claudius: analytic computations for scattering

calculs analytiques pour la diffusion des ondes

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Abstract

We solve scattering problem for cases where we have analytical expressions. All cavities consider will be invariant by rotation. The shape consider are disk, annulus, ball and spherical shell where the permittivity and permeability are radial function.

Contents

1 General settings

We consider the scattering by spherical cavities, disk or annulus in dimension 2, and ball or spherical shell in dimension 3. We call $|\cdot|_d$ the euclidean norm in dimension $d = 2, 3$. Disk and ball are denoted by $\mathbb{B}^d_\rho = \{\boldsymbol{x} \in \mathbb{R}^d \mid |\boldsymbol{x}|_d < \rho\}$ for $\rho > 0$, and annulus and spherical shell are denoted by $\mathbb{A}^d_{\rho,\sigma} = \{\boldsymbol{x} \in \mathbb{R}^d \mid \rho < |\boldsymbol{x}|_d < \sigma\}$ for $0 < \rho < \sigma$. The cavities will be compose of concentric spherical shell with impenetrable or penetrable core and denoted by Ω. For a cavity with impenetrable core and $N \geq 0$ layers, we have $\Omega = \mathbb{A}^d_{\rho_1,\rho_2} \cup \mathbb{A}^d_{\rho_2,\rho_3} \cup \cdots \cup \mathbb{A}^d_{\rho_N,\rho_{N+1}}$ with $0 < \rho_1 < \rho_2 < \cdots < \rho_{N+1}$. For a cavity with penetrable core and $N \geq 0$ layers, we have $\Omega = \mathbb{B}^d_{\rho_1} \cup \mathbb{A}^d_{\rho_1,\rho_2} \cup \cdots \cup \mathbb{A}^d_{\rho_N,\rho_{N+1}}$ with $0 = \rho_0 < \rho_1 < \cdots < \rho_{N+1}$. The boundary/interfaces Γ of the different layers of the cavity Ω is compose of concentric spheres $\Gamma = \rho_1\mathbb{S}^{d-1} \cup \rho_2\mathbb{S}^{d-1} \cup \cdots \cup \rho_N\mathbb{S}^{d-1}$.

We define the function $\varepsilon \in \mathrm{L}^{\infty}(\mathbb{R}^d)$ and $\mu \in \mathrm{L}^{\infty}(\mathbb{R}^d)$ as

$$\varepsilon(\boldsymbol{x}) = \begin{cases} \varepsilon_{\mathsf{c}}\left(|\boldsymbol{x}|_d\right) & \text{if } \boldsymbol{x} \in \Omega \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mu(\boldsymbol{x}) = \begin{cases} \mu_{\mathsf{c}}\left(|\boldsymbol{x}|_d\right) & \text{if } \boldsymbol{x} \in \Omega \\ 1 & \text{otherwise} \end{cases}$$

where

$$\varepsilon_{\mathsf{c}} = \sum_{n=0}^{N} \varepsilon_n \,\mathbf{1}_{(\rho_n,\rho_{n+1})} \qquad \text{and} \qquad \mu_{\mathsf{c}} = \sum_{n=0}^{N} \mu_n \,\mathbf{1}_{(\rho_n,\rho_{n+1})}$$

with $\varepsilon_n, \mu_n \in \mathscr{C}^{\infty}([\rho_n, \rho_{n+1}], \mathbb{R}^*)$ for $0 \leq n \leq N$. For the impenetrable case the index $n = 0$ is not use. In some context the function can be view as the permittivity (ε) and the permeability (μ) of a material.

A positive wavenumber is noted k. We only consider a plane wave incident field, we have $u^{\mathsf{in}} : \boldsymbol{x} \mapsto \exp(\mathrm{i}\, k\, \boldsymbol{\nu} \cdot \boldsymbol{x})$ with $\boldsymbol{\nu} \in \mathbb{S}^{d-1}$ the direction of the plane wave. In the following, the scattered field is noted u^{sc} and the total field is noted u.

2 Helmholtz's equation

2.1 The scattering problem

We consider three type of problems: a cavity

- with a penetrable core;
- with an impenetrable core and an homogeneous Dirichlet condition on the inner radius;
- with an impenetrable core and an homogeneous Neumann condition on the inner radius.

