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Source: *The Journal of Finance*, Vol. 61, No. 1 (Feb., 2006), pp. 341-378
Published by: Wiley for the American Finance Association
Stable URL: http://www.jstor.org/stable/3699343

THE JOURNAL OF FINANCE • VOL. LXI, NO. 1 • FEBRUARY 2006

Unspanned Stochastic Volatility: Evidence from Hedging Interest Rate Derivatives

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ABSTRACT

Most existing dynamic term structure models assume that interest rate derivatives are redundant securities and can be perfectly hedged using solely bonds. We find that the quadratic term structure models have serious difficulties in hedging caps and cap straddles, even though they capture bond yields well. Furthermore, at-the-money straddle hedging errors are highly correlated with cap-implied volatilities and can explain a large fraction of hedging errors of all caps and straddles across moneyness and maturities. Our results strongly suggest the existence of systematic unspanned factors related to stochastic volatility in interest rate derivatives markets.

INTEREST RATE CAPS AND SWAPTIONS are among the most widely traded interest rate derivatives in the world. According to the Bank for International Settlements, their combined notional value has been more than 10 trillion dollars in recent years, which is many times larger than that of exchange-traded options. Because of the size of these markets, accurate and efficient pricing and hedging of caps and swaptions have enormous practical importance. Prices of interest rate derivatives may also contain richer information about term structure dynamics than bond yields or swap rates. Over the last decade, a huge literature (both theoretical and empirical) on multifactor dynamic term structure models (hereafter DTSMs) has been developed.[1] While numerous studies have fit these models to bond yields or swap rates, their performance in pricing and hedging interest rate derivatives is largely unknown. Therefore, studying existing DTSMs from a derivatives perspective may shed new light on our understanding of these models. Indeed, Dai and Singleton (2003) emphasize that there is an "enormous potential for new insights from using (interest rate) derivatives data in (dynamic term structure) model estimations."

*Li is from the Stephen M. Ross School of Business, University of Michigan. Zhao is from Rutgers Business School, Rutgers University. We would like to thank Warren Bailey, Alexander David, Bob Dittmar, Bob Goldstein, Bob Jarrow, Hong Liu, Mark Ready, Robert Stambaugh (the editor), Liuren Wu, Xiaoyan Zhang, Guofu Zhou, an anonymous referee, and seminar participants at Cornell University, Fordham University, Rutgers University, Singapore Management University, SUNY at Binghamton, University of California at Riverside, University of Massachusetts at Amherst, University of Michigan, University of Wisconsin–Madison, and Washington University in St. Louis for helpful comments and discussions. We thank David George for editorial assistance. We are responsible for any remaining errors.

[1] Dai and Singleton (2003) and Piazzesi (2003) provide excellent surveys of the literature.

One of the key issues in the fast-growing literature on London Interbank Offered Rate (LIBOR)-based interest rate derivatives is the so-called unspanned stochastic volatility puzzle (USV).[2] Interest rate caps and swaptions are derivatives written on LIBOR and swap rates, and their prices should be determined by the same set of risk factors that determine LIBOR and swap rates. However, several recent studies have shown that there seem to be risk factors that affect the prices of caps and swaptions but that are not spanned by the underlying LIBOR and swap rates.

Heidari and Wu (2003a) show that while the three common term structure factors (i.e., the level, slope, and curvature of the yield curve) can explain 99.5% of the variations of bond yields, they explain less than 60% of swaption implied volatilities. After including three additional volatility factors, the explanatory power is increased to over 97%. Similarly, Collin-Dufresne and Goldstein (2002a) show that there is a very weak correlation between changes in swap rates and returns on at-the-money (ATM) cap straddles: The R^2's of regressions of straddle returns on changes of swap rates are typically less than 20%. Furthermore, one principal component explains 80% of regression residuals of straddles with different maturities. As straddles are approximately delta neutral and are mainly exposed to volatility risk, they refer to the factor that drives straddle returns but is not affected by the term structure factors as USV. Therefore, these studies suggest that interest rate derivatives are not redundant securities and thus cannot be hedged using bonds alone. In other words, bonds do not span interest rate derivatives.

The existence of USV has profound implications for term structure modeling, especially for the relevance of existing DTSMs for pricing and hedging interest rate derivatives. One of the main reasons for the popularity of these models is their tractability. They provide closed-form solutions for the prices of not only zero-coupon bonds, but also of a wide range of interest rate derivatives (see, for example, Duffie, Pan, and Singleton (2000), Chacko and Das (2002), and Leippold and Wu (2002)). The closed-form formulas significantly reduce the computational burden of implementing these models and simplify their applications in practice. However, almost all existing DTSMs assume that bonds and derivatives are driven by the same set of risk factors, which implies that derivatives are redundant and can be perfectly hedged using solely bonds. Hence, the presence of USV in the derivatives market implies that one fundamental assumption underlying all DTSMs does not hold, and these models need to be substantially extended to incorporate the unspanned factors before they can be applied to derivatives.

The evidence on USV is not without controversy. Fan, Gupta, and Ritchken (2003) (hereafter, FGR) challenge the findings of Heidari and Wu (2003a) and Collin-Dufresne and Goldstein (2002a), arguing that the linear regression approach used in these two studies could give misleading results of USV due to the

[2] Another issue is the relative pricing between caps and swaptions. Although both caps and swaptions are derivatives on LIBOR rates, existing models calibrated to one set of prices tend to significantly misprice the other set of prices. For a more detailed review of the literature, see Dai and Singleton (2003).

highly nonlinear dependence of straddle returns on the underlying yield factors. Instead, FGR show that multifactor models with state variables linked solely to underlying LIBOR and swap rates can hedge swaptions and even swaption straddles very well. Consequently, they conclude that "the potential benefits of looking outside the LIBOR market for factors that might impact swaptions prices without impacting swap rates" are minor. However, the models considered in FGR (2003) belong to the Heath, Jarrow, and Morton (1992) (hereafter, HJM) class of models. Unlike DTSMs, HJM models take the whole yield curve as given and thus have no predictions on the cross-sectional and time-series properties of bond yields.[3] Therefore, FGR does not answer the key question addressed in this paper, namely, whether multifactor DTSMs can price both bonds and interest rate derivatives.

Our paper contributes to the literature by providing a comprehensive empirical analysis of multifactor DTSMs in hedging interest rate derivatives. If there are indeed USV factors, then DTSMs estimated using bonds alone should not be able to hedge derivatives well. Thus, our model-based hedging avoids the problems facing the linear regression approach of previous studies and helps resolve the controversy on USV. We choose the quadratic term structure models (QTSMs) of Ahn, Dittmar, and Gallant (2002) (hereafter, ADG) and Leippold and Wu (2002) over the affine term structure models (ATSMs) of Duffie and Kan (1996) in our analysis because of their superior performance in capturing the conditional volatility of bond yields, which is important for pricing derivatives.[4] We estimate the canonical forms of the three-factor QTSMs using LIBOR bond yields via an extended Kalman filter and use the estimated model parameters and latent state variables to hedge interest rate derivatives. While previous studies mainly focus on ATM caps and swaptions, we use a unique data set of caps with different strikes and maturities in our analysis. As shown in Jarrow, Li, and Zhao (2003), there is a pronounced volatility smile/skew in cap-implied volatilities. Thus, our data set makes it possible to study the cross-sectional performance of QTSMs in hedging caps.

Our empirical analysis documents several important findings. First, we find that consistent with the existing literature, the QTSMs can capture term structure dynamics very well. The maximal flexible model has a great fit with various aspects of bond yields, such as the unconditional mean, variance, skewness, kurtosis, and first-order autocorrelation of both the level and changes of bond yields. The models also have excellent performance in hedging zero-coupon bonds: Model-based hedging in all the QTSMs can reduce more than 95% of the variations of bond returns.

[3] Heidari and Wu (2003b) show that yield residuals in DTSMs are important for pricing interest rate derivatives. Thus, by taking the whole yield curve as an input, one may already include some of the unspanned factors in the model.

[4] ADG (2002) and Brandt and Chapman (2002) show that the QTSMs can capture the conditional volatility of bond yields better than the ATSMs. In addition, there is also preliminary evidence that the ATSMs may not be able to price interest rate derivatives well (see, for example, Jagannathan, Kaplin, and Sun (2003)).

Second, however, the QTSMs have serious difficulties in hedging long-term and out-of-the-money (OTM) caps, and especially ATM cap straddles. In fact, the QTSMs can explain only a small percentage of the variations of ATM cap straddle returns. Principal component analysis of hedging errors of caps and straddles across moneyness and maturities shows that there are additional factors affecting cap prices that are not spanned by the yield factors.

Third, we find that the unspanned factors are mainly due to stochastic volatility but not liquidity risk. For example, there is a strong correlation between ATM straddle hedging errors and changes in ATM cap-implied volatilities.[5] However, the correlation between ATM straddle hedging errors and changes of percentage bid-ask spreads of ATM caps is very weak. Moreover, if the unspanned factors are due to liquidity risk, then deep in-the-money (ITM) and OTM options, which are less liquid than ATM options, should have larger hedging errors. The fact that ITM and OTM straddles can be hedged better than ATM straddles also indicates that it is highly unlikely that the unspanned factors are due to liquidity risk.

Finally, we find that the USV factors have a systematic impact on the prices of all caps and straddles. The first few principal components of ATM straddle hedging errors can explain a large percentage of hedging errors of caps and straddles across moneyness and maturities. To the extent that the USV factors can be proxied by ATM straddle hedging errors, our results show that the impact of USV on cap prices is systematic. While Collin-Dufresne and Goldstein (2002a) find that one principal component explains more than 85% of the variations of straddle regression residuals, we find multiple principal components in ATM straddle hedging errors. Interestingly, the hedging errors of short-, medium-, and long-term ATM straddles are highly correlated within each group. This evidence will be useful for future development of term structure models that explicitly incorporate USV factors.

In summary, our hedging analysis of the QTSMs shows that there are indeed unspanned factors in the cap market. Our paper further extends the existing literature by demonstrating that the unspanned factors are: (1) mainly due to stochastic volatility but not liquidity risk; and, (2) systematic factors that affect all caps and straddles across moneyness and maturities. The evidence strongly suggests that existing DTSMs need to incorporate the USV factors for pricing and hedging interest rate derivatives.[6]

The rest of this paper is organized as follows. In Section I, we introduce the data and provide preliminary analysis of USV in the cap market using linear regression. Section II introduces the QTSMs and their empirical performance for capturing term structure dynamics. In Section III, we study the pricing and hedging of caps in the QTSMs. Section IV concludes the paper. The Appendix provides more detailed discussion on the implementations of the QTSMs.

[5] If LIBOR rates do exhibit stochastic volatility, then the Black (1976) implied volatilities of ATM caps should provide a proxy of the unobserved instantaneous stochastic volatility.

[6] Other studies using different models and data sets, such as Heidari and Wu (2003b) and Bikbov and Chernov (2004), also find that yield factors alone cannot fully capture the prices of interest rate derivatives.

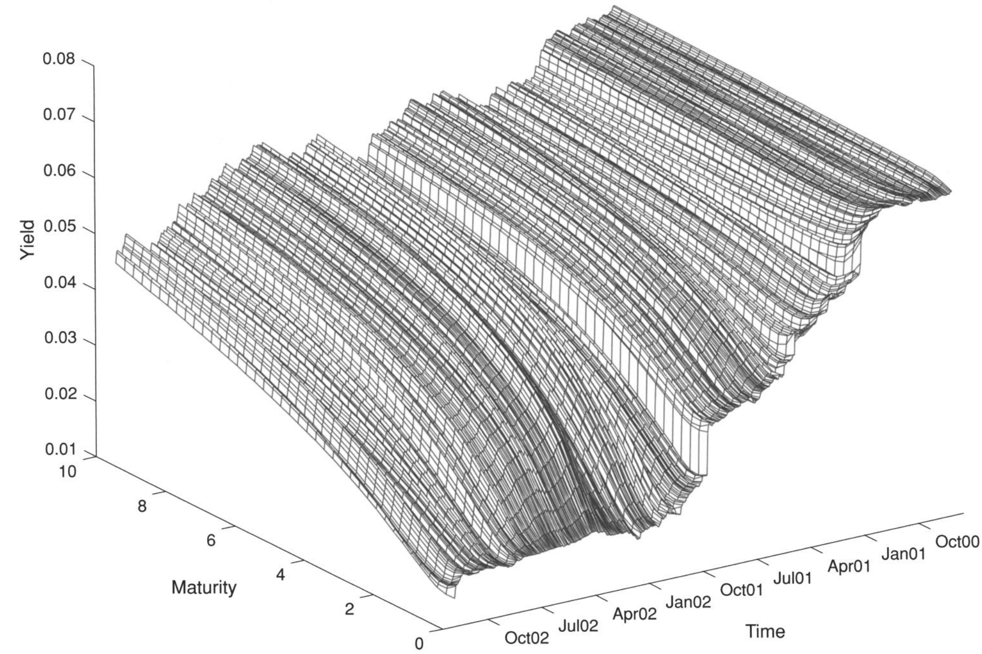


Figure 1. The term structure of LIBOR zero-coupon bonds. This figure contains the term structure of LIBOR zero-coupon bonds between August 1, 2000 and November 7, 2002. The daily yield curve of LIBOR zero-coupon bonds is constructed by linearly interpolating the 3-month LIBOR forward rates at nine different maturities (3 and 6 month, 1, 2, 3, 4, 5, 7, and 10 year).

I. Data and Preliminary Analysis of Unspanned Stochastic Volatility

In this section, we introduce the data used in our study and provide preliminary analysis of USV in the cap market using the linear regression approach of Collin-Dufresne and Goldstein (2002a). The data are obtained from SwapPX and contain LIBOR and swap rates, and prices of caps with different strikes and maturities.[7] We make the same assumption as Collin-Dufresne and Goldstein (2002a), namely, that the quoted swap rate is equivalent to a par-bond rate for an issuer with LIBOR-credit quality and cap, floor, and swap markets have homogeneous credit quality. The data cover the period from August 1, 2000 to November 7, 2002. After excluding weekends, holidays, and missing data, we have a total of 557 trading days in our sample. The data are collected for each day the market was open between 3:30 p.m. and 4:00 p.m.

As the caps in our data are written on 3-month LIBOR rates, we linearly interpolate the 3-month LIBOR spot and forward rates at nine different maturities (3 and 6 month, 1, 2, 3, 4, 5, 7, and 10 year) to construct the yield curve of LIBOR zero-coupon bonds each day.[8] As shown in Figure 1, the yield curve

[7] For more detailed descriptions of SwapPX, see http://www.govpx.com/mkting/start_swappx.html.

