Computational Statistics HW 4

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Problem 1

1

At time step n we suppose there are i balls in the left bin, ie. $X_n = i$. Then at time step $n+1$ we can have $X = i+1$ or $X = i-1$.

$$\begin{aligned}
P(X_{n+1} = i+1 | X_{n-1} = i) &= P(\text{a ball on the left side is chosen}) \\
&= P(\bigcup_{k=1}^{i} \text{Ball k Chosen}) \\
&= \sum_{k=1}^{m} \frac{1}{m} \\
&= \frac{i}{m} \\
\implies P(X_{n+1} = i-1) = 1 - \frac{i}{m}
\end{aligned}$$

From this we can see that the dependency of $n+1$ is strictly upon the state of X_n and therefore X_n is a Markov chain, with a state space $\mathcal{X} = \{0, 1, 2, \dots, m\}$

2

The transition probabilities $p_{i+1,i} = P(X_n = i | X_{n-1} = i+1) = \frac{m-i+1}{m}$, $p_{i,i-1} = P(X_n = i | X_{n-1} = i+1) = \frac{i-1}{m}$ and 0 otherwise.

3

We write our transition matrix P as

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1/m & 0 & (m-1)/m & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$\pi(i)$ in vector notation is,

$$\begin{pmatrix} 1/2^m \binom{m}{0} & 1/2^m \binom{m}{1} & \dots & 1/2^m \binom{m}{m} \end{pmatrix}$$

Then each entry of the vector $P^T \pi^T$ can be written as,

$$(P^T \pi^T)_i = \sum_{j=0}^{m} P^T_{i,j} \pi(j) = P^T_{i,i-1} \pi(i-1) + P^T_{i,i+1} \pi(i+1)$$

Where $i+1$ and $i-1$ are defined, else those terms are 0. Then we have for $i = 0$,

$$(P^T \pi^T)_0 = P^T_{0,1} \pi(1) = \frac{1}{2^m} \left(\frac{1}{m} \frac{m!}{(m-1)!}\right) = \frac{1}{2^m} \binom{m}{0} = (P\pi)_0$$

Similarly for $i = m$

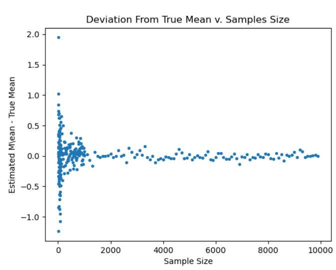
$$(P^T \pi^T)_m = P^T_{m,m-1} \pi(m-1) = \frac{1}{2^m} \left(\frac{1}{m} \frac{m!}{(m-1)!}\right) = \frac{1}{2^m} \binom{m}{m} = (P\pi)_m$$

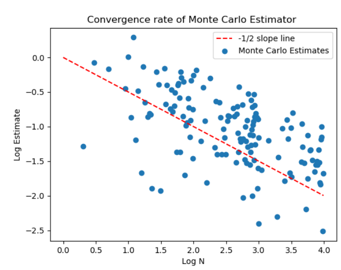
And finally for $0 < i < m$ we get,

$$(P^T \pi^T)_i = \frac{1}{2^m} \frac{i-1}{m} \binom{m}{i-1} + \frac{m-i+1}{m} \binom{m}{i+1} = \frac{1}{2^m} \binom{m}{i} = (\pi P)_i$$

Therefore π is the stationary distribution.

Problem 2



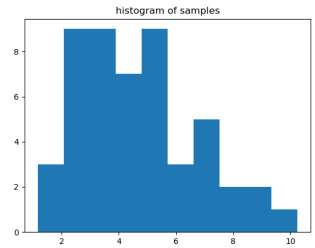


The expectation of this random variable is just the expectation of the expectation which is the expectation of $Exp(1)$ which is 1.

Problem 3

1

We generate 50 samples and construct a histogram.



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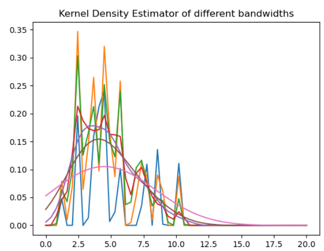

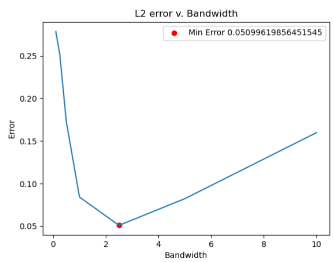

Then we use our h's to construct kernel density estimators and plot them and their errors.

3

Finally we compute the optimized error which is approximately 3.33.