-Nikodym derivative is given by

$$\frac{d\mathbb{Q}^\tau}{d\mathbb{Q}} = \frac{1}{P_0^\tau e^{\int_0^\tau ds\, r_s}} \tag{1.32}$$

such that the prices of any derivative, discounted by P_t^τ, are martingales with respect to $\mathbb{Q}^\tau$. Taking conditional expectations in (1.32) and applying the martingale condition with respect to $\mathbb{Q}$, we have

$$\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{d\mathbb{Q}^\tau}{d\mathbb{Q}}\right|\mathcal{F}_t\right] = \frac{P_t^\tau}{P_0^\tau e^{\int_0^t ds\, r_s}}, \quad t \leq \tau$$

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Note that, for each maturity τ, we will have the corresponding numeraire and probability measure.

If we now apply the procedure implemented in Subsection 1.4.2 to obtain $\mathbb{Q}$ to the new measure $\mathbb{Q}^\tau$, we obtain that there exists, for each time to maturity u, a process h_t^u verifying

$$d\widetilde{Z}_t^y(\tau) = d\widetilde{Z}_t^y + dt \int_{u=0}^{\infty} duc_t^{u,y} h_t^u \tag{1.33}$$

where $d\widetilde{Z}_t^y(\tau)$ represents the stochastic string shock under the τ-forward measure. To identify the process h_t^u it is enough to require that $P_t^\upsilon (P_t^\tau)^{-1}$ be a martingale under $\mathbb{Q}^\tau$ for any maturity υ. Applying Itô's rule for continuous semimartingales in the dynamics of the bond return (see (1.22)), we obtain that

$$\frac{d (P_t^\tau)^{-1}}{(P_t^\tau)^{-1}} = \left[-r_t + \int_{x=0}^{\tau-t} \int_{y=0}^{\tau-t} dxdyc_t^{x,y}\sigma_t^x\sigma_t^y \right] dt + \int_{y=0}^{\tau-t} d\widetilde{Z}_t^y dy\sigma_t^y \tag{1.34}$$

Using the rule of integration by parts (Protter (2004, p. 68)) in (1.22) and (1.34), we obtain

$$
\begin{aligned}
\frac{d (P_t^\upsilon / P_t^\tau)}{P_t^\upsilon / P_t^\tau} &= dt \int_{x=0}^{\tau-t} \int_{y=0}^{\tau-t} dxdyc_t^{x,y}\sigma_t^x\sigma_t^y + \int_{y=0}^{\tau-t} d\widetilde{Z}_t^y dy\sigma_t^y \\
&\quad - dt \int_{x=0}^{\tau-t} \int_{y=0}^{\upsilon-t} dxdyc_t^{x,y}\sigma_t^x\sigma_t^y - \int_{y=0}^{\upsilon-t} d\widetilde{Z}_t^y dy\sigma_t^y
\end{aligned}
\tag{1.35}
$$

Substituting (1.33) in (1.35), we see that the discounted price process, $P_t^\upsilon (P_t^\tau)^{-1}$, is a martingale under $\mathbb{Q}^\tau$ if and only if

$$h_t^u = \begin{cases} \sigma_t^u & \text{if} \quad 0 \leq u \leq \tau - t \\ 0 & \text{if} \quad u > \tau - t \end{cases}$$

from which we obtain the relationship between the stochastic shocks under the measures $\mathbb{Q}$ and $\mathbb{Q}^\tau$

$$d\widetilde{Z}_t^y(\tau) = d\widetilde{Z}_t^y + dt \int_{u=0}^{\tau-t} duc_t^{u,y}\sigma_t^u$$

expression that is analogous to that obtained in Goldstein (2000). Substituting this expression in (1.22), we obtain the dynamics of the bond price under the τ-forward measure

$$\frac{dP_t^\upsilon}{P_t^\upsilon} = \left[r_t + \int_{y=0}^{\upsilon-t} \int_{u=0}^{\tau-t} dyduc_t^{u,y}\sigma_t^u\sigma_t^y \right] dt - \int_{y=0}^{\upsilon-t} d\widetilde{Z}_t^y(\tau) dy\sigma_t^y \tag{1.36}$$

Using these results, in a similar way to Collin-Dufresne and Goldstein (2001), we can price European call options on zero-coupon bonds, as stated in the next theorem.

Theorem 1.8 *In the Gaussian case, the price at time t of an European call option with exercise date μ and strike K on a zero-coupon bond maturing at time $\tau \geq \mu$ is given by*

$$C(t,\mu,\tau) = P_t^\tau \Phi (d_1) - K P_t^\mu \Phi (d_2) \tag{1.37}$$

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where $\Phi(\cdot)$ denotes the distribution function of a standard normal variable, with

$$d_1 = \frac{\ln\left(\frac{P_t^\tau}{KP_t^\mu}\right) + \frac{1}{2}\Omega(t, \mu, \tau)}{\sqrt{\Omega(t, \mu, \tau)}}, \quad d_2 = d_1 - \sqrt{\Omega(t, \mu, \tau)}$$

and

$$\Omega(t, \mu, \tau) = \int_{s=t}^{\mu} ds \left[\int_{y=\mu-s}^{\tau-s} \int_{u=\mu-s}^{\tau-s} dy du c_s^{u,y} \sigma_s^u \sigma_s^y \right] \tag{1.38}$$

■

Proof. See the appendix. ■

Remark 1.1 *In contrast to the pricing of zero-coupon bonds, the modeling of stochastic strings affects the pricing of interest rate derivatives because the correlation function $c_t^{u,y}$ appears in the expression of the variance $\Omega(t, \mu, \tau)$.*

■

The next remark illustrates the consistency of the results we have obtained.

Remark 1.2 *If $c_t^{u,y} = 1$, $\forall t$, then expression (1.38) simplifies to*

$$\Omega(t, \mu, \tau) = \int_{s=t}^{\mu} ds \left[\int_{y=\mu-s}^{\tau-s} dy \sigma_s^y \right]^2$$

■

With the pertinent changes in the parameterization, this result coincides with that obtained for one-factor Gaussian HJM models (see, for instance, Björk (2004), p. 364 or Musiela and Rutkowski (2006)).

1.7 Relationship with Other Models

In this section we analyze the relationship between our model and other related TSIR models. Our analysis excludes Goldstein (2000) and Santa-Clara and Sornette (2001) as these papers are the most inspiring for our model and we have previously mentioned the relationships between these papers and our this chapter.

1.7.1 Kennedy (1994)

This paper models the instantaneous forward interest rate at time s for maturity t as $\hat{f}_s^t = \mu_s^t + X_s^t$ where μ_s^t is a deterministic and continuous function and X_s^t is a Gaussian random field, centered and continuous, with covariance

$$cov\left(X_{s_1}^{t_1}, X_{s_2}^{t_2}\right) = c\left(s_1 \wedge s_2, t_1, t_2\right) \tag{1.39}$$

The main result consists of showing the equivalence between the martingale condition for the discounted bond price, the risk-neutral bond pricing and the no-arbitrage condition

$$\mu_s^t = \mu_0^t + \int_{v=0}^{t} dvc\,(s \wedge v, v, t)\,, \quad \mu_0^t = \hat{f}_0^t \tag{1.40}$$

Additionally, this author obtains a closed-form expression for the price at time s of a call option with maturity t and strike K on a bond maturing at time $t + \Delta$

$$C\,(s, t, t + \Delta) = P_s^{t+\Delta} \Phi\left(d_1^{Kennedy}\right) - P_s^t K \Phi\left(d_2^{Kennedy}\right)$$

with

$$d_1^{Kennedy} = \frac{\ln\left(\frac{P_s^{t+\Delta}}{KP_s^t}\right)}{\sigma\,(s, t, \Delta)} + \frac{\sigma\,(s, t, \Delta)}{2}, \quad d_2^{Kennedy} = d_1^{Kennedy} - \sigma\,(s, t, \Delta)$$

and

$$\sigma^2\,(s, t, \Delta) = 2 \int_{u=t}^{t+\Delta} \int_{v=t}^{u} dudv\,[c\,(t, u, v) - c\,(s, u, v)] \tag{1.41}$$

To see the relationship between the results in Kennedy (1994) and our results, we need to know the role that the function $c\,(s, t_1, t_2)$ would play if the Kennedy (1994) model could be included in our model. To this aim, if we make

$$X_s^t \equiv \int_{v=0}^{s} d\widetilde{Z}_v^{t-v} \sigma_v^{t-v}$$

and apply (1.39), we obtain

$$c\,(s, t_1, t_2) = cov\left(X_s^{t_1}, X_s^{t_2}\right) = \int_{u=0}^{s} duc_u^{t_1-u, t_2-u} \sigma_u^{t_1-u} \sigma_u^{t_2-u} \tag{1.42}$$

Substituting this expression in the integral in (1.40), we get

$$\int_{v=0}^{t} dvc\,(s \wedge v, v, t) = \int_{v=0}^{t} \int_{u=0}^{s \wedge v} dvduc_u^{v-u, t-u} \sigma_u^{v-u} \sigma_u^{t-u} = \int_{u=0}^{s} \int_{v=u}^{t} dudvc_u^{v-u, t-u} \sigma_u^{v-u} \sigma_u^{t-u}$$

$$= \int_{u=0}^{s} du\sigma_u^{t-u} \int_{v=u}^{t} dvc_u^{v-u, t-u} \sigma_u^{v-u}$$

that corresponds to the drift in the expression (1.21), with the pertinent changes due to the different parameterization.

Substituting (1.42) in (1.41) and applying that the integrand is symmetric, it is easy to check that $\sigma^2\,(s, t, \Delta) = \Omega\,(s, t, t + \triangle)$ where $\Omega\,(\cdot, \cdot, \cdot)$ is given by (1.38).

Then, if the Kennedy (1994) model satisfies our conditions, its pricing expression for European bond call options coincides with that obtained in our Gaussian case.

1.7.2 Chu (1996)

This author models the bond return by using a SDE driven by a infinite-dimensional Brownian motion. With our notation, this SDE can be written as

$$\frac{dP_s^T}{P_s^T} = M\left(s, T, P\right) ds + \sum_{k=0}^{\infty} \sigma^{(k)}\left(s, T, P\right) dW_s^{(k)}$$

Working under absence of arbitrage and using functional derivatives, Chu obtains a PDE to price bond derivatives and solves this equation for different derivatives whose pay-offs are homogeneous of degree one with deterministic parameters (Gaussian model).

The price at time s of an European call option with maturity T_0 and strike K on a bond maturing at $T > T_0$ is given by

$$C\left(s, P_s^{T_0}, P_s^T\right) = P_s^T \Phi\left(d_1^{Chu}\right) - P_s^{T_0} K \Phi\left(d_2^{Chu}\right) \tag{1.43}$$

with

$$d_1^{Chu} = \frac{\ln\left(\frac{P_s^T}{KP_s^{T_0}}\right) + \frac{1}{2}\xi^2\left(s, T_0, T\right)}{\xi\left(s, T_0, T\right)}, \quad d_2^{Chu} = d_1^{Chu} - \xi\left(s, T_0, T\right)$$

and

$$\xi^2\left(s, T_0, T\right) = \int_{t=s}^{T_0} dt \left[Z\left(t, T, T\right) + Z\left(t, T_0, T_0\right) - 2Z\left(t, T_0, T\right)\right] \tag{1.44}$$

where $Z\left(s, u_1, u_2\right)$ is defined by $cov\left[dP_s^{u_1}, dP_s^{u_2}\right] \equiv Z\left(s, u_1, u_2\right) P_s^{u_1} P_s^{u_2} ds$ and satisfies

$$Z\left(s, u_1, u_2\right) = \sum_{k=0}^{\infty} \sigma^{(k)}\left(s, u_1\right) \sigma^{(k)}\left(s, u_2\right) \tag{1.45}$$

To determine the relationship between the expression (1.43) and our expression (1.37), it is enough to rewrite $Z\left(s, u_1, u_2\right)$ in our framework. To this aim, using (1.22), we have

$$cov\left[dP_s^{u_1}, dP_s^{u_2}\right] = d\left[P_\cdot^{u_1}, P_\cdot^{u_2}\right]_s = \left[\int_{x=0}^{u_1-s} \int_{y=0}^{u_2-s} dx dy c_s^{x,y} \sigma_s^x \sigma_s^y\right] P_s^{u_1} P_s^{u_2} ds$$

Then, we can write the identity

$$Z\left(s, u_1, u_2\right) = \int_{x=0}^{u_1-s} \int_{y=0}^{u_2-s} dx dy c_s^{x,y} \sigma_s^x \sigma_s^y \tag{1.46}$$

that, replaced in (1.44), leads to $\xi^2\left(s, T_0, T\right) = \Omega\left(s, T_0, T\right)$ where $\Omega\left(\cdot, \cdot, \cdot\right)$ is given by (1.38).

Then, the price of an European call option in our model coincides with that obtained in Chu (1996), showing that the one-factor string model is equivalent (at least for European option pricing purposes) to a HJM model with infinite factors. Moreover, if we make $\sigma^{(k)} = 0$, $k > n$ in (1.45), we obtain that the multi-factor HJM models can be seen as particular cases of our framework.

1.7.3 Collin-Dufresne and Goldstein (2001)

These authors introduce the so called *generalized affine models* which consists of a random field model with stochastic volatility. In short, under the measure $\mathbb{Q}$, the dynamics of the bond price obeys the SDE

$$\frac{dP_s^\tau}{P_s^\tau} = r_s ds - \hat{\sigma}_s^\tau \sqrt{\Sigma(s)} d\widehat{\widetilde{Z}}_s^\tau \tag{1.47}$$

where $\hat{\sigma}_s^\tau$ is an arbitrary deterministic function and the *volatility state variable*, $\Sigma(s)$, evolves as

$$d\Sigma(s) = \varkappa (\theta - \Sigma) ds + \vartheta \sqrt{\Sigma} d\widetilde{W}_s \tag{1.48}$$

where the Brownian field $d\widehat{\widetilde{Z}}_s^\tau$ and the Brownian motion $d\widetilde{W}_s$ are mutually independent.

The (deterministic) correlation structure of the Brownian field is given by $d\widehat{\widetilde{Z}}_s^\tau d\widehat{\widetilde{Z}}_s^\nu \equiv \hat{c}_s^{\tau,\nu} ds$. If we compare expressions (1.22) and (1.47), we find that our formulation matches that in Collin-Dufresne and Goldstein (2001) if we include the factor $\sqrt{\Sigma(s)}$ described by (1.48) in our volatility. Moreover, we can make the formal identification

$$\int_{y=0}^{\tau-s} d\widetilde{Z}_s^y dy \sigma_s^y \equiv \hat{\sigma}_s^\tau d\widehat{\widetilde{Z}}_s^\tau \tag{1.49}$$

Collin-Dufresne and Goldstein (2001) also present an example of comparative statistics with a price dynamics as that given by (1.47) but without the stochastic factor $\sqrt{\Sigma(s)}$ (Gaussian case), which allows them to price analytically caplets. This expression coincides with (1.37) but with a different volatility, which is given by

$$\overline{\Omega}(t, \mu, \tau) = \int_{s=t}^{\mu} ds \left[(\hat{\sigma}_s^\tau)^2 - 2\hat{\sigma}_s^\tau \hat{\sigma}_s^\mu \hat{c}_s^{\tau,\mu} + (\hat{\sigma}_s^\mu)^2 \right]$$

This volatility is the same as our volatility if we substitute the identity (1.49) in the expression for $\ln(P_\mu^\tau)$ (see (1.67) in the appendix).

1.7.4 Baaquie (2001)

This author uses tools extracted from the quantum field theory to model the TSIR. The dynamics of the instantaneous forward interest rate is given by the SPDE

$$\frac{\partial \hat{f}_t^s}{\partial t} = \hat{\alpha}_t^s + \hat{\sigma}_t^s A_t^s$$

where A_t^s is a two-dimensional *quantum field*. This expression coincides formally with our expression (1.9) in Assumption 1.2, making $A_t^s dt \equiv dZ_t^{s-t}$.

Working with the Feynman path integral in a Gaussian model, this author obtains the no-arbitrage condition

$$\hat{\alpha}_t^s = \hat{\sigma}_t^s \int_{s'=t}^{s} ds' \widehat{D}\left(s, s'; t, T_{FR}\right) \sigma_t^{s'} \tag{1.50}$$

where $\widehat{D}(s, s'; t, T_{FR})$ is the *propagator* and where T_{FR} is the maximum duration for forward rates.

This author also obtains that the price of a European call option is given by a Black-Scholes-type formula with variance

$$q^2(t, \mu, \tau) = \int_{r=t}^{\mu} \int_{s=\mu}^{\tau} \int_{s'=\mu}^{\tau} dr ds ds' \hat{\sigma}_r^s \widehat{D}(s, s'; r, T_{FR}) \hat{\sigma}_r^{s'} \tag{1.51}$$

Baaquie and Srikant (2004) obtain the following expression for the propagator, that depends just on $\theta = s - t$, $\theta' = s' - t$ and T_{FR}

$$D(\theta, \theta'; T_{FR}) = \widehat{D}(s, s'; t, T_{FR})$$
$$= \frac{\mu}{2 \sinh \mu T_{FR}} \left[\cosh \mu \left(T_{FR} - |\theta - \theta'| \right) + \cosh \mu \left[T_{FR} - (\theta + \theta') \right] \right] \tag{1.52}$$

where μ is a parameter that quantifies the intensity of the fluctuations in the direction of time to maturity. Using the equalities $2(t \wedge s) = t + s - |s - t|$, $2(t \vee s) = t + s + |s - t|$ and hyperbolic identities, the expression (1.52) becomes

$$D(\theta, \theta'; T_{FR}) = \mu \left[\coth(\mu T_{FR}) \cosh \mu (\theta \vee \theta') - \sinh \mu (\theta \vee \theta') \right] \cosh \mu (\theta \wedge \theta') \tag{1.53}$$

If $\mu \to 0$, we recover the one-factor HJM model, that is, $D(\theta, \theta'; T_{FR}) = 1$. To see this, applying a Taylor expansion to the hyperbolic functions leads to $\lim_{\mu \to 0} D(\theta, \theta'; T_{FR}) = 1/T_{FR}$. The extra factor T_{FR} is irrelevant because of the freedom in scaling existing in $\hat{\sigma}$ and $\widehat{D}$.

We now want to see under which conditions the Baaquie (2001) model is equivalent to our model. To this aim, looking at the expressions (1.50)-(1.51), we see that it is enough to require that the propagator be independent of T_{FR} and be an admissible correlation function.

For the dependence on T_{FR}, departing from the expression (1.53), it is easy to see that

$$\lim_{T_{FR} \to \infty} D(\theta, \theta'; T_{FR}) \equiv D(\theta, \theta') = \mu e^{-\mu(\theta \vee \theta')} \cosh \mu (\theta \wedge \theta')$$

but $D(\theta, \theta')$ defined in this way is not a correlation function because $D(\theta, \theta) \neq 1$. This problem is solved making use of the previously mentioned freedom in scaling, defining the *normalized propagator*

$$\widetilde{D}(\theta, \theta') = \frac{D(\theta, \theta')}{\sqrt{D(\theta, \theta)} \sqrt{D(\theta', \theta')}}$$

that is a correlation function. With this propagator (and the corresponding rescaling of $\hat{\sigma}$), the expressions (1.50) and (1.51) are equivalent to (1.21) and (1.38). Then, when $T_{FR} \to \infty$, the Baaquie (2001) model coincides with our Gaussian case. In this case, our correlation function between shocks coincides with the normalized propagator, with the concrete value

$$c^{x,y} = \widetilde{D}(x, y) = \sqrt{\frac{e^{-\mu(x \vee y)} \cosh \mu (x \wedge y)}{e^{-\mu(x \wedge y)} \cosh \mu (x \vee y)}}$$

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To finish the analysis of the Baaquie (2001) model and going back to the stochastic string interpretation of the forward curve, we can interpret $1/\mu^2$ as the string "tension". Thus, we can interpret the one-factor HJM model ($\mu \to 0$) as describing a string with infinite tension (rigid). The shocks that affect at each time the string move it in the same magnitude in all its points. Baaquie (2002) states that $1/\hat{\sigma}_t^s$ can be interpreted as the "effective mass" or inertia of the string. Then, the stochastic string modeling can introduce a flexibility in the string allowing for different (although correlated) changes in different points.

1.7.5 Kimmel (2004)

This author develops a random field model in which the volatilities of the innovations in the instantaneous forward interest rate are state-dependent. This dependence is achieved introducing latent variables that follow a joint diffusion process. This assumption is able to keep a key property in models with deterministic volatilities, such as Kennedy (1994): each instantaneous forward rate belongs to a finite-dimensional diffusion.

In short, the instantaneous forward rate at time t for maturity τ follows the dynamics

$$d\hat{f}_t^\tau = \mu_f (X_t, \tau - t)\, dt + \sigma_{fZ}^T (X_t, \tau - t)\, dZ_t + \int_{s=0}^\infty d_s W_{s,t}\, \sigma_{fW} (X_t, \tau - t, s) \qquad (1.54)$$

where μ_f and σ_{fW} are scalar-valued functions, σ_{fZ} is an $N \times 1$ vector-valued function, $W_{s,t}$ is a two-parameter Brownian sheet, independent of Z_t, and X_t is an $N \times 1$ vector that includes latent variables and obeys the diffusion process $dX_t = \mu_X (X_t)\, dt + \sigma_X (X_t)\, dZ_t$ where Z_t is an N-dimensional standard Brownian motion, μ_X is an $N \times 1$ vector-valued function, and σ_X is an $N \times N$ matrix-valued function. This model with N latent variables is known as $LV - N$ model.

Departing from (1.54) we obtain

$$cov \left[d\hat{f}_t^{\tau_1}, d\hat{f}_t^{\tau_2} \right] = \left[\sigma_{fZ}^T (X_t, \tau_1 - t)\, \sigma_{fZ} (X_t, \tau_2 - t) + C_{WW}(X_t, \tau_1 - t, \tau_2 - t) \right] dt \qquad (1.55)$$

where

$$C_{WW}(X_t, \tau_1 - t, \tau_2 - t) = \int_{s=0}^\infty ds \left[\sigma_{fW} (X_t, \tau_1 - t, s)\, \sigma_{fW} (X_t, \tau_2 - t, s) \right] \qquad (1.56)$$

Note that, if $N = 0$, we have $\mu_X = \sigma_X = \sigma_{fZ} \equiv 0$, $\mu_f (X_t, \tau - t) \equiv \mu_f (t, \tau - t)$ and $\sigma_{fW}(X_t, \tau - t, s) \equiv \sigma_{fW}(t, \tau - t, s)$. Then, for a $LV - 0$ model, (1.55) becomes $cov \left[d\hat{f}_t^{\tau_1}, d\hat{f}_t^{\tau_2} \right] = C_{WW}(t, \tau_1 - t, \tau_2 - t)dt$. Departing from Assumptions 1.1 and 1.2, it is easy to check that we have $cov \left[df_t^x, df_t^y \right] = c_t^{x,y} \sigma_t^x \sigma_t^y dt$. Then, our model is equivalent to a $LV - 0$ model if we make the identification $C_{WW} (t, x, y) = c_t^{x,y} \sigma_t^x \sigma_t^y$. Moreover, considering (1.46) and the function defined in (1.56),

we have

$$Z(s, u_1, u_2) = \int_{x=0}^{u_1-s} \int_{y=0}^{u_2-s} dx dy C_{WW}(s, x, y)$$
$$= \int_{k=0}^{\infty} dk \left[\int_{x=0}^{u_1-s} dx \sigma_{fW}(s, x, k) \right] \left[\int_{y=0}^{u_2-s} dy \sigma_{fW}(s, y, k) \right]$$

Comparing this expression with expression (1.45), we can interpret the Kimmel (2004) model not as a stochastic string model but as a HJM one with a continuum of factors that depend on the parameter k.

1.8   Conclusions

Santa-Clara and Sornette (2001) developed a model of the TSIR that generalizes the HJM framework, by introducing a different stochastic shock for each point of the forward curve. Surprisingly, despite the importance of this work, few papers have proposed extensions or further analysis based on this model, possibly because of two deficiencies. The first one is the lack of a stochastic integration theory that provides rigor to the model. The second is the absence of concrete results to be used in the proposed framework.

This chapter tries to palliate these deficiencies. We reformulate the stochastic string model of Santa-Clara and Sornette using continuous semimartingales and we obtain some new results, such as a) an expression for the dynamics of the short-term interest rate, b) a differential bond pricing equation, and c) a closed-form expression for the price of an European call option on a zero-coupon bond.

We think that, in this way, the stochastic string framework can be more useful to model the TSIR. There are many ways in which this work can be extended, such as to go deeper in the relationship with the multi-factor HJM models, to obtain pricing expressions for other derivative assets, to analyze the consistency of the model in relation with the different families of parametric forward curves, or to apply the model to the relative pricing of *caps* and *swaptions*. All these topics are object of ongoing research.

1.9 Appendix

Proof of Lemma 1.1

Continuity of the process is easily obtained from its definition (see (1.14)).

To show that the process X^τ is a semimartingale for each τ, we will start by expressing (1.15) in the most convenient form $X_t^\tau = X_0^\tau + \overline{A}_t^\tau - \overline{M}_t^\tau$ where

$$\overline{A}_t^\tau \equiv \int_{v=0}^t dv \left[f_v^{\tau-v} - \int_{y=0}^{\tau-v} dy \alpha_v^y \right], \quad \overline{M}_t^\tau \equiv \int_{v=0}^t \int_{y=0}^{\tau-v} dZ_v^y dy \sigma_v^y \tag{1.57}$$

Then, it is enough to show that, for all τ, $\overline{A}^\tau$ is a finite variation process and $\overline{M}_t^\tau$ is a local martingale.

By definition, $\overline{A}_t^\tau$ has continuous and differentiable paths for all τ. Moreover, by the continuity of the processes, we have

$$\int_0^t ds \left| \frac{d\overline{A}_s^\tau}{ds} \right| = \int_0^t ds \left| f_s^{\tau-s} - \int_{y=0}^{\tau-s} dy \alpha_s^y \right| < \infty, \ \forall t$$

Applying Theorem 1.8 in Klebaner (2005), we have that $\overline{A}^\tau$ is a finite variation process for all τ.