The domain of the operator $u \mapsto -\mu^{-1} \operatorname{div}(\varepsilon^{-1} \nabla u)$ of domain depend on the type of problem:

$$\begin{aligned} D^{\mathcal{D}} &:= \{u \in \mathrm{L}^2(\mathbb{R}^d \setminus \mathbb{B}_{\rho_1}) \mid \operatorname{div}(\varepsilon^{-1} \nabla u) \in \mathrm{L}^2(\mathbb{R}^d \setminus \mathbb{B}_{\rho_1}) \text{ and } u|_{\rho_1 \mathbb{S}^{d-1}} = 0\}, \\ D^{\mathcal{N}} &:= \{u \in \mathrm{L}^2(\mathbb{R}^d \setminus \mathbb{B}_{\rho_1}) \mid \operatorname{div}(\varepsilon^{-1} \nabla u) \in \mathrm{L}^2(\mathbb{R}^d \setminus \mathbb{B}_{\rho_1})\}, \\ D^{\mathcal{P}} &:= \{u \in \mathrm{L}^2(\mathbb{R}^d) \mid \operatorname{div}(\varepsilon^{-1} \nabla u) \in \mathrm{L}^2(\mathbb{R}^d)\} \end{aligned}$$

and we define the "locale" version

$$\begin{aligned} D^{\mathcal{D}/\mathcal{N}}_{\mathrm{loc}} &:= \{u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \mathbb{B}_{\rho_1}) \mid \forall \chi \in \mathscr{C}^{\infty}_{\mathrm{comp}}(\mathbb{R}^d),\ \chi u \in D^{\mathcal{D}/\mathcal{N}}\}, \\ D^{\mathcal{P}}_{\mathrm{loc}} &:= \{u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d) \mid \forall \chi \in \mathscr{C}^{\infty}_{\mathrm{comp}}(\mathbb{R}^d),\ \chi u \in D^{\mathcal{P}}\}. \end{aligned}$$

We define the following scattering problem: Given a wavenumber $k > 0$ and an incident field u^{in}, find the scattering field $u^{\mathsf{sc}} \in D^{\mathcal{D}/\mathcal{N}/\mathcal{P}}_{\mathrm{loc}}$ such that the total field $u = u^{\mathsf{in}} + u^{\mathsf{sc}}$ satisfy

$$(\mathcal{D}/\mathcal{N}) \begin{cases} -\mu^{-1} \operatorname{div}\left(\varepsilon^{-1} \nabla u\right) - k^2 u = 0 & \text{in } \Omega \text{ and } \mathbb{R}^2 \setminus \overline{\mathbb{B}^d_{\rho_{N+1}}} \\ u = 0 \qquad \text{or} \qquad \partial_{\boldsymbol{n}} u = 0 & \text{on } \rho_1 \mathbb{S}^{d-1} \\ [u] = 0 \text{ and } \left[\varepsilon^{-1} \partial_{\boldsymbol{n}} u\right] = 0 & \text{across } \rho_2 \mathbb{S}^{d-1} \cup \cdots \cup \rho_{N+1} \mathbb{S}^{d-1} \\ u^{\mathsf{sc}} \text{ is } k\text{-outgoing} \end{cases} \tag{1}$$

in the Dirichlet ($\mathcal{D}$) or Neumann ($\mathcal{N}$) case, or

$$(\mathcal{P}) \begin{cases} -\mu^{-1} \operatorname{div}\left(\varepsilon^{-1}\nabla u\right) - k^2 u = 0 & \text{in } \Omega \text{ and } \mathbb{R}^2 \setminus \overline{\Omega} \\ [u] = 0 \text{ and } \left[\varepsilon^{-1}\partial_{\boldsymbol{n}} u\right] = 0 & \text{across } \rho_1\mathbb{S}^{d-1} \cup \cdots \cup \rho_{N+1}\mathbb{S}^{d-1} \\ u^{\mathsf{sc}} \text{ is } k\text{-outgoing} \end{cases} \tag{2}$$

in the penetrable case, where $\boldsymbol{n} : \Gamma \to \mathbb{S}^{d-1}$ are the outward unit normal and u^{sc} is k-outgoing mean that, for $|\boldsymbol{x}| \geq \rho_N$, we have