[8] We also consider other interpolation schemes and obtain very similar results.

Table I

Summary Information of LIBOR Zero-Coupon Bond Yields

This table reports the summary statistics and principal component analysis of the levels and changes of LIBOR zero-coupon bond yields, which are constructed by linearly interpolating the 3-month LIBOR forward rates at nine different maturities (3- and 6-month, 1-, 2-, 3-, 5-, 7-, and 10-year). The sample period is from August 1, 2000 to November 7, 2002. Excluding holidays, weekends, and missing data, we have 557 trading days in total.

	Maturity (year)					
	0.5	1	2	5	7	10
Panel A: Summary Statistics of the Levels of LIBOR Zero-Coupon Bond Yields						
Mean (%)	3.536	3.691	4.159	5.049	5.361	5.666
Standard deviation (%)	1.791	1.631	1.359	0.924	0.777	0.638
Skewness	0.613	0.601	0.461	0.082	−0.011	−0.055
Kurtosis	1.876	2.003	2.155	2.518	2.672	2.799
First-order partial autocorrelation	0.998	0.998	0.997	0.997	0.996	0.994
Panel B: Summary Statistics of the Changes of LIBOR Zero-Coupon Bond Yields						
Mean (%)	−0.010	−0.010	−0.009	−0.007	−0.006	−0.005
Standard deviation (%)	0.048	0.061	0.070	0.072	0.069	0.068
Skewness	−8.388	−3.320	−0.981	−0.101	0.184	0.371
Kurtosis	130.510	44.748	14.980	6.491	4.839	3.921
First-order partial autocorrelation	0.240	0.141	0.116	0.066	0.085	0.082

Panel C: Principal Component Analysis of the Levels and Changes of LIBOR Zero-Coupon Bond Yields.

	Principal Component					
	1	2	3	4	5	6
Level	96.83%	3.10%	0.045%	0.019%	0.002%	0.000%
Change	87.72%	9.84%	1.51%	0.74%	0.13%	0.07%

Note: The entries represent the percentages of the variations of the levels and changes of LIBOR zero-coupon bond yields explained by each of their first six principal components.

is relatively flat at the beginning of the sample and declines over time, with the short end declining more than the long end. As a result, the yield curve becomes upward sloping in later part of the sample. Table I reports the summary statistics of the levels and changes of 6-month, 1-, 2-, 5-, 7-, and 10-year yields. Consistent with the upward sloping yield curve, long-term bonds tend to have higher yields than short-term bonds. On average, all yields exhibit negative changes, consistent with the declining interest rates during our sample period. Changes of short-term yields have higher standard deviation and kurtosis and are more negatively skewed than changes of long-term yields. Yield levels are highly persistent with first-order autoregressive coefficients close to one. In contrast, yield changes are much less persistent and the first-order autoregressive coefficients decline with maturity. Principal component analysis shows that consistent with the existing literature, the first three principal components can explain more than 99% of the variations in both the levels and changes of bond yields.

Interest rate caps are portfolios of call options on LIBOR rates. Specifically, a cap gives its holder a series of European call options, called caplets, on LIBOR forward rates. Each caplet has the same strike price as the others, but with different expiration dates. Suppose $L(t, T)$ is the 3-month LIBOR forward rate at $t \leq T$, for the interval from T to $T + \frac{1}{4}$. A caplet for the period $[T, T + \frac{1}{4}]$ struck at K pays $\frac{1}{4}(L(T, T) - K)^{+}$ at $T + \frac{1}{4}$.[9] Note that while the cash flow of this caplet is received at time $T + \frac{1}{4}$, the LIBOR rate is determined at time T. Hence, there is no uncertainty about the caplet's cash flow after the LIBOR rate is set at time T. A cap is just a portfolio of these caplets whose maturities are 3 months apart. For example, a 5-year cap on 3-month LIBOR struck at 6% represents a portfolio of 19 separately exercisable caplets with quarterly maturities ranging from 6 months to 5 years, where each caplet has a strike price of 6%.

Existing literature on interest rate derivatives has mainly focused on ATM contracts. One advantage of our data is that we observe prices of caps over a wide range of strikes and maturities.[10] For example, every day for each maturity, there are 10 different strike prices, which are 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 8.0, 9.0, and 10.0% between August 1, 2000 and October 17, 2001, and 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, and 7.0% between November 2, 2001 and November 7, 2002.[11] Throughout the whole sample, caps have 15 different maturities, which are 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0, 9.0, and 10.0 years. The rich cross-sectional information on cap prices allows us to study the performance of existing term structure models in pricing and hedging caps for different maturities and moneyness.

Ideally, we would like to study caplet prices because they provide clear predictions of model performance across maturities. Unfortunately, we only observe cap prices. To simplify empirical analysis, we consider the difference between the prices of caps with adjacent maturities, which we refer to as *difference caps* in the rest of the paper. Thus, our analysis deals with only the sum of the few caplets between two neighboring maturities and the same strike. For example, in the rest of the paper, 1.5-year difference caps represent the sum of the 1.25- and 1.5-year caplet.

Due to daily changes in LIBOR rates, difference caps have a different set of moneyness (defined as the ratio between the strike price and the average LIBOR forward rates underlying the few caplets that form the difference cap) on each day. Therefore, throughout our analysis, we focus on the prices of difference caps at fixed moneyness. That is, each day we interpolate difference cap prices

[9] As shown in Appendix A, a caplet behaves like a put option on a zero-coupon bond.

[10] While Gupta and Subrahmanyam (2003) and Deuskar, Gupta, and Subrahmanyam (2003) also consider caps with different strikes, the data set used in our paper is probably the most comprehensive data set in the existing literature for caps written on dollar LIBOR rates.

[11] The strike prices were lowered to 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, and 5.5% between October 18, 2001 and November 1, 2001.

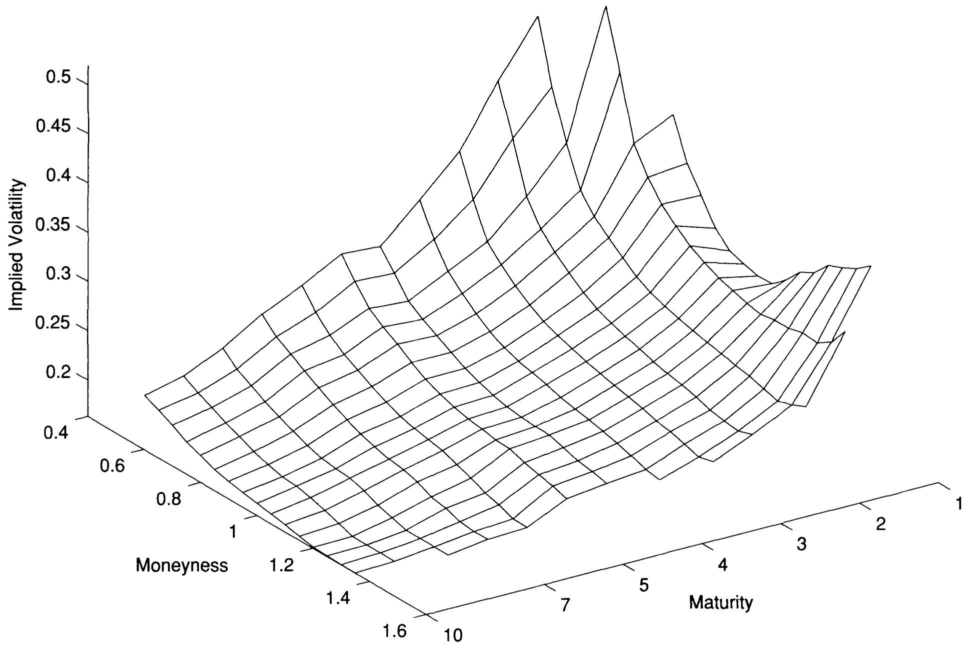


Figure 2. The average Black implied volatilities of difference caps. This figure contains the implied volatilities of difference caps across moneyness and maturities averaged over the period between August 1, 2000 and November 7, 2002. A difference cap is a portfolio of caplets with the same strike and maturities between two neighboring cap maturities, which are 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0, 9.0, and 10.0 years in our sample. Thus, the price of a difference cap equals the difference between the prices of two corresponding caps with adjacent maturities. Moneyness is defined as the ratio between the strike and the average LIBOR forward rates underlying the few caplets that form the difference cap. We interpolate difference cap prices with respect to strike to obtain the prices of difference caps at fixed moneyness. We refrain from extrapolation and interpolation for grid points without nearby observations, and eliminate all observations that violate various arbitrage restrictions.

with respect to strike price to obtain prices at fixed moneyness. Specifically, we use local cubic polynomials to preserve the shape of the original curves and to attain smoothing over the grid points. We refrain from extrapolation and interpolation for grid points without observations nearby, and eliminate all observations that violate various arbitrage restrictions.

Figure 2 plots the average implied volatilities of difference caps across moneyness and maturities over the whole sample period. Consistent with the existing literature, the implied volatilities of difference caps with a moneyness between 0.8 and 1.2 have a humped shape. However, the implied volatilities of all other difference caps decline with maturity. There is also a pronounced volatility skew for difference caps with all maturities, with the skew being stronger for short-term difference caps. The pattern is similar to that of equity options: ITM difference caps have higher implied volatilities than OTM difference caps. The implied volatilities of the very short-term difference caps are more of a symmetric smile than a skew. Figure 3 plots the time series of Black implied volatilities for 2.5- and 8-year difference caps across moneyness. It is clear that the implied volatilities are much higher in the second half of our sample. As Black implied volatility measures the volatility of percentage changes of LIBOR rates, the volatility increase during the second half of the sample is mainly due to the drop in the LIBOR rates (see Figure 1). As a result of changing interest rates and strike prices, there are more ITM caps in the second half of our sample.

Figure 4 contains the time-series plots of absolute and percentage bid-ask spreads of ATM caps during the whole sample period. Bid-ask spreads of ATM caps are quoted in implied volatilities. We convert the implied volatilities into bid and ask prices and the difference between the two is the absolute spread. The percentage spread is the ratio between the absolute spread and the cap

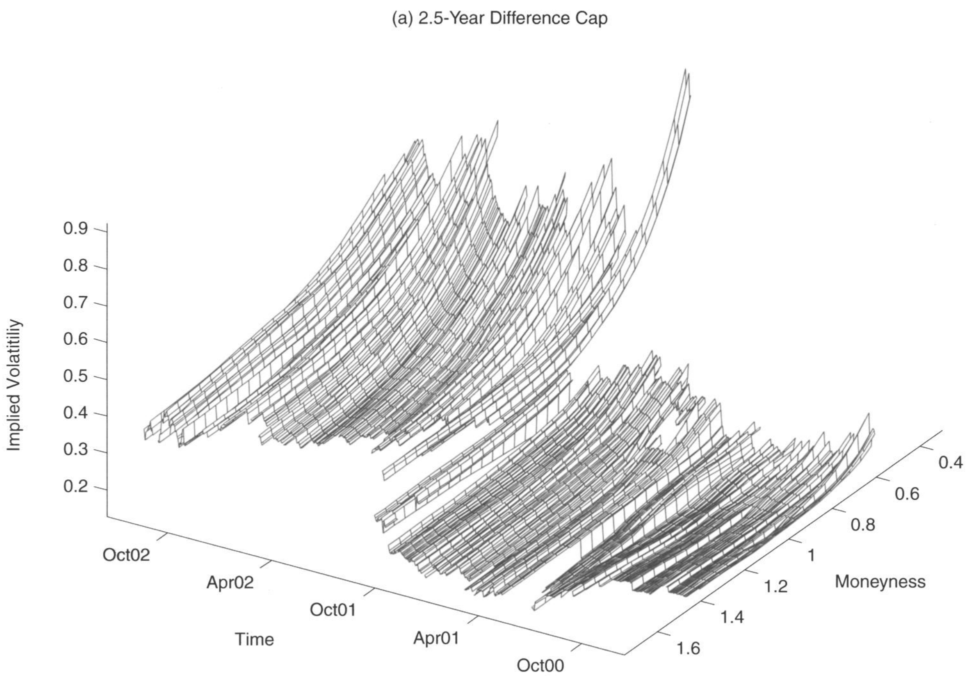


Figure 3. The implied volatilities of 2.5- and 8-year difference caps. This figure contains the time-series observations of the Black implied volatilities of 2.5- and 8-year difference caps across moneyness between August 1, 2000 and November 7, 2003. A 2.5-year difference cap consists of two caplets with the same strike and maturities of 2.25 and 2.5 years, while a 8-year difference cap consists of four caplets with the same strike and maturities of 7.25, 7.5, 7.75, and 8 years. Moneyness is defined as the ratio between the strike and the average LIBOR forward rates underlying the few caplets that form the difference cap. We interpolate difference cap prices with respect to strike to obtain the prices of difference caps at fixed moneyness. We refrain from extrapolation and interpolation for grid points without observations nearby, and eliminate all observations that violate various arbitrage restrictions.

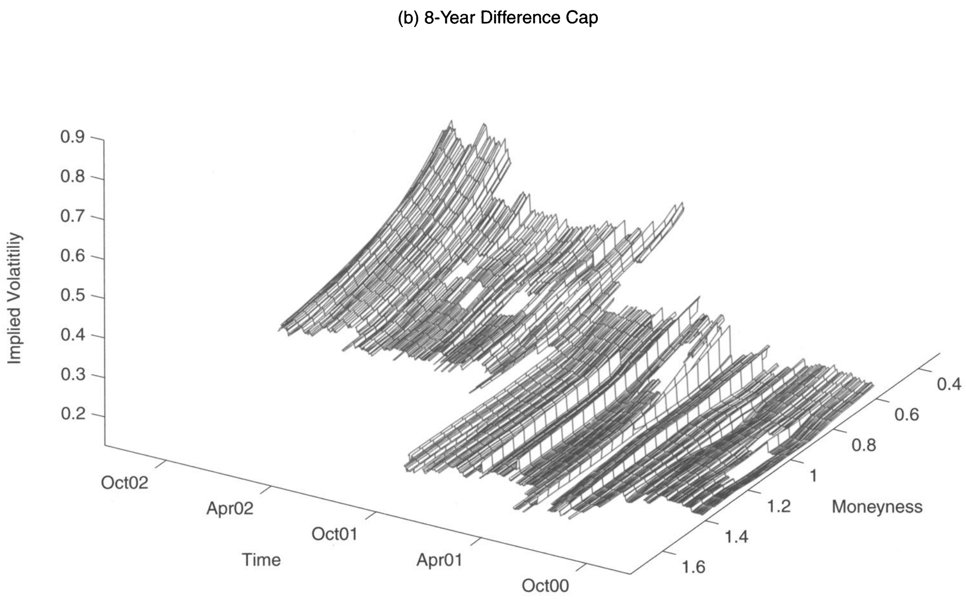


Figure 3 —*Continued*

price (the midpoint of the bid and ask prices). We measure the spread in prices (instead of implied volatilities) because the analyses in later sections focus on explaining cap prices. Percentage spreads may be more appropriate given the dramatic differences in cap prices across maturities. Both the absolute and percentage bid-ask spreads decline consistently over the sample period, while the percentage spreads decline more given the rising implied volatility. We emphasize that the bid-ask spreads may not be a perfect measure of liquidity of the OTC cap market. In fact, the quoted spreads (in implied volatilities) for all caps equal 1% throughout the whole sample. On the one hand, we could interpret this fact as evidence that the liquidity of the cap market does not change very much during the sample period. On the other hand, one could also argue that the spread may not be a very informative measure of the liquidity of the cap market. Fortunately, as shown in later sections, we have other ways of examining the impact of liquidity that do not depend on bid-ask spreads.

