claudius: analytic computations of scattering
calculs analytiques pour la diffusion

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Abstract

We solve scattering problem for cases where we have analytical expressions. All cavities consider will be invariant by rotation. The shape consider are disk, annulus, ball and spherical shell where the permittivity and permeability are radial function.

Contents

1 General settings

We consider the scattering by spherical cavities, disk or annulus in dimension 2, and ball or spherical shell in dimension 3. We call $|\cdot|_d$ the euclidean norm in dimension $d = 2, 3$. Disk and ball are denoted by $\mathbb{B}^d_\rho = \{\boldsymbol{x} \in \mathbb{R}^d \mid |\boldsymbol{x}|_d < \rho\}$ for $\rho > 0$, and annulus and spherical shell are denoted by $\mathbb{A}^d_{\rho,\sigma} = \{\boldsymbol{x} \in \mathbb{R}^d \mid \rho < |\boldsymbol{x}|_d < \sigma\}$ for $0 < \rho < \sigma$. The cavities will be compose of concentric spherical shell with impenetrable or penetrable core and denoted by Ω. For a cavity with impenetrable core and $N \geq 0$ layers, we have $\Omega = \mathbb{A}^d_{\rho_1,\rho_2} \cup \mathbb{A}^d_{\rho_2,\rho_3} \cup \cdots \cup \mathbb{A}^d_{\rho_N,\rho_{N+1}}$ with $0 < \rho_1 < \rho_2 < \cdots < \rho_{N+1}$. For a cavity with penetrable core and $N \geq 0$ layers, we have $\Omega = \mathbb{B}^d_{\rho_1} \cup \mathbb{A}^d_{\rho_1,\rho_2} \cup \cdots \cup \mathbb{A}^d_{\rho_N,\rho_{N+1}}$ with $0 = \rho_0 < \rho_1 < \cdots < \rho_{N+1}$. The boundary/interfaces Γ of the different layers of the cavity Ω is compose of concentric spheres $\Gamma = \rho_1\mathbb{S}^{d-1} \cup \rho_2\mathbb{S}^{d-1} \cup \cdots \cup \rho_N\mathbb{S}^{d-1}$.

We define the function $\varepsilon \in \mathrm{L}^\infty(\mathbb{R}^d)$ and $\mu \in \mathrm{L}^\infty(\mathbb{R}^d)$ as

$$\varepsilon(\boldsymbol{x}) = \begin{cases} \varepsilon_{\mathsf{c}}\left(|\boldsymbol{x}|_d\right) & \text{if } \boldsymbol{x} \in \Omega \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mu(\boldsymbol{x}) = \begin{cases} \mu_{\mathsf{c}}\left(|\boldsymbol{x}|_d\right) & \text{if } \boldsymbol{x} \in \Omega \\ 1 & \text{otherwise} \end{cases}$$

where

$$\varepsilon_{\mathsf{c}} = \sum_{n=0}^{N} \varepsilon_n \, \mathbf{1}_{(\rho_n,\rho_{n+1})} \qquad \text{and} \qquad \mu_{\mathsf{c}} = \sum_{n=0}^{N} \mu_n \, \mathbf{1}_{(\rho_n,\rho_{n+1})}$$

with $\varepsilon_n, \mu_n \in \mathscr{C}^\infty([\rho_n, \rho_{n+1}], \mathbb{R}^*)$ for $0 \leq n \leq N$. For the impenetrable case the index $n = 0$ is not use. In some context the function can be view as the permittivity (ε) and the permeability (μ) of a material.

A positive wavenumber is noted k. We only consider a plane wave incident field, we have $u^{\mathsf{in}} : \boldsymbol{x} \mapsto \exp(\mathrm{i}\, k\, \boldsymbol{\nu} \cdot \boldsymbol{x})$ with $\boldsymbol{\nu} \in \mathbb{S}^{d-1}$ the direction of the plane wave. In the following, the scattered field is noted u^{sc} and the total field is noted u.

