Mathematics Review

For Applications In Physics

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Statement of Purpose

I am writing this as a review for myself of mathematial concepts that I have learned. I will not try and build from the ground up the concepts here. For example, I will just write about how to find the eigenvectors of a matrix, not how eigenvectors are defined.

Notation I Like

- I try to stay as consistent as I can with variable and function letters; no one needs u *and* v for velocity. That being said, different fields do regularly use different notations, and it would be ridiculous of me to ignore them entirely, I'll try to include some "translations" for important things.

- I like Euler, Leibniz, Newton notation for differentiation. all of the following can be considered equivalent:

$$\frac{\mathrm{d}}{\mathrm{d}x}f = \frac{\mathrm{d}f}{\mathrm{d}x} = \mathrm{D}_x f = f'(x)$$
$$\frac{\partial}{\partial x}f = \frac{\partial f}{\partial x} = \partial_x f = f_x(x, y)$$

- Bold notation will be used over arrow notation for vectors.

$$\mathbf{A} = A_i = \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = A_r\hat{\boldsymbol{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$$

- A function f is linear if you can take out constants and add/subtract without changing the function:[1]

$$f(cx) = cf(x) \quad c \in \mathbb{R}$$
$$f(x \pm y) = f(x) \pm f(y)$$

[1] Examples of functions that aren't linear are: $f(x) = \sin(x)$, $f(x) = x^2$

1 Differential Calculus

Differentiation in One Dimension (1-D)

The derivative is the proportionality factor of how rapidly the function $f(x)$ varies when the argument x is changed by dx; f changes by an amount df:

$$df = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)dx$$

Multiplying and dividing functions $f = f(x),\ g = g(x)$, in derivatives
Product Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = (f')g + f(g') \tag{1.1}$$

Quotient Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f}{g}\right) = \frac{(f')g - f(g')}{(g)^2} \tag{1.2}$$

Differentiation in Three Dimensions (3-D)

For 3-variable functions:

$$df = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)dx + \left(\frac{\mathrm{d}f}{\mathrm{d}y}\right)dy + \left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)dz$$

The derivative of $f(x, y, z)$ tells one how f changes when one alters all three variables by dx, dy, dz.

Gradient

$$\boldsymbol{\nabla} f(x,y,z) = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} f(x,y,z) = \boxed{\frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}}$$

The del operator (∇) is a **vector derivative operator**. It can take in a scalar field $f(x, y, z)$ and returns a vector field. The vectors **points in the direction of f's maximum increase of the scalar field**, moreover, the magnitude of ∇f gives the magnitude of each vector along this maximal direction.

Just like 1-D derivatives, you can find the extrema of a function with three variables by observing it at a stationary point (x, y, z):

$$\nabla f = 0$$

Gradients obey the following product rules:

$$\nabla(fg) = (\nabla f)g + f(\nabla g)$$
$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

Divergence

Divergence is a measure of how much a vector field spreads out from a point or volume. The divergence of a vector field **A** is caluclated by taking a dot product between the del operator and a **A**. **The divergence of a vector field is a scalar.**

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \cdot \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \boxed{\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \tag{1.3}$$

Divergences obey the following Product Rules:

$$\boldsymbol{\nabla} \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$
$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

When the divergence of a vector field is zero everywhere it is called **solenoidal**. Any closed surface has no net flux across it in a solenoidal field.

Curl

Curl is a measure of how much a vector "swirls" around the point in question. The curl of a vector field **A** is found by taking the cross product of the del operator with **A**. One can find the curl conveniently as the determinant of the following matrix:

$$\boldsymbol{\nabla} \times \mathbf{A} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \boxed{\begin{bmatrix} (\partial_y A_z - \partial_z A_y) \\ (\partial_z A_x - \partial_x A_z) \\ (\partial_x A_y - \partial_y A_x) \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{bmatrix}} \tag{1.4}$$

Curls obey the following Product Rules:

$$\boldsymbol{\nabla} \times (f\mathbf{A}) = f(\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$
$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

The curl of a vector field is a vector field.[2] When the curl of a vector field is zero, the field is called **irrotational** and the field is conservative.