We now show that $\overline{M}_t^\tau$ is a martingale for all τ. First, applying (1.57), changing the order of integration and using the condition (1.12) in Assumption 1.2, we have

$$\mathbb{E}\left[|\overline{M}_t^\tau|\right] = \mathbb{E}\left[\left|\int_{v=0}^t \int_{y=0}^{\tau-v} dZ_v^y dy \sigma_v^y\right|\right] = \mathbb{E}\left[\left|\int_{y=0}^\tau \int_{v=0}^{t\wedge(\tau-y)} dy dZ_v^y \sigma_v^y\right|\right]$$

$$= \mathbb{E}\left[\left|\int_0^\tau dy M_{t\wedge(\tau-y)}^y\right|\right] \leq \mathbb{E}\left[\int_0^\tau dy \left|M_{t\wedge(\tau-y)}^y\right|\right] < \infty \tag{1.58}$$

Moreover, for all τ and for all $s < t$, we have

$$\mathbb{E}\left[\overline{M}_t^\tau \,|\mathcal{F}_s\right] = \mathbb{E}\left[\int_{v=0}^t \int_{y=0}^{\tau-v} dZ_v^y dy \sigma_v^y \,|\mathcal{F}_s\right]$$

$$= \mathbb{E}\left[\int_{v=0}^s \int_{y=0}^{\tau-v} dZ_v^y dy \sigma_v^y \,|\mathcal{F}_s\right] + \mathbb{E}\left[\int_{v=s}^t \int_{y=0}^{\tau-v} dZ_v^y dy \sigma_v^y \,|\mathcal{F}_s\right]$$

$$= \int_{v=0}^s \int_{y=0}^{\tau-v} dZ_v^y dy \sigma_v^y = \overline{M}_s^\tau \tag{1.59}$$

where we have applied the martingale property of Z^y and the definition of $\mathcal{F}_s$ (see (1.13)). Now, using (1.58)-(1.59) we have that $\overline{M}_t^\tau$ is a martingale for all τ. ∎

Proof of Theorem 1.5

Using Lemma 1.1, we can apply Itô's formula for continuous semimartingales (Protter (2004, p. 81)) to $P_t^\tau = \exp\{X_t^\tau\}$, to obtain that P^τ is a semimartingale for all τ and

$$dP_t^\tau = P_t^\tau d_t X_t^\tau + \frac{1}{2} P_t^\tau d\left[X_\cdot^\tau, X_\cdot^\tau\right]_t \tag{1.60}$$

To compute the last term in this expression, taking into account the bilinearity of the bracket and using that, if X is a continuous semimartingale of finite variation, then $[X, Y]_t = 0$ for any semimartingale Y, departing from (1.15), it is easy to see that

$$[X_{\cdot}^{\tau}, X_{\cdot}^{\tau}]_t = \int_{s=0}^{t} \int_{x=0}^{\tau-s} \int_{y=0}^{\tau-s} d\left[Z_{\cdot}^x, Z_{\cdot}^y\right]_s dx dy \sigma_s^x \sigma_s^y$$

or, in differential form,

$$d[X_{\cdot}^{\tau}, X_{\cdot}^{\tau}]_t = \int_{x=0}^{\tau-t} \int_{y=0}^{\tau-t} d\left[Z_{\cdot}^x, Z_{\cdot}^y\right]_t dx dy \sigma_t^x \sigma_t^y$$

Substituting this expression in (1.60) and using (1.7) and (1.16), we get

$$\frac{dP_t^{\tau}}{P_t^{\tau}} = \left[f_t^{\tau-t} - \int_{y=0}^{\tau-t} dy \alpha_t^y + \frac{1}{2} \int_{x=0}^{\tau-t} \int_{y=0}^{\tau-t} dx dy c_t^{x,y} \sigma_t^x \sigma_t^y\right] dt - \int_{y=0}^{\tau-t} dZ_t^y dy \sigma_t^y \qquad (1.61)$$

Given the bank account process B_t, the discounted price process, $P_t^{\tau} B_t^{-1}$, is a semimartingale. Moreover, using (1.2) and taking into account that B^{-1} is a continuous semimartingale of finite variation as r is continuous (and, then, bounded) in $[0, T]$, we have

$$d\left(P_t^{\tau} B_t^{-1}\right) = -r_t P_t^{\tau} B_t^{-1} dt + B_t^{-1} dP_t^{\tau} \qquad (1.62)$$

Substituting (1.61) in (1.62), we obtain the dynamics for the discounted price process under the real probability measure. ∎

Proof of Lemma 1.2

Applying the Girsanov-Meyer Theorem (Protter (2004, p. 132)) to the integral version of expression (1.17), we have

$$-\int_{s=0}^{t} \int_{y=0}^{\tau-s} d\widetilde{Z}_s^y dy \sigma_s^y = -\int_{s=0}^{t} \int_{y=0}^{\tau-s} dZ_s^y dy \sigma_s^y - \int_{s=0}^{t} \frac{1}{\eta_s} d\left[\eta_{\cdot}, -\int_{p=0}^{\cdot} \int_{y=0}^{\tau-p} dZ_p^y dy \sigma_p^y\right]_s$$

that, using (1.18) and some algebra, can be rewritten as

$$-\int_{s=0}^{t} \int_{y=0}^{\tau-s} d\widetilde{Z}_s^y dy \sigma_s^y = -\int_{s=0}^{t} \int_{y=0}^{\tau-s} dZ_s^y dy \sigma_s^y - \int_{s=0}^{t} \int_{u=0}^{\infty} \int_{y=0}^{\tau-s} ds du dy c_s^{u,y} \lambda_s^u \sigma_s^y$$

Then, we obtain the formal equality

$$d\widetilde{Z}_t^y = dZ_t^y + dt \int_{u=0}^{\infty} du c_t^{u,y} \lambda_t^u$$

∎

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Proof of Theorem 1.6

Applying Lemma 1.2 to expression (1.17), we obtain the dynamics for the discounted price process under the measure $\mathbb{Q}$

$$\frac{d\left(P_t^\tau B_t^{-1}\right)}{P_t^\tau B_t^{-1}} = \left[-r_t + f_t^{\tau-t} - \int_{y=0}^{\tau-t} dy\alpha_t^y + \frac{1}{2}\int_{x=0}^{\tau-t}\int_{y=0}^{\tau-t} dxdy c_t^{x,y}\sigma_t^x\sigma_t^y \right.$$
$$\left. + \int_{u=0}^{\infty}\int_{y=0}^{\tau-t} dudy c_y^{u,y}\lambda_t^u\sigma_t^y\right]dt - \int_{y=0}^{\tau-t} dy d\widetilde{Z}_t^y\sigma_t^y$$

As $\mathbb{Q}$ is an equivalent martingale measure, the drift of this process must be null, that is,

$$-r_t + f_t^x - \int_{y=0}^{x} dy\alpha_t^y + \frac{1}{2}\int_{u=0}^{x}\int_{y=0}^{x} dudy c_t^{u,y}\sigma_t^u\sigma_t^y + \int_{u=0}^{\infty}\int_{y=0}^{x} dudy c_t^{u,y}\lambda_t^u\sigma_t^y = 0, \ x = \tau - t \quad (1.63)$$

Using that the second integrand is symmetric and differentiating (1.63) with respect to x, we obtain the no-arbitrage condition in the stochastic string model. $\blacksquare$

Proof of Theorem 1.8

The price at time t of this option is given by

$$C(t,\mu,\tau) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^\mu dsr_s}\left(P_\mu^\tau - K\right)\mathbf{1}_{\left(P_\mu^\tau > K\right)}|\mathcal{F}_t\right]$$

Applying that discounted prices are martingales under $\mathbb{Q}$, we have

$$C(t,\mu,\tau) = P_t^\tau\mathbb{E}^{\mathbb{Q}}\left[\frac{e^{-\int_t^\tau dsr_s}}{P_t^\tau}\mathbf{1}_{\left(P_\mu^\tau > K\right)}|\mathcal{F}_t\right] - KP_t^\mu\mathbb{E}^{\mathbb{Q}}\left[\frac{e^{-\int_t^\mu dsr_s}}{P_t^\mu}\mathbf{1}_{\left(P_\mu^\tau > K\right)}|\mathcal{F}_t\right]$$

Applying the Bayes formula (Klebaner (2005, Theorem 10.8)) with the Radon-Nikodym derivative (see expression (1.32)) and using the monotonicity of the logarithmic function, it is easy to check that

$$C(t,\mu,\tau) = P_t^\tau\mathbb{E}^{\mathbb{Q}^\tau}\left[\mathbf{1}_{\left(\ln\left(P_\mu^\tau\right) > \ln(K)\right)}|\mathcal{F}_t\right] - KP_t^\mu\mathbb{E}^{\mathbb{Q}^\mu}\left[\mathbf{1}_{\left(\ln\left(P_\mu^\tau\right) > \ln(K)\right)}|\mathcal{F}_t\right] \quad (1.64)$$

where $\mathbb{E}^{\mathbb{Q}^\omega}$, $\omega = \mu, \tau$ represents the expectation with respect to the ω-forward measure.

Departing from expression (1.36) and using the stochastic exponential, we can write the price at time μ of a bond maturing at time τ, with respect to the ω-forward measure as

$$P_\mu^\tau = P_t^\tau\exp\left\{\int_{s=t}^{\mu} ds\left[r_s + \int_{y=0}^{\tau-s}\int_{u=0}^{\omega-s} dyduc_s^{u,y}\sigma_s^u\sigma_s^y - \frac{1}{2}\int_{y=0}^{\tau-s}\int_{u=0}^{\tau-s} dyduc_s^{u,y}\sigma_s^u\sigma_s^y\right]\right.$$
$$\left. - \int_{s=t}^{\mu}\int_{y=0}^{\tau-s} d\widetilde{Z}_s^y(\omega)dy\sigma_s^y\right\} \quad (1.65)$$

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If we make $\tau = \mu$ in this expression, we have

$$P_\mu^\mu = 1 = P_t^\mu \exp\left\{\int_{s=t}^\mu ds\left[r_s + \int_{y=0}^{\mu-s}\int_{u=0}^{\omega-s} dyduc_s^{u,y}\sigma_s^u\sigma_s^y - \frac{1}{2}\int_{y=0}^{\mu-s}\int_{u=0}^{\mu-s} dyduc_s^{u,y}\sigma_s^u\sigma_s^y\right]\right.$$
$$\left. - \int_{s=t}^\mu\int_{y=0}^{\mu-s} d\widetilde{Z}_s^y(\omega)\,dy\sigma_s^y\right\} \tag{1.66}$$

Using (1.65)-(1.66), we get

$$\ln\left(P_\mu^\tau\right) = \int_{s=t}^\mu ds\left[\int_{y=0}^{\tau-s}\int_{u=0}^{\omega-s} dyduc_s^{u,y}\sigma_s^u\sigma_s^y - \frac{1}{2}\int_{y=0}^{\tau-s}\int_{u=0}^{\tau-s} dyduc_s^{u,y}\sigma_s^u\sigma_s^y\right.$$
$$\left. - \int_{y=0}^{\mu-s}\int_{u=0}^{\omega-s} dyduc_s^{u,y}\sigma_s^u\sigma_s^y + \frac{1}{2}\int_{y=0}^{\mu-s}\int_{u=0}^{\mu-s} dyduc_s^{u,y}\sigma_s^u\sigma_s^y\right]$$
$$+ \ln\left(\frac{P_t^\tau}{P_t^\mu}\right) - \int_{s=t}^\mu\int_{y=0}^{\tau-s} d\widetilde{Z}_s^y(\omega)\,dy\sigma_s^y + \int_{s=t}^\mu\int_{y=0}^{\mu-s} d\widetilde{Z}_s^y(\omega)\,dy\sigma_s^y \tag{1.67}$$

If we consider now that σ_t^x are $c_t^{x,y}$ deterministic, then $\ln\left(P_\mu^\tau\right)$ is Gaussian with variance

$$\Omega(t,\mu,\tau) = var_t\left[\ln\left(P_\mu^\tau\right)\right] = \int_{s=t}^\mu ds\left[\int_{y=\mu-s}^{\tau-s}\int_{u=\mu-s}^{\tau-s} dyduc_s^{u,y}\sigma_s^u\sigma_s^y\right]$$

that is independent of the measure ω.

Making $\omega = \mu, \tau$ in (1.67) and taking conditional expectations under $\mathbb{Q}^\mu$ and $\mathbb{Q}^\tau$, we get

$$\mathbb{E}^{\mathbb{Q}^\mu}\left[\ln\left(P_\mu^\tau\right)|\mathcal{F}_t\right] = \ln\left(\frac{P_t^\tau}{P_t^\mu}\right) - \frac{1}{2}\Omega(t,\mu,\tau) \tag{1.68}$$

$$\mathbb{E}^{\mathbb{Q}^\tau}\left[\ln\left(P_\mu^\tau\right)|\mathcal{F}_t\right] = \ln\left(\frac{P_t^\tau}{P_t^\mu}\right) + \frac{1}{2}\Omega(t,\mu,\tau) \tag{1.69}$$

By standard calculus with the Gaussian distribution, replacing expressions (1.68)-(1.69) in (1.64) we obtain the mentioned result. ∎

Chapter 2

The Stochastic String Model as a Unifying Theory of the Term Structure of Interest Rates

2.1   Introduction

Since its publication, the HJM model has been an important reference for modeling the TSIR. This is proven by the large number of papers published related to their work. Some papers develop particular one- and multi-factor HJM models (Amin and Morton (1994), Jeffrey (1995), Ritchken and Sankarasubramanian (1995), Inui and Kijima (1998), Mercurio and Moraleda (2000), Driessen *et al.* (2003), and Kuo and Paxson (2006)). Other works focus on obtaining HJM volatilities using Principal Component Analysis (PCA) (Driessen *et al.* (2003) and Angelini and Herzel (2005)). Some papers study the consistency of HJM models with families of parametric forward curves (Björk and Christensen (1999), Filipovic (2001), La Chioma and Piccoli (2007), and Roncoroni *et al.* (2010)). Finally, there are papers that value derivatives with HJM models (Brace and Musiela (1994a, b), Mercurio and Moraleda (2000), and Kramin *et al.* (2008)). Recently, a number of papers have concentrated on extending or generalizing the original HJM model. More concretely, these papers propose *infinite-dimensional models* of the TSIR to develop infinite-dimensional HJM models and random field or stochastic string models.

One of the greatest achievements of the HJM modeling was to provide a framework that encompassed most of the previous models. However, from today's perspective, the topics mentioned above are seen as unrelated. Moreover, there are some unsolved issues with the HJM framework:

1. The number of factors in the multi-factor HJM model is determined arbitrarily. Some authors (Amin and Morton (1994), Moraleda and Vorst (1997), Driessen *et al.* (2003), Kuo and Paxson (2006)) choose the functional form and the number of volatilities and then make comparisons with different alternatives to try to determine, from an empirical point of view, which model provides the best fit to market derivative prices. We will see that this procedure has weak

theoretical foundation since it is more appropriate to consider the number of factors as the desired degree of approximation in a model with infinite factors.

2. The model does not explain the relationship between the volatilities and the factors obtained from PCA. Some multi-factor HJM models obtain the first factors from PCA and use them as parametric proxies for the volatilities. The main disadvantage of this approach is that the relationship between the factors from PCA (obtained from a discrete data set) and the HJM volatilities (with a continuum of maturities) is unknown. Another related disadvantage is that the orthogonality in $\mathbb{R}^n$ of the empirical factors for a PCA with n data points has no counterpart in a similar condition for volatilities. These drawbacks disappear when we use stochastic string models.

3. The exact relationship among multi-factor HJM models, infinite-dimensional HJM models, and stochastic string models is unknown. Although some authors have shown that there are some links among these models (Pang (1999), Goldstein (2000)), to the best of our knowledge, none of them has offered a full explanation of the relaionship between these models. In this chapter, we will show that the stochastic string model is able to nest the other two models.

4. The HJM model is not consistent with Nelson and Siegel (1987) (NS) models. NS models are among the most commonly used parametric models when estimating forward curves (BIS, (2005)). However, the HJM model is not consistent with them as, given an initial NS forward curve, the dynamic of HJM rates does not guarantee that future forward rates still belong to the parametric family of NS models. We will show that stochastic string models are consistent with a family of NS generalized models. Nonetheless, this consistency will be obtained at the cost of substituting a no-arbitrage model by a sequence of models with an approximated no-arbitrage condition. This sequence converges, under some conditions, to the no-arbitrage stochastic string model when we increase the number of factors.

The main objective of this chapter is to propose the stochastic string model of Santa-Clara and Sornette (2001), as reformulated in the previous chapter, as a theory that unifies and explains the most important models of the TSIR developed so far. This chapter makes the following contributions to the literature:

a) Obtains an orthogonality condition for the volatilities in HJM models (both multi-factor and infinite-dimensional).

b) Interprets multi-factor HJM models as approximations to a full (infinite-dimensional) stochastic string model.

c) Proposes a covariance function between forward curve shocks with good properties with respect to the HJM modeling, the PCA, and the consistency problem.

d) Reinterprets the stochastic string modeling as a PCA with a continuum of sample points. As a result, this chapter shows that the use of volatilities obtained from stylized parametric functions coming from PCA is correct.

e) Provides a result of consistency based on Hilbert spaces. In more detail, we show that there is consistency between a sequence of models with arbitrage opportunities and a sequence of extended NS families. We also show that, under some conditions, this sequence of models with arbitrage opportunities converges to a no-arbitrage model.

f) Provides a theorem for the valuation of interest rate options with a payoff function that is homogeneous of degree one. This new result allows for the valuation of European bond options, caps, and swaptions.

The rest of the chapter is organized as follows. Section 2.2 presents a classification of infinite-dimensional models of the TSIR and proposes an approximation to each member class. It also relates models of different classes. Section 2.3 provides the main mathematical result used in this chapter, Mercer's Theorem, and adapts it to the objectives of the chapter.

Section 2.4 analyzes, from the perspective of the stochastic string modeling, two related topics: the HJM model and the PCA. The instantaneous conditional covariance between forward curve shocks, obtained with Mercer's Theorem, will allow us to develop an infinite-dimensional HJM model as a particular case of the stochastic string model. The sorting in decreasing order of the eigenvalues in the Mercer Theorem will allow us to justify the reduction of the infinite-dimensional model to multi-factor models. It is also proposed an example of covariance function that will be very important for the rest of the chapter. This section ends up with the relationship between stochastic strings and the PCA.

Section 2.5 examines in depth the problem of consistency, using the theory of Hilbert spaces. We reformulate the definitions of Björk and Christensen (1999) to obtain sufficient consistency conditions (similar to the drift and volatility conditions of the paper just mentioned) that are valid for infinite-dimensional models. These conditions will turn necessary for finite-dimensional subspaces. We show that the model is inconsistent for a very general class of finite-dimensional manifolds, including NS, and that this inconsistency holds even after reducing the number of terms in the drift. This section concludes with the main results of consistency with approximated no-arbitrage opportunities and convergence to a no-arbitrage model. The reduction of these results to infinite-dimensional and multi-factor HJM models is also shown.

Section 2.6 provides a fundamental valuation theorem which gives a closed formula for the price of European options with payoff functions that are homogeneous of degree one. As an application, a European bond call option is priced, obtaining the same result as in Chu (1996) and in Chapter 1. Finally, Section 2.7 summarizes the main findings and concludes.

2.2   Infinite-Dimensional Models of the TSIR

Infinite-dimensional models solve some of the problems of multi-factor HJM models by introducing an infinite number of stochastic shocks simultaneously acting over the TSIR. Roncoroni *et al.* (2010) divide these models into infinite-dimensional HJM models and random field or stochastic string models.

Infinite-dimensional HJM models extend the original HJM model to the case of infinite-dimensional Brownian motion and modelize the dynamic of the forward curve with stochastic differential equations (SDEs) with values in Hilbert spaces (Da Prato and Zabczyk, (1992)). Examples of these models are Chu (1996), Filipovic (2001), Carmona and Tehranchi (2004), Aihara and Bagchi (2005), and Roncoroni *et al.* (2010).

Random field models or stochastic string models use at each time a continuum of stochastic shocks, one for each point of the forward curve. These shocks are imperfectly correlated in order to maintain the forward curve continuous. The dependence of the shocks in instantaneous forward rates on time to maturity is the main difference with infinite-dimensional HJM models. Random field and stochastic string models are used by Kennedy (1994, 1997), Goldstein (2000), Santa-Clara and Sornette (2001), and in the previous chapter, for example.[1]

In this chapter we show that infinite-dimensional (and, therefore, multi-factor) HJM models are particular cases of stochastic string models. More concretely, we are able to obtain the infinite-dimensional HJM models of Chu (1996) and Roncoroni *et al.* (2010) using a specific stochastic string process within the framework of the Chapter 1. Moreover, we provide an explicit relationship between the two mentioned models. We start by reviewing in some detail the models of Chu (1996) and our model of Chapter 1 and the relation between them.

2.2.1   The Infinite-Dimensional HJM Model of Chu (1996)

Chu models the dynamic of the bond return with the SDE

$$\frac{dP(s,T)}{P(s,T)} = M(s,T)\,ds - \sum_{k=0}^{\infty} \overline{\sigma}_{CHU}^{(k)}(s,T)\,dW_k(s) \tag{2.1}$$

---

[1] An analysis of the relationship between models can be found in Chapter 1.

where $P(s,T)$ is the price at time s of a bond with maturity T and $W_k(s), k = 0, 1, \ldots$ are independent Brownian motions. For a multi-factor HJM model we have that

$$\frac{dP(s,T)}{P(s,T)} = r(s)\,ds - \sum_{k=0}^{n} \left[\int_{v=s}^{T} dv\,\overline{\sigma}_{HJM}^{(k)}(s,v) \right] d\widetilde{W}_k(s)$$

where $\overline{\sigma}_{HJM}^{(k)}$ are the volatilities, $r(s)$ is the instantaneous short-term interest rate, and $\widetilde{W}_k(s)$ are Brownian motions under the equivalent martingale measure, $\mathbb{Q}$. Thus, we can interpret (2.1) as an infinite-dimensional HJM model written under the equivalent martingale measure. The reduction of the model of Chu (1996) to the $(n+1)$-factor HJM case can be obtained making $\overline{\sigma}_{HJM}^{(k)}(s,T) = \frac{\partial}{\partial T} \overline{\sigma}_{CHU}^{(k)}(s,T)$ for $k \leq n$ and the rest of volatilities equal to zero.

Working under no-arbitrage conditions and without using the martingale approximation, Chu obtains a partial differential equation for the valuation of bond derivatives as the Kolmogorov field equation for the bond price. She then solves this equation for the case in which the covariance function $Z(s,u_1,u_2)$ between bond returns, defined by

$$cov\left[dP(s,u_1), dP(s,u_2)\right] \equiv Z(s,u_1,u_2)\,P(s,u_1)\,P(s,u_2)\,ds \tag{2.2}$$

where

$$Z(s,u_1,u_2) = \sum_{k=0}^{\infty} \overline{\sigma}_{CHU}^{(k)}(s,u_1)\,\overline{\sigma}_{CHU}^{(k)}(s,u_2) \tag{2.3}$$

is deterministic (Gaussian model). We next reproduce his fundamental result with a different notation adapted to this chapter.

Theorem 2.1 *(Chu (1996)) Let us consider the dynamics (2.1) of the bond return and let us assume that the market is efficient. Moreover, let us suppose that*

a) *The final payoff of a contingent claim at time T_0 is given by $C[T_0, \mathbf{P}_{T_0}] = \max\left(\Phi\left(\mathbf{P}_{T_0}\right), 0\right)$ where $\mathbf{P}_{T_0} \equiv \left(P(T_0, T_0), P(T_0, T_1), \ldots, P(T_0, T_N)\right)$ with $0 \leq T_0 < T_1 < \cdots < T_N$ and $\Phi : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ is a homogeneous function of degree one.*

b) *The correlation matrix W with elements*

$$W_{ij}(s, T_0) = \frac{1}{b_i(s, T_0)\, b_j(s, T_0)} \int_{t=s}^{T_0} dt\, Z(t, T_i, T_j), \quad i, j = 0, 1, \ldots, N$$

is deterministic and non-singular, where $b_i^2(s, T_0) = \int_{t=s}^{T_0} dt\, Z(t, T_i, T_i)$.

Then, the price at time s of this contingent claim is given by

$$C[s, \mathbf{P}_s] = \int_{\mathbb{R}^{N+1}} d\boldsymbol{x}\, g(x_0, x_1, \ldots, x_N; W)$$
$$\times \left(\Phi\left[P(s, T_0)\, e^{b_0 x_0 - \frac{1}{2} b_0^2}, P(s, T_1)\, e^{b_1 x_1 - \frac{1}{2} b_1^2}, \ldots, P(s, T_N)\, e^{b_N x_N - \frac{1}{2} b_N^2} \right] \right)_+$$

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where

$$g(x_0, x_1, \ldots, x_N; W) = \frac{1}{\sqrt{(2\pi)^{N+1} |W|}} \exp\left(-\frac{1}{2} \sum_{i,j=0}^{N} x_i \left(W^{-1}\right)_{ij} x_j\right)$$

is the density function of a multivariate normal random variable. ∎

Chu (1996) applies this theorem to obtain valuation formulas for different European derivatives (standard bond options, compound options, swaps, swaptions, caps, and captions). For example, the price at time s of a European call option with maturity T_0 and strike K on a bond that matures at time $T > T_0$ is given by[2]

$$C\left(s, P\left(s, T_0\right), P\left(s, T\right)\right) = P\left(s, T\right) \mathcal{N}\left(d_1^{CHU}\right) - KP\left(s, T_0\right) \mathcal{N}\left(d_2^{CHU}\right) \tag{2.4}$$

with

$$d_1^{CHU} = \frac{\ln\left(\frac{P(s,T)}{KP(s,T_0)}\right) + \frac{1}{2}\xi^2\left(s, T_0, T\right)}{\xi\left(s, T_0, T\right)}, \quad d_2^{CHU} = d_1^{CHU} - \xi\left(s, T_0, T\right)$$

and

$$\xi^2\left(s, T_0, T\right) = \int_{t=s}^{T_0} dt \left[Z\left(t, T, T\right) + Z\left(t, T_0, T_0\right) - 2Z\left(t, T_0, T\right)\right] \tag{2.5}$$

Finally, given its importance for our work, we point out that, in the model of Chu (1996), the conditional covariance between the logarithms of bond prices is given by

$$\Delta_{ij}^{CHU}(s, T_0) \equiv cov\left[\ln P\left(T_0, T_i\right), \ln P\left(T_0, T_j\right)\middle| \mathcal{F}_s\right]$$

$$= W_{ij}\left(s, T_0\right) b_i\left(s, T_0\right) b_j\left(s, T_0\right) = \int_{t=s}^{T_0} dt Z\left(t, T_i, T_j\right)$$

2.2.2 The Stochastic String Model

In Chapter 1 we assumed the following dynamics for the instantaneous forward rate in the Musiela (1993) parameterization

$$df(t, x) = \alpha(t, x) dt + \sigma(t, x) dZ(t, x), \ 0 \le t \le \Upsilon, \ x \ge 0 \tag{2.6}$$

where $Z(t, x)$ is the so-called *stochastic string process* and Υ is the finite time horizon for trading risk-free zero-coupon bonds. Different assumptions about this process and about the filtration of the probability space as well as the consideration of a martingale representation property allows us to obtain the no-arbitrage dynamics of the forward rate

$$df(t, x) = \left[\frac{\partial f(t, x)}{\partial x} + \int_{y=0}^{x} dy R_t(x, y)\right] dt + \sigma(t, x) d\widetilde{Z}(t, x) \tag{2.7}$$

---

[2]This expression is also obtained by Roncoroni *et al.* (2010) (using the infinite-dimensional HJM analysis) and by Kennedy (1994) (using the random field approach). Moreover, this expression is consistent with the one obtained for one-factor Gaussian HJM models (see Chapter 1).

where $d\widetilde{Z}(t,x)$ is the stochastic string shock under the equivalent martingale measure $\mathbb{Q}$. Moreover,

$$R_t(x,y) \equiv c(t,x,y)\,\sigma(t,x)\,\sigma(t,y) = \frac{cov\,[df\,(t,x)\,,df\,(t,y)]}{dt} \tag{2.8}$$

where

$$c(t,x,y) = \frac{d\,[Z\,(\cdot,x)\,,Z\,(\cdot,y)]_t}{dt} = corr\,[dZ\,(t,x)\,,dZ\,(t,y)] \tag{2.9}$$

is the correlation function between the stochastic string shocks. Under the $\mathbb{Q}$ measure, the dynamics of bond returns with maturity T is given by

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt - \int_{y=0}^{T-t} d\widetilde{Z}(t,y)\,dy\sigma(t,y) \tag{2.10}$$

from where we obtain, under the ω-forward measure, the following expression for the logarithm of the price, at time T_0, of a bond with maturity $T \geq T_0$

$$\begin{aligned}
\ln P(T_0,T) = {} & \int_{t=s}^{T_0} dt\left[\int_{y=0}^{T-t}\int_{u=0}^{\omega-t} dydu R_t(u,y) - \frac{1}{2}\int_{y=0}^{T-t}\int_{u=0}^{T-t} dydu R_t(u,y)\right.\\
& \left. - \int_{y=0}^{T_0-t}\int_{u=0}^{\omega-t} dydu R_t(u,y) + \frac{1}{2}\int_{y=0}^{T_0-t}\int_{u=0}^{T_0-t} dydu R_t(u,y)\right] + \ln\frac{P(s,T)}{P(s,T_0)}\\
& - \int_{t=s}^{T_0}\int_{y=0}^{T-t} d\widetilde{Z}^\omega(t,y)\,dy\sigma(t,y) + \int_{t=s}^{T_0}\int_{y=0}^{T_0-t} d\widetilde{Z}^\omega(t,y)\,dy\sigma(t,y). \tag{2.11}
\end{aligned}$$

Here $d\widetilde{Z}^\omega(t,y)$ represents the stochastic string shock under the ω-forward measure. Using (2.11) and taking deterministic $\sigma(t,x)$ and $c(t,x,y)$ (Gaussian model), we obtain that the price of a European bond call option is given by (2.4) with

$$d_1 = \frac{\ln\left(\frac{P(s,T)}{KP(s,T_0)}\right) + \frac{1}{2}\Omega(s,T_0,T)}{\sqrt{\Omega(s,T_0,T)}}, \quad d_2 = d_1 - \sqrt{\Omega(s,T_0,T)}$$

and

$$\Omega(s,T_0,T) = \int_{v=s}^{T_0} dv\left[\int_{y=T_0-v}^{T-v}\int_{w=T_0-v}^{T-v} dydw R_v(w,y)\right] \tag{2.12}$$

2.2.3 Relationship between both types of Models

In Chapter 1 we made a first approximation to determine the relationship between infinite-dimensional HJM models and stochastic string models. This approximation uses (2.9), (2.10), and the invariance of the quadratic covariation to changes of equivalent measures to write

$$\begin{aligned}
cov\,[dP\,(s,u_1)\,,dP\,(s,u_2)] &= d\,[P\,(\cdot,u_1)\,,P\,(\cdot,u_2)]_s\\
&= \left[\int_{x=0}^{u_1-s}\int_{y=0}^{u_2-s} dxdy c\,(s,x,y)\,\sigma\,(s,x)\,\sigma\,(s,y)\right]P\,(s,u_1)\,P\,(s,u_2)\,ds\\
&= \left[\int_{x=0}^{u_1-s}\int_{y=0}^{u_2-s} dxdy R_s\,(x,y)\right]P\,(s,u_1)\,P\,(s,u_2)\,ds
\end{aligned}$$

Compared to (2.2), this expression allows us to obtain the covariance function between bond returns in the stochastic string model

$$Z\left(s, u_1, u_2\right) = \int_{x=0}^{u_1-s} \int_{y=0}^{u_2-s} dx dy R_s\left(x, y\right)$$

Substituting this expression in (2.5), we easily get that

$$\xi^2\left(s, T_0, T\right) = \Omega\left(s, T_0, T\right)$$

Thus, the models of Chu (1996) and our model of Chapter 1 are equivalent for pricing European call options in the Gaussian case.

