# 1 Differential Equations

Differential equations are equalities made from functions and their derivatives. Ordinary differential equations (ODEs) only have 1-D derivatives, while partial differential equations (PDEs) have partial derivatives in multiple dimensions. The order of a differential equation is the determined by the highest order derivative present in the differential equation. Since this is physics oriented, the highest order I'll need to write about is second order. Sections on PDEs and nonlinear equations will also be physics-oriented.
Introduction: $F(x, y, y', y'')$
Laplace and Poisson equations, Fourier analysis, linear ODES (non/homo first sec order)
Dynamics: $F(t, x, \dot{x}, \ddot{x}) = 0$
heat and wave equations, laplace analysis, nonlinear stuffs

## Table of Contents

## 1.1 First Order ODEs

### 1.1.1 Homogeneous

A function of the form $f(x, y, y'...) = 0$. If the equation as a whole is linear[1], the solution will be of the form $y = e^{mx}$ where m is some complex number ($m = a + bi$). Solutions start by plugging in this guess of $y = e^{mx}$ and finding $m$:

$$ay'' + by' + cy = 0$$
$$a\big(m^2e^{mx}\big) + b(me^{mx}) + c(e^{mx}) = 0$$
$$\big(am^2 + bm + c\big)e^{mx} = 0$$
$$am^2 + bm + c = 0$$
$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve for $m$, since $m$ is a complex root there are three cases:

$$\underset{a_1 \neq a_2 \ \& \ b=0}{\textbf{Real Distinct Roots:}} \quad y_c = c_1e^{a_1x} + c_2e^{a_2x} \tag{1.1}$$

$$\underset{a_1 = a_2 \ \& \ b=0}{\textbf{Real Repeating Roots:}} \quad y_c = c_1e^{ax} + c_2xe^{ax} \tag{1.2}$$

$$\underset{b \neq 0}{\textbf{Complex Roots:}} \quad y_c = e^{ax}\left(c_1\cos(bx) + c_2\sin(bx)\right) \tag{1.3}$$

### 1.1.2 Non-Homogeneous

Takes the form: $y' + P(x)y = g(x)$.

$$\mu(x) \equiv e^{\int P(x)dx}$$
$$\frac{d}{dx}(\mu(x)y) = \mu(x)g(x) \xrightarrow{Move\ dx,\ integrate} \mu(x)y + c_1 = \int \mu(x)g(x)dx$$
$$y(x) = \frac{\int \mu(x)g(x)dx - c_1}{\mu(x)}$$

**Variation of Parameters**

Not limited by non-constant coefficients.

$$y_p = y_c \int \frac{g(x)}{y_c}dx$$

### 1.1.3 Reduction of Order

## 1.2 Second Order ODEs

### 1.2.1 Homogeneous

### 1.2.2 Non-Homogeneous

Takes the form: $y'' + Q(x)y' + P(x)y = g(x)$

[1] An example of a nonlinear differential equation could be $f(x, y, y') = \sin(y'(x)) + y^2(x) = 0$

### 1.2.3 Variation of Parameters

The general idea is to replace the coefficients $(c_1, c_2)$ with functions.

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

For the case of a second order, the relations of $u(x)$ are depicted below

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$

In summary: Find Wronskian (determinant represented by W). Find $u_i'$. Integrate to get $u_i$ and plug in to $y_p$, then add to $y_c$.

### 1.2.4 Undetermined Coefficients

There are two subcategories of this method: The superposition approach and the annihilator approach. Superposition: Solve for the complementary function $y_c$ (shown in the 'Homogeneous' section), then find a particular solution $y_p$.