If the cap market is well integrated with the LIBOR and swap market, then the three common term structure factors that explain more than 99% of the variations of bond yields should also explain cap prices well. Low explanatory power would suggest that there could be factors affecting cap prices that are not spanned by bonds. To test this hypothesis, we regress weekly returns of difference caps and straddles with fixed moneyness and maturity on weekly changes of the three yield factors.

As caps are traded over the counter, we only observe their prices with fixed time to maturity, but not fixed maturity dates. To calculate weekly returns at a fixed moneyness, we need the price of a difference cap 1 week later that has

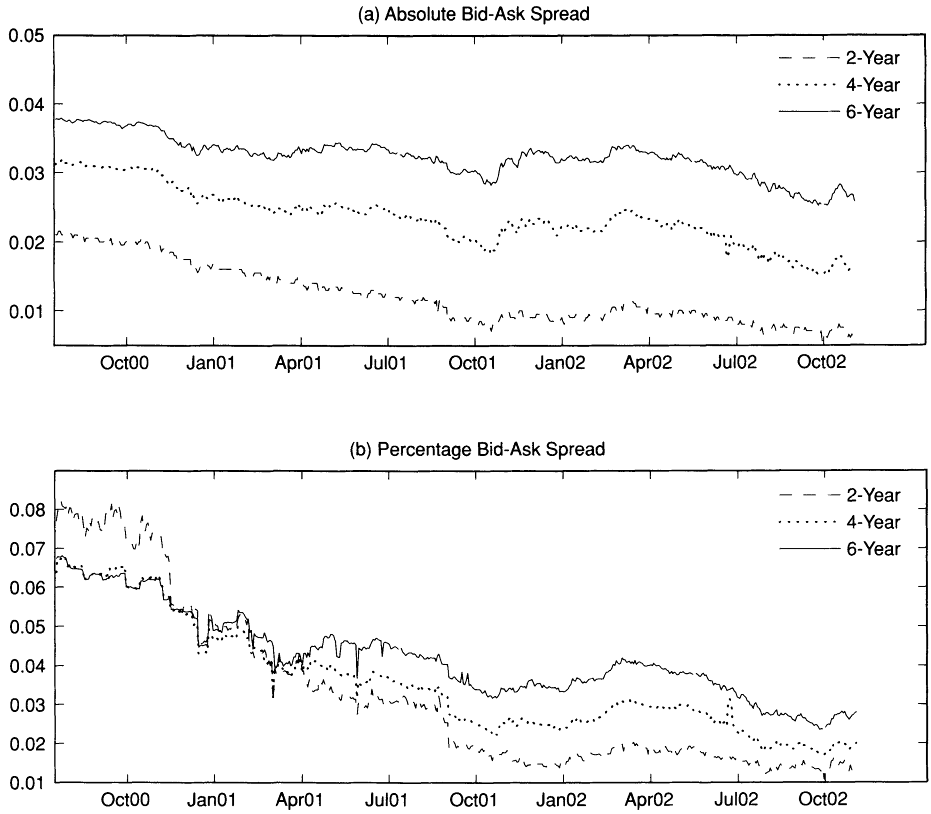


Figure 4. The absolute and percentage bid-ask spreads of ATM caps. This figure contains the time-series observations of the absolute and percentage bid-ask spreads of at-the-money (ATM) caps between August 10, 2000 and November 7, 2002. The bid-ask spreads of ATM caps are quoted in implied volatilities. We convert the implied volatilities into bid and ask prices and the difference between the two is the absolute spread. To fit the three series (2, 4, and 6 year) into the same plot, we rescale the absolute spread by maturity. The percentage spread is the ratio between the absolute spread and the cap price (the midpoint of the bid and ask prices).

the same strike price and a maturity that is 1 week shorter. Following previous studies such as FGR (2003) and Collin-Dufresne and Goldstein (2002a), we linearly interpolate with respect to maturity the prices of difference caps with the same strike price a week later.[12] Through the above interpolation, we obtain a series of weekly difference cap returns for each moneyness and maturity. We also obtain difference floor prices from difference cap prices using put–call parity and we construct weekly straddle returns for each moneyness and maturity.

Panels A and B of Table II report the R^2's of regressions of weekly returns of difference caps and straddles on weekly changes of the three yield factors

[12] We also consider other interpolation schemes and obtain similar results.

Table II

Regression Analysis of Unspanned Stochastic Volatility in the Cap Market

This table reports the R^2's of the regressions of weekly returns of difference caps and cap straddles across moneyness and maturities on weekly changes of the three yield factors (level, slope, and curvature). A difference cap is a portfolio of caplets with the same strike and maturities between two neighboring cap maturities, which are 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0, 9.0, and 10.0 years in our sample. Moneyness (K/F) is defined as the ratio between the strike and the average LIBOR forward rates underlying the few caplets that form the difference cap. We interpolate difference cap prices with respect to strike to obtain the prices of difference caps at fixed moneyness. For each moneyness, we construct difference straddle prices by obtaining difference floor prices from difference cap prices using put–call parity. Due to changes in interest rates and strike prices, we do not have the same number of observations for each moneyness/maturity group. The bold entries represent moneyness/maturity groups that have less than 10% of missing values and the rest are the ones with 10% to 50% of missing values.

Moneyness	Maturity												
(K/F)	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
Panel A: R^2's of Regressions of Weekly Returns of Difference Caps on Weekly Changes of the Three Yield Factors (Level, Slope, and Curvature)													
0.60					0.918	0.896	0.683		0.904	0.789	0.784	0.594	0.566
0.70			0.926	0.914	0.902	**0.860**	**0.673**	0.497	0.829	0.661	0.681	0.491	0.380
0.80		0.902	**0.920**	**0.904**	**0.875**	**0.869**	**0.667**	**0.493**	**0.780**	**0.647**	**0.642**	0.441	0.339
0.90	0.954	**0.860**	**0.875**	**0.874**	**0.834**	**0.813**	**0.628**	**0.517**	**0.699**	**0.611**	**0.539**	**0.425**	**0.318**
1.00	**0.871**	**0.663**	**0.807**	**0.786**	**0.785**	**0.766**	**0.570**	**0.453**	**0.570**	**0.545**	**0.494**	**0.369**	**0.300**
1.10	**0.761**	**0.542**	**0.714**	**0.729**	**0.714**	**0.723**	**0.507**	0.406	0.446	0.487	0.461	0.309	0.240
1.20	0.642	**0.430**	**0.599**	**0.624**	**0.592**	0.603	0.457	0.424	0.298	0.367	0.421		
1.30	0.802	**0.279**	**0.428**	0.453	0.427	0.487	0.504	0.504	0.126				
1.40	0.726	0.086	0.265	0.284	0.298	0.274							
Panel B. R^2's of Regressions of Weekly Returns of Difference Cap Straddles on Weekly Changes of the Three Yield Factors (Level, Slope, and Curvature)													
0.60					0.720	0.696	0.285		0.672	0.332	0.477	0.238	0.258
0.70			0.769	0.690	0.615	**0.521**	**0.258**	0.142	0.404	0.173	0.184	0.102	0.080
0.80		0.557	**0.624**	**0.577**	**0.418**	**0.481**	**0.174**	**0.090**	**0.225**	**0.128**	**0.112**	0.064	0.073
0.90	0.507	**0.242**	**0.364**	**0.322**	**0.208**	**0.223**	**0.078**	**0.060**	**0.094**	**0.056**	**0.035**	**0.027**	**0.066**
1.00	**0.300**	**0.061**	**0.209**	**0.160**	**0.083**	**0.031**	**0.044**	**0.021**	**0.025**	**0.014**	**0.046**	**0.024**	**0.035**
1.10	**0.567**	**0.218**	**0.366**	**0.256**	**0.263**	**0.171**	**0.147**	0.065	0.101	0.057	0.140	0.044	0.024
1.20	0.751	**0.543**	**0.608**	**0.510**	**0.552**	0.476	0.302	0.167	0.251	0.209	0.228		
1.30	0.842	**0.739**	**0.775**	0.708	0.737	0.660	0.502	0.249	0.390				
1.40	0.888	0.821	0.879	0.832	0.845	0.851							

for each moneyness/maturity group, respectively. Because of changing interest rates and strike prices, we do not have the same number of observations throughout the whole sample for all moneyness/maturity groups. The bold entries represent observations with less than 10% of missing values and the rest with 10% to 50% of missing values. In total, we have 111 weeks of nonoverlapping observations if there are no missing data.

The R^2's in Panel A of Table II show that the three yield factors can explain a large percentage of returns of ITM and short-term difference caps. But the explanatory power is significantly worsened for OTM and long-term difference caps. This is because deep ITM difference caps behave like the underlying bonds, which should be hedged reasonably well using the yield factors. However, if there is indeed USV in the cap market, then its impact on the prices of ATM and OTM and long-term caps is likely to be more significant, because their prices depend more on the probability that the option will be in the money at maturity. Therefore, our evidence suggests that bonds may not be able to span caps.

Collin-Dufresne and Goldstein (2002a) argue that the unspanned factor is mainly related to stochastic volatility because changes of swap rates can explain little variation in ATM straddle returns, which are mostly sensitive to volatility risk. We also regress weekly straddle returns at different moneyness and maturities on weekly changes in the three yield factors and obtain very similar results. In general, the R^2's in Panel B of Table II are very small for straddles that are close to the money. For deep ITM and OTM straddles, the R^2's increase significantly. This is consistent with the fact that the three yield factors can explain deep ITM options well. Therefore, the results from our linear regression analysis are consistent with that of Collin-Dufresne and Goldstein (2002a). As pointed out by FGR (2003), however, linear regression could be misleading due to the nonlinear dependence of straddle returns on underlying yield factors. To rigorously address this issue, we examine the performance of multifactor DTSMs in hedging caps and cap straddles.

II. Quadratic Term Structure Models

A. Quadratic Term Structure Models

In the last decade, a large number of multifactor DTSMs have been developed and their performance for capturing both the cross-sectional and time-series properties of bond yields has been extensively studied. Even though one of the main advantages of these models is that they provide closed-form solutions for the prices of a wide range of interest rate derivatives, their empirical performance for pricing and hedging interest rate derivatives is largely unknown. The existence of USV suggests that these models may not be directly applicable to derivatives because they all rely on the fundamental assumption that bonds and derivatives are driven by the same set of risk factors. While pricing mainly focuses on modeling the terminal distribution of the underlying asset price, hedging focuses on modeling the evolution of the underlying price dynamics and thus imposes more stringent requirements on model performance. In this paper, we provide a comprehensive empirical analysis of DTSMs in hedging interest rate derivatives and hope to resolve the controversy on USV through this exercise.

The ATSMs of Duffie and Kan (1996) and the QTSMs of ADG (2002) and Leippold and Wu (2002) are probably the most widely studied models in the existing term structure literature.[13] In our empirical analysis, we choose to use

[13] The affine models include the completely affine models of Dai and Singleton (2000), the essentially affine models of Duffee (2002), and the semiaffine models of Duarte (2004). Other DTSMs include the hybrid models of Ahn et al. (2003), the regime-switching models of Bansal and Zhou (2002) and Dai, Singleton, and Yang (2003), and models with macroeconomic jump effects, such as Piazzesi (2004), and many others.

the QTSMs because they have several advantages over the ATSMs. First, since the nominal spot interest rate is a quadratic function of the state variables, it is guaranteed to be positive in the QTSMs. On the other hand, in the ATSMs, the spot rate, an affine function of the state variables, is guaranteed to be positive only when all the state variables follow square root processes. Second, the QTSMs do not have the limitations that face the ATSMs in simultaneously fitting interest rate volatility and correlations among the state variables. That is, in the ATSMs, increasing the number of factors that follow square root processes improves the modeling of volatility clustering in bond yields, but reduces the flexibility in modeling correlations among the state variables. Third, the QTSMs have the potential to capture observed nonlinearity in term structure data (see, for example, Ahn and Gao (1999)). Indeed, ADG (2002) and Brandt and Chapman (2002) show that the QTSMs can capture conditional volatility of bond yields better than the ATSMs.

In the rest of this section, we briefly introduce the QTSMs, their estimation, and their performance in capturing term structure dynamics. The economy is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$, where $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the augmented filtration generated by an N-dimensional standard Brownian motion, W, on this probability space. We assume $\{\mathcal{F}_t\}_{0\leq t\leq T}$ satisfies the usual hypothesis (see Protter (1990)). The QTSMs rely on the following assumptions:

- The instantaneous interest rate r_t is a quadratic function of the N-dimensional state variables X_t,

$$r(X_t) = X_t'\Psi X_t + \beta' X_t + \alpha. \tag{1}$$

- The state variables follow a multivariate Gaussian process,

$$dX_t = [\mu + \xi X_t]dt + \Sigma dW_t. \tag{2}$$

- The market price of risk is an affine function of the state variables,

$$\zeta(X_t) = \eta_0 + \eta_1 X_t. \tag{3}$$

Note that in the above equations, Ψ, ξ, Σ, and η_1 are N-by-N matrices, β, μ, and η_0 are vectors of length N, and α is a scalar. The quadratic relation between r_t and X_t has the desired property that r_t is guaranteed to be positive if Ψ is positive semidefinite and $\alpha - \frac{1}{4}\beta'\Psi\beta \geq 0$. Although X_t follows a Gaussian process in (2), interest rate r_t exhibits conditional heteroskedasticity because of the quadratic relationship between r_t and X_t. As a result, the QTSMs are more flexible in modeling volatility clustering in bond yields and correlations among the state variables than the ATSMs.