In this chapter we will achieve a much stronger result, since we will present a theorem for the valuation of bond options, analogous to the theorem of Chu (1996) but obtained with methods completely different in the context of the stochastic string modeling.

2.3 Mercer's Theorem

This section presents the Mercer Theorem, the main mathematical tool that we will use in the rest of the chapter. The original form of this theorem (Mercer (1909)) does not apply to our work because it refers to compact domains and we use here the extension of Sun (2005) to non-compact domains.

Definition 2.1 *Let (X, d) be a metric space and let $K : X \times X \to \mathbb{R}$ be continuous and symmetric. We say that K is a **Mercer kernel** if it is positive semidefinite, that is, if for each finite set $\{x_1, \ldots, x_m\} \subset X$ and for $\{c_1, \ldots, c_m\} \subset \mathbb{R}$, we have that $\sum_{i,j=1}^{m} c_i c_j K\left(x_i, x_j\right) \geq 0$.* ∎

Proposition 2.1 *Let K be a Mercer kernel and let*

$$\mathcal{H}_K \equiv \overline{span\left\{K_x \equiv K\left(x, \cdot\right) : x \in X\right\}}.$$

Then, the set $\mathcal{H}_K$ is a Hilbert space with the scalar product

$$\langle f, g \rangle_{\mathcal{H}_K} = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_j K\left(x_i, x_j\right)$$

for $f = \sum_{i=1}^{n} c_i K_{x_i}$ and $g = \sum_{j=1}^{m} d_j K_{y_j}$. The space $\mathcal{H}_K$ is known as the Reproducing Kernel Hilbert Space (RKHS) associated to the kernel K. ∎

Let us consider now a non-degenerated Borel measure μ, defined on X and let us assume that X verifies

$$X = \bigcup_{n=1}^{\infty} X_n \quad \text{with} \quad X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$$

where each X_i is compact, has finite measure, and verifies that any compact subset of X is contained in X_i for some i.

Theorem 2.2 (Mercer): *Let K be a Mercer kernel and let us define the integral operator L_K on $L^2(X, \mu)$ as*

$$L_K f(x) = \int_X K(x, y) f(y) \, d\mu(y), \; x \in X$$

Let us suppose that the following conditions hold

i) $K_x \in L^2(X, \mu), \; \forall x \in X.$

ii) $\int_X \int_X K^2(x, y) \, d\mu(x) \, d\mu(y) < +\infty.$

Let $\{\lambda_i\}_{i=0}^{\infty}$ be the (positive) eigenvalues of L_K and let $\{\Phi_i\}_{i=0}^{\infty}$ be the corresponding orthonormal eigenfunctions in $L^2(X, \mu)$. Then

$$K(x, y) = \sum_{i=0}^{\infty} \lambda_i \Phi_i(x) \Phi_i(y)$$

This series converges absolutely and uniformly on $Y_1 \times Y_2$, where Y_1, Y_2 are any compact subsets of X. ∎

Remark 2.1 *In particular, these conditions hold if $K(x, y) \in L^2(X^2, \mu \times \mu)$, that is, if L_K is an integral operator of the Hilbert-Schmidt type on $L^2(X, \mu)$.* ∎

2.4 HJM Models and Principal Components Analysis

In this section we will use Mercer's Theorem to gain insights into the relationship between HJM models and stochastic string models, and to relate multi-factor HJM models with PCA.

From the analysis of Section 2.2, we know that the key function to be taken into account in the infinite-dimensional modeling of the TSIR is the instantaneous conditional covariance between shocks to the forward curve

$$R_s(x, y) \equiv c(s, x, y) \sigma(s, x) \sigma(s, y) \tag{2.13}$$

This function verifies

$$Z(s, u_1, u_2) = \int_{w=0}^{u_1 - s} \int_{z=0}^{u_2 - s} dw dz R_s(w, z) \tag{2.14}$$

$$cov[df(s, x), df(s, y)] = R_s(x, y) \, ds \tag{2.15}$$

$$cov\left[\frac{dP(s, s+x)}{P(s, s+x)}, \frac{dP(s, s+y)}{P(s, s+y)}\right] = \left[\int_{w=0}^{x} \int_{z=0}^{y} dw dz R_s(w, z)\right] ds \tag{2.16}$$

In Chapter 1 we have assumed that, for each $s \in [0, \Upsilon]$, $\sigma(s, \cdot)$ is continuous and $c(s, \cdot, \cdot)$ is continuous and symmetric. Then, $R_s : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and symmetric. Moreover, expression (2.15) implies that R_s is positive semidefinite. Thus, we can state the following result.

Proposition 2.2 R_s *is a Mercer kernel in* $\mathbb{R}^+$ *for each* s. ∎

To apply the Mercer Theorem to R_s, we consider $X = \mathbb{R}^+$ and $d\mu(x) = p(x)dx$, with p non-negative, continuous in $\mathbb{R}^+$ and with $\int_{x=0}^{\infty} dx\, p(x) < +\infty$. Under these conditions $L^2(X, \mu) = L_p^2(\mathbb{R}^+)$, which is a separable Hilbert space with $\langle f, g \rangle_{L_p^2(\mathbb{R}^+)} = \int_{x=0}^{\infty} dx\, p(x)\, f(x)\, g(x)$ that verifies $f \in L_p^2(\mathbb{R}^+) \Leftrightarrow p^{\frac{1}{2}} f \in L^2(\mathbb{R}^+)$.[3] We now make a fundamental assumption that will allow us to apply Mercer's Theorem to the stochastic string modeling.

Assumption 2.1 R_s *is a Hilbert-Schmidt kernel in* $L_p^2(\mathbb{R}^+)$ *for each* s. ∎

Direct application of the Theorem 2.2 to the kernel R_s will lead us to the following theorem.

Theorem 2.3 *Let* $\{\lambda_{s,i}\}_{i=0}^{\infty}$ *be the (positive) eigenvalues of* L_{R_s} *and let* $\{g_{s,i}\}_{i=0}^{\infty}$ *be their respective eigenfunctions, orthonormal in* $L_p^2(\mathbb{R}^+)$. *Then for each* $s \in \mathbb{R}^+$, *we have*

$$R_s(x, y) = \sum_{i=0}^{\infty} \lambda_{s,i} g_{s,i}(x)\, g_{s,i}(y) \tag{2.17}$$

The series converges absolutely and uniformly in $Y_1 \times Y_2$, *where* Y_1, Y_2 *are any compact subsets of* $\mathbb{R}^+$. ∎

In the Mercer Theorem for compact sets, the sum of the eigenvalues is finite. In our case, as in Roncoroni *et al.* (2010), we have to assume the following.

Assumption 2.2 $\sum_{i=0}^{\infty} \lambda_{s,i} < +\infty \quad \forall s \in [0, \Upsilon]$.

Remark 2.2 *Without loss of generality, we will assume that the eigenvalues are sorted in decreasing order:* $\lambda_{s,i} > \lambda_{s,i+1}$.

As a first application of the previous theorem, we can use expression (2.17) to rewrite expression (2.7) and obtain the following result.

Proposition 2.3 *The no-arbitrage dynamics of the instantaneous forward rate in the stochastic string model can be written as*

$$df(t, x) = \left[\frac{\partial f(t, x)}{\partial x} + \sum_{i=0}^{\infty} \lambda_{t,i} g_{t,i}(x) \int_{y=0}^{x} dy\, g_{t,i}(y) \right] dt + \sigma(t, x)\, d\widetilde{Z}(t, x) \tag{2.18}$$

∎

---

[3]We choose the space $L_p^2(\mathbb{R}^+)$ as it is the correct one to study the consistency problem of Section 2.5. In the rest of the chapter, it is not difficult to check that everything could be done working on $L^2(\mathbb{R}^+)$.

The last result allows us to recover the HJM models within the framework of stochastic strings. We start by incorporating a particular case of the stochastic string process, which is an infinite-dimensional extension of the process proposed by Pang (1999). Previously, we need from Chapter 1 the definition of the stochastic string process.

Definition 2.2 *The infinite-dimensional process $Z(t,x)$ is a **stochastic string process** if it verifies the following properties:*

a) *The stochastic processes $Z(\cdot, x)$ and $Z(t, \cdot)$ are continuous for each $x \geq 0$ and for each $t \in [0, \Upsilon]$, respectively.*

b) *The process $Z(\cdot, x)$ is a martingale for each $x \geq 0$.*

c) *The process $Z(t, \cdot)$ is differentiable for each $t \in [0, \Upsilon]$.*

d) *For each $x, y \geq 0$, it is verified that*

$$d[Z(\cdot, x), Z(\cdot, y)]_t = c(t, x, y)\, dt \tag{2.19}$$

where $c(t, x, y)$ is an admissible, continuous, and differentiable correlation function for each t.

Proposition 2.4 *Assume that, for all $i \in \mathbb{N}$, $t \in [0, \Upsilon]$, and $x, y \geq 0$, the following conditions hold:*

i) $\sigma_{CHU,t}^{'(i)}(\cdot) \in C^1(\mathbb{R}^+)$.

ii) $\left|\sigma_{CHU,t}^{'(i)}(x)\right| \leq M_i$, $\left|\sigma_{CHU,t}^{''(i)}(x)\right| \leq N_i$ with $\sum_{i=0}^{\infty} M_i, \sum_{i=0}^{\infty} N_i < +\infty$.

iii) $\sigma_{CHU,\cdot}^{'(i)}(x) \in C^0([0, \Upsilon])$.

iv) $\sum_{i=0}^{\infty} \sigma_{CHU,t}^{'(i)}(x)\sigma_{CHU,t}^{'(i)}(y)$ *is positive semidefinite.*

Then, the process $Z_P(t,x)$ determined by

$$dZ_P(t,x) = \sum_{i=0}^{\infty} \frac{\sigma_{CHU,t}^{'(i)}(x)}{\sigma(t,x)}\, dW_i(t) \tag{2.20}$$

with $\mathbf{W}(t) = (W_0(t), W_1(t), \ldots)$ a l^2-valued cylindrical Wiener process, is a stochastic string process. The correlation function between shocks is given by

$$c_P(t,x,y) = \sum_{i=0}^{\infty} \frac{\sigma_{CHU,t}^{'(i)}(x)\sigma_{CHU,t}^{'(i)}(y)}{\sigma(t,x)\sigma(t,y)} \tag{2.21}$$

Proof. See the appendix. ∎

Corollary 2.1 *Under the assumptions of Proposition 2.4 it is verified that*

$$d\widetilde{Z}_P(t,x) = \sum_{i=0}^{\infty} \frac{\sigma_{CHU,t}^{\prime(i)}(x)}{\sigma(t,x)} d\widetilde{W}_i(t) \tag{2.22}$$

∎

Proof. See the appendix. ∎

Proposition 2.5 *If we take $\widetilde{Z}(t,x) \equiv \widetilde{Z}_P(t,x)$ in the stochastic string model, we recover the Chu (1996) model where, in the Musiela parameterization,[4] the volatilities are given by*

$$\sigma_{CHU,s}^{(k)}(x) = \sqrt{\lambda_{s,k}} \int_{w=0}^{x} dw g_{s,k}(w) \tag{2.23}$$

∎

Proof. See the appendix. ∎

The previous results lead us to important findings on the modeling of the TSIR from the point of view of stochastic strings. For example, taking expressions (2.13) and (2.17) and making $x = y$, we can rewrite the volatility in the string model.

Proposition 2.6 *The relationship between the volatility in the stochastic string model and the eigenfunctions of L_{R_s} is given by*

$$\sigma^2(s,x) = \sum_{i=0}^{\infty} \lambda_{s,i} g_{s,i}^2(x) \tag{2.24}$$

∎

Integrating (2.24) and using the orthonormality of the eigenfunctions $g_{s,i}$ in $L_p^2(\mathbb{R}^+)$, we easily obtain

$$\int_{x=0}^{\infty} dx p(x) \sigma^2(s,x) = \sum_{i=0}^{\infty} \lambda_{s,i} < +\infty \tag{2.25}$$

Then, each eigenvector $g_{s,i}$ explains a proportion $\lambda_{s,i}$ of the "total weighted variance" in the forward curve $\sum_{i=0}^{\infty} \lambda_{s,i}$. Moreover, from expressions (2.23)-(2.24) we trivially get the following result (whose finite-dimensional version can be found in Goldstein (2000)).

Proposition 2.7 *The volatilities in the infinite-dimensional HJM model of Chu (1996) and the volatility of the stochastic string model are related by*

$$\sigma^2(s,x) = \sum_{i=0}^{\infty} \left[\sigma_{CHU,s}^{\prime(i)}(x) \right]^2 \tag{2.26}$$

∎

---

[4]We adopt the notation $\overline{\sigma}(s,T)$ and $\sigma_s(x) \equiv \overline{\sigma}(s,s+x)$ for volatilities in the parameterization of the maturity time T and the Musiela parameterization, respectively.

Replacing (2.22) in (2.18) and using (2.23) together with the orthonormality of the eigenfunctions, we readily obtain the dynamics of the forward rate in the model of Chu (1996).

Theorem 2.4 *Under the conditions of Proposition 2.4, the no-arbitrage dynamics of the instantaneous forward rate can be written as*

$$df(t,x) = \left[\frac{\partial f(t,x)}{\partial x} + \sum_{i=0}^{\infty} \sigma_{CHU,t}^{'(i)}(x) \int_{y=0}^{x} dy \sigma_{CHU,t}^{'(i)}(y) \right] dt + \sum_{i=0}^{\infty} \sigma_{CHU,t}^{'(i)}(x) d\widetilde{W}_i(t) \qquad (2.27)$$

with

$$\left\langle \sigma_{CHU,t}^{'(i)}, \sigma_{CHU,t}^{'(j)} \right\rangle_{L_p^2(\mathbb{R}^+)} = \lambda_{t,i} \delta_{i,j} \qquad (2.28)$$

∎

Using (2.23), equation (2.27) can be rewritten as

$$df(t,x) = \left[\frac{\partial f(t,x)}{\partial x} + \sum_{i=0}^{\infty} \lambda_{t,i} g_{t,i}(x) \int_{y=0}^{x} dy g_{t,i}(y) \right] dt + \sum_{i=0}^{\infty} \sqrt{\lambda_{t,i}} g_{t,i}(x) d\widetilde{W}_i(t) \qquad (2.29)$$

This expression coincides (except for the time dependence of the eigenfunctions), with that obtained in the infinite-dimensional HJM model of Roncoroni *et al.* (2010). Moreover, expressions (2.27) and (2.29) clearly indicate the relationship between this model and the model of Chu (1996) and the fact that the infinite-dimensional HJM modeling is a particular case of the stochastic string framework.

As an application of the results in this section we now propose a covariance function between forward rates, R_s^*, defined in $\mathbb{R}^+ \times \mathbb{R}^+$. To this end, we consider the functions $h_k(x) = e^{-\tau x} L_k(x)$, $x \geq 0$, $0 < \tau < 1/2$. These functions and their derivatives are bounded for each $k \in \mathbb{N}$. Thus, for each s, we can find a strictly decreasing sequence of positive numbers, $\{\lambda_{s,k}\}_{k=0}^{\infty}$, such that $\sum_{k=0}^{\infty} \lambda_{s,k} < +\infty$ that verifies condition ii) of Proposition 2.4 for $\sigma_{CHU,s}^{'(k)*} = \sqrt{\lambda_{s,k}} h_k$.

Then, the function

$$R_s^*(x,y) = \sum_{k=0}^{\infty} \sigma_{CHU,s}^{'(k)*}(x) \sigma_{CHU,s}^{'(k)*}(y) = e^{-\tau(x+y)} \sum_{k=0}^{\infty} \lambda_{s,k} L_k(x) L_k(y) \qquad (2.30)$$

is well defined because the series converges uniformly in $\mathbb{R}^+ \times \mathbb{R}^+$ as in the proof of Proposition 2.4. Moreover, if $\lambda_{s,k}$ is continuous in s, R_s^* fits perfectly into our framework as stated in the following result.

Proposition 2.8 *If $\{\lambda_{s,k}\}_{k=0}^{\infty}$ is such that $\left\{ \sigma_{CHU,s}^{'(k)*} \right\}_{k=0}^{\infty}$ verifies condition iii) of Proposition 2.4, then the following results hold:*

   i) R_s^* is a Mercer kernel.

ii) R_s^* *is a Hilbert-Schmidt integral kernel in* $L_{p_\tau}^2(\mathbb{R}^+)$ *with* $p_\tau(x) = e^{(2\tau-1)x}$.

iii) R_s^* *generates the dynamics of the forward rate (2.27), with volatilities* $\sigma_{CHU,t}^{'(k)*}(x) = \sqrt{\lambda_{s,k}}e^{-\tau x}L_k(x)$, *i.e., it is verified that*

$$df(t,x) = \left[\frac{\partial f(t,x)}{\partial x} + e^{-\tau x}\sum_{i=0}^{\infty}\lambda_{t,i}L_i(x)\int_{y=0}^{x}dy e^{-\tau y}L_i(y)\right]dt \qquad (2.31)$$
$$+ e^{-\tau x}\sum_{i=0}^{\infty}\sqrt{\lambda_{t,i}}L_i(x)d\widetilde{W}_i(t)$$

∎

Proof. See the appendix. ∎

Remark 2.3 *The first volatilities* $\sigma_{CHU,t}^{'(i)*}$ *are given by*

$$\sigma_{CHU,t}^{'(0)*}(x) = \sqrt{\lambda_{t,0}}\ e^{-\tau x}$$
$$\sigma_{CHU,t}^{'(1)*}(x) = (1-x)\sqrt{\lambda_{t,1}}\ e^{-\tau x}$$
$$\sigma_{CHU,t}^{'(2)*}(x) = \frac{1}{2}\left(2-4x+x^2\right)\sqrt{\lambda_{t,2}}\ e^{-\tau x}$$

and correspond to the stylized forms (exponentially smoothed) of the first three factors (level, slope, and curvature) commonly found in the empirical analysis of the TSIR (see, for example, Litterman and Scheikman (1991)). ∎

Remark 2.4 *If we consider the one-factor model with* $\sigma_{CHU,t}^{'(0)*}(x)$, *(2.31) becomes*

$$df(t,x) = \left[\frac{\partial f(t,x)}{\partial x} + \lambda_{t,0}e^{-\tau x}\int_{y=0}^{x}dy e^{-\tau y}\right]dt + \sqrt{\lambda_{t,0}}e^{-\tau x}d\widetilde{W}(t) \qquad (2.32)$$

which is the model of Hull and White (1990) written within the HJM scheme in the parameterization of Musiela. Furthermore, this is the only order of approximation that makes the short-term interest rate to be Markovian (see Chapter 1). ∎

Based on the previous results for infinite-dimensional HJM models, it is easy to get their $(n+1)$-factor versions making $\lambda_{s,j} = 0$ i.e. $\sigma_{HJM,t}^{(j)} \equiv 0$ for $j > n$, and $\sigma_{HJM,t}^{(j)} \equiv \sigma_{CHU,t}^{'(j)}$ for $j \leq n$. It is important to notice that in the multi-factor case not all the regularity conditions are needed, since some of them were used to ensure the uniform convergence of the series. Concretely, we should eliminate condition ii) of Proposition 2.4 and, as a consequence, the conditions about the eigenvalues of R_s^*.

Notice that while Chu (1996) gets multi-factor HJM models canceling the volatilities, we obtain these models canceling the eigenvalues. This approximation is justified because the eigenvalues are

positive and are arranged in decreasing order. What is more, making $\lambda_{s,j} = 0$ for $j > 0$ there will only be one non-zero volatility and by (2.21) we will have that $c_P(t, x, y) = 1$, which is the usual condition for reducing the results of stochastic string models to the one-factor HJM case (Santa-Clara and Sornette (2001)).

Additionally, our model allows us to reinterpret the $(n + 1)$-factor HJM models not as full and final models, but as approximated models with the covariance function

$$R_s^{(n)}(x, y) = \sum_{i=0}^{n} \lambda_{s,i} g_{s,i}(x) g_{s,i}(y) = \sum_{i=0}^{n} \sigma_{HJM,s}^{(i)}(x) \sigma_{HJM,s}^{(i)}(y) \tag{2.33}$$

which is an approximation of order n to the true covariance (see (2.17)). The ordering of the eigenvalues makes that higher-order approximations produce smaller and smaller corrections to the covariance. The expression (2.33) appears in Goldstein (2000) but, in contrast to this chapter, it does not imply an approximation.

Other important result is the orthogonality of the HJM volatilities in (2.28). This supposes a restriction to the set of volatilities allowed in both multi-factor and infinite-dimensional HJM models. To the best of our knowledge, this condition is not provided in any previous work related with HJM models.

2.4.1 Relationship with the Principal Component Analysis

Expressions (2.24)-(2.25) motivate the search for the relationship between the stochastic string approximation and the PCA. We start reviewing the main features of the PCA applied to the TSIR, following Rebonato (1996) partially.

Let us consider n values of the increments of the forward curve, corresponding to different times to maturity and imperfectly correlated, given by

$$df_i(t) = \alpha_i(t)dt + \sigma_i(t)dW_i(t), \ i = 1, \ldots, n$$

where W_i are standard Brownian motions with $dW_i(t)dW_j(t) = \rho_{ij}(t)dt, \ i, j = 1, \ldots, n$.

The structure of the covariance between increments in the forward curve is given by the matrix Σ with elements $\Sigma_{ij}(t) = cov[df_i(t), df_j(t)] = \rho_{ij}(t)\sigma_i(t)\sigma_j(t)dt, \ i, j = 1, \ldots, n$. Since Σ is real and symmetric,[5] it is always possible to find an orthogonal matrix $A = (a_{ij})$ and a diagonal matrix $\Lambda = (\lambda_i)$ that verify $\Sigma = A\Lambda A^T$. For $i = 1, \ldots, n$, the terms λ_i are the eigenvalues of Σ, that is, solutions of the equation $\Sigma A_i = \lambda_i A_i$ where $A_i = (a_{1i}, \ldots, a_{ni})^T$ is the eigenvector associated with the eigenvalue λ_i. Without loss of generality, we will assume λ_i are ordered decreasingly. By the

---

[5]To simplify notation, from now on we consider $dt = 1$ or, what is the same, we will work with covariances per unit of time. Moreover, we will eliminate the dependence on t.

orthogonality of A, the eigenvectors A_i verify

$$\langle A_i, A_j \rangle_{\mathbb{R}^n} = A_i^T A_j = \sum_{k=1}^n a_{ki} a_{kj} = \|A_i\|^2 \delta_{ij} \qquad (2.34)$$

The matrix A allows us to define n new variables (called *principal components*) y_i, $i = 1, \ldots, n$, given by $dy_i = \sum_{k=1}^n a_{ki} df_k$. Hence, each principal component y_i is determined uniquely by the eigenvector A_i. If we require the eigenvectors to have unit length, $\|A_i\| = 1$, then the expression (2.34) becomes the orthonormality condition $\langle A_i, A_j \rangle_{\mathbb{R}^n} = \delta_{ij}$. In this way, the principal components explain the same total variability as the original variables, that is,

$$\|dy\|^2 = \|df\|^2 = \sum_{i=1}^n \sigma_i^2 = \sum_{i=1}^n \lambda_i$$

Additionally, $\lambda_k / \sum_{i=1}^n \lambda_i$ indicates the proportion (decreasing in k) of the total variance explained by the factor k.

The next table compares the PCA of the forward curve with the stochastic string modeling:

PCA	Stochastic String
$a_{ij}(t)$	$g_{t,j}(x)$
$\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$	$\langle \cdot, \cdot \rangle_{L_p^2(\mathbb{R}^+)}$
$df_i(t) = \alpha_i(t)\,dt + \sigma_i(t)\,dW_i(t)$	$df(t,x) = \alpha(t,x)\,dt + \sigma(t,x)\,dZ(t,x)$
$dW_i(t)\,dW_j(t) = \rho_{ij}(t)\,dt$	$d[Z_\cdot^x, Z_\cdot^y]_t = c(t,x,y)\,dt$
$cov\,[df_i(t), df_j(t)] = \rho_{ij}(t)\,\sigma_i(t)\,\sigma_j(t)\,dt$	$cov\,[df(t,x), df(t,y)] = c(t,x,y)\,\sigma(t,x)\,\sigma(t,y)\,dt$
$\Sigma(t)\,A_i(t) = \lambda_i(t)\,A_i(t)$	$L_{R_t} g_{t,i}(x) = \lambda_{t,i} g_{t,i}(x)$

Hence, we can state that the application of Mercer's Theorem to the stochastic string modeling allows us to interpret it as a generalization of the PCA. This result stems from considering a continuous of points in the forward curve.

The analysis with stochastic strings has an important advantage over the PCA. In the latter, we go from the description of the forward curve using n points to the description of the curve using n factors, which explain a decreasing proportion of the variance. However, in the former, we go from the description of the forward curve using an uncountable infinite set of points to the description of the curve using a countable infinite number of factors, with the same interpretation with respect to the proportion of the variance explained.

The contents of this subsection provides a theoretical foundation for the common practice of taking the volatilities of multi-factor HJM models as factors obtained from PCA. Additionally,

we have obtained that the PCA may be a reasonable alternative for the empirical analysis of the stochastic string model.[6]

2.5   The Consistency Problem

As we indicated in the introduction, one of the methods most used in the estimation of the forward curve is the method proposed by NS, which consists of adjusting the following parametric curve to the data

$$f_{NS}(t,x) = z_{1,t} + (z_{2,t} + z_{3,t}x) e^{-z_{4,t}x}, \; z_{i,t} \in \mathbb{R} \tag{2.35}$$

However, it has been shown (Björk and Christensen (1999), Filipovic (2001)) that, in general, this parameterization is inconsistent with the HJM model. That is, under no-arbitrage conditions, if the initial forward curve is of the NS-type, it is not ensured that the evolution of this curve with HJM dynamics continues to produce curves of NS. The stochastic string model cannot solve this problem since it generalizes HJM model, but it can provide another perspective to the problem of consistency, in line with what was seen before about the relationship between both types of modeling.

We start our approximation to the consistency problem by adapting the fundamental features of the work of Björk and Christensen (1999) to the formalism of the stochastic string.

The no-arbitrage dynamics of the forward rates (see (2.7)) can be rewritten as

$$df(t,x) = \alpha(t,x)\,dt + \sigma(t,x)\,d\widetilde{Z}(t,x) \tag{2.36}$$

with

$$\alpha(t,x) = \frac{\partial f(t,x)}{\partial x} + \int_{y=0}^{x} dy R_t(x,y) \tag{2.37}$$

This will be from now on our interest rate model, $\mathcal{M}$.

We take $L_p^2(\mathbb{R}^+)$ as the space for the forward curves. Then, given an orthonormal basis $\{\psi_n\}_{n=0}^{\infty}$ of $L_p^2(\mathbb{R}^+)$, it is possible to express the instantaneous forward interest rates in the form

$$f(t,x) = \sum_{i=0}^{\infty} \alpha_i(t)\,\psi_i(x), \quad \sum_{i=0}^{\infty} \alpha_i^2(t) < +\infty$$

which will allow us to parameterize the forward curves using their Fourier coefficients in the basis given.

Let $\mathcal{Z} \subset l^2 = \left\{ \mathbf{z} = \{z_n\}_{n=0}^{\infty} : \sum_{i=0}^{\infty} z_n^2 < +\infty \right\}$. We will assume the existence of a parametric family of forward curves $G : \mathcal{Z} \to L_p^2(\mathbb{R}^+)$ such that at each $\mathbf{z} \in \mathcal{Z}$ corresponds $G(\mathbf{z}) = \sum_{i=0}^{\infty} z_i \psi_i \in L_p^2(\mathbb{R}^+)$. For simplicity, we will denote the forward curve by $x \mapsto G(x; \mathbf{z}) = \sum_{i=0}^{\infty} z_i \psi_i(x)$, where $\mathbf{z} = \{z_n\}_{n=0}^{\infty}$ is one realization of the infinite parameters.

