**MECH0026 – Elasticity and Plasticity** **Autumn Term**

# Handout 2: Introduction to the Finite Element Method

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# Contents

# 1 Introduction

Often, engineering problems require solutions that are mathematically described by differential equations. However, closed-form or exact solutions (e.g. of the types we saw in the *Elasticity* part of this module) cannot always be obtained, due to the complexity of geometries or material properties. Numerical methods are often required to find the solutions to such problems. The following four numerical methods are most widely used to solve engineering problems:

1. The finite difference method
2. The finite element method
3. The control volume (or finite volume) method
4. The boundary element method.

The Finite Element Method (FEM) is a powerful technique originally developed for numerical solution of complex problems in structural mechanics, and it remains the method of choice for complex systems. It originated from the need to solve complex elasticity and structural analysis problems in civil and aeronautical engineering, and it's development can be traced back to the work of Alexander Hrennikoff (1941) and Richard Courant (1941). While the approaches used by these pioneers are different, they share one essential characteristic: mesh discretisation of a continuous domain into a set of discrete sub-domains, usually called elements. In the FEM, the structural system is modelled by a set of appropriate finite elements interconnected at points called nodes. Elements may have physical properties such as thickness, coefficient of thermal expansion, density, Young's modulus, shear modulus and Poisson's ratio *etc*.

The origin of FEMs can be traced to the matrix analysis of structures, and were further developed based on engineering methods in the 1950s (for example, work in the aircraft industry introduced practicing engineers to FEM). The name "*finite element*" was first used in the 1960*s*, during which era the mathematical validity of the method was recognised, and the method was expanded from its structural beginnings to include heat transfer, groundwater flow, magnetic fields and other areas. NASA issued a request for proposals for the development of the finite element software NASTRAN in 1965. In the 1970s large general purpose FEM packages became available, and in the 1980s FEM software became available on microcomputers, complete with colour graphics pre- and post- processors. As of 1995, approximately $40,000$ papers and many books on the FEM and its applications had been published – currently, a PubMed search (which focuses on academic research in the life sciences and biomedical topics) returns $> 10000$ journal papers under the search "*finite element method*". Current commercially available software packages include ANSYS, COMSOL Multiphysics, Flexcom (FEM package used in the offshore oil and gas industry), and many more....

The FEM is a numerical technique for finding approximate solutions to partial differential equations (PDEs) and their systems, as well as (less often) integral equations. In simple terms, FEM is a method for dividing up a very complicated problem into small elements that can be solved in relation to each other – it is particularly useful for complex geometries.

We start with a particularly simple element - the *spring* element. The stiffness of a single spring element, $k$, is already known, whereas for several springs in a system, the global stiffness matrix needs to be *assembled*. This procedure can easily be explained using spring elements. We then move onto beam elements where we again calculate the stiffness matrix - the process of assembling the element stiffness matrices into a global stiffness matrix is the same for any type of element.

# 2 Spring Elements

## 2.1 Single Element

A single spring element is depicted in Figure 1.

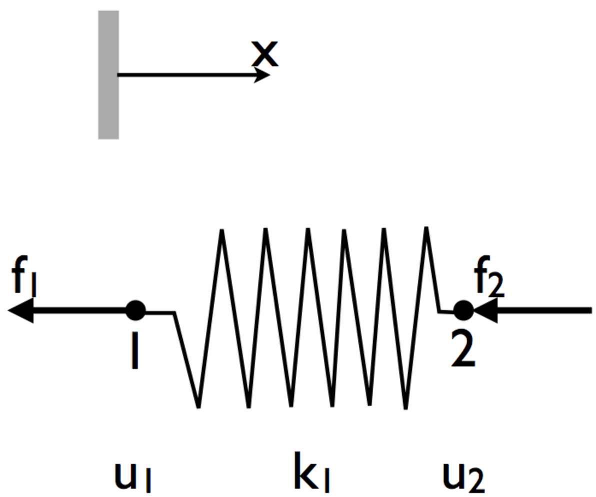


Figure 1: Single spring element with stiffness $k$

Assuming that the spring is linear and elastic, it follows Hooke's law $F = -kx$, where $x$ is the displacement of the spring's end from its equilibrium position, $F$ is the restoring force exerted by the spring on that end, and $k$ is the spring constant. Let node 2 be displaced by a distance $u_2$ and node 1 by a distance $u_1$ with $u_2 > u_1$. Consider the equilibrium of forces for the spring under these displacements,

$$f_1 = -k\left(u_2 - u_1\right), \quad f_2 = -f_1 = k\left(u_2 - u_1\right). \tag{1}$$

Equation (1) can be written in matrix notation as

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{2}$$

which corresponds to *stiffness* $\times$ *displacement vector* =*force vector*. We may analyse equation (2) by setting $u_2 = 0$ to give,

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} u_1 \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} k \\ -k \end{pmatrix} u_1, \tag{3}$$

and similarly by setting $u_1 = 0$ to give

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} 0 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -k \\ k \end{pmatrix} u_2. \tag{4}$$

From this it can be seen that the first column in the stiffness matrix gives the force at each node required to generate a unit displacement at node 1 and zero displacement at all the other nodes. Similarly, the second column in the stiffness matrix gives the nodal forces with a unit displacement at node 2 and zero displacement at all the other nodes.

## 2.2 System of Spring Elements

Next we consider two spring elements in series and connecting three nodes, as depicted in Figure 2.

