MECH0026 – Elasticity and Plasticity **Autumn Term**

Handout 1: Theory of Elasticity

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Contents

1 Introduction

Illustrative Examples:

- Inflation of a balloon - why is it hard to start blowing up a balloon?
- Weight on a wire - if we twist the weight, will the wire get longer/ shorter/ stay the same?

The materials in these examples (such as rubber and steel) are each elastic and can be described/ modelled using elasticity theory, even though their chemical compositions are very different.

Elasticity theory is the central model of solid mechanics, and it is divided into:

1. nonlinear theory (large deformations);
2. linear theory (small deformation).

Brief History of Elasticity:

1678	Hooke's Law (relates the extension of a spring to the tensile force it is placed under)
1705	Jacob Bernoulli (elastic rods)
1742	Daniel Bernoulli (elastic rods)
1744	Leonard Euler (elastic rods)
1821	Navier derived the special case of linear elasticity via a molecular model
1822	Cauchy considered stress in a material, and derived equations for linear and nonlinear elasticity
1927	A.E.H. Love wrote his Treatise on Linear Elasticity
1950s	R. Rublin derived exact solutions for an incompressible, nonlinear elastic solid (e.g. rubber)
1960s-1980s	Nonlinear theory developed by J.L. Erickson & C. Truesdell
1980s - current	Applications to material science, biology....

1.1 Physical Motivation

Robert Hooke (1678) wrote...

> ... it is ... evident that the rule or law of nature in every springing body is that the force or power thereof to restore itself to its natural position is always proportionate to the distance or space it is removed there from, whether it be by rarefaction, or separation of its parts the one from the other, or by condensation, or crowding of those parts nearer together.

Consider a simple physics experiment: a tensile force T is applied to a spring with natural length L. If we denote the spring constant by k and the new spring length by l, then the extension of the spring is given by $(l - L)$ and

$$T = k(l - L). \tag{1}$$

Equation (1) summarises the basic assumption of linear elasticity: stress is linear in strain.

Indeed, the solution of any two-dimensional problem in elasticity requires the following fundamentals:

1. equilibrium of forces;
2. compatibility of displacements;
3. laws of material behaviour (e.g. Hooke's Law).

These ideas form the basis for the material in this Section of the course.

1.2 Notation for Forces and Stresses

Consider a small cubic element of dimension h; forces in the x-, y- and z- coordinates are denoted X, Y, Z respectively. For each pair of parallel sides of the cubic element, one symbol is needed to denote the normal component of stress, and two more to denote the shearing components of stress. We denote:

Normal components: σ_x, σ_y, σ_z
Shear components: τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, τ_{zy}

Consider moments of force acting on the element about a line through the midpoint C of such a cubic element, and parallel to the z-axis (so we need to consider the surface stresses only). Consider shrinking the element to a point, then:

1. body forces (such as weight of the element) reduce $\sim h^3$;
2. surface forces reduce $\sim h^2$.

Hence, for a small element, surface forces dominate over body forces. Denoting the small dimensions of the element by δx, δy, δz we derive the equation of equilibrium (taking moments about C),

$$\tau_{yx}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \tau_{xy}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{2}$$

Hence

$$\boxed{\tau_{yx} = \tau_{xy},} \tag{3}$$

and similarly

$$\boxed{\tau_{zx} = \tau_{xz}, \quad \tau_{zy} = \tau_{yz}.} \tag{4}$$

1.3 Components of Strain

We assume that the body does not move as a rigid body, so that no displacements of particles of the body are possible without a deformation of it. We also consider small deformations only, as is relevant to many engineering applications.

We denote the x-, y- and $z-$ components of the displacement by $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$, where each of u, v and w vary continuously over the body. A 2D example is shown in Figure 1 where the point $A = (x_1, y_1)$ is displaced to $A' = (x_1', y_1')$, and similarly $B = (x_2, y_2)$ to $B' = (x_2', y_2')$. The relationship between the spatial coordinates and the deformation components are:

$$x_1' = x_1 + u(x_1, y_1), \quad y_1' = y_1 + v(x_1, y_1), \tag{5}$$
$$x_2' = x_2 + u(x_2, y_2), \quad y_2' = y_2 + v(x_2, y_2). \tag{6}$$

