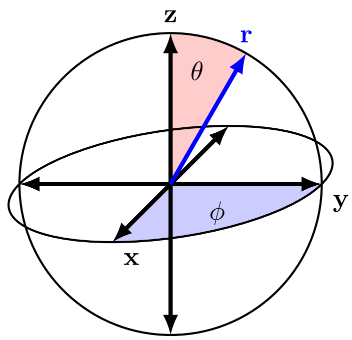
1 Curvilinear Coordinates

1.1 Spherical coordinates

In this system, the following describe the space's basis set:



Distance from the origin: r

Angle from z-axis, the polar angle: θ

Angle around z-axis, the azimuthal angle: ϕ

Important relationships:

$$\begin{bmatrix} \hat{\mathbf{e}}_\mathbf{r} \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\phi) \\ -\sin(\phi) & \cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{e}}_\mathbf{x} \\ \hat{\mathbf{e}}_\mathbf{y} \\ \hat{\mathbf{e}}_\mathbf{z} \end{bmatrix}$$

Matrix is orthogonal, transpose to find $\hat{\mathbf{e}}_\mathbf{x}$ in terms of $\hat{\mathbf{e}}_\mathbf{r}$

Position, velocity, and acceleration:

$$\mathbf{r}(t) = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_\mathbf{r} \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{bmatrix}$$

$$\mathbf{v}(t) = \begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ r\dot{\phi}\sin(\theta) \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_\mathbf{r} \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{bmatrix}$$

$$\mathbf{a}(t) = \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2(\theta) \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin(\theta)\cos(\theta) \\ 2r\dot{\theta}\dot{\phi}\cos(\theta) + 2r\dot{\phi}\sin(\theta) + r\ddot{\phi}\sin(\theta) \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_\mathbf{r} \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{bmatrix}$$

Infinitesimal Displacement:

$$d\mathbf{l} = dr\ \hat{\mathbf{e}}_{\mathbf{r}} + rd\theta\ \hat{\mathbf{e}}_{\theta} + r\sin(\theta)d\phi\ \hat{\mathbf{e}}_{\phi}$$

Infinitesimal Areas:

Held Constant	$d\mathbf{a}$
r	$r^2\sin(\theta)d\theta d\phi\ \hat{\mathbf{e}}_{\mathbf{r}}$
θ	$r\sin(\theta)drd\phi\ \hat{\mathbf{e}}_{\theta}$
ϕ	$rdrd\theta\ \hat{\mathbf{e}}_{\phi}$

Infinitesimal volume:

$$dV = r^2\sin(\theta)drd\theta d\phi$$

Spherical Vector Derivatives:

$$\text{Gradient:} \quad \nabla f = \begin{bmatrix} \partial_r f \\ \frac{1}{r}\partial_\theta f \\ \frac{1}{r\sin(\theta)}\partial_\phi f \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_{\mathbf{r}} \\ \hat{\mathbf{e}}_{\theta} \\ \hat{\mathbf{e}}_{\phi} \end{bmatrix}$$

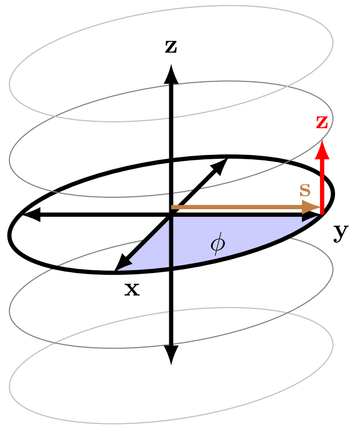
$$\text{Divergence:} \quad \nabla \cdot \mathbf{A} = \frac{1}{r^2}\partial_r\big(r^2 A_r\big) + \frac{1}{r\sin(\theta)}\partial_\theta(\sin(\theta)A_\theta) + \frac{1}{r\sin(\theta)}\partial_\phi A_\phi$$

$$\text{Curl:} \quad \nabla \times \mathbf{A} = \begin{bmatrix} \frac{1}{r\sin(\theta)}\big(\partial_\theta A_\phi \sin(\theta) - \partial_\phi A_\theta\big) \\ \frac{1}{r}\left(\frac{1}{\sin(\theta)}\partial_\phi A_r - \partial_r(rA_\phi)\right) \\ \frac{1}{r}(\partial_r(rA_\theta) - \partial_\theta A_r) \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_{\mathbf{r}} \\ \hat{\mathbf{e}}_{\theta} \\ \hat{\mathbf{e}}_{\phi} \end{bmatrix}$$

$$\text{Scalar Laplacian:} \quad \nabla^2 f = \frac{1}{r^2}\partial_r\big(r^2\partial_r f\big) + \frac{1}{r^2\sin(\theta)}\partial_\theta(\sin(\theta)\partial_\theta f) + \frac{1}{r^2\sin^2(\theta)}\partial_\phi^2 f$$