To guarantee the stationarity of the state variables, we assume that ξ permits the eigenvalue decomposition

$$\xi = U\Lambda U^{-1}, \tag{4}$$

where Λ is the diagonal matrix of the eigenvalues that take negative values, $\Lambda \equiv \mathrm{diag}[\lambda_i]_N$, and U is the matrix of the eigenvectors of ξ, $U \equiv [u_1, u_2, \ldots, u_N]$. The conditional distribution of the state variables X_t is multivariate Gaussian with conditional mean

$$E[X_{t+\Delta t} \mid X_t] = U\Lambda^{-1}[\Phi - I_N]U^{-1}\mu + U\Lambda^{-1}[\Phi - I_N]U^{-1}X_t, \tag{5}$$

and conditional variance

$$\operatorname{var}[X_{t+\Delta t} \mid X_t] = U\Theta U', \tag{6}$$

where Φ is a diagonal matrix with elements $\exp(\lambda_i \Delta t)$ for $i = 1, \ldots, N$, and Θ is an N-by-N matrix with elements $[\frac{v_{ij}}{\lambda_i + \lambda_j}(e^{\Delta t(\lambda_i+\lambda_j)} - 1)]$, where $[v_{ij}]_{N\times N} = U^{-1}\Sigma\Sigma' U'^{-1}$.

With the specification of the market price of risk, we can relate the risk-neutral measure Q to the physical measure P as follows:

$$E\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right] = \exp\left[-\int_0^t \zeta(X_s)'\,dW_s - \frac{1}{2}\int_0^t \zeta(X_s)'\zeta(X_s)\,ds\right], \quad \text{for } t \le T. \tag{7}$$

Applying Girsanov's theorem, we obtain the risk-neutral dynamics of the state variables

$$dX_t = [\delta + \gamma X_t]\,dt + \Sigma dW_t^Q, \tag{8}$$

where $\delta = \mu - \Sigma\eta_0, \gamma = \xi - \Sigma\eta_1$, and W_t^Q is an N-dimensional standard Brownian motion under measure Q.

Under the above assumptions, a large class of fixed-income securities can be priced in (essentially) a closed form (see Leippold and Wu (2002)). We discuss the pricing of zero-coupon bonds below and the pricing of caps in Appendix A. Let $V(t, \tau)$ be the time-t value of a zero-coupon bond that pays \$1 at time $T(\tau = T - t)$. In the absence of arbitrage, the discounted value process $\exp(-\int_0^t r(X_s)\,ds)V(t,\tau)$ is a Q martingale. Thus, the value function must satisfy the fundamental partial differential equation, which requires that the bond's instantaneous return equal the risk-free rate,

$$\frac{1}{2}tr\left(\Sigma\Sigma'\frac{\partial^2 V(t,\tau)}{\partial X_t \partial X_t'}\right) + \frac{\partial V(t,\tau)}{\partial X_t'}(\delta + \gamma X_t) + \frac{\partial V(t,\tau)}{\partial t} = r_t V(t,\tau), \tag{9}$$

with the terminal condition $V(t, 0) = 1$. The solution takes the form

$$V(t,\tau) = \exp\left[-X_t' A(\tau) X_t - b(\tau)' X_t - c(\tau)\right], \tag{10}$$

where $A(\tau), b(\tau)$, and $c(\tau)$ satisfy the following system of ordinary differential equations (ODEs),

$$\frac{\partial A(\tau)}{\partial \tau} = \Psi + A(\tau)\gamma + \gamma' A(\tau) - 2A(\tau)\Sigma\Sigma' A(\tau), \tag{11}$$

$$\frac{\partial b(\tau)}{\partial \tau} = \beta + 2A(\tau)\delta + \gamma' b(\tau) - 2A(\tau)\Sigma\Sigma' b(\tau), \tag{12}$$

and

$$\frac{\partial c(\tau)}{\partial \tau} = \alpha + b(\tau)'\delta - \frac{1}{2}b(\tau)'\Sigma\Sigma'b(\tau) + tr[\Sigma\Sigma'A(\tau)],$$
$$\text{with } A(0) = 0_{N\times N},\ b(0) = 0_N,\ \text{and } c(0) = 0. \tag{13}$$

Consequently, the yield-to-maturity, $y(t, \tau)$, is a quadratic function of the state variables

$$y(t,\tau) = \frac{1}{\tau}[X_t'A(\tau)X_t + b(\tau)'X_t + c(\tau)]. \tag{14}$$

In contrast, in the ATSMs, the yields are linear in the state variables and therefore the correlations among the yields are solely determined by the correlations of the state variables. Although the state variables in the QTSMs follow a multivariate Gaussian process, the quadratic form of the yields helps to model the time varying volatility and correlation of bond yields.

B. Estimation Method

To price and hedge caps in the QTSMs, we need to estimate both model parameters and latent state variables. Due to the quadratic relationship between bond yields and the state variables, the state variables are not identified by the observed yields even in the univariate case in the QTSMs. Previous studies, such as ADG (2002), use the efficient method of moments (EMM) of Gallant and Tauchen (1996) to estimate the QTSMs. However, in our analysis, we need to estimate not only model parameters, but also the latent state variables. Hence, we choose the extended Kalman filter to estimate model parameters and extract the latent state variables. Duffee and Stanton (2001) show that the extended Kalman filter has excellent finite sample performance in estimating DTSMs. Previous studies that use the extended Kalman filter in estimating the ATSMs include Duan and Simonato (1999), De Jong and Santa-Clara (1999), and Lund (1997), among many others.

To implement the extended Kalman filter, we first recast the QTSMs into a state-space representation. Suppose we have a time series of observations of yields of L zero-coupon bonds with maturities $\Gamma = (\tau_1, \tau_2, \ldots, \tau_L)$. Let Ξ be the set of parameters for QTSMs, and $Y_k = f(X_k, \Gamma; \Xi)$ be the vector of the L observed yields at the discrete time points $k\Delta t$, for $k = 1, 2, \ldots, K$, where Δt is the sample interval (one day in our case). After the following change of variable:

$$Z_k = U^{-1}\left(\xi^{-1}\mu + X_k\right), \tag{15}$$

we have the state equation

$$Z_k = \Phi Z_{k-1} + w_k, \quad w_k \sim N(0, \Theta), \tag{16}$$

where Φ and Θ are first introduced in (4) and (5), and the measurement equation

$$Y_k = d_k + H_k Z_k + v_k, \ \ v_k \sim N(0, R^v), \tag{17}$$

where the innovations in the state and measurement equations w_k and v_k follow serially independent Gaussian processes and are independent of each other. The time-varying coefficients of the measurement equation d_k and H_k are determined at the ex ante forecast of the state variables,

$$H_k = \left.\frac{\partial f(Uz - \xi^{-1}\mu, \Gamma)}{\partial z}\right|_{z=Z_{k|k-1}} \tag{18}$$

and

$$d_k = f\left(UZ_{k|k-1} - \xi^{-1}\mu, \Gamma\right) - H_k Z_{k|k-1} + B_k, \tag{19}$$

where $Z_{k|k-1} = \Phi Z_{k-1}$.

In the QTSMs, the transition density of the state variables is multivariate Gaussian under the physical measure. Thus, the transition equation in the Kalman filter is exact. The only source of approximation error is due to the linearization of the quadratic measurement equation. As our estimation uses daily data, the approximation error, which is proportional to one day ahead forecast error, is likely to be minor. In Appendix B, we further discuss how to minimize the approximation error by introducing the correction term B_k.[14] The Kalman filter starts with the initial state variable $Z_0 = E(Z_0)$ and the variance–covariance matrix P_0^Z,

$$P_0^Z = E[(Z_0 - E(Z_0))(Z_0 - E(Z_0))']. \tag{20}$$

The unconditional mean and variance have closed-form expressions that can be derived using (5) and (6) by letting Δt go to infinity. Given the set of filtering parameters, $\{\Xi, R^v\}$, we can write the log-likelihood of observations based on the Kalman filter,

$$\log \mathcal{L}(Y; \{\Xi, R^v\}) = \sum_{k=1}^{K} \log f\left(Y_k; \mathcal{Y}_{k-1}, \{\Xi, R^v\}\right) = -\frac{LK}{2}\log(2\pi) - \frac{1}{2}\sum_{k=1}^{K}\log\left|P^Y_{k|k-1}\right|$$
$$-\frac{1}{2}\sum_{k=1}^{K}\left[(Y_k - \hat{Y}_{k|k-1})'\left(P^Y_{k|k-1}\right)^{-1}(Y_k - \hat{Y}_{k|k-1})\right], \tag{21}$$

where $\mathcal{Y}_{k-1}$ is the information set at time $(k-1)\Delta t$, and $P^Y_{k|k-1}$ is the time $(k-1)\Delta t$ conditional variance of Y_k,

$$P^Y_{k|k-1} = H_k P^Z_{k|k-1} H_k' + R^v, \tag{22}$$

and

$$P^Z_{k|k-1} = \Phi P^Z_{k-1} \Phi' + \Theta. \tag{23}$$

Parameters are obtained by maximizing the above likelihood function. To avoid local minimum, in our estimation procedure, we use many different starting values and search for the optimal point using the simplex method. Then, we

[14] The differences between parameter estimates with and without the correction term are very small; and we report those estimates with the correction term B_k.

use the gradient-based optimization method to refine those estimates until they cannot be further improved. This is the standard technique in the literature (see, for example, Duffee (2002)).

C. Parameter Estimates and Model Performance

Following ADG (2002), we consider the canonical form of the three-factor QTSMs, which nests most other specifications of the three-factor quadratic models. In all models, the following restrictions are imposed for identification purpose. We assume that Ψ is a symmetric semipositive definite matrix with diagonal elements of 1

$$\Psi = \begin{bmatrix} 1 & \Psi_{12} & \Psi_{13} \\ \Psi_{12} & 1 & \Psi_{23} \\ \Psi_{13} & \Psi_{23} & 1 \end{bmatrix}. \tag{24}$$

We also assume $\mu \geq 0, \alpha > 0, \beta = 0_N, \xi$ and δ_1 are lower triangular matrices, and Σ is a diagonal matrix. We consider the following three models with a decreasing order of complexity:

- QTSM1. This is the maximal flexible model that has a fully specified covariance matrix of the state variables and allows interactions among state variables in the determination of r_t. For this model, we need to estimate α, three off-diagonal elements of Ψ, three elements of μ, six elements of ξ, three elements of Σ, three elements of δ_0, and six elements of δ_1 . The total number of parameters is 25.
- QTSM2. This model has orthogonal state variables, but allows interactions among the state variables in determining r_t. That is, in QTSM2, ξ and δ_1 are diagonal, so the state variables are orthogonal under both measure P and Q. However, Ψ is nondiagonal, allowing interactions in the determination of r_t . The total number of parameters of this model is 19.
- QTSM3. It has orthogonal state variables and no interactions in the determination of r_t. Thus, the additional restriction in this model relative to QTSM2 is that Ψ is diagonal. In total, we have 16 parameters for QTSM3.

The increasing complexity of these models allows us to understand the contributions of correlations among the state variables in pricing and hedging interest rate caps. We estimate the above three-factor models using 6-month, 1-, 2-, 5-, 7- and 10-year yields. Over the sample period, we have 557 observations, where we drop the likelihood of the first one for initializing the Kalman filter and the observation after September 11, 2001 as an extreme outlier.[15] In implementing the extended Kalman filter, we assume that all yields are observed with independent measurement errors that follow the normal distribution of zero mean and standard deviation of $\sigma_{1/2}, \sigma_1, \sigma_2, \sigma_5, \sigma_7$, and σ_{10} for

[15] The market was closed for a couple of weeks after September 11, 2001, and reopened on September 24, 2001. It is difficult for any model to capture the dramatic decline in interest rates between September 11 and 24. Hence, we drop September 24 as an outlier in our estimation.

each maturity, respectively. We thus have six additional parameters for each of the three models.

Parameter estimates and log-likelihood values for each model are reported in Table III. The likelihood ratios between different models indicate that correlations among the state variables and their interactions in determining r_t are important for better model performance. Figure 5 plots the time-series

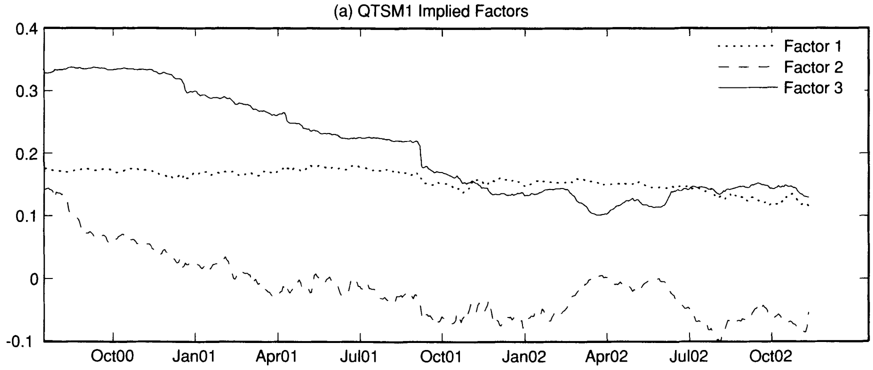
Table III

Parameter Estimates of Three-Factor QTSMs

This table reports the parameter estimates and standard errors (in parentheses) of the three canonical three-factor QTSMs using an extended Kalman filter based on 6-month, 1-, 2-, 5-, 7-, and 10-year LIBOR zero-coupon bond yields observed between August 1, 2000 and November 7, 2002. In implementing the extended Kalman filter, we assume that all yields are observed with independent measurement errors, which follow a normal distribution of zero mean and standard deviation of $\sigma_{1/2}, \sigma_1, \sigma_2, \sigma_5, \sigma_7$, and σ_{10} for each maturity, respectively. We thus have six additional parameters for each of the three models.