$$u^{\mathsf{sc}}(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} \beta_m \, \mathsf{H}^{(1)}_m(k\,r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta}, \qquad \beta_m \in \mathbb{C}, \qquad d = 2, \tag{3a}$$

$$u^{\mathsf{sc}}(\boldsymbol{x}) = \sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell} \beta^m_\ell \, \mathsf{h}^{(1)}_\ell(k\,r)\,Y^m_\ell(\theta,\phi), \qquad \beta^m_\ell \in \mathbb{C}, \qquad d = 3, \tag{3b}$$

with $(r,\theta) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ the polar coordinate, $(r,\theta,\phi) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times [0,\pi]$ the spherical coordinate, Y^m_ℓ the spherical harmonic, and $z \mapsto \mathsf{H}^{(1)}_m(z)$ (resp. $z \mapsto \mathsf{h}^{(1)}_m(z)$) is the cylindrical (resp. spherical) Hankel function of the first kind of order m.

2.2 The 1d reduction

We look for solution of problem of the form

$$u(\boldsymbol{x}) = \sum_{m\in\mathbb{Z}} w_m(r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta} \qquad \text{where } w_m(r) := \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\,\mathsf{e}^{-\mathsf{i}\,m\,\theta}\,\mathrm{d}\theta \qquad d = 2$$

$$u(\boldsymbol{x}) = \sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell} w_{\ell,m}(r)\,Y^m_\ell(\theta,\phi) \qquad \text{where } w_{\ell,m}(r) := \int_{\mathbb{S}^2} u(r,\theta,\phi)\,\overline{Y^m_\ell(\theta,\phi)}\,\mathrm{d}\omega \qquad d = 3$$

with $(r,\theta) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ the polar coordinate, $(r,\theta,\phi) \in \mathbb{R}_+ \times [0,\pi] \times \mathbb{R}/2\pi\mathbb{Z}$ the spherical coordinate. Similarly, we write the same kind of expansion for u^{in} and u^{sc}. Using Eqs. (9) and (10), we deduce that, in dimension 2, we have $w^{\mathsf{in}}_m(r) = \mathsf{J}_m(k\,r)$ for $m \in \mathbb{Z}$ and, in dimension 3, we have $w^{\mathsf{in}}_{\ell,0}(r) = \mathsf{i}^\ell\sqrt{\frac{2\ell+1}{4\pi}}\,\mathsf{j}_\ell(k\,r)$ for $\ell \in \mathbb{N}$ and $w^{\mathsf{in}}_{\ell,m}(r) = 0$ for $\ell \in \mathbb{N}^*$ and $m \in \{-\ell,\dots,-1,1,\dots,\ell\}$. To treat dimension 2 and 3 in a unified manner, we set $w^{\mathsf{in}}_p(r) = c_p\,\mathsf{j}_p(k\,r)$ and $\widehat{p}_d \in \mathbb{R}_+$ where

$$\begin{array}{lllll} p = m \in \mathbb{Z}, & \mathsf{j}_p = \mathsf{J}_m, & c_p = 1, & \widehat{p}_d = p^2, & \text{for } d = 2; \\ p = \ell \in \mathbb{N}, & \mathsf{j}_p = \mathsf{j}_\ell, & c_p = \mathsf{i}^\ell\sqrt{4\pi(2\ell+1)}, & \widehat{p}_d = p(p+1), & \text{for } d = 3. \end{array}$$

So Eqs. (1) to (3) reduce to a family of problem index by p: Given $k > 0$ find $w^{\mathsf{sc}}_p \in D^{\mathcal{D}_p/\mathcal{N}_p/\mathcal{P}_p}_{\mathrm{loc}}$ such that $w_p = w^{\mathsf{in}}_p + w^{\mathsf{sc}}_p$ and