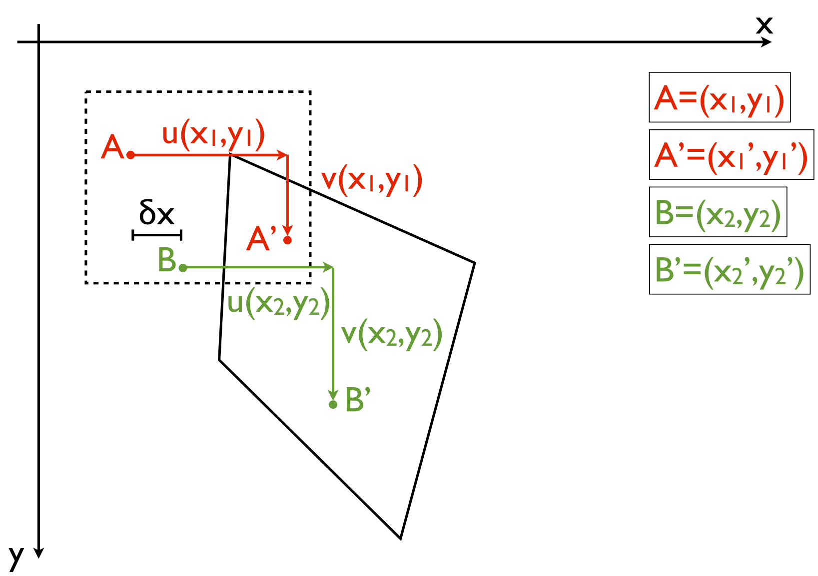

Figure 1: The definition of the deformation components $u(x, y)$ and $v(x, y)$

We note that, if B only represents a small perturbation to A in the x-direction, then we can write

$$x_2 = x_1 + \delta x, \tag{7}$$

where δx is that small perturbation. Therefore, the x- deformation component associated with the point B is

$$u(x_2, y_2) = u(x_1 + \delta x, y_2). \tag{8}$$

Given that δx is small, we may Taylor expand about this small parameter to give

$$u(x_2, y_2) = u(x_1, y_2) + \delta x \frac{\partial u}{\partial x}(x_1, y_2) + \mathcal{O}\left(\delta x^2\right), \tag{9}$$

Therefore, $\delta x \frac{\partial u}{\partial x}$ represents the increase in the function u with a small increase in the x-coordinate, and so $\frac{\partial u}{\partial x}$ is the unit elongation in the x- direction. Extrapolating to small perturbations in the y- and z- directions, we can see that, in general

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},} \tag{10}$$

where ε denotes the unit elongation.

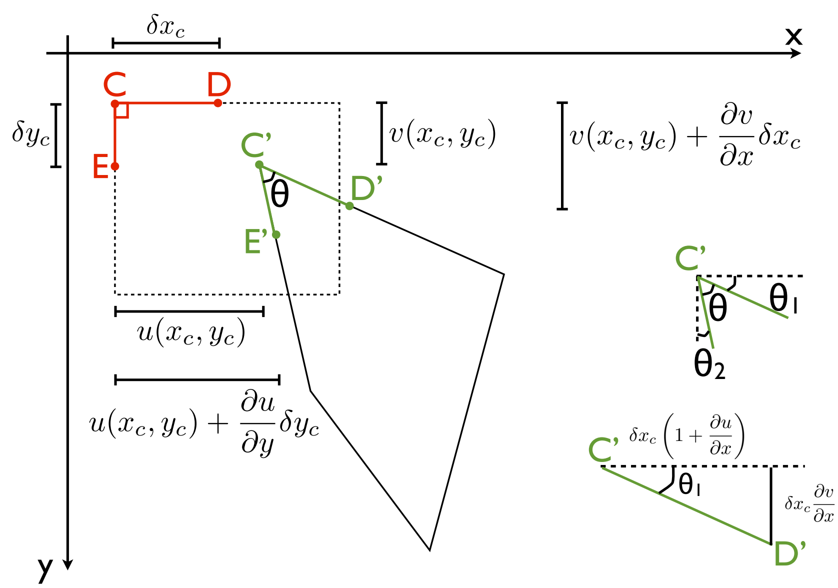

Figure 2: The definition of shearing strains

Next we consider the shearing strains. Under the distortion ECD → E'C'D' shown in Figure 2, a right angle is distorted to the angle θ. We denote the coordinates of the points C, D and E by:

$$\mathrm{C} = (x_c, y_c), \quad \mathrm{D} = (x_c + \delta x_c, y_c), \quad \mathrm{E} = (x_c, y_c + \delta y_c), \tag{11}$$

where δx_c and δy_c are both small. Then the displacement of the point D in the y-direction is

$$v(x_c + \delta x_c, y_c) = v(x_c, y_c) + \delta x_c \frac{\partial v}{\partial x} + \mathcal{O}(\delta x_c^2), \tag{12}$$

whereas the displacement of the point E in the x-direction is

$$u(x_c, y_c + \delta y_c) = u(x_c, y_c) + \delta y_c \frac{\partial u}{\partial y} + \mathcal{O}(\delta y_c^2). \tag{13}$$

Owing to these displacements the new direction C'D' of the element CD is inclined to the initial direction by the small angle θ_1 indicated on Figure 2. Using trigonometry, we can see that

$$\tan\theta_1 = \frac{\partial v/\partial x}{1 + \partial u/\partial x}. \tag{14}$$

Assuming that θ_1 is small so that $\tan\theta_1 \approx \theta_1$ and that displacement gradients are small (so that we can neglect product terms) we see that

$$\theta_1 \approx \frac{\partial v}{\partial x}\left(1+\frac{\partial u}{\partial x}\right)^{-1} \approx \frac{\partial v}{\partial x}. \tag{15}$$

Similarly we can show that the direction C'E' is inclined to CE by the small angle $\theta_2 \approx \frac{\partial u}{\partial y}$. Therefore, the initial right angle ECD is diminished by the angle $\partial v/\partial x + \partial u/\partial y$, which is the shearing strain between the x-z and y-z planes. Denoting the shearing strain by γ then

$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.} \tag{16}$$

The six quantities ε_x, ε_y, ε_z, γ_{xy}, γ_{xz}, γ_{yz} are the components of strain.