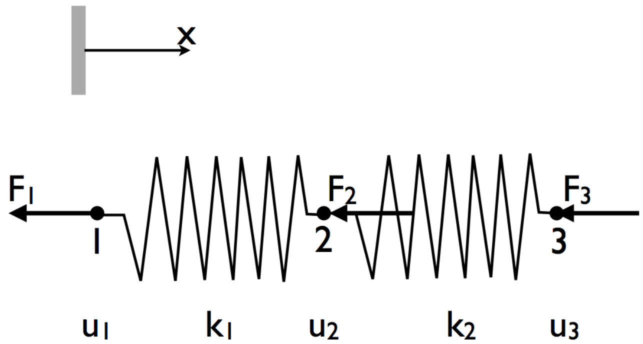


Figure 2: System of single spring elements

We let the element connecting nodes 1 and 2 have spring constant $k_1$, and that connecting nodes 2 and 3 have spring constant $k_2$. For element 1, the element nodes are 1 and 2 as in Figure 1. For this element, the equilibrium equation is given by

$$\begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1^{ele1} \\ f_2^{ele1} \end{pmatrix}. \tag{5}$$

For element 2

$$\begin{pmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_2^{ele2} \\ f_3^{ele2} \end{pmatrix}. \tag{6}$$

Equilibrium of nodal forces can be written as

$$F_1 = f_1^{ele1}, \quad F_2 = f_2^{ele1} + f_2^{ele2}, \quad F_3 = f_3^{ele2}, \tag{7}$$

where $F$ indicates global nodal forces and $f$ element nodal forces. Making use of equation (5) in (7) yields

$$F_1 = f_1^{ele1} = k_1 u_1 - k_1 u_2, \tag{8}$$
$$F_2 = f_2^{ele1} + f_2^{ele2} = -k_1 u_1 + (k_1 + k_2)\, u_2 - k_2 u_3, \tag{9}$$
$$F_3 = f_3^{ele2} = -k_2 u_2 + k_2 u_3. \tag{10}$$

This can be written in matrix form as

$$\begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \tag{11}$$

or $\mathbf{Kx} = \mathbf{F}$, where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{x}$ is the global nodal displacement vector and $\mathbf{F}$ is the global nodal force vector.

## 2.3 Boundary Conditions

In order to solve equation (11) for the nodal displacements, it would be necessary to calculate the inverse of the stiffness matrix and then calculate $\mathbf{x} = \mathbf{K}^{-1}\mathbf{F}$. However, the determinant of the stiffness matrix is given by

$$|\mathbf{K}| = k_1 \left( (k_1 + k_2)\, k_2 - k_2^2 \right) - k_1^2 k_2 = 0, \tag{12}$$

and so the stiffness matrix is singular and the nodal displacements can not be solved for.

In physical terms this means that the system (springs) are not constrained, *i.e.* rigid body motion is possible. To avoid this, constraints should be applied – the most common constraint is to "fix" one or more of the nodes.

For example, let's fix node 1 by setting $u_1 = 0$ and $F_2 = F_3 = P$. The external force applied at node 1 would be the reaction force, an unknown. Equation (11) can be written as

$$\begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix} \begin{pmatrix} 0 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ P \\ P \end{pmatrix}, \tag{13}$$

which reduces to

$$\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} P \\ P \end{pmatrix}, \tag{14}$$

with $F_1 = -k_1 u_2$, the reaction force. This is equivalent to deleting row 1 and column 1 in the stiffness matrix. If it was node 2 that was constrained, it would have been row 2 and column 2 that would have been deleted.

The *reduced* stiffness matrix is no longer singular, and the system can be solved for the unknown nodal displacements at node 2 and node 3 (node 1 is fixed):

$$\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1/k_1 & 1/k_1 \\ 1/k_1 & 1/k_1 + 1/k_2 \end{pmatrix} \begin{pmatrix} P \\ P \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 2P/k_1 \\ 2P/k_1 + P/k_2 \end{pmatrix}, \tag{15}$$

and the reaction force follows $F_1 = -k_1 u_2 = -2P$, which ensures equilibrium.

**Note**: The unconstrained system has 3 degrees of freedom (one displacement per node). The constrained system has only 2 degrees of freedom.

With this knowledge of assembling stiffness matrices and boundary conditions, we move on to explore more complex elements. The process of assembling a global stiffness matrix and applying boundary conditions are the same for any type of element, it is just the formulation (and size) of the stiffness matrix that may differ.

# 3 Beam Elements

Figure 3 shows a simple plane beam element. The elastic modulus $E$ and moment of inertia $I$ are constant along the length of the beam. Only lateral displacements are assumed, $v = v(x)$. We consider a beam only loaded at the end nodes with forces and moments.

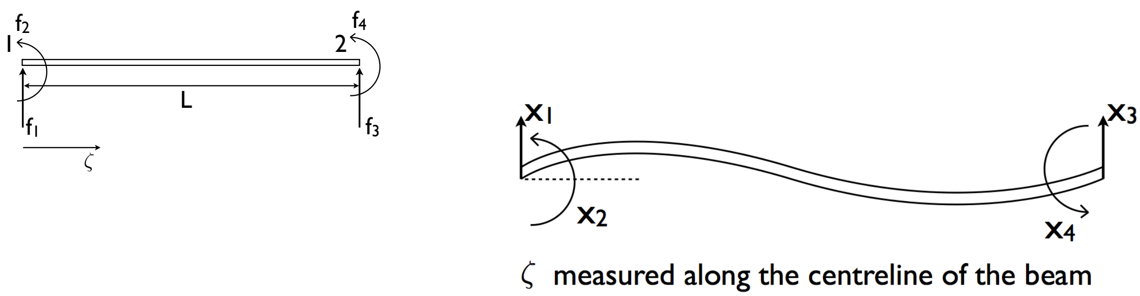


Figure 3: A single beam element

Equilibrium of the system depicted in Figure 3 requires that

$$f_1 + f_3 = 0, \quad f_2 + f_4 = L f_1, \tag{16}$$

where the first condition derives from the sum of the forces in the $y$–direction, and the second from the sum of moments around node 2. Using classical beam theory (specifically the moment-curvature relationship),

$$M = \frac{EI}{\rho}, \tag{17}$$

where $M$ is the moment and

$$\frac{1}{\rho} = \frac{\mathrm{d}^2 v}{\mathrm{d}\zeta^2}, \tag{18}$$

is the curvature of the beam (see Figure 4).