$$(\mathcal{D}_p/\mathcal{N}_p) : \begin{cases} -\dfrac{\mu^{-1}}{r^{d-1}}\partial_r\left(\dfrac{r^{d-1}}{\varepsilon}\,\partial_r w_p\right) + \dfrac{\widehat{p}_d}{r^2\,\varepsilon\mu} w_p - k^2\,w_p = 0 & \text{in } \bigcup_{n=1}^{N}(\rho_n,\rho_{n+1}) \text{ and } (\rho_{N+1},+\infty) \\ w_p(\rho_1) = 0 \quad \text{or} \quad w_p'(\rho_1) = 0 & \text{on } \{\rho_1\} \\ [w_p] = 0 \quad \text{and} \quad \left[\varepsilon^{-1}\,w_p'\right] = 0 & \text{across } \{\rho_2,\dots,\rho_{N+1}\} \\ w^{\mathsf{sc}}_p(r) \propto \mathfrak{h}^{(1)}_p(k\,r) & r \geq \rho_{N+1} \end{cases} \tag{4}$$

in the Dirichlet ($\mathcal{D}$) or Neumann ($\mathcal{N}$) case, or

$$(\mathcal{P}_p) : \begin{cases} -\dfrac{\mu^{-1}}{r^{d-1}}\partial_r\left(\dfrac{r^{d-1}}{\varepsilon}\,\partial_r w_p\right) + \dfrac{\widehat{p}_d}{r^2\,\varepsilon\mu} w_p - k^2\, w_p = 0 & \text{in } \displaystyle\bigcup_{n=0}^{N}(\rho_n,\rho_{n+1}) \text{ and } (\rho_{N+1},+\infty) \\ w_0'(0) = 0 & \text{on } \{0\} \\ [w_p] = 0 \quad \text{and} \quad \left[\varepsilon^{-1}\, w_p'\right] = 0 & \text{across } \{\rho_1,\dots,\rho_{N+1}\} \\ w_p^{\mathsf{sc}}(r) \propto \mathfrak{h}_p^{(1)}(k\,r) & r \geq \rho_{N+1} \end{cases} \tag{5}$$

in the penetrable case, where $\propto$ meaning "up to a multiplicative constant", $\mathfrak{h}_p^{(1)} = \mathsf{H}_m^{(1)}$ if $d = 2$, $\mathfrak{h}_p^{(1)} = \mathsf{h}_\ell^{(1)}$ if $d = 3$ and

$$\begin{aligned} \mathcal{D}_{\mathrm{loc}}^{\mathcal{D}_p/\mathcal{N}_p} &\coloneqq \{w \in \mathrm{L}^2_{\mathrm{loc}}((\rho_1,+\infty)) \mid \forall\chi \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}),\ \chi w \in D^{\mathcal{D}_p/\mathcal{N}_p}\}, \\ \mathcal{D}_{\mathrm{loc}}^{\mathcal{P}_p} &\coloneqq \{w \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+^*) \mid \forall\chi \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}),\ \chi w \in D^{\mathcal{P}_p}\} \end{aligned}$$

with $A_p w = \partial_r(\varepsilon^{-1}\, r^{d-1}\partial_r w) - \widehat{p}_d\, r^{d-3}\,\varepsilon^{-1}\, w$ and

$$\begin{aligned} D^{\mathcal{D}_p} &\coloneqq \{w \in \mathrm{L}^2((\rho_1,+\infty), r^{d-1}\,\mathrm{d}r) \mid A_p w \in \mathrm{L}^2((\rho_1,+\infty)) \text{ and } w(\rho_1) = 0\}, \\ D^{\mathcal{N}_p} &\coloneqq \{w \in \mathrm{L}^2((\rho_1,+\infty), r^{d-1}\,\mathrm{d}r) \mid A_p w \in \mathrm{L}^2((\rho_1,+\infty))\}, \\ D^{\mathcal{P}_p} &\coloneqq \{w \in \mathrm{L}^2(\mathbb{R}_+^*, r^{d-1}\,\mathrm{d}r) \mid A_p w \in \mathrm{L}^2(\mathbb{R}_+^*)\}. \end{aligned}$$

In each layers n or in the core we assume that we have a solution basis of the ordinary differential equation

$$-\frac{1}{r^{d-1}\,\mu_n}\partial_r\left(\frac{r^{d-1}}{\varepsilon_n}\,\partial_r w_p\right) + \frac{\widehat{p}_d}{r^2\,\varepsilon_n\,\mu_n} w_p - k^2 w_p = 0$$

noted $f_{n,p}$ and $g_{n,p}$, we also assume that $f_{n,p}$ is continues at 0.