Parameter	QTSM3		QTSM2		QTSM1	
α	0.0001*	(0.0051)	0.2584*	(4.0608)	0.0034*	(0.0563)
Ψ_{12}			−0.9033	(0.0066)	0.8373	(0.0078)
Ψ_{13}			−0.2723	(0.0287)	−0.5525	(0.0024)
Ψ_{23}			0.0745	(0.0177)	−0.7585	(0.0073)
μ_1	0.1120	(0.0184)	0.7359	(0.0822)	0.3566	(0.0017)
μ_2	0.0059	(0.0419)	0.1153	(0.0167)	0.2265	(0.0071)
μ_3	0.1565	(0.0382)	1.5268	(0.0150)	0.6136	(0.0088)
ξ_{11}	−1.2234	(0.0428)	−0.0468	(0.0049)	−0.0144	(0.0004)
ξ_{21}					4.5979	(0.1614)
ξ_{31}					1.0728	(0.1238)
ξ_{22}	−0.6142	(0.1631)	−0.7578	(0.0789)	−3.4952	(0.0978)
ξ_{32}					3.2078	(0.3519)
ξ_{33}	−0.0083	(0.0020)	−0.0002	(0.0000)	−2.2678	(0.1608)
Σ_{11}	0.0479	(0.0008)	0.0519	(0.0025)	0.0222	(0.0003)
Σ_{22}	0.0725	(0.0042)	0.0801	(0.0016)	0.0728	(0.0003)
Σ_{33}	0.0468	(0.0019)	0.0235	(0.0024)	0.0220	(0.0004)
δ_1	0.0094	(0.0007)	0.0359	(0.0002)	0.0104	(0.0001)
δ_2	−0.1903	(0.0019)	0.0190	(0.0041)	−0.0021	(0.0005)
δ_3	−0.0438	(0.0034)	−0.0108	(0.0009)	−0.0378	(0.0003)
γ_{11}	−0.0530	(0.0039)	−0.1295	(0.0003)	−0.0518	(0.0004)
γ_{21}					1.0130	(0.0073)
γ_{31}					0.0276	(0.0012)
γ_{22}	−1.1378	(0.0200)	−1.1219	(0.0068)	−1.1698	(0.0001)
γ_{32}					0.3018	(0.0013)
γ_{33}	−0.5544	(0.0155)	0.0133	(0.0044)	−0.0558	(0.0023)
$\sigma_{1/2}$	4.2749*	(0.2026)	0.0128*	(0.6710)	0.0002*	(0.1130)
σ_1	2.8876*	(0.1358)	2.2165*	(0.1003)	2.0417*	(0.1043)
σ_2	1.5794*	(0.1481)	1.8485*	(0.3384)	1.9916*	(0.0660)
σ_5	1.7779*	(0.0677)	2.1867*	(0.0496)	1.9257*	(0.0744)
σ_7	0.8568*	(0.0861)	0.0049*	(0.2917)	0.0005*	(0.0958)
σ_{10}	2.9730*	(0.1153)	2.7741*	(0.2328)	2.7830*	(0.0998)
Log-likelihood	21,243		22,043		22,300	

$^*1 \times 10^{-4}$.



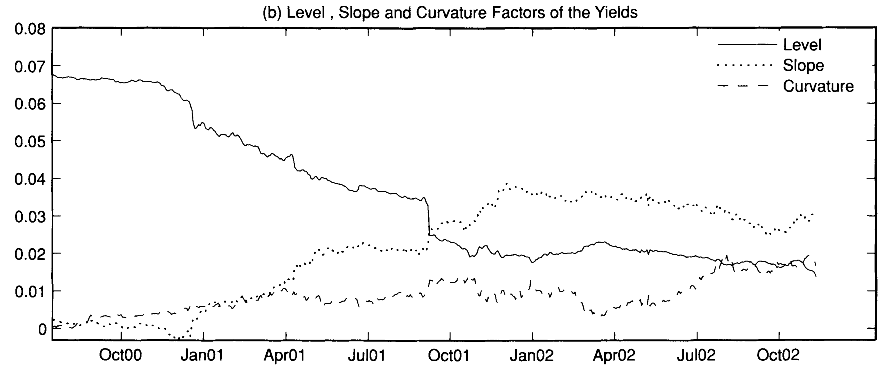


Figure 5. The QTSM1 implied factors and the three yield level factors. This figure contains the time-series observations of QTSM1 implied state variables and the three yield level factors obtained from principal component analysis. The parameters and state variables of QTSM1 are estimated using an extended Kalman filter based on six-month, 1-, 2-, 5-, 7-, and 10-year LIBOR zero-coupon bond yields between August 1, 2000 and November 7, 2002.

observations of QTSM1 model-implied state variables and the three yield factors obtained from principal component analysis. There are clear differences between the two sets of variables due to the nonlinear relationship between bond yields and the state variables in the QTSMs.

We examine the performance of the QTSMs in capturing yield curve dynamics from several different perspectives. Given the estimated model parameters and state variables, we can compute the one-day-ahead projection of yields based on the estimated model. Figure 6 shows that QTSM1 projected yields are almost indistinguishable from the corresponding observed yields. Panels A and B of Table IV report the summary statistics of the levels and changes of QTSM1 projected yields, respectively. A comparison with the summary statistics of the actual yields in Panels A and B of Table I shows that QTSM1 can capture the mean, standard deviation, skewness, kurtosis, and persistence (first-order autoregressive coefficients) of bond yields very well.

In later sections, we study whether bonds span caps by testing whether the QTSMs estimated using bond data alone can price and hedge caps well. For

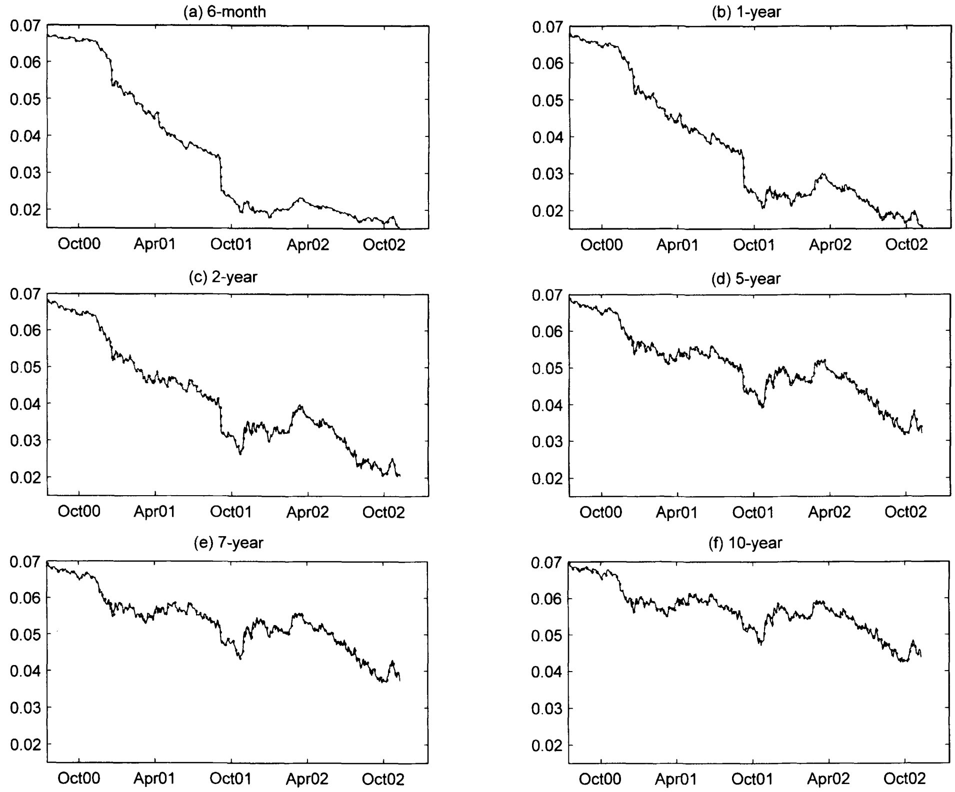


Figure 6. The observed yields (dot) and the QTSM1 projected yields (solid). This figure contains the time-series observations of QTSM1 projected yields and the actual yields of LIBOR zero-coupon bonds with 6-month, 1-, 2-, 5-, 7-, and 10-year maturities between August 1,2000 and November 7, 2002. The parameters and state variables of QTSM1 are estimated using an extended Kalman filter based on the same six yields. Due to the excellent model fit, the differences between the two sets of yields are not very significant in the figure.

comparison, we first examine the performance of the QTSMs in pricing and hedging zero-coupon bonds in Panels C and D of Table IV, respectively. The three QTSMs have excellent performance in pricing LIBOR bonds: The root-mean-squared-errors (RMSEs) of percentage pricing errors (measured as the difference between market and model-implied yields divided by market yield) for most bonds in Panel C are less than 1%. For hedging analysis, we assume that the filtered state variables are traded and we use them as hedging instruments.[16] We conduct a delta-neutral hedge for the six zero-coupon bonds on a daily basis. Hedging performance is measured by variance ratio, which is defined as the percentage of the variations of an unhedged position that can

[16] Alternatively, we could select a few bonds as hedging instruments and match their hedge ratios with respect to the three state variables with that of the bonds to be hedged. This approach, however, would introduce another layer of model misspecification when computing the hedge ratios of the bonds used as hedging instruments. Later, we show that even under this favorable condition, the QTSM models still have serious difficulties in hedging caps and cap straddles.

Table IV

The Performance of QTSMs in Modeling Bond Yields

This table reports the performance of the three QTSMs in capturing bond yields. The parameters and state variables of the QTSMs are estimated using an extended Kalman filter based on 6-month, 1-, 2-, 5-, 7-, and 10-year LIBOR zero-coupon bond yields observed between August 1, 2000 and November 7, 2002. For brevity, we only report results for the best model, QTSM1, in Panels A and B.

	Maturity (year)					
	0.5	1	2	5	7	10
Panel A: Summary Statistics of QTSM1 Predicted Levels of LIBOR Zero-Coupon Bond Yields						
Mean (%)	3.529	3.683	4.154	5.049	5.358	5.666
Standard Deviation (%)	1.787	1.632	1.350	0.918	0.774	0.622
Skewness	0.616	0.598	0.468	0.093	−0.017	−0.059
Kurtosis	1.882	1.997	2.163	2.534	2.676	2.783
First-order partial autocorrelation	0.998	0.998	0.998	0.997	0.996	0.995
Panel B: Summary Statistics of QTSM1 Predicted Changes of LIBOR Zero-Coupon Bond Yields						
Mean (%)	−0.010	−0.009	−0.009	−0.007	−0.006	−0.005
Standard Deviation (%)	0.047	0.055	0.065	0.072	0.069	0.062
Skewness	−7.463	−3.222	−0.653	0.280	0.343	0.366
Kurtosis	111.082	42.979	12.058	4.714	4.279	4.132
First-order partial autocorrelation	0.286	0.262	0.186	0.109	0.094	0.080
Panel C: RMSEs of Percentage Pricing Errors of LIBOR Zero-Coupon Bond Yields under QTSMs						
QTSM3	1.894	0.917	0.384	0.419	0.111	0.538
QTSM2	0.205	0.889	0.530	0.512	0.073	0.519
QTSM1	0.150	0.762	0.546	0.434	0.077	0.535
Panel D: Hedging Effectiveness of QTSMs for LIBOR Zero-Coupon Bonds Measured by Variance Ratio, Which Is the Percentage of the Variations of an Unhedged Position That Can Be Reduced through Hedging						
QTSM3	0.717	0.948	0.982	0.980	0.993	0.930
QTSM2	0.990	0.956	0.963	0.975	0.997	0.934
QTSM1	0.994	0.962	0.969	0.976	0.997	0.932

Note. Percentage pricing error is measured as the difference between market and model-implied yield divided by market yield. All entries are in percentage terms.

be reduced by hedging. The results on the hedging performance in Panel D of Table IV show that in most cases the variance ratios are higher than 95%. The pricing and hedging performance of LIBOR bonds reflects the excellent fit of bond yields by the QTSMs.

III. Pricing and Hedging Interest Rate Caps in QTSMs

If both the LIBOR and swap market and the cap market are well integrated, then the estimated three-factor QTSMs should be able to price and hedge caps well. Otherwise, it would be a strong indication that there are risk factors affecting cap prices that are not spanned by bonds. With the estimated parameters and the state variables of the three QTSMs, we re-examine the issue of USV in the cap market.

A. Pricing Interest Rate Caps

We first study the performance of the QTSMs in pricing caps. Panels A, B, and C of Table V report the RMSE of percentage pricing errors of caps with different moneyness and maturities for QTSM3, QTSM2, and QTSM1, respectively.[17] Percentage pricing error, defined as the difference between market and model price divided by market price, is a better measure of model performance because caps with different moneyness and maturities can have dramatically different prices. This measure has been used in previous studies, such as Longstaff, Santa-Clara, and Schwartz (2001) and FGR (2003). As pointed out before, we interpolate cap prices with respect to strike price to obtain prices at fixed moneyness. Similar to Table II, the bold entries are moneyness/maturity groups that have less than 10% of missing values and the rest have between 10% to 50% of missing values.

All three QTSMs have smaller percentage pricing errors for ITM and long-term caps than OTM and short-term caps. For example, in QTSM1, while the percentage pricing errors for ITM caps are less than 10%, they can be over 40% for short-term OTM caps. QTSM2 and QTSM3 have especially high percentage pricing errors for short-term and OTM caps. A more careful analysis of pricing errors shows that the poor pricing performance of short-term caps is mainly due to the first 100 days of the data. The high percentage bid-ask spreads during this period indicate that the liquidity in the caps market is likely to be low. One possible reason for the low liquidity is that during this period, the yield curve was flat and interest rates were quite stable, which may reduce the demands for short-term caps for hedging purposes. In general, QTSM1 has smaller percentage pricing errors across moneyness and maturities than the other two models, except QTSM2 has slightly lower percentage pricing errors for deep ITM caps. This indicates that the additional flexibility provided by the correlations among the state variables improves the model's pricing performance. QTSM2 has lower pricing errors than QTSM3 for short-term caps, but higher pricing errors for long-term caps.

While the QTSMs have significant pricing errors, the RMSE of percentage pricing error does not tell the direction of mispricing. Panel D of Table V reports the average percentage pricing errors of the best model, QTSM1. It is clear that QTSM1 underprices ITM caps and overprices OTM caps. This is consistent with Jarrow et al. (2003) who show that it is difficult to capture the pronounced volatility skew in cap data.

The pricing analysis shows that although the QTSMs can capture the level of bond yields pretty well (see Table IV), they still have systematic errors for

[17] Caps are priced under the QTSMs using the closed-form formula provided in Appendix A.

