Mathematics Review

For Applications In Physics

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Notation/How I write things

- I try to stay as consistent as I can with variable and function letters; no one needs u *and* v for velocity. That being said, different fields do regularly use different notations, and it would be ridiculous of me to ignore them entirely, I'll try to include some "translations" for important things.

- Different types of derivative notation will be used when convenient (Euler, Leibniz, Newton), all of the following can be considered equivalent:

$$\frac{\mathrm{d}}{\mathrm{d}x}f = \frac{\mathrm{d}f}{\mathrm{d}x} = \mathrm{D}_x f = f'(x)$$
$$\frac{\partial}{\partial x}f = \frac{\partial f}{\partial x} = \partial_x f = f_x(x, y)$$

- Bold notation will be used over arrow notation for vectors.

$$\mathbf{A} = A_i = \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = A_r\hat{\boldsymbol{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$$

- A function f is linear if you can take out constants and add/subtract without changing the function:[1]

$$f(cx) = cf(x) \quad c \in \mathbb{R}$$
$$f(x \pm y) = f(x) \pm f(y)$$

[1]Examples of functions that aren't linear are: $f(x) = \sin(x)$, $f(x) = x^2$

1 Differential Calculus

Differentiation in One Dimension (1-D)

The derivative is the proportionality factor of how rapidly the function $f(x)$ varies when the argument x is changed by dx; f changes by an amount df:

$$df = \left(\frac{df}{dx}\right)dx$$

Multiplying and dividing functions in derivatives

$$f = f(x),\ \ g = g(x)$$

Product Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = (f')g + f(g') \tag{1.1}$$

Quotient Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f}{g}\right) = \frac{(f')g - f(g')}{(g)^2} \tag{1.2}$$

Differentiation in Three Dimensions (3-D)

For 3-variable functions:

$$df = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)dx + \left(\frac{\mathrm{d}f}{\mathrm{d}y}\right)dy + \left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)dz$$

The derivative of $f(x, y, z)$ tells one how f changes when one alters all three variables by dx, dy, dz.

Gradient

$$\boxed{\nabla f = \begin{bmatrix}\partial_x f\\ \partial_y f\\ \partial_z f\end{bmatrix} = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}} \tag{1.3}$$

The gradient of f is a vector field that assigns a vector to each point on f that **points in the direction of f's maximum increase**, moreover, the magnitude of ∇f gives the magnitude of each vector along this maximal direction.

Just like 1-D derivatives, you can find the extrema of a function with three variables by observing if at a stationary point (x, y, z):

$$\nabla f = 0$$

Gradients obey the following Product Rules:

$$\nabla(fg) = (\nabla f)g + f(\nabla g)$$
$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

Divergence

Divergence is a measure of how much a vector field spreads out from a point or volume. Similar to a dot product, it takes a vector to a number, **the divergence of a vector field is a scalar.**

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z} \tag{1.4}$$

Divergences obey the following Product Rules:

$$\boldsymbol{\nabla} \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$
$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

When the divergence of a vector field is zero everywhere it is called **solenoidal**. Any closed surface has no net flux across it in a solenoidal field.

Curl

Curl is a measure of how much a vector "swirls" around the point in question. One can find the curl conveniently as the determinant of the following matrix:

$$\boldsymbol{\nabla} \times \mathbf{A} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \boxed{\begin{bmatrix} (\partial_y A_z - \partial_z A_y) \\ (\partial_z A_x - \partial_x A_z) \\ (\partial_x A_y - \partial_y A_x) \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{bmatrix}} \tag{1.5}$$

Curls obey the following Product Rules:

$$\boldsymbol{\nabla} \times (f\mathbf{A}) = f(\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$
$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

The curl of a vector field is a vector field.[2] When the curl of a vector field is zero, the field is called **irrotational** and the field is conservative.

Laplacian

The laplace operator (denoted by ∇^2) is a kind of second derivative for scalars and vectors.[3] It can be thought of as taking whichever two vector derivatives are possible.

$$\nabla^2 f = \nabla \cdot (\nabla f) \tag{1.6}$$
$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \tag{1.7}$$

The following is also true for second derivatives based on the nature of first order vector derivatives:

$$\nabla \times (\nabla f) = 0 \tag{1.8}$$
$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{1.9}$$

[2] *Technically* a pseudo-vector field.

[3] Some people use Δ instead of ∇^2, but that seems goofy, so I don't use it.