---

[6]This issue is particularly relevant because there are only a finite number of maturities available in practice. See, for example, Pang (1999).

Definition 2.3 *Given $G : \mathcal{Z} \to L_p^2(\mathbb{R}^+)$, the **forward curve manifold** $\mathcal{G}$ is defined as*

$$\mathcal{G} = \left\{ G(\cdot, \mathbf{z}) \in L_p^2(\mathbb{R}^+) : \mathbf{z} \in \mathcal{Z} \right\} = \mathrm{Im}\, G$$

$\blacksquare$

In the rest of this section we will work only with deterministic volatilities. Then, we can rewrite expression (2.36) in the formulation of Stratonovich[7] as

$$df(t, x) = \alpha(t, x)dt + \sigma(x) \circ d\widetilde{Z}(t, x) \tag{2.38}$$

As $\widetilde{Z}(t, x)$ is a standard Brownian motion under $\mathbb{Q}$ for each x (see Chapter 1), we have

$$\int_{x=0}^{\infty} dx p(x) \left[\sigma(x) \circ d\widetilde{Z}(t, x) \right]^2 = \int_{x=0}^{\infty} dx p(x) \sigma^2(x) dt$$

and then we obtain the following lemma.

Lemma 2.1 $\sigma(\cdot) \in L_p^2(\mathbb{R}^+) \Leftrightarrow \sigma(\cdot) \circ d\widetilde{Z}(t, \cdot) \in L_p^2(\mathbb{R}^+)$

$\blacksquare$

Definition 2.4 *The stochastic string model $\mathcal{M}$ is **consistent** with the forward curve manifold $\mathcal{G}$ if there exists a stochastic process $\mathbf{z}(t) = \{z_n(t)\}_{n=0}^{\infty}$ with state space $\mathcal{Z}$ and with Stratonovich differentials of the form*

$$dz_n(t) = \gamma_n(t, \mathbf{z}(t))dt + \int_{x=0}^{\infty} dx \Psi_n(t, \mathbf{z}(t), x) \circ d\widetilde{Z}(t, x) \tag{2.39}$$

such that, for each fixed initial time s, as long as $y(s, \cdot) \in \mathcal{G}$, the stochastic process defined by

$$y(t, x) = G(x, \mathbf{z}(t)) = \sum_{i=0}^{\infty} z_i(t) \psi_i(x), \ \forall t \geq s, \ x \geq 0$$

is the solution to equation (2.38) with initial condition $f(s, \cdot) = y(s, \cdot)$.

$\blacksquare$

Remark 2.5 *If $z_n(t) = 0$ for $n > N$ and if $\widetilde{Z}(t, x) = \widetilde{Z}_P^{(n)}(t, x)$ is the finite-dimensional version of $\widetilde{Z}_P(t, x)$, then, the concept of r-invariance of Björk and Christensen (1999) is recovered.*

$\blacksquare$

Our first result on consistency requires an assumption, which is the extension to the infinite-dimensional case of Itô's formula for n-tuple semimartingales (Protter (2004), Theorem 21, p. 278).

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[7] For details on the Stratonovich calculus see, for example, Karatzas and Shreve (1998) or Protter (2004). As with the ordinary stochastic calculus, all the calculations that we will do with the differentials of Stratonovich must be understood in formal sense, and will acquire full sense only in their integral form.

Assumption 2.3 *Let $\mathbf{X}^i$, $i = 0, 1, \ldots$ be real-valued semimartingales with continuous paths such that $\mathbf{X} = \left\{ X^i \right\}_{i=0}^{\infty}$ is a l^2-valued semimartingale and consider $f : l^2 \longrightarrow \mathbb{R}$ with second-order continuous path derivatives. Then $f(\mathbf{X})$ is a semimartingale and the following formula holds*

$$df(\mathbf{X}) = \sum_{i=0}^{\infty} \frac{\partial f(\mathbf{X})}{\partial X^i} \circ dX^i$$

∎

Remark 2.6 *In this assumption, the partial derivatives $\dfrac{\partial f(\mathbf{X})}{\partial X^i}$ have to be intended in the Gâteaux sense $\dfrac{\partial f(\mathbf{X})}{\partial X^i} = \dfrac{d}{d\alpha} f(\mathbf{X} + \alpha \mathbf{e}_i)|_{\alpha=0}$ where $\mathbf{e}_i$ is the i-th vector of the standard orthonormal basis of l^2.*

Theorem 2.5 *Assume that, for any $(t, \mathbf{z}) \in [0, \Upsilon] \times \mathcal{Z}$, it is verified that $\partial_x G(\cdot, \mathbf{z}) + \int_{y=0}^{\cdot} dy R(\cdot, y)$ and $\sigma(\cdot) \circ d\widetilde{Z}(t, \cdot)$ belong to $\mathcal{G}$. Then, the stochastic string model $\mathcal{M}$, determined by (2.36)-(2.37), is consistent with $\mathcal{G} \subset L_p^2(\mathbb{R}^+)$ for any weight function p.*

∎

Proof. See the appendix. ∎

Björk and Christensen (1999) achieve a result similar to Theorem 2.5, but with necessary and sufficient conditions. To obtain the necessary conditions we need $\mathcal{G}$ to be a finite-dimensional subspace of $L_p^2(\mathbb{R}^+)$. Concretely, if we consider subspaces of the type

$$L_{p,n}^2(\mathbb{R}^+) = \left\{ f \in L_p^2(\mathbb{R}^+) : f = \sum_{i=0}^{n} z_i \psi_i, \ z_i \in \mathbb{R} \right\}, \ n \in \mathbb{N}$$

as forward curve manifolds, we can state the following result.

Theorem 2.6 *The stochastic string model $\mathcal{M}$ determined by (2.36)-(2.37), is consistent with $\mathcal{G} = L_{p,n}^2(\mathbb{R}^+)$ if and only if $\partial_x G(\cdot, \mathbf{z}) + \int_{y=0}^{\cdot} dy R(\cdot, y)$ and $\sigma(\cdot) \circ d\widetilde{Z}(t, \cdot)$ belong to $L_{p,n}^2(\mathbb{R}^+)$ for each $(t, \mathbf{z}) \in [0, \Upsilon] \times \mathcal{Z}$.*

∎

Proof. See the appendix. ∎

Returning to the analysis of the consistency of the NS family, we follow Roncoroni *et al.* (2010), and consider, as in Section 2.4, $p_\tau(x) \equiv e^{(2\tau - 1)x}$, $0 < \tau < 1/2$; moreover, we choose $h_n(x) = e^{-\tau x} L_n(x)$, $n = 0, \cdots, \infty$ as an orthonormal basis of $L_{p_\tau}^2(\mathbb{R}^+)$. Then, we have that

$$L_{p_\tau,1}^2(\mathbb{R}^+) = \left\{ f \in L_{p_\tau}^2(\mathbb{R}^+) : f(x) = [\alpha_0 + \alpha_1(-x+1)] e^{-\tau x}, \ \alpha_0, \alpha_1 \in \mathbb{R} \right\}$$

If we make $z_1 = 0$, $z_2 = \alpha_0 + \alpha_1$, $z_3 = -\alpha_1$, $z_4 = \tau$ in (2.35), then, we have that $L_{p_\tau,1}^2(\mathbb{R}^+)$ coincides with the NS family without independent term and with constant coefficient in the exponent. The

case of fixed z_4 is considered in the original work of Nelson and Siegel (1987) to facilitate the estimation of the parameters; it is also used in other works, such as Björk and Christensen (1999), Diebold and Li (2006), La Chioma and Piccoli (2007), and Roncoroni *et al.* (2010). There are also theoretical results that place restrictions on these coefficients in order to guarantee the consistency of the families of exponential-polynomial curves (Filipovic (2001), Theorem 3.6.1).

For all the above, to simplify and abusing a little of notation,[8] from now on we shall consider that the NS family is given by $L^2_{p_\tau,1}(\mathbb{R}^+)$.

2.5.1 Consistency with Approximated Absence of Arbitrage

Given the functional form of the forward curves of $L^2_{p_\tau,n}(\mathbb{R}^+)$, it seems logical to ask whether we will obtain consistency or not using finite-order approximations to the model (2.18), as given by covariances of the type $R^{*(n)} = e^{-\tau(x+y)} \sum_{i=0}^n \lambda_i L_i(x) L_i(y) \equiv \sum_{i=0}^n \lambda_i h_i(x) h_i(y)$. The following proposition gives us a (negative) response.

Proposition 2.9 *If the approximated stochastic string model with arbitrage opportunities defined by*

$$df_k(t,x) = \left[\frac{\partial f_k(t,x)}{\partial x} + \sum_{i=0}^k \lambda_i h_i(x) \int_{y=0}^x dy h_i(y) \right] dt + \sqrt{\sum_{i=0}^k \lambda_i h_i^2(x)} \circ d\widetilde{Z}(t,x) \qquad (2.40)$$

is consistent with $L^2_{p_\tau,n}(\mathbb{R}^+)$ for some values of k and n, then it is trivial. ∎

Proof. See the appendix. ∎

This proposition implies the inconsistency of the NS family even with models with approximated drift.

Corollary 2.2 *A stochastic string model of the form (2.40) and non trivial is inconsistent with $L^2_{p_\tau,1}(\mathbb{R}^+)$ for any value of k.* ∎

To obtain a positive result of consistency we have to extend the forward curve manifold. The next theorem is similar to Corollary 7.2 of Björk and Christensen (1999).

Theorem 2.7 *For each $n \in \mathbb{N}$, we consider the subspaces of $L^2_{p_\tau}(\mathbb{R}^+)$ defined by*

$$\mathcal{G}_n = span\left\{ h_0, \ldots, h_n, \widetilde{h}_0, \ldots, \widetilde{h}_n \right\}, \ h_i(x) = e^{-\tau x} L_i(x), \ \widetilde{h}_j(x) = e^{-2\tau x} L_j(x)$$

---

[8]Using the notation of Björk and Christensen (1999), it would actually be $\mathcal{G}_\tau = z_1 + L^2_{p_\tau,1}(\mathbb{R}^+)$, a restricted NS family.

and let P_n be the orthogonal projector onto $L^2_{p_\tau,n}(\mathbb{R}^+) \subset \mathcal{G}_n$. Then the approximated stochastic string model with arbitrage opportunities given by

$$df^{(n)}(t,x) = \left\{ \frac{\partial f^{(n)}(t,x)}{\partial x} + \sum_{i=0}^{[n/2]} \lambda_i h_i(x) \int_{y=0}^x dy h_i(y) \right\} dt + P_n \left[\sqrt{\sum_{i=0}^{\infty} \lambda_i h_i^2(x)} \circ d\widetilde{Z}(t,x) \right] \quad (2.41)$$

with $[m] = \max\{n \in \mathbb{N} : n \leq m\}$, is consistent with $\mathcal{G}_n$. ∎

Proof. See the appendix. ∎

Remark 2.7 *We can interpret (2.41) as a model $\mathcal{M}_n$ with arbitrage opportunities, which is an approximation of order n of the no-arbitrage full model $\mathcal{M}$ given by the covariance (see Proposition 2.8)*

$$R^*(x,y) = \sum_{i=0}^{\infty} \lambda_i h_i(x) h_i(y) = e^{-\tau(x+y)} \sum_{i=0}^{\infty} \lambda_i L_i(x) L_i(y)$$

If in this theorem we take the approximations corresponding to infinite-dimensional and multi-factor HJM models (see Theorem 2.4 and the subsequent comments), we obtain the following results of consistency.

Corollary 2.3 *Under the conditions of Proposition 2.8, the approximated infinite-dimensional HJM model with arbitrage opportunities given by*

$$df_\infty^{(n)}(t,x) = \left\{ \frac{\partial f_\infty^{(n)}(t,x)}{\partial x} + e^{-\tau x} \sum_{i=0}^{[n/2]} \lambda_i L_i(x) \int_{y=0}^x dy e^{-\tau y} L_i(y) \right\} dt \qquad (2.42)$$

$$+ e^{-\tau x} \sum_{i=0}^{\infty} \sqrt{\lambda_i} L_i(x) d\widetilde{W}_i(t)$$

and the $(m+1)$-factor approximated HJM model with arbitrage opportunities given by

$$df_m^{(n)}(t,x) = \left\{ \frac{\partial f_m^{(n)}(t,x)}{\partial x} + e^{-\tau x} \sum_{i=0}^{[n/2]} \lambda_i L_i(x) \int_{y=0}^x dy e^{-\tau y} L_i(y) \right\} dt \qquad (2.43)$$

$$+ e^{-\tau x} \sum_{i=0}^{m} \sqrt{\lambda_i} L_i(x) d\widetilde{W}_i(t)$$

are consistent with $\mathcal{G}_n$. ∎

Proof. See the appendix. ∎

The model given by (2.43) is an approximated version of that in (2.31) and, for $n = m = 0$, it matches the Hull and White (1990) model.[9] This fact allows us to get directly the following consistency result, already obtained by Björk and Christensen (1999, Proposition 5.3 and Remark 5.1).

---

[9]This model is arbitrage-free and is the only one with this property within Corollary 2.3.

Corollary 2.4 *The model of Hull and White (1990) is consistent with $\mathcal{G}_0 = span\left\{e^{-\tau x}, e^{-2\tau x}\right\}$.* ∎

With this interpretation of (2.41), it is interesting to know whether we get closer, in some sense, to the model $\mathcal{M}$ by increasing n. Roncoroni *et al.* (2010) prove that the weak solutions of (2.41) converge to the solution of (2.29). We will work with strong solutions and with convergence to the most general dynamics of the stochastic string, but we will need additional requirements on the series $\left\{f^{(n)}\right\}_{n=0}^{\infty}$.

Before stating the theorem we need the following result, taken from Frink (1935).

Lemma 2.2 *If $\{g_n(x)\}_{n=0}^{\infty}$ converges to $g(x)$ on $[a, b]$ and the second derivatives $g_n''(x)$ exist and are uniformly bounded on $[a, b]$, then $g'(x)$ exists and the sequence $\{g_n'(x)\}_{n=0}^{\infty}$ converges uniformly to $g'(x)$ on $[a, b]$.*

Theorem 2.8 *Let us consider the dynamics of the no-arbitrage stochastic string forward rate given by*

$$df(t, x) = \left[\frac{\partial f(t, x)}{\partial x} + \sum_{i=0}^{\infty} \lambda_i h_i(x) \int_{y=0}^{x} dy h_i(y)\right] dt + \sqrt{\sum_{i=0}^{\infty} \lambda_i h_i^2(x)} \circ d\widetilde{Z}(t, x) \tag{2.44}$$

We also consider a family of dynamics that approximate the previous ones and that, for each $n \in \mathbb{N}$, admits arbitrage opportunities. This family is given by

$$df^{(n)}(t, x) = \left[\frac{\partial f^{(n)}(t, x)}{\partial x} + \sum_{i=0}^{[n/2]} \lambda_i h_i(x) \int_{y=0}^{x} dy h_i(y)\right] dt + P_n \left[\sqrt{\sum_{i=0}^{\infty} \lambda_i h_i^2(x)} \circ d\widetilde{Z}(t, x)\right] \tag{2.45}$$

Let $f^{(n)}(t, x)$ be the solution of (2.45) for each $n \in \mathbb{N}$. If for each $t \in \mathbb{R}^+$ it is verified that

i) $\left\{f^{(n)}(t, x)\right\}_{n=0}^{\infty}$ converges on $[a, b] \subset \mathbb{R}^+$ with $\lim_{n \to \infty} f^{(n)}(t, x) = \overline{f}(t, x)$

ii) $\left\{\frac{\partial^2 f^{(n)}(t, x)}{\partial x^2}\right\}_{n=0}^{\infty}$ exists and it is uniformly bounded on $[a, b]$

Then, $\overline{f}(t, x)$ is solution of (2.44) on $[a, b]$. ∎

Proof. See the appendix. ∎

Remark 2.8 *We can obtain a result similar to Corollary 2.3 by replacing expression (2.45) by equations (2.42)-(2.43) in Theorem 2.8 and $d\widetilde{Z}(t, x)$ by $d\widetilde{Z}_P^*(t, x)$ in (2.44).* ∎

2.6 Option Pricing

Before stating the main theorem in this section, we need some lemmas. The first one will give us the conditional covariance between the logarithms of bond prices in the stochastic string model. The second one will allow us to write the price of an option (with homogeneous payoff function of degree one) in terms of expected values under the forward measure. Finally, the third lemma will give us the distribution of the bond log-price in the stochastic string model for the Gaussian case.

Lemma 2.3 *Consider two bonds maturing at times T_i and T_j. Conditioned on information available at time s, the covariance at time T_0 between the log-prices of these two bonds is given by*

$$\Delta_{ij}\left(s, T_0\right) \equiv cov\left[\ln P\left(T_0, T_i\right), \ln P\left(T_0, T_j\right)\middle| \mathcal{F}_s\right]$$
$$= \int_{t=s}^{T_0} dt\left[\int_{y=T_0-t}^{T_i-t}\int_{u=T_0-t}^{T_j-t} dydu R_t\left(u, y\right)\right] \equiv \int_{t=s}^{T_0} dt\overline{Z}\left(t, T_0, T_i, T_j\right) \qquad (2.46)$$

This covariance is independent of the probability measure and verifies $\Delta_{ij}\left(s, T_0\right) = \Delta_{ji}\left(s, T_0\right)$ and $\Delta_{0j}\left(s, T_0\right) = 0$. ∎

Proof. See the appendix. ∎

Remark 2.9 *If we reduce expression (2.46) to the multi-factor HJM case, replacing R_t by $R_t^{(n)}$, we get*

$$\Delta_{ij}^{(n)}\left(s, T_0\right) = \int_{t=s}^{T_0} dt \sum_{k=0}^{n}\left[\int_{y=T_0-t}^{T_i-t} dy\sigma_{HJM,t}^{(k)}\left(y\right)\right]\left[\int_{u=T_0-t}^{T_j-t} du\sigma_{HJM,t}^{(k)}\left(u\right)\right]$$

This expression is identical to that in Brace and Musiela (1994a) for a multi-factor HJM model. ∎

Lemma 2.4 *Let $\mathbb{Q}^{T_j}$ be the T_j-forward measure and let $\Phi : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ be a homogeneous function of degree one. Then it is verified that*

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t=s}^{T_0} dtr(t)}\left[\Phi\left(\mathbf{P}_{T_0}\right)\right]_+\middle| \mathcal{F}_s\right] = \sum_{j=0}^{N} P\left(s, T_j\right)\mathbb{E}^{\mathbb{Q}^{T_j}}\left[\left.\frac{\partial\Phi(\mathbf{y})}{\partial y_j}\right|_{y=\mathbf{P}_{T_0}}\mathbf{1}_{\Phi\left(\mathbf{P}_{T_0}\right)>0}\middle| \mathcal{F}_s\right] \qquad (2.47)$$

∎

Proof. See the appendix. ∎

Lemma 2.5 *If $\sigma(t, x)$ and $c(t, x, y)$ are deterministic, then, under the probability measure $\mathbb{Q}^{T_j}$, the bond price has a conditional lognormal distribution with mean*

$$\mathbb{E}^{\mathbb{Q}^{T_j}}\left[\ln P\left(T_0, T_i\right)\middle| \mathcal{F}_s\right] = \ln\frac{P\left(s, T_i\right)}{P\left(s, T_0\right)} + \Delta_{ij}\left(s, T_0\right) - \frac{1}{2}\Delta_{ii}\left(s, T_0\right)$$

and variance

$$Var\left[\ln P\left(T_0, T_i\right)\middle| \mathcal{F}_s\right] = \Delta_{ii}\left(s, T_0\right)$$

that is independent of the probability measure. ∎

Proof. See the appendix. ■

Now we can state and prove the theorem equivalent to Theorem 2.1 of Chu (1996) in the stochastic string modeling.

Theorem 2.9 *If, in the stochastic string model of the Chapter 1, it is verified that*

a) *The final payoff of a contingent claim at time T_0 is given by $C[T_0, \mathbf{P}_{T_0}] = \max(\Phi(\mathbf{P}_{T_0}), 0)$ where Φ is a homogeneous function of degree one.*

b) *The correlation matrix M with elements $M_{ij}(s, T_0) = \frac{\Delta_{ij}(s,T_0)}{\sqrt{\Delta_{ii}(s,T_0)}\sqrt{\Delta_{jj}(s,T_0)}}$, $i, j = 1, \ldots, N$ is deterministic and non-singular.*

Then, the price at time s of this contingent claim is given by

$$
\begin{aligned}
C[s, \boldsymbol{P}_s] &= \int_{\mathbb{R}^N} d\mathbf{x} \, g(x_1, \ldots, x_N; M) \\
&\times \left(\Phi\left[P(s, T_0), P(s, T_1) e^{\sqrt{\Delta_{11}} x_1 - \frac{1}{2}\Delta_{11}}, \ldots, P(s, T_N) e^{\sqrt{\Delta_{NN}} x_N - \frac{1}{2}\Delta_{NN}} \right] \right)_+
\end{aligned}
\tag{2.48}
$$

where $g(x_1, \ldots, x_N; M)$ is the density function of a multivariate random variable. ■

Proof. See the appendix. ■

This theorem is very similar to Theorem 2.1 but it requires one integration less. Note that in the model of Chu (1996) the instantaneous conditional covariance between bond log-prices, $\frac{\partial}{\partial s}\Delta_{ij}^{CHU}(s, T_0) = -Z(s, T_i, T_j)$, is independent of T_0, while in our model $\frac{\partial}{\partial s}\Delta_{ij}(s, T_0) = -\overline{Z}(s, T_0, T_i, T_j)$. Hence, our theorem is equivalent to the theorem of Chu if we substitute $Z(t, T_i, T_j)$ by $\overline{Z}(t, T_0, T_i, T_j)$ and use the property of dimensionality reduction (Miller (1964)) $\int_{\mathbb{R}^k} d\mathbf{y} \, g(\mathbf{x}; M) = g(\mathbf{z}; Z)$, where $\mathbf{x} = (\mathbf{y}|\mathbf{z}) = (y_1, \ldots, y_k, z_{k+1}, \ldots, z_N)$ and $M_{N \times N} = \left(\begin{array}{c|c} Y_{k \times k} & V_{k \times (N-k)} \\ \hline V_{(N-k) \times k}^T & Z_{(N-k) \times (N-k)} \end{array} \right)$. Since $\overline{Z}(s, s, T_i, T_j) = Z(s, T_i, T_j)$, we can assert that Theorem 2.9 is the stochastic string generalization of the theorem of Chu, which is a valid approximation to Theorem 2.9 for exercise times close to the current time.

The next example illustrates the applicability of Theorem 2.9 to the valuation of options.

Example 2.1 *Consider a European call option that matures at time T_0 with strike K written on a zero-coupon bond that expires at $T > T_0$. Its final payoff at T_0 is $(\Phi[P(T_0, T_0), P(T_0, T)])_+ = (P(T_0, T) - KP(T_0, T_0))_+$. Then, expression (2.48) becomes*

$$
\begin{aligned}
C_{Call}[s, P(s, T_0), P(s, T)] &= \int_{-\infty}^{+\infty} dx \, g(x; 1) \left(P(s, T) e^{\sqrt{\Delta} x - \frac{1}{2}\Delta} - KP(s, T_0) \right)_+ \\
&= P(s, T) \mathcal{N}(d_1) - KP(s, T_0) \mathcal{N}(d_2)
\end{aligned}
$$

with

$$d_1 = \frac{\ln\left(\frac{P(s,T)}{KP(s,T_0)}\right) + \frac{1}{2}\Delta}{\sqrt{\Delta}}, \ d_2 = d_1 - \sqrt{\Delta}$$

This expression coincides with (2.4) since, applying (2.46), we have that

$$
\begin{aligned}
\Delta(s,T_0) &= \int_{t=s}^{T} dt \overline{Z}(t,T_0,T,T) = \int_{t=s}^{T} dt \left[Z(t,T,T) + Z(t,T_0,T_0) - 2Z(t,T_0,T) \right] \\
&= \Omega(s,T_0,T)
\end{aligned}
$$

■

Remark 2.10 *Using the results of Section 2.4, we can rewrite Ω as*

$$
\begin{aligned}
\Omega(s,T_0,T) &= \int_{v=s}^{T_0} dv \left[\int_{x=T_0-v}^{T-v} \int_{y=T_0-v}^{T-v} dx dy R(x,y) \right] \\
&= \int_{v=s}^{T_0} dv \sum_{i=0}^{\infty} \left[\int_{x=T_0-v}^{T-v} dx \sigma_{CHU}^{\prime(i)}(x) \right]^2 = \int_{v=s}^{T_0} dv \sum_{i=0}^{\infty} \lambda_i \left[\int_{x=T_0-v}^{T-v} dx g_i(x) \right]^2 \quad (2.49)
\end{aligned}
$$

which coincides with that obtained by Roncoroni et al. (2010).

The HJM $(n+1)$-factor reduction of expression (2.49) provides

$$j\Omega^{(n)}(s,T_0,T) = \int_{v=s}^{T_0} dv \left[\int_{x=T_0-v}^{T-v} \int_{y=T_0-v}^{T-v} dx dy R^{(n)}(x,y) \right] = \int_{v=s}^{T_0} dv \sum_{i=0}^{n} \left[\int_{x=T_0-v}^{T-v} dx \sigma_{HJM}^{(i)}(x) \right]^2$$

which, with the appropriate changes in the parameterization, coincides with the corresponding expression for the valuation of call options for HJM Gaussian models (Björk (2004)).

Finally, if we take the zero-order approximation and make $R^{(0)} = R^{*(0)}$, we have

$$\Omega^{(0)}(s,T_0,T) = \frac{1}{\tau^2} \int_{v=s}^{T_0} dv \lambda_0 \left[e^{-\tau(T_0-v)} - e^{-\tau(T-v)} \right]^2$$

that is the variance in the Hull and White model (Jamshidian (1989), Hull and White (1990)). ■

2.7 Conclusions

In this chapter we have presented the stochastic string model of Santa-Clara and Sornette (2001) in the reformulation of the previous chapter as a truly unifying and explanatory theory of most of the continuous-time models of the TSIR done so far. We have shown that the (multi-factor and infinite-dimensional) HJM approximations are particular cases of the stochastic string analysis. We have also proven that it is possible to reinterpret the PCA, the study of consistency, and the valuation of options in terms of our unifying framework, which allows us to recover existing results in the literature and to provide new ones.

Among the new results that we obtain, we can mention the following: a) an orthogonality condition for the volatilities of HJM models, b) the interpretation of multi-factor HJM models as approximations of different order to a full infinite-dimensional model, c) a result of consistency based on Hilbert spaces, and d) a theorem for the valuation of interest rate options with homogeneous payoff function of degree one.

Roncoroni *et al.* (2010) points out two important drawbacks of stochastic string models that make them inappropriate to capture the relationship between interest rate risk and shapes of the term structure: 1) they do not take into account the difference between the parameters t (current time) and x (time to maturity) and 2) they do not allow to sort the sources of uncertainty in terms of their relative importance in explaining the interest rate risk.

Our framework eliminates these shortcomings. The difference between t and x is clearly specified in the conditions that a stochastic string process must meet and, as seen before, the second drawback is solved by the Mercer's Theorem and the ordering of the eigenvalues.

2.8 Appendix

Proof of Proposition 2.4

For each fixed x, the process $Z_P(t,x)$ is continuous in $[0,\Upsilon]$, since $\sum_{i=0}^{\infty}\int_{s=0}^{t}\sigma_{CHU,s}^{\prime(i)}(x)\,dW_i(t)$ is uniformly convergent in probability on $[0,\Upsilon]$ (Filipovic (2001, Proposition 2.2.1)) and their terms are continuous functions in $[0,\Upsilon]$. The differentiability in x (and thus, the continuity) is obtained because $\sigma_{CHU,t}^{\prime(i)}(\cdot)$ is differentiable $\forall i \in \mathbb{N}$ (by i)) and because the series $\sum_{i=0}^{\infty}\sigma_{CHU,t}^{\prime(i)}(x)\,dW_i(t)$ and $\sum_{i=0}^{\infty}\sigma_{CHU,t}^{\prime\prime(i)}(x)\,dW_i(t)$ are uniformly convergent by the Weierstrass test, (from ii)) and the fact that $W_i(t)$ is a standard Brownian motion for each i. Definition (2.20) makes that $Z_P(t,x)$ is a local martingale in t. The boundedness of this process implies the martingale property. Moreover,

$$d\left[Z_P(\cdot,x),Z_P(\cdot,y)\right]_t = \frac{\sum_{i=0}^{\infty}\sigma_{CHU,t}^{\prime(i)}(x)\,\sigma_{CHU,t}^{\prime(i)}(y)}{\sigma(t,x)\,\sigma(t,y)}dt \equiv c_P(t,x,y)\,dt$$

The numerator of c_P is well defined because $\left|\sigma_{CHU,t}^{\prime(i)}(x)\,\sigma_{CHU,t}^{\prime(i)}(y)\right| \le M_i^2$, with $\sum_{i=0}^{\infty}M_i^2 < +\infty$, and because this is a series absolutely and uniformly convergent (by the Weierstrass test).