1.4 Hooke's Law

Hooke's Law proposes a linear relationship between the components of stress and the components of strain:

$$\boxed{\begin{aligned}
\varepsilon_x &= \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right], && (17)\\
\varepsilon_y &= \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right], && (18)\\
\varepsilon_z &= \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right], && (19)
\end{aligned}}$$

where the constants of proportionality are E (the modulus of elasticity in tension) and ν (Poisson's ratio).

Note that, comparing against the standard tensile test where the sides parallel to the x-axis are submitted to a normal stress uniformally distributed over the two opposite sides, then the extension in the x-direction is accompanied by lateral strain components (contractions), and this is consistent with equations (17)–(19).

For standard materials used in engineering (such as steel), we expect unit elongations to be small (i.e. the moduli E are very large compared to the allowable stresses). The Poisson ratio $\nu \approx 0.25$ for most materials (or 0.30 for structured steel). Note that (17)–(19) apply for both extension and compression, and have been tested against a large number of materials.

The corresponding relationships between the shearing strains and stresses are:

$$\boxed{\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{xz} = \frac{\tau_{xz}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G}, \qquad (20)}$$

where

$$G = \frac{E}{2\left(1+\nu\right)}, \tag{21}$$

is the modulus of rigidity (or modulus of elasticity in shear).

1.5 Equations of Equilibrium (for 2D problems)

Consider the equilibrium of a small rectangular block of dimension h x k with stresses acting on the different faces. Note that there are variations in the stresses through the material, so $(\sigma_x)_1 \neq (\sigma_x)_3$, etc. Note that the faces are small, so the forces are obtained by multiplying the stress values by the areas of the faces on which they act.

As the stress components can now vary through the block, body forces must be taken into account; we denote the body force per unit area by

$$\mathbf{F} = (X, Y)\,. \tag{22}$$

Resolving forces in the x-direction yields

$$k\,(\sigma_x)_1 - k\,(\sigma_x)_3 + h\,(\tau_{xy})_2 - h\,(\tau_{xy})_4 + Xhk = 0. \tag{23}$$

Hence

$$\frac{(\sigma_x)_1 - (\sigma_x)_3}{h} + \frac{(\tau_{xy})_2 - (\tau_{xy})_4}{k} + X = 0, \tag{24}$$

so that, taking the limit as $h \to 0$, $k \to 0$,

$$\boxed{\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X &= 0, && (25)\\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y &= 0. && (26) \end{aligned}}$$

Equations (25)–(26) are referred to as the *Differential Equations of Motion of Equilibrium for 2D Problems*. Note that if the body force was due to gravity acting in the y-direction, then $(X, Y) = (0, -\rho g)$.

Note that we have two equations ((25) and (26)) in three unknowns (σ_x, σ_y and τ_{xy}), and therefore need one more equation in order to close the problem. Considering the relationship between the strain components and displacements,

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{27}$$

we see by differentiating that

$$\boxed{\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.} \tag{28}$$

This compatibility condition must always be satisfied.

Note that for plane stress, Hooke's law applies and so

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right), \quad \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right), \quad \gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2\,(1+\nu)}{E}\tau_{xy}. \tag{29}$$

Substituting the relationships (29) into (28) yields

$$\frac{\partial^2}{\partial y^2}\left(\sigma_x - \nu\sigma_y\right) + \frac{\partial^2}{\partial x^2}\left(\sigma_y - \nu\sigma_x\right) = 2\left(1+\nu\right)\frac{\partial^2 \tau_{xy}}{\partial x \partial y}. \tag{30}$$

However, (25)–(26) provides another relationship between τ_{xy} and σ_x, σ_y; cross-differentiating to eliminate the term in τ_{xy} yields

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y\right) = -\left(1+\nu\right)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right).} \tag{31}$$

For the example of weight as the body force, this reduces to

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y\right) = 0.} \tag{32}$$

Equations (25), (26) and (32) provide three equations in the three unknowns σ_x, σ_y and τ_{xy}, and must be solved subject to appropriate boundary conditions.

2 Stress Functions

The solution of two-dimensional problems is composed of the integration of the differential equations of equilibrium (25)–(26), together with the compatibility equation and appropriate boundary conditions on the physical domain.

Consider a structure which has only its own weight as a body force. The equations to satisfy are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \tag{33}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} - \rho g = 0, \tag{34}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0. \tag{35}$$

We can see that equations (33)–(35) are satisfied by defining a stress function $\phi = \phi(x, y)$ such that

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} + \rho g y, \tag{36}$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} + \rho g y, \tag{37}$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \tag{38}$$

This stress function is analogous to the concept of a velocity potential in fluid mechanics.