2 Integral Calculus

Remember that $d\mathbf{l}$, $d\mathbf{a}$, and dV are different in different coordinate systems.[4]

$$\textbf{Curve Integral} \quad \int_C \mathbf{A} \cdot d\mathbf{l} = \iiint \mathbf{A} \cdot (dx\ \hat{\boldsymbol{x}} + dy\ \hat{\boldsymbol{y}} + dz\ \hat{\boldsymbol{z}}) \tag{2.1}$$

$$\textbf{Surface Integral} \quad \int_S \mathbf{A} \cdot d\mathbf{a} = \iint_D (\mathbf{A} \cdot \hat{\boldsymbol{z}}) dx dy = \iint_D A_k dx_i dx_j \tag{2.2}$$

$$\textbf{Volume Integral} \quad \int_{\mathcal{V}} \mathbf{A} dV = \int A_x dV\ \hat{\boldsymbol{x}} + \int A_y dV\ \hat{\boldsymbol{y}} + \int A_z dV\ \hat{\boldsymbol{z}} \tag{2.3}$$

The Fundamental Theorem for Gradients

Similar to the fundamental theorem of calculus, the curve integral of the gradient of a scalar function is equal to the difference of values of that scalar function at the endpoints.

$$\boxed{\int_C (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})} \tag{2.4}$$

Divergence Theorem

The divergence of $\mathbf{A}$ over a volume is equal to the components of $\mathbf{A}$ that are normal to the surface that bounds the volume.

$$\boxed{\oint_S \mathbf{A} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{A}) dV} \tag{2.5}$$

The Fundamental Theorem for Curls: Stokes' Theorem

The integral of a derivative over a region is equal to the value of the function at the boundary. That is, the curl over a surface is equal to the value of the function at the perimeter P.

$$\boxed{\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a}} \tag{2.6}$$

Integration by Parts in Vector Calculus

$$\int_{\mathcal{V}} f(\nabla \cdot \mathbf{A}) dV = \oint_S f\mathbf{A} \cdot d\mathbf{a} - \int_{\mathcal{V}} \mathbf{A} \cdot (\nabla f) dV$$

[4] $\oint$ is used to say that the thing you're integrating is either a closed path for a curve integral or a closed surface for a surface integral.

3 Theory of Vector Fields

The Helmholtz Theorem

A field is uniquely determined by its divergence and curl when boundary conditions are applied. For a vector field $\mathbf{A}$, if :

$$\left.\begin{aligned}\nabla \cdot \mathbf{A} &= \phi \\ \nabla \times \mathbf{A} &= \mathbf{C}\end{aligned}\right\} \implies \nabla \cdot \mathbf{C} = 0$$

Then $\mathbf{A}$ can be determined uniquely from ϕ and $\mathbf{C}$

Potentials

If the curl of a vector field $\mathbf{E}$ vanishes everywhere, then the field is conservative, meaning that the curve integral between any two points is path independent (so if the path is closed, the curve integral is zero) and by definition of conservative fields, $\mathbf{E}$ can be represented as the gradient of some scalar function V: [5]

$$\boxed{\nabla \times \mathbf{E} = 0} \iff \boxed{\oint_C \mathbf{E} \cdot d\mathbf{l} = 0} \iff \boxed{\mathbf{E} = -\nabla V}$$

If the divergence of a vector field, $\mathbf{B}$, vanishes everywhere, then the surface integral of $\mathbf{B}$ is independent of the surface for any given boundary line.

$$\boxed{\nabla \cdot \mathbf{B} = 0} \iff \boxed{\oint_S \mathbf{B} \cdot d\mathbf{s} = 0} \iff \boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

In general, the following is always true for a vector field $\mathbf{F}$:

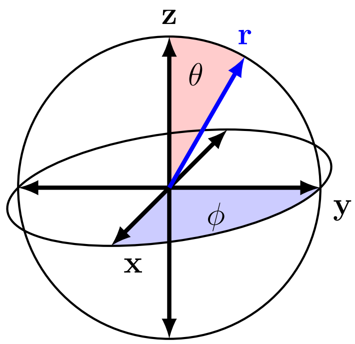
$$\mathbf{F} = -\nabla V + \boldsymbol{\nabla} \times \mathbf{A} \tag{3.1}$$

[5]The negative of the gradient is **used by convention** to make physics easier. Think about the gravitational force compared to gravitational potential, the force field points from from high (scalar) potential to low (scalar) potential.

