1 Differential Equations

Since this section is kind of big I made a seperate table of contents here. Differential equations are equalities made from functions and their derivatives. Ordinary differential equations (ODEs) only have 1-D derivatives, while partial differential equations (PDEs) have partial derivatives in multiple dimensions.

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1.1 Homogeneous ODEs

A function of the form $f(x, y, y'...) = 0$. If the equation as a whole is linear [1], the solution will be of the form $y = e^{mx}$ where m is some complex number ($m = a + bi$). Solutions start by plugging in this guess of $y = e^{mx}$ and finding m:

$$ay'' + by' + cy = 0$$

$$a\big(m^2 e^{mx}\big) + b(me^{mx}) + c(e^{mx}) = 0$$

$$\big(am^2 + bm + c\big)e^{mx} = 0$$

$$am^2 + bm + c = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve for m, since m is a complex root there are three cases:

$$\underset{a_1 \neq a_2 \text{ \& } b=0}{\textbf{Real Distinct Roots:}} \quad y_c = c_1 e^{a_1 x} + c_2 e^{a_2 x} \tag{1}$$

$$\underset{a_1 = a_2 \text{ \& } b=0}{\textbf{Real Repeating Roots:}} \quad y_c = c_1 e^{ax} + c_2 x e^{ax} \tag{2}$$

$$\underset{b \neq 0}{\textbf{Complex Roots:}} \quad y_c = e^{ax}\left(c_1 \cos(bx) + c_2 \sin(bx)\right) \tag{3}$$

1.2 Non-Homogeneous First order ODEs

Takes the form: $y' + P(x)y = g(x)$.

$$\mu(x) \equiv e^{\int P(x)dx}$$

$$\frac{d}{dx}(\mu(x)y) = \mu(x)g(x) \xrightarrow{Move\ dx,\ integrate} \mu(x)y + c_1 = \int \mu(x)g(x)dx$$

$$y(x) = \frac{\int \mu(x)g(x)dx - c_1}{\mu(x)}$$

[1] An example of a nonlinear differential equation could be $f(x, y, y') = \sin(y'(x)) + y^2(x) = 0$

Variation of Parameters

Not limited by non-constant coefficients.

$$y_p = y_c \int \frac{g(x)}{y_c} dx$$

1.3 Non-Homogeneous Higher order ODEs

Takes the form: $y'' + Q(x)y' + P(x)y = g(x)$

Variation of Parameters

The general idea is to replace the coefficients (c_1, c_2) with functions.

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

For the case of a second order, the relations of $u(x)$ are depicted below

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$

In summary: Find Wronskian (determinant represented by W). Find u_i'. Integrate to get u_i and plug in to y_p, then add to y_c.

Undetermined Coefficients

There are two subcategories of this method: The superposition approach and the annihilator approach. Superposition: Solve for the complementary function y_c (shown in the 'Homogeneous' section), then find a particular solution y_p.

1.4 System

Multiple differential equations that are related to each other. They are typically solved by putting them into matrices and using eigenvalues/eigenvectors.

$$\textbf{First Order:} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies \mathbf{X'} = M\mathbf{X}$$

$$\textbf{Solving:} \ |M - \lambda I| = 0 \quad \Big| \quad (M - \lambda_i I)\mathbf{e}_i = 0 \quad \Big| \quad \mathbf{X} = \sum_i \vec{c}_i e^{\lambda_i x}$$

$$\textbf{Normal Modes:} \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies \mathbf{X''} = M\mathbf{X}$$

$$\textbf{Solving:} \ |M - \omega^2 I| = 0$$

Initial Value Problem (IVP)

A differential equation that has an exact solution. After solving for the general case (c_i terms), plug in the initial conditions and solve for the constant coefficients.

1.5 Laplace Transform

A powerful method for solving IVPs using an integral transform. The general method is to transform a differential equation from the t domain to the s domain using the transform, where the equation becomes a simple algebra system. After solving for $Y(s)$, use the inverse transform to turn the obtained function into the complete solution. Using Partial Fraction Decomposition is often useful when solving these. Use tables.

$$\begin{aligned}
\mathcal{L}[f'](s) &= s\mathcal{L}[f](s) - f(0) \\
\mathcal{L}[f''](s) &= s^2\mathcal{L}[f](s) - sf(0) - f'(0) \\
\mathcal{L}[f'''](s) &= s^3\mathcal{L}[f](s) - s^2f(0) - sf'(0) - f''(0) \\
&\vdots
\end{aligned}$$

Laplace Convolution of two functions f, g is defined to be

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

If $\mathcal{L}[f](s) = F(s)$ & $\mathcal{L}[g](s) = G(s)$ exists, then $\mathcal{L}^{-1}[FG] = (f * g)$ and $\mathcal{L}[f * g](s) = FG$. This is useful for when we want to recover $h(t)$ from $H(s) = FG$ for a known FG.

1.6 Partial Fraction Decomposition

1.7 Bernoulli Equations: $y' + P(x)y = g(x)y^n$

Take the form: $y' + P(x)y = g(x)y^n$ for $n \in \mathbb{R}$ When $n \neq 0, 1$ solve by substituting $u = y^{1-n}$

$$y' + \frac{1}{x}y = xy^2 \xrightarrow{n=2 \therefore u=y^{-1} \therefore y=u^{-1}} \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx}$$

$$-u^{-2}\frac{du}{dx} + \frac{1}{ux} = xu^{-2} \xrightarrow{rearrange} \frac{du}{dx} - \frac{1}{x}u = -x$$

$$\mu(x) = e^{-\int 1/x \ dx} = e^{-\ln(x)} = \frac{1}{x}$$

$$\int d(\mu(x)u) = \int \mu(x)g(x)dx \ ; \ \int d\left(\frac{u}{x}\right) = \int (-1)dx$$

$$u = -x^2 + c_1 x \ \therefore \ \boxed{y = \frac{1}{-x^2 + c_1 x}}$$

Cauchy-Euler: $x^n y^{(n)} + ... + x^2 y'' + axy' + by = 0$

Assume solution of $y = x^m$ and plug in. Solve for m.

$$\underset{m_1 \neq m_2}{\textbf{Real Distinct Roots:}} \quad y = c_1 x^{m_1} + c_1 x^{m_2} \tag{4}$$

$$\underset{m_1 = m_2}{\textbf{Real Repeating Roots:}} \quad y = c_1 \ln(x) x^m + c_2 x^m \tag{5}$$

$$\underset{b \neq 0}{\textbf{Complex Roots:}} \quad y = c_1 x^\alpha \cos(\beta \ln(x)) + c_2 x^\alpha \sin(\beta \ln(x)) \tag{6}$$

1.8 Numerical

What types of numerical differential equations do you do again. . . ?