Example 1. If ε_n and μ_n are non zero constants, we have

$$f_{n,p}(r) = \begin{cases} \mathsf{j}_p(\sqrt{\varepsilon_n\mu_n}\,k\,r) & \text{if } \Re(\varepsilon_n\mu_n) > 0 \\ \mathsf{i}_p(\sqrt{-\varepsilon_n\mu_n}\,k\,r) & \text{if } \Re(\varepsilon_n\mu_n) < 0 \end{cases}$$

and

$$g_{n,p}(r) = \begin{cases} \mathfrak{y}_p(\sqrt{\varepsilon_n\mu_n}\,k\,r) & \text{if } \Re(\varepsilon_n\mu_n) > 0 \\ \mathfrak{K}_p(\sqrt{-\varepsilon_n\mu_n}\,k\,r) & \text{if } \Re(\varepsilon_n\mu_n) < 0 \end{cases}$$

where $\mathsf{j}_p, \mathsf{i}_p, \mathfrak{y}_p, \mathfrak{K}_p$ are $\mathsf{J}_m, \mathsf{I}_m, \mathsf{Y}_m, \mathsf{K}_m$ for the dimension 2 and $\mathsf{j}_\ell, \mathsf{i}_\ell, \mathsf{y}_\ell, \mathsf{k}_\ell$ for the dimension 3.

2.3 The solution

We write the solution w_p has

$$w_p(r) = \begin{cases} \alpha_{0,p}\, f_{0,p}(r) & \text{if } 0 \leq r \leq \rho_1 \\ \alpha_{n,p}\, f_{n,p}(r) + \beta_{n,p}\, g_{n,p}(r) & \text{if } 1 \leq n \leq N \text{ and } \rho_n \leq r \leq \rho_{n+1} \\ \beta_{N+1,p}\, \mathfrak{h}_p^{(1)}(k\,r) + c_p\, \mathsf{j}_p(k\,r) & \text{if } r \geq \rho_{N+1} \end{cases} \tag{6}$$

The first coefficient $\alpha_{0,p}$ is not use in the impenetrable cases. To compute the coefficient $\alpha_{0,p}$, $\alpha_{1,p}, \beta_{1,p}, \dots, \alpha_{N,p}, \beta_{N,p}$, and $\beta_{N+1,p}$, we use the transmission conditions at the interfaces. For $n \in \{1,\dots,N\}$, they give $T_{n-1,p}(\rho_n)(\alpha_{n-1,p}, \beta_{n-1,p})^\intercal = T_{n,p}(\rho_n)(\alpha_{n,p}, \beta_{n,p})^\intercal$ where

$$T_{0,p}(\rho) = \begin{pmatrix} f_{0,p}(\rho) \\[2ex] \dfrac{f_{0,p}'(\rho)}{\varepsilon_n(\rho)} \end{pmatrix} \in \mathcal{M}_{2,1}(\mathbb{C}) \quad \text{and} \quad T_{n,p}(\rho) = \begin{pmatrix} f_{n,p}(\rho) & g_{n,p}(\rho) \\[2ex] \dfrac{f_{n,p}'(\rho)}{\varepsilon_n(\rho)} & \dfrac{g_{n,p}'(\rho)}{\varepsilon_n(\rho)} \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{C}) \tag{7}$$

and the last interface $T_{N,p}(\rho_{N+1})(\alpha_{N,p},\beta_{N,p})^\intercal = H_p\beta_{N+1,p} + (c_p\,\mathfrak{j}_p(k\,r), c_pk\,\mathfrak{j}'_p(k\,r))^\intercal$ where

$$H_p = \begin{pmatrix} \mathfrak{h}_p^{(1)}(k\,\rho_{N+1}) \\ k\mathfrak{h}_p^{(1)\prime}(k\,\rho_{N+1}) \end{pmatrix} \in \mathcal{M}_{2,1}(\mathbb{C}). \tag{8}$$

In the impenetrable cases the Dirichlet or Neumann conditions give $T_{1,p}^{\mathcal{D}/\mathcal{N}}(\alpha_{1,p},\beta_{1,p})^\intercal = \mathbf{0}_{2,1}$ where

$$T_{1,p}^{\mathcal{D}} = \begin{pmatrix} f_{0,p}(\rho_1) & g_{0,p}(\rho_1) \end{pmatrix} \in \mathcal{M}_{1,2}(\mathbb{C}), \quad \text{and} \quad T_{1,p}^{\mathcal{N}} = \begin{pmatrix} f'_{0,p}(\rho_1) & g'_{0,p}(\rho_1) \end{pmatrix} \in \mathcal{M}_{1,2}(\mathbb{C}).$$