2 Helmholtz's equation

2.1 The scattering problem

We consider the operator $u \mapsto -\mu^{-1} \operatorname{div}(\varepsilon^{-1} \nabla u)$ of domain

$$D^{d,\varepsilon} := \{u \in \mathrm{L}^2(\mathbb{R}^d) \mid \operatorname{div}(\varepsilon^{-1} \nabla u) \in \mathrm{L}^2(\mathbb{R}^d)\}$$

and we define the "locale" version

$$D^{d,\varepsilon}_{\mathrm{loc}} := \{u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d) \mid \forall \chi \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}^d),\ \chi u \in D^{d,\varepsilon}\}.$$

We define the following scattering problem: Given a wavenumber $k > 0$ and an incident field u^{in}, find the scattering field $u^{\mathsf{sc}} \in D^{d,\varepsilon}_{\mathrm{loc}}$ such that the total field $u = u^{\mathsf{in}} + u^{\mathsf{sc}}$ satisfy

$$(\mathcal{D}/\mathcal{N}) \begin{cases} -\mu^{-1} \operatorname{div}\left(\varepsilon^{-1} \nabla u\right) - k^2 u = 0 & \text{in } \Omega \text{ and } \mathbb{R}^2 \setminus \overline{\Omega} \\ u = 0 \quad \text{or} \quad \partial_{\boldsymbol{n}} u = 0 & \text{on } \rho_1 \mathbb{S}^{d-1} \\ [u] = 0 \text{ and } \left[\varepsilon^{-1} \partial_{\boldsymbol{n}} u\right] = 0 & \text{across } \rho_2 \mathbb{S}^{d-1} \cup \cdots \cup \rho_N \mathbb{S}^{d-1} \\ u^{\mathsf{sc}} \text{ is } k\text{-outgoing} \end{cases} \tag{1}$$

in the Dirichlet ($\mathcal{D}$) or Neumann ($\mathcal{N}$) case, or

$$(\mathcal{P}) \begin{cases} -\mu^{-1}\operatorname{div}\left(\varepsilon^{-1}\nabla u\right) - k^2 u = 0 & \text{in } \Omega \text{ and } \mathbb{R}^2 \setminus \overline{\Omega} \\ [u] = 0 \text{ and } \left[\varepsilon^{-1}\partial_{\boldsymbol{n}} u\right] = 0 & \text{across } \rho_1 \mathbb{S}^{d-1} \cup \cdots \cup \rho_N \mathbb{S}^{d-1} \\ u^{\mathsf{sc}} \text{ is } k\text{-outgoing} \end{cases} \tag{2}$$

in the penetrable case, where $\boldsymbol{n} : \Gamma \to \mathbb{S}^{d-1}$ are the outward unit normal and u^{sc} is k-outgoing mean that for $|\boldsymbol{x}| \geq \rho_N$, there exist β such that

$$u^{\mathsf{sc}}(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} \beta_m \ \mathsf{H}_m^{(1)}(k\,r)\, \mathsf{e}^{\mathsf{i}\,m\,\theta} \qquad d = 2 \tag{3a}$$

$$u^{\mathsf{sc}}(\boldsymbol{x}) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} \beta_\ell^m \ \mathsf{h}_\ell^{(1)}(k\,r)\, Y_\ell^m(\theta,\phi) \qquad d = 3 \tag{3b}$$

with $(r,\theta) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ the polar coordinate, $(r,\theta,\phi) \in \mathbb{R}_+ \times [0,\pi] \times \mathbb{R}/2\pi\mathbb{Z}$ the spherical coordinate, Y_ℓ^m the spherical harmonic, and $z \mapsto \mathsf{H}_m^{(1)}(z)$ (resp. $z \mapsto \mathsf{h}_m^{(1)}(z)$) is the cylindrical (resp. spherical) Hankel function of the first kind of order m.