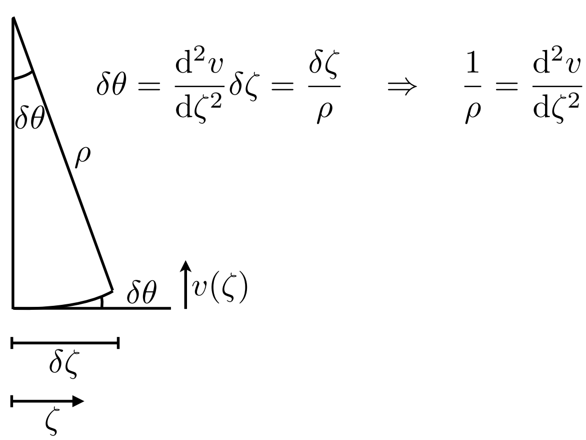


Figure 4: Moment-curvature relationships

Evaluating the moment at a distance $\zeta$ from node 1,

$$M(\zeta) = \zeta f_1 - f_2 \quad \Rightarrow EI\frac{\mathrm{d}^2 v(\zeta)}{\mathrm{d}\zeta^2} = \zeta f_1 - f_2. \tag{19}$$

Integrating with respect to $\zeta$ gives

$$EI\frac{\mathrm{d}v(\zeta)}{\mathrm{d}\zeta} = \frac{1}{2}\zeta^2 f_1 - f_2\zeta + A, \quad \Rightarrow \quad EIv(\zeta) = \frac{1}{6}\zeta^3 f_1 - \frac{1}{2}f_2\zeta^2 + A\zeta + B, \tag{20}$$

and therefore the lateral displacement is a cubic function in $x$.

Next we apply boundary conditions to prescribe the displacement and rotation at each node:

$$v\big|_{\zeta=0} = x_1, \quad \left.\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right|_{\zeta=0} = x_2, \tag{21}$$

$$v\big|_{\zeta=L} = x_3, \quad \left.\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right|_{\zeta=L} = x_4. \tag{22}$$

The conditions on $\zeta = 0$ give $B = EIx_1$ and $A = EIx_2$, whereas the conditions at $\zeta = L$ yield

$$EIx_3 = \frac{L^3}{6}f_1 - \frac{L^2}{2}f_2 + EIx_2L + EIx_1, \tag{23}$$

$$EIx_4 = \frac{L^2}{2}f_1 - f_2L + EIx_2. \tag{24}$$

Multiplying equation (23) by $6/L^3$ and equation (24) by $3/2L^2$ gives

$$\frac{6EI}{L^3}x_3 = f_1 - \frac{3}{L}f_2 + \frac{6EI}{L^2}x_2 + \frac{6EI}{L^3}x_1, \tag{25}$$

$$\frac{3EI}{L^2}x_4 = \frac{3}{2}f_1 - \frac{3}{L}f_2 + \frac{3EI}{L^2}x_2. \tag{26}$$

Computing (26)-(25) eliminates $f_2$ and provides an expression for $f_1$ given by

$$f_1 = \frac{EI}{L^3}\left(12x_1 + 6Lx_2 - 12x_3 + 6Lx_4\right). \tag{27}$$

Computing $(2/3)\times$ (26)+(25) provides an expression for $f_2$ given by

$$f_2 = \frac{EI}{L^3}\left(6Lx_1 + 4L^2x_2 - 6Lx_3 + 2L^2x_4\right). \tag{28}$$

The vertical equilibrium relationship of (16) provides

$$f_3 = -f_1 = \frac{EI}{L^3}\left(-12x_1 - 6Lx_2 + 12x_3 - 6Lx_4\right), \tag{29}$$

whereas the moment equilibrium enables $f_4$ to be determined through

$$f_4 = Lf_1 - f_2 = \frac{EI}{L^3}\left(6Lx_1 + 2L^2x_2 - 6Lx_3 + 4L^2x_4\right). \tag{30}$$

The system of equations for $f_1$, $f_2$, $f_3$ and $f_4$ may be rewritten in matrix form as

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \tag{31}$$

or equivalently $\mathbf{F} = \mathbf{K}\mathbf{x}$ where $\mathbf{K}$ again denotes the stiffness matrix of the element.

**Note**: This system has four degrees of freedom (two per node).

## 3.1 Multiple Beam Elements - Finite Element Formulation

A beam is divided into elements connecting nodes, where each node has two degrees of freedom (as shown below). We denote the degrees of freedom of node $j$ by $x_{2j-1}$ and $x_{2j}$, where $x_{2j-1}$ is the transverse displacement and $x_{2j}$ is the slope or rotation (see Figure 5).