Laplacian

The laplace operator (denoted by ∇^2) is a kind of second derivative for scalars and vectors.[3] It can be thought of as taking whichever vector derivatives are possible.

$$\nabla^2 f = \nabla \cdot (\nabla f) \tag{1.5}$$
$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \tag{1.6}$$

[2] *Technically* a pseudo-vector field.
[3] Some people use Δ instead of ∇^2, but that seems goofy, so I don't use it.

The following is also true for second derivatives based on the nature of first order vector derivatives:

$$\nabla \times (\nabla f) = 0 \tag{1.7}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{1.8}$$

2 Integral Calculus

Remember that $d\mathbf{l}$, $d\mathbf{a}$, and dV are different in different coordinate systems.

$$\textbf{Curve Integral} \quad \int_C \mathbf{A} \cdot d\mathbf{l} = \iiint \mathbf{A} \cdot (dx\ \hat{\boldsymbol{x}} + dy\ \hat{\boldsymbol{y}} + dz\ \hat{\boldsymbol{z}}) \tag{2.1}$$

$$\textbf{Surface Integral} \quad \int_S \mathbf{A} \cdot d\mathbf{a} = \iint_D (\mathbf{A} \cdot \hat{\boldsymbol{z}}) dx dy = \iint_D A_k dx_i dx_j \tag{2.2}$$

$$\textbf{Volume Integral} \quad \int_{\mathcal{V}} \mathbf{A} dV = \int A_x dV\ \hat{\boldsymbol{x}} + \int A_y dV\ \hat{\boldsymbol{y}} + \int A_z dV\ \hat{\boldsymbol{z}} \tag{2.3}$$

The Fundamental Theorem for Gradients

Similar to the fundamental theorem of calculus, the curve integral of the gradient of a scalar function is equal to the difference of values of that scalar function at the endpoints.

$$\boxed{\int_C (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})} \tag{2.4}$$

Divergence Theorem

The divergence of $\mathbf{A}$ over a volume is equal to the components of $\mathbf{A}$ that are normal to the surface that bounds the volume.[4]

$$\boxed{\oint_S \mathbf{A} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{A}) dV} \tag{2.5}$$

The Fundamental Theorem for Curls: Stokes' Theorem

The integral of a derivative over a region is equal to the value of the function at the boundary. That is, the curl over a surface is equal to the value of the function at the perimeter P.

$$\boxed{\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a}} \tag{2.6}$$

Integration by Parts in Vector Calculus

$$\int_{\mathcal{V}} f(\nabla \cdot \mathbf{A}) dV = \oint_S f\mathbf{A} \cdot d\mathbf{a} - \int_{\mathcal{V}} \mathbf{A} \cdot (\nabla f) dV$$

[4] $\oint$ is the closed path/closed surface integral sign.

3 Theories with Vector Fields

The Helmholtz Theorem

A field is uniquely determined by its divergence and curl when boundary conditions are applied. If there exists a scalar field ϕ, and vector field $\mathbf{A}$, C, and: This means that a vector field $\mathbf{F}$ can be :

$$\mathbf{F} = -\nabla V + \boldsymbol{\nabla} \times \mathbf{A} \tag{3.1}$$

$$\left.\begin{array}{r} \nabla \cdot \mathbf{A} = \phi \\ \nabla \times \mathbf{A} = \mathbf{C} \end{array}\right\} \implies \nabla \cdot \mathbf{C} = 0$$