Substituting, we see that (33)–(35) are automatically satisfied; the compatibility condition, however, provides an equation to solve for ϕ:

$$\boxed{\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0,} \tag{39}$$

or, in short hand,

$$\boxed{\nabla^4 \phi = 0.} \tag{40}$$

Equations (39), (40) are referred to as the *biharmonic equation*.

We note that for plane stress conditions $\sigma_z = 0$ (i.e. the normal stress acts in 2 directions only), and so Hooke's law reduces to

$$\boxed{\varepsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y), \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu \sigma_x), \quad \gamma_{xy} = \frac{\tau_{xy}}{G}.} \tag{41}$$

In summary the strategy is:

1. Solve for ϕ using (40) together with appropriate boundary conditions;
2. Calculate the stresses σ_x, σ_y and τ_{xy} from (36)–(38);
3. Calculate the elongations and shearing strain ε_x, ε_y, γ_{xy} from Hooke's law (41);
4. Calculate the displacements by integrating the relationships between the strain components and the displacements, which in 2D simplify to

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.} \tag{42}$$

Finally, in preparation for solving the equation sets in different geometries, we note their form for plane polar co-ordinate systems (refer to "Handout 2 - Rectangular and Plane Polar Coordinates" for further details) as would be relevant to cylindrical geometries. In this scenario, $\phi = \phi(r, \theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Now,

$$\boxed{\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}, \quad \sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right),} \tag{43}$$

and the new form of the biharmonic equation is

$$\boxed{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0.} \tag{44}$$

Finally, Hooke's Law becomes:

$$\boxed{\varepsilon_r = \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right), \quad \varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right), \quad \gamma_{r\theta} = \frac{1}{G}\tau_{r\theta},} \tag{45}$$

and finally the displacements $u(r, \theta)$ and $v(r, \theta)$ may be calculated by integrating

$$\boxed{\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}.} \tag{46}$$

2.1 Polynomial Solutions in Rectangular Coordinates

The stress function ϕ solves the biharmonic equation

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0, \tag{47}$$

subject to appropriate boundary conditions. In a rectangular coordinate system, we can see that polynomials of powers up to three satisfy (47), for arbitrary values of the coefficients.

2.1.1 Second-Order Polynomials

We see that a second-order polynomial of the form

$$\phi = \frac{1}{2}Ax^2 + Bxy + \frac{1}{2}Cy^2, \tag{48}$$

will always satisfy the biharmonic equation (47), independent of the values of the constant coefficients A, B, C. By differentiating we see that the stresses (neglecting body forces) are given by

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = C, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = A, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -B, \tag{49}$$

and hence the stress function (48) represents a state of uniform tension or compression in each of the two perpendicular directions (characterised by C and A), as well as uniform shear (characterised by $-B$). Hence, if we wanted to describe a rectangular plate in pure shear, substituting $A = C = 0$ into the stress function (48) would be appropriate, so that $\phi = Bxy$.

2.1.2 Third-Order Polynomials

We consider a third-order polynomial of the form

$$\phi_3 = \frac{A}{3\cdot 2}x^3 + \frac{B}{2}x^2y + \frac{C}{2}xy^2 + \frac{D}{3\cdot 2}y^3, \tag{50}$$

which again satisfies the biharmonic equation (47) for arbitrary values of the constant coefficients. Again neglecting body forces, the stresses are given by

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = Cx + Dy, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = Ax + By, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -Bx - Cy. \tag{51}$$

Hence, a third order polynomial stress function characterises a stress state varying linearly with the coordinates.

For example, if $A = B = C = 0$, then

$$\sigma_x = Dy, \quad \sigma_y = 0, \quad \tau_{xy} = 0, \quad \phi = \frac{D}{6}y^3, \tag{52}$$

which corresponds to pure bending.

If $A = C = D = 0$, then

$$\sigma_x = 0, \quad \sigma_y = By, \quad \tau_{xy} = -Bx, \quad \phi = \frac{B}{2}x^2 y, \tag{53}$$

which corresponds to a combination of tensile and compressive loading, together with shear.

2.1.3 Fourth-Order Polynomials

In the case of second- and third- order polynomials, the choice of coefficients was arbitrary, as the biharmonic equation for the stress function was automatically satisfied. However this is not the case for polynomials of order four or higher, and hence the biharmonic equation imposes a restriction on the coefficients. Consider an arbitrary fourth-order polynomial of the form

$$\phi_4 = \frac{A}{4\cdot 3}x^4 + \frac{B}{3\cdot 2}x^3 y + \frac{C}{2}x^2 y^2 + \frac{D}{3\cdot 2}xy^3 + \frac{E}{4\cdot 3}y^4. \tag{54}$$

Now the biharmonic equation is satisfied providing

$$E = -(2C + A), \tag{55}$$

so A, B, C, D are arbitrary, but E is determined from the compatibility condition. The stresses are given by

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = Cx^2 + Dxy - (2C + A)y^2, \tag{56}$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = Ax^2 + Bxy + Cy^2, \tag{57}$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{B}{2}x^2 - 2Cxy - \frac{D}{2}y^2. \tag{58}$$

Consider, for example, the situation where $A = B = C = 0$, so that $E = 0$ also from the compatibility condition. Then,

$$\phi = \frac{D}{6}xy^3, \quad \sigma_x = Dxy, \quad \sigma_y = 0, \quad \tau_{xy} = -\frac{D}{2}y^2, \tag{59}$$

so there is

- Constant shear on $y = \pm h$;
- Parabolic shear on $x = 0, l$;
- Linear distribution of σ_x on $x = l$.