## 1.3 System of Linear Equations

Multiple differential equations that are related to each other. They are typically solved by putting them into matrices and using eigenvalues/eigenvectors.

$$\textbf{First Order:} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies \textbf{X'} = M\mathbf{X}$$

$$\textbf{Solving:}\ |M - \lambda I| = 0 \quad \Big| \quad (M - \lambda_i I)\mathbf{e}_i = 0 \quad \Big| \quad \mathbf{X} = \sum_i \vec{c}_i e^{\lambda_i x}$$

$$\textbf{Normal Modes:} \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies \textbf{X''} = M\mathbf{X}$$

$$\textbf{Solving:}\ |M - \omega^2 I| = 0$$

## 1.4 The Laplace Transform

A powerful method for solving IVPs using an integral transform. The general method is to transform a differential equation from the $t$ domain to the $s$ domain using the transform, where the equation becomes a simple algebra system. After solving for $Y(s)$, use the inverse transform to turn the obtained function into the complete solution. Using Partial Fraction Decomposition is often useful when solving these. Use tables.

$$\begin{aligned}
\mathcal{L}[f'](s) &= s\mathcal{L}[f](s) - f(0) \\
\mathcal{L}[f''](s) &= s^2\mathcal{L}[f](s) - sf(0) - f'(0) \\
\mathcal{L}[f'''](s) &= s^3\mathcal{L}[f](s) - s^2f(0) - sf'(0) - f''(0) \\
&\vdots
\end{aligned}$$

Laplace Convolution of two functions $f, g$ is defined to be

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

If $\mathcal{L}[f](s) = F(s)$ & $\mathcal{L}[g](s) = G(s)$ exists, then $\mathcal{L}^{-1}[FG] = (f * g)$ and $\mathcal{L}[f * g](s) = FG$. This is useful for when we want to recover $h(t)$ from $H(s) = FG$ for a known $FG$.

## 1.5 Partial Fraction Decomposition

Useful for re-writing some of the results of a Laplace transform. It involves decomposing the denominator of some difficult fraction into multiple seperate fractions.

$$\frac{1}{x^2 - x + 1} = \frac{A}{\phantom{x}}$$

## 1.6 Bernoulli Equations: $y' + P(x)y = g(x)y^n$

Take the form: $y' + P(x)y = g(x)y^n$ for $n \in \mathbb{R}$ When $n \neq 0, 1$ solve by substituting $u = y^{1-n}$

$$y' + \frac{1}{x}y = xy^2 \xrightarrow{n=2 \ \therefore \ u=y^{-1} \ \therefore \ y=u^{-1}} \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx}$$

$$-u^{-2}\frac{du}{dx} + \frac{1}{ux} = xu^{-2} \xrightarrow{rearrange} \frac{du}{dx} - \frac{1}{x}u = -x$$

$$\mu(x) = e^{-\int 1/x \ dx} = e^{-\ln(x)} = \frac{1}{x}$$

$$\int d(\mu(x)u) = \int \mu(x)g(x)dx \ ; \ \int d\left(\frac{u}{x}\right) = \int (-1)dx$$

$$u = -x^2 + c_1x \ \therefore \ \boxed{y = \frac{1}{-x^2 + c_1x}}$$

## Cauchy-Euler: $x^ny^{(n)} + ... + x^2y'' + axy' + by = 0$

Also called The "Equidimensional" Equation. Assume solution of $y = x^m$ and plug in. Solve for $m$.

$$\underset{m_1 \neq m_2}{\textbf{Real Distinct Roots:}} \quad y = c_1x^{m_1} + c_1x^{m_2} \tag{1.4}$$

$$\underset{m_1 = m_2}{\textbf{Real Repeating Roots:}} \quad y = c_1\ln(x)x^m + c_2x^m \tag{1.5}$$

$$\underset{b \neq 0}{\textbf{Complex Roots:}} \quad y = c_1x^\alpha\cos(\beta\ln(x)) + c_2x^\alpha\sin(\beta\ln(x)) \tag{1.6}$$

## 1.7 Numerical

What types of numerical differential equations do you do again...?

## 1.8 PDEs in Physics

### 1.8.1 The Heat Equation

### 1.8.2 The Wave Equation

### 1.8.3 Laplace's Equation

### 1.8.4 Poisson's Equation