To see that it is an acceptable correlation function, it suffices with condition iv).[10]

The continuity of the correlation function is obtained from the continuity of $\sigma_{CHU,t}^{\prime(i)}(\cdot)$ and from the uniform convergence of the series.

For the differentiability of $c_P(t,x,y)$ with respect to x and y, the choice of $\bar{c}_P^{(i)}(t,x,y) \equiv \sigma_{CHU,t}^{\prime(i)}(x)\sigma_{CHU,t}^{\prime(i)}(y)$ implies that $\dfrac{\partial\bar{c}_P^{(i)}(t,x,y)}{\partial x} = \sigma_{CHU,t}^{\prime\prime(i)}(x)\sigma_{CHU,t}^{\prime(i)}(y)$. Taking absolute values we obtain $\left|\dfrac{\partial\bar{c}_P^{(i)}(t,x,y)}{\partial x}\right| \le NM_i \equiv \overline{M}_i$ with $N = \sup_{i\in\mathbb{N}}N_i$ and $\sum_{i=0}^{\infty}\overline{M}_i < +\infty$, from where $\sum_{i=0}^{\infty}\dfrac{\partial\bar{c}_P^{(i)}(t,x,y)}{\partial x}$ uniformly converges for each fixed y. Since $\bar{c}_P(t,x,y) \equiv \sum_{i=0}^{\infty}\bar{c}_P^{(i)}(t,x,y)$ is (uniformly) convergent, we have that $\dfrac{\partial\bar{c}_P(t,x,y)}{\partial x} = \sum_{i=0}^{\infty}\dfrac{\partial\bar{c}_P^{(i)}(t,x,y)}{\partial x}$ and since $\dfrac{\partial\bar{c}_P^{(i)}(t,x,y)}{\partial x}$ is continuous in x, $\forall i \in \mathbb{N}$, $\dfrac{\partial\bar{c}_P(t,x,y)}{\partial x}$ is also continuous. By symmetry, the same reasoning applies to the dependence on y, getting that $\dfrac{\partial\bar{c}_P(t,x,y)}{\partial y}$ exists and is continuous, so that we obtain that $\bar{c}_P(t,x,y)$ is differentiable and therefore $c_P(t,x,y)$ is also differentiable. ∎

Proof of Corollary 2.1

Under the assumptions of the model of the Chapter 1 and using the Girsanov-Meyer Theorem, we get

$$d\widetilde{W}_j(t) = dW_j(t) + dt\int_{u=0}^{\infty}du\,\frac{\sigma_{CHU,t}^{\prime(j)}(u)}{\sigma(t,u)}\lambda(t,u) \tag{2.50}$$

---

[10]The correlation function is bounded because it is semidefinite and $c_P(t,x,x)=1$.

where $\lambda(t, u)$ is the specific market risk premium associated to a bond with maturity u.[11] Taking (2.50) together with the similar expression for $Z(t, x)$ from the paper just mentioned

$$d\widetilde{Z}(t, x) = dZ(t, x) + dt \int_{u=0}^{\infty} duc(t, u, x)\lambda(t, u)$$

and applying them in (2.20) with the correlation function c_P, we finally obtain

$$d\widetilde{Z}_P(t, x) = \sum_{i=0}^{\infty} \frac{\sigma_{CHU,t}^{'(i)}(x)}{\sigma(t, x)} d\widetilde{W}_i(t)$$

$\blacksquare$

Proof of Proposition 2.5

Substituting (2.22) in (2.10) we have

$$\frac{dP(t, T)}{P(t, T)} = r(t)\,dt - \sum_{i=0}^{\infty}\left[\int_{y=0}^{T-t} dy\sigma_{CHU,t}^{'(i)}(y)\right] d\widetilde{W}_i(t)$$

$$= r(t)\,dt - \sum_{i=0}^{\infty} \overline{\sigma}_{CHU}^{(i)}(t, T)\, d\widetilde{W}_i(t)$$

which coincides with equation (2.1). Considering expressions (2.3), (2.14) and Theorem 2.3 we obtain equation (2.23). $\blacksquare$

Proof of Proposition 2.8

To prove i) and ii), it suffices to check that R_s^* is continuous, that it is a Hilbert-Schmidt kernel and that it is positive semidefinite. The continuity of R_s^* is obtained immediately from the uniform convergence and from the continuity of the terms. Let us show that it is an integral Hilbert-Schmidt kernel. Using (3.8) we have

$$\int_{x=0}^{\infty} \int_{y=0}^{\infty} dxdy\,[R_s^*(x, y)]^2\, p_\tau(x)\, p_\tau(y)$$

$$= \int_{x=0}^{\infty} \int_{y=0}^{\infty} dxdye^{-(x+y)} \left[\sum_{i=0}^{\infty} \lambda_{s,i} L_i(x) L_i(y)\right] \left[\sum_{j=0}^{\infty} \lambda_{s,j} L_j(x) L_j(y)\right]$$

and using the Cauchy product of series[12] and the orthonormality relation

$$\int_{x=0}^{\infty} dxe^{-x} L_i(x) L_j(x) = \delta_{ij}$$

---

[11]The expression (2.50) allows us to obtain the market price of risk associated with $W_j(t)$ in our model as $\lambda_j(t) = \int_{u=0}^{\infty} du \frac{\sigma_{CHU,t}^{'(j)}(u)}{\sigma(t,u)} \lambda(t, u)$. In the one-factor model, this expression reduces to $\lambda_j(t) = \int_{u=0}^{\infty} du\lambda(t, u)$, which is the equation obtained in Chapter 1.

[12]The Cauchy product of series converges because the series are absolutely convergent.

we get

$$\int_{x=0}^{\infty} \int_{y=0}^{\infty} dx dy \left[R_s^* (x,y) \right]^2 p_\tau (x) p_\tau (y) = \sum_{i=0}^{\infty} \lambda_{s,i}^2 < +\infty$$

Since R_s^* is a Hilbert-Schmidt kernel, to prove the condition of positive semidefinite it would suffice to show that $L_{R_s^*}$ is a positive operator (Sun (2005)). For $f \in L_{p_\tau}^2 (\mathbb{R}^+)$ we have that

$$\begin{aligned}
\langle f, L_{R_s^*} f \rangle &= \int_{x=0}^{\infty} dx p_\tau (x) f (x) \left[\int_{y=0}^{\infty} dy R_s^* (x,y) f (y) p_\tau (y) \right] \\
&= \int_{x=0}^{\infty} dx p_\tau (x) f (x) \left[\int_{y=0}^{\infty} dy \sum_{i=0}^{\infty} \lambda_{s,i} h_i (x) h_i (y) f (y) p_\tau (y) \right] \\
&= \sum_{i=0}^{\infty} \lambda_{t,i} \left[\int_{x=0}^{\infty} dx p_\tau (x) f (x) h_i (x) \right]^2 \geq 0 \\
&= \sum_{i=0}^{\infty} \lambda_{t,i} \langle f, h_i \rangle^2 \geq 0
\end{aligned}$$

from where $L_{R_s^*}$ is positive and R_s^* is positive semidefinite. Furthermore, the volatilities $\sigma_{CHU,s}^{\prime(k)} (x) = \sqrt{\lambda_{s,k}} e^{-\tau x} L_k(x)$ corresponding to R_s^* clearly verify the conditions of Proposition 2.4; thus, we can apply Theorem 2.4, which concludes the proof. ∎

Proof of Theorem 2.5

Let $y (s, \cdot) \in \mathcal{G} \subset L_p^2 (\mathbb{R}^+)$. Since this is a separable Hilbert space, there will be $\mathbf{z}_0 = \{z_{i,0}\}_{i=0}^{\infty}$ such that

$$y (s, x) = \sum_{i=0}^{\infty} z_{i,0} \psi_i (x) = \sum_{i=0}^{\infty} \langle \psi_i, y (s, \cdot) \rangle \psi_i (x) = G (x, \mathbf{z}_0)$$

Now let $\mathbf{z} (t) = \{z_i (t)\}_{i=0}^{\infty}$ be defined as[13]

$$dz_i (t) = \langle \psi_i, \widetilde{\alpha} (\mathbf{z} (t), \cdot) \rangle dt + \left\langle \psi_i, \sigma (\cdot) \circ d\widetilde{Z} (t, \cdot) \right\rangle, t \geq s, \quad \mathbf{z} (s) = \mathbf{z}_0 \tag{2.51}$$

with $\widetilde{\alpha} (\mathbf{z}, x) = \dfrac{\partial G (x; \mathbf{z} (t))}{\partial x} + \int_{u=0}^{x} du R (x, u)$. This definition makes sense under the hypothesis of this theorem. The process $\mathbf{z} (t)$ is of the form (2.39) with $\gamma_n (t, \mathbf{z}) \equiv \langle \psi_n, \widetilde{\alpha} (\mathbf{z} (t), \cdot) \rangle$ and $\Psi_n (t, \mathbf{z}, x) \equiv \psi_n (x) \sigma (x) p (x)$.

---

[13]For simplicity, we assume that equation (2.51) has a solution that is a continuous semimartingale.

Making $y(t,x) = G(x, \mathbf{z}(t)) = \sum_{i=0}^{\infty} z_i(t)\psi_i(x)$ and applying Assumption 2.3 we have

$$
\begin{aligned}
dy(t,x) &= \sum_{i=0}^{\infty} \psi_i(x) \circ dz_i(t) = \sum_{i=0}^{\infty} \psi_i(x)\, dz_i(t) \\
&= \sum_{i=0}^{\infty} \langle \psi_i, \widetilde{\alpha}(\mathbf{z}(t),\cdot) \rangle \psi_i(x)\, dt + \sum_{i=0}^{\infty} \left\langle \psi_i, \sigma(\cdot) \circ d\widetilde{Z}(t,\cdot) \right\rangle \psi_i(x) \\
&= \widetilde{\alpha}(\mathbf{z}(t),x)\, dt + \sigma(x) \circ d\widetilde{Z}(t,x) \\
&= \left[\frac{\partial y(t,x)}{\partial x} + \int_{u=0}^{x} du R(x,u) \right] dt + \sigma(x) \circ d\widetilde{Z}(t,x)
\end{aligned}
$$

and, then, $y(t,x)$ solves (2.38) with the initial condition $f(s,\cdot) = y(s,\cdot)$. ∎

Proof of Theorem 2.6

The necessary condition is directly obtained from Theorem 2.5. For the sufficient condition, taking differentials in $y(t,x) = G(x; \mathbf{z}(t)) = \sum_{i=0}^{n} z_i(t)\psi_i(x)$ we obtain

$$
\begin{aligned}
dy(t,x) &= \sum_{i=0}^{n} \psi_i(x) \circ dz_i(t) = \sum_{i=0}^{n} \psi_i(x)\, dz_i(t) \\
&= \sum_{i=0}^{n} \psi_i(x)\gamma_i(t,\mathbf{z}(t))\, dt + \sum_{i=0}^{n} \psi_i(x) \int_{u=0}^{\infty} du\, \Psi_i(t,\mathbf{z}(t),u) \circ d\widetilde{Z}(t,u)
\end{aligned}
$$

Then, applying the consistency, we get that, $\forall t \geq s$, $\partial_x G(\cdot,\mathbf{z}(t)) + \int_{u=0}^{\cdot} dy R(\cdot,u)$ and $\sigma(\cdot) \circ d\widetilde{Z}(t,\cdot)$ belong to $L^2_{p,n}(\mathbb{R}^+)$. Since s can be arbitrarily chosen, and then $\mathbf{z}(s)$, concludes the proof. ∎

Proof of Proposition 2.9

Let us assume that consistency holds. By Theorem 2.6, for each $\mathbf{z} \in \mathcal{Z}$ it is verified that

$$
\frac{\partial G(\cdot,\mathbf{z})}{\partial x} + \sum_{i=0}^{k} \lambda_i h_i(\cdot) \int_{y=0}^{\cdot} dy h_i(y) \in L^2_{p,n}(\mathbb{R}^+) \tag{2.52}
$$

For the first term of this equation we have

$$
\frac{\partial G(x,\mathbf{z})}{\partial x} = \frac{\partial}{\partial x}\left[\sum_{i=0}^{n} z_i h_i(x) \right] = e^{-\tau x} \sum_{i=0}^{n} z_i \left[L_i^{(1)}(x) - \tau L_i(x) \right] = e^{-\tau x} \sum_{i=0}^{n} \overline{z}_i L_i(x)
$$

where we have made $L_i^{(n)}(x) = \dfrac{d^n L_i(x)}{dx^n}$ and we have applied that $L_i^{(1)}$ can be expressed as a linear combination of $\{L_j\}_{j=0}^{i-1}$.

Thus, condition (2.52) is equivalent to the existence of $\{\gamma_i\}_{i=0}^{n}$ such that

$$
\sum_{i=0}^{k} \lambda_i h_i(x) \int_{y=0}^{x} dy h_i(y) = \sum_{i=0}^{n} \gamma_i h_i(x) \tag{2.53}
$$

Using that $h_i(x) = e^{-\tau x} L_i(x)$ we can rewrite (2.53) as

$$\sum_{i=0}^{k} \lambda_i \left[\sum_{l=0}^{i} \frac{1}{\tau^{l+1}} L_i^{(l)}(0) \right] L_i(x) - e^{-\tau x} \sum_{i=0}^{k} \lambda_i \left[\sum_{l=0}^{i} \frac{1}{\tau^{l+1}} L_i^{(l)}(x) \right] L_i(x) = e^{-\tau x} \sum_{i=0}^{n} \gamma_i L_i(x)$$

from where we conclude that $\lambda_i = 0$, $\forall i = 0, \ldots, k$, which makes the model trivial. $\blacksquare$

Proof of Theorem 2.7

By Theorem 2.5 it is sufficient to show that the following two conditions hold with $G_n(\cdot, \mathbf{z}) \in \mathcal{G}_n$, $\forall \mathbf{z} \in \mathcal{Z}$.

- $\partial_x G_n(\cdot, \mathbf{z}) + \sum_{i=0}^{[n/2]} \lambda_i h_i(\cdot) \int_{y=0}^{\cdot} dy h_i(y) \in \mathcal{G}_n$:

    For each $\mathbf{z} \in \mathcal{Z}$, there exist $\{\alpha_i\}_{i=0}^{n}$ and $\{\beta_i\}_{i=0}^{n}$ such that

    $$G_n(x, \mathbf{z}) = \sum_{i=0}^{\infty} z_i h_i(x) = \sum_{i=0}^{n} \alpha_i h_i(x) + \sum_{i=0}^{n} \beta_i \widetilde{h}_i(x)$$

    from where

    $$\frac{\partial G_n(x, \mathbf{z})}{\partial x} = e^{-\tau x} \sum_{i=0}^{n} \alpha_i \left[L_i^{(1)}(x) - \tau L_i(x) \right] + e^{-2\tau x} \sum_{j=0}^{n} \beta_j \left[L_j^{(1)}(x) - 2\tau L_j(x) \right]$$

    $$= \sum_{i=0}^{n} \alpha_i \sum_{k=0}^{i} \nu_k h_k(x) + \sum_{j=0}^{n} \beta_j \sum_{l=0}^{j} \gamma_l \widetilde{h}_l(x)$$

    $$= \sum_{k=0}^{n} \left(\nu_k \sum_{i=k}^{n} \alpha_i \right) h_k(x) + \sum_{l=0}^{n} \left(\gamma_l \sum_{j=l}^{n} \beta_j \right) \widetilde{h}_l(x)$$

    and, then $\partial_x G_n(\cdot, \mathbf{z}) \in \mathcal{G}_n$.

    Moreover, we have

    $$\sum_{i=0}^{[n/2]} \lambda_i h_i(x) \int_{y=0}^{x} dy h_i(y) = e^{-\tau x} \sum_{i=0}^{[n/2]} \lambda_i L_i(x) \left[\sum_{l=0}^{i} \frac{1}{\tau^{l+1}} L_i^{(l)}(0) - e^{-\tau x} \sum_{l=0}^{i} \frac{1}{\tau^{l+1}} L_i^{(l)}(x) \right]$$

    $$= e^{-\tau x} \sum_{i=0}^{[n/2]} \overline{\lambda}_i L_i(x) - e^{-2\tau x} \sum_{i=0}^{[n/2]} \widetilde{\lambda}_i \left[\sum_{j=0}^{2i} \beta_j L_j(x) \right]$$

    $$= \sum_{i=0}^{[n/2]} \overline{\lambda}_i h_i(x) - \sum_{k=0}^{n} \widehat{\lambda}_k \widetilde{h}_k(x)$$

    from where $\sum_{i=0}^{[n/2]} \lambda_i h_i(\cdot) \int_{y=0}^{\cdot} dy h_i(y) \in \mathcal{G}_n$.

- $P_n\left[\sqrt{\sum_{i=0}^\infty \lambda_i h_i^2\left(\cdot\right)}\circ d\widetilde{Z}(t,\cdot)\right]\in\mathcal{G}_n$:

  Since $\sum_{i=0}^\infty \lambda_i < +\infty$, Lemma 2.1 implies that $\sqrt{\sum_{i=0}^\infty \lambda_i h_i^2\left(\cdot\right)}\circ d\widetilde{Z}(t,\cdot)\in L_{p_\tau}^2\left(\mathbb{R}^+\right)$. After projecting we remain in $L_{p_\tau,n}^2\left(\mathbb{R}^+\right)\subset\mathcal{G}_n$. $\blacksquare$

Proof of Corollary 2.3

For the first expression it suffices to take Theorem 2.7 and make $d\widetilde{Z}\left(t,x\right)=d\widetilde{Z}_P^*\left(t,x\right)$, use the chain of equalities

$$P_n\left[\sqrt{\sum_{i=0}^\infty \lambda_i h_i^2\left(x\right)}\circ d\widetilde{Z}_P^*\left(t,x\right)\right]=P_n\left[\sigma^*\left(x\right)d\widetilde{Z}_P^*\left(t,x\right)\right]$$

$$=P_n\left[\sum_{i=0}^\infty \sigma_{CHU}^{\prime(i)*}\left(x\right)d\widetilde{W}_i\left(t\right)\right]$$

$$=\sum_{j=0}^n\left\langle h_j,\sum_{i=0}^\infty \sigma_{CHU}^{\prime(i)*}\left(\cdot\right)d\widetilde{W}_i\left(t\right)\right\rangle h_j\left(x\right)$$

$$=\sum_{i=0}^\infty\left\{\sum_{j=0}^n\left\langle h_j,\sigma_{CHU}^{\prime(i)*}\left(\cdot\right)\right\rangle h_j\left(x\right)\right\}d\widetilde{W}_i\left(t\right)$$

$$=\sum_{i=0}^\infty \sqrt{\lambda_i}h_i\left(x\right)d\widetilde{W}_i\left(t\right),$$

and substitute $h_i\left(x\right)=e^{-\tau x}L_i\left(x\right)$.

The second expression is obtained from the first one, making $\lambda_i=0$ for $i>m$. $\blacksquare$

Proof of Theorem 2.8

Taking limits in (2.45) and writing $\sigma^*\left(x\right)=\sqrt{\sum_{i=0}^\infty \lambda_i h_i^2\left(x\right)}$ we have for the last two terms on the right side of the equation that

$$\lim_{n\to\infty}P_n\left[\sigma^*\left(x\right)\circ d\widetilde{Z}\left(t,x\right)\right]=\lim_{n\to\infty}\sum_{i=0}^n\left\langle h_i,\sigma^*\left(\cdot\right)\circ d\widetilde{Z}\left(t,\cdot\right)\right\rangle h_i\left(x\right)$$

$$=\sum_{i=0}^\infty\left\langle h_i,\sigma^*\left(\cdot\right)\circ d\widetilde{Z}\left(t,\cdot\right)\right\rangle h_i\left(x\right)$$

$$=\sigma^*\left(x\right)\circ d\widetilde{Z}\left(t,x\right)$$

$$\lim_{n\to\infty}\sum_{i=0}^{[n/2]}\lambda_i h_i\left(x\right)\int_{y=0}^x dy h_i\left(y\right)=\sum_{i=0}^\infty \lambda_i h_i\left(x\right)\int_{y=0}^x dy h_i\left(y\right)$$

For the first term it suffices to apply Lemma 2.2 to see that $\lim_{n\to\infty}\dfrac{\partial f^{(n)}(t,x)}{\partial x}=\dfrac{\partial\overline{f}(t,x)}{\partial x}$ on $[a,b]$.

With the limit of the left side of the equation we conclude that

$$d\overline{f}(t,x) = \left[\frac{\partial \overline{f}(t,x)}{\partial x} + \sum_{i=0}^{\infty} \lambda_i h_i(x) \int_{y=0}^{x} dy h_i(y)\right] dt + \sqrt{\sum_{i=0}^{\infty} \lambda_i h_i^2(x)} \circ d\widetilde{Z}(t,x)$$

that is, $\overline{f}(t,x)$ is solution of (2.44). ∎

Proof of Lemma 2.3

With the interpretation of conditional covariances in terms of quadratic covariations of semimartingales of Chapter 1 and using expression (2.11) we have

$$cov\left[\ln P(T_0,T_i),\ln P(T_0,T_j)|\,\mathcal{F}_s\right] = \left[\ln P(T_0,T_i),\ln P(T_0,T_j)\right]_s$$

$$= \left[-\int_{t=\cdot}^{T_0}\int_{y=0}^{T_i-t} d\widetilde{Z}^\omega(t,y)dy\sigma(t,y) + \int_{t=\cdot}^{T_0}\int_{y=0}^{T_0-t} d\widetilde{Z}^\omega(t,y)dy\sigma(t,y),\right.$$

$$\left.-\int_{t=\cdot}^{T_0}\int_{u=0}^{T_j-t} d\widetilde{Z}^\omega(t,u)du\sigma(t,u) + \int_{t=\cdot}^{T_0}\int_{u=0}^{T_0-t} d\widetilde{Z}^\omega(t,u)du\sigma(t,u)\right]_s$$

Using (2.9) and the invariance of the quadratic covariation with respect to changes of equivalent measures concludes the proof. ∎

Proof of Lemma 2.4

Applying the homogeneity of Φ, the law of iterated expectations and the relationship between conditioned expectations with respect to $\mathbb{Q}$ and $\mathbb{Q}_{T_j}$ (Brace and Musiela, 1994a), we have the following chain of equalities that proves the lemma.

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t=s}^{T_0} dtr(t)}\left[\Phi(\mathbf{P}_{T_0})\right]_+\Big|\,\mathcal{F}_s\right] = B(s)\mathbb{E}^{\mathbb{Q}}\left[\left[\sum_{j=0}^{N}\left.\frac{\partial \Phi(\mathbf{y})}{\partial y_j}\right|_{y=\mathbf{P}_{T_0}}\overline{P}(T_0,T_j)\right]_+\Bigg|\,\mathcal{F}_s\right]$$

$$= B(s)\sum_{j=0}^{N}\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{\partial \Phi(\mathbf{y})}{\partial y_j}\right|_{y=\mathbf{P}_{T_0}}\frac{P(T_0,T_j)}{B(T_0)}\mathbf{1}_{\Phi(\mathbf{P}_{T_0})>0}\Bigg|\,\mathcal{F}_s\right]$$

$$= \frac{B(s)}{B(0)}\sum_{j=0}^{N}\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{\partial \Phi(\mathbf{y})}{\partial y_j}\right|_{y=\mathbf{P}_{T_0}}e^{-\int_{t=s}^{T_j} dtr(t)}\mathbf{1}_{\Phi(\mathbf{P}_{T_0})>0}\Bigg|\,\mathcal{F}_s\right]$$

$$= \sum_{j=0}^{N}P(s,T_j)\mathbb{E}^{\mathbb{Q}_{T_j}}\left[\left.\frac{\partial \Phi(\mathbf{y})}{\partial y_j}\right|_{y=\mathbf{P}_{T_0}}\mathbf{1}_{\Phi(\mathbf{P}_{T_0})>0}\Bigg|\,\mathcal{F}_s\right]$$

∎

Proof of Lemma 2.5

Making $T = T_i$ and $\omega = T_j$ in expression (1.67), taking conditional expectation, and using the martingale property of the stochastic string process (see Chapter 1), we have

$$\mathbb{E}^{\mathbb{Q}_{T_j}}\left[\ln P\left(T_0, T_i\right) | \mathcal{F}_s\right]$$

$$= \ln\left(\frac{P\left(s, T_i\right)}{P\left(s, T_0\right)}\right) + \int_{t=s}^{T_0} dt \left[\int_{y=0}^{T_i-t} \int_{u=0}^{T_j-t} dy du R\left(u, y\right) - \frac{1}{2}\int_{y=0}^{T_i-t}\int_{u=0}^{T_i-t} dy du R\left(u, y\right)\right.$$

$$\left. - \int_{y=0}^{T_0-t}\int_{u=0}^{T_j-t} dy du R\left(u, y\right) + \frac{1}{2}\int_{y=0}^{T_0-t}\int_{u=0}^{T_0-t} dy du R\left(u, y\right)\right]$$

$$= \ln\left(\frac{P\left(s, T_i\right)}{P\left(s, T_0\right)}\right) + \Delta_{ij}\left(s, T_0\right) - \frac{1}{2}\Delta_{ii}\left(s, T_0\right)$$

where the last equality is a consequence of Lemma 2.3. The variance is obtained trivially from the same lemma. ∎

Proof of Theorem 2.9

Lemma 2.4 allows us to write the price at time s of the contingent claim as

$$C\left[s, \boldsymbol{P}_s\right] = \sum_{j=0}^{N} P\left(s, T_j\right) \mathbb{E}^{\mathbb{Q}_{T_j}} \left[\left. \frac{\partial \Phi\left(\mathbf{y}\right)}{\partial y_j}\right|_{y=\mathbf{P}_{T_0}} \mathbf{1}_{\Phi\left(\mathbf{P}_{T_0}\right)>0} \middle| \mathcal{F}_s\right] \tag{2.54}$$

For $i = 1, \ldots, N$, $j = 0, 1, \ldots, N$, we define

$$x'_{ij}\left(s, T_0\right) \equiv \frac{\ln P\left(T_0, T_i\right) - \ln\left(\frac{P\left(s, T_i\right)}{P\left(s, T_0\right)}\right) - \Delta_{ij}\left(s, T_0\right) + \frac{1}{2}\Delta_{ii}\left(s, T_0\right)}{\sqrt{\Delta_{ii}\left(s, T_0\right)}}$$

By Lemma 2.5 we have that, under $\mathbb{Q}_{T_j}$, $x'_{ij}\left(s, T_0\right)$ has a conditional standard normal distribution. This new variable, together with the condition $P\left(T_0, T_0\right) = 1$, allow us to write

$$\Phi\left(\mathbf{P}_{T_0}\right) = \Phi\left(P\left(T_0, T_0\right), P\left(T_0, T_1\right), \ldots, P\left(T_0, T_N\right)\right)$$

$$= \Phi\left[1, e^{\sqrt{\Delta_{11}}x'_{1j}+\Delta_{1j}-\frac{1}{2}\Delta_{11}}\frac{P\left(s, T_1\right)}{P\left(s, T_0\right)}, \cdots, e^{\sqrt{\Delta_{NN}}x'_{Nj}+\Delta_{Nj}-\frac{1}{2}\Delta_{NN}}\frac{P\left(s, T_N\right)}{P\left(s, T_0\right)}\right] \tag{2.55}$$

The homogeneity property implies that $\Phi\left(\mathbf{P}_{T_0}\right) > 0$ iff

$$\Phi\left[P\left(s, T_0\right), e^{\sqrt{\Delta_{11}}x'_{1j}+\Delta_{1j}-\frac{1}{2}\Delta_{11}}P\left(s, T_1\right), \cdots, e^{\sqrt{\Delta_{NN}}x'_{Nj}+\Delta_{Nj}-\frac{1}{2}\Delta_{NN}}P\left(s, T_N\right)\right] > 0$$

Writing the expectation in (2.54) in terms of the new variables x'_{ij} we have

$$C\left[s, \boldsymbol{P}_s\right] = \sum_{j=0}^{N} P\left(s, T_j\right) \int_{\mathbb{R}^N} d\boldsymbol{x}'_j\, g\left(x'_{1j}, \ldots, x'_{Nj}; M\right) \left. \frac{\partial \Phi\left(\mathbf{y}\right)}{\partial y_j}\right|_{\mathbf{y}=\mathbf{P}_{T_0}} \mathbf{1}_{\Phi\left(\mathbf{P}_{T_0}\right)>0}$$

where M is the correlation matrix given by $(M)_{kl} = corr\left(x'_{kj}, x'_{lj}\right) = \dfrac{\Delta_{kl}}{\sqrt{\Delta_{kk}}\sqrt{\Delta_{ll}}}$, $k, l = 1, \ldots, N$ and $\Phi\{P\} > 0$ is explicitly expressed in terms of x'_{ij} by (2.55). Making the change of variable

$x_i \equiv x'_{ij} + \frac{\Delta_{ij}}{\sqrt{\Delta_{ii}}}$, $i = 1, \ldots, N$ and taking into account (2.55), we finally obtain

$$C\left[s, \boldsymbol{P}_s\right] = P\left(s, T_0\right) \int_{\mathbb{R}^N} d\boldsymbol{x} g\left(x_1, \ldots, x_N; M\right) \left.\frac{\partial \Phi\left(\mathbf{y}\right)}{\partial y_0}\right|_{\mathbf{y}=\mathbf{P}_{T_0}} \mathbf{1}_{\Phi\left(\mathbf{P}_{T_0}\right)>0}$$

$$+ \sum_{j=1}^{N} P\left(s, T_j\right) \int_{\mathbb{R}^N} d\boldsymbol{x} g\left(x_1, \ldots, x_N; M\right) e^{\sqrt{\Delta_{jj}} x_j - \frac{1}{2}\Delta_{jj}} \left.\frac{\partial \Phi\left(\mathbf{y}\right)}{\partial y_j}\right|_{\mathbf{y}=\mathbf{P}_{T_0}} \mathbf{1}_{\Phi\left(\mathbf{P}_{T_0}\right)>0}$$

$$= P\left(s, T_0\right) \int_{\mathbb{R}^N} d\boldsymbol{x} g\left(x_1, \ldots, x_N; M\right) \left.\frac{\partial \Phi\left(\mathbf{y}\right)}{\partial y_0}\right|_{\mathbf{y}=\mathbf{P}_{T_0}} \mathbf{1}_{\Phi\left(\mathbf{P}_{T_0}\right)>0}$$

$$+ P\left(s, T_0\right) \sum_{j=1}^{N} \int_{\mathbb{R}^N} d\boldsymbol{x} g\left(x_1, \ldots, x_N; M\right) P\left(T_0, T_j\right) \left.\frac{\partial \Phi\left(\mathbf{y}\right)}{\partial y_j}\right|_{\mathbf{y}=\mathbf{P}_{T_0}} \mathbf{1}_{\Phi\left(\mathbf{P}_{T_0}\right)>0}$$

$$= P\left(s, T_0\right) \int_{\mathbb{R}^N} d\boldsymbol{x} g\left(x_1, \ldots, x_N; M\right) \left[\Phi\left(P\left(T_0, T_0\right), P\left(T_0, T_1\right), \ldots, P\left(T_0, T_N\right)\right)\right]_+$$

$$= \int_{\mathbb{R}^N} d\boldsymbol{x} g\left(x_1, \ldots, x_N; M\right)$$

$$\times \left(\Phi\left[P\left(s, T_0\right), P\left(s, T_1\right) e^{\sqrt{\Delta_{11}} x_1 - \frac{1}{2}\Delta_{11}}, \ldots, P\left(s, T_N\right) e^{\sqrt{\Delta_{NN}} x_N - \frac{1}{2}\Delta_{NN}}\right]\right)_+$$

Chapter 3

Valuation of Caps and Swaptions under a Stochastic String Model

3.1   Introduction

During the last years, the market of interest rate derivatives has experienced an uninterrupted increase from an average monthly trading volume of 2,177 in 2005 to 6,729 in 2012 (ISDA (2012a)).[1] The dynamism of this market and its high trading volume ($341.2 trillions of total notional outstanding in 2012, ISDA (2012b)) have created a great interest among researchers. We can mention as specially fruitful the theoretical and empirical analysis of *caps* and *swaptions*.[2] See Brace and Musiela (1994a, b), Brace *et al.* (1997), Jamshidian (1997), Navas (1999), De Jong *et al.* (2001, 2004), Longstaff *et al.* (2001a, b), Collin-Dufresne and Goldstein (2001, 2002), Klaasen *et al.* (2003), Eberlein and Kluge (2006), among others.