2.1.4 Fifth-Order Polynomials and Higher

We consider an arbitrary fifth-order polynomial of the form

$$\phi_5 = \frac{A}{5.4}x^5 + \frac{B}{4.3}x^4 y + \frac{C}{3.2}x^3 y^2 + \frac{D}{3.2}x^2 y^3 + \frac{E}{4.3}xy^4 + \frac{F}{5.4}y^5. \tag{60}$$

Now the biharmonic equation requires that

$$E = -(2C + 3A), \tag{61}$$
$$F = -(B + 2D)/3. \tag{62}$$

We see that, as the order of the polynomial increases from four upwards, the number of conditions imposed by the biharmonic equation (compatibility condition) increases.

2.1.5 Superposition of Polynomial Solutions

The elementary solutions that we have considered thus far may be superimposed to give solutions of practical interest; this is a consequence of the biharmonic equation $\nabla^4\phi = 0$ being a linear differential equation. For a given practical scenario (e.g. with a combination of loading and shearing effects), combinations of the various polynomial solution that satisfy the appropriate boundary conditions can be added together to model the physical situation.

2.2 Example: Cantilever Loaded at the Free End

We consider an example scenario of a cantilever loaded with a force P at the free end, and with no shear forces acting on the top and bottom of the cantilever. We note that the fourth-order polynomial solution for the stress function with $A = B = C = 0$ and $E = -(2C + A) = 0$ (i.e. $\phi = Dxy^3/6$) gave us:

- Constant shear on $y = \pm h$,
- Parabolic shear on $x = 0, l$,
- Linear distribution of σ_x on $x = l$,

Further, the second-order polynomial solution $\phi = Bxy$ gives rise to a uniform shear on $y = \pm h$. To describe the cantilever setup, we therefore consider a linear sum of these two stress functions:

$$\phi = Bxy + \frac{D}{6}xy^3, \tag{63}$$

with corresponding stresses

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = Dxy, \tag{64}$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0, \tag{65}$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -B - \frac{1}{2}Dy^2. \tag{66}$$

The values of the coefficients B and D are determined from the boundary conditions on the cantilever.

1. The longitudinal sides must be free of shear:

$$\tau_{xy}\big|_{y=\pm h} = 0 \quad \Rightarrow \quad -B - \frac{Dh^2}{2} = 0, \quad \Rightarrow \quad D = -\frac{2B}{h^2}. \tag{67}$$

2. The shear force distribution over the free end should add up to the applied force P. Now,

$$\tau_{xy} = -B - \frac{D}{2}y^2 = -B + \frac{By^2}{h^2}. \tag{68}$$

   The applied force is related to the shear through

$$\begin{aligned} P &= -t\int_{-h}^{h} \tau_{xy}\mathrm{d}y, \\ &= -t\int_{-h}^{h} \left(\frac{By^2}{h^2} - B\right)\mathrm{d}y, \\ &= -t\left[\frac{By^3}{2h^2} - By\right]_{-h}^{h}, \\ &= -2\left[\frac{Bh}{3} - Bh\right]t, \\ &= \frac{4}{3}Bht. \end{aligned} \tag{69}$$

Therefore,

$$B = \frac{3}{4}\frac{P}{ht}, \quad D = -\frac{3}{2}\frac{P}{h^3 t}. \tag{70}$$

hence, substituting the coefficients back into the defined stress function gives

$$\phi = -\frac{P}{4h^3 t}\left(xy^3 - 3xyh^2\right), \tag{71}$$

and the stresses are given by

$$\sigma_x = -\frac{3}{2}\frac{P}{h^3 t}xy, \tag{72}$$

$$\sigma_y = 0, \tag{73}$$

$$\tau_{xy} = -\frac{3}{4}\frac{P}{ht} + \frac{3}{4}\frac{P}{h^3 t}y^2. \tag{74}$$

We may also calculate the moment of inertia, I, for the cross-section of the cantilever beam:

$$\begin{aligned} I &= t\int_{-h}^{h} y^2 \, \mathrm{d}y, \\ &= t\left[\frac{y^3}{3}\right]_{-h}^{h}, \\ &= \frac{2th^3}{3}, \end{aligned} \tag{75}$$

and

$$\sigma_x = -\frac{P}{I}xy, \quad \sigma_y = 0, \quad \tau_{xy} = -\frac{P}{2I}\left(h^2 - y^2\right). \tag{76}$$