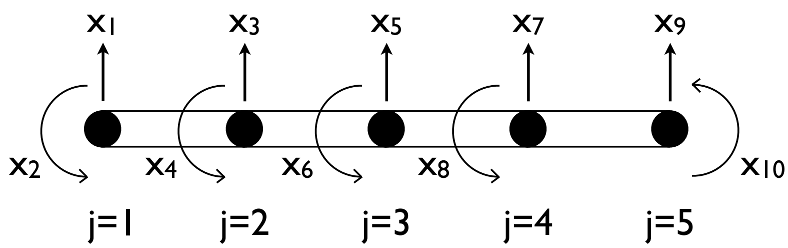


Figure 5: Finite element formulation

We model an individual beam element within Figure 5 as a uniform beam without transverse loading along its length, so that

$$EI\frac{\mathrm{d}^4 v}{\mathrm{d}\zeta^4} = 0, \tag{32}$$

so that

$$v(\zeta) = a\zeta^3 + b\zeta^2 + c\zeta + d. \tag{33}$$

The constants $a$, $b$, $c$ and $d$ may be chosen to satisfy specific boundary conditions – we consider four *base cases* as detailed below. Each base case corresponds to forcing one of the degrees of freedom (either displacement or rotation, at either of the nodes). A solution for an arbitrary combination of boundary conditions can then be obtained by linear superposition of the base states derived below. The four base cases are summarised in Figure 6.

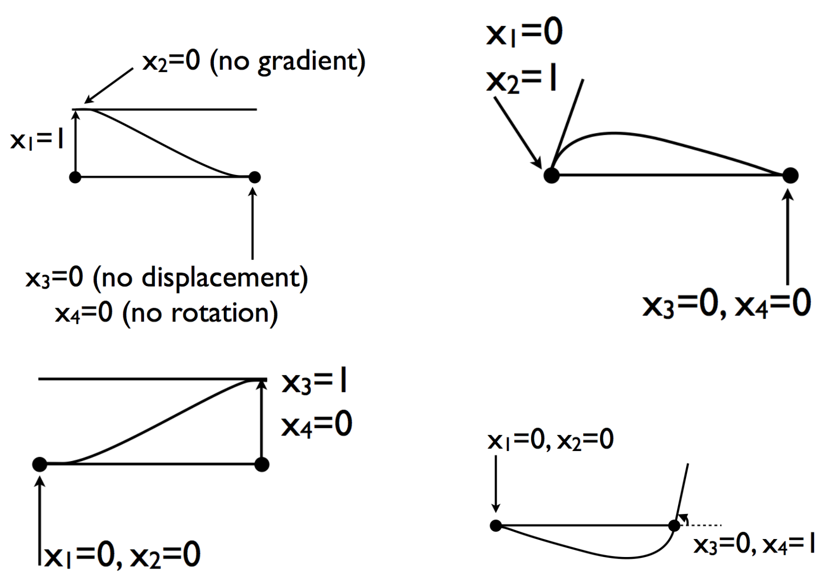


Figure 6: The four shape functions: case 1 (top left), case 2 (top right), case 3 (bottom left), case 4 (bottom right)

**Case 1 $\mathbf{x_1 = 1}$ ($\mathbf{x_2 = x_3 = x_4 = 0}$)**
In this case there is a fixed displacement at node 1, but no rotation (*i.e.* flat gradient) at node 1, and neither displacement nor rotation at node 2:

$$v_1(0) = d = 1, \quad v_1'(0) = c = 0, \tag{34}$$
$$v_1(L) = aL^3 + bL^2 + 1 = 0, \quad v_1'(L) = 3aL^2 + 2bL = 0. \tag{35}$$

Solving this system for $a$ and $b$ yields $a = 2/L^3$ and $b = -3/L^2$ so that

$$v_1(\zeta) = 1 - \frac{3\zeta^2}{L^2} + \frac{2\zeta^3}{L^3}. \tag{36}$$

**Case 2 $\mathbf{x_2 = 1}$ ($\mathbf{x_1 = x_3 = x_4 = 0}$)**
In this case there is a fixed rotation (*i.e.* prescribed gradient) at node 1:

$$v_2(0) = d = 0, \quad v_2'(0) = c = 1, \tag{37}$$
$$v_2(L) = aL^3 + bL^2 + L = 0, \quad v_2'(L) = 3aL^2 + 2bL + 1 = 0. \tag{38}$$

Solving this system for $a$ and $b$ yields $a = 1/L^2$ and $b = -2/L$ so that

$$v_2(\zeta) = \zeta - \frac{2\zeta^2}{L} + \frac{\zeta^3}{L^2}. \tag{39}$$

**Case 3 $\mathbf{x_3 = 1}$ ($\mathbf{x_1 = x_2 = x_4 = 0}$)**
In this case there is a prescribed displacement at node 2:

$$v_3(0) = d = 0, \quad v_3'(0) = c = 0, \tag{40}$$
$$v_3(L) = aL^3 + bL^2 = 1, \quad v_3'(L) = 3aL^2 + 2bL = 0. \tag{41}$$

Solving this system for $a$ and $b$ yields $a = -2/L^3$ and $b = 3/L^2$ so that

$$v_3(\zeta) = \frac{3\zeta^2}{L^2} - \frac{2\zeta^3}{L^3}. \tag{42}$$

**Case 4 $\mathbf{x_4 = 1}$ ($\mathbf{x_1 = x_2 = x_3 = 0}$)**
In this case there is a prescribed rotation at node 2:

$$v_4(0) = d = 0, \quad v_4'(0) = c = 0, \tag{43}$$
$$v_4(L) = aL^3 + bL^2 = 0, \quad v_4'(L) = 3aL^2 + 2bL = 1. \tag{44}$$

Solving this system for $a$ and $b$ yields $a = 1/L^2$ and $b = -1/L$ so that

$$v_4(\zeta) = -\frac{\zeta^2}{L} + \frac{\zeta^3}{L^2}. \tag{45}$$

The functions $v_i(\zeta)$ for $i = 1, 2, 3, 4$ are referred to as *shape functions* or *basis functions*.

