1 Differential Equations

Differential equations are equalities made from functions and their derivatives. Since a differential equation relates functions to their rates of change, it can be used to model virtually anything that changes. Some specific examples where differential equations have been useful are: analyzing heat and fluid flow, throwing things, population growth, lasers, rockets, planetary motion, particle beam trajectories (and much more! :D).

Differential equations can be categorized in many different ways. The main distinctons that I make here are between Ordinary, Partial, and Non-Linear Differential Equations (ODEs, PDEs, and NLDEs). In each section I'll try and talk about the types of equations, how they show up in physics, and methods used to analyze them. Since this is physics oriented, the highest order derivative I'll need to consider is second order. PDEs and nonlinear differential equations are also large fields in their own right, and general analysis of them is difficult at best (and impossible for most other cases), so the secitons on PDEs and nonlinear equations will be physics-oriented as well.

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1.1 Ordinary Differential Equations (ODEs)

ODEs are differential equations involving only one independent variable. In physics, they typically appear as functions of space or of time, so a function $y(x)$ or $x(t)$:

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy(x) = g(x) \qquad (1.1)$$
$$a\ddot{x} + b\dot{x} + cx(t) = h(t)$$

Where $g(x), h(t)$ are functions related to the problem.[1] For ODEs, a, b, c can be constants or functions of x. Mathematically speaking, this is just two ways to write the same differential equation, changing notaiton and what you call the variables. Physically however, there are slight differences; Equations of time usually start at time $t = 0$ and only have positive values for t, while equations of space can have negative values for x. In analysis sections, I'll write about the $y(x)$ case since the $x(t)$ case can be covered in examples.

1.1.1 Second Order

In Eqn. (1.1), if $a, b, c \neq 0$, the equation is said to be second order. If $g(x) = 0$ then the equation is also said to be homogeneous. If $g(x) \neq 0$ then it is non-homogeneous (or inhomogeneous). The answer is assumed to look like $e^{\lambda x}$, so that is what is substituted in.[2]

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$
$$a\big(\lambda^2 e^{\lambda x}\big) + b\big(\lambda e^{\lambda x}\big) + c\big(e^{\lambda x}\big) = 0$$
$$\big(a\lambda^2 + b\lambda + c\big)e^{\lambda x} = 0$$

$$a\lambda^2 + b\lambda + c = 0 \qquad (1.2)$$
$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Eqn. (1.2) is called the characteristic equation. There are three solution outcomes for y, depending on the value of λ, assuming that it can be a complex number $\lambda = \alpha + \beta i$:

Real Distinct Roots: $_{\alpha_1 \neq \alpha_2 \ \& \ \beta = 0}$
$$y_c = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} \qquad (1.3)$$

Real Repeating Roots: $_{\alpha_1 = \alpha_2 \ \& \ \beta = 0}$
$$y_c = c_1 e^{\alpha x} + c_2 x e^{\alpha x} \qquad (1.4)$$

Complex Roots: $_{\beta \neq 0}$
$$y_c = e^{\alpha x}(c_1 \cos(\beta x) + c_2 i \sin(\beta x)) \qquad (1.5)$$

Eqn. (1.4) is found due to linearity; (EXAMPLE ABOUT MULTIPLYING BY CONSTANT) and Eqn. (1.5) is found through Euler's Identity. Note that this solution algorithm works for first order equations as well.

If $g(x) \neq 0$ then the equation is said to be nonhomogeneous (or inhomogeneous, they are synonyms). Since the equation is still linear, then by definition any two linearly independent solutions can be used to construct a third. To solve this, we look for a homogeneous solution[3] y_h and particular solution y_p. y_h is the solution to the homogeneous case, and y_p is found

1.1.2 Shortcut for First Order

In Eqn. (1.1), when $a = 0$ the equation is said to be first order. The significant remaining cases depend on $g(x)$. If $g(x) = 0$, then the equation is also said to be homogeneous, and if $g(x) \neq 0$ then it is non-

[1] If instead of $g(x), h(t)$ there was $g(f(x)), h(x(t))$, then the equations would be nonlinear, and you would be in the wrong section.

[2] The use of λ is to highlight the fact that e^x is an eigenfunction of the derivative, and λ is the eigenvalue.

[3] sometimes called the complimentary solution y_c

homogeneous. For homogeneous equations, the general form is:

$$b\frac{\mathrm{d}y}{\mathrm{d}x} + cy(x) = 0 \tag{1.6}$$

Just like polynomials, it can be helpful to remove coefficients from the highest order term:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{c}{b}y(x) = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y(x) = 0$$

This is assuming $c \to c(x),\ b \to b(x)$ and making $P(x) \equiv \frac{c(x)}{b(x)}$. This equation is called seperable, because all of the x and y terms can be separated to either side of the equation.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -P(x)y$$
$$\frac{dy}{y} = -P(x)dx$$
$$\int \frac{dy}{y} = -\int P(x)dx$$

$$\ln(y) + C_{\text{onst}} = -\int P(x)dx$$
$$e^{\ln(y)} = e^{-C_{\text{onst}}}e^{-\int P(x)dx}$$
$$\boxed{y(x) = Ae^{-\int P(x)dx}} \tag{1.7}$$

Examples:
Numerical solve:

1.1.3 System

1.2 Partial Differential Equations (PDEs)

1.3 Nonlinear Differential Equations (NLDEs)

1.4 First Order ODEs

1.4.1 Non-Homogeneous

Takes the form: $y' + P(x)y = g(x)$.