Then $\mathbf{A}$ can be determined uniquely from ϕ and $\mathbf{C}$

Potentials

If the curl of a vector field $\mathbf{E}$ vanishes everywhere, then the field is conservative, meaning that the curve integral between any two points is path independent (so if the path is closed, the curve integral is zero) and by definition of conservative fields, $\mathbf{E}$ can be represented as the gradient of some scalar function V: [5]

$$\boxed{\nabla \times \mathbf{E} = 0} \iff \boxed{\oint_C \mathbf{E} \cdot d\mathbf{l} = 0} \iff \boxed{\mathbf{E} = -\nabla V}$$

If the divergence of a vector field, $\mathbf{B}$, vanishes everywhere, then the surface integral of $\mathbf{B}$ is independent of the surface for any given boundary line.

$$\boxed{\nabla \cdot \mathbf{B} = 0} \iff \boxed{\oint_S \mathbf{B} \cdot d\mathbf{s} = 0} \iff \boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

[5] The negative of the gradient is **used by convention** to make physics easier. Think about the gravitational force compared to gravitational potential, the force field points from from high (scalar) potential to low (scalar) potential.

4 Linear Algebra

4.1 Eigenvalue, Eigenvector, Eigenfunction

By definition an eigenvector $\mathbf{v}$ of a matrix $\mathbf{M}$ must satisfy:

$$\mathbf{M}\mathbf{v} = \lambda\mathbf{v} \quad \exists\lambda \in \mathbb{C}$$

Rearranging to find λ gives

$$\mathbf{M}\mathbf{v} - \lambda\mathbf{v} = (\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = 0$$

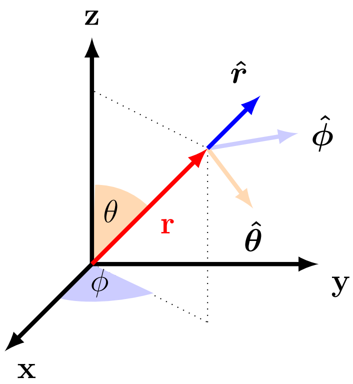
The eigenvector $\mathbf{v} = 0$ is a trivial solution, but if the matrix $(\mathbf{M} - \lambda\mathbf{I})$ is invertible, then it's inverse can be applied to both sides, leaving us with no choice but $\mathbf{v} = 0$. Because of this, nontrivial values of λ are therefore values that make $(\mathbf{M} - \lambda\mathbf{I})$ singular (i.e. non-invertible). A matrix is singular if the determinant is zero, so to find nontrivial λ, you solve:

$$\det(\mathbf{M} - \lambda\mathbf{I}) = |\mathbf{M} - \lambda\mathbf{I}| = 0$$

5 Curvilinear Coordinates

5.1 Spherical Coordinates

In this system of coordinates, the following describe the space's basis set:



Distance from the origin: r
Polar angle: θ
Azimuthal angle: ϕ

Important relationships:

$$\begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

Matrix is orthogonal, transpose to find $\hat{\boldsymbol{x}}$ in terms of $\hat{\boldsymbol{r}}$

Position, velocity, and acceleration:

$$\mathbf{r}(t) = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}$$

$$\mathbf{v}(t) = \begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ r\dot{\phi}\sin(\theta) \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}$$

$$\mathbf{a}(t) = \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2(\theta) \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin(\theta)\cos(\theta) \\ 2r\dot{\theta}\dot{\phi}\cos(\theta) + 2\dot{r}\dot{\phi}\sin(\theta) + r\ddot{\phi}\sin(\theta) \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}$$

Infinitesimal Displacement:

$$d\mathbf{l} = dr\ \hat{\boldsymbol{r}} + rd\theta\ \hat{\boldsymbol{\theta}} + r\sin(\theta)d\phi\ \hat{\boldsymbol{\phi}}$$

Infinitesimal Areas:

Held Constant	$d\mathbf{a}$
r	$r^2\sin(\theta)d\theta d\phi\ \hat{\boldsymbol{r}}$
θ	$r\sin(\theta)drd\phi\ \hat{\boldsymbol{\theta}}$
ϕ	$rdrd\theta\ \hat{\boldsymbol{\phi}}$