Table V

The Performance of QTSMs in Pricing Interest Rate Caps

This table reports the performance of the three QTSMs in pricing difference caps. Percentage pricing error is measured as the difference between market and model-implied price divided by market price. The price of a difference cap equals the difference between the prices of two corresponding caps with adjacent maturities. Moneyness (K/F) is defined as the ratio between the strike and the average LIBOR forward rates underlying the few caplets that form the difference cap. We interpolate difference cap prices with respect to strike to obtain the prices of difference caps at fixed moneyness. The bold entries represent moneyness/maturity groups that have less than 10% of missing values and the rest are the ones with 10% to 50% of missing values. All entries are in percentage terms.

Moneyness	Maturity												
K/F	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
Panel A: RMSEs of Percentage Pricing Errors of Difference Caps under QTSM3													
0.60				11.3	10.5	9.7	9.2	11.0	10.9	10.3	9.4	8.8	9.4
0.70		15.6	13.8	11.4	10.3	**9.5**	**9.5**	11.3	11.1	11.3	10.1	9.9	10.9
0.80		16.7	**15.5**	**13.2**	**12.7**	**11.5**	**11.3**	**12.6**	**12.6**	**12.1**	**11.2**	10.6	11.3
0.90	25.3	**27.2**	**22.4**	**17.9**	**17.3**	**15.1**	**13.7**	**13.9**	**14.9**	**12.9**	**13.6**	**11.8**	**12.0**
1.00	**69.0**	**50.9**	**35.0**	**26.1**	**23.6**	**19.1**	**18.9**	**17.5**	**18.3**	**15.2**	**15.7**	**13.1**	**13.6**
1.10	**174.0**	**84.6**	**54.6**	**38.0**	**33.5**	**26.8**	**25.1**	22.2	24.5	19.8	20.3	20.4	17.5
1.20	411.1	**135.5**	**81.7**	**55.7**	48.9	40.2	34.8	30.7	36.2	25.7	24.6	25.1	19.4
1.30	207.6	**364.1**	**116.7**	80.9	70.5	57.2	42.9	34.5	53.5	32.6	27.3		
1.40	200.0	344.5	167.2	116.9	94.5	76.3	50.6						
Panel B: RMSEs of Percentage Pricing Errors of Difference Caps under QTSM2													
0.60				5.0	4.0	3.8	4.5	4.9	4.3	4.2	4.4	6.0	7.2
0.70		9.0	6.0	5.0	4.5	**5.0**	**5.8**	5.6	5.1	5.4	7.4	8.7	10.1
0.80		9.1	**6.8**	**7.5**	**8.0**	**9.5**	**9.9**	**8.9**	**9.5**	**9.8**	**12.6**	14.6	16.5
0.90	11.5	**13.7**	**11.1**	**13.5**	**14.6**	**16.6**	**16.9**	**15.7**	**16.4**	**16.7**	**20.0**	**21.3**	**23.0**
1.00	**30.9**	**25.4**	**19.5**	**21.6**	**22.5**	**25.0**	**26.1**	**24.3**	**24.6**	**24.6**	**27.3**	**28.6**	**30.2**
1.10	**75.8**	**39.7**	**31.2**	**32.5**	**33.6**	**36.3**	**35.5**	32.0	32.8	30.8	33.0	34.8	33.3
1.20	169.2	**59.5**	**46.8**	**47.5**	**45.5**	44.7	37.8	32.5	42.0	38.8	40.6	43.6	40.1
1.30	86.9	**134.6**	**65.5**	59.6	56.8	53.7	43.0	37.1	59.3	51.3	48.0		
1.40	91.2	130.6	91.6	77.6	75.5	70.8	53.3						
Panel C: RMSEs of Percentage Pricing Errors of Difference Caps under QTSM1													
0.60				8.1	6.9	6.1	5.6	7.1	6.5	5.5	4.2	4.0	4.7
0.70		13.0	9.6	7.1	6.2	**5.5**	**5.6**	7.0	5.9	5.9	4.6	4.9	6.0
0.80		12.7	**9.4**	**6.9**	**6.1**	**5.6**	**6.1**	**7.0**	**5.3**	**5.1**	**4.9**	5.8	7.2
0.90	10.7	**14.2**	**9.4**	**7.1**	**7.0**	**7.2**	**7.7**	**7.9**	**6.2**	**6.1**	**7.5**	**7.9**	**9.2**
1.00	**16.2**	**17.3**	**10.7**	**9.1**	**9.2**	**10.0**	**11.0**	**10.4**	**8.7**	**8.6**	**9.6**	**10.2**	**11.5**
1.10	**34.3**	**20.9**	**13.8**	**12.8**	**13.5**	**14.8**	**14.5**	13.2	11.7	11.0	11.4	13.9	14.8
1.20	77.1	**27.4**	**19.7**	**19.0**	**18.6**	17.9	14.4	11.7	15.7	14.1	13.8		17.8
1.30	45.2	**55.2**	**29.1**	24.7	24.2	21.4	16.2	15.5	24.5	19.1	14.9		
1.40	47.2	57.4	43.2	34.0	33.4	28.8	20.9						
Panel D: Average Percentage Pricing Errors of Difference Caps under QTSM1													
0.60				6.6	5.5	4.4	3.4	5.1	4.7	4.0	2.6	1.5	1.4
0.70		11.2	7.4	5.1	4.5	**3.7**	**3.1**	4.9	3.8	4.0	1.3	0.4	0.1
0.80		10.1	**7.1**	**4.2**	**3.3**	**1.7**	**1.7**	**3.5**	**1.8**	**1.8**	**−0.9**	−1.9	−2.5
0.90	6.3	**8.3**	**5.3**	**1.8**	**0.6**	**−1.2**	**−1.3**	**0.6**	**−1.1**	**−1.0**	**−4.0**	**−4.6**	**−4.6**
1.00	**−0.7**	**4.6**	**3.0**	**−0.4**	**−1.6**	**−3.7**	**−4.6**	**−2.6**	**−4.0**	**−3.8**	**−6.3**	**−6.6**	**−6.4**
1.10	**−10.9**	**1.4**	**0.3**	**−3.4**	**−4.6**	**−−7.1**	**−6.9**	−3.8	−6.0	−4.9	−7.5	−8.6	−6.5
1.20	−23.0	**−3.1**	**−3.1**	**−7.7**	**−7.1**	−7.9	−3.8	−0.5	−7.5	−5.9	−8.9		−6.2
1.30	−11.6	**−14.8**	**−6.4**	−8.6	−7.0	−7.0	−1.7	3.0	−10.6	−7.7	−8.6		
1.40	−2.6	−13.7	−9.6	−9.3	−8.5	−8.9	−0.9						

pricing caps, especially OTM caps. A deep ITM option behaves almost like the underlying asset because the probability that the option will eventually be in the money is high. Thus, it is not surprising that the QTSMs have relatively better performance in pricing ITM caps. However, deep OTM caps are much more sensitive to higher-order moments and even the tail distribution of the underlying interest rates. Therefore, to accurately price OTM caps, the QTSMs need to capture not only the level but also the whole distribution of bond yields well.

B. Hedging Interest Rate Caps

Pricing analysis mainly focuses on whether a model can capture the terminal distribution of the underlying asset price on the maturity date of an option. On the other hand, hedging analysis reveals whether a model can capture the evolution dynamics of the underlying price process. In this section, we study the performance of the three-factor QTSMs in hedging caps.

Based on the estimated model parameters, we conduct a delta-neutral hedge of weekly changes of difference cap prices using filtered state variables as hedging instruments. We could also use LIBOR zero-coupon bonds as hedging instruments by matching the hedge ratios of a difference cap with that of zero-coupon bonds. However, using deltas of zero-coupon bonds introduces an additional layer of potential model misspecification in calculating the hedge ratios of the bonds used as hedging instruments. If our approach introduces any biases, they should make it more difficult to find USV. To improve hedging performance, we allow daily re-balancing, that is, we adjust the hedge ratios every day given changes in market conditions. Therefore, daily changes of a hedged position are equal to the difference between daily changes of the unhedged position and the hedging portfolio, where the latter is equal to the sum of the products of a difference cap's hedge ratios with respect to the state variables and changes in the corresponding state variables. Weekly changes are just the accumulation over daily positions. Over the sample period, there are 111 nonoverlapping hedged and unhedged changes for each moneyness/maturity group if there are no missing data.

Again, we measure hedging effectiveness by variance ratio, the percentage of the variations of an unhedged position that can be reduced by hedging. This measure is similar in spirit to R^2 in linear regression.[18] The variance ratios of the three QTSMs in Table VI show that all models have better hedging performance for ITM, short-term (maturities from 1.5 to 4 years) difference caps than OTM, medium, and long-term difference caps (maturities longer than 4 years). There is a high percentage of variations in long-term and OTM difference cap prices that cannot be hedged. The maximal flexible model QTSM1 again has better hedging performance than the other two models.

Interestingly, the variance ratios of model-based hedging in Table VI are very similar to the R^2's of linear regressions Table II. In general, linear regression

[18] FGR (2003) also consider RMSE of hedging errors because their hedging errors have significant biases. Since the hedging bias in our case is very small, we only report the variance ratios.

Table VI

The Performance of QTSMs in Hedging Interest Rate Caps

This table reports the performance of the three QTSMs in hedging difference caps. A difference cap is a portfolio of caplets with the same strike and maturities between two neighboring cap maturities, which are 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0, 9.0, and 10.0 years in our sample. Moneyness (K/F) is defined as the ratio between the strike and the average LIBOR forward rates underlying the few caplets that form the difference cap. We interpolate difference cap prices with respect to strike to obtain the prices of difference caps at fixed moneyness. Hedging effectiveness is measured by variance ratio, the percentage of the variations of an unhedged position that can be reduced through hedging. The bold entries represent moneyness/maturity groups that have less than 10% of missing values and the rest are the those with 10% to 50% of missing values.

Moneyness	Maturity												
(K/F)	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
Panel A: Hedging Effectiveness of QTSM3 for Difference Caps Measured by Variance Ratio													
0.60					0.909				0.857	0.574	0.688	0.485	0.290
0.70			0.884	0.906	0.916	**0.866**	**0.647**	0.455	0.811	0.547	0.592	0.347	0.250
0.80		0.821	**0.880**	**0.897**	**0.871**	**0.868**	**0.649**	**0.474**	**0.781**	**0.597**	**0.581**	0.331	0.216
0.90	0.878	**0.810**	**0.838**	**0.870**	**0.836**	**0.832**	**0.598**	**0.488**	**0.732**	**0.565**	**0.504**	**0.323**	**0.187**
1.00	**0.855**	**0.662**	**0.781**	**0.809**	**0.801**	**0.802**	**0.521**	**0.416**	**0.649**	**0.502**	**0.447**	**0.250**	**0.162**
1.10	**0.805**	**0.538**	**0.694**	**0.741**	**0.713**	**0.733**	**0.440**	0.353	0.539	0.451	0.315	0.189	
1.20	0.630	**0.440**	**0.566**	**0.628**	**0.566**	0.602	0.436	0.412	0.359	0.372			
1.30	0.597	**0.328**	**0.424**	0.488	0.277	0.446	0.421		0.170				
1.40		0.180	0.220	0.277	0.049								
Panel B: Hedging Effectiveness of QTSM2 for Difference Caps Measured by Variance Ratio													
0.60					0.916				0.853	0.665	0.668	0.487	0.246
0.70			0.899	0.908	0.914	**0.856**	**0.657**	0.478	0.818	0.612	0.557	0.332	0.207
0.80		0.834	**0.891**	**0.898**	**0.873**	**0.857**	**0.661**	**0.494**	**0.781**	**0.591**	**0.537**	0.296	0.175
0.90	0.893	**0.815**	**0.852**	**0.868**	**0.849**	**0.827**	**0.624**	**0.505**	**0.745**	**0.571**	**0.491**	**0.304**	**0.158**
1.00	**0.875**	**0.682**	**0.802**	**0.819**	**0.820**	**0.807**	**0.566**	**0.446**	**0.695**	**0.533**	**0.475**	**0.277**	**0.153**
1.10	**0.844**	**0.571**	**0.722**	**0.763**	**0.742**	**0.755**	**0.510**	0.402	0.625	0.491	0.416	0.189	
1.20	0.741	**0.446**	**0.593**	**0.645**	**0.604**	0.639	0.519	0.480	0.531	0.421			
1.30	0.681	**0.274**	**0.410**	0.516	0.343	0.539	0.548		0.338				
1.40		0.063	0.189	0.330	0.178								
Panel C: Hedging Effectiveness of QTSM1 for Difference Caps Measured by Variance Ratio													
0.60					0.917				0.862	0.679	0.665	0.494	0.257
0.70			0.903	0.913	0.916	**0.862**	**0.666**	0.487	0.822	0.619	0.565	0.355	0.218
0.80		0.831	**0.890**	**0.900**	**0.876**	**0.864**	**0.670**	**0.504**	**0.785**	**0.594**	**0.537**	0.305	0.185
0.90	0.890	**0.810**	**0.853**	**0.872**	**0.851**	**0.832**	**0.631**	**0.514**	**0.748**	**0.577**	**0.491**	**0.314**	**0.171**
1.00	**0.875**	**0.677**	**0.803**	**0.824**	**0.826**	**0.815**	**0.575**	**0.456**	**0.695**	**0.533**	**0.476**	**0.287**	**0.164**
1.10	**0.851**	**0.575**	**0.737**	**0.779**	**0.763**	**0.773**	**0.523**	0.411	0.623	0.490	0.415	0.204	
1.20	0.756	**0.489**	**0.645**	**0.692**	**0.654**	0.673	0.521	0.470	0.533	0.415			
1.30	0.724	**0.393**	**0.534**	0.591	0.444	0.602	0.540		0.334				
1.40		0.260	0.373	0.464	0.319								
Panel D: Hedging Effectiveness of QTSM1 Combined with the Three Yield Factors (Level, Slope, and Curvature) for Difference Caps Measured by Variance Ratio													
0.60					0.921				0.912	0.788	0.815	0.658	0.579
0.70			0.927	0.922	0.919	**0.872**	**0.675**	0.507	0.847	0.679	0.666	0.462	0.353
0.80		0.886	**0.915**	**0.912**	**0.882**	**0.873**	**0.675**	**0.510**	**0.811**	**0.653**	**0.639**	0.417	0.324
0.90	0.942	**0.853**	**0.876**	**0.882**	**0.855**	**0.837**	**0.633**	**0.524**	**0.776**	**0.632**	**0.573**	**0.412**	**0.310**
1.00	**0.923**	**0.746**	**0.841**	**0.836**	**0.839**	**0.820**	**0.578**	**0.462**	**0.709**	**0.566**	**0.541**	**0.361**	**0.269**
1.10	**0.895**	**0.659**	**0.799**	**0.804**	**0.794**	**0.786**	**0.530**	0.421	0.630	0.508	0.480	0.278	
1.20	0.848	**0.611**	**0.741**	**0.746**	**0.729**	0.701	0.530	0.486	0.572	0.427			
1.30	0.796	**0.560**	**0.680**	0.687	0.626	0.679	0.559		0.378				
1.40		0.455	0.582	0.638	0.573								

has higher explanatory power than model-based hedging for ITM difference caps, and model-based hedging has better performance for OTM difference caps. This is consistent with the fact that ITM difference caps behave more like the underlying bonds and OTM difference caps depend more on the higher-order moments and even the tail distribution of the underlying process. The fact that linear regression can have higher R^2's for some difference caps is also partly because we run a separate regression with different parameters for difference caps within each moneyness/maturity group. On the other hand, the hedge ratios of all difference caps in model-based hedging are determined by the same set of parameters estimated using bond data. Thus, the number of parameters and degrees of freedom are much larger in the regression analysis in Table II. Therefore, both regression and model-based hedging suggest that bond market factors cannot satisfactorily hedge difference caps, especially OTM and long-term difference caps.