In the previous two chapters we have enlarged considerably the stochastic string modeling for the term structure of interest rates (TSIR) initiated by Santa-Clara and Sornette (2001). The main goal of this chapter is to apply the stochastic string framework to price caps and swaptions and to shed light on three fundamental issues: a) the use of Black (1976) formulas by practitioners, b) the relative valuation of caps and swaptions, as proposed in Longstaff, Santa-Clara and Schwartz (2001a) (LSS from now on), and c) the observational equivalence obtained by Kerkhof and Pelsser (2002).

In the industry, practitioners typically use the Black formula for pricing options on futures to value caps and swaptions given their implied volatilities. However, it is well known that this approach is inconsistent (see, for instance, Björk (2004) for a more detailed discussion). Several theoretical papers have tried to explain this behavior (Goldys *et al.* (1994), Musiela (1994), Miltersen *et al.* (1997), Sandmann and Sondermann (1997)) and the culmination of these efforts arrives with the LIBOR market model of Brace *et al.* (1997) (BGM from now on). Almost simultaneously,

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[1]Except a decrease in the period 2009-2010 that could be due to the financial crisis.

[2]For an introduction to these derivatives see, for instance, Longstaff *et al.* (2001a).

Jamshidian (1997) re-obtains by other means the BGM model and applies his methodology to the valuation of swaps, obtaining the so called swap market model (SMM).

The main achievement of the BGM model consists of obtaining the Black formula for caplets within the HJM scheme. This is attained by imposing the log-normality of the LIBOR rate through a relationship between its volatility and the HJM volatilities. In a similar way, dealing with the forward swap rate, the Black (1976) formula for swaptions in the swap market model is obtained.

We pose three fundamental objections to market models. First, these models do not obtain the Black formula within a general framework for interest rates. Instead, they impose *ad hoc* the log-normality of forward rates to obtain the desired results.

Second, although the main idea underlying market models is that the Black formulas reflect the market view on caps and swaptions prices, the use of these formulas in the market is just a way of determining the price of the derivative through its implied volatility (as emphasized by LSS). Thus, it does not imply that market participants perceive the Black (1976) model as the most appropriate one for caps and swaptions.

Thirdly, although each market model justifies the use of the Black formula for each derivative, both models (BGM and SMM) are incompatible (see Jamshidian (1997)). Hence, using Black formulas for caps and swaptions is logically inconsistent.

We aim to apply stochastic strings for modeling the term structure of interest rates, as discussed in the previous two chapters, and pricing interest rate derivatives. We consider two alternative ways. The first one is a Gaussian model where the cap valuation formula is the infinite-dimensional extension of that obtained in Gaussian HJM models. The Black's formula is obtained as an approximation when the day-count fraction tends to infinity. Similarly, an exact swaption pricing formula is derived, which can be approximated by Black formula for swaptions under the assumption of equal covariances when considering different bond prices. Our second alternative leads us to a stochastic string LIBOR market model that provides an exact Black formula.

The relative valuation of caps and swaptions is the second problem we are interested in. As we shall see later, a cap can be seen as a portfolio of bond put options and a swaption is equivalent to a put option on a bond portfolio. Obviously, the valuation of any portfolio is affected by the correlations between its constituents. Then, as shown by Merton (1973) and emphasized by LSS, the relation between prices of caps and swaptions is driven by the correlation structure between forward rates.

As we will see later, using different assumptions, LSS price caps with real swaption prices to obtain information on this correlation structure. They obtain that: a) the hypothesis that the real cap prices equate the values implied by the swaptions market is rejected for all the maturities and b) caps are undervalued with respect to swaptions.

Several solutions to this problem have been proposed in the literature. LSS suggest to include a time-varying covariance structure in the model. Following this idea and based on empirical evidence, Collin-Dufresne and Goldstein (2001) propose a random field model with stochastic volatility and correlation and they obtain closed-form expressions for cap prices. They also develop some efficient methods for pricing swaptions.

We will be able to solve the problem of the relative valuation of caps and swaptions maintaining time-independent covariances. We will provide closed-form expressions and we will show that one of the assumptions in LSS is inconsistent with our framework. Hence, the problem of the relative valuation of caps and swaptions could arise from a bad specification of the LSS model.

Our third issue of interest is related to the number of parameters to be estimated in the BGM and LSS models. LSS argue that we need to estimate just $n(n+1)/2$ parameters in a model with n forward rates. In contrast, the BGM model requires to estimate n parameters for each forward rate and, hence, it is less parsimonious than the LSS model. However, in a posterior paper, Kerkhof and Pelsser (2002) show that both models are observationally equivalent and, then, the same number of parameters ($nk - k(k-1)/2$ parameters for k-factor models) must be estimated. We will show that this observational equivalence remains in our framework and that our model is the most parsimonious one in almost all the cases as it requires to estimate just k parameters.

The remaining of the chapter is as follows. Section 3.2 includes the main results of the stochastic string modeling and applies this method to price European options on bonds and bond portfolios. We present a type of covariance function between shocks to the forward curve, with important properties, introduced in the previous chapter. For this covariance function, the short-term interest rate follows the Hull and White (1990) dynamics and, then, we can generalize the pricing expressions for portfolio options obtained in Jamshidian (1989) to any number of factors.

Section 3.3 provides a closed-form expression for the cap price that is the infinite-dimensional generalization of that obtained in Gaussian HJM models. We show that a certain approximation of this closed-form expression constitutes the Black (1976) formula commonly used in the industry. In a similar way, we obtain analytical expressions for swaption prices, generalizing those obtained by Brace and Musiela (1994b) and Musiela and Rutkowski (1995). Another approximation allows us to recover the Black formula for swaptions. This section ends pricing swaptions with the generalization of the Jamshidian (1989) formula for portfolio options.

Section 3.4 presents a stochastic string model for the LIBOR forward rate that allows us to recover explicitly the Black formula for caps. This model nests the BGM model and an approximated one that is compatible with Gaussian models. We show that the swap forward rate cannot follow a lognormal distribution (eliminating the possibility of including the swap market model).

Section 3.5 analyzes in detail the assumptions of LSS from our modeling point of view and tries

to explain the problem of the relative valuation of caps and swaptions. We find that the assumption related to the equality of the factors for the historical and implied covariances is wrong. Then, the problem of the relative valuation of caps and swaptions can arise due to a bad specification of the LSS model. Section 3.6 proves that the observational equivalence between the LSS and BGM models remains true in our setup. We also show that, within the stochastic string framework, the LSS and BGM models become more parsimonious in almost all the cases. Section 3.7 summarizes the main conclusions. Two final technical appendices include the expressions of the approximated stochastic string LIBOR market model and the proofs, respectively.

3.2   Preliminary Results

In Chapter 1 we assumed the following dynamics for the instantaneous forward rate in the Musiela (1993) parameterization

$$df\left(t,x\right) = \alpha\left(t,x\right)dt + \sigma\left(t,x\right)dZ\left(t,x\right), \quad 0 \leq t \leq \Upsilon, \quad x \geq 0$$

where $Z\left(t,x\right)$ is the so called *stochastic string process* and Υ is the finite time horizon for trading risk-free zero-coupon bonds. Several assumptions on this process and on the filtration of the probability space jointly with a martingale representation property lead to the no-arbitrage dynamics

$$df\left(t,x\right) = \left[\frac{\partial f\left(t,x\right)}{\partial x} + \int_{y=0}^{x} dy R_t\left(x,y\right)\right]dt + \sigma\left(t,x\right)d\widetilde{Z}\left(t,x\right)$$

where $d\widetilde{Z}\left(t,x\right)$ is the stochastic string shock under the equivalent martingale measure $\mathbb{Q}$. Moreover,

$$R_t\left(x,y\right) \equiv c\left(t,x,y\right)\sigma\left(t,x\right)\sigma\left(t,y\right) = \frac{cov\left[df\left(t,x\right),df\left(t,y\right)\right]}{dt}$$

where

$$c\left(t,x,y\right) = \frac{d\left[Z\left(\cdot,x\right),Z\left(\cdot,y\right)\right]_t}{dt} = corr\left[dZ\left(t,x\right),dZ\left(t,y\right)\right]$$

is the correlation function between stochastic string shocks. Taking $\sigma\left(t,x\right)$ and $c\left(t,x,y\right)$ as deterministic (Gaussian model), in Chapter 2 we provided the following result for the pricing of European-type derivatives whose payoff function is homogeneous of degree one.

Theorem 3.1 *Consider the following assumptions:*

a) *The final payoff of a contingent claim at time T_0 is given by $\boldsymbol{C}\left[T_0,\mathbf{P}_{T_0}\right] = \max\left(\Phi\left(\mathbf{P}_{T_0}\right),0\right)$ where $\mathbf{P}_{T_0} = \left(P\left(T_0,T_0\right),P\left(T_0,T_1\right),\ldots,P\left(T_0,T_n\right)\right)$, $P\left(T_0,T_i\right)$ denotes the price at time T_0 of a bond maturing at time T_i, with $T_0 < T_1 < \cdots < T_n$, and $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}$ is a homogeneous function of degree one.*

b) *The correlation matrix M with elements $M_{ij}(s, T_0) = \frac{\Delta_{ij}(s,T_0)}{\sqrt{\Delta_{ii}(s,T_0)}\sqrt{\Delta_{jj}(s,T_0)}}$, $i, j = 1, \ldots, n$ is*
deterministic and non-singular with

$$\Delta_{ij}(s, T_0) \equiv cov\left[\ln P(T_0, T_i), \ln P(T_0, T_j)|\mathcal{F}_s\right] = \int_{t=s}^{T_0} dt \left[\int_{y=T_0-t}^{T_i-t}\int_{u=T_0-t}^{T_j-t} dy du R_t(u, y)\right].$$

(3.1)

Then, the price at time s of this contingent claim is given by

$$\boldsymbol{C}\left[s, \boldsymbol{P}_s\right] = \int_{\mathbb{R}^n} d\mathbf{x} g\left(x_1, \ldots, x_n; M\right)$$
$$\times \left(\Phi\left[P(s, T_0), P(s, T_1) e^{\sqrt{\Delta_{11}}x_1 - \frac{1}{2}\Delta_{11}}, \ldots, P(s, T_N) e^{\sqrt{\Delta_{nn}}x_n - \frac{1}{2}\Delta_{nn}}\right]\right)_+$$

(3.2)

where

$$g\left(x_1, \ldots, x_n; M\right) = \frac{1}{\sqrt{(2\pi)^n |M|}} \exp\left(-\frac{1}{2}\sum_{i,j=1}^n x_i \left(M^{-1}\right)_{ij} x_j\right)$$

is the density function of a multivariate random variable. ∎

For completeness, we next present a result (Example 1 in Chapter 2) that follows from Theorem 3.1.

Corollary 3.1 *Consider a European call option that matures at time T_0 with strike K written on a zero-coupon bond that matures at time $T > T_0$. The price at time t of this option is given by*

$$\textbf{Call}_K(t, T_0, T) = P(t, T)\mathcal{N}(d_1) - KP(t, T_0)\mathcal{N}(d_2)$$

(3.3)

where $\mathcal{N}(\cdot)$ denotes the distribution function of a standard normal random variable with

$$d_1 = \frac{\ln\left(\frac{P(t,T)}{KP(t,T_0)}\right) + \frac{1}{2}\Omega(t, T_0, T)}{\sqrt{\Omega(t, T_0, T)}}, \quad d_2 = d_1 - \sqrt{\Omega(t, T_0, T)}$$

and

$$\Omega(t, T_0, T) = \int_{v=t}^{T_0} dv \left[\int_{y=T_0-v}^{T-v}\int_{w=T_0-v}^{T-v} dy dw R_v(u, y)\right]$$

∎

Applying the put-call parity and the properties of the distribution function for a standard normal variable allows us to obtain explicitly the put price, as stated in the next corollary.

Corollary 3.2 *Consider a European put option that matures at time T_0 with strike K written on a zero-coupon bond that matures at $T > T_0$. The price at time t of this option is given by*

$$\textbf{Put}_K(t, T_0, T) = KP(t, T_0)\mathcal{N}(-d_2) - P(t, T)\mathcal{N}(-d_1)$$

(3.4)

∎

The pricing of swaptions requires pricing a European put option on a bond portfolio. This price is obtained as a corollary of the following theorem that prices a European call portfolio option.

Theorem 3.2 *Consider a European call option that matures at time T_0 with strike K written on the cash-flows $C_1, \ldots, C_n$ received at times $T_1, \ldots, T_n$. The price at time t of this option, $\mathbf{Call}_K[t, \boldsymbol{P}_t]$, is given by*

$$\int_{\mathbb{R}^n} d\boldsymbol{x} \left[\sum_{i=1}^n C_i P(t, T_i) g\left(x_1 - \frac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \frac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right) - KP(t, T_0) g(x_1, \ldots, x_n; M) \right]_+ \quad (3.5)$$

or, alternatively,

$$\int_{\mathbb{R}^n} d\boldsymbol{y} \left[\sum_{i=1}^n C_i P(t, T_i) g(y_1 + \gamma_{1i}, \ldots, y_n + \gamma_{ni}; I_n) - KP(t, T_0) g(y_1, \ldots, y_n; I_n) \right]_+$$

with $\gamma_i = (\gamma_{1i}, \ldots, \gamma_{ni})'$ verifying $\gamma_i' \gamma_j = \Delta_{ij}$. ∎

Proof. See Appendix B. ∎

Remark 3.1 *The previous theorem for $n = 1$ and $C_1 = 1$ provides expression (3.3) for the price of a European call on a zero-coupon bond.* ∎

Similarly to Theorem 3.2, we now provide put option prices.

Theorem 3.3 *Consider a European put option that matures at time T_0 with strike K written on the cash-flows $C_1, \ldots, C_n$ received at times $T_1, \ldots, T_n$. The price at time t of this option, $\mathbf{Put}_K[t, \boldsymbol{P}_t]$, is given by*

$$\int_{\mathbb{R}^n} d\boldsymbol{x} \left[KP(t, T_0) g(x_1, \ldots, x_n; M) - \sum_{i=1}^n C_i P(t, T_i) g\left(x_1 - \frac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \frac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right) \right]_+$$

or, alternatively,

$$\int_{\mathbb{R}^n} d\boldsymbol{y} \left[KP(t, T_0) g(y_1, \ldots, y_n; I_n) - \sum_{i=1}^n C_i P(t, T_i) g(y_1 + \gamma_{1i}, \ldots, y_n + \gamma_{ni}; I_n) \right]_+ \quad (3.6)$$

with $\gamma_i = (\gamma_{1i}, \ldots, \gamma_{ni})'$ verifying $\gamma_i' \gamma_j = \Delta_{ij}$. ∎

Remark 3.2 *Expression (3.6) is formally identical to that in Theorem 3.1 in Brace and Musiela (1994a). The only difference is that, in our expression, the covariance is given by*

$$\Delta_{ij}(t, T_0) = \int_{s=t}^{T_0} ds \sum_{k=0}^{\infty} \left[\int_{y=T_0-s}^{T_i-s} dy \sigma^{(k)}(s, y) \right] \left[\int_{y=T_0-s}^{T_j-s} du \sigma^{(k)}(s, u) \right] \quad (3.7)$$

where $\left\{ \sigma^{(k)}(s, y) \right\}_{k=0}^{\infty}$ is the volatilities set in an infinite-dimensional HJM model. Moreover, in Brace and Musiela (1994a), the sum in (3.7) includes just a finite number of terms, as it corresponds to a multi-factor HJM model (see Chapter 2). ∎

Previous expressions for portfolio options are quite computationally cumbersome as they involve solving multiple integrals, as many as bonds included in the portfolio. Jamshidian (1989) avoids this problem introducing what is known as the *Jamshidian's trick*: in one-factor models, portfolio options can be interpreted as a portfolio of options (with appropriate strikes) on the same underlying bonds.

We will use a very concrete type of covariance function, proposed in Chapter 2 to show that, in our framework, we can extend the Jamshidian's idea to models with any number of factors. In this way, we can obtain a simple expression for the price of a portfolio option avoiding the need of solving complex numerical integrals.

We start by presenting a proposition that shows an important property of the covariance presented in the previous chapter in its time-independent version.

Proposition 3.1 *Consider the stochastic string model given by the homogeneous covariance*

$$R^{*(n)}(x,y) = e^{-\tau(x+y)} \sum_{k=0}^{n} \lambda_k L_k(x) L_k(y),\ 0 < \tau < \frac{1}{2},\ \forall n \in \mathbb{N} \tag{3.8}$$

with $\lambda_k \geq \lambda_{k+1} > 0$, where L_n are Laguerre polynomials $L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n} (e^{-x} x^n)$. Then, the short-term interest rate $r(t)$ follows the dynamics specified in Hull and White (1990) under the equivalent martingale measure, that is,

$$dr(t) = [\omega(t) - \varkappa r(t)]\, dt + \sigma d\widetilde{Z}(t,0) \tag{3.9}$$

with

$$\varkappa = \frac{\sum_{k=0}^{n} \lambda_k (k+\tau)}{\sum_{k=0}^{n} \lambda_k}$$

$$\sigma = \sqrt{\sum_{k=0}^{n} \lambda_k}$$

$$\omega(t) = \varkappa \left[\int_{u=0}^{t} du \left. \frac{\partial f(u,x)}{\partial x} \right|_{x=0} + r(0) \right] + \left. \frac{\partial f(0,x)}{\partial x} \right|_{x=0} + \int_{u=0}^{t} du \left. \frac{\partial^2 f(u,x)}{\partial x^2} \right|_{x=0} + \sigma^2 t$$

Proof. See Appendix B. ∎

Remark 3.3 *According to this proposition, all the multi-factor HJM models obtained with the covariance (3.8) are Markovian . Moreover, the bond price is given by*

$$P(r(t),t,T) = e^{A(t,T)-B(t,T)r(t)} \tag{3.10}$$

with

$$A(t,T) = \frac{1}{2}\sigma^2 \int_{s=t}^{T} ds B^2(s,T) - \int_{s=t}^{T} ds \omega(s) B(s,T)$$

$$B(t,T) = \frac{1 - e^{-\varkappa(T-t)}}{\varkappa}$$

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It is important to note that if we had considered time-dependent coefficients in (3.8) only the model with $n = 0$ would be Markovian (see Chapter 2). ∎

Using the previous results, the Jamshidian's trick, and expressions (3.3) and (3.4), straightforward algebra leads to the final call and put prices.

Proposition 3.2 *Consider European call and put options with maturity T_0 and strike K written on the cash-flows $C_1, \ldots, C_n$ received at times $T_1, \ldots, T_n$. Then, if $R_t(x,y) = R^{*(m)}(x,y)$, their prices are given by*

$$\mathbf{Call}_K^{*(m)}[t, \boldsymbol{P}_t] = \sum_{i=1}^{n} C_i \left[P(t, T_i) \mathcal{N}\left(d_{1,i}^{*(m)}\right) - K_i P(t, T_0) \mathcal{N}\left(d_{2,i}^{*(m)}\right) \right]$$

$$\mathbf{Put}_K^{*(m)}[t, \boldsymbol{P}_t] = \sum_{i=1}^{n} C_i \left[K_i P(t, T_0) \mathcal{N}\left(-d_{2,i}^{*(m)}\right) - P(t, T_i) \mathcal{N}\left(-d_{1,i}^{*(m)}\right) \right]$$

where

$$d_{1,i}^{*(m)} = \frac{\ln\left(\frac{P(t,T_i)}{K_i P(t,T_0)}\right) + \frac{1}{2}\Omega^{*(m)}(t, T_0, T_i)}{\sqrt{\Omega^{*(m)}(t, T_0, T_i)}}, \quad d_{2,i}^{*(m)} = d_{1,i}^{*(m)} - \sqrt{\Omega^{*(m)}(t, T_0, T_i)}$$

with

$$\Omega^{*(m)}(t, T_0, T_i) = \int_{v=t}^{T_0} dv \left[\int_{y=T_0-v}^{T_i-v} \int_{w=T_0-v}^{T_i-v} dy\,dw\,R^{*(m)}(w, y) \right]$$

where $K_i = P(r^*, T_0, T_i)$ and r^* solves $\sum_{i=1}^{n} C_i P(r^*, T_0, T_i) = K$, for $P(r, t, T)$ given by (3.10). ∎

Proof. See Appendix B. ∎

3.3 Valuation of Caps and Swaptions under the Gaussian Stochastic String Model

From now on we consider the interval $[T_0, T_n]$ and the partition $\{T_j = T_0 + \delta j\}_{j=1}^{n}$, $\delta = \frac{T_n - T_0}{n}$. A *cap* that starts at time T_0 and ends at time T_n on a \$1 principal is constituted by a set of n contingent claims, $\left\{\mathbf{Cpl}_{T_{j-1}, T_j}(t)\right\}_{j=1}^{n}$, with maturities T_j, named *caplets*, whose payoff function is given by

$$\mathbf{Cpl}_{T_{j-1}, T_j}(T_j) = \delta\left[L(T_{j-1}) - K\right]_+$$

where K denotes the *cap rate* and $L(T_{j-1})$ is the LIBOR rate determined at time T_{j-1} for the period $[T_{j-1}, T_j]$ and defined by

$$P(T_{j-1}, T_j) = \left[1 + \delta L(T_{j-1})\right]^{-1}$$

As is well known (Hull and White (1990)), the caplet $\mathbf{Cpl}_{T_{j-1}, T_j}(T_j)$ can be interpreted as $(1 + \delta K)$ European puts maturing at T_{j-1} with strike $(1 + \delta K)^{-1}$ on a \$1 face value discount bond that matures at time T_j.

Using the expression for the European put price (see (3.4)) and summing the caplet prices we obtain the cap price, as stated in the following result.

Proposition 3.3 *The price at time t of a cap is given by* $\mathbf{Cap}(t) = \sum_{j=1}^{n} \mathbf{Cpl}_{T_{j-1},T_j}(t)$ *where*

$$\mathbf{Cpl}_{T_{j-1},T_j}(t) = P(t,T_{j-1})\mathcal{N}(h_{j,1}(t)) - (1+\delta K)P(t,T_j)\mathcal{N}(h_{j,2}(t)) \tag{3.11}$$

with

$$h_{j,1}(t) = \frac{-\ln\left(\frac{(1+\delta K)P(t,T_j)}{P(t,T_{j-1})}\right) + \frac{1}{2}\Omega(t,T_{j-1},T_j)}{\sqrt{\Omega(t,T_{j-1},T_j)}}, \quad h_{j,2}(t) = h_{j,1}(t) - \sqrt{\Omega(t,T_{j-1},T_j)}$$

∎

Remark 3.4 *Expression (3.11) for the cap price appeared previously in Chu (1996). If we consider the following approximation (see Chapter 2)*

$$\Omega(t,T_{j-1},T_j) = \Omega^{(n)}(t,T_{j-1},T_j) = \int_{v=t}^{T_{j-1}} dv \left[\int_{y=T_{j-1}-v}^{T_j-v} \int_{w=T_{j-1}-v}^{T_j-v} dy\,dw \sum_{l=0}^{n} \sigma_{HJM,t}^{(l)}(y)\,\sigma_{HJM,t}^{(l)}(w) \right]$$

we recover the cap price in Gaussian HJM models (see Brace and Musiela (1994a, b)).

∎

Corollary 3.3 *If $\delta \gg$, an approximated expression for the cap price is given as*

$$\mathbf{Cap}^B(t) = \delta \sum_{j=1}^{n} P(t,T_j) \left[L(t,T_{j-1})\mathcal{N}(h_{j,1}^B(t)) - K\mathcal{N}(h_{j,2}^B(t)) \right] \tag{3.12}$$

with

$$h_{j,1}^B(t) = \frac{\ln L(t,T_{j-1}) - \ln K + \frac{1}{2}\Omega(t,T_{j-1},T_j)}{\sqrt{\Omega(t,T_{j-1},T_j)}}, \quad h_{j,2}^B(t) = h_{j,1}^B(t) - \sqrt{\Omega(t,T_{j-1},T_j)}$$

where $L(t,T_{j-1})$ denotes the process for the LIBOR rate, defined as

$$1 + \delta L(t,T_{j-1}) = \frac{P(t,T_{j-1})}{P(t,T_j)} \tag{3.13}$$

∎

Proof. See Appendix B. ∎

Remark 3.5 *Expression (3.12) is the infinite-dimensional extension of that obtained in the LIBOR market model by Brace et al. (1997) and corresponds to the Black formula for caps (see, for instance, Björk (2004)).*

∎

We consider now a *payer swaption* with strike K, \$1 principal, and maturing at time T_0. At maturity T_0, its holder has the option to enter into a swap with settlement times $\{T_j\}_{j=1}^n$, with $T_0 < T_1 < \cdots < T_n$, $T_j - T_{j-1} = \delta$ where he has to pay the fixed amount $K\delta$ and receives $\delta L(T_{j-1})$.