Infinitesimal volume:

$$dV = r^2\sin(\theta)drd\theta d\phi$$

Spherical Vector Derivatives:

Gradient: $$\nabla f = \begin{bmatrix} \partial_r f \\ \frac{1}{r}\partial_\theta f \\ \frac{1}{r\sin(\theta)}\partial_\phi f \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}$$

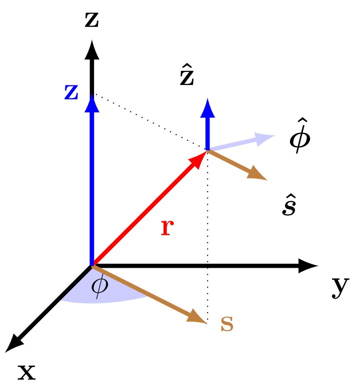
Divergence: $$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{r^2}\partial_r\left(r^2 A_r\right) + \frac{1}{r\sin(\theta)}\partial_\theta(\sin(\theta)A_\theta) + \frac{1}{r\sin(\theta)}\partial_\phi A_\phi$$

Curl: $$\boldsymbol{\nabla} \times \mathbf{A} = \begin{bmatrix} \frac{1}{r\sin(\theta)}\left[\partial_\theta(A_\phi\sin(\theta)) - \partial_\phi A_\theta\right] \\ \frac{1}{r\sin(\theta)}\partial_\phi A_r - \frac{1}{r}\partial_r(rA_\phi) \\ \frac{1}{r}(\partial_r(rA_\theta) - \partial_\theta A_r) \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}$$

Scalar Laplacian: $$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial\phi^2}$$

5.2 Cylindrical Coordinates

In this system, the following describe the space's basis set:



Distance from the z-axis: s
Azimuthal angle: ϕ
Distance on z-axis: z

Important relationships:

$$\begin{bmatrix} \hat{\boldsymbol{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

Matrix is orthogonal, transpose to find $\hat{\boldsymbol{x}}$ in terms of $\hat{\boldsymbol{s}}$

Position, velocity, and acceleration:

$$\mathbf{r}(t) = \begin{bmatrix} s \\ 0 \\ z \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

$$\mathbf{v}(t) = \begin{bmatrix} \dot{s} \\ s\dot{\phi} \\ \dot{z} \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

$$\mathbf{a}(t) = \begin{bmatrix} \ddot{s} - r\dot{\phi}^2 \\ s\ddot{\phi} + 2\dot{s}\dot{\phi} \\ \ddot{z} \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

Infinitesimal Length:

$$d\mathbf{l} = ds\ \hat{\boldsymbol{s}} + sd\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\boldsymbol{z}}$$

Infinitesimal Areas:

Held Constant	$d\mathbf{a}$
s	$sd\phi dz\ \hat{\boldsymbol{s}}$
ϕ	$dsdz\ \hat{\boldsymbol{\phi}}$
z	$sdsd\phi\ \hat{\boldsymbol{z}}$

Infinitesimal Volume:

$$dV = sdsd\phi dz$$

Cylindrical Vector Derivatives:

Gradient: $$\nabla f = \left(\frac{\partial f}{\partial s}\right)\hat{\boldsymbol{s}} + \left(\frac{1}{s}\frac{\partial f}{\partial \phi}\right)\hat{\boldsymbol{\phi}} + \left(\frac{\partial f}{\partial z}\right)\hat{\boldsymbol{z}}$$

Divergence: $$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Curl: $$\boldsymbol{\nabla}\times\mathbf{A} = \begin{bmatrix} \frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \\ \\ \frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} \\ \\ \frac{1}{s}\frac{\partial}{\partial s}(sA_\phi) - \frac{1}{s}\frac{\partial A_s}{\partial \phi} \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{s}} \\ \\ \hat{\boldsymbol{\phi}} \\ \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

Scalar Laplacian: $$\nabla^2 f = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

6 Differential Equations

Differential equations are equalities made from functions and their derivatives. Since a differential equation relates functions to their rates of change, it can be used to model virtually anything that changes.