To control for the fact that the QTSMs may be misspecified, in Panel D of Table VI we further regress hedging errors of each moneyness/maturity group on changes of the three yield factors. While the three yield factors can explain some additional hedging errors, their incremental explanatory power is not very significant. Thus, even excluding hedging errors that can be captured by the three yield factors, there is still a large fraction of difference cap prices that cannot be explained by the QTSMs. We conduct principal component analysis of hedging errors of difference caps with different moneyness in Table VII, focusing on those moneyness groups for which we have enough observations throughout the whole sample period. We also repeat the analysis by combining these difference caps together. It is clear that the first principal component explains about 50–60% of the hedging errors of all difference caps and difference caps within each moneyness group. Each of the next two components explains an additional 10% of hedging errors. Our analysis of hedging errors suggests that there could be multiple unspanned factors in the cap market.

Table VII

Principal Component Analysis of Cap Hedging Errors

This table reports the principal component analysis of difference cap hedging errors across moneyness. Moneyness (K/F) is defined as the ratio between the strike and the average LIBOR forward rates underlying the few caplets that form the difference cap.

Moneyness	Principal Component				
(K/F)	1	2	3	4	5
0.80	60.6%	11.9%	9.4%	4.6%	4.5%
0.85	58.3%	11.8%	9.6%	6.2%	4.4%
0.90	57.6%	11.9%	10.0%	5.5%	5.2%
0.95	57.5%	10.7%	9.8%	6.4%	4.7%
1.00	56.0%	12.1%	9.8%	7.0%	5.3%
1.05	40.0%	25.0%	21.1%	5.9%	3.8%
1.10	49.0%	31.7%	8.6%	5.2%	3.3%
1.15	67.1%	14.9%	8.5%	6.5%	1.8%
Overall	51.5%	12.1%	9.7%	6.6%	5.6%

C. Hedging Cap Straddles: Evidence of Unspanned Stochastic Volatility

Hedging analysis based on the QTSMs confirms the findings of Collin-Dufresne and Goldstein (2002a) that there are unspanned factors in the cap market. Collin-Dufresne and Goldstein (2002a) show that changes in swap rates in general can explain less than 20% of ATM cap straddle returns, which are most sensitive to volatility risk. Therefore, they argue that the unspanned factor is a stochastic volatility factor that significantly affects cap prices but is not affected by bond yields. However, as pointed out by FGR (2003), linear regression results could be misleading because straddle returns are highly nonlinear in underlying yield factors. They show that although linear regression can explain little variation in swaption straddle returns, a three-factor HJM model can hedge swaption straddles pretty well.

In Table VIII, we re-examine the issue of USV in the cap market by testing the performance of the QTSMs in hedging cap straddles. We obtain difference floor prices from difference cap prices using put–call parity and we construct weekly straddle returns. As straddles are highly sensitive to volatility risk, we conduct both delta- and gamma-neutral hedges. The variance ratios of QTSM1 are as low as the R^2's of linear regressions of straddle returns on the yield factors in Table II, suggesting that neither approach can explain much of the variation in straddle returns. While FGR (2003) show that linear regression and model-based hedging have dramatically different performance for swaption straddles, we find that the difference between the two approaches for cap straddles is very small. The fact that a model-based delta- and gamma-neutral hedge does not significantly outperform linear regression is a strong indication of the existence of USV. This is because in the presence of USV, the advantages of a nonlinear model and a gamma-neutral hedge could be greatly diminished due to the huge hedging errors caused by USV. Even though we do not have a rigorous statistical criteria that specifies what levels of R^2 would indicate the existence of unspanned factors, the fact that the QTSMs can hedge more than 95% of bond returns but only less than 10% of ATM straddle returns strongly suggests that the QTSMs may be missing risk factors that are important for the cap market. We could provide a more rigorous statistical test of USV using the GMM type of tests developed in Buraschi and Jackwerth (2001) and Coval and Shumway (2001). For example, we could include factors from the cap market in the stochastic discount factor (SDF), which should include only factors from the swap market if there is no USV. The existence of USV can be tested by examining whether the coefficients of the factors from the cap market in the SDF are statistically significant.

Collin-Dufresne and Goldstein (2002a) show that 80% of straddle regression residuals can be explained by one additional factor. Principal component analysis of ATM straddle hedging errors in Panel B of Table VIII shows that the first factor can explain about 60% of the total variation of hedging errors. The second and third factors each explain about 10% of hedging errors and two additional factors combined can explain another 10% of hedging errors. The correlation matrix of the ATM straddle hedging errors across maturities in

Table VIII
Hedging Interest Rate Cap Straddles

This table reports the performance of QTSM1 in hedging difference cap straddles. A difference cap is a portfolio of caplets with the same strike and maturities between two neighboring cap maturities, which are 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0, 9.0, and 10.0 years in our sample. Moneyness (K/F) is defined as the ratio between the strike and the average LIBOR forward rates underlying the few caplets that form the difference cap. We interpolate difference cap prices with respect to strike to obtain the prices of difference caps at fixed moneyness. For each moneyness, we construct straddle prices by obtaining difference floor prices from difference cap prices using put–call parity. Hedging effectiveness is measured by variance ratio, the percentage of the variations of an unhedged position that can be reduced through hedging. The bold entries represent moneyness/maturity groups that have less than 10% of missing values and the rest are the those with 10% to 50% of missing values.

Panel A: Hedging Effectiveness of QTSM1 for Difference Cap Straddles Measured by Variance Ratio

	Maturity												
(K/F)	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.60					0.711				0.709	0.329	0.596	0.362	0.250
0.70			0.776	0.723	0.674	**0.557**	**0.250**	0.152	0.473	0.206	0.187	0.096	0.074
0.80		0.560	**0.652**	**0.615**	**0.437**	**0.488**	**0.179**	**0.093**	**0.293**	**0.126**	**0.113**	0.053	0.070
0.90	0.558	**0.278**	**0.405**	**0.339**	**0.248**	**0.265**	**0.066**	**0.049**	**0.138**	**0.052**	**0.032**	**0.016**	**0.060**
1.00	**0.364**	**0.081**	**0.210**	**0.142**	**0.149**	**0.142**	**0.024**	**0.006**	**0.045**	**0.006**	**0.047**	**0.009**	**0.002**
1.10	**0.622**	**0.212**	**0.368**	**0.226**	**0.314**	**0.283**	**0.146**	0.065	0.133	0.054	0.091	0.018	
1.20	0.788	**0.527**	**0.633**	**0.515**	**0.593**	0.481	0.368	0.201	0.343	0.256			
1.30	0.851	**0.727**	**0.808**	0.728	0.781	0.729	0.525		0.454				
1.40		0.817	0.894	0.863	0.880								

Panel B: Principal Component Analysis of ATM Straddle Hedging Errors under QTSM1

Principal Component					
1	2	3	4	5	6
59.3%	12.4%	9.4%	6.7%	4.0%	2.8%

Panel C: Correlation Matrix of ATM Straddles Hedging Errors Across Maturity

	Maturity												
Maturity	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
1.5	1.00												
2.0	0.38	1.00											
2.5	0.28	0.66	1.00										
3.0	0.03	0.33	0.73	1.00									
3.5	0.27	0.52	0.63	0.59	1.00								
4.0	0.13	0.44	0.37	0.37	0.77	1.00							
4.5	0.20	0.21	−0.04	−0.08	−0.05	−0.06	1.00						
5.0	0.10	0.11	−0.12	−0.13	−0.16	−0.15	0.96	1.00					
6.0	0.21	0.16	0.19	0.13	0.25	0.05	0.27	0.23	1.00				
7.0	0.30	0.34	0.33	0.35	0.46	0.38	0.28	0.22	0.08	1.00			
8.0	0.10	0.12	0.30	0.30	0.25	0.11	0.36	0.34	0.29	0.29	1.00		
9.0	0.14	0.11	0.25	0.29	0.26	0.12	0.39	0.37	0.32	0.38	0.83	1.00	
10.0	0.08	−0.01	0.17	0.14	0.12	0.01	0.32	0.35	0.26	0.28	0.77	0.86	1.00

Table IX

Straddle Hedging Errors and Cap-Implied Volatilities

This table reports the relationship between straddle hedging errors under QTSM1 and ATM cap-implied volatilities.

Panel A. Principal Component Analysis of ATM Cap-Implied Volatilities

Principal Component					
1	2	3	4	5	6
85.73%	7.91%	1.85%	1.54%	0.72%	0.67%

Panel B: R^2's of the Regressions of ATM Straddle Hedging Errors on Changes of the Three Yield Factors (row one); Changes of the Three Yield Factors and the First Four Principal Components of the ATM Cap-Implied Volatilities (row two); and Changes of the Three Yield Factors and Maturity-Wise ATM Cap-Implied Volatility (row three)

Maturity												
1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
0.10	0.06	0.02	0.01	0.01	0.04	0.00	0.00	0.01	0.01	0.00	0.01	0.04
0.29	0.49	0.54	0.43	0.63	0.47	0.95	0.96	0.21	0.70	0.68	0.89	0.96
0.68	0.70	0.81	0.87	0.85	0.90	0.95	0.98	0.95	0.98	0.97	0.98	0.99

Panel C shows that the hedging errors of short-term (2.0-, 2.5-, 3.0-, 3.5-, and 4.0-year), medium-term (4.5- and 5.0-year), and long-term (8.0-, 9.0-, and 10.0-year) straddles are highly correlated within each group, again suggesting that there could be multiple unspanned factors.

To further understand whether the unspanned factors are related to stochastic volatility, we study the relationship between ATM cap-implied volatilities and straddle hedging errors. Principal component analysis in Panel A of Table IX shows that the first component explains 85% of the variations of cap-implied volatilities. In Panel B, we regress straddle hedging errors on changes of the three yield factors and obtain R^2's that are close to zero. However, if we include the weekly changes of the first few principal components of cap-implied volatilities, the R^2's increase significantly: For some maturities, the R^2's are above 90%. Although the time series of implied volatilities are very persistent, their differences are not and we do not suffer from the well-known problem of spurious regression. In the extreme case in which we regress straddle hedging errors of each maturity on changes of the yield factors and cap-implied volatilities with the same maturity, the R^2's in most cases are above 90%. These results show that straddle returns are mainly affected by volatility risk but not term structure factors.

Given the fact that the cap market is generally less liquid than the swap market, the straddle hedging errors could also be due to liquidity risk, a market force that affects derivatives prices but is not captured by the existing term structure models. Following the existing market microstructure literature, we

Table X

Straddle Hedging Errors and Bid-Ask Spreads

This table reports the relationship between the ATM straddle hedging errors under QTSM1 and the absolute and percentage bid-ask spreads of ATM caps, which serve as a proxy of liquidity risk. The bid-ask spread is quoted in implied volatilities. We convert the implied volatilities into bid and ask prices and the difference between the two is the absolute bid-ask spread. The percentage bid-ask spread is the ratio between the absolute spread and the cap price (the midpoint of the bid and ask prices). The entries are the R^2's of the regressions of ATM straddle hedging errors on weekly changes of the absolute/percentage bid-ask spreads of ATM caps with: (1) corresponding maturities (first row); (2) all maturities (second row).

Maturity												
1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
Panel A: R^2's of Regressions of ATM Straddle Hedging Errors on Weekly Changes of the Absolute Bid-Ask Spreads of ATM Caps												
0.007	0.003	0.024	0.008	0.015	0.020	0.010	0.018	0.011	0.004	0.002	0.000	0.000
0.150	0.240	0.160	0.150	0.140	0.150	0.230	0.170	0.140	0.230	0.090	0.090	0.070
Panel B: R^2's of Regressions of ATM Straddle Hedging Errors on Weekly Changes of the Percentage Bid-Ask Spreads of ATM Caps												
0.037	0.008	0.004	0.034	0.020	0.036	0.014	0.006	0.005	0.002	0.031	0.001	0.000
0.150	0.130	0.140	0.160	0.100	0.210	0.080	0.050	0.050	0.070	0.170	0.090	0.070

use both the absolute and percentage bid-ask spread of ATM caps as a proxy of liquidity risk and we study how much of the ATM straddle hedging errors can be explained by liquidity risk. We regress the weekly ATM straddle hedging errors on weekly changes of absolute and percentage bid-ask spreads. By including the bid-ask spreads of caps of all maturities in the regression, the regression R^2 can serve as the upper bound of the explanatory power of the liquidity risk factors. In Table X, the R^2's range from 5% to 20% with an average around 10%. Comparing with the regression of the implied volatilities above, we can see that the major source of these hedging errors is the USV factors. Another important indication that the unspanned factors are not due to liquidity risk is the fact that the QTSMs can hedge ITM and OTM straddles, which are much less liquid, much better than ATM straddles.

Thus, the poor hedging performance of the QTSMs is mainly due to the USV in the cap market. If the USV is indeed systematic, including this factor should significantly improve the hedging performance of all caps. As ATM straddles are mainly exposed to volatility risk, their hedging errors can serve as a proxy of the USV. Panels A and B of Table XI report the R^2's of regressions of hedging errors of difference caps and cap straddles across moneyness and maturities on changes of the three yield factors and the first five principal components of straddle hedging errors. In contrast to the regressions in Panel D of Table VI, which only include the three yield factors, the additional factors from straddle hedging errors significantly improve the R^2's of the regressions: For most

Table XI

ATM Straddle Hedging Errors as Proxies for Systematic Unspanned Stochastic Volatility

This table reports the contribution of unspanned stochastic volatility (USV) proxied by the first few principal components of ATM straddle hedging errors under QTSM1 in explaining the hedging errors of difference caps and cap straddles across moneyness and maturities. Moneyness (K/F) is defined as the ratio between the strike and the average LIBOR forward rates underlying the few caplets that form the difference cap. We interpolate difference cap prices with respect to strike to obtain the prices of difference caps at fixed moneyness. For each moneyness, we construct straddle prices by obtaining difference floor prices from difference cap prices using put–call parity. The entries are the R^2's of the regressions of hedging errors of difference caps and straddles across moneyness and maturities on changes of the three yield factors (level, slope, and curvature) and the first five principal components of straddle hedging errors. The bold entries represent moneyness/maturity groups that have less than 10% of missing values and the rest are the those with 10% to 50% of missing values.