In addition, we define

$$J_{q,p} = (0,\ldots,0,c_p\,\mathfrak{j}_p(k\,r), c_pk\,\mathfrak{j}'_p(k\,r))^\intercal \in \mathbb{C}^q.$$

For the Dirichlet or Neumann case, the coefficient $\alpha_{1,p},\beta_{1,p},\ldots,\alpha_{N,p},\beta_{N,p}$, and $\beta_{N+1,p}$ are solution of the system

$$M_{N,p}^{\mathcal{D}/\mathcal{N}}(\alpha_{1,p},\beta_{1,p},\ldots,\alpha_{N,p},\beta_{N,p},\beta_{N+1,p})^\intercal = J_{2N+1,p}$$

where

$$M_{N,p}^{\mathcal{D}/\mathcal{N}} = \begin{pmatrix} T_{1,p}^{\mathcal{D}/\mathcal{N}}(\rho_1) & & & & & \\ T_{1,p}(\rho_2) & -T_{2,p}(\rho_2) & & & & \\ & T_{2,p}(\rho_3) & -T_{3,p}(\rho_3) & & & \\ & & \ddots & \ddots & & \\ & & & T_{N-1,p}(\rho_N) & -T_{N,p}(\rho_N) & \\ & & & & T_{N,p}(\rho_{N+1}) & H_p \end{pmatrix} \in \mathcal{M}_{2N+1}(\mathbb{C}).$$

For the penetrable case, the coefficient $\alpha_{0,p}$, $\alpha_{1,p},\beta_{1,p},\ldots,\alpha_{N,p},\beta_{N,p}$, and $\beta_{N+1,p}$ are solution of the system

$$M_{N,p}^{\mathcal{P}}(\alpha_{0,p},\alpha_{1,p},\beta_{1,p},\ldots,\alpha_{N,p},\beta_{N,p},\beta_{N+1,p})^\intercal = J_{2N+1,p}$$

where

$$M_{N,p}^{\mathcal{P}} = \begin{pmatrix} T_{0,p}(\rho_1) & -T_{1,p}(\rho_1) & & & & & \\ & T_{1,p}(\rho_2) & -T_{2,p}(\rho_2) & & & & \\ & & T_{2,p}(\rho_3) & -T_{3,p}(\rho_3) & & & \\ & & & \ddots & \ddots & & \\ & & & & T_{N-1,p}(\rho_N) & -T_{N,p}(\rho_N) & \\ & & & & & T_{N,p}(\rho_{N+1}) & H_p \end{pmatrix} \in \mathcal{M}_{2N+2}(\mathbb{C}).$$

3 Maxwell's equation

A Miscellaneous

A.1 Coordinates

A.1.1 Polar

In dimension two, the Cartesian coordinate $\boldsymbol{x} = (x,y) \in \mathbb{R}^2$ are describe in term of polar coordinate $(r,\theta) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ by

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arg(x+\mathrm{i}y) \end{cases}$$

A.1.2 Spherical

In dimension three, the Cartesian coordinate $\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3$ are describe in term of spherical coordinate $(r, \theta, \phi) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times [0, \pi]$ by

$$\begin{cases} x = r\cos(\theta)\sin(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\phi) \end{cases} \qquad \text{and} \qquad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arg(x + \mathrm{i}y) \\ \phi = \arccos\left(\dfrac{z}{r}\right) \end{cases}.$$

