Scattering by rotationally invariant cavities

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Abstract

We solve scattering problem by rotationally invariant cavities. The shape consider are disk and annulus where the permittivity and permeability are radial function.

Contents

1 The scattering problem

1.1 Problem statement

Let $\mathbb{D} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid |\boldsymbol{x}| < 1\}$ be the unit disk and $\mathbb{A}_\delta = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \delta < |\boldsymbol{x}| < 1\}$ an annulus of width $\delta > 0$. The cavity Ω denote either $\mathbb{D}$ or $\mathbb{A}_\delta$ and the interface $\Gamma = \partial\Omega$ is the boundary of Ω. We define the function $\varepsilon \in \mathrm{L}^\infty(\mathbb{R}^2)$ (permittivity) and $\mu \in \mathrm{L}^\infty(\mathbb{R}^2)$ (permeability) as

$$\varepsilon(\boldsymbol{x}) = \begin{cases} \varepsilon_{\mathsf{c}}(|\boldsymbol{x}|) & \text{if } \boldsymbol{x} \in \overline{\Omega} \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \mu(\boldsymbol{x}) = \begin{cases} \mu_{\mathsf{c}}(|\boldsymbol{x}|) & \text{if } \boldsymbol{x} \in \overline{\Omega} \\ 1 & \text{otherwise} \end{cases}$$

where $\varepsilon_{\mathsf{c}}, \mu_{\mathsf{c}} \in \mathscr{C}^\infty(\overline{I}, \mathbb{R}^*)$ with $I = (0,1)$ or $I = (\delta, 1)$. Let $\mathcal{D}^2 := \{u \in \mathrm{L}^2(\mathbb{R}^2) \mid \operatorname{div}(\varepsilon^{-1}\nabla u) \in \mathrm{L}^2(\mathbb{R}^2)\}$ be the domain of the operator $u \mapsto -\mu^{-1}\operatorname{div}(\varepsilon^{-1}\nabla u)$ and we define the "loc" version

$$\mathcal{D}^2_{\mathrm{loc}} := \{u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2) \mid \forall \chi \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}^2),\ \chi u \in \mathcal{D}^2\}.$$

We define the following scattering problem: Given a wavenumber $k > 0$ and an incident field $u^{\mathsf{in}} : \boldsymbol{x} \mapsto \mathsf{e}^{\mathsf{i}\,k\,y}$, find the scattering field $u^{\mathsf{sc}} \in \mathcal{D}^2_{\mathrm{loc}}$ such that the total field $u = u^{\mathsf{in}} + u^{\mathsf{sc}}$ satisfy

$$\begin{cases} -\mu^{-1}\,\mathrm{div}\left(\varepsilon^{-1}\nabla u\right) - k^2 u = 0 & \text{in } \Omega \text{ and } \mathbb{R}^2 \setminus \overline{\Omega} \\ [u]_\Gamma = 0 \text{ and } \left[\varepsilon^{-1}\partial_{\boldsymbol{\nu}} u\right]_\Gamma = 0 & \text{across } \Gamma \\ u^{\mathsf{sc}} \text{ is } k\text{-outgoing} \end{cases} \tag{1a}$$

where $\boldsymbol{\nu} : \Gamma \to \mathbb{S}^1$ is the exterior unit normal and u^{sc} is k-outgoing mean that for $|\boldsymbol{x}| \geq 1$, there exist $(c_m)_{m\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$u^{\mathsf{sc}}(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} c_m\, \mathsf{H}^{(1)}_m(k\,r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta} \tag{1b}$$

with $(r,\theta) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ the polar coordinate associated to the Cartesian coordinates $\boldsymbol{x} \in \mathbb{R}^2$ and $z \mapsto \mathsf{H}^{(1)}_m(z)$ is the Hankel function of the first kind of order m.

1.2 1D reduction

We look for solution of problem (1) of the form

$$u(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} w_m(r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta} \qquad \text{where } w_m(r) := \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\,\mathsf{e}^{-\mathsf{i}\,m\,\theta}\,\mathrm{d}\theta.$$

Similarly we write $u^{\mathsf{in}}(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} w^{\mathsf{in}}_m(r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta}$ and $u^{\mathsf{sc}}(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} w^{\mathsf{sc}}_m(r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta}$. For the incident field, we have $w^{\mathsf{in}}_m(r) = \mathsf{J}_m(k\,r)$ because the Jacobi-Anger expansion [DLMF, Eq. 10.12.1] states that

$$u^{\mathsf{in}}(\boldsymbol{x}) = \mathsf{e}^{\mathsf{i}k\,y} = \sum_{m\in\mathbb{Z}} \mathsf{J}_m(k\,r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta} \tag{2}$$

where $z \mapsto \mathsf{J}_m(z)$ is the Bessel function of the first kind. The series in equation (2) converges absolutely on every compact set of $\mathbb{R}^2$.

