Analytic computations for scattering

Zoïs Moitier

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Abstract

We solve scattering problem for cases where we have analytical expressions. All cavities consider will be invariant by rotation. The shape consider are disk, annulus, ball and spherical shell where the permittivity and permeability are radial function.

Contents

1 The scattering problems

1.1 Problem statement

Let $\mathbb{D} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid |\boldsymbol{x}| < 1\}$ be the unit disk and $\mathbb{A}_\delta = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \delta < |\boldsymbol{x}| < 1\}$ an annulus of width $1-\delta$ where $\delta \in (0,1)$. The cavity Ω denote either $\mathbb{D}$ or $\mathbb{A}_\delta$ and the interface $\Gamma = \partial\Omega$ is the boundary of Ω. We define the function $\varepsilon \in \mathrm{L}^\infty(\mathbb{R}^2)$ (permittivity) and $\mu \in \mathrm{L}^\infty(\mathbb{R}^2)$ (permeability) as

$$\varepsilon(\boldsymbol{x}) = \begin{cases} \varepsilon_{\mathsf{c}}(|\boldsymbol{x}|) & \text{if } \boldsymbol{x} \in \overline{\Omega} \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mu(\boldsymbol{x}) = \begin{cases} \mu_{\mathsf{c}}(|\boldsymbol{x}|) & \text{if } \boldsymbol{x} \in \overline{\Omega} \\ 1 & \text{otherwise} \end{cases}$$

where $\varepsilon_{\mathsf{c}}, \mu_{\mathsf{c}} \in \mathscr{C}^\infty(\overline{I}, \mathbb{R}^*)$ with $I = (0,1)$ or $I = (\delta, 1)$. Let $\mathcal{D}^2 := \{u \in \mathrm{L}^2(\mathbb{R}^2) \mid \operatorname{div}(\varepsilon^{-1}\nabla u) \in \mathrm{L}^2(\mathbb{R}^2)\}$ be the domain of the operator $u \mapsto -\mu^{-1}\operatorname{div}(\varepsilon^{-1}\nabla u)$ and we define the "loc" version

$$\mathcal{D}^2_{\mathrm{loc}} := \{u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2) \mid \forall \chi \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}^2),\ \chi u \in \mathcal{D}^2\}.$$

We define the following scattering problem: Given a wavenumber $k > 0$ and an incident field $u^{\mathsf{in}} : \boldsymbol{x} \mapsto \mathsf{e}^{\mathsf{i}\,k\,y}$, find the scattering field $u^{\mathsf{sc}} \in \mathcal{D}^2_{\mathrm{loc}}$ such that the total field $u = u^{\mathsf{in}} + u^{\mathsf{sc}}$ satisfy

$$\begin{cases} -\mu^{-1}\,\mathrm{div}\left(\varepsilon^{-1}\nabla u\right) - k^2 u = 0 & \text{in } \Omega \text{ and } \mathbb{R}^2 \setminus \overline{\Omega} \\ [u]_\Gamma = 0 \text{ and } \left[\varepsilon^{-1}\partial_{\boldsymbol{\nu}} u\right]_\Gamma = 0 & \text{across } \Gamma \\ u^{\mathsf{sc}} \text{ is } k\text{-outgoing} \end{cases} \tag{1a}$$

where $\boldsymbol{\nu} : \Gamma \to \mathbb{S}^1$ is the exterior unit normal and u^{sc} is k-outgoing mean that for $|\boldsymbol{x}| \geq 1$, there exist $(\beta_m)_{m\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$u^{\mathsf{sc}}(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} \beta_m\ \mathsf{H}^{(1)}_m(k\,r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta} \tag{1b}$$

with $(r, \theta) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ the polar coordinate associated to the Cartesian coordinates $\boldsymbol{x} \in \mathbb{R}^2$ and $z \mapsto \mathsf{H}^{(1)}_m(z)$ is the Hankel function of the first kind of order m.