It is well known (Chance (2003)) that the fixed payer swaption can be interpreted as a European put with strike \$1 on a sequence of cash-flows $C_1, \ldots, C_n$ obtained at times $T_1, \ldots, T_n$ with $C_j = \delta K$ for $j = 1, \ldots, n-1$ and $C_n = 1 + \delta K$. Therefore, we can apply Theorem 3.3 to obtain the following result.

Proposition 3.4 *The price at time t of a fixed payer swaption with strike K that matures at time T_0 is given by*

$$\mathbf{Swn}(t) = \int_{\mathbb{R}^n} d\boldsymbol{x} \left[P(t, T_0) g(x_1, \ldots, x_n; M) - \sum_{i=1}^n C_i P(t, T_i) g\left(x_1 - \tfrac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \tfrac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right) \right]_+ \tag{3.14}$$

with $C_j = \delta K$ for $j = 1, \ldots, n-1$ and $C_n = 1 + \delta K$. Equivalently, this price can be written as

$$\mathbf{Swn}(t) = \int_{\mathbb{R}^n} d\boldsymbol{y} \left[P(t, T_0) g(y_1, \ldots, y_n; I_n) - \sum_{i=1}^n C_i P(t, T_i) g(y_1 + \gamma_{1i}, \ldots, y_n + \gamma_{ni}; I_n) \right]_+ \tag{3.15}$$

with $\gamma_i = (\gamma_{1i}, \ldots, \gamma_{ni})'$ verifying $\gamma_i' \gamma_j = \Delta_{ij}$. ∎

Remark 3.6 *The finite-dimensional version of expression (3.15) can be found in Brace and Musiela (1994b) and Musiela and Rutkowsky (1995).* ∎

Corollary 3.4 *If $\Delta_{ij} = \Delta <<$, $i, j = 1, \ldots, n$, an approximated expression for the price of a fixed payer swaption is*

$$\mathbf{Swn}^B(t) = \delta \sum_{i=1}^n P(t, T_i) \left[\varkappa(t, T_0, n) \mathcal{N}\left(g_1^B(t, T_0)\right) - K\mathcal{N}\left(g_2^B(t, T_0)\right) \right] \tag{3.16}$$

with

$$g_1^B(t, T_0) = \frac{\ln \varkappa(t, T_0, n) - \ln K + \tfrac{1}{2}\Delta}{\sqrt{\Delta}}, \quad g_2^B(t, T_0) = g_1^B(t, T_0) - \sqrt{\Delta}$$

where

$$\varkappa(t, T_0, n) = [P(t, T_0) - P(t, T_n)] \left(\delta \sum_{i=1}^n P(t, T_i) \right)^{-1} \tag{3.17}$$

is the forward swap rate. ∎

Proof. See Appendix B. ∎

Remark 3.7 *Expression (3.16) coincides with the Black formula for swaptions (see, for instance, Björk (2004)).* ∎

To end this section, applying Proposition 3.2 to the pricing of swaptions provides the following result that, to our knowledge, is new in the literature.

Proposition 3.5 *If we consider $R_t(x, y) = R^{*(m)}(x, y)$, the price at time t of a fixed payer swaption with strike K that matures at time T_0 is given by*

$$\mathbf{Swn}^{*(m)}(t) = \sum_{i=1}^{n} C_i \left[K_i P(t, T_0) \mathcal{N}\left(-d_{2,i}^{*(m)}\right) - P(t, T_i) \mathcal{N}\left(-d_{1,i}^{*(m)}\right) \right] \qquad (3.18)$$

where $C_j = \delta K$ for $j = 1, \ldots, n - 1$, $C_n = 1 + \delta K$, and

$$d_{1,i}^{*(m)} = \frac{\ln\left(\frac{P(t,T_i)}{K_i P(t,T_0)}\right) + \frac{1}{2}\Omega^{*(m)}(t, T_0, T_i)}{\sqrt{\Omega^{*(m)}(t, T_0, T_i)}}, \quad d_{2,i}^{*(m)} = d_{1,i}^{*(m)} - \sqrt{\Omega^{*(m)}(t, T_0, T_i)}$$

with

$$\Omega^{*(m)}(t, T_0, T_i) = \int_{v=t}^{T_0} dv \left[\int_{y=T_0-v}^{T_i-v} \int_{w=T_0-v}^{T_i-v} dy \, dw \, R^{*(m)}(w, y) \right]$$

where $K_i = P(r^*, T_0, T_i)$ and r^* solves $\sum_{i=1}^{n} C_i P(r^*, T_0, T_i) = 1$, for $P(r, t, T)$ given by (3.10). ∎

It is important to note the savings in the computational cost of expression (3.18) relative to expressions (3.14) or (3.15). Moreover, our expression is not the result of a mathematical approximation but it is exact until the desired order.[3] This order is determined by the number of factors considered in a multi-factor HJM model (see Chapter 2).

3.4 The Stochastic String Market Model

In this section we introduce a market model under the stochastic string framework that nests the BGM and LSS models.

In Chapter 1 we provided the following expression for the dynamics of the bond price ratio

$$\frac{d\left[P(t, \nu) / P(t, \tau)\right]}{P(t, \nu) / P(t, \tau)} = dt \int_{x=0}^{\tau-t} \int_{y=0}^{\tau-t} dx \, dy \, R_t(x, y) + \int_{y=0}^{\tau-t} d\widetilde{Z}(t, y) \, dy \sigma(t, y)$$

$$- dt \int_{x=0}^{\tau-t} \int_{y=0}^{\nu-t} dx \, dy \, R_t(x, y) - \int_{y=0}^{\nu-t} d\widetilde{Z}(t, y) \, dy \sigma(t, y) \qquad (3.19)$$

and this relationship between stochastic string shocks under the martingale equivalent measure, $d\widetilde{Z}(t, y)$, and under the τ-forward measure, $d\widetilde{Z}^\tau(t, y)$

$$d\widetilde{Z}^\tau(t, y) = d\widetilde{Z}(t, y) + dt \int_{u=0}^{\tau-t} du \, c(t, u, y) \, \sigma(t, u) \qquad (3.20)$$

---

[3]Approximate expressions for the price of a swaption appear, for instance, in Brace and Musiela (1994b), Brace *et al.* (1997), Pang (1999), Collin-Dufresne and Goldstein (2002), and Han (2007).

Making $\nu = T_{j-1}$, $\tau = T_j$ and joining (3.19) and (3.20), we obtain the following expression under the T_j-forward measure

$$\frac{d\left[P\left(t,T_{j-1}\right)/P\left(t,T_j\right)\right]}{P\left(t,T_{j-1}\right)/P\left(t,T_j\right)} = \int_{y=T_{j-1}-t}^{T_j-t} d\widetilde{Z}^{T_j}\left(t,y\right) dy\sigma\left(t,y\right) \tag{3.21}$$

Differentiating (3.13) and using (3.21), we obtain the stochastic string dynamics for the LIBOR rate

$$\frac{dL\left(t,T_{j-1}\right)}{L\left(t,T_{j-1}\right)} = \frac{1+\delta L\left(t,T_{j-1}\right)}{\delta L\left(t,T_{j-1}\right)} \int_{y=T_{j-1}-t}^{T_j-t} d\widetilde{Z}^{T_j}\left(t,y\right) dy\sigma\left(t,y\right) \tag{3.22}$$

Our market model is completely general as we have not made any assumption on the volatility $\sigma\left(t,y\right)$. We consider now two alternative volatility structures that will lead to two concrete models within our framework.

3.4.1 The Stochastic String LIBOR Market Model

Expression (3.22) can be rewritten as

$$\frac{dL\left(t,T_{j-1}\right)}{L\left(t,T_{j-1}\right)} = \int_{y=T_{j-1}-t}^{T_j-t} d\widetilde{Z}^{T_j}\left(t,y\right) dy\overline{\sigma}\left(t,y\right) \tag{3.23}$$

with $\overline{\sigma}\left(t,y\right) = \frac{1+\delta L\left(t,T_{j-1}\right)}{\delta L\left(t,T_{j-1}\right)}\sigma\left(t,y\right)$. If we take a deterministic $\overline{\sigma}\left(t,y\right)$, we obtain a lognormal stochastic string dynamics for the LIBOR forward rate that will be named stochastic string LIBOR market model. From now on, functions that depend on σ will be overlined to indicate that we have replaced σ by $\overline{\sigma}$ in all the volatilities. As an application we will obtain again explicitly expression (3.12) that corresponds to the Black (1976) formula for the cap price.

Using the stochastic exponential in (3.23), we have

$$\frac{L\left(t,T_{j-1}\right)}{L\left(0,T_{j-1}\right)} = \exp\left\{\int_{s=0}^{t}\int_{y=T_{j-1}-s}^{T_j-s} d\widetilde{Z}^{T_j}\left(s,y\right) dy\overline{\sigma}\left(s,y\right) - \frac{1}{2}\int_{s=0}^{t}\int_{x=T_{j-1}-s}^{T_j-s}\int_{y=T_{j-1}-s}^{T_j-s} dsdxdy\overline{R}_s\left(x,y\right)\right\} \tag{3.24}$$

Making $t = T_{j-1}$ in (3.24), we obtain

$$\frac{L\left(T_{j-1},T_{j-1}\right)}{L\left(0,T_{j-1}\right)} = \exp\left\{\int_{s=0}^{T_{j-1}}\int_{y=T_{j-1}-s}^{T_j-s} d\widetilde{Z}^{T_j}\left(s,y\right) dy\overline{\sigma}\left(s,y\right)\right.$$
$$\left. - \frac{1}{2}\int_{s=0}^{T_{j-1}}\int_{x=T_{j-1}-s}^{T_j-s}\int_{y=T_{j-1}-s}^{T_j-s} dsdxdy\overline{R}_s\left(x,y\right)\right\} \tag{3.25}$$

Dividing (3.25) by (3.24) and taking logarithms, we get

$$\ln\left(\frac{L\left(T_{j-1},T_{j-1}\right)}{L\left(t,T_{j-1}\right)}\right) = \int_{s=t}^{T_{j-1}}\int_{y=T_{j-1}-s}^{T_j-s} d\widetilde{Z}^{T_j}\left(s,y\right) dy\overline{\sigma}\left(s,y\right) - \frac{1}{2}\int_{s=t}^{T_{j-1}}\int_{x=T_{j-1}-s}^{T_j-s}\int_{y=T_{j-1}-s}^{T_j-s} dsdxdy\overline{R}_s\left(x,y\right)$$

Then, under the T_j-forward measure, $\mathbb{Q}^{T_j}$, $L(T_{j-1}, T_{j-1})$ follows a conditional lognormal distribution with

$$\mathbb{E}^{\mathbb{Q}^{T_j}}\left[\ln L(T_{j-1}, T_{j-1})\middle|\mathcal{F}_t\right] = \ln L(t, T_{j-1}) - \frac{1}{2}\overline{\Omega}(t, T_{j-1}, T_j)$$

$$Var^{\mathbb{Q}^{T_j}}\left[\ln L(T_{j-1}, T_{j-1})\middle|\mathcal{F}_t\right] = \int_{s=t}^{T_{j-1}}\int_{x=T_{j-1}-s}^{T_j-s}\int_{y=T_{j-1}-s}^{T_j-s} ds\,dx\,dy\,\overline{R}_s(x,y) = \overline{\Omega}(t, T_{j-1}, T_j)$$

Hence, the caplet price is given by

$$\begin{aligned}
\mathbf{Cpl}_{T_{j-1}, T_j}(t) &= \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_{s=t}^{T_j} ds\,r(s)}\delta\left[L(T_{j-1}) - K\right]_+\middle|\mathcal{F}\right)\\
&= \delta P(t, T_j)\,\mathbb{E}^{\mathbb{Q}^{T_j}}\left(\left[L(T_{j-1}) - K\right]_+\middle|\mathcal{F}_t\right)\\
&= \delta P(t, T_j)\left[L(t, T_{j-1})\mathcal{N}\left(\overline{h}_{j,1}^B(t)\right) - K\mathcal{N}\left(\overline{h}_{j,2}^B(t)\right)\right] \tag{3.26}
\end{aligned}$$

where we have moved from measure Q to measure Q^{T_j} (Brace and Musiela (1994a)) and we have applied the properties of the lognormal distribution. Adding the caplet prices leads to expression (3.12).

An interesting particular case of (3.23) arises with the process

$$d\widetilde{Z}^{T_j}(t, x) = \sum_{i=0}^{n}\frac{\sigma_{HJM,t}^{(i)}(x)}{\sigma(t, x)}d\widetilde{W}_i^{T_j}(t)$$

where $\sigma_{HJM,t}^{(i)}(x)$ are the volatilities of a multi-factor HJM model with

$$\sigma^2(t, x) = \sum_{i=0}^{n}\left(\sigma_{HJM,t}^{(i)}(x)\right)^2$$

This process was proposed originally by Pang (1999). In the previous chapter we showed that, under certain regularity conditions, this process is an admissible stochastic string shock. Replacing it into (3.23) we obtain

$$\frac{dL(t, T_{j-1})}{L(t, T_{j-1})} = \gamma'(t, T_{j-1} - t)\cdot d\widetilde{\mathbf{W}}^{T_j}(t) \tag{3.27}$$

with

$$\gamma(t, T - t) = \frac{1 + \delta L(t, T)}{\delta L(t, T)}\begin{pmatrix}\int_{y=T-t}^{T-t+\delta} dy\,\sigma_{HJM,t}^{(0)}(y)\\ \vdots\\ \int_{y=T-t}^{T-t+\delta} dy\,\sigma_{HJM,t}^{(n)}(y)\end{pmatrix} \tag{3.28}$$

Expression (3.27) with deterministic γ equates expression (3.6) in Brace *et al.* (1997). Hence, we have shown explicitly that the BGM model is nested in our framework. Moreover, this construction can be done in infinite-dimensional terms just taking the appropriate regularity conditions for the volatilities (see Chapter 2).

However, obtaining a lognormal model for the LIBOR rate that is compatible with HJM, the BGM model, implies that the HJM volatilities must be state dependent (see (3.28)).[4] This fact supposes the disadvantage, extensible to the stochastic string LIBOR model (see (3.23)), of the incompatibility of the BGM model with the Gaussian ones, with all their important results (see Section 3.3). The next subsection proposes a way of avoiding this incompatibility at the cost of losing the exactness of the model and the absence of arbitrage.

3.4.2 An Approximated Stochastic String LIBOR Market Model

Taking $\sigma(t, y)$ as deterministic in (3.22), we work under the Gaussian stochastic string modeling and obtain the same results for pricing caps as in Section 3.3. As said before, the Gaussian modeling is incompatible with the Black formula, but we will obtain it as an approximate result. Thus, we can explain the use of this formula in the market (as an approximation) with the advantages of the Gaussian framework discussed in the previous subsection.

As in Corollary 3.3, the approximation consists of taking $\delta >>$. In this case, (3.22) can be rewritten as

$$\frac{dL(t, T_{j-1})}{L(t, T_{j-1})} = \int_{y=T_{j-1}-t}^{T_j-t} d\widetilde{Z}^{T_j}(t, y) \, dy \sigma(t, y) \tag{3.29}$$

As σ is deterministic, we recover the lognormality of the LIBOR rate although in a approximated way. Note that, although useful for pricing caps, this approximation is conceptually inappropriate as the usual values for δ are 0.25, 0.5, or 1. Moreover, this approximation implies arbitrage opportunities. In fact, taking into account (3.13), the approximation $\frac{1+\delta L(t, T_{j-1})}{\delta L(t, T_{j-1})} \simeq 1$ is equivalent to $P(t, T_j) = 0$.[5]

It is important to remark that (3.23) and (3.29) are formally identical and, then, will provide similar results. However, the results obtained from (3.23) will be exact and incompatible with the Gaussian models while those obtained from (3.29) are approximated and compatible with the Gaussian models.

From now on we will work just with the exact stochastic string LIBOR market model (see (3.23)) and relegate to the Appendix A all the expressions corresponding to the approximated model obtained from (3.29). For instance, expression (3.41) in this appendix includes the caplet price in the approximated model and is formally identical to (3.26) but replacing $\overline{h}_{j,\cdot}^B(t)$ by $h_{j,\cdot}^B(t)$.

Thus, from the point of view of our modeling, we can see the Black formula for cap prices alternatively as a) an exact expression incompatible with Gaussian models or b) an approximation (with arbitrage opportunities) of expression (3.11), valid in Gaussian stochastic string models.

---

[4]This problem is shared with other previous market models included in the HJM scheme. See, for instance, Miltersen *et al.* (1997).

[5]Alternatively, note that $\delta >>$ implies $T_j >>$ and, then, $P(t, T_j) \simeq 0$. Regarding this issue, see the conditions imposed on bond prices in Chapter 1.

3.4.3 The Dynamics of the Forward Swap Rate

We now analyze whether we could extend the swap market model (Jamshidian (1997)) to the stochastic string setting in order to recover the Black formula for swaptions that is used in the market. Or, at least, recover this formula through an approximation similar to that previously used. The key point of the swap market model (Björk (2004)) consists of proposing a lognormal dynamics for the forward swap rate $\varkappa(t, T_0, n)$ under the measure $\mathbb{Q}^X$ with numeraire $X(t) = \sum_{i=1}^{n} P(t, T_i)$.

In the stochastic string setup and working as in expression (3.21), we obtain

$$\frac{d\left[P\left(t, T_{j-1}\right) / X\left(t\right)\right]}{P\left(t, T_{j-1}\right) / X\left(t\right)} = \frac{1}{X\left(t\right)} \sum_{j=1}^{n} P\left(t, T_j\right) \int_{y=0}^{T_j-t} d\widetilde{Z}^X\left(t, y\right) dy\sigma\left(t, y\right) - \int_{y=0}^{T_{j-1}-t} d\widetilde{Z}^X\left(t, y\right) dy\sigma(t, y)$$

where $d\widetilde{Z}^X\left(t, y\right) = d\widetilde{Z}\left(t, y\right) + \frac{dt}{X(t)} \sum_{i=1}^{n} P\left(t, T_i\right) \int_{x=0}^{T_i-t} duc\left(t, u, y\right) \sigma\left(t, u\right)$ is the stochastic string shock under $\mathbb{Q}^X$. Applying this expression to the forward swap rate (see (3.17)), we obtain the following result.

Proposition 3.6 *In the stochastic string model, the dynamics of the forward swap rate under $\mathbb{Q}^X$ is given by*

$$\frac{d\varkappa\left(t, T_0, n\right)}{\varkappa\left(t, T_0, n\right)} = \frac{1}{X\left(t\right)} \sum_{j=1}^{n} P\left(t, T_j\right) \int_{y=0}^{T_j-t} d\widetilde{Z}^X(t, y)dy\sigma(t, y)$$

$$+ \frac{1}{P\left(t, T_0\right) - P\left(t, T_n\right)}$$

$$\times \left[P\left(t, T_n\right) \int_{y=0}^{T_n-t} d\widetilde{Z}^X(t, y)dy\sigma(t, y) - P\left(t, T_0\right) \int_{y=0}^{T_0-t} d\widetilde{Z}^X(t, y)dy\sigma(t, y)\right]$$

This proposition shows that the forward swap rate does not follow a lognormal distribution except in the simplest case $(n = 1)$ where, making $\varkappa(t, T_0, 1) = L(t, T_0)$, we recover the exact dynamics (3.22) of the LIBOR rate.

3.5 The Problem of the Relative Valuation of Caps and Swaptions

This section analyzes the LSS model from the point of view of the stochastic string modeling. Our aim is to provide a theoretical solution to the problem of the relative valuation of caps and swaptions, i.e., the caps mispricing with information extracted from the swaptions market. We start by analyzing the assumptions of the LSS model related to this issue and will see that some of them do not hold in our model.

Assumption 3.1 *The dynamics of the LIBOR rate is given by*

$$\frac{dL\left(t, T_{j-1}\right)}{L\left(t, T_{j-1}\right)} = \widehat{\alpha}_{j-1}\left(t\right) dt + \overline{\sigma}_{j-1}\left(t\right) d\widetilde{Z}_{j-1}\left(t\right), \ j = 1, \cdots, n \tag{3.30}$$

where $\widehat{\alpha}_{j-1}$ is undetermined, $\overline{\sigma}_{j-1}$ is deterministic, and $\widetilde{Z}_{j-1}$ are (correlated and specific for each forward rate) Brownian motions under $\mathbb{Q}$.

If we rewrite the exact dynamics of the LIBOR rate (see (3.23)) under the equivalent martingale measure and we make

$$\int_{y=T_{j-1}-t}^{T_j-t} d\widetilde{Z}(t,y)\, dy \overline{\sigma}(t,y) \equiv \overline{\sigma}_{j-1}(t)\, d\widetilde{Z}_{j-1}(t) \tag{3.31}$$

and

$$\frac{1+\delta L(t,T_{j-1})}{\delta L(t,T_{j-1})} \int_{y=T_{j-1}-t}^{T_j-t} \int_{u=0}^{Tj-t} dy\, du\, R_t(u,y) \equiv \widehat{\alpha}_{j-1}(t) \tag{3.32}$$

we recover the LIBOR dynamics (3.30) of the previous assumption.

Applying the correlation condition on $\widetilde{Z}_{j-1}$ and the identification (3.31), we get

$$dt = d\left[\widetilde{Z}_{i-1}(\cdot), \widetilde{Z}_{i-1}(\cdot)\right]_t = \frac{dt}{\overline{\sigma}_{i-1}^2(t)} \int_{x=T_{i-1}-t}^{T_i-t} \int_{y=T_{i-1}-t}^{Ti-t} dy\, du\, \overline{R}_t(x,y)$$

and, then,

$$\overline{\sigma}_{i-1}^2(t) = \int_{x=T_{i-1}-t}^{T_i-t} \int_{y=T_{i-1}-t}^{Ti-t} dy\, du\, \overline{R}_t(x,y) = \left(\frac{1+\delta L(t,T_{j-1})}{\delta L(t,T_{j-1})}\right)^2 \sum_{l=0}^{n} \left(\int_{x=T_{i-1}-t}^{T_i-t} dx\, \sigma_{HJM,t}^{(l)}(x)\right)^2$$

where we have applied that $R_t^{(n)}(x,y) = \sum_{l=0}^{n} \sigma_{HJM,t}^{(l)}(x)\, \sigma_{HJM,t}^{(l)}(y)$ is the n-order approximation to R_t in the stochastic string modeling (see Chapter 2).

Hence, we have identified the volatilities $\overline{\sigma}_i(t)$ in (3.30) as a function of the multi-factor HJM volatilities $\sigma_{HJM,t}^{(l)}(x)$. [6] Then, the dynamics (3.30) of the LSS model is compatible with the exact model for the LIBOR rate (see (3.23)) and, applying Subsection 3.4.1, its multi-factor HJM reduction is also compatible with the BGM model. Moreover, making the necessary changes in (3.31)-(3.32), it is also compatible with the approximated model (see Appendix A).

Assumption 3.2 *The dynamics of the bond price is given by*

$$d\mathbf{P}(t) = r(t)\mathbf{P}(t)\, dt + \mathbf{J}^{-1}(t)\overline{\boldsymbol{\sigma}}(t)\mathbf{L}(t)\, d\widetilde{\mathbf{Z}}(t) \tag{3.33}$$

where

$$\mathbf{P}(t) = (P(t,T_1),\ldots,P(t,T_{n-1}))'$$

$$\overline{\boldsymbol{\sigma}}(t)\mathbf{L}(t)\, d\widetilde{\mathbf{Z}}(t) = \left(\overline{\sigma}_0(t)L(t,T_0)\, d\widetilde{Z}_0(t),\ldots,\overline{\sigma}_{n-2}(t)L(t,T_{n-2})\, d\widetilde{Z}_{n-2}(t)\right)'$$

---

[6] Appropriate regularity conditions allow us to extend this analysis to infinite-dimensional HJM volatilities.

and $\mathbf{J}(t)$ *is the Jacobian matrix*

$$\mathbf{J}(t) = \frac{\partial\left[L(t,T_0),\ldots,L(t,T_{n-2})\right]}{\partial\left[P(t,T_1),\ldots,P(t,T_{n-1})\right]}$$

with $J_{ii}(t) = -\dfrac{1}{\delta}\dfrac{P(t,T_{i-1})}{P^2(t,T_i)}$, $J_{i,i-1}(t) = \dfrac{1}{\delta P(t,T_i)}$ *for* $i = 1,\cdots,n-1$, *and zero in the remaining cases.*

The introduction of the short-term interest rate in the drift of (3.33) is really the way of imposing the no-arbitrage condition as LSS just applies the Itô's rule to obtain the diffusion term because $\widehat{\alpha}_{j-1}(t)$ is unknown.[7] However, with our approximation, we can apply this rule to (3.30) and check if (3.33) is right.

From the definition of the LIBOR rate process (see (3.13)) we obtain

$$P(t,T_{j-1}) = \frac{P(t,T_0)}{\displaystyle\prod_{k=0}^{j-2}\left[1+\delta L(t,T_k)\right]}, \quad j = 2,\ldots,n$$

So we can apply the Itô's rule to $P(t,T_{j-1})$ in the form

$$
\begin{aligned}
dP(t,T_{j-1}) =& \sum_{i=0}^{j-2}\frac{\partial P(t,T_{j-1})}{\partial L(t,T_i)}dL(t,T_i) + \frac{1}{2}\sum_{i,l=0}^{j-2}\frac{\partial^2 P(t,T_{j-1})}{\partial L(t,T_i)\,\partial L(t,T_l)}d\left[L(\cdot,T_i),L(\cdot,T_l)\right]_t \\
&+ \frac{\partial P(t,T_{j-1})}{\partial P(t,T_0)}dP(t,T_0) + \sum_{i=0}^{j-2}\frac{\partial^2 P(t,T_{j-1})}{\partial L(t,T_i)\,\partial P(t,T_0)}d\left[L(\cdot,T_i),P(\cdot,T_0)\right]_t \\
&+ \frac{1}{2}\frac{\partial^2 P(t,T_{j-1})}{\partial P^2(t,T_0)}d\left[P(\cdot,T_0),P(\cdot,T_0)\right]_t
\end{aligned}
$$

For the two first terms of $\frac{dP(t,T_{j-1})}{P(t,T_{j-1})}$, working with the dynamics (3.30), using (3.13) and the symmetry of $R_t(x,y)$ and some algebra leads to

$$-\int_{y=T_0-t}^{T_{j-1}-t}\int_{u=0}^{T_0-t}dy\,du\,R_t(u,y)dt - \sum_{i=0}^{j-2}\frac{\delta P(t,T_{i+1})}{P(t,T_i)}L(t,T_i)\,\overline{\sigma}_i(t)\,d\widetilde{Z}_i(t) \qquad (3.34)$$

For the remaining terms of $\frac{dP(t,T_{j-1})}{P(t,T_{j-1})}$, substituting (3.31) in (3.30) we have

$$\frac{dL(t,T_{j-1})}{L(t,T_{j-1})} = \widehat{\alpha}_{j-1}(t)\,dt + \int_{y=T_{j-1}-t}^{T_j-t}d\widetilde{Z}(t,y)\,dy\,\overline{\sigma}(t,y)$$

that jointly with the dynamics of the bond return in the stochastic string modeling (see Chapter 1)

$$\frac{dP(t,T_0)}{P(t,T_0)} = r(t)\,dt - \int_{y=0}^{T_0-t}d\widetilde{Z}(t,y)\,dy\,\sigma(t,y)$$

---

[7]Note that this no-arbitrage condition does not eliminate the trivial arbitrage possibility previously discussed when considering $\delta \gg$.

leads to the contribution

$$r\left(t\right)dt - \int_{y=0}^{T_0-t} d\widetilde{Z}\left(t,y\right)dy\sigma\left(t,y\right) + \int_{x=T_0-t}^{T_{j-1}-t}\int_{y=0}^{T_0-t} dxdyR_t(x,y)dt - \frac{1}{2}\int_{x=0}^{T_0-t}\int_{y=0}^{T_0-t} dxdyR_t(x,y)dt$$

(3.35)

Adding up (3.34) and (3.35) we obtain

$$\frac{dP\left(t,T_{j-1}\right)}{P\left(t,T_{j-1}\right)} = \left[r\left(t\right) - \frac{1}{2}\int_{x=0}^{T_0-t}\int_{y=0}^{T_0-t} dxdyR_t(x,y)\right]dt$$

$$- \sum_{i=0}^{j-2}\frac{\delta P\left(t,T_{i+1}\right)}{P\left(t,T_i\right)}L\left(t,T_i\right)\overline{\sigma}_i\left(t\right)d\widetilde{Z}_i\left(t\right) - \int_{y=0}^{T_0-t} d\widetilde{Z}\left(t,y\right)dy\sigma\left(t,y\right)$$

Finally, as in LSS (footnote 12), for $y \leq T_0 - t$ we assume that the volatility $\sigma\left(t,y\right)$ vanishes (and so $R_t(x,y)$). Taking into account that

$$\left[\mathbf{J}^{-1}\left(t\right)\right]_{ij} = \begin{cases} -\delta\dfrac{P\left(t,T_i\right)P\left(t,T_j\right)}{P\left(t,T_{j-1}\right)} & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases}$$

we get expression (3.33) of Assumption 3.2.