Differential equations can be categorized in many different ways. The main distinctons that I make here are between Ordinary, Partial, and Non-Linear Differential Equations (ODEs, PDEs, and NLDEs). Since this is physics oriented, I'll stick to differential equations that are relevant to physics (first and second order equations, etc.)

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6.1 Ordinary Differential Equations (ODEs)

ODEs are differential equations involving only one independent variable. In physics, they typically appear as functions of space or of time, so a function $y(x)$ or $x(t)$:

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy(x) = g(x) \tag{6.1}$$
$$a\ddot{x} + b\dot{x} + cx(t) = h(t)$$

Where $g(x), h(t)$ are functions related to the problem.[6] For ODEs, a, b, c can be constants or functions of x. Mathematically speaking, this is just two ways to write the same differential equation, changing notaiton and what you call the variables. Physically however, there are slight differences; Equations of time usually start at time $t = 0$ and only have positive values for t, while equations of space can have negative values for x. In analysis sections, I'll write about the $y(x)$ case since the $x(t)$ case can be covered in examples.

6.1.1 Second Order

In Eqn. (6.1), if $a(x), b(x), c(x) \neq 0$, the equation is said to be second order. If $g(x) = 0$ then the equation is also said to be homogeneous. If $g(x) \neq 0$ then it is non-homogeneous (or inhomogeneous). The answer is assumed to look like $e^{\lambda x}$, so that is what is substituted in.[7]

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$
$$a\left(\lambda^2 e^{\lambda x}\right) + b\left(\lambda e^{\lambda x}\right) + c\left(e^{\lambda x}\right) = 0$$
$$\left(a\lambda^2 + b\lambda + c\right)e^{\lambda x} = 0$$

$$a\lambda^2 + b\lambda + c = 0 \tag{6.2}$$
$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Eqn. (6.2) is called the characteristic equation. There are three solution outcomes for y, depending on the value of λ, assuming that it can be a complex number $\lambda = \alpha + \beta i$:

Real Distinct Roots: $\alpha_1 \neq \alpha_2$ & $\beta = 0$
$$y_c = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} \tag{6.3}$$

Real Repeating Roots: $\alpha_1 = \alpha_2$ & $\beta = 0$
$$y_c = c_1 e^{\alpha x} + c_2 x e^{\alpha x} \tag{6.4}$$

Complex Roots: $\beta \neq 0$
$$y_c = e^{\alpha x}(c_1 \cos(\beta x) + c_2 i \sin(\beta x)) \tag{6.5}$$

Eqn. (6.4) is found due to linearity; (EXAMPLE ABOUT MULTIPLYING BY CONSTANT) and Eqn. (6.5) is found through Euler's Identity. Note that this solution algorithm works for first order equations as well.

[6] If instead of $g(x), h(t)$ there was $g(f(x)), h(x(t))$, then the equations would be nonlinear, and you would be in the wrong section.

[7] The use of λ is to highlight the fact that e^x is an eigenfunction of the derivative, and λ is the eigenvalue.

If $g(x) \neq 0$ then the equation is said to be nonhomogeneous (or inhomogeneous, they are synonyms). Since the equation is still linear, then by definition any two linearly independent solutions can be used to construct a third. To solve this, we look for a homogeneous solution[8] y_h and particular solution y_p. y_h is the solution to the homogeneous case, and y_p is found