1.2 1D reduction

We look for solution of problem (1) of the form

$$u(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} w_m(r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta} \qquad \text{where } w_m(r) \coloneqq \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\,\mathsf{e}^{-\mathsf{i}\,m\,\theta}\,\mathrm{d}\theta.$$

Similarly we write $u^{\mathsf{in}}(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} w^{\mathsf{in}}_m(r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta}$ and $u^{\mathsf{sc}}(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} w^{\mathsf{sc}}_m(r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta}$. For the incident field, we have $w^{\mathsf{in}}_m(r) = \mathsf{J}_m(k\,r)$ because the Jacobi-Anger expansion [DLMF, Eq. 10.12.1] states that

$$u^{\mathsf{in}}(\boldsymbol{x}) = \mathsf{e}^{\mathsf{i}k\,y} = \sum_{m\in\mathbb{Z}} \mathsf{J}_m(k\,r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta} \tag{2}$$

where $z \mapsto \mathsf{J}_m(z)$ is the Bessel function of the first kind. The series in equation (2) converges absolutely on every compact set of $\mathbb{R}^2$.

The domain of the operator $w \mapsto -r^{-1}\,\mu^{-1}\partial_r(r\,\varepsilon^{-1}\partial_r w) + m^2\,r^{-2}\,\varepsilon^{-1}\mu^{-1}w$ and its "loc" version are define by

$$\begin{aligned} \mathcal{D}^{1,m} &\coloneqq \{w \in \mathrm{L}^2(\mathbb{R}^*_+, r\,\mathrm{d}r) \mid \partial_r(r\,\varepsilon^{-1}\,\partial_r w) - m^2\,r^{-1}\varepsilon^{-1}w \in \mathrm{L}^2(\mathbb{R}^*_+)\} \\ \mathcal{D}^{1,m}_{\mathrm{loc}} &\coloneqq \{w \in \mathrm{L}^2(\mathbb{R}^*_+, r\,\mathrm{d}r) \mid \forall\chi \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}),\ \chi w \in \mathcal{D}^{1,m}\}. \end{aligned}$$

Problem 1 reduce to a family of problem index by $m \in \mathbb{Z}$: Given $k > 0$ find $w^{\mathsf{sc}}_m \in \mathcal{D}^{1,m}_{\mathrm{loc}}$ such that $w_m = w^{\mathsf{in}}_m + w^{\mathsf{sc}}_m$ and

$$\begin{cases} -\dfrac{1}{r\,\mu}\partial_r\left(\dfrac{r}{\varepsilon}\,\partial_r w_m\right) + \dfrac{m^2}{r^2\,\varepsilon\mu} w_m - k^2\,w_m = 0 & \text{in } I \text{ and } \mathbb{R}^* \setminus \overline{I} \\ w_0'(0) = 0 & \text{on } \{0\} \\ [w_m]_{\{1\}} = 0 \quad \text{and} \quad \left[\varepsilon^{-1}\,w_m'\right]_{\{1\}} = 0 & \text{across } \{1\} \\ w^{\mathsf{sc}}_m(r) \propto \mathsf{H}^{(1)}_m(k\,r) & r \geq 1 \end{cases} \tag{3}$$

with $\propto$ meaning "up to a multiplicative constant".

2 The resonances problem

2.1 Problem statement

2.2 1D reduction

3 Disk cavities with constant optical parameters

Solution of the scattering problem. In this section, we assume that the cavity is the unit disk $\mathbb{D}$ and that ε_{c} and μ_{c} are non zero constant in $[0,1]$. The solution of problem (3) for $0<r<1$ depend of the sign of $\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}$. We denote by C_m the function

$$C_m(z) = \begin{cases} \mathsf{J}_m(\sqrt{\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) > 0 \\ \mathsf{I}_m(\sqrt{-\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) < 0 \end{cases}$$

where $z \mapsto \mathsf{I}_m(z)$ is the modified Bessel of the first kind. The solution of problem (3) is

$$w_m(r) = \begin{cases} a_m\, C_m(k\,r) & \text{if } r \le 1 \\ b_m\ \mathsf{H}^{(1)}_m(k\,r) + \mathsf{J}_m(k\,r) & \text{if } r > 1 \end{cases} \tag{4}$$

where (a_m, b_m) are solution of

$$\begin{pmatrix} C_m(k) & -\mathsf{H}^{(1)}_m(k) \\ \varepsilon_{\mathsf{c}}^{-1}\, C_m'(k) & -\mathsf{H}^{(1)\prime}_m(k) \end{pmatrix} \begin{pmatrix} a_m \\ b_m \end{pmatrix} = \begin{pmatrix} \mathsf{J}_m(k) \\ \mathsf{J}_m'(k) \end{pmatrix}. \tag{5}$$