2.2 The 1d reduction

3 Maxwell's equation

A Miscellaneous

A.1 Coordinates

A.1.1 Polar

In dimension two, the Cartesian coordinate $\boldsymbol{x} = (x,y) \in \mathbb{R}^2$ are describe in term of polar coordinate $(r,\theta) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ by

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arg(x+\mathsf{i}y) \end{cases}$$

A.1.2 Spherical

In dimension three, the Cartesian coordinate $\boldsymbol{x} = (x,y,z) \in \mathbb{R}^3$ are describe in term of spherical coordinate $(r,\theta,\phi) \in \mathbb{R}_+ \times [0,\pi] \times \mathbb{R}/2\pi\mathbb{Z}$ by

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arccos\left(\dfrac{z}{r}\right) \\ \phi = \arg(x+\mathsf{i}y) \end{cases} .$$

A.2 Bessel's functions

A.2.1 Cylindrical

The cylindrical Bessel's function J_m, Y_m, $\mathsf{H}_m^{(1)}$, and $\mathsf{H}_m^{(2)}$ are define in [DLMF, Sec. 10.2] and they are solutions of the ODE

$$-\frac{1}{z}\left(z\,w'\right)' + \frac{m^2}{z^2} w - w = 0.$$

The cylindrical modified Bessel's function I_m and K_m are define in [DLMF, Sec. 10.25] and they are solutions of the ODE

$$-\frac{1}{z}\,(z\,w')' + \frac{m^2}{z^2}w + w = 0.$$

A.2.2 Spherical

The spherical Bessel's function j_m, y_m, $\mathsf{h}_m^{(1)}$, $\mathsf{h}_m^{(2)}$ $(-)$ and i_m, k_m $(+)$ are define in [DLMF, Sec. 10.47] and they are solutions of the ODE

$$-\frac{1}{z^2}\left(z^2\,w'\right)' + \frac{\ell(\ell+1)}{z^2}w \mp w = 0$$

We have the following relation between the cylindrical and spherical Bessel function

$$\mathsf{f}_\ell(z) = \sqrt{\frac{\pi}{2z}}\mathsf{F}_{\ell+\frac{1}{2}}(z)$$

where the couples (f,F) are (j,J), (y,Y), $(\mathsf{h}^{(1)},\mathsf{H}^{(1)})$, $(\mathsf{h}^{(2)},\mathsf{H}^{(2)})$, (i,I), and (k,K).

A.3 Spherical harmonic

For $\ell \in \mathbb{N}$ and $m \in \{-\ell,\dots,0,\dots,\ell\}$, the spherical harmonic Y_ℓ^m in the spherical coordinate is define by

$$Y_\ell^m(\theta,\phi) = (-1)^m\sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos(\theta))\,\mathsf{e}^{\mathsf{i}\,m\,\phi}$$

where P_ℓ^m are the associated Legendre polynomial define by

$$P_\ell^m(x) = \frac{1}{2^\ell\,\ell!}\,(1-x^2)^{\frac{m}{2}}\,\partial_x^{\ell+m}(x^2-1)^\ell.$$

A.4 Spherical wave expansion of plane wave

The Jacobi-Anger expansion [DLMF, Eq. 10.12.1] states that, for $z \in \mathbb{C}$ and $t \in \mathbb{C}^*$,

$$\mathsf{e}^{\frac{z}{2}(t-t^{-1})} = \sum_{m\in\mathbb{Z}} \mathsf{J}_m(z)\,t^m. \tag{4}$$

Which give the following expansion for a 2d plane wave

$$\mathsf{e}^{\mathsf{i}\,k\,y} = \sum_{m\in\mathbb{Z}} \mathsf{J}_m(k\,r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta} \tag{5}$$

where (x,y) are the Cartesian coordinates and (r,θ) the corresponding polar coordinates.

We have the following expansion for a 3d plane wave

$$\mathsf{e}^{\mathsf{i}\,k\,\boldsymbol{\nu}\cdot\boldsymbol{x}} = 4\pi\sum_{\ell=0}^{+\infty}\mathsf{i}^\ell\ \mathsf{j}_\ell(k\,r)\sum_{m=-\ell}^{\ell}Y_\ell^m(\boldsymbol{\nu})\overline{Y_\ell^m(\omega)} \tag{6}$$

where (x,y) are the Cartesian coordinates and (r,θ) the corresponding polar coordinates.