Moneyness	Maturity												
(K/F)	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
	Panel A: R^2's of Regressions of Difference Cap Hedging Errors on USV Proxies and the Three Yield Factors (Level, Slope, and Curvature)												
0.60					0.945				0.948	0.880	0.884	0.786	0.880
0.70			0.934	0.944	0.943	**0.911**	**0.934**	0.936	0.940	0.885	0.839	0.791	0.776
0.80		0.917	**0.934**	**0.938**	**0.935**	**0.909**	**0.950**	**0.946**	**0.951**	**0.898**	**0.862**	0.821	0.840
0.90	0.949	**0.900**	**0.908**	**0.922**	**0.924**	**0.896**	**0.961**	**0.969**	**0.969**	**0.920**	**0.871**	**0.856**	**0.871**
1.00	**0.932**	**0.859**	**0.905**	**0.909**	**0.939**	**0.902**	**0.988**	**0.989**	**0.984**	**0.973**	**0.910**	**0.894**	**0.907**
1.10	**0.913**	**0.793**	**0.890**	**0.894**	**0.928**	**0.880**	**0.979**	0.976	0.974	0.967	0.913	0.921	
1.20	0.881	**0.749**	**0.844**	**0.848**	**0.894**	0.846	0.966	0.963	0.954	0.957			
1.30	0.861	**0.702**	**0.802**	0.808	0.836	0.802	0.920		0.908				
1.40		0.640	0.725	0.743	0.761	0.536							
	Panel B: R^2's of Regressions of Difference Cap Straddle Hedging Errors on USV Proxies and the Three Yield Factors (Level, Slope, and Curvature)												
0.60					0.851				0.839	0.589	0.724	0.712	0.744
0.70			0.874	0.897	0.825	**0.775**	**0.854**	0.886	0.800	0.621	0.688	0.749	0.730
0.80		0.812	**0.839**	**0.828**	**0.800**	**0.734**	**0.872**	**0.886**	**0.751**	**0.607**	**0.740**	0.805	0.805
0.90	0.883	**0.718**	**0.801**	**0.791**	**0.763**	**0.690**	**0.919**	**0.930**	**0.721**	**0.622**	**0.777**	**0.859**	**0.840**
1.00	**0.901**	**0.745**	**0.861**	**0.822**	**0.851**	**0.745**	**0.971**	**0.976**	**0.738**	**0.703**	**0.849**	**0.922**	**0.894**
1.10	**0.896**	**0.757**	**0.883**	**0.769**	**0.880**	**0.735**	**0.953**	0.960	0.758	0.788	0.892	0.931	
1.20	0.920	**0.831**	**0.883**	**0.785**	**0.890**	0.808	0.947	0.935	0.809	0.843			
1.30	0.945	**0.890**	**0.913**	0.859	0.921	0.872	0.920		0.830				
1.40		0.924	0.941	0.912	0.946								

moneyness/maturity groups, the R^2's are above 90%. Interestingly, for long-term caps, the R^2's of ATM and OTM caps are actually higher than that of ITM caps. Therefore, a combination of the yield factors and the USV factors can explain cap prices across moneyness and maturities very well.

D. Discussions

While our analysis is mainly based on the QTSMs, we believe that the evidence on USV is so compelling that the results should be robust to potential model misspecification. The fact that the QTSMs provide excellent fit of bond yields but can explain only a small percentage of the variations of ATM straddle returns is a strong indication that the models miss some risk factors that are important for the cap market. Other than introducing USV factors as we propose, it is not clear that any other modification of the QTSMs could improve their cross-sectional hedging performance of caps and straddles across moneyness and maturities.[19]

While we estimate the QTSMs using only bond prices, we could also include cap prices in model estimation. We do not choose the second approach for several reasons. First, the current approach is consistent with the main objective of our study: Use risk factors extracted from the swap market to explain cap prices. Second, it is not clear that modifications of model parameters without changing the fundamental structure of the model could remedy the poor cross-sectional hedging performance of the QTSMs. In fact, if the QTSMs indeed miss some important factors, then no matter how they are estimated (using bonds or derivatives data), they are unlikely to have good hedging performance.[20] Finally, Jagannathan et al. (2003) do not find significant differences between parameters of ATSMs estimated using LIBOR/swap rates and cap/swaption prices.

The existence of USV strongly suggests that existing DTSMs need to relax their fundamental assumption that derivatives are redundant securities by explicitly incorporating USV factors. It also suggests that it might be more convenient to consider derivative pricing in the forward rate models of HJM (1992) or the random field models of Goldstein (2000) and Santa-Clara and Sornette (2001) because it is generally very difficult to introduce USV in DTSMs. For example, Collin-Dufresne and Goldstein (2002a) show that highly restrictive assumptions on model parameters need to be imposed to guarantee that some

[19] While jumps could partly account for the poor hedging performance of short-term caps, they are unlikely to have a large impact on the hedging errors of long-term caps, which are the main challenges facing the QTSMs.

[20] The following example illustrates this point. Suppose the true model of stock prices is the stochastic volatility model of Heston (1993). Then the Black-Scholes model estimated using option data might be able to fit option prices better than the same model estimated using the underlying stock price. But unless we allow the volatility parameter to be re-estimated each day, both approaches are unlikely to have good hedging performance because both ignore the stochastic volatility factor. On the other hand, allowing for a time-varying volatility parameter is another (although less efficient) way to introduce stochastic volatility.

state variables that are important for derivative pricing do not affect bond prices. In contrast, they show that it is much easier to introduce USV in the HJM and random field class of models: Any HJM or random field model in which the forward rate has a stochastic volatility exhibits USV. While it has always been argued that HJM and random field models are more appropriate for pricing derivatives than DTSMs, the reasoning given here is quite different. That is, in addition to the commonly known advantages of these models (such as they can perfectly fit the initial yield curve while DTSMs generally cannot), another advantage of HJM and random field models is that they can easily accommodate USV (see Collin-Dufresne and Goldstein (2002b) for illustration). Of course, the tradeoff here is that in HJM and random field models, the yield curve becomes an input to, not a prediction of, the model. Therefore, the real challenge is as it has always been, to develop term structure models that can successfully price both bonds and derivatives simultaneously.

IV. Conclusion

In this paper, we examine whether bonds and interest rate derivatives are driven by the same set of risk factors, an important assumption underlying most existing DTSMs. One direct implication of this assumption is that interest rate derivatives are redundant securities and can be perfectly hedged using solely bonds. We test this prediction by studying the performance of multifactor QTSMs in hedging interest rate derivatives. We find that these models have serious difficulties in hedging caps and cap straddles, even though they capture bond yields well. Furthermore, ATM straddle hedging errors are highly correlated with cap-implied volatilities and can explain a large fraction of hedging errors of all caps and straddles across moneyness and maturities. Our results strongly suggest the existence of systematic USV factors in the cap market, and existing DTSMs need to explicitly incorporate such factors in pricing and hedging interest rate derivatives.

Appendix A: Closed-Form Pricing Formula for Interest Rate Caps

Leippold and Wu (2002) show that a large class of fixed-income securities can be priced in closed form in the QTSMs using the transform analysis of Duffie et al. (2001). They show that the time-t value of a contract that has an exponential quadratic payoff structure at terminal time T,

$$\exp(-q(X_T)) = \exp(-X_T'\bar{A}X_T - \bar{b}'X_T - \bar{c}) \tag{A1}$$

has the following form:

$$\begin{aligned}\psi(q, X_t, t, T) &= E_Q\left(e^{-\int_t^T r(X_s)\,ds} e^{-q(X_T)} \middle| \mathcal{F}_t\right)\\ &= \exp[-X_t A(T-t)X_t - b(T-t)'X_t - c(T-t)],\end{aligned} \tag{A2}$$

where $A(\cdot)$, $b(\cdot)$, and $c(\cdot)$ satisfy the ODEs (4)–(6) with the initial conditions $A(0) = \bar{A}$, $b(0) = \bar{b}$, and $c(0) = \bar{c}$.

The time-t price of a call option with payoff $(e^{-q(X_T)}-y)^+$ at $T = t+\tau$ equals

$$\begin{aligned} C(q, y, X_t, \tau) &= E_Q\big(e^{-\int_t^T r(X_s)\,ds}\big(e^{-q(X_T)} - y\big)^+\big|\mathcal{F}_t\big) \\ &= E_Q\big(e^{-\int_t^T r(X_s)\,ds}\big(e^{-q(X_T)} - y\big)\mathbf{1}_{\{-q(X_T)\geq \ln(y)\}}\big|\mathcal{F}_t\big) \\ &= G_{q,q}(-\ln(y), X_t, \tau) - yG_{0,q}(-\ln(y), X_t, \tau), \end{aligned} \tag{A3}$$

where $G_{q_1,q_2}(y, X_t, \tau) = E_Q[e^{-\int_t^T r(X_s)ds}e^{-q_1(X_T)}\mathbf{1}_{\{q_2(X_T)\leq y\}}\mid \mathcal{F}_t]$, and can be computed by the inversion formula

$$\begin{aligned} G_{q_1,q_2}(y, X_t, \tau) = {}& \frac{\psi(q_1, X_t, t, T)}{2} \\ & -\frac{1}{\pi}\int_0^\infty \frac{e^{ivy}\psi(q_1 + ivq_2) - e^{-ivy}\psi(q_1 - ivq_2)}{iv}\,dv. \end{aligned} \tag{A4}$$

Similarly, the price of a put option is

$$P(q, y, \tau, X_t) = yG_{0,-q}(\ln(y), X_t, \tau) - G_{q,-q}(\ln(y), X_t, \tau). \tag{A5}$$

We are interested in pricing a cap that is a portfolio of European call options on future interest rates with a fixed strike price. For simplicity, we assume the face value is one and the strike price is $\bar{r}$. At time 0, let $\tau, 2\tau, \ldots, n\tau$ be the fixed dates for future interest payments. At each fixed date $k\tau$, the $\bar{r}$-capped interest payment is given by $\tau(\mathcal{R}((k-1)\tau, k\tau) - \bar{r})^+$, where $\mathcal{R}((k-1)\tau, k\tau)$ is the τ-year floating interest rate at time $(k-1)\tau$, defined by

$$\begin{aligned} \frac{1}{1+\tau\mathcal{R}((k-1)\tau, k\tau)} &= \varrho((k-1)\tau, k\tau) \\ &= E^Q\left(\exp\left(-\int_{(k-1)\tau}^{k\tau} r(X_s)\,ds\right)\middle|\mathcal{F}_{(k-1)\tau}\right). \end{aligned} \tag{A6}$$

The market value at time 0 of the caplet paying at date $k\tau$ can be expressed as

$$\begin{aligned} \text{Caplet}(k) &= E^Q\left[\exp\left(-\int_0^{k\tau} r(X_s)\,ds\right)\tau(\mathcal{R}((k-1)\tau, k\tau) - \bar{r})^+\right] \\ &= (1+\tau\bar{r})E^Q\left[\exp\left(-\int_0^{(k-1)\tau} r(X_s)\,ds\right)\left(\frac{1}{(1+\tau\bar{r})} - \varrho((k-1)\tau, k\tau)\right)^+\right]. \end{aligned} \tag{A7}$$

Hence, the pricing of the k^{th} caplet is equivalent to the pricing of a $(k-1)\,\tau$-for-τ put struck at $K = \frac{1}{(1+\tau\bar{r})}$. Therefore,

$$\begin{aligned} \text{Caplet}(k) = {}& G_{0,-q_\tau}\big(\ln K, X_{(k-1)\tau}, (k-1)\tau\big) \\ & -\frac{1}{K}G_{q_\tau,-q_\tau}\big(\ln K, X_{(k-1)\tau}, (k-1)\tau\big). \end{aligned} \tag{A8}$$

Similarly, for the k^{th} – floorlet,

$$\text{Floorlet}(k) = -G_{0,q_\tau}\big(-\ln K, X_{(k-1)\tau}, (k-1)\tau\big) + \frac{1}{K}G_{q_\tau,q_\tau}\big(-\ln K, X_{(k-1)\tau}, (k-1)\tau\big). \tag{A9}$$

Appendix B: Quadratic Measurement Bias Correction in Kalman Filter

The linearized measurement equation generally introduces a bias term. For a quadratic measurement equation, the bias term could be corrected (see Grewal and Andrews (2001)). Specifically, the yield with maturity τ_j, Y_{jk}, is a quadratic function of the state variables Z_k in the form

$$Y_{jk} = Z_k' A Z_k + b' Z_k + c \equiv q(Z_k), \tag{B1}$$

for some parameters A, b, and c. Using Taylor series expansion at the ex ante forecast of the state variables $Z_{k|k-1}$,

$$Y_{jk} = q(Z_{k|k-1}) + [b' + Z_{k|k-1}'(A + A')](Z_k - Z_{k|k-1}) + (Z_k - Z_{k|k-1})' A (Z_k - Z_{k|k-1}). \tag{B2}$$

The extended Kalman filter omits the quadratic term in the above expression and thus introduces the bias term B_k to the measurement equation, that is,

$$\begin{aligned} B_k &= E_{k-1}[(Z_k - Z_{k|k-1})' A (Z_k - Z_{k|k-1})] \\ &= E_{k-1}[\text{trace}((Z_k - Z_{k|k-1})' A (Z_k - Z_{k|k-1}))] \\ &= [\text{trace}(A(Z_k - Z_{k|k-1})'(Z_k - Z_{k|k-1}))] \\ &= \text{trace}\{A E_{k-1}[(Z_k - Z_{k|k-1})'(Z_k - Z_{k|k-1})]\} \\ &= \text{trace}\{A P_{k|k-1}^{Z}\}. \end{aligned} \tag{B3}$$

However, we should note that this does not eliminate the linearization approximation error of the measurement equation since the Kalman gain is still computed with first derivatives of the measurement function.

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