4 Curvilinear Coordinates

4.1 Spherical Coordinates

In this system, the following describe the space's basis set:



Distance from the origin: r

Angle from z-axis, the polar angle: θ

Angle around z-axis, the azimuthal angle: ϕ

Important relationships:

$$\begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

Matrix is orthogonal, transpose to find $\hat{\boldsymbol{x}}$ in terms of $\hat{\boldsymbol{r}}$

Position, velocity, and acceleration:

$$\mathbf{r}(t) = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}$$

$$\mathbf{v}(t) = \begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ r\dot{\phi}\sin(\theta) \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}$$

$$\mathbf{a}(t) = \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2(\theta) \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin(\theta)\cos(\theta) \\ 2r\dot{\theta}\dot{\phi}\cos(\theta) + 2\dot{r}\dot{\phi}\sin(\theta) + r\ddot{\phi}\sin(\theta) \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}$$

Infinitesimal Displacement:

$$d\mathbf{l} = dr\ \hat{\boldsymbol{r}} + rd\theta\ \hat{\boldsymbol{\theta}} + r\sin(\theta)d\phi\ \hat{\boldsymbol{\phi}}$$

Infinitesimal Areas:

Held Constant	$d\mathbf{a}$
r	$r^2\sin(\theta)d\theta d\phi\ \hat{\boldsymbol{r}}$
θ	$r\sin(\theta)drd\phi\ \hat{\boldsymbol{\theta}}$
ϕ	$rdrd\theta\ \hat{\boldsymbol{\phi}}$

Infinitesimal volume:

$$dV = r^2\sin(\theta)drd\theta d\phi$$

Spherical Vector Derivatives:

Gradient: $$\nabla f = \begin{bmatrix} \partial_r f \\ \frac{1}{r}\partial_\theta f \\ \frac{1}{r\sin(\theta)}\partial_\phi f \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}$$

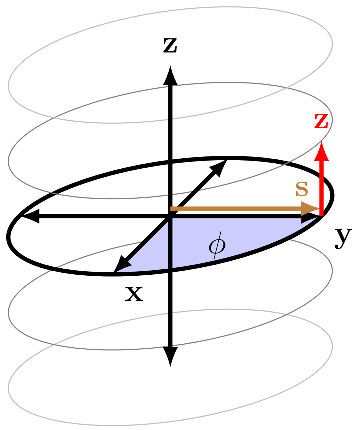
Divergence: $$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\partial_r\big(r^2 A_r\big) + \frac{1}{r\sin(\theta)}\partial_\theta(\sin(\theta)A_\theta) + \frac{1}{r\sin(\theta)}\partial_\phi A_\phi$$

Curl: $$\nabla \times \mathbf{A} = \begin{bmatrix} \dfrac{1}{r\sin(\theta)}\big(\partial_\theta A_\phi \sin(\theta) - \partial_\phi A_\theta\big) \\ \dfrac{1}{r}\left(\dfrac{1}{\sin(\theta)}\partial_\phi A_r - \partial_r(rA_\phi)\right) \\ \dfrac{1}{r}(\partial_r(rA_\theta) - \partial_\theta A_r) \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}$$

Scalar Laplacian: $$\nabla^2 f = \frac{1}{r^2}\partial_r\big(r^2\partial_r f\big) + \frac{1}{r^2\sin(\theta)}\partial_\theta(\sin(\theta)\partial_\theta f) + \frac{1}{r^2\sin^2(\theta)}\partial_\phi^2 f$$

4.2 Cylindrical Coordinates

In this system, the following describe the space's basis set:



Distance from the z-axis: s

Angle around x-axis, the azimuthal angle: ϕ

Distance on z-axis: z

Important relationships:

$$\begin{bmatrix} \hat{\boldsymbol{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

Matrix is orthogonal, transpose to find $\hat{\boldsymbol{x}}$ in terms of $\hat{\boldsymbol{s}}$

Position, velocity, and acceleration:

$$\mathbf{r}(t) = \begin{bmatrix} s \\ 0 \\ z \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

$$\mathbf{v}(t) = \begin{bmatrix} \dot{s} \\ s\dot{\phi} \\ \dot{z} \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

$$\mathbf{a}(t) = \begin{bmatrix} \ddot{s} - r\dot{\phi}^2 \\ s\ddot{\phi} + 2\dot{s}\dot{\phi} \\ \ddot{z} \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

Infinitesimal Length:

$$d\mathbf{l} = ds\ \hat{\boldsymbol{s}} + sd\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\boldsymbol{z}}$$