References

[DLMF] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.28 of 2020-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

B The scattering problems

B.1 1D reduction

We look for solution of problem of the form

$$u(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} w_m(r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta} \qquad \text{where } w_m(r) \coloneqq \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\,\mathsf{e}^{-\mathsf{i}\,m\,\theta}\,\mathrm{d}\theta.$$

Similarly we write $u^{\mathsf{in}}(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} w^{\mathsf{in}}_m(r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta}$ and $u^{\mathsf{sc}}(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} w^{\mathsf{sc}}_m(r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta}$. The series in equation (4) converges absolutely on every compact set of $\mathbb{R}^2$.

The domain of the operator $w \mapsto -r^{-1}\,\mu^{-1}\partial_r(r\,\varepsilon^{-1}\partial_r w) + m^2\,r^{-2}\,\varepsilon^{-1}\mu^{-1}w$ and its "loc" version are define by

$$\begin{aligned}
\mathcal{D}^{1,m} &\coloneqq \{w \in \mathrm{L}^2(\mathbb{R}^*_+, r\,\mathrm{d}r) \mid \partial_r(r\,\varepsilon^{-1}\,\partial_r w) - m^2\,r^{-1}\varepsilon^{-1}w \in \mathrm{L}^2(\mathbb{R}^*_+)\}\\
\mathcal{D}^{1,m}_{\mathrm{loc}} &\coloneqq \{w \in \mathrm{L}^2(\mathbb{R}^*_+, r\,\mathrm{d}r) \mid \forall\chi \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}),\ \chi w \in \mathcal{D}^{1,m}\}.
\end{aligned}$$

Problem reduce to a family of problem index by $m\in\mathbb{Z}$: Given $k>0$ find $w^{\mathsf{sc}}_m \in \mathcal{D}^{1,m}_{\mathrm{loc}}$ such that $w_m = w^{\mathsf{in}}_m + w^{\mathsf{sc}}_m$ and

$$\begin{cases}
-\dfrac{1}{r\,\mu}\partial_r\left(\dfrac{r}{\varepsilon}\,\partial_r w_m\right) + \dfrac{m^2}{r^2\,\varepsilon\mu}w_m - k^2\,w_m = 0 & \text{in } I \text{ and } \mathbb{R}^*\setminus\overline{I}\\
w_0'(0) = 0 & \text{on } \{0\}\\
[w_m]_{\{1\}} = 0 \quad\text{and}\quad \left[\varepsilon^{-1}\,w_m'\right]_{\{1\}} = 0 & \text{across } \{1\}\\
w^{\mathsf{sc}}_m(r) \propto \mathsf{H}^{(1)}_m(k\,r) & r \geq 1
\end{cases} \tag{7}$$

with $\propto$ meaning "up to a multiplicative constant".

C The resonances problem

C.1 Problem statement

C.2 1D reduction

D Disk cavities with constant optical parameters

Solution of the scattering problem. In this section, we assume that the cavity is the unit disk $\mathbb{D}$ and that ε_{c} and μ_{c} are non zero constant in $[0,1]$. The solution of problem (7) for $0<r<1$ depend of the sign of $\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}$. We denote by C_m the function

$$C_m(z) = \begin{cases}
\mathsf{J}_m(\sqrt{\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\,z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) > 0\\
\mathsf{I}_m(\sqrt{-\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\,z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) < 0
\end{cases}$$

where $z\mapsto \mathsf{I}_m(z)$ is the modified Bessel of the first kind. The solution of problem (7) is

$$w_m(r) = \begin{cases}
a_m\,C_m(k\,r) & \text{if } r\leq 1\\
b_m\,\mathsf{H}^{(1)}_m(k\,r) + \mathsf{J}_m(k\,r) & \text{if } r>1
\end{cases} \tag{8}$$

where (a_m, b_m) are solution of

$$\begin{pmatrix} C_m(k) & -\mathsf{H}^{(1)}_m(k)\\ \varepsilon_{\mathsf{c}}^{-1}\,C_m'(k) & -\mathsf{H}^{(1)\prime}_m(k)\end{pmatrix}\begin{pmatrix} a_m\\ b_m\end{pmatrix} = \begin{pmatrix}\mathsf{J}_m(k)\\ \mathsf{J}_m'(k)\end{pmatrix}. \tag{9}$$

