An Infinite Series About π

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Theorem 1.

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k+1}(2k+3)k!(k+1)!}$$

Proof. By Binomial Theorem,

$$\begin{aligned} \sqrt{1-z} &= (1-z)^{\frac{1}{2}} \\ &= 1 - \frac{1}{2}z - \frac{1}{2^2 \cdot 2!}z^2 - \frac{1 \cdot 3}{2^3 \cdot 3!}z^3 - \frac{1 \cdot 3 \cdot 5}{2^4 \cdot 4!}z^4 - \cdots \end{aligned}$$

Since

$$\begin{aligned} 1 \cdot 3 \cdot \ \cdots \ \cdot (2k-1) &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ \cdots \ \cdot (2k-1)(2k)}{2 \cdot 4 \cdot \ \cdots \ \cdot (2k)} \\ &= \frac{(2k)!}{2^k \cdot k!} \end{aligned}$$

we have

$$\begin{aligned} \sqrt{1-z} &= 1 - \frac{1}{2}z - \sum_{k=1}^{\infty} \frac{(2k)!}{2^k \cdot k!} \frac{1}{2^{k+1} \cdot (k+1)!} z^{k+1} \\ &= 1 - \frac{1}{2}z - \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k+1} \cdot k!(k+1)!} z^{k+1} \end{aligned}$$

Also

$$\frac{(2k)!}{2^{2k+1} \cdot k!(k+1)!} = \frac{1}{2}$$

when $k = 0$. So

$$\sqrt{1-z} = 1 - \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k+1} \cdot k!(k+1)!} z^{k+1} \tag{1}$$

In a Cartesian coordinate system (x, y), the equation $x^2 + y^2 = 1$ describes a circle with radius 1 and center $(0, 0)$. Let $\{x, y\} \subset [0, 1]$, then $y = \sqrt{1-x^2}$, and the equation describes a quarter of the circle in this range. The area of the quarter of this circle is $\pi/4$, thus

$$\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$$

Taking $z = x^2$ in equation (1), we get

$$\sqrt{1-x^2} = 1 - \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k+1} \cdot k!(k+1)!} x^{2k+2}$$

Therefore,

$$\begin{aligned}
\frac{\pi}{4} &= \int_0^1 \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k+1} \cdot k!(k+1)!} x^{2k+2}\,\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k+1} \cdot k!(k+1)!} \int_0^1 x^{2k+2}\,\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k+1} \cdot k!(k+1)!} \frac{1}{2k+3} \\
&= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k+1}(2k+3)k!(k+1)!}
\end{aligned}$$

$\square$

We can also use a *recursive* form for faster computation:

$$\frac{\pi}{4} = 1 - \sum_{k=0}^{\infty} \frac{A_k}{2k+3}$$

where

$$A_k = \begin{cases} 1/2 & \text{if } k = 0 \\ \dfrac{2k(2k-1)}{4k(k+1)} A_{k-1} & \text{if } k > 0 \end{cases}$$