Infinitesimal Areas:

Held Constant	$d\mathbf{a}$
s	$sd\phi dz\ \hat{\boldsymbol{s}}$
ϕ	$dsdz\ \hat{\boldsymbol{\phi}}$
z	$sdsd\phi\ \hat{\boldsymbol{z}}$

Infinitesimal Volume:

$$dV = sdsd\phi dz$$

Cylindrical Vector Derivatives:

Gradient:
$$\nabla f = \begin{bmatrix} \partial_s f \\ \frac{1}{s}\partial_\phi f \\ \partial_z f \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

Divergence:
$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{s}\partial_s(sA_s) + \frac{1}{s}\partial_\phi A_\phi + \partial_z A_z$$

Curl:
$$\boldsymbol{\nabla} \times \mathbf{A} = \begin{bmatrix} \frac{1}{s}\partial_\phi A_z - \partial_z A_\phi \\ \partial_z A_s - \partial_s A_z \\ \frac{1}{s}\partial_s(sA_\phi) - \frac{1}{s}\partial_\phi A_s \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} \end{bmatrix}$$

Scalar Laplacian:
$$\nabla^2 f = \frac{1}{s}\partial_s(s\partial_s F) + \frac{1}{s^2}\partial_\phi^2 F + \partial_z^2 F$$

5 Differential Equations

Differential equations are equalities made from functions and their derivatives. Ordinary differential equations (ODEs) only have 1-D derivatives, while partial differential equations (PDEs) have partial derivatives in multiple dimensions. The order of a differential equation is the determined by the highest order derivative present in the differential equation. Since this is physics oriented, the highest order I'll need to write about is second order.

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5.1 First Order ODEs

5.1.1 Homogeneous

A function of the form $f(x, y, y'...) = 0$. If the equation as a whole is linear[6], the solution will be of the form $y = e^{mx}$ where m is some complex number ($m = a + bi$). Solutions start by plugging in this guess of $y = e^{mx}$ and finding m:

$$\begin{gathered}
ay'' + by' + cy = 0 \\
a\left(m^2 e^{mx}\right) + b(me^{mx}) + c(e^{mx}) = 0 \\
\left(am^2 + bm + c\right)e^{mx} = 0 \\
am^2 + bm + c = 0 \\
m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{gathered}$$

Solve for m, since m is a complex root there are three cases:

$$\underset{a_1 \neq a_2\ \&\ b=0}{\textbf{Real Distinct Roots:}} \quad y_c = c_1 e^{a_1 x} + c_2 e^{a_2 x} \tag{5.1}$$

$$\underset{a_1 = a_2\ \&\ b=0}{\textbf{Real Repeating Roots:}} \quad y_c = c_1 e^{ax} + c_2 x e^{ax} \tag{5.2}$$

$$\underset{b \neq 0}{\textbf{Complex Roots:}} \quad y_c = e^{ax}\left(c_1 \cos(bx) + c_2 \sin(bx)\right) \tag{5.3}$$

5.1.2 Non-Homogeneous

Takes the form: $y' + P(x)y = g(x)$.

$$\begin{gathered}
\mu(x) \equiv e^{\int P(x)dx} \\
\frac{d}{dx}(\mu(x)y) = \mu(x)g(x) \xrightarrow{Move\ dx,\ integrate} \mu(x)y + c_1 = \int \mu(x)g(x)dx \\
y(x) = \frac{\int \mu(x)g(x)dx - c_1}{\mu(x)}
\end{gathered}$$

Variation of Parameters

Not limited by non-constant coefficients.

$$y_p = y_c \int \frac{g(x)}{y_c} dx$$

[6] An example of a nonlinear differential equation could be $f(x, y, y') = \sin(y'(x)) + y^2(x) = 0$

5.1.3 Reduction of Order

5.2 Second Order ODEs

5.2.1 Homogeneous

5.2.2 Non-Homogeneous

Takes the form: $y'' + Q(x)y' + P(x)y = g(x)$

5.2.3 Variation of Parameters

The general idea is to replace the coefficients (c_1, c_2) with functions.

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

For the case of a second order, the relations of $u(x)$ are depicted below

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$

In summary: Find Wronskian (determinant represented by W). Find u_i'. Integrate to get u_i and plug in to y_p, then add to y_c.

5.2.4 Undetermined Coefficients

There are two subcategories of this method: The superposition approach and the annihilator approach.
Superposition: Solve for the complementary function y_c (shown in the 'Homogeneous' section), then find a particular solution y_p.

