Postage Stamp Problem

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1 Introduction

This is an annotation of the proof of the postage stamp problem presented in CS70 Note 3 as Theorem 3.6. The proof is copied straight from the note, but the annotations are added from information provided by Steven Bi on Piazza, as well as some of my own intuition. This problem serves as an example of proof by strong induction with multiple base cases.

2 Proof

Theorem 3.6. *For every natural number $n \geq 12$, it holds that $n = 4x + 5y$ for some $x, y \in \mathbb{N}$.*

Proof. We proceed by induction on n.

Base Case ($n = 12$): We have $12 = 4 \cdot 3 + 5 \cdot 0$.

Base Case ($n = 13$): We have $13 = 4 \cdot 2 + 5 \cdot 1$.

Base Case ($n = 14$): We have $14 = 4 \cdot 1 + 5 \cdot 2$.

Base Case ($n = 15$): We have $15 = 4 \cdot 0 + 5 \cdot 3$.

We will see why there are multiple base cases, and why these are the chosen ones, soon. But first we must understand the inductive hypothesis.

Inductive Hypothesis: Assume that the claim holds for all $12 \leq n \leq k$ for $k \geq 15$.

This is where we start seeing strong induction take place. This is the difference between simple induction and strong induction.

Simple Induction
$(P(0) \wedge (\forall k : P(k) \Rightarrow P(k+1))) \Rightarrow (\forall n \in \mathbb{N} : P(n))$
Given a base case $P(0)$, assume a single $P(k)$ to be true to prove that $P(k+1)$ is true. The important part here is that simple induction only allows you to assume the inductive hypothesis for a single k.

Strong Induction
$(P(0) \wedge (\forall k : (\forall x \leq k : P(x)) \Rightarrow P(k+1))) \Rightarrow (\forall n \in \mathbb{N} : P(n))$
Given a base case $P(0)$ we are allowed to assume $P(x)$ for all $x \leq k$. This introduces another variable x to enumerate all the values that we can assume P for so long as those values are less than k. This is different from simple induction where you could only assume one k. In strong induction assuming P for all the values less than k allows us to prove $P(k+1)$.

Now, let's go back to our inductive hypothesis (Assume that the claim holds for all $12 \leq n \leq k$ for $k \geq 15$). Instead of x used in the example of strong induction, we are using n to enumerate all the values between 12 and k for which we can assume P to be true. Note that $k \geq 15$ because of the base cases which we will get to soon.

Inductive Step: We prove the claim for $n = k+1 \geq 16$. You get this by adding 1 to both sides of $k \geq 15$. In any induction we are always trying to prove the claim for some $n = k+1$ by using the inductive hypothesis. In this case we have actually proved the claim manually up to and including $n = 15$.

$$\begin{aligned} k+1 &\geq 16 \\ (k+1)-4 &\geq 16-4 \\ (k+1)-4 &\geq 12 \end{aligned}$$

We have now arrived at the inductive hypothesis, because $(k+1)-4 \geq 12$ and we can do some algebraic manipulation.

$$\begin{aligned} (k+1)-4 = 4x' + 5y' &\text{ for some } x', y' \in \mathbb{N} \\ (k+1) = 4x' + 5y' + 4 \\ (k+1) = 4(x'+1) + 5y' \\ x = x'+1,\, y = y' \end{aligned}$$

The proof is done because we have shown that $k+1 \geq 12$ can be written in the form of some $4x+5y$ for some $x, y \in \mathbb{N}$ by assuming that all n between 12 and k for $k \geq 15$ can be written in the same form. Now we can finally talk about those base cases.

3 Multiple Base Cases

Using 12 is obvious because the theorem claims "For every natural number $n \geq 12$". Imagine we stopped there and said that we prove the claim for $n = k + 1 \geq 12$. This is the same as saying $P \Rightarrow P \equiv \neg P \vee P$ which is a tautology. So we've established that we need more base cases, but which ones? You could have ∞ base cases but then you're not using strong induction or any induction for that matter. In this particular example, all multiples of 4 have some $x > 0 \in \mathbb{Z}$ and $y = 0$. This is covered by the case $n = 12$. However, every 4 values, you reach a multiple of 4 so only 4 base cases are needed before you can start invoking the inductive hypothesis. So, let's think about $n = 16$. This wasn't a base case because $n = 16 - 4 = 12$ is a base case that shows how a multiple of 4 can be represented in the right form. Same goes for $n = 20$. $n = (20 - 4) - 4 = 12$, so 20 is only 2 4's away from the base case 12. In other words $P(12) \Rightarrow P(16) \Rightarrow P(20) \Rightarrow ....$ So, if you find yourself at $n = 20$ without any information about $n = 16$, you can trace yourself back by subtracting 4 until you hit the base case of $n = 12$. We can repeat this process for any number, say $n = 21$. $n = (21 - 4) = 17$, okay.. doesn't tell us much, $n = (17 - 4) = 13$, another base case! That's because $P(13) \Rightarrow P(17) \Rightarrow P(21) \Rightarrow ....$ So only 4 base cases are necessary to prove the claim for all $n \geq 12$.