Therefore, by linear superposition, we may write the deflection of a uniform beam (with no transverse load) in terms of the end displacements and rotations as:

$$v(\zeta) = \left(1 - \frac{3\zeta^2}{L^2} + \frac{2\zeta^3}{L^3}\right) x_1 + \left(\zeta - \frac{2\zeta^2}{L} + \frac{\zeta^3}{L^2}\right) x_2 + \left(\frac{3\zeta^2}{L^2} - \frac{2\zeta^3}{L^3}\right) x_3 + \left(-\frac{\zeta^2}{L} + \frac{\zeta^3}{L^2}\right) x_4. \tag{46}$$

The transverse deflection can also be written in the following conventional format for finite element analysis:

$$v(\zeta) = \mathbf{N}(\zeta)\,\mathbf{x} = \sum_{i=1}^{4} N_i(\zeta)\, x_i, \tag{47}$$

for $i = 1, \cdots 4$ where $\mathbf{N}(\zeta) = [N_1(\zeta), N_2(\zeta), N_3(\zeta), N_4(\zeta)]$ is a row vector of shape functions and $\mathbf{x} = [x_1, x_2, x_3, x_4]^T$ is a column vector of displacements/rotations.

The components $N_i$ for $i = 1, 2, 3, 4$ are given by

$$N_1(\zeta) = 1 - \frac{3\zeta^2}{L^2} + \frac{2\zeta^3}{L^3}, \quad N_2(\zeta) = \zeta - \frac{2\zeta^2}{L} + \frac{\zeta^3}{L^2}, \quad N_3(\zeta) = \frac{3\zeta^2}{L^2} - \frac{2\zeta^3}{L^3}, \quad N_4(\zeta) = -\frac{\zeta^2}{L} + \frac{\zeta^3}{L^2}. \tag{48}$$

## 3.2 Work Equivalent End Loads

Next we consider the scenario where a beam carries a load $p(\zeta)$ and is in equilibrium with its end loads. The Principle of Virtual Displacements states that:

*"An elastic structure is in equilibrium under a given system of loads and temperature distribution if, for any virtual displacement from a compatible state of deformation, the virtual work is equal to the virtual strain energy"*

Consider a virtual displacement $\delta v(\zeta)$ that is compatible with nodal displacements $\delta x_i$ (see Figure 7).

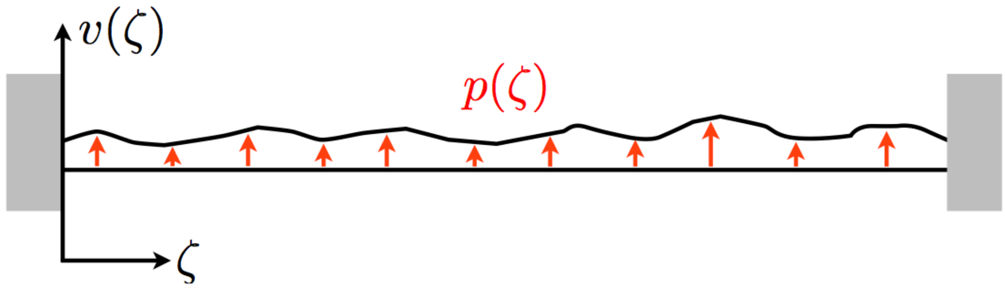


Figure 7: A distributed load $p(\zeta)$ per unit width of the beam, and the corresponding displaced shape $v(\zeta)$

The same virtual work is obtained when the load $p(\zeta)$ is replaced by a set of work equivalent loads $A_{mE_i}$ applied at the ends of the beam (here $m$ denotes the $m^{th}$ element, $E$ stands for *"Equivalent"*, and $i$ denotes the $i^{th}$ degree of freedom in element $m$), such that

$$\int_0^L p(\zeta)\delta v(\zeta)\,\mathrm{d}\zeta = \sum_{i=1}^{4}\int_0^L p\left(\zeta\right)N_i\left(\zeta\right)\mathrm{d}\zeta\delta x_i, \tag{49}$$

$$= \sum_{i=1}^{4} A_{mE_i}\delta x_i. \tag{50}$$

We note that expression (47) has been used to write the virtual displacements in terms of the displacements $\delta x_i$ in (49). Equation (50) applies for arbitrary $\delta x_i$ and so the work equivalent end loads are given by

$$A_{mE_i} = \int_0^L p(\zeta)N_i(\zeta)\,\mathrm{d}\zeta. \tag{51}$$

It may be shown that these work equivalent end loads are equal and opposite to the reactions obtained with the load $p(\zeta)$ when the ends of the beam are fully fixed. If $\mathbf{A}_{mE}$ is a column vector of the work equivalent loads, then

$$\mathbf{A}_{mE} = \int_0^L p\left(\zeta\right)\mathbf{N}^T\left(\zeta\right)\,\mathrm{d}\zeta, \tag{52}$$

where the $^T$ denotes transpose and transforms the row vector of shape functions to a column vector.

## 3.3 Assembly of Beam Elements

For the configuration of a beam element system, see Figure 8.