The domain of the operator $w \mapsto -r^{-1}\,\mu^{-1}\partial_r(r\,\varepsilon^{-1}\partial_r w) + m^2\,r^{-2}\,\varepsilon^{-1}\mu^{-1}w$ and its “loc” version are define by

$$\begin{aligned} \mathcal{D}^{1,m} &:= \{w \in \mathrm{L}^2(\mathbb{R}^*_+, r\,\mathrm{d}r) \mid \partial_r(r\,\varepsilon^{-1}\,\partial_r w) - m^2\,r^{-1}\varepsilon^{-1}w \in \mathrm{L}^2(\mathbb{R}^*_+)\} \\ \mathcal{D}^{1,m}_{\mathrm{loc}} &:= \{w \in \mathrm{L}^2(\mathbb{R}^*_+, r\,\mathrm{d}r) \mid \forall\chi \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}),\ \chi w \in \mathcal{D}^{1,m}\}. \end{aligned}$$

Problem 1 reduce to a family of problem index by $m \in \mathbb{Z}$: Given $k > 0$ find $w^{\mathsf{sc}}_m \in \mathcal{D}^{1,m}_{\mathrm{loc}}$ such that $w_m = w^{\mathsf{in}}_m + w^{\mathsf{sc}}_m$ and

$$\begin{cases} -\dfrac{1}{r\,\mu}\partial_r\left(\dfrac{r}{\varepsilon}\,\partial_r w_m\right) + \dfrac{m^2}{r^2\,\varepsilon\mu}w_m - k^2\,w_m = 0 & \text{in } I \text{ and } \mathbb{R}^* \setminus \overline{I} \\ w_0'(0) = 0 & \text{on } \{0\} \\ [w_m]_{\{1\}} = 0 \quad \text{and} \quad \left[\varepsilon^{-1}\,w_m'\right]_{\{1\}} = 0 & \text{across } \{1\} \\ w^{\mathsf{sc}}_m(r) \propto \mathsf{H}^{(1)}_m(k\,r) & r \geq 1 \end{cases} \tag{3}$$

with $\propto$ meaning “up to a multiplicative constant”.

2 The resonances problem

2.1 Problem statement

2.2 1D reduction

3 Disk cavities with constant optical parameters

3.1 Solution of the scattering problem

In this section, we assume that the cavity is the unit disk $\mathbb{D}$ and that ε_{c} and μ_{c} are non zero constant in $[0,1]$. The solution of problem (3) for $0 < r < 1$ depend of the sign of $\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}$. We denote by C_m the function

$$C_m(z) = \begin{cases} \mathsf{J}_m(\sqrt{\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \varepsilon_{\mathsf{c}}\mu_{\mathsf{c}} > 0 \\ \mathsf{I}_m(\sqrt{-\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \varepsilon_{\mathsf{c}}\mu_{\mathsf{c}} < 0 \end{cases}$$

where $z \mapsto \mathsf{I}_m(z)$ is the modified Bessel of the first kind. The solution of problem (3) is

$$w_m(r) = \begin{cases} \alpha_m\, C_m(k\, r) & \text{if } r \leq 1 \\ \beta_m\ \mathsf{H}^{(1)}_m(k\, r) + \mathsf{J}_m(k\, r) & \text{if } r > 1 \end{cases} \tag{4}$$

where (α_m, β_m) are solution of

$$\begin{pmatrix} C_m(k) & -\mathsf{H}^{(1)}_m(k) \\ \varepsilon_{\mathsf{c}}^{-1}\, C'_m(k) & -\mathsf{H}^{(1)\prime}_m(k) \end{pmatrix} \begin{pmatrix} \alpha_m \\ \beta_m \end{pmatrix} = \begin{pmatrix} \mathsf{J}_m(k) \\ \mathsf{J}'_m(k) \end{pmatrix}. \tag{5}$$

3.2 Solution of the resonance problem

A resonances ℓ is a complex satisfying

$$\varepsilon_{\mathsf{c}}^{-1}\, C'_m(\ell)\, \mathsf{H}^{(1)}_m(\ell) - C_m(\ell)\mathsf{H}^{(1)\prime}_m(\ell) = 0 \tag{6}$$

with the associated mode

$$w_\ell(r) = \begin{cases} C_m(k\, r) & \text{if } r \leq 1 \\ \frac{C_m(k)}{\mathsf{H}^{(1)}_m(k)}\, \mathsf{H}^{(1)}_m(k\, r) & \text{if } r > 1 \end{cases}. \tag{7}$$

4 Annulus cavities

4.1 Constant optical parameters

4.2 The flat well version

4.3 The full flat version

References

[DLMF] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.28 of 2020-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.