Evaluation of Displacements

As a first step towards evaluating the displacements, we relate the stresses to the strains via Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right), \quad \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right), \quad \gamma_{xy} = \frac{\tau_{xy}}{G}. \tag{77}$$

Given that $\sigma_y = 0$ for the cantilever setup, this reduces to

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} = -\frac{P}{EI}xy, \tag{78}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -\frac{\nu\sigma_x}{E} = \frac{\nu P}{EI}xy \tag{79}$$

Integrating yields

$$u = -\frac{1}{2}\frac{P}{EI}x^2 y + f(y), \quad v = \frac{1}{2}\frac{\nu P}{EI}xy^2 + g(x), \tag{80}$$

where $f(y)$ and $g(x)$ are unknown functions of x, y. We relate f and g to each other through the shear strain relationship and Hooke's law:

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = -\frac{1}{2}\frac{P}{IG}(h^2 - y^2) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{81}$$

Hence, substituting equation (80) into (81) yields

$$-\frac{1}{2}\frac{P}{EI}x^2 + \frac{\mathrm{d}f}{\mathrm{d}y} + \frac{1}{2}\frac{\nu P}{EI}y^2 + \frac{\mathrm{d}g}{\mathrm{d}x} = -\frac{1}{2}\frac{P}{IG}\left(h^2 - y^2\right). \tag{82}$$

The only way in which this type of equation can hold is if the terms involving x and y are separately constants:

$$\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{Px^2}{2EI} + p, \tag{83}$$

$$\frac{\mathrm{d}f}{\mathrm{d}y} = -\frac{\nu}{2}\frac{P}{EI}y^2 + \frac{1}{2}\frac{P}{IG}y^2 + q, \tag{84}$$

where

$$p + q = -\frac{1}{2}\frac{P}{IG}h^2. \tag{85}$$

Hence,

$$f(y) = -\frac{1}{6}\frac{\nu P}{EI}y^3 + \frac{1}{6}\frac{P}{IG}y^3 + qy + r, g(x) = \frac{1}{6}\frac{Px^3}{EI} + px + s. \tag{86}$$

Therefore,

$$u = -\frac{1}{2}\frac{P}{EI}x^2y - \frac{1}{6}\frac{\nu P}{EI}y^3 + \frac{1}{6}\frac{Py^3}{IG} + qy + r, \tag{87}$$

$$v = \frac{1}{2}\frac{\nu P}{EI}xy^2 + \frac{1}{6}\frac{Px^3}{EI} + px + s. \tag{88}$$

Since p and q satisfy (85), there are only three unknowns, requiring three conditions to determine them (i.e. three conditions preventing rigid body movement, in the x-y plane).

At the fixed end, we assume there is no displacement or rotation so that

$$u = v = \frac{\partial v}{\partial x} = 0, \quad \text{on} \quad x = L, y = 0. \tag{89}$$

Therefore, $r = 0$ and

$$s = -\frac{1}{6}\frac{PL^3}{EI} - pL, \tag{90}$$

with

$$0 = \frac{PL^2}{2EI} + p \quad \Rightarrow \quad p = -\frac{PL^2}{2EI}, \tag{91}$$

so that q may be calculated from (85)

$$q = -\frac{1}{2}\frac{P}{IG}h^2 + \frac{PL^2}{2EI}. \tag{92}$$

Substituting these constants back into the displacement relationships (87)–(88) yields

$$u = -\frac{1}{2}\frac{P}{EI}x^2y - \frac{1}{6}\frac{\nu P}{EI}y^3 + \frac{1}{6}\frac{Py^3}{IG} + \left(-\frac{1}{2}\frac{P}{IG}h^2 + \frac{1}{2}\frac{P}{EI}L^2\right)y, \tag{93}$$

$$v = +\frac{1}{2}\frac{\nu P}{EI}xy^2 + \frac{1}{6}\frac{Px^3}{EI} - \frac{PL^2x}{2EI} + \frac{1}{3}\frac{PL^3}{EI}. \tag{94}$$

Finally, note that the deflections at the loaded end $x = 0$, $y = 0$ are

$$v = \frac{1}{3}\frac{PL^3}{EI}, \tag{95}$$

as in simple beam theory. However, it should also be noted that distortion of plane sections is predicted. For example, consider u at the fixed end $x = L$,

$$u_L = -\frac{1}{6}\frac{\nu P}{EI}y^3 + \frac{1}{6}\frac{Py^3}{IG} - \frac{1}{2}\frac{P}{IG}h^2 y, \tag{96}$$

which is cubic in y, thus showing that plane sections do not remain plane in bending.