6.1.2 Shortcut for First Order

In Eqn. (6.1), when $a = 0$ the equation is said to be first order. The significant remaining cases depend on $g(x)$. If $g(x) = 0$, then the equation is also said to be homogeneous, and if $g(x) \neq 0$ then it is non-homogeneous. For homogeneous equations, the general form is:

$$b\frac{\mathrm{d}y}{\mathrm{d}x} + cy(x) = 0 \tag{6.6}$$

Just like polynomials, it can be helpful to remove coefficients from the highest order term:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{c}{b}y(x) = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y(x) = 0$$

This is assuming $c \to c(x)$, $b \to b(x)$ and making $P(x) \equiv \frac{c(x)}{b(x)}$. This equation is called seperable, because all of the x and y terms can be separated to either side of the equation.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -P(x)y$$
$$\frac{dy}{y} = -P(x)dx$$
$$\int \frac{dy}{y} = -\int P(x)dx$$

$$\ln(y) + C_{\text{onst}} = -\int P(x)dx$$
$$e^{\ln(y)} = e^{-C_{\text{onst}}}e^{-\int P(x)dx}$$
$$\boxed{y(x) = Ae^{-\int P(x)dx}} \tag{6.7}$$

[8]sometimes called the complimentary solution y_c

6.1.3 System

6.2 Second Order ODEs

6.2.1 Homogeneous

6.2.2 Non-Homogeneous

Takes the form: $y'' + Q(x)y' + P(x)y = g(x)$

6.2.3 Variation of Parameters

The general idea is to replace the coefficients (c_1, c_2) with functions.

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

For the case of a second order, the relations of $u(x)$ are depicted below

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$

In summary: Find Wronskian (determinant represented by W). Find u_i'. Integrate to get u_i and plug in to y_p, then add to y_c.

6.2.4 Undetermined Coefficients

There are two subcategories of this method: The superposition approach and the annihilator approach.
Superposition: Solve for the complementary function y_c (shown in the 'Homogeneous' section), then find a particular solution y_p.

6.3 System of Linear Equations

Multiple differential equations that are related to each other.

$$\textbf{First Order:} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies \textbf{X'} = M\textbf{X}$$

They are typically solved by putting them into matrices and finding eigenvalues/eigenvectors.
By definition an eigenvector $\mathbf{v}$ of a matrix $\mathbf{M}$ must satisfy:

$$\mathbf{Mv} = \lambda \mathbf{v} \quad \exists \lambda \in \mathbb{C}$$

Rearranging to find λ gives

$$\mathbf{Mv} - \lambda\mathbf{v} = (\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = 0$$

The eigenvector $\mathbf{v} = 0$ is a trivial solution, but if the matrix $(\mathbf{M} - \lambda\mathbf{I})$ is invertible, then it's inverse can be applied to both sides, leaving us with no choice but $\mathbf{v} = 0$. Because of this, nontrivial values of λ are therefore values that make $(\mathbf{M} - \lambda\mathbf{I})$ singular (i.e. non-invertible). A matrix is singular if the determinant is zero, so to find nontrivial λ, you solve:

$$\det(\mathbf{M} - \lambda\mathbf{I}) = |\mathbf{M} - \lambda\mathbf{I}| = 0$$

Which could yield

$$\textbf{Solving: } |M - \lambda I| = 0 \quad \Big| \quad (M - \lambda_i I)\mathbf{e}_i = 0 \quad \Big| \quad \mathbf{X} = \sum_i \vec{c}_i e^{\lambda_i x}$$

$$\textbf{Normal Modes: } \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies \mathbf{X"} = M\mathbf{X}$$

$$\textbf{Solving: } |M - \omega^2 I| = 0$$

6.4 The Laplace Transform

A powerful method for solving IVPs using an integral transform. The general method is to transform a differential equation from the t domain to the s domain using the transform, where the equation becomes a simple algebra system. After solving for $Y(s)$, use the inverse transform to turn the obtained function into the complete solution. Using Partial Fraction Decomposition is often useful when solving these. Use tables.