$$\mu(x) \equiv e^{\int P(x)dx}$$

$$\frac{d}{dx}(\mu(x)y) = \mu(x)g(x) \xrightarrow{Move\ dx,\ integrate} \mu(x)y + c_1 = \int \mu(x)g(x)dx$$

$$y(x) = \frac{\int \mu(x)g(x)dx - c_1}{\mu(x)}$$

Variation of Parameters

Not limited by non-constant coefficients.

$$y_p = y_c \int \frac{g(x)}{y_c} dx$$

1.4.2 Reduction of Order

1.5 Second Order ODEs

1.5.1 Homogeneous

1.5.2 Non-Homogeneous

Takes the form: $y'' + Q(x)y' + P(x)y = g(x)$

1.5.3 Variation of Parameters

The general idea is to replace the coefficients (c_1, c_2) with functions.

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

For the case of a second order, the relations of $u(x)$ are depicted below

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$

In summary: Find Wronskian (determinant represented by W). Find u_i'. Integrate to get u_i and plug in to y_p, then add to y_c.

1.5.4 Undetermined Coefficients

There are two subcategories of this method: The superposition approach and the annihilator approach.
Superposition: Solve for the complementary function y_c (shown in the 'Homogeneous' section), then find a particular solution y_p.

1.6 System of Linear Equations

Multiple differential equations that are related to each other. They are typically solved by putting them into matrices and using eigenvalues/eigenvectors.

$$\textbf{First Order: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies \textbf{X'} = M\mathbf{X}$$

$$\textbf{Solving: } |M - \lambda I| = 0 \quad \Big| \quad (M - \lambda_i I)\mathbf{e}_i = 0 \quad \Big| \quad \mathbf{X} = \sum_i \vec{c}_i e^{\lambda_i x}$$

$$\textbf{Normal Modes: } \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies \textbf{X''} = M\mathbf{X}$$

$$\textbf{Solving: } |M - \omega^2 I| = 0$$

1.7 The Laplace Transform

A powerful method for solving IVPs using an integral transform. The general method is to transform a differential equation from the t domain to the s domain using the transform, where the equation becomes a simple algebra system. After solving for $Y(s)$, use the inverse transform to turn the obtained function into the complete solution. Using Partial Fraction Decomposition is often useful when solving these. Use tables.

$$\begin{aligned}
\mathcal{L}[f'](s) &= s\mathcal{L}[f](s) - f(0) \\
\mathcal{L}[f''](s) &= s^2\mathcal{L}[f](s) - sf(0) - f'(0) \\
\mathcal{L}[f'''](s) &= s^3\mathcal{L}[f](s) - s^2f(0) - sf'(0) - f''(0) \\
&\vdots
\end{aligned}$$

Laplace Convolution of two functions f, g is defined to be

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

If $\mathcal{L}[f](s) = F(s)$ & $\mathcal{L}[g](s) = G(s)$ exists, then $\mathcal{L}^{-1}[FG] = (f * g)$ and $\mathcal{L}[f * g](s) = FG$. This is useful for when we want to recover $h(t)$ from $H(s) = FG$ for a known FG.

1.8 Partial Fraction Decomposition

Useful for re-writing some of the results of a Laplace transform. It involves decomposing the denominator of some difficult fraction into multiple seperate fractions.

$$\frac{1}{x^2 - x + 1} = \frac{A}{}$$

1.9 Bernoulli Equations

Take the form: $y' + P(x)y = g(x)y^n$ for $n \in \mathbb{R}$ When $n \neq 0, 1$ solve by substituting $u = y^{1-n}$

$$y' + \frac{1}{x}y = xy^2 \xrightarrow{n=2 \therefore u=y^{-1} \therefore y=u^{-1}} \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx}$$

$$-u^{-2}\frac{du}{dx} + \frac{1}{ux} = xu^{-2} \xrightarrow{rearrange} \frac{du}{dx} - \frac{1}{x}u = -x$$

$$\mu(x) = e^{-\int 1/x \ dx} = e^{-\ln(x)} = \frac{1}{x}$$

$$\int d(\mu(x)u) = \int \mu(x)g(x)dx \ ; \ \int d\left(\frac{u}{x}\right) = \int (-1)dx$$

$$u = -x^2 + c_1 x \ \therefore \ \boxed{y = \frac{1}{-x^2 + c_1 x}}$$

Cauchy-Euler

Also called The "Equidimensional" Equation. Assume solution of $y = x^m$ and plug in. Solve for m.

$$\underset{m_1 \neq m_2}{\textbf{Real Distinct Roots:}} \quad y = c_1 x^{m_1} + c_1 x^{m_2} \tag{1.8}$$

$$\underset{m_1 = m_2}{\textbf{Real Repeating Roots:}} \quad y = c_1 \ln(x) x^m + c_2 x^m \tag{1.9}$$

$$\underset{b \neq 0}{\textbf{Complex Roots:}} \quad y = c_1 x^\alpha \cos(\beta \ln(x)) + c_2 x^\alpha \sin(\beta \ln(x)) \tag{1.10}$$

1.10 PDEs in Physics

1.10.1 The Heat Equation

1.10.2 The Wave Equation

1.10.3 Laplace's Equation

1.10.4 Poisson's Equation