2.3 Problems in Polar Coordinates

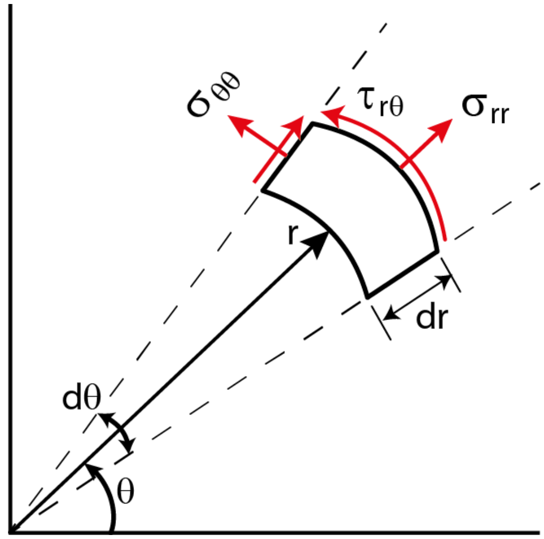

Figure 3: Stress component in a material point with coordinates (r, θ), and dimensions dr, $d\theta$.

The biharmonic equation is written in polar coordinates as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0. \tag{97}$$

For a radially-symmetric setup, ϕ is independent of θ and (97) reduces to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}^2 \phi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = 0, \tag{98}$$

which in turn can be written as

$$\frac{\mathrm{d}^4 \phi}{\mathrm{d}r^4} + \frac{2}{r}\frac{\mathrm{d}^3 \phi}{\mathrm{d}r^3} - \frac{1}{r^2}\frac{\mathrm{d}^2 \phi}{\mathrm{d}r^2} + \frac{1}{r^3}\frac{\mathrm{d}\phi}{\mathrm{d}r} = 0. \tag{99}$$

Equation (99) has the following general solution,

$$\phi = A \ln r + B r^2 \ln r + C r^2 + D, \tag{100}$$

with constant coefficients A, B, C and D determined by the boundary conditions.

2.4 Example: Hollow Cylinder Subjected to Internal and External Pressures

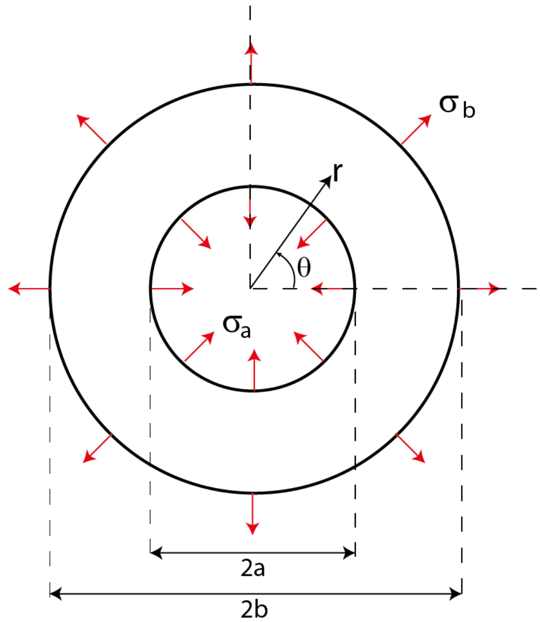

Figure 4: Cylinder of internal and external radii a, b, subjected to internal and external pressure stresses σ_a, σ_b.

We consider the case when $B = D = 0$ so that

$$\phi = A \ln r + Cr^2, \tag{101}$$

and consequently

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{1}{r}\frac{\partial \phi}{\partial r} = \frac{A}{r^2} + 2C, \tag{102}$$

$$\sigma_\theta = \frac{\mathrm{d}^2 \phi}{\mathrm{d}r^2} = -\frac{A}{r^2} + 2C, \tag{103}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = 0. \tag{104}$$

This could represent a hollow cylinder subjected to internal and external stresses, σ_a and σ_b respectively. The boundary conditions for such a problem are:

1. $\sigma_r = \sigma_a$ on $r = a$;
2. $\sigma_r = \sigma_b$ on $r = b$.

Using these boundary conditions the coefficients A and C may be determined from

$$\frac{A}{a^2} + 2C = \sigma_a, \quad \frac{A}{b^2} + 2C = \sigma_b, \tag{105}$$

which together yield

$$A = \frac{a^2 b^2 \left(\sigma_b - \sigma_a\right)}{b^2 - a^2}, \quad 2C = \frac{-\sigma_a a^2 + \sigma_b b^2}{b^2 - a^2}. \tag{106}$$

Substituting these values into (102)–(103) gives the expressions for the stress components.

$$\sigma_r = \frac{a^2 b^2 \left(\sigma_a - \sigma_b\right)}{b^2 - a^2} \frac{1}{r^2} + \frac{-\sigma_a a^2 + \sigma_b b^2}{b^2 - a^2}, \tag{107}$$

$$\sigma_\theta = -\frac{a^2 b^2 \left(\sigma_a - \sigma_b\right)}{b^2 - a^2} \frac{1}{r^2} + \frac{-\sigma_a a^2 + \sigma_b b^2}{b^2 - a^2}. \tag{108}$$

For the simplified case with internal pressure only (i.e. $\sigma_b = 0$),

$$\sigma_r = \frac{a^2 \sigma_a}{b^2 - a^2} \left(\frac{b^2}{r^2} - 1\right), \tag{109}$$

$$\sigma_\theta = \frac{-a^2 \sigma_a}{b^2 - a^2} \left(1 + \frac{b^2}{r^2}\right). \tag{110}$$

From this it can be seen that in the case of internal pressure ($\sigma_a < 0$), σ_r is always compressive ($a \leq r \leq b$ so $\sigma_r \leq 0$) and σ_θ is always tensile ($\sigma_\theta \geq 0$). The maximum value of σ_θ occurs at the inner surface $r = a$ and is

$$\sigma_{\theta_{\max}} = -\sigma_a \frac{\left(a^2 + b^2\right)}{\left(b^2 - a^2\right)} > -\sigma_a, \tag{111}$$

so the maximum value of σ_θ is always greater than the internal pressure.