$$\begin{aligned}
\mathcal{L}[f'](s) &= s\mathcal{L}[f](s) - f(0) \\
\mathcal{L}[f''](s) &= s^2\mathcal{L}[f](s) - sf(0) - f'(0) \\
\mathcal{L}[f'''](s) &= s^3\mathcal{L}[f](s) - s^2f(0) - sf'(0) - f''(0) \\
&\vdots
\end{aligned}$$

Laplace Convolution of two functions f, g is defined to be

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

If $\mathcal{L}[f](s) = F(s)$ & $\mathcal{L}[g](s) = G(s)$ exists, then $\mathcal{L}^{-1}[FG] = (f * g)$ and $\mathcal{L}[f * g](s) = FG$. This is useful for when we want to recover $h(t)$ from $H(s) = FG$ for a known FG.

6.5 Partial Fraction Decomposition

Useful for re-writing some of the results of a Laplace transform. It involves decomposing the denominator of some difficult fraction into multiple seperate fractions.

$$\frac{1}{x^2 - x + 1} = \frac{A}{}$$

6.6 Bernoulli Equations

Take the form: $y' + P(x)y = g(x)y^n$ for $n \in \mathbb{R}$ When $n \neq 0, 1$ solve by substituting $u = y^{1-n}$

$$y' + \frac{1}{x}y = xy^2 \xrightarrow{n=2\ \therefore\ u=y^{-1}\ \therefore\ y=u^{-1}} \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx}$$

$$-u^{-2}\frac{du}{dx} + \frac{1}{ux} = xu^{-2} \xrightarrow{rearrange} \frac{du}{dx} - \frac{1}{x}u = -x$$

$$\mu(x) = e^{-\int 1/x\ dx} = e^{-\ln(x)} = \frac{1}{x}$$

$$\int d(\mu(x)u) = \int \mu(x)g(x)dx\ ;\ \int d\left(\frac{u}{x}\right) = \int(-1)dx$$

$$u = -x^2 + c_1 x\ \therefore\ \boxed{y = \frac{1}{-x^2 + c_1 x}}$$

Cauchy-Euler

Also called The "Equidimensional" Equation. Assume solution of $y = x^m$ and plug in. Solve for m.

$$\underset{m_1 \neq m_2}{\textbf{Real Distinct Roots:}}\quad y = c_1 x^{m_1} + c_1 x^{m_2} \tag{6.8}$$

$$\underset{m_1 = m_2}{\textbf{Real Repeating Roots:}}\quad y = c_1 \ln(x)x^m + c_2 x^m \tag{6.9}$$

$$\underset{b \neq 0}{\textbf{Complex Roots:}}\quad y = c_1 x^\alpha \cos(\beta \ln(x)) + c_2 x^\alpha \sin(\beta \ln(x)) \tag{6.10}$$

6.7 PDEs in Physics

6.7.1 The Heat Equation

6.7.2 The Wave Equation

6.7.3 Laplace's Equation

6.7.4 Poisson's Equation

6.8 Nonlinear Differential Equations (NLDEs)

7 Fourier Analysis

7.1 Fourier Series

If a function f and it's derivative f' are both piece-wise continuous on the interval $[-L, L]$, then f can be written as a weighted sum of sines and cosines in a Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

If f is odd: $a_n = 0$ and if f is even: $b_n = 0$

7.2 Fourier Transform

This integral transform that transforms a (piece-wise continuous) function into it's component frequencies. Along frequency $\omega \equiv n\pi/l$

$$\mathcal{F}[f](\omega) \equiv \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

8 Special Functions

Special functions functions that are defined to have established names and properties because of their importance. Simple examples of special functions are the **sin and log** functions. Some of them are unintuitive, and the notation is often weird, but analysis becomes easier if you use them.