5.3 System of Linear Equations

Multiple differential equations that are related to each other. They are typically solved by putting them into matrices and using eigenvalues/eigenvectors.

$$\textbf{First Order: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies \mathbf{X'} = M\mathbf{X}$$

$$\textbf{Solving: } |M - \lambda I| = 0 \quad \Big| \quad (M - \lambda_i I)\mathbf{e}_i = 0 \quad \Big| \quad \mathbf{X} = \sum_i \vec{c}_i e^{\lambda_i x}$$

$$\textbf{Normal Modes: } \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies \mathbf{X''} = M\mathbf{X}$$

$$\textbf{Solving: } |M - \omega^2 I| = 0$$

5.4 The Laplace Transform

A powerful method for solving IVPs using an integral transform. The general method is to transform a differential equation from the t domain to the s domain using the transform, where the equation becomes a simple algebra system. After solving for $Y(s)$, use the inverse transform to turn the obtained function into the complete solution. Using Partial Fraction Decomposition is often useful when solving these. Use tables.

$$\begin{aligned}
\mathcal{L}[f'](s) &= s\mathcal{L}[f](s) - f(0) \\
\mathcal{L}[f''](s) &= s^2\mathcal{L}[f](s) - sf(0) - f'(0) \\
\mathcal{L}[f'''](s) &= s^3\mathcal{L}[f](s) - s^2f(0) - sf'(0) - f''(0) \\
&\vdots
\end{aligned}$$

Laplace Convolution of two functions f, g is defined to be

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

If $\mathcal{L}[f](s) = F(s)$ & $\mathcal{L}[g](s) = G(s)$ exists, then $\mathcal{L}^{-1}[FG] = (f * g)$ and $\mathcal{L}[f * g](s) = FG$. This is useful for when we want to recover $h(t)$ from $H(s) = FG$ for a known FG.

5.5 Partial Fraction Decomposition

Useful for re-writing some of the results of a Laplace transform. It involves decomposing the denominator of some difficult fraction into multiple seperate fractions.

$$\frac{1}{x^2 - x + 1} = \frac{A}{}$$

5.6 Bernoulli Equations: $y' + P(x)y = g(x)y^n$

Take the form: $y' + P(x)y = g(x)y^n$ for $n \in \mathbb{R}$ When $n \neq 0, 1$ solve by substituting $u = y^{1-n}$

$$y' + \frac{1}{x}y = xy^2 \xrightarrow{n=2 \ \therefore \ u=y^{-1} \ \therefore \ y=u^{-1}} \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx}$$
$$-u^{-2}\frac{du}{dx} + \frac{1}{ux} = xu^{-2} \xrightarrow{rearrange} \frac{du}{dx} - \frac{1}{x}u = -x$$
$$\mu(x) = e^{-\int 1/x \ dx} = e^{-\ln(x)} = \frac{1}{x}$$
$$\int d(\mu(x)u) = \int \mu(x)g(x)dx \ ; \ \int d\left(\frac{u}{x}\right) = \int (-1)dx$$
$$u = -x^2 + c_1x \ \therefore \ \boxed{y = \frac{1}{-x^2 + c_1x}}$$

Cauchy-Euler: $x^n y^{(n)} + ... + x^2y'' + axy' + by = 0$

Also called The “Equidimensional” Equation. Assume solution of $y = x^m$ and plug in. Solve for m.

$$\underset{m_1 \neq m_2}{\textbf{Real Distinct Roots:}} \quad y = c_1x^{m_1} + c_1x^{m_2} \tag{5.4}$$

$$\underset{m_1 = m_2}{\textbf{Real Repeating Roots:}} \quad y = c_1 \ln(x)x^m + c_2x^m \tag{5.5}$$

$$\underset{b \neq 0}{\textbf{Complex Roots:}} \quad y = c_1x^\alpha \cos(\beta \ln(x)) + c_2x^\alpha \sin(\beta \ln(x)) \tag{5.6}$$

5.7 Numerical

What types of numerical differential equations do you do again. . . ?