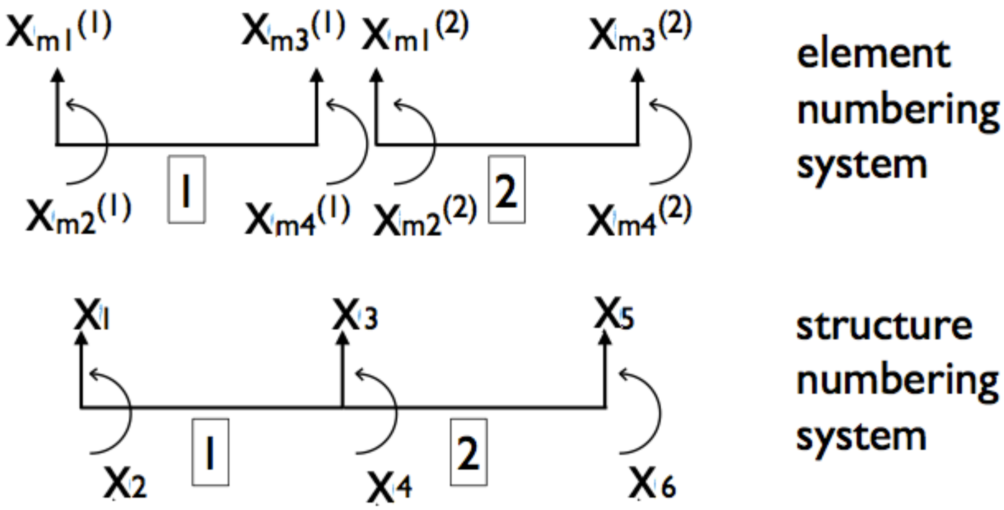


Figure 8: Assembly of beam elements

The strain energy of a member or element is given by $U_m = \frac{1}{2}\mathbf{x}_m^T\mathbf{K}_m\mathbf{x}_m$, whereas the strain energy of the complete structure is given by

$$U = \sum_m U_m = \sum_m \frac{1}{2}\mathbf{x}_m^T\mathbf{K}_m\mathbf{x}_m, \tag{53}$$

where $m$ is the number of elements.

We may write the strain energy of the complete structure as $U = \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x}$ where $\mathbf{x}$ contains terms of the vectors $\mathbf{x}_m$.

The total potential energy of the applied loads is dependent on the joint loads $\mathbf{A}_J$ and on the work equivalent end loads $\mathbf{A}_{mE}$ due to loads on individual members, thus

$$V = -\sum_m \mathbf{x}_m^T\mathbf{A}_{mE} - \mathbf{x}^T\mathbf{A}_J = -\mathbf{x}^T\left(\mathbf{A}_E + \mathbf{A}_J\right), \tag{54}$$

where $\mathbf{A}_E$ comprises the work equivalent loads from all members of the structure.

Setting $\delta\left(U + V\right) = 0$ we obtain

$$\delta\left[\frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} - \mathbf{x}^T\left(\mathbf{A}_E + \mathbf{A}_J\right)\right] = 0. \tag{55}$$

The total potential energy is stationary for arbitrary variations $\delta\mathbf{x}$ about the equilibrium condition, so we have

$$\mathbf{K}\mathbf{x} = \mathbf{A}_E + \mathbf{A}_J \quad \text{or} \quad \mathbf{K}\mathbf{x} = \mathbf{A}_C. \tag{56}$$

Now, the unknown degrees of freedom in $\mathbf{x}$ can be obtained. The solution procedure is to apply the boundary conditions (i.e. known loads and known degrees of freedom), and solve the set of linear equations. The unknown degrees of freedom should expressed in terms of material properties, geometrical parameters, and applied loads.

## 3.4 Examples

**Example 1** Consider the example of Figure 9 where a point load is applied at the mid-point of a beam of length $L$.

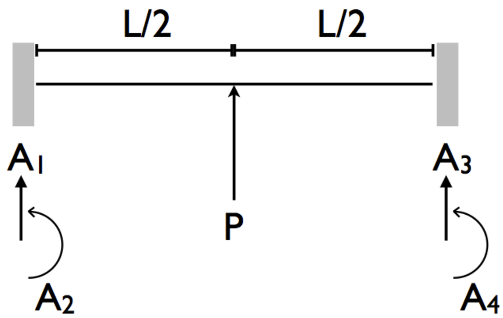


Figure 9: Example 1

The work equivalent end loads are given by (52) which reduces to

$$\mathbf{A}_{mE} = \int_0^L p\left(\zeta\right) \mathbf{N}^T\left(\zeta\right) \mathrm{d}\zeta = P\mathbf{N}^T\left(L/2\right), \tag{57}$$

as the function takes the value $p\left(L/2\right) = P$ with $p\left(\zeta\right) = 0$ for all other values of $\zeta$ in the range $[0, L]$. We evaluate the shape functions by substituting $\zeta = L/2$ into equation (48) to give

$$\mathbf{A}_{mE} = P \begin{bmatrix} 1 - \frac{3}{4} + \frac{1}{4} \\ \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{8}\right) L \\ \frac{3}{4} - \frac{1}{4} \\ \left(-\frac{1}{4} + \frac{1}{8}\right) L \end{bmatrix} = \begin{bmatrix} P/2 \\ PL/8 \\ P/2 \\ -PL/8 \end{bmatrix}. \tag{58}$$

**Example 2** Consider instead a uniform load $w$ distributed along the length of the beam, as depicted in Figure 10.

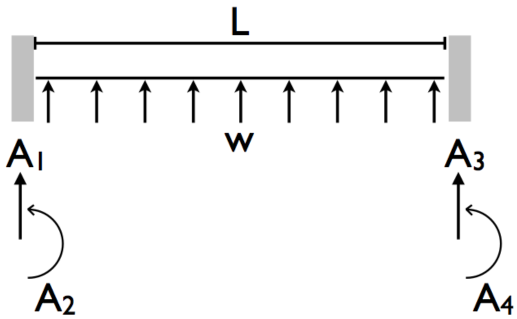


Figure 10: Example 2

Now (52) reduces to

$$\mathbf{A}_{mE} = w \int_0^L \mathbf{N}^T(\zeta)\,\mathrm{d}\zeta, \tag{59}$$

$$= w \int_0^L \begin{pmatrix} 1 & 0 & -3/L^2 & 2/L^3 \\ 0 & 1 & -2/L & 1/L^2 \\ 0 & 0 & 3/L^2 & 2/L^3 \\ 0 & 0 & -1/L & 1/L^2 \end{pmatrix} \begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \end{pmatrix} \mathrm{d}\zeta \tag{60}$$