Solution of the resonance problem. A resonances ℓ is a complex satisfying

$$\varepsilon_{\mathsf{c}}^{-1}\, C_m'(\ell)\, \mathsf{H}^{(1)}_m(\ell) - C_m(\ell)\mathsf{H}^{(1)\prime}_m(\ell) = 0 \tag{6}$$

with the associated mode

$$w_\ell(r) = \begin{cases} C_m(k\,r) & \text{if } r \le 1 \\ \frac{C_m(k)}{\mathsf{H}^{(1)}_m(k)}\, \mathsf{H}^{(1)}_m(k\,r) & \text{if } r > 1 \end{cases}. \tag{7}$$

4 Annulus cavities

Solution of the scattering problem. In this section, we assume that the cavity is the annulus $\mathbb{A}_\delta = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \delta < |\boldsymbol{x}| < 1\}$ with $0<\delta<1$ and that ε_{c} and μ_{c} are non zero constant in $[0,1]$. The solution of problem (3) for $\delta < r < 1$ depend of the sign of $\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}$ and are denoted by C_m and D_m. The solution of problem (3) is

$$w_m(r) = \begin{cases} a_m\ \mathsf{J}_m(k\,r) & \text{if } 0 \le r < \delta \\ b_m\, C_m(k\,r) + c_m\, D_m(k\,r) & \text{if } \delta \le r \le 1 \\ d_m\ \mathsf{H}^{(1)}_m(k\,r) + \mathsf{J}_m(k\,r) & \text{if } r > 1 \end{cases} \tag{8}$$

where (a_m, b_m, c_m, d_m) are solution of

$$\begin{pmatrix} -\mathsf{J}_m(k\delta) & C_m(k\delta) & D_m(k\delta) & 0 \\ -\mathsf{J}_m'(k\delta) & \varepsilon_{\mathsf{c}}^{-1}\, C_m'(k\delta) & \varepsilon_{\mathsf{c}}^{-1}\, D_m'(k\delta) & 0 \\ 0 & C_m(k) & D_m(k) & -\mathsf{H}^{(1)}_m(k) \\ 0 & \varepsilon_{\mathsf{c}}^{-1}\, C_m'(k) & \varepsilon_{\mathsf{c}}^{-1}\, D_m'(k) & -\mathsf{H}^{(1)\prime}_m(k) \end{pmatrix} \begin{pmatrix} a_m \\ b_m \\ c_m \\ d_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathsf{J}_m(k) \\ \mathsf{J}_m'(k) \end{pmatrix}. \tag{9}$$

Solution of the resonance problem. A resonances ℓ is a complex satisfying

$$\varepsilon_{\mathsf{c}}^{-1}\, C_m'(\ell)\, \mathsf{H}_m^{(1)}(\ell) - C_m(\ell)\mathsf{H}_m^{(1)\prime}(\ell) = 0 \tag{10}$$

with the associated mode

$$w_\ell(r) = \begin{cases} C_m(k\, r) & \text{if } r \leq 1 \\ \frac{C_m(k)}{\mathsf{H}_m^{(1)}(k)}\, \mathsf{H}_m^{(1)}(k\, r) & \text{if } r > 1 \end{cases}. \tag{11}$$

4.1 Constant optical parameters

$$C_m(z) = \begin{cases} \mathsf{J}_m(\sqrt{\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) > 0 \\ \mathsf{I}_m(\sqrt{-\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) < 0 \end{cases} \quad \text{and} \quad D_m(z) = \begin{cases} \mathsf{Y}_m(\sqrt{\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) > 0 \\ \mathsf{K}_m(\sqrt{-\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}}\, z) & \text{if } \Re(\varepsilon_{\mathsf{c}}\mu_{\mathsf{c}}) < 0 \end{cases}$$

where $z \mapsto \mathsf{K}_m(z)$ is the modified Bessel of the second kind.

4.2 The flat well version

References

[DLMF] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.28 of 2020-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.