5.8 PDEs in Physics

5.8.1 The Heat Equation

5.8.2 The Wave Equation

5.8.3 Laplace’s Equation

5.8.4 Poisson’s Equation

6 Fourier Analysis

6.1 Fourier Series

If a function f and it's derivative f' are both piece-wise continuous on the interval $[-L, L]$, then f can be written as a weighted sum of sines and cosines in a Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right]$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

If f is odd: $a_n = 0$ and if f is even: $b_n = 0$

6.2 Fourier Transform

Along frequency $\omega \equiv n\pi/l$

$$\mathcal{F}[f](\omega) \equiv \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

7 Special Functions

Special functions functions that are defined to have established names and properties because of their importance. Simple examples of special functions are the **sin and log** functions. Some of them are unintuitive, and the notation is often weird, but analysis becomes easier if you use them.

7.1 Bessel Functions

Bessel functions are defined to be solutions to Bessel's differential equation:

$$x^2y''(x) + xy'(x) + \left(\lambda x^2 - n^2\right)y(x) = 0 \quad \begin{cases} n \in \mathbb{N} \\ \lambda > 0 \\ x \geq 0 \end{cases} \tag{7.1}$$

$$y(x) = c_1 J_n(\sqrt{\lambda}x) \tag{7.2}$$

Where J_n is a Bessel function of the first kind.

7.2 Dirac Delta Function

Given that the product $f(x)\delta(x)$ is zero everywhere except at x = 0:

$$\delta(x) \equiv \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)dx & = 1 \end{cases}$$

As a result of the definition:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

Three-dimensional Dirac Delta Function

The Dirac Delta Function picks out the value of the function f at the location of the space in question.

$$\delta^3(r) = \delta(x)\delta(y)\delta(z)$$

$$\int_{-\infty}^{\infty} f(r)\delta^3(r-a)dV = f(a)$$

A useful version of this, re-casted for use in electrodynamics:

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r})$$

7.3 Error Function

7.4 Gamma Function

7.5 Hermite Polynomials

7.6 Laguerre Polynomials

7.7 Legendre Polynomials

They are solutions to *Legendre's Differential Equation*:

$$\left(1 - x^2\right)y''(x) - 2xy'(x) + n(n+1)y(x) = 0 \quad \begin{cases} n \in \mathbb{N} \\ -1 \leq x \leq 1 \end{cases} \tag{7.3}$$

They can also be generated by the Rodriguez formula, a much easier way to find Legendre Polynomials:[**pinsky˙2011**]

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[\left(x^2 - 1\right)^n\right] \tag{7.4}$$

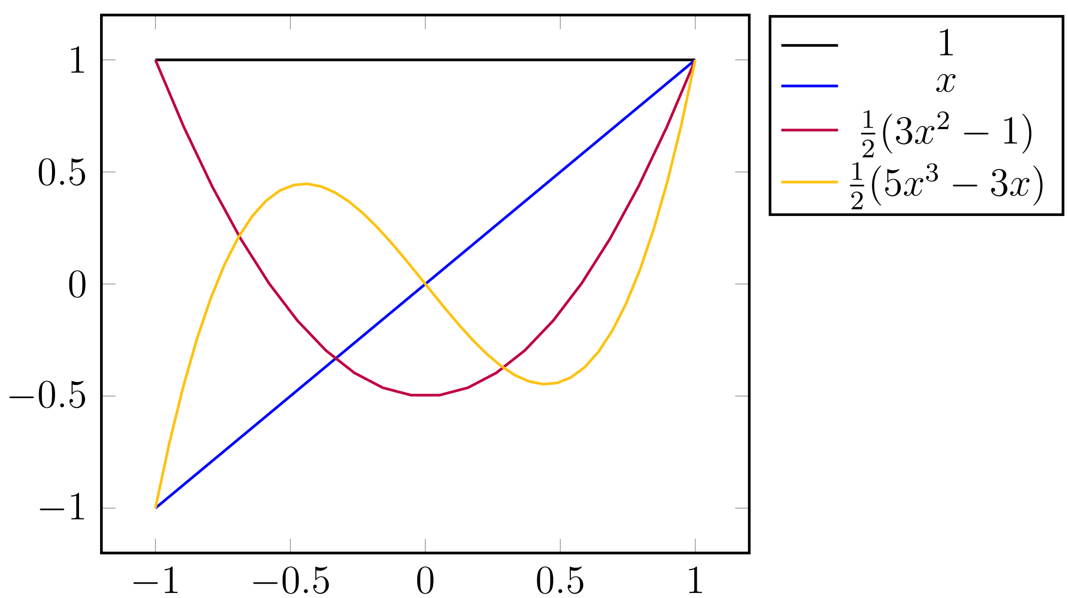

Figure 1: The First Five Legendre Polynomials ($n = 0, 1, 2, 3, 4$)