A.2 Bessel's functions

A.2.1 Cylindrical

The cylindrical Bessel's function J_m, Y_m, $\mathsf{H}^{(1)}_m$, and $\mathsf{H}^{(2)}_m$ are define in [DLMF, Sec. 10.2] and they are solutions of the ODE

$$-\frac{1}{z}\left(z\, w'\right)' + \frac{m^2}{z^2} w - w = 0.$$

The cylindrical modified Bessel's function I_m and K_m are define in [DLMF, Sec. 10.25] and they are solutions of the ODE

$$-\frac{1}{z}\left(z\, w'\right)' + \frac{m^2}{z^2} w + w = 0.$$

A.2.2 Spherical

The spherical Bessel's function j_m, y_m, $\mathsf{h}^{(1)}_m$, $\mathsf{h}^{(2)}_m$ $(-)$ and i_m, k_m $(+)$ are define in [DLMF, Sec. 10.47] and they are solutions of the ODE

$$-\frac{1}{z^2}\left(z^2\, w'\right)' + \frac{\ell(\ell+1)}{z^2} w \mp w = 0$$

We have the following relation between the cylindrical and spherical Bessel function

$$\mathsf{f}_\ell(z) = \sqrt{\frac{\pi}{2z}}\, \mathsf{F}_{\ell+\frac{1}{2}}(z)$$

where the couples (f, F) are (j, J), (y, Y), $(\mathsf{h}^{(1)}, \mathsf{H}^{(1)})$, $(\mathsf{h}^{(2)}, \mathsf{H}^{(2)})$, (i, I), and (k, K).

A.3 Spherical harmonic

For $\ell \in \mathbb{N}$ and $m \in \{-\ell, \dots, 0, \dots, \ell\}$, the spherical harmonic Y^m_ℓ in the spherical coordinate is define by

$$Y^m_\ell(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\, \mathrm{e}^{\mathrm{i}\, m\, \theta}\, P^m_\ell(\cos(\phi))$$

where P^m_ℓ are the associated Legendre polynomial define by

$$P^m_\ell(x) = \frac{(-1)^m}{2^\ell\, \ell!}\,(1-x^2)^{\frac{m}{2}}\, \partial_x^{\ell+m}(x^2-1)^\ell.$$

A.4 Spherical wave expansion of plane wave

The Jacobi-Anger expansion [DLMF, Eq. 10.12.1] states that, for $z \in \mathbb{C}$ and $t \in \mathbb{C}^*$,

$$\mathsf{e}^{\frac{z}{2}(t-t^{-1})} = \sum_{m\in\mathbb{Z}} \mathsf{J}_m(z)\, t^m.$$

For a 2d plane wave of direction $(0,1)^\intercal$ and wavenumber $k > 0$, we get

$$\mathsf{e}^{\mathsf{i}\,k\,y} = \sum_{m\in\mathbb{Z}} \mathsf{J}_m(k\,r)\,\mathsf{e}^{\mathsf{i}\,m\,\theta} = \mathsf{J}_0(k\,r) + \sum_{m=1}^{+\infty} \mathsf{J}_m(k\,r)\,\left(\mathsf{e}^{\mathsf{i}\,m\,\theta} + (-1)^m \mathsf{e}^{-\mathsf{i}\,m\,\theta}\right) \tag{9}$$

where (x, y) are the Cartesian coordinates and (r, θ) the corresponding polar coordinates.

For a 3d plane wave of direction $\boldsymbol{\nu} = (1, \nu_\theta, \nu_\phi)^\intercal \in \mathbb{S}^2$ and wavenumber $k > 0$, we have the following expansion

$$\mathsf{e}^{\mathsf{i}\,k\,\boldsymbol{\nu}\cdot\boldsymbol{x}} = 4\pi \sum_{\ell=0}^{+\infty} \mathsf{i}^\ell\, \mathsf{j}_\ell(k\,r) \sum_{m=-\ell}^{\ell} \overline{Y_\ell^m(\nu_\theta, \nu_\phi)}\, Y_\ell^m(\theta, \phi)$$

where $\boldsymbol{x} = (x, y, z)$ are the Cartesian coordinates and (r, θ, ϕ) the corresponding spherical coordinates. For the particular direction $\boldsymbol{\nu} = (0, 0, 1)^\intercal$ in Cartesian coordinates or $\boldsymbol{\nu} = (1, 0, 0)^\intercal$ in spherical coordinates, we obtain

$$\mathsf{e}^{\mathsf{i}\,k\,z} = \sum_{\ell=0}^{+\infty} \mathsf{i}^\ell \sqrt{4\pi(2\ell+1)}\, \mathsf{j}_\ell(k\,r)\, Y_\ell^0(\theta, \phi). \tag{10}$$

References

[DLMF] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.28 of 2020-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.