Assumption 3.3 *The factors that generate the historical covariance matrix also generate the implied covariance matrix, but the implied variances of such factors can differ from their historical values.*[8]

If we take

$$\boldsymbol{\Sigma}_{ij}(t) = \frac{1}{dt}cov\left(\frac{dL\left(t,T_{i-1}\right)}{L\left(t,T_{i-1}\right)}, \frac{dL\left(t,T_{j-1}\right)}{L\left(t,T_{j-1}\right)}\right)$$

as the historical covariance matrix and use the expression (3.23) of the exact LIBOR model, we obtain

$$\boldsymbol{\Sigma}_{ij}\left(t\right) = \int_{x=T_{i-1}-t}^{T_i-t}\int_{y=T_{j-1}-t}^{T_j-t} dxdy\overline{R}_t\left(x,y\right)$$

(3.36)

In Chapter 2 we showed that, under very general conditions, we can write

$$R_t(x,y) = \sum_{k=0}^{\infty}\lambda_{t,k}f_{t,k}(x)f_{t,k}(y)$$

where $\lambda_{t,k}$ and $f_{t,k}$ are, respectively, the eigenvalues and eigenvectors of an integral operator associated to R_t. As $\lambda_{t,k} > 0$ and $\lambda_{t,k+1} < \lambda_{t,k}$, we can approximate $R_t(x,y)$ considering a finite number

---

[8]To simplify the calculations, LSS considered the historical correlation matrix, verifying that the obtained factors are almost identical to those achieved with the covariance matrix. For a comparative analysis of both matrices, see Kerkhof and Pelsser (2002).

of terms. Using this approximation jointly with the time homogeneity (following LSS) and replacing in (3.36), we obtain

$$\mathbf{\Sigma}_{ij}(t) = \frac{1 + \delta L\left(t, T_{i-1}\right)}{\delta L\left(t, T_{i-1}\right)} \frac{1 + \delta L\left(t, T_{j-1}\right)}{\delta L\left(t, T_{j-1}\right)} \int_{x=T_{i-1}-t}^{T_i-t} \int_{y=T_{j-1}-t}^{T_j-t} dx dy \sum_{k=0}^{n} \lambda_k f_k(x) f_k(y)$$

$$= \frac{1 + \delta L\left(t, T_{i-1}\right)}{\delta L\left(t, T_{i-1}\right)} \frac{1 + \delta L\left(t, T_{j-1}\right)}{\delta L\left(t, T_{j-1}\right)} \sum_{k=0}^{n} \lambda_k \int_{x=T_{i-1}-t}^{T_i-t} dx f_k(x) \int_{x=T_{j-1}-t}^{T_j-t} dx f_k(y) \qquad (3.37)$$

Furthermore, assuming that the market swaption price is given by the (exact in our Gaussian modeling) expression (3.14), we have that the covariance implied in this price corresponding to $\mathbf{\Sigma}_{ij}$ is

$$\mathbf{\Theta}_{ij}\left(t, T_0\right) = -\frac{\partial}{\partial t} \Delta_{ij}\left(t, T_0\right) = \int_{x=T_0-t}^{T_i-t} \int_{y=T_0-t}^{T_j-t} dx dy R(x, y)$$

$$= \sum_{k=0}^{n} \lambda_k \int_{x=T_0-t}^{T_i-t} dx f_k(x) \int_{y=T_0-t}^{T_j-t} dy f_k(y) \qquad (3.38)$$

This covariance is the same that could be obtained taking $cov\left(\frac{dP(t,T_i)}{P(t,T_i)}, \frac{dP(t,T_j)}{P(t,T_j)}\right)$ in (3.33). This agrees with the LSS procedure, consisting in taking the implied covariance obtained from Assumption 3.3 to generate sample paths for the dynamics (3.33). This covariance is robust with respect to the model specification, being the covariance between changes in the log-prices identical in the stochastic string Gaussian, exact LIBOR, and approximated LIBOR models. Hence, the result to be obtained will not depend on the model that really describes the swaption price.

It is easy to check that a necessary condition for Assumption 3.3 to be verified is the commutativity of the implied and historical covariance matrices, that is, $\mathbf{\Sigma}\mathbf{\Theta} = \mathbf{\Theta}\mathbf{\Sigma}$.[9] However, this is not the case, as can be seen by considering expressions (3.37)-(3.38) and applying some algebra. This is true even for the approximated model.

Therefore, Assumption 3.3 is incompatible with our model. However, it is important to note that, although the historical and implied covariance matrices are diagonalized by different eigenvectors, both matrices are determined by f_k, although these values are not obtained by the LSS procedure. The importance of these *inner factors* is emphasized by Roncoroni *et al.* (2010) that assigned them the name *shape factors* and in Chapter 2, that shows the relation with the factors obtained from a Principal Component Analysis. The knowledge of $\{\lambda_i, f_i\}_{i=0}^{k}$ is equivalent to the knowledge of $R^{(k)}(x, y)$ that drives the dynamics of the interest rates in a $(k+1)$-factor model.

Assumption 3.4 *The variance used in the Black's (1976) formula to price a caplet is the average variance for the corresponding LIBOR rate.*

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[9]See LSS, footnote 17, p. 2080.

According to our exact model, the variance associated to a caplet in the Black's (1976) formula is given by

$$\overline{\Omega}\left(t, T_{j-1}, T_j\right) = \int_{v=t}^{T_{j-1}} dv \left(\int_{x=T_{j-1}-v}^{T_j-v} \int_{y=T_{j-1}-v}^{T_j-v} dy du \overline{R}(x, y) \right)$$

or, alternatively,

$$\overline{\Omega}\left(t, T_{j-1}, T_j\right) = \int_{v=t}^{T_{j-1}} dv \overline{\sigma}_{j-1}^2\left(v\right) \tag{3.39}$$

If we take into account that LSS assumes constant variances for each interval, (3.39) can be interpreted as the time average of the variances between changes in LIBOR rates.

We have just seen that Assumption 3.3 of LSS does not hold in the stochastic string framework, which could explain the problem of the relative valuation of caps and swaptions. A possible way of avoiding the need of making this Assumption and maintaining the remaining ones in LSS consists of taking a $(k+1)$-factor parametric form of the covariance $R\left(x, y\right)$ with *good* properties (for instance, $R^{*(k)}$ in (3.8)) and estimating $\lambda_0, \cdots, \lambda_k$ from swaptions prices.[10] With these values we can build $\overline{\Omega}$ and apply the Black formula.

It is important to mention that, given the generality of the stochastic string modeling, there exist other alternatives to perform the relative valuation of caps and swaptions. For example, we could use the Gaussian scheme of Section 3.3 or we could estimate the eigenvalues directly from the TSIR. However, these possibilities involve an empirical study that is beyond the scope of this thesis.

3.6 The Observational Equivalence Problem

As mentioned earlier, LSS argue that their model is more parsimonious than the multi-factor models for the LIBOR forward rate. However, Kerkhof and Pelsser (2002) reject this argument and show that both models require the estimation of the same number of parameters. The key point in their argument is that the covariance matrices of the log-changes in the forward rates are equal in the LSS and the BGM models. We will show that this is still true in our framework, validating then the observational equivalence.

Kerkhof and Pelsser obtain the aforementioned covariance matrix for the LSS model that, with our notation and eliminating the functional dependence, can be written as

$$\mathbf{\Sigma}_{ij}^{LSS} = \rho_{ij} \overline{\sigma}_{i-1} \overline{\sigma}_{j-1} \tag{3.40}$$

where $\rho_{ij} = d\left[\widetilde{Z}_{i-1}\left(\cdot\right), \widetilde{Z}_{j-1}\left(\cdot\right)\right]_t / dt$. In our model, using (3.31), we have that

$$\rho_{ij} = \frac{1}{\overline{\sigma}_{i-1} \overline{\sigma}_{j-1}} \int_{x=T_{i-1}-t}^{T_i-t} \int_{y=T_{j-1}-t}^{T_j-t} dx dy \overline{R}_t\left(x, y\right)$$

---

[10]See Roncoroni *et al.* (2010) for more details on the estimation of the eigenvalues.

Replacing this expression in (3.40), we get

$$\mathbf{\Sigma}_{ij}^{LSS} = \int_{x=T_{i-1}-t}^{T_i-t} \int_{y=T_{j-1}-t}^{T_j-t} dxdy \overline{R}_t(x,y)$$

that coincides with the expression obtained for the LSS model in our framework, see (3.36).

We obtain now explicitly the value of $\mathbf{\Sigma}$ for the BGM model under our setup. Using expressions (3.27)-(3.28), we get

$$\begin{aligned}
\mathbf{\Sigma}_{ij}^{BGM} &= \frac{1}{dt} cov\left[\frac{dL(t,T_{i-1})}{L(t,T_{i-1})}, \frac{dL(t,T_{j-1})}{L(t,T_{j-1})}\right] \\
&= \frac{\delta L(t,T_{i-1})}{1+\delta L(t,T_{i-1})} \frac{\delta L(t,T_{j-1})}{1+\delta L(t,T_{j-1})} \int_{x=T_{i-1}-t}^{T_i-t} \int_{y=T_{j-1}-t}^{T_j-t} dxdy R_t^{(k)}(x,y) \\
&= \int_{x=T_{i-1}-t}^{T_i-t} \int_{y=T_{j-1}-t}^{T_j-t} dxdy \overline{R}_t^{(k)}(x,y)
\end{aligned}$$

where $R_t^{(k)}(x,y) = \sum_{l=0}^k \sigma_{HJM,t}^{(l)}(x)\sigma_{HJM,t}^{(l)}(y)$ is an approximation of order k to the covariance $R_t(x,y)$. Then, in concordance with Kerkhof and Pelsser (2002), we see that the covariances of the LSS and BGM models coincide if we consider a factorial version of the LSS model with the same number of factors as the BGM model. This coincidence remains even for the LSS and BGM models in our approximated framework. Then, we can conclude that our framework maintains the observational equivalence between the LSS and BGM models and, then, both models need to estimate the same number of parameters.

However, considering both models under the stochastic string modeling modifies the number of parameters to estimate, making these models more parsimonious. The reason is that, as we have seen, our framework is completely defined in a $(k+1)$-factor approximation by $\{\lambda_i, f_i\}_{i=0}^k$. The eigenfunctions f_i usually belong to a uniparametric family.[11] Hence, in general, in a k-factor approximation, we need to estimate just k parameters, a value independent of the number of forward rates, n. Then, the stochastic string approximation is more parsimonious if $k < nk - k(k-1)/2$, the number of parameters obtained by Kerkhof and Pelsser (2002). This is equivalent to $k < 2n - 1$, an inequality that is verified in most cases.

3.7 Conclusions

This chapter has applied recent developments in the stochastic string modeling to value caps and swaptions. We have obtained closed-form expressions for caps and swaptions prices and we have shown how these expressions reduce to Black formulas under certain approximations. We have also

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[11] For instance, in the case of the covariance R^*, we have $f_i(x) = L_i(x)e^{-\tau x}$ and the parameter τ is usually estimated a priori and maintained fixed (see Roncoroni *et al.* (2010) and Chapter 2).

developed a stochastic string LIBOR market model that nests the LSS and BGM models. With the same approximation applied previously to caps, we have obtained an approximated model that is compatible with the Black formula for caps.

Under our framework, we have shown that the LSS model is not well specified, which could explain the problem of the relative valuation of caps and swaptions. We have proposed a possible solution to this problem. We have also corroborated the observational equivalence of the LSS and BGM models obtained by Kerkhof and Pelsser (2002). Additionally, the number of parameters to estimate is smaller than what was stated by these authors in almost all the cases.

The generality of the stochastic string framework allows us to consider several concrete models, namely, Gaussian and (exact or approximated) lognormal ones. Hence, a possible line of future research could be to analyze empirically which model fits better to market prices of caps and swaptions. Two possible alternatives are: a) a Gaussian model compatible with deterministic HJM volatilities and with an approximated Black formula for caps or b) a model that is compatible with lognormal LIBOR rates (and state-dependent HJM volatilities) and exact Black formula. Regarding this issue, we can mention the existence of certain indirect empirical evidence in favor of Gaussian models (BGM, De Jong *et al.* (2004), Han (2007)).

3.8   Appendix A

This appendix provides the main expressions corresponding to the approximated stochastic string LIBOR model that is obtained making $\delta >>$ in the exact stochastic string LIBOR model.

- Dynamics of the LIBOR rate:

$$\frac{dL\left(t,T_{j-1}\right)}{L\left(t,T_{j-1}\right)} = \int_{y=T_{j-1}-t}^{T_j-t} d\widetilde{Z}^{T_j}\left(t,y\right) dy \sigma\left(t,y\right), \quad \text{deterministic } \sigma$$

- Caplet price:

$$\mathbf{Cpl}_{T_{j-1},T_j}\left(t\right) = \delta P\left(t,T_j\right)\left[L\left(t,T_{j-1}\right)\mathcal{N}\left(h_{j,1}^B(t)\right) - K\mathcal{N}\left(h_{j,2}^B(t)\right)\right] \tag{3.41}$$

- BGM dynamics:

$$\frac{dL\left(t,T_{j-1}\right)}{L\left(t,T_{j-1}\right)} = \boldsymbol{\gamma}'\left(t,T_{j-1}-t\right) \cdot d\widetilde{\mathbf{W}}^{T_j}(t)$$

with deterministic $\boldsymbol{\gamma}\left(t,T-t\right)$ given by

$$\begin{pmatrix} \int_{y=T-t}^{T-t+\delta} dy\sigma_{HJM,t}^{(0)}\left(y\right) \\ \vdots \\ \int_{y=T-t}^{T-t+\delta} dy\sigma_{HJM,t}^{(n)}\left(y\right) \end{pmatrix}$$

- LSS dynamics:

$$\frac{dL\left(t,T_{j-1}\right)}{L\left(t,T_{j-1}\right)} = \alpha_{j-1}\left(t\right)dt + \sigma_{j-1}\left(t\right)d\widetilde{Z}_{j-1}\left(t\right)$$

with

$$\alpha_{j-1}\left(t\right) \equiv \int_{y=T_{j-1}-t}^{T_j-t} \int_{u=0}^{T_j-t} dydu R_t(u,y)$$

$$\sigma_{j-1}(t)d\widetilde{Z}_{j-1}\left(t\right) \equiv \int_{y=T_{j-1}-t}^{T_j-t} d\widetilde{Z}(t,y)dy\sigma(t,y)$$

- LSS volatility:

$$\sigma_{i-1}^2\left(t\right) = \int_{x=T_{i-1}-t}^{T_i-t} \int_{y=T_{i-1}-t}^{T_i-t} dydu R_t\left(x,y\right) = \sum_{l=0}^{n}\left(\int_{x=T_{i-1}-t}^{T_i-t} dx\sigma_{HJM,t}^{(l)}\left(x\right)\right)^2$$

- Bond return in the LSS model:

$$\frac{dP\left(t,T_{j-1}\right)}{P\left(t,T_{j-1}\right)} = r(t)dt - \sum_{i=0}^{j-2}\sigma_i(t)d\widetilde{Z}_i(t)$$

- Historical covariance matrix:

$$\boldsymbol{\Sigma}_{ij}(t) = \int_{x=T_{i-1}-t}^{T_i-t} \int_{y=T_{j-1}-t}^{T_j-t} dx dy R(x,y)$$

- Implied covariance matrix:

$$\boldsymbol{\Theta}_{ij}(t, T_0) = \int_{x=T_0-t}^{T_i-t} \int_{y=T_0-t}^{T_j-t} dx dy R(x,y)$$

3.9   Appendix B

Proof of Theorem 3.2

The pay-off at time T_0 from the option is

$$\mathbf{Call}_K[T_0, \boldsymbol{P}_{T_0}] = \left[\sum_{i=1}^n C_i P(T_0, T_i) - K\right]_+ = \left[\sum_{i=1}^n C_i P(T_0, T_i) - KP(T_0, T_0)\right]_+ = [\Phi(\boldsymbol{P}_{T_0})]_+$$

with Φ homogeneous of degree one. Applying Theorem 3.1 we get

$$\mathbf{Call}_K[t, \boldsymbol{P}_t]$$

$$= \int_{\mathbb{R}^n} d\boldsymbol{x}\, g(x_1, \ldots, x_n; M) \left[\sum_{i=1}^n C_i P(t, T_i) e^{\sqrt{\Delta_{ii}}x_i - \frac{1}{2}\Delta_{ii}} - KP(t, T_0)\right]_+$$

$$= \int_{\mathbb{R}^n} d\boldsymbol{x} \left[\sum_{i=1}^n C_i P(t, T_i) g\left(x_1 - \frac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \frac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right) - KP(t, T_0) g(x_1, \ldots, x_n; M)\right]_+ \tag{3.42}$$

where the last equation arises from applying the identity

$$g(x_1, \ldots, x_n; M) e^{\sqrt{\Delta_{ii}}x_i - \frac{1}{2}\Delta_{ii}} = g\left(x_1 - \frac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \frac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right) \tag{3.43}$$

Consider the matrix $Q = (q_{ij})$ that verifies $M = Q'Q$ and define $\mathbf{y} = \left(Q'\right)^{-1}\mathbf{x}$. We will have $d\boldsymbol{x}\, g(x_1, \ldots, x_n; M) = d\boldsymbol{y}\, g(y_1, \ldots, y_n; I_n)$. Using (3.42) and (3.43) we can write

$$\mathbf{Call}_K[t, \boldsymbol{P}_t] = \int_{\mathbb{R}^n} d\boldsymbol{y}\, g(y_1, \ldots, y_n; I_n) \left[\sum_{i=1}^n C_i P(t, T_i) e^{\sqrt{\Delta_{ii}}\sum_{k=1}^n q_{ki}y_k - \frac{1}{2}\Delta_{ii}} - KP(t, T_0)\right]_+ \tag{3.44}$$

In addition, applying $\sum_{k=1}^n q_{ki}^2 = M_{ii} = 1$, it is easy to obtain the identity

$$g(y_1 + \gamma_{1i}, \ldots, y_n + \gamma_{ni}; I_n) = g(y_1, \ldots, y_n; I_n) e^{\sqrt{\Delta_{ii}}\sum_{k=1}^n q_{ki}y_k - \frac{1}{2}\Delta_{ii}}$$

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where we have done $\gamma_{ki} \equiv -\sqrt{\Delta_{ii}} q_{ki}$. Using this identity in (3.44) we have

$$\mathbf{Call}_K [t, \boldsymbol{P}_t] = \int_{\mathbb{R}^n} d\boldsymbol{y} \left[\sum_{i=1}^n C_i P(t, T_i) g(y_1 + \gamma_{1i}, \ldots, y_n + \gamma_{ni}; I_n) - KP(t, T_0) g(y_1, \ldots, y_n; I_n) \right]_+$$

with $\gamma_i' \gamma_j = \sum_{k=1}^n \gamma_{ki} \gamma_{kj} = \sqrt{\Delta_{ii}} \sqrt{\Delta_{jj}} \sum_{k=1}^n q_{ki} q_{kj} = \sqrt{\Delta_{ii}} \sqrt{\Delta_{jj}} M_{ij} = \Delta_{ij}$. ∎

Proof of Proposition 3.1

In the stochastic string modeling, under the equivalent martingale measure, the short-term interest rate and its dynamics are given by (see Chapter 1)

$$r(t) = r(0) + \int_{u=0}^t du \left. \frac{\partial f(u, x)}{\partial x} \right|_{x=0} + \int_{u=0}^t d\widetilde{Z}(u, 0) \sigma(u, 0) \tag{3.45}$$

and

$$dr(t) = \left(\left. \frac{\partial f(0, x)}{\partial x} \right|_{x=0} + \int_{u=0}^t du \left[\left. \frac{\partial^2 f(u, x)}{\partial x^2} \right|_{x=0} + \sigma^2(u, 0) \right] + \int_{u=0}^t d\widetilde{Z}(u, 0) \left. \frac{\partial \sigma(u, x)}{\partial x} \right|_{x=0} \right) dt$$

$$+ \sigma(t, 0) d\widetilde{Z}(t, 0) \tag{3.46}$$

Moreover, from (3.8), we obtain the HJM volatilities $\sigma_{HJM}^{(k)}(x) = \sqrt{\lambda_k} e^{-\tau x} L_k(x)$ and the stochastic string volatility $\sigma(x) = \sqrt{\sum_{k=0}^n \left(\sigma_{HJM}^{(k)}(x) \right)^2}$ (see Chapter 2). Differentiating and making $x = 0$, we get

$$\left. \frac{\partial \sigma(x)}{\partial x} \right|_{x=0} = -\frac{\sum_{k=0}^n \lambda_k (k + \tau)}{\sqrt{\sum_{k=0}^n \lambda_k}} = -\varkappa \sigma$$

where we have used $L_k(0) = 1$ and $L_k'(0) = -k$ (see Abramowitz and Stegun (1972)). Replacing in (3.46) and using (3.45), we obtain

$$dr(t) = \left(\left. \frac{\partial f(0, x)}{\partial x} \right|_{x=0} + \int_{u=0}^t du \left. \frac{\partial^2 f(u, x)}{\partial x^2} \right|_{x=0} + \sum_{k=0}^n \lambda_k t - \varkappa \left[r(t) - r(0) - \int_{u=0}^t du \left. \frac{\partial f(u, x)}{\partial x} \right|_{x=0} \right] \right) dt$$

$$+ \sqrt{\sum_{k=0}^n \lambda_k} \, d\widetilde{Z}(t, 0)$$

which implies (3.9). ∎

Proof of Corollary 3.3

Apply the definition of $L(t, T_{j-1})$ in (3.11) and approximate $1 + \delta L(t, T_{j-1})$ by $\delta L(t, T_{j-1})$ and $1 + \delta K$ by δK. ∎

Proof of Corollary 3.4

Considering (3.14) and applying (3.17), we get

$$
\mathbf{Swn}\left(t\right) = \int_{\mathbb{R}^n} d\boldsymbol{x} \left[P\left(t, T_n\right) g\left(x_1, \ldots, x_n; M\right) + \delta \varkappa\left(t, T_0, n\right) g\left(x_1, \ldots, x_n; M\right) \sum_{i=1}^{n} P\left(t, T_i\right) \right.
$$
$$
\left. - \sum_{i=1}^{n} C_i P\left(t, T_i\right) g\left(x_1 - \tfrac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \tfrac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right) \right]_+
$$
$$
\simeq \delta \sum_{i=1}^{n} P\left(t, T_i\right) \int_{\mathbb{R}^n} d\boldsymbol{x} \left[\varkappa\left(t, T_0, n\right) g\left(x_1, \ldots, x_n; M\right) - K g\left(x_1 - \tfrac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \tfrac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right) \right]_+
$$
$$
\simeq \delta \sum_{i=1}^{n} P\left(t, T_i\right) \int_{\mathbb{R}^n} d\boldsymbol{x} g\left(x_1, \ldots, x_n; M\right) \left[\varkappa\left(t, T_0, n\right) - K e^{\sqrt{\Delta} x_i - \frac{1}{2}\Delta} \right]_+
$$
$$
= \delta \sum_{i=1}^{n} P\left(t, T_i\right) \int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{g_1^B} dx_i \cdots \int_{-\infty}^{+\infty} dx_n g\left(x_1, \ldots, x_n; M\right) \left[\varkappa\left(t, T_0, n\right) - K e^{\sqrt{\Delta} x_i - \frac{1}{2}\Delta} \right]
$$

$$(3.47)$$

Calculating the corresponding integrals, we obtain

$$
\int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{g_1^B} dx_i \cdots \int_{-\infty}^{+\infty} dx_n g\left(x_1, \ldots, x_n; M\right) = \mathcal{N}\left(g_1^B\right) \tag{3.48}
$$

and

$$
\int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{g_1^B} dx_i \cdots \int_{-\infty}^{+\infty} dx_n g\left(x_1, \ldots, x_n; M\right) e^{\sqrt{\Delta} x_i - \frac{1}{2}\Delta}
$$
$$
= \int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{g_1^B} dx_i \cdots \int_{-\infty}^{+\infty} dx_n g\left(x_1 - \sqrt{\Delta}, \ldots, x_n - \sqrt{\Delta}; M\right)
$$
$$
= \int_{-\infty}^{+\infty} dy_1 \cdots \int_{-\infty}^{g_2^B} dy_i \cdots \int_{-\infty}^{+\infty} dy_n g\left(x_1, \ldots, x_n; M\right)
$$
$$
= \mathcal{N}\left(g_2^B\right) \tag{3.49}
$$

where we have applied the dimensional reduction property of the multivariate normal distribution (Miller (1964)). Substituting (3.48)-(3.49) into (3.47) provides finally

$$
\mathbf{Swn}(t) \simeq \delta \sum_{i=1}^{n} P\left(t, T_i\right) \left[\varkappa\left(t, T_0, n\right) \mathcal{N}\left(g_1^B\left(t, T_0\right)\right) - K \mathcal{N}\left(g_2^B\left(t, T_0\right)\right) \right]
$$

∎

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Conclusions

One of the ways in which scientific theories advance consists of the generalization of existent mathematical models. When a new theoretical framework that includes another commonly accepted is proposed, not only the knowledge of the described phenomena is enlarged but also new results not included in previous models usually emerge. Finance is not an exception to this situation.

In this doctoral dissertation a model has been proposed, the stochastic string model of Santa-Clara and Sornette (2001) in its reformulation of the first chapter, as a framework that generalizes the HJM model. This allows a better understanding of the continuous-time dynamics of the TSIR, showing that the infinite-dimensional HJM models are particular cases of our framework and that the multifactorial ones are approximations to the whole model. It also allows to relate issues previously unrelated or with an unclear relation as the relationship between HJM volatilities and the factors obtained from a Principal Components analysis.

Another important advance achieved is the appearance in the model, in a natural way, of the covariance function between forward curve shocks. This function is very important for option pricing as has been noted in the second chapter and, as a consequence, in the study of the caps and swaptions market and of some related problems, in the third chapter.

Relative to concrete results, keeping apart what was said in detail in the summary, we hope that subsequent developments bring other ones. Additionally to those included in the conclusions of the chapters, we can highlight two that are subject to current research: the completeness of the market (and thus, the design of hedging strategies with derivatives) with the stochastic string model and bond portfolio immunization in the infinite-dimensional framework presented.

Conclusiones

Una de las formas de avance de las teorías científicas consiste en la generalización de los modelos matemáticos existentes. Cuando se propone un nuevo marco de trabajo teórico que incluye otro comúnmente aceptado no sólo se amplía el conocimiento acerca de los fenómenos descritos, sino que suelen surgir nuevos resultados no incluidos en los modelos previos. El campo de las finanzas no es ajeno a esta situación.

En esta tesis doctoral se ha propuesto un modelo, el de cuerda estocástica de Santa-Clara y Sornette (2001) en su reformulación del primer capítulo, como un marco de trabajo que generaliza el modelo HJM. Ello permite comprender mejor la dinámica en tiempo continuo de la ETTI, mostrando los modelos HJM infinito-dimensionales como casos particulares dentro del nuevo marco y los HJM multifactoriales, a su vez, como aproximaciones al modelo completo. También permite relacionar aspectos anteriormente no relacionados o con una relación no demasiado clara como la relación entre las volatilidades HJM y los factores obtenidos de un análisis de componentes principales.

Otro de los avances importantes conseguidos es la aparición de forma natural en el modelo de la función de covarianza entre perturbaciones a la curva forward. Esta función es clave en la valoración de opciones como se ha puesto de manifiesto en el capítulo segundo y, como consecuencia, en el estudio del mercado de caps y swaptions así como de algunos problemas relacionados, ya en el tercer capítulo.

En cuanto a los resultados concretos obtenidos del modelo, además de los ya presentados en esta tesis y detallados en el resumen, es de esperar que se obtengan nuevos en los subsiguientes desarrollos. Además de los ya indicados en las conclusiones de los distintos capítulos, podemos destacar dos que son objeto de investigación en la actualidad: el estudio de la completitud del mercado (y, por tanto, el diseño de estrategias de cobertura de derivados) con el modelo de cuerda estocástica y la inmunización de carteras de bonos en el marco de trabajo infinito-dimensional presentado.

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