8.1 Bessel Functions

Bessel functions are defined to be solutions to Bessel's differential equation:

$$x^2y''(x) + xy'(x) + \left(\lambda x^2 - n^2\right)y(x) = 0 \quad \begin{cases} n \in \mathbb{N} \\ \lambda > 0 \\ x \geq 0 \end{cases} \tag{8.1}$$

$$y(x) = c_1 J_n(\sqrt{\lambda} x) \tag{8.2}$$

Where J_n is a Bessel function of the first kind.

8.2 Dirac Delta Function

The Dirac delta function $\delta(x)$ is a generalized distribution defined by Paul Dirac to have the following properties:

$$\delta(x) \equiv \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)dx & = 1 \end{cases}$$

The Dirac Delta Function picks out the value of the function f at a single point.

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \int_{-\infty}^{\infty} f(a)\delta(x-a)dx = f(a)$$

Note that the interval of integration only needs to contain a for this to be true.

Three-dimensional Dirac Delta Function

$$\delta^3(r) = \delta(x)\delta(y)\delta(z)$$

$$\int_{-\infty}^{\infty} f(r)\delta^3(r-a)dV = f(a)$$

A useful version of this, re-casted for use in electrodynamics:

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r})$$

8.3 Error Function

8.4 Gamma Function

8.5 Hermite Polynomials

8.6 Laguerre Polynomials

8.7 Legendre Polynomials

These are a set of complete and orthogonal polynomials which are handy in electrodynamics and quantum mechanics. They can be generated by the Rodriguez formula[3]:

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(x^2-1\right)^n\right] \tag{8.3}$$

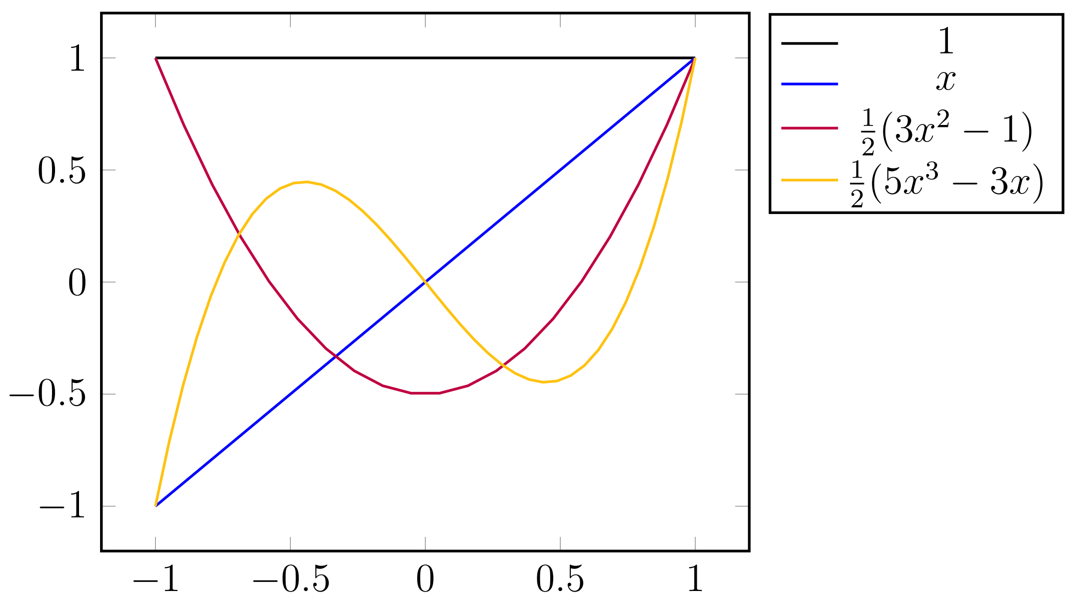

Figure 1: The First four Legendre Polynomials ($n = 0, 1, 2, 3$)

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[3] Mark A. Pinsky. *Partial Differential Equations and Boundary-Value Problems with Applications. I read an unreleased version published around 2003 that I found in a free book pile, but I'm citing the most recent version here.* American Mathematical Society, 2011. ISBN: 978-0821868898.