$$= w \begin{pmatrix} 1 & 0 & -3/L^2 & 2/L^3 \\ 0 & 1 & -2/L & 1/L^2 \\ 0 & 0 & 3/L^2 & 2/L^3 \\ 0 & 0 & -1/L & 1/L^2 \end{pmatrix} \begin{pmatrix} [\zeta]_0^L \\ [\zeta^2/2]_0^L \\ [\zeta^3/3]_0^L \\ [\zeta^4/4]_0^L \end{pmatrix} \tag{61}$$

and so

$$\mathbf{A}_{mE} = \begin{pmatrix} wL/2 \\ wL^2/12 \\ wL/2 \\ -wL^2/12 \end{pmatrix}. \tag{62}$$

**Example 3** When a beam element is subjected to loading along its length, the equivalent joint loads can be evaluated using the principal of virtual work (as described above). Alternatively, it is useful to consider the stiffness method of analysis as the superposition of the following:

1. the solution of a fully restrained structure for the applied loading;
2. the solution of the unrestrained structure for the case when joint loads are applied which are equal and opposite to the restraints applied in 1.

For example, the solution for the setup in Figure x A may be obtained by linear superposition of the solutions for the setups shown in Figure 11 B, C.

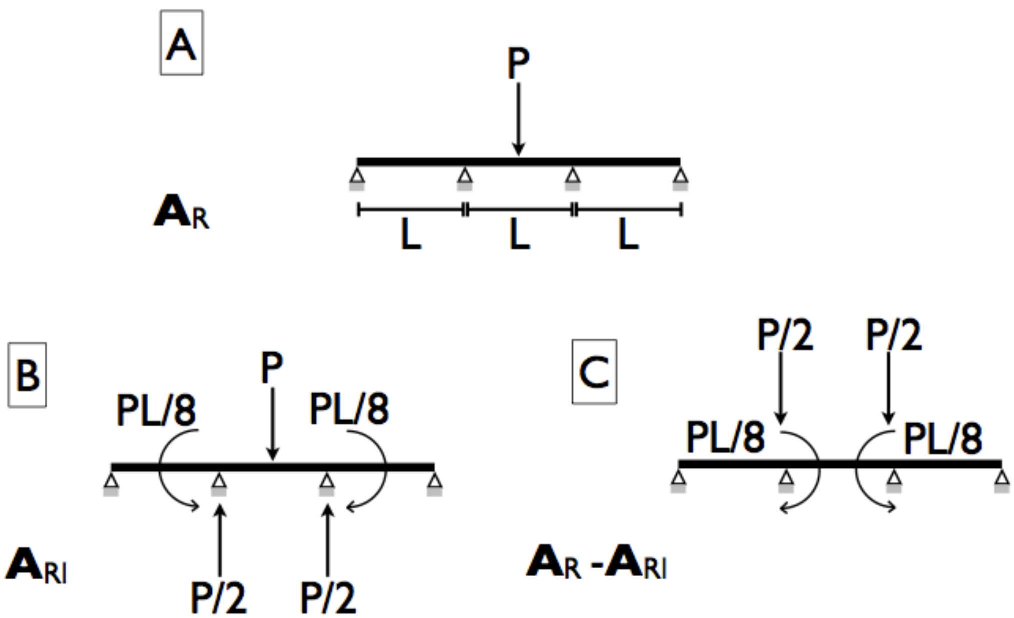


Figure 11: Example 3: A=B+C

**Example 4** This example is taken from Gere J.M. and Weaver W., "*Matrix Analysis of Framed Structures*", 3rd ed.; New York: Van Nostrand Reinhold; 1990.

A continuous beam is shown in Figure 12.

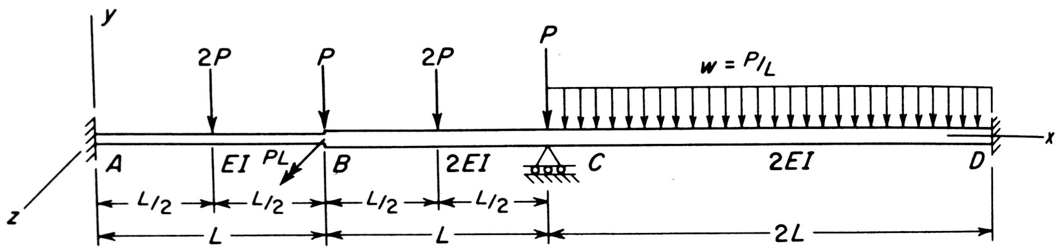


Figure 12: Example 4: A beam of variable rigidity

It is restrained against translation at support C, and against both translations and rotations at points A and D. At point B, the flexural rigidity of the beam changes from $EI$ to $2EI$; this point is therefore taken as a node. The number of unknown degrees of freedom of the system is twice the number of nodes minus the number of support restraints $= 2 \times 4 - 5 = 3$.