Solution of the resonance problem. A resonances ℓ is a complex satisfying

$$\varepsilon_{\mathsf{c}}^{-1}\, C'_m(\ell)\, \mathsf{H}^{(1)}_m(\ell) - C_m(\ell)\mathsf{H}^{(1)\,\prime}_m(\ell) = 0 \tag{10}$$

with the associated mode

$$w_\ell(r) = \begin{cases} C_m(k\,r) & \text{if } r \leq 1 \\ \frac{C_m(k)}{\mathsf{H}^{(1)}_m(k)}\, \mathsf{H}^{(1)}_m(k\,r) & \text{if } r > 1 \end{cases}. \tag{11}$$

E Annulus cavities

Solution of the scattering problem. In this section, we assume that the cavity is the annulus $\mathbb{A}_\delta = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \delta < |\boldsymbol{x}| < 1\}$ with $0 < \delta < 1$ and that ε_{c} and μ_{c} are non zero constant in $[0, 1]$. The solution of problem (7) for $\delta < r < 1$ depend of the sign of $\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}$ and are denoted by C_m and D_m. The solution of problem (7) is

$$w_m(r) = \begin{cases} a_m\ \mathsf{J}_m(k\,r) & \text{if } 0 \leq r < \delta \\ b_m\, C_m(k\,r) + c_m\, D_m(k\,r) & \text{if } \delta \leq r \leq 1 \\ d_m\ \mathsf{H}^{(1)}_m(k\,r) + \mathsf{J}_m(k\,r) & \text{if } r > 1 \end{cases} \tag{12}$$

where (a_m, b_m, c_m, d_m) are solution of

$$\begin{pmatrix} -\mathsf{J}_m(k\delta) & C_m(k\delta) & D_m(k\delta) & 0 \\ -\mathsf{J}'_m(k\delta) & \varepsilon_{\mathsf{c}}^{-1}\, C'_m(k\delta) & \varepsilon_{\mathsf{c}}^{-1}\, D'_m(k\delta) & 0 \\ 0 & C_m(k) & D_m(k) & -\mathsf{H}^{(1)}_m(k) \\ 0 & \varepsilon_{\mathsf{c}}^{-1}\, C'_m(k) & \varepsilon_{\mathsf{c}}^{-1}\, D'_m(k) & -\mathsf{H}^{(1)\,\prime}_m(k) \end{pmatrix} \begin{pmatrix} a_m \\ b_m \\ c_m \\ d_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathsf{J}_m(k) \\ \mathsf{J}'_m(k) \end{pmatrix}. \tag{13}$$

Solution of the resonance problem. A resonances ℓ is a complex satisfying

$$\varepsilon_{\mathsf{c}}^{-1}\, C'_m(\ell)\, \mathsf{H}^{(1)}_m(\ell) - C_m(\ell)\mathsf{H}^{(1)\,\prime}_m(\ell) = 0 \tag{14}$$

with the associated mode

$$w_\ell(r) = \begin{cases} C_m(k\,r) & \text{if } r \leq 1 \\ \frac{C_m(k)}{\mathsf{H}^{(1)}_m(k)}\, \mathsf{H}^{(1)}_m(k\,r) & \text{if } r > 1 \end{cases}. \tag{15}$$

E.1 Constant optical parameters

$$C_m(z) = \begin{cases} \mathsf{J}_m(\sqrt{\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) > 0 \\ \mathsf{I}_m(\sqrt{-\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) < 0 \end{cases} \quad \text{and} \quad D_m(z) = \begin{cases} \mathsf{Y}_m(\sqrt{\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) > 0 \\ \mathsf{K}_m(\sqrt{-\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) < 0 \end{cases}$$

where $z \mapsto \mathsf{K}_m(z)$ is the modified Bessel of the second kind.

E.2 The flat well version