How Lagrange Interpolation Works

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April 2nd, 2017

1 Introduction

This guide is meant to serve as a supplement to Note 8 of CS70. It provides some more explicit explanations on why Lagrange Interpolation works because I didn't understand it the first time I read it. Credit goes to the authors of the original Note 8, as most of this material is a rehash.

2 Lagrange Interpolation

Given $d+1$ points, we can find the unique degree d polynomial that fits all the $d+1$ points.

3 Proofs

Correctness First let's find a polynomial such that $y_1 = 1$ and $y_j = 0$ for $2 \leq j \leq d+1$. This roughly translates to, "$y_1 = 1$ for a single j, and $y = 0$ for the rest". Let's take a look at $q(x)$ and see if it does what we want.

$$q(x) = (x - x_2)(x - x_3)...(x - x_{d+1})$$

So, if we are given $x_1....x_{d+1}$ we know that only x_1 will evaluate to a non-zero number. We also know that $q(x)$ is of degree d since it has exactly d x's which get multiplied together. So, what is $q(x_1)$?

$$q(x_1) = (x_1 - x_2)(x_1 - x_2)...(x_1 - x_{d+1})$$

Again, all we know about $q(x_1)$ is that it is some non-zero number. Now, our conditions in the beginning said that we need to find $p(x)$ such that $y_1 = 1$ and $y_j = 0$ for $2 \leq j \leq d+1$. So, if we let $p(x) = \frac{q(x)}{q(x_1)}$, we get that $p(x_1) = 1$. This is because $q(x_1)$ is evaluated to some number already, and applying $q(x)$ to x_1 will give us $q(x_1)$. Since the numerator and the denominator of the fraction are equal, we get $p(x) = 1$.

Here's an example where we are given $(1, 1)$, $(2, 0)$, $(3, 0)$ in order to find $d = 2$ polynomial $p(x)$.

$$\begin{aligned}
q(x) &= (x-2)(x-3) \\
q(1) &= (1-2)(1-3) \\
p(x) &= \frac{q(x)}{q(1)} \\
p(x) &= \frac{x^2 - 5x + 6}{2} \\
p(1) &= \frac{1^2 - 5(1) + 6}{2} \\
p(1) &= \frac{2}{2} \\
p(1) &= 1
\end{aligned}$$

Now that we have done found $p(x)$ for $y_1 = 1$ and $y_j = 0$ for $2 \leq j \leq d+1$, we can do the same for an arbitrary y_i and y_j where $i \neq j$. We can use the following notation:

$$\Delta_i = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

This function isn't as confusing as it looks. The numerator is simply $q(x)$ like we defined above. Instead of choosing $j \neq 1$ we are choosing some arbitrary i instead. The denominator is simply $q(x_1)$, evaluate $q(x)$ at the x_i you omitted in $q(x)$, or the numerator. Instead of $q(1)$ we are choosing some arbitrary $q(x_i)$.

This is where the magic happens. We now know how to construct $\Delta_1(x), ..., \Delta_{d+1}(x)$. If we add them all up we certainly have a polynomial of degree d, but we need for them to evaluate to y_i given x_i. So, the point of our delta functions $\Delta_i(x)$, is to return 1 given x_i. How can we exploit this fact to have it return y_i given x_i? Simply multiply $y_i\Delta_i(x)$!

$$p(x) = \sum_{i=1}^{d+1} y_i \Delta_i(x)$$

If we expand this summation to look like this:

$$p(x) = y_1 \Delta_1(x_1) + ...y_2 \Delta_2(x_2) + ...y_{d+1} \Delta_{d+1}(x_{d+1})$$

We can see that given any x_i, exactly d terms evaluates to 0 and the $i-th$ term evaluates to y_i times 1, since $\Delta_i(x_i)$ always returns 1 and $\Delta_{i \neq j}(x_{i \neq j})$ always returns 0.