2.5 Example: Stress Concentration Around a Hole in a Plate

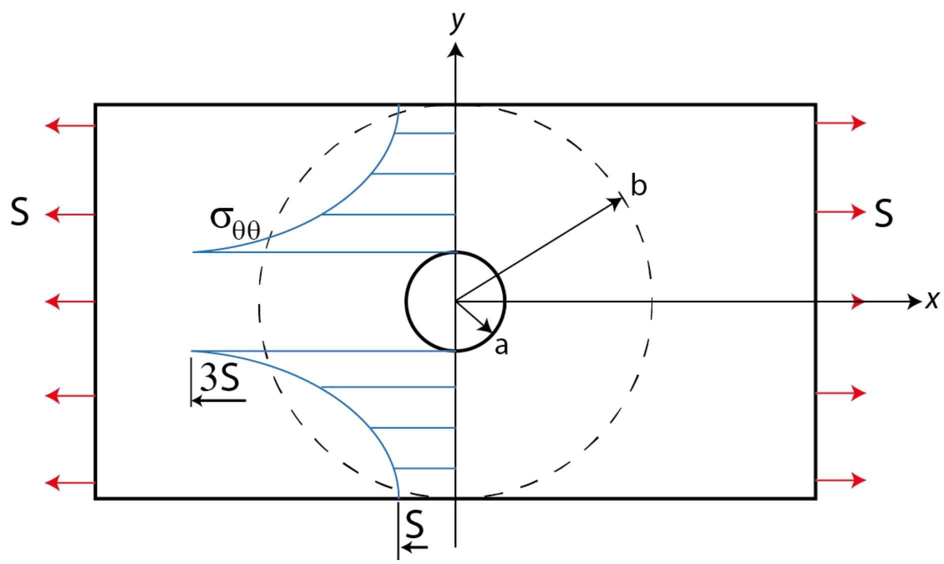

Figure 5: Stress concentration around a hole in a large plate, subjected to a uniform tensile stress S

The solution of the biharmonic equation in cylindrical polar coordinates (97) may be obtained using separation of variables of the form

$$\phi = R(r)\Theta(\theta). \tag{112}$$

One such solution is

$$\phi = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\cos 2\theta, \tag{113}$$

where, again, the coefficients may be determined from the boundary conditions.

A linear combination of equation (101) and (113) of the form

$$\phi = \frac{S}{2}\left(2r^2 - a^2\ln r\right) - \frac{S}{4}\left(r^2 + \frac{a^4}{r^2} - 2a^2\right)\cos 2\theta, \tag{114}$$

can be shown to represent the stress distribution in a **large** plate with a circular hole when subjected to a uniform tensile stress S (see Figure 5). Note that in this geometry,

$$\mathbf{e}_r = \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y, \quad \mathbf{e}_\theta = -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y. \tag{115}$$

Now

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta, \tag{116}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta, \tag{117}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = -\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta. \tag{118}$$

Examining the above equations it can be seen that as $r \to \infty$ (i.e. as we move far away from the hole),

$$\sigma_r \to \frac{S}{2}(1+\cos 2\theta) = S\cos^2\theta\sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau_{xy}\sin 2\theta, \tag{119}$$

$$\sigma_\theta \to \frac{S}{2}(1-\cos 2\theta) = S\sin^2\theta = \sigma_x\sin^2\theta + \sigma_y\cos^2\theta - \tau_{xy}\sin 2\theta, \tag{120}$$

$$\tau_{r\theta} \to -\frac{S}{2}\sin 2\theta = (\sigma_y - \sigma_x)\frac{\sin 2\theta}{2} + \tau_{xy}\cos 2\theta. \tag{121}$$

At the boundary,

$$\sigma_x = S, \quad \sigma_y = 0, \quad \tau_{xy} = 0. \tag{122}$$

Along the line of the hole ($r = a$) using (116)–(117)

$$\sigma_r = 0, \quad \sigma_\theta = S - 2S\cos 2\theta, \quad \tau_{r\theta} = 0. \tag{123}$$

Therefore, normal and shear stress are zero at $r = a$, but the circumferential normal stress is maximum at $\theta = \pi/2, 3\pi/2$ (see this by differentiating σ_θ with respect to θ and setting it equal to zero), where

$$\sigma_\theta = 3S, \tag{124}$$

and the minimum at $\theta = 0, \pi$ where

$$\sigma_\theta = -S. \tag{125}$$