1.2 Cylindrical coordinates

In this system, the following describe the space's basis set:



Distance from the z-axis: s

Angle around x-axis, the azimuthal angle: ϕ

Distance on z-axis: z

Important relationships:

$$\begin{bmatrix} \hat{\mathbf{e}}_{\mathbf{s}} \\ \hat{\mathbf{e}}_{\phi} \\ \hat{\mathbf{e}}_{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{e}}_{\mathbf{x}} \\ \hat{\mathbf{e}}_{\mathbf{y}} \\ \hat{\mathbf{e}}_{\mathbf{z}} \end{bmatrix}$$

Matrix is orthogonal, transpose to find $\hat{\mathbf{e}}_{\mathbf{x}}$ in terms of $\hat{\mathbf{e}}_{\mathbf{s}}$

Position, velocity, and acceleration:

$$\mathbf{r}(t) = \begin{bmatrix} s \\ 0 \\ z \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_{\mathbf{s}} \\ \hat{\mathbf{e}}_{\phi} \\ \hat{\mathbf{e}}_{\mathbf{z}} \end{bmatrix} \qquad \mathbf{v}(t) = \begin{bmatrix} \dot{s} \\ s\dot{\phi} \\ \dot{z} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_{\mathbf{s}} \\ \hat{\mathbf{e}}_{\phi} \\ \hat{\mathbf{e}}_{\mathbf{z}} \end{bmatrix}$$

$$\mathbf{a}(t) = \begin{bmatrix} \ddot{s} - r\dot{\phi}^2 \\ s\ddot{\phi} + 2\dot{s}\dot{\phi} \\ \ddot{z} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_{\mathbf{s}} \\ \hat{\mathbf{e}}_{\phi} \\ \hat{\mathbf{e}}_{\mathbf{z}} \end{bmatrix}$$

Infinitesimal length:

$$d\mathbf{l} = ds\ \hat{\mathbf{e}}_{\mathbf{s}} + sd\phi\ \hat{\mathbf{e}}_{\phi} + dz\ \hat{\mathbf{e}}_{\mathbf{z}}$$

Infinitesimal Areas:

Held Constant	$d\mathbf{a}$
s	$sd\phi dz\ \hat{\mathbf{e}}_{\mathbf{s}}$
ϕ	$dsdz\ \hat{\mathbf{e}}_{\phi}$
z	$sdsd\phi\ \hat{\mathbf{e}}_{\mathbf{z}}$

Infinitesimal volume:

$$dV = sdsd\phi dz$$

Cylindrical Vector Derivatives:

$$\text{Gradient:} \quad \nabla F = \begin{bmatrix} \partial_s F \\ \\ \frac{1}{s}\partial_\phi F \\ mathlarger\partial_z F \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_\mathbf{s} \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_\mathbf{z} \end{bmatrix}$$

$$\text{Divergence:} \quad \boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{s}\partial_s(sA_s) + \frac{1}{s}\partial_\phi A_\phi + \partial_z A_z$$

$$\text{Curl:} \quad \boldsymbol{\nabla} \times \mathbf{A} = \begin{bmatrix} \frac{1}{s}\partial_\phi A_z - \partial_z A_\phi \\ \partial_z A_s - \partial_s A_z \\ \frac{1}{s}\partial_s(sA_\phi) - \frac{1}{s}\partial_\phi A_s \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_\mathbf{s} \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_\mathbf{z} \end{bmatrix}$$

$$\text{Scalar Laplacian:} \quad \nabla^2 F = \frac{1}{s}\partial_s(s\partial_s F) + \frac{1}{s^2}\partial_\phi^2 F + \partial_z^2 F$$