The node data is provided in Table 1.

| Node Number | Indices for Possible Displacements | Restraint |
|---|---|---|
| 1 | 1 | ✓ |
| | 2 | ✓ |
| 2 | 3 | ✗ |
| | 4 | ✗ |
| 3 | 5 | ✓ |
| | 6 | ✗ |
| 4 | 7 | ✓ |
| | 8 | ✓ |

Table 1: Node Data

The element data is summarised in Table 2.

| Member Number | Joint Numbers at Member Ends | | Indices for possible joint displacements | | | | $I_z$ | Length |
|---|---|---|---|---|---|---|---|---|
| i | j | k | j1 | j2 | k1 | k2 | | |
| 1 | 1 | 2 | 1 | 2 | 3 | 4 | I | L |
| 2 | 2 | 3 | 3 | 4 | 5 | 6 | 2I | L |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 2I | 2L |

Table 2: Element Data

The element stiffness matrices are given below. To help identify the elements of each matrix, it is helpful to label the rows and columns to correspond to the labelling of the possible displacements at the ends of the member.

$$\mathbf{K}_m^1 = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6l & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}, \tag{63}$$

$$\mathbf{K}_m^2 = \frac{EI}{L^3}\begin{bmatrix} 24 & 12L & -24 & 12L \\ 12L & 8L^2 & -12L & 4L^2 \\ -24 & -12L & 24 & -12L \\ 12L & 4L^2 & -12L & 8L^2 \end{bmatrix}, \tag{64}$$

$$\mathbf{K}_m^3 = \frac{EI}{L^3}\begin{bmatrix} 3 & 3L & -3 & 3L \\ 3L & 4L^2 & -3L & 2L^2 \\ -3 & -3L & 3 & -3L \\ 3L & 2L^2 & -3L & 4L^2 \end{bmatrix}. \tag{65}$$

The joint stiffness matrix is:

$$\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 & 0 \\ -12 & -6L & (12+24) & (-6L+12L) & -24 & 12L & 0 & 0 \\ 6L & 2L^2 & (-6L+12L) & (4L^2+8L^2) & -12L & 4L^2 & 0 & 0 \\ 0 & 0 & -24 & -12L & (24+3) & (-12L+3L) & -3 & 3L \\ 0 & 0 & 12L & 4L^2 & (-12L+3L) & (8L^2+4L^2) & -3L & 2L^2 \\ 0 & 0 & 0 & 0 & -3 & -3L & 3 & -3L \\ 0 & 0 & 0 & 0 & 3L & 2L^2 & -3L & 4L^2 \end{bmatrix} \tag{66}$$

$$= \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 & 0 \\ -12 & -6L & 36 & 6L & -24 & 12L & 0 & 0 \\ 6L & 2L^2 & 6L & 12L^2 & -12L & 4L^2 & 0 & 0 \\ 0 & 0 & -24 & -12L & 27 & -9L & -3 & 3L \\ 0 & 0 & 12L & 4L^2 & -9L & 12L^2 & -3L & 2L^2 \\ 0 & 0 & 0 & 0 & -3 & -3L & 3 & -3L \\ 0 & 0 & 0 & 0 & 3L & 2L^2 & -3L & 4L^2 \end{bmatrix} \tag{67}$$

The load data is as follows:

| Node | $F_y$ | $M_z$ |
|---|---|---|
| 1 | 0 | 0 |
| 2 | $-P$ | $PL$ |
| 3 | $-P$ | 0 |
| 4 | 0 | 0 |

,

| Element | $A^i_{mL1}$ | $A^i_{mL2}$ | $A^i_{mL3}$ | $A^i_{ml4}$ |
|---|---|---|---|---|
| 1 | $-P$ | $-PL/4$ | $-P$ | $PL/4$ |
| 2 | $-P$ | $-PL/4$ | $-P$ | $PL/4$ |
| 3 | $-P$ | $-PL/3$ | $-P$ | $PL/3$ |

,

| Node | $RF_y$ | $RM_z$ |
|---|---|---|
| 1 | $RF_1$ | $RM_1$ |
| 2 | 0 | 0 |
| 3 | $RF_3$ | 0 |
| 4 | $RF_4$ | $RM_4$ |

, (68)

where the first table corresponds to the actions at nodes, the second to the fixed-end actions due to distributed loads, and the third to reaction forces and moments from the supports.

The vector of actions at nodes is:

$$\mathbf{A}_J = \begin{pmatrix} 0 & 0 & -P & PL & -P & 0 & 0 & 0 \end{pmatrix}^T \tag{69}$$

The vector of equivalent node loads is:

$$\mathbf{A}_E = \begin{pmatrix} -P & -PL/4 & (-P-P) & (PL/4-PL/4) & (-P-P) & (PL/4-PL/3) & -P & PL/3 \end{pmatrix}^T \tag{70}$$

The vector reaction loads is:

$$\mathbf{A}_R = \begin{pmatrix} RF_1 & RM_1 & 0 & 0 & RF_2 & 0 & RF_4 & RM_4 \end{pmatrix}^T \tag{71}$$

The combined node load vector is:

$$\mathbf{A}_C = \begin{pmatrix} -P + RF_1 & -PL/4 + RM_2 & -3P & PL & -3P + RF_3 & -PL/12 & -P + RF_4 & PL/3 + RM_4 \end{pmatrix}^T \tag{72}$$

Solving the system $\mathbf{Kx} = \mathbf{A}_C$, we find the unknown degrees of freedom:

$$\mathbf{x} = \frac{PL^2}{3024EI} \begin{pmatrix} 0 & 0 & -398L & 366 & 0 & 255 & 0 & 0 \end{pmatrix}^T \tag{73}$$

Substituting them back, we can solve for the reaction forces and moments:

$$\mathbf{A}_R = \frac{P}{1008} \begin{pmatrix} 332 & 1292L & 0 & 0 & 3979 & 0 & 753 & -166L \end{pmatrix}^T \tag{74}$$

Thus, we